Cases of the Agashe–Ribet–Stein Conjecture

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November 20, 2026

Abstract. We prove the Agashe–Ribet–Stein inequality for optimal elliptic curves over $\mathbb{Q}$ at every prime $p \geq 5$, at $p = 3$ when $v_3(N) \leq 4$, and at $p = 2$ when $v_2(N) \leq 3$, where N is the conductor.

hexagon:2611.00723v1

1 Introduction

Let $E/\mathbb{Q}$ be an elliptic curve of conductor N, and let $\pi : J_0(N) \to E$ be an optimal quotient, meaning that $\ker \pi$ is connected. The modular parametrization

$$\varphi : X_0(N) \longrightarrow E, \qquad P \longmapsto \pi([P] - [\infty]),$$

has degree m_E. Let f be the normalized weight-two newform attached to E. Its congruence number r_E is the largest positive integer r for which an integral Fourier cusp form g, Petersson orthogonal to f, satisfies $f \equiv g \pmod{r}$ coefficient by coefficient. Both invariants belong to the optimal quotient at its conductor level; this is the convention of [CK04, §1] and [ARS12, §2.1].

The comparison of these invariants arose from the study of modular parametrizations and congruence primes. Zagier [Zag85] related the degree of a parametrization to periods and the Petersson norm, and discussed the congruence interpretation of the projection onto an elliptic factor. The divisibility argument attributed there to Ribet was developed through integral Hecke algebras and their idempotents. Cojocaru and Kani [CK04, Theorem 1.1 and §2] gave a proof of

$$m_E \mid r_E.$$

Their argument compares the denominator of the same rational idempotent in the endomorphism ring of $J_0(N)$ and in the integral Hecke algebra. At prime level, the corresponding equality of these denominators also gives equality of the two invariants.

A second approach compares integral differential lattices. Abbes and Ullmo [AU96, §§1,3] studied congruences in the Fourier lattice and in the cotangent space of the Néron model of $J_0(N)$. Their comparison proved that the Manin constant is a unit at every prime of good reduction and established the Manin conjecture for odd squarefree conductor. Edixhoven's work on minimal and stable models [Edi90], and subsequently on integral structures in weight two [Edi06], made the geometry of the special fibre central to these questions. In particular, Fourier integrality at ∞ can be expressed as a condition along one vertical component, whereas the Néron lattice imposes a global integrality condition. This distinction is especially visible at the small primes, where wild stabilizers and weight-one forms affect the comparison.

Frey and Müller asked whether $m_E = r_E$ in general [FM99, Question 4.4]. Agashe, Ribet and Stein answered this negatively and proposed a conductor bound for the discrepancy [ARS12, §2.1]. Their examples include an optimal curve of conductor 54 with $(m_E, r_E) = (2, 6)$, and

one of conductor 242 with $(m_E, r_E) = (16, 176)$. They proved that $v_p(m_E) = v_p(r_E)$ whenever $v_p(N) \leq 1$, and related the discrepancy to failures of multiplicity one in the modular Jacobian [ARS12, Theorem 2.1 and Proposition 2.3]. Their proposed bound is the following, with $v_p(p) = 1$.

Conjecture 1.1 (Agashe–Ribet–Stein)**.** *For every optimal elliptic curve $E/\mathbb{Q}$ of conductor N and every prime p,*

$$v_p(r_E/m_E) \leq \left\lfloor \frac{v_p(N)}{2} \right\rfloor. \tag{1.1}$$

This is [ARS12, Conjecture 2.2], with an equivalent integer right side. The examples just given attain this bound at 3 and 11, respectively. The result of this paper is:

Theorem 1.2. *Let $E/\mathbb{Q}$ be an optimal elliptic curve of conductor N, with modular degree m_E and congruence number r_E. Inequality* (1.1) *holds in each of the following cases:*

(i) $p \geq 5$*;*

(ii) $p = 3$ *and* $v_3(N) \leq 4$*;*

(iii) $p = 2$ *and* $v_2(N) \leq 3$*.*

The prime-to-p part of N is arbitrary in each case. Throughout the paper $d = v_p(N)$ is the conductor exponent, and $\delta = v_p(\Delta_{\min}(E))$ is the valuation of the minimal discriminant. For $p \geq 5$, one has $d \leq 2$; the depth-two argument therefore completes all such primes.

Corollary 1.3. *If $16 \nmid N$ and $243 \nmid N$, then Conjecture 1.1 holds for E at every prime. Equivalently,*

$$\frac{r_E}{m_E} \mid \prod_{p \mid N} p^{\lfloor v_p(N)/2 \rfloor}.$$

Proof. The two hypotheses place the small primes in the ranges of Theorem 1.2. Its first assertion covers every other prime. Since $m_E \mid r_E$, the local inequalities are equivalent to the displayed divisibility. □

To describe the differential comparison used in the proof, put $R = \mathbb{Z}_p$ and let $S \subset S_2(\Gamma_0(N), \mathbb{Q}_p)$ be the full R-lattice defined by integral Fourier coefficients at ∞. We identify $g \in S$ with $g(q)\, dq/q$. Let $i : E \hookrightarrow J_0(N)$ be the polarized dual of π, and let ω_E be a Néron differential. The Manin constant c_E is defined by $\pi^* \omega_E = c_E f(q)\, dq/q$. Projection onto the line of f, together with $\pi i = [m_E]$, gives

$$i^* S = \frac{m_E}{c_E r_E} R\omega_E. \tag{1.2}$$

We prove this identity in Lemma 2.1. It makes (1.1) equivalent to

$$p^{\lfloor d/2 \rfloor} c_E i^* S \subseteq R\omega_E.$$

Thus the relevant denominator is that of the projection of the entire Fourier lattice onto the original elliptic direction.

Work on the Manin constant provides related integral comparisons. Agashe, Ribet and Stein [ARS06] established several bounds and integrality results for the constant. Česnavičius

[Ces18b] proved the Manin conjecture in the semistable case. Česnavičius, Neururer and Saha [CNS24, Theorem 5.15 and Proposition 6.2] proved canonical integrality of normalized newform differentials and compared canonical and Néron lattices through rational singularities. These results supply integral-model tools for the present argument. Recent work also treats adjacent comparisons: Hamidi [Ham26, Theorems A and B] bounds localized congruence–modular exponent ratios by a multiplicity-one defect; Jeon and Kwon [JK26] study degree divisibility and old homomorphism lattices at higher levels; and the preprint of Chi, Liu and Nie [CLN26, Theorem 1.1] states the general Manin-constant theorem. Here c_E is retained in (1.2), and the proof constructs the required inclusion for every element of S.

The local arguments begin with global old differentials. An old differential has zero pullback by i, so it can be subtracted from a given element of the Fourier lattice. At odd conductor depth two, intersection theory bounds its vertical poles on a fine cyclic model. Divided pullback from depth one identifies the whole polar sheaf. A Rosenlicht differential with the prescribed restrictions then lifts to a global old differential. For $p \geq 5$, retaining all components of a full level-three marking gives a trace of degree 24, which is a unit at p. At 3, the descent instead uses residues and the nodal special fibre of the coarse depth-one model.

At ternary conductor depth three, the possible discriminant valuations are $3, 5, 9, 11$. Canonical orders and the original good-reduction scaling treat the latter two. For the first two, explicit normalizations locate all positive-genus components of the supersingular domains. Proper de Rham pairings compute their contributions. The calculation at $\delta = 3$ includes the fractional deformation of the cubic centers, its full polar trace, and the common third coefficient of the target translation. Global Cartier and Hecke relations, followed by the oriented Igusa Artin–Schreier class and its relative boundary, annihilate the resulting principal parts. At ternary depth four, the proof identifies the components supporting the elliptic quotient and evaluates the remaining trace by a global meromorphic primitive on the Igusa quotient.

At two, the depth-two construction lifts the two forms arising from weight-one doubling and uses their stack canonical orders to descend an old differential to the coarse model. The depth-three argument combines the four-branch local model with an explicit integral comparison at level 40. A residue-matched old lift controls the poles, and the cotangent pairing with the local unipotent Picard groups supplies the last factor of two. All these constructions use the fixed quotient E, its Néron differential, and the full lattice S.

Proof of Theorem 1.2. The cases $v_p(N) \leq 1$ are [ARS12, Theorem 2.1]. Theorem 3.4 treats all the remaining primes $p \geq 5$. At three, Theorems 3.6, 4.1, and 5.1 treat conductor depths two, three, and four, respectively. At two, Theorems 6.1 and 7.1 treat depths two and three. Their lattice conclusions imply (1.1) by Lemma 2.1 and (2.4). □

2 Congruence numbers and differential lattices

We first relate the original congruence number to the pullback of the entire Fourier-integral lattice. All the local arguments in this paper will be applied to that lattice.

2.1 The original lattice

Let $f = \sum_{n\geq 1} a_n q^n$ be the normalized newform associated with the optimal elliptic quotient $\pi : J_0(N) \to E$. Put

$$S_{\mathbb{Z}} = \left\{ \sum_{n\geq 1} b_n q^n \in S_2(\Gamma_0(N), \mathbb{Q}) : b_n \in \mathbb{Z} \text{ for every } n \right\}.$$

We identify a cusp form g with the differential $g(q)\,dq/q$. The Petersson orthogonal complement of $\mathbb{Q}f$ is defined over $\mathbb{Q}$; write it as W. In the notation of [ARS12, §2.1],

$$r_E = \#(S_{\mathbb{Z}}/(\mathbb{Z}f + (S_{\mathbb{Z}} \cap W))). \tag{2.1}$$

The quotient is cyclic. This will also follow from the proof below.

Fix a prime p, and initially let $R = \mathbb{Z}_p$ and $K = \mathbb{Q}_p$. Write $S = S_{\mathbb{Z}} \otimes_{\mathbb{Z}} R$. Equivalently, S is the lattice of rational weight-two cusp forms over K whose expansion at the original cusp ∞ belongs to $R[[q]]$. A finite unramified extension of R, or the completed strict henselization when local coordinates are needed, may be used throughout: all lattice inclusions under consideration descend by faithful flatness.

Let $i : E \hookrightarrow J_0(N)$ be the dual of π, using the canonical principal polarizations. Let ω_E be a Néron differential, and define the Manin constant by

$$\pi^*\omega_E = c_E f(q)\frac{dq}{q}. \tag{2.2}$$

Changing the sign of ω_E changes the sign of c_E, which is irrelevant to every valuation below. We use the known integrality $c_E \in \mathbb{Z} \setminus \{0\}$; see [CNS24, §1].

Lemma 2.1. *With the preceding definitions,*

$$i^*S = \frac{m_E}{c_E r_E}\, R\omega_E. \tag{2.3}$$

Proof. Let $\varphi : X_0(N) \to E$ be the modular parametrization obtained from π by sending the cusp ∞ to the origin. Its norm and Picard pullback are π and i, respectively. The norm–pullback identity for a finite map of curves gives $\pi i = [\deg \varphi] = [m_E]$.

The rational endomorphism

$$e = \frac{1}{m_E} i\pi \in \mathrm{End}(J_0(N)) \otimes \mathbb{Q}$$

is an idempotent fixed by the Rosati involution. On complex holomorphic differentials, the polarization pairing is the Petersson Hermitian pairing. Thus e^* is the orthogonal projection onto the line $\mathbb{Q}f$, and $\ker(i^*) = W$. Both statements are rational and therefore remain valid over K.

Write $g = \alpha(g)f + w$, where $w \in W \otimes K$. Since the first Fourier coefficient of f is 1, $S \cap Kf = Rf$. Projection consequently induces an isomorphism

$$S/(Rf + (S \cap (W \otimes K))) \;\simeq\; \alpha(S)/R.$$

The image $\alpha(S)$ is a rank-one fractional ideal of R containing R, so the right side is cyclic. Its length is $v_p(r_E)$ by (2.1); hence $\alpha(S) = r_E^{-1}R$. Finally (2.2) and $\pi i = [m_E]$ give $i^*f = (m_E/c_E)\omega_E$. Applying this to the decomposition of g proves (2.3). □

In particular, for a nonnegative integer a, the ARS bound $v_p(r_E/m_E) \le a$ is equivalent to

$$p^a c_E i^* S \subseteq R\omega_E. \tag{2.4}$$

In several cases we prove the stronger assertion

$$p^a i^* S \subseteq R\omega_E, \tag{2.5}$$

which implies

$$v_p(c_E r_E/m_E) \le a, \qquad v_p(r_E/m_E) \le a. \tag{2.6}$$

The first inequality is stronger than the required ARS inequality. This distinction keeps the Manin constant in the comparison.

2.2 Canonical and Néron lattices

For a proper normal flat curve $\mathcal{X}/R$ with smooth generic fibre X, let $\omega_{\mathcal{X}/R}$ denote its relative dualizing sheaf. When $\mathcal{X}$ is regular, this is invertible. For a normal model it is understood as the rank-one reflexive dualizing sheaf. Let $\mathcal{J}$ be the Néron model of $\mathrm{Jac}(X)$, and write

$$\mathcal{H}_X = H^0(\mathcal{J}, \Omega^1_{\mathcal{J}/R})^{\mathrm{inv}} \subset H^0(X, \Omega^1_{X/K})$$

for its invariant differential lattice.

We shall use the following precise comparison from [CNS24, Proposition 6.2]. If R is an excellent discrete valuation ring, $\mathcal{X}/R$ is proper, normal and flat with geometrically irreducible smooth generic fibre, and its special fibre has a smooth point, then

$$\mathcal{H}_X \subseteq H^0(\mathcal{X}, \omega_{\mathcal{X}/R}). \tag{2.7}$$

Equality holds if and only if $\mathcal{X}$ has rational singularities; in this case its relative Picard identity component is the Néron identity component. In particular equality holds for a regular model. The smooth-point hypothesis will always be provided by a smooth outer cusp or a smooth point of a multiplicity-one outer component.

Lemma 2.2. *Let $g : Y \to X$ be a finite morphism of smooth proper curves over K. Canonical trace maps $\mathcal{H}_Y$ into $\mathcal{H}_X$. Also $i^*\mathcal{H}_{X_0(N)} \subseteq R\omega_E$.*

Proof. The Picard pullback $g^* : \mathrm{Jac}(X) \to \mathrm{Jac}(Y)$ extends to a homomorphism of Néron models. Its cotangent map is $\mathrm{Tr}_g : H^0(Y, \Omega^1) \to H^0(X, \Omega^1)$, by Serre duality: its tangent map is pullback on $H^1(\mathcal{O})$. Pullback of invariant differentials on Néron models preserves their integral lattices. This proves the trace assertion with the stated direction. The same argument applied to the homomorphism $i : E \to J_0(N)$ proves the second assertion. □

For the normal modular model, integrality of the q-expansion at ∞ is equivalent to nonnegative differential order on the irreducible component containing that cusp [Edi06, Proposition 2.5], for arbitrary N. After adding prime-to-p auxiliary level, every outer component with all-connected p-power subgroup maps to the original infinity component. Pullback is therefore integral on each of these components. This is a divisorial assertion on the original model, followed by pullback, and applies even when the auxiliary marking space is disconnected.

2.3 Old differentials and global lifts

For $p \mid N$, a differential pulled back from $X_0(N/p)$ by either degeneracy map belongs to the p-old subspace. Newform orthogonality and Lemma 2.1 give

$$i^*\eta_{\mathrm{old}} = 0. \tag{2.8}$$

Equation (2.8) holds for every element of the characteristic-zero old space, including rational scalar multiples. We construct the needed old differentials by globally lifting their prescribed special-fibre restrictions.

Our convention for the operator V_ℓ is

$$(V_\ell g)(q) = g(q^\ell).$$

The degeneracy map $\tau \mapsto \ell\tau$ pulls back the *differential* $g(q)\,dq/q$ to $\ell(V_\ell g)(q)\,dq/q$. These factors are retained explicitly when a pullback is divided by p.

Lemma 2.3. *Suppose $\mathcal{Z}/R$ is a proper flat normal curve whose special fibre is a reduced, geometrically connected nodal curve. Assume $H^0(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}}) = R$. Every Rosenlicht differential on $\mathcal{Z}_k$ lifts to $H^0(\mathcal{Z}, \omega_{\mathcal{Z}/R})$. In particular, for two smooth components meeting at nodes, a pair of meromorphic differentials with at most simple poles at the nodes and opposite residues there has such a lift.*

Proof. The special fibre is proper, geometrically connected and reduced, so $H^0(\mathcal{Z}_k, \mathcal{O}) = k$. The exact sequence for multiplication by a uniformizer on $\mathcal{O}_{\mathcal{Z}}$ then shows that $H^1(\mathcal{Z}, \mathcal{O})$ has no torsion. Relative duality gives $H^1(\mathcal{Z}, \omega_{\mathcal{Z}/R}) = R$. In the reduction sequence for the dualizing sheaf, the obstruction to lifting lies in the uniformizer torsion of this last module, which is zero.

For a nodal fibre, its dualizing sheaf is the sheaf of Rosenlicht differentials on the normalization: at a node the two simple-pole residues must sum to zero. Finally, prescribed simple-pole residues on a smooth proper component are realizable exactly when their sum, with residue-field traces, is zero. This follows from the residue sequence and Serre duality, whose last map is the sum of residues. It supplies the pairs asserted in the lemma. □

2.4 Trace after a good-reduction extension

We record a useful normalization for the small-prime arguments. Suppose E has potentially good reduction, and put $\delta = v_p(\Delta_{\min})$. Over a finite extension L/K where E has good reduction, choose a minimal good differential ω_{good}. With the valuation extended by $v_p(p) = 1$,

$$\omega_{\mathrm{good}} = a\omega_E, \qquad v_p(a) = \delta/12. \tag{2.9}$$

Indeed an admissible change of Weierstrass coordinates scales the discriminant by the twelfth power and the invariant differential by the inverse first power.

For the original parametrization $\varphi : X_0(N) \to E$, the map i^* is its canonical differential trace. After a semistable extension and regular resolution, the coefficient of that trace on the good target is computed from the nonconstant specialization maps of components. Components contracted by φ have no contribution at the generic point of the target special fibre. A source bound $v_p(\eta) \geq r$, relative to the canonical lattice at all nonconstant components, consequently gives

$$i^*\eta = b\omega_E, \qquad v_p(b) \geq r + \delta/12. \tag{2.10}$$

To see that no ramification factor is suppressed, multiply η by an element of L that clears its canonical order at the relevant generic points. Finite duality maps the resulting integral canonical lattice into the integral canonical lattice of the good target. Divide by that same element and then use (2.9). This is a codimension-one calculation, so the proper map may be replaced by its finite restriction near the generic target point. The coefficient b belongs to the original unramified field K. Thus $v_p(b) > -1$ implies $b \in R$.

We shall also use finite duality in positive characteristic, where the canonical trace need not be the field trace on coefficients. If $h : C \to D$ is finite between smooth proper curves, $P \in C$, $Q = h(P)$, and e_P is the local degree, then projection formula gives the pole estimate

$$\mathrm{Tr}_h(\omega_C(nP)) \subseteq \omega_D(\lceil n/e_P \rceil Q) \tag{2.11}$$

locally at Q, with the largest of these bounds when several points lie above Q. In fact $nP \leq h^*(\lceil n/e_P \rceil Q)$, and ordinary canonical trace preserves regular differentials. This proof applies also to inseparable maps. On a smooth proper curve over an algebraically closed field, a differential with only one possible simple pole is regular: the residue theorem makes its sole residue zero, and the coefficient of a simple pole is that residue.

3 Odd primes at depth two

This section constructs a global old differential that removes the polar part of pS. The construction on a fine model works for every odd p. At $p \geq 5$ it descends by a trace of unit degree. The separate descent needed at $p = 3$ is treated afterwards.

3.1 The fine cyclic tower

Write $N = p^2M$, with $p \nmid M$, and choose a prime-to-p auxiliary level that makes the moduli problem fine. After an unramified extension, work on one geometrically connected component of the resulting proper tame curve B/R. Let $\mathcal{L}$ be its Hodge bundle, let $w = \deg(\mathcal{L}_k) > 0$, and let $\kappa > 0$ be the degree of its reduced cusp divisor. The Kodaira–Spencer and Hasse identities give

$$2g(B_k) - 2 = 2w - \kappa, \qquad \deg(\mathrm{SS}) = (p-1)w. \tag{3.1}$$

Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be the normalized compact fine models obtained by adding cyclic subgroups of order p and p^2, respectively. We use the regular normalized models, including their normalized cusp charts. The fine cyclic model is regular by [KM85, Theorem 6.6.1]; its compactification is covered by [Ces17, Theorem 5.13]. For a representable tame moduli problem finite étale over the moduli stack of elliptic curves in characteristic p, the supersingular crossing equation at depth n is

$$(x - y^{p^n})(x^{p^n} - y) \prod_{\substack{a+b=n \\ a,b \geq 1}} (x^{p^{a-1}} - y^{p^{b-1}})^{p-1} = 0; \tag{3.2}$$

see [KM85, Theorem 13.4.7]. The chosen auxiliary level makes our moduli problem representable, and its prime-to-p subgroup and basis markings are finite étale on the elliptic locus. This theorem therefore applies at every odd prime. At depths one and two it gives

$$(\mathcal{X}_1)_k = B_{\mathrm{et}} \cup B_{\mathrm{conn}}, \qquad (\mathcal{X}_2)_k = C_0 + (p-1)C_1 + C_2. \tag{3.3}$$

The first fibre is nodal. The reduced components in the second fibre are smooth copies or Frobenius twists of B_k. The component C_2 contains the all-connected subgroup and lies over the original infinity component.

Here is the particular supersingular local information that will be used. The completed fine local ring has the form $R[[x,y]]/(F)$, with $F = F_0 + pF_1$, F_1 a unit, and

$$\overline{F}_0 = (x^{p^2} - y)(x - y^{p^2})(x - y)^{p-1}.$$

The reduced tangent lines are $y = 0$, $x = 0$, and $x - y = 0$. Thus the three reduced branches meet pairwise transversely at each supersingular point. There are no other intersections. The total space is regular; its maximal ideal is generated by x, y. The local equation just displayed is an equation modulo p, not an assertion that the mixed-characteristic equation is its product of tangent lines.

Let $\beta : \mathcal{X}_2 \to \mathcal{X}_1$ delete the first isogeny in the cyclic chain. Its generic degree is p, and its component data are

component	image	vertical index	residual degree	canonical pullback order
C_0	B_{et}	1	1	0
C_1	B_{et}	$p-1$	1	$p-2$
C_2	B_{conn}	1	p	1.

(3.4)

The residual map in the last row is Frobenius. The different formula of [CNS24, §5.9 and Proposition 5.12], with source and target exchanged, gives orders $0, p-2, 2$ for the full target projection at depth two. Its factorization through β subtracts the pullback of the lower different, of orders $0, 0, 1$, giving the last column of (3.4). The different tower formula used here includes inseparable residue extensions.

All component maps are nonconstant by the ordinary moduli description, so the proper morphism β is finite. Above the generic point of B_{et}, the degree formula reads

$$f_0 + (p-1)f_1 = p.$$

Both residual degrees are positive, hence $f_0 = f_1 = 1$. The two finite maps between smooth reduced components are therefore isomorphisms. We identify both C_0 and C_1 with the *same* B_{et} using these actual maps.

3.2 The integer pole bound

The intersection matrix on $\mathcal{X}_2$, divided by w, is

$$\begin{pmatrix} -p(p-1) & p-1 & p-1 \\ p-1 & -2 & p-1 \\ p-1 & p-1 & -p(p-1) \end{pmatrix}. \tag{3.5}$$

Indeed its off-diagonal entries follow from (3.1), and each row has zero product with the fibre multiplicity vector $(1, p-1, 1)$.

Lemma 3.1. *If η is a holomorphic differential on the generic fibre of $\mathcal{X}_2$, and $v_2 = \mathrm{ord}_{C_2}(\eta) \geq 0$, then*

$$\mathrm{ord}_{C_0}(\eta) \geq -2, \qquad \mathrm{ord}_{C_1}(\eta) \geq -p.$$

Consequently, for $Y = C_0 + C_1$ and $Z = C_2$,

$$p\eta \in H^0(\mathcal{X}_2, \omega_{\mathcal{X}_2/R}(Y - Z)). \tag{3.6}$$

Proof. Write $v_j = \mathrm{ord}_{C_j}(\eta)$ and $b = (2g(B_k) - 2)/w = 2 - \kappa/w < 2$. The horizontal part of the divisor of η is effective. Its intersection with C_0 and C_1 is nonnegative. Adjunction and (3.5) therefore give

$$-p(p-1)v_0 + (p-1)v_1 \le p(p-1) + b - (p-1)v_2, \tag{3.7}$$

$$(p-1)v_0 - 2v_1 \le 2 + b - (p-1)v_2. \tag{3.8}$$

The inverse of the coefficient matrix on the left has nonpositive entries. Multiplication gives

$$v_0 \ge -2 - \frac{b}{p-1} + v_2, \qquad v_1 \ge -p - b + (p-1)v_2.$$

Since $p \ge 3$, the first lower bound is strictly greater than -3; integrality gives $v_0 \ge -2$. Substitute this into (3.8). It yields

$$2v_1 \ge -2p - b + (p-1)v_2 > -2p - 2,$$

so $v_1 \ge -p$. Multiplication by p adds the fibre multiplicities $1, p-1, 1$ to the three orders. The resulting orders are at least $(-1,-1,1)$, proving (3.6). □

3.3 The polar sheaf and its global sections

Write $\omega_2 = \omega_{\mathcal{X}_2/R}$. Adjunction supplies the polar exact sequence

$$0 \longrightarrow \omega_2(-Z) \longrightarrow \omega_2(Y - Z) \longrightarrow \omega_Y(-Z) \longrightarrow 0. \tag{3.9}$$

Since $\beta^* B_{\mathrm{et}} = C_0 + (p-1)C_1$ contains Y, the restriction of β factors through a map $\phi : Y \to B_{\mathrm{et}}$. Its restrictions to both normalization components are the isomorphisms already fixed.

Lemma 3.2. *Divided pullback by β defines an isomorphism*

$$\phi^* \omega_{B_{\mathrm{et}}/k} \xrightarrow{\ \sim\ } \omega_Y(-Z). \tag{3.10}$$

Every global section on the right is the image of a unique $h \in H^0(B_{\mathrm{et}}, \omega_{B_{\mathrm{et}}/k})$.

Proof. Let h be a local regular differential on B_{et}. The pair $(h, 0)$ is a dualizing differential on the nodal special fibre of $\mathcal{X}_1$: its residues on both branches at every node are zero. Locally lift it to a section η_h of $\omega_{\mathcal{X}_1/R}$. The table (3.4), and the vanishing of η_h on B_{conn}, show that

$$\frac{\beta^* \eta_h}{p}$$

has orders at least $(-1,-1,1)$ on (C_0, C_1, C_2). It has no horizontal pole. Regularity of the total space and the codimension-one criterion for a section of an invertible sheaf extend this assertion over the supersingular closed points.

Changing η_h by $p\zeta$ changes the divided pullback by $\beta^*\zeta$, whose orders are at least $(0, p-2, 1)$. The change lies in $\omega_2(-Z)$; its polar class in (3.9) is zero. The polar class is therefore independent of the lift. Multiplication by a local function on B_{et} commutes with this construction: lift that function to the lower nodal model, and use the same independence calculation to remove the difference between two lifts. Thus the construction is $\mathcal{O}_{B_{\mathrm{et}}}$-linear and induces an actual sheaf morphism (3.10).

On each component the exact different orders in (3.4) show that the morphism is generically nonzero. The degree of ω_Y on that component is $2g(B_k) - 2 + \deg(\mathrm{SS})$, and the twist by $-Z$ subtracts $\deg(\mathrm{SS})$. Both sides of (3.10) consequently have degree $2g(B_k) - 2$. The restriction is an isomorphism on each smooth component. Its local multiplier is then a unit on each branch at a node, hence a unit in the nodal local ring. This proves the isomorphism on the whole of Y, including its gluing points.

The normalization sequence for the two copies of B_{et}, tensored with $\omega_{B_{\mathrm{et}}}$, identifies a global section of the left side with a pair (h_0, h_1) of regular differentials whose values agree along SS. Their difference belongs to $H^0(B_{\mathrm{et}}, \omega_{B_{\mathrm{et}}}(-\mathrm{SS}))$, which is zero because

$$\deg(\omega_{B_{\mathrm{et}}}(-\mathrm{SS})) = (3-p)w - \kappa < 0.$$

Hence $h_0 = h_1$. Conversely a single h pulls back to such a pair, proving the last assertion. □

Proposition 3.3. *For every generic holomorphic differential η integral on C_2, there is $\eta_h \in H^0(\mathcal{X}_1, \omega_{\mathcal{X}_1/R})$ such that*

$$p\eta - \frac{\beta^*\eta_h}{p} \in H^0(\mathcal{X}_2, \omega_{\mathcal{X}_2/R}). \tag{3.11}$$

Proof. Lemma 3.1 places $p\eta$ in the middle sheaf of (3.9). Lemma 3.2 identifies its polar restriction with a unique regular h on B_{et}. Lemma 2.3, applied to the reduced nodal depth-one fibre, lifts the pair $(h, 0)$ to a *global* canonical differential η_h on $\mathcal{X}_1$. By its construction in (3.10), $\beta^*\eta_h/p$ has exactly the same polar restriction as $p\eta$. Their difference belongs to $H^0(\omega_2(-Z))$, which is contained in $H^0(\omega_2)$. □

3.4 Descent and the primes at least five

Theorem 3.4. *If $p \geq 5$ and $v_p(N) = 2$, then*

$$pi^*S \subseteq R\omega_E, \qquad v_p(c_E r_E / m_E) \leq 1.$$

Consequently the Agashe–Ribet–Stein inequality holds for every prime $p \geq 5$.

Proof. Add a full level-three marking and retain every determinant component and every relative-position component of the marking space. The generic degree of the forgetful map is

$$|\mathrm{GL}_2(\mathbb{F}_3)|/2 = 24.$$

This remains the degree when $3 \mid M$: a generic pair (A, C_M) has automorphisms $\{\pm 1\}$, and its 48 ordered bases of $A[3]$ give 24 isomorphism classes. After an unramified extension, apply Proposition 3.3 on every connected component of the fine tower.

Let g_2, g_1 be the two marking maps, and $\widetilde{\beta}$ the fine degeneracy map. For $\eta \in S$, the divisorial interpretation of its original q_∞-integrality shows that $g_2^*\eta$ is integral on every all-connected outer component. Thus the proposition gives, componentwise,

$$pg_2^*\eta = h + \frac{\widetilde{\beta}^* h_1}{p}, \qquad h \in H^0(\mathcal{X}_2, \omega_2), \quad h_1 \in H^0(\mathcal{X}_1, \omega_1).$$

The fine models are regular, proper and have smooth outer cusps. Their canonical lattices are their Néron lattices by (2.7).

The generic square of marking and degeneracy maps is Cartesian. Indeed a level-three basis on a quotient by a p-isogeny pulls back uniquely since p is prime to 3. The Weil pairing is raised to the p-th power, which permutes the determinant components that we have retained. The cyclic M-subgroup is transported uniquely because $p \nmid M$. Finite trace base change therefore gives

$$\mathrm{Tr}_{g_2}\, \widetilde{\beta}^* = \beta^* \,\mathrm{Tr}_{g_1}\,.$$

Tracing the preceding equality and using Lemma 2.2, we obtain

$$24p\eta = h_0 + \frac{\beta^* \widetilde{h}_1}{p}, \qquad h_0 \in \mathcal{H}_{X_0(N)},$$

where $\widetilde{h}_1$ is a generic lower-level holomorphic differential. Since $24 \in R^\times$,

$$pS \subseteq \mathcal{H}_{X_0(N)} + V_{\mathrm{old}}.$$

Apply i^*. The Néron term is integral by Lemma 2.2, and the actual old term vanishes by (2.8). This proves the first inclusion; Lemma 2.1 proves the valuation bound.

For an elliptic curve over $\mathbb{Q}$, the conductor exponent at $p \geq 5$ is at most two. When it is zero or one, [ARS12, Theorem 2.1] gives $v_p(r_E/m_E) = 0$. When it is two, the preceding bound and the integrality of c_E give $v_p(r_E/m_E) \leq 1$. These are exactly all the required cases. □

3.5 Descent at three

The polar construction also proves the depth-two assertion at three, but its descent must be made on the original coarse curve.

Lemma 3.5. *For $3 \nmid M$, the reduced special fibre of the normal model $X_0(3M)_{\mathbb{Z}_3}$ is nodal. At a supersingular point its two branches are the normalizations of the two ordinary components.*

Proof. Work over a strictly henselian unramified base. The universal full-two depth-one curve $X(\Gamma(2) \cap \Gamma_0(3))$ is the integral split-Cartan conjugate of $X_0(12)$; the integral conjugacy is the one in [CNS24, Example 6.3]. Its generic genus is zero. The two reduced special components are projective lines, meeting at the single supersingular full-two point.

Let $H \subseteq S_3$ be the stabilizer of a chosen tame marking, and take the actual normal proper H-quotient. The action on either ordinary component function field is faithful. The associated vertical extension has ramification index one, so the two image components still have multiplicity one. A normal two-dimensional local ring is Cohen–Macaulay; its special fibre is therefore Cohen–Macaulay and, being generically reduced, reduced. Each normalized image component is $\mathbf{P}^1$. Away from their common supersingular point the quotient is smooth; cusp inertia here is tame and the normalized cusp charts are smooth.

The generic quotient again has genus zero. Constancy of Euler characteristic in a proper flat family gives $\chi(\mathcal{O}_{\mathrm{special}}) = 1$, whereas the normalization has Euler characteristic two. The normalization defect at the only possible singular point is thus one. A reduced two-branch curve singularity with normalization defect one is the node: its completed local ring consists of the pairs in $k[[u]] \oplus k[[v]]$ with equal constant terms.

Prime-to-three subgroup markings deform uniquely over the completed elliptic deformation. Hence the completed coarse ring with any cyclic M-marking is precisely the quotient by its stabilizer H just considered. This transfers the nodal assertion to $X_0(3M)$. The argument uses the actual normal quotient and does not commute wild invariants with reduction modulo three. □

Theorem 3.6. *If $v_3(N) = 2$, then*

$$3i^*S \subseteq R\omega_E, \qquad v_3(c_E r_E/m_E) \leq 1.$$

Proof. Use the full-two marking on the tame elliptic curve. The central automorphism -1 is ineffective on its deformation curve; it has order prime to three. One may first add full level four to obtain a representable fine problem, apply Lemmas 3.1–3.2, and descend through the effective kernel on the marking curves from full-four to full-two level, retaining both determinant components. Its order is eight: the matrix kernel has order sixteen, and its central subgroup $\{\pm 1\}$ acts ineffectively. This is a tame quotient and division by eight is permitted. In particular the polar construction is valid on the full-two deformation curve. Only canonical differentials and even-weight bundles are used there. The degree calculation can equivalently be written with $w = (2g(B) - 2 + \kappa)/2$, so it requires no descent of an odd-weight Hodge bundle.

For $g \in S$, the polar section of $3g$ gives a unique regular differential h on the full-two tame curve B. The construction is natural under permutations of the marking, hence h is invariant under the relevant marking group. It descends as a rational differential h_0 on the original coarse tame component. At tame points an invariant regular differential descends regularly. At a wild supersingular point put $u = \lambda + 1$, where λ is the Legendre parameter. Parameters for the quotients by C_3 and S_3, respectively, are

$$z = \frac{2u^3}{u^2 - 1}, \qquad j = \frac{u^6}{(u^2 - 1)^2}.$$

In characteristic three their valuations and derivative valuations are

$$(e, d_{\mathrm{diff}}) = (3, 4), \qquad (6, 7).$$

For example dz/du has order four and dj/du order seven at $u = 0$; these are the differents of the separable local function-field extensions. If h_0 has order r below such a point, the order of its pullback is $er + d_{\mathrm{diff}}$. Regularity above therefore implies $r \geq -1$ in both cases. Thus the only possible poles of h_0 are simple poles at wild supersingular points.

By Lemma 3.5, these points are nodes on the original lower-level fibre. The residues of h_0 sum to zero. Choose a differential h_1 on the other normalization component with the opposite residue at each node and no other poles. Lemma 2.3 supplies both this differential and an actual global canonical lift η_h of the pair (h_0, h_1) on $X_0(3M)_R$.

Set

$$\Theta = \frac{\beta^*\eta_h}{3}.$$

This is a characteristic-zero old differential at the original level. Pull it to the full-two model. On the two polar branches its polar class is the same diagonal h as that of $3g$. More explicitly, compare this lift with a fine lift of $(h, 0)$. Their difference reduces to $(0, h_1^{\mathrm{fine}})$, so vanishes on the lower etale branch. Its divided pullback has orders at least $(0, 1, 0)$. It is canonical and cannot change the polar class. Hence $3g - \Theta$ is canonical on the full-two model.

At each ordinary vertical generic point the fine-to-coarse map has vertical index one, separable residual extension and different zero. The preceding integrality therefore implies nonnegative order on every coarse vertical component. The generic differential is holomorphic, so there is no horizontal pole. The codimension-one criterion for the reflexive dualizing sheaf shows that

$$3g - \Theta \in H^0(X_0(9M)_R, \omega).$$

By [CNS24, Theorem 6.12(b)], this actual coarse model has rational singularities. Proposition 6.2 of the same paper identifies its canonical lattice with its Néron lattice. Apply Lemma 2.2 and (2.8) to obtain $3i^*g \in R\omega_E$. Lemma 2.1 gives the asserted valuation bound. □

4 The prime three at conductor depth three

Throughout this section $N = 27M$, where $3 \nmid M$. The elliptic curve, the quotient, its polarized dual, and the Néron differential are the original E, π, i, ω_E of Section 2. In particular c_E always denotes the Manin constant. A differential multiplier of any other map will be denoted by μ.

Theorem 4.1. *If $v_3(N) = 3$, then*

$$3c_E i^* S \subseteq \mathbb{Z}_3 \omega_E.$$

Consequently $v_3(r_E/m_E) \leq 1$. This includes all four possible minimal discriminant valuations

$$v_3(\Delta_{\min}) = 3, 5, 9, 11.$$

For the last two valuations the stronger inclusion $3i^*S \subseteq \mathbb{Z}_3\omega_E$ holds.

We construct an integral old correction and compute its remaining proper periods. Throughout these calculations the source is an arbitrary element of S, paired with the pullback of ω_E.

4.1 The marked modular model and the selected diagonal

We may extend $\mathbb{Z}_3$ unramifiedly to split the markings and write the resulting ring as R. When M is odd, use the intermediate auxiliary group

$$H_{27M} = \Gamma_0(27M) \cap \mathrm{red}_2^{-1}(A_3).$$

Its curve has degree two over $X_0(27M)$; adding full level two above it has degree three. When M is even, choose the component of the full-two marking through the original infinity cusp. Its group is $\Gamma_0(27M) \cap \Gamma(2)$, and its degree over the original curve is again two. There is then no stabilizer of order three. Additional full-four rigidification has degree a power of two and can be removed by trace. All such unit degrees are retained until the last step.

Here is the underlying elliptic moduli interpretation. Start with

$$\mathcal{E}_a : \quad v^2 = t^3 + at^2 + t, \qquad \mathcal{E}'_a = \mathcal{E}_a/\langle(0,0)\rangle : \quad v^2 = t(t-a-2)(t-a+2).$$

The modular object for the full-two marking is $\mathcal{E}'_a$. The cyclic M-subgroup stays on this curve, also when M is even. The selected halving subgroup on $\mathcal{E}_a$ has nonzero points with coordinate $t = 1$. In the coordinate above a supersingular point, the original and auxiliary Tate parameters satisfy $q_{\text{original}} = q_{\text{aux}}^2$.

If $r : X \to X_0(N)$ is this auxiliary map of unit degree h, set $i_X = r^*i$ and $\varphi_X = \varphi r$. Then

$$i_X^* r^* \eta = h\, i^* \eta.$$

We prove the claimed inclusion on pulled-back elements of S; the factor h is a unit and can be cancelled. Below i, φ, g denote these pulled-back objects when working on X. In particular the Fourier multiplier on the original cusp remains c_E, and both numerator and denominator acquire the same factor two in every ratio involving $d \log j$.

On the original complex modular curve put

$$\mathsf{U}_0 = \begin{pmatrix} 1 & 1/3 \\ 0 & 1 \end{pmatrix}, \qquad \mathsf{L}_0 = \begin{pmatrix} 1 & 0 \\ -9M & 1 \end{pmatrix}.$$

On the marked curves use their lifts

$$\mathsf{U} = \mathsf{U}_0^4 = \begin{pmatrix} 1 & 4/3 \\ 0 & 1 \end{pmatrix}, \qquad \mathsf{L} = \mathsf{L}_0^4 = \begin{pmatrix} 1 & 0 \\ -36M & 1 \end{pmatrix}.$$

Both matrices are the identity modulo four. Their cubes lie in $\Gamma_0(27M) \cap \Gamma(4)$, and their commutator is

$$\begin{pmatrix} 1 - 48M + 2304M^2 & 64M \\ 1728M^2 & 1 + 48M \end{pmatrix} \in \Gamma_0(27M) \cap \Gamma(4).$$

They therefore induce commuting automorphisms of order three on every auxiliary cover used here, and project to the original $\mathsf{U}_0, \mathsf{L}_0$ actions. Normalize them by

$$\mathsf{S} = \mathsf{L}^M, \qquad \mathsf{T} = \mathsf{U}.$$

They are the two coordinate deck transformations of the nonbacktracking chain of three isogenies. The coordinate norms annihilate the new elliptic embedding:

$$(1 + \mathsf{S} + \mathsf{S}^2)i = (1 + \mathsf{T} + \mathsf{T}^2)i = 0. \tag{4.1}$$

Indeed each norm is pullback followed by the actual degree-three old quotient, and the quotient of i is zero by (2.8).

There is one diagonal of this group that fixes i. Here are details that also fix the arithmetic normalization. Let χ be the quadratic character of $\mathbb{Q}(\sqrt{-3})$. Twisting E by χ preserves its conductor. Away from three the character is unramified. At three wild inertia acts nontrivially, since otherwise the conductor exponent of a potentially good two-dimensional representation would be at most two. Its determinant is trivial on inertia. A one-dimensional wild-invariant space would therefore make both characters trivial, so its wild-invariant space is zero. A tame quadratic twist changes neither that space nor the Swan conductor. This proves the conductor assertion.

Write $A = \mathsf{T} - \mathsf{T}^2$. Equation (4.1) gives $A^2 i = -3i$. The homomorphism Ai, composed with the twisting isomorphism $E^\chi \to E$, descends to $\mathbb{Q}$; its image is the newform factor of $f \otimes \chi$. The partial Fricke w_{27} acts on these two rational newform lines by signs λ, λ'. Thus, if $\epsilon_0 = \lambda\lambda'$,

$$2\mathsf{T}i = -i + Ai, \qquad 2\mathsf{S}i = -i + \epsilon_0 Ai.$$

The group $\mathrm{Hom}(E, J)$ has no torsion, so cancellation of two shows that $\mathsf{S}i = \mathsf{T}i$ or $\mathsf{T}^2 i$. One order-three diagonal fixes i, and the other has zero norm on it. This uses newform multiplicity one, as recalled in [ARS12, §3], together with the displayed normalizer identities.

Use the partial Fricke w_{27}, so that even M causes no problem with the auxiliary four-marking. A matrix

$$W_{27} = \begin{pmatrix} 27a & b \\ 27Mc & 27d \end{pmatrix}, \qquad 27ad - Mbc = 1$$

satisfies, modulo $\Gamma_0(27M)$,

$$W_{27}\mathsf{U}W_{27}^{-1} = \mathsf{L}^M, \qquad W_{27}\mathsf{L}W_{27}^{-1} = \mathsf{U}^M.$$

For example the lower-left entry after multiplying the first conjugate by L^{-n} is divisible by $27M$ exactly when $n = Mc^2 = M \pmod 3$; the second congruence follows from $n = Mb^2 = M \pmod 3$. Since $M^2 = 1 \pmod 3$, w_{27} exchanges S, T. These conjugation relations also hold on the four-marked curve: their differences lie in $\Gamma_0(27M)$, and reduction modulo four puts them in $\Gamma(4)$, since $\det W_{27} = 27$ is a unit modulo four. Choose the primitive cube root ζ_3 to be the multiplier of S on the left Kummer function u in (4.4); this action is verified below.

Call the selected diagonal *same* if it multiplies both Kummer coordinates by ζ_3, and *opposite* if it multiplies them by ζ_3, ζ_3^{-1}. Put $\sigma = 1, -1$, respectively. Denote it by $\mathsf{K} = \mathsf{ST}^\sigma$. The original map satisfies

$$\varphi\mathsf{K} = \varphi + P_K, \quad 3P_K = 0, \qquad \varphi w_{27} = \epsilon\varphi + P, \quad \epsilon \in \{1, -1\}. \tag{4.2}$$

The differences are constant because their differentials vanish; their values at infinity specify the displayed points. The cusp $w_{27}(\infty)$ has denominator M and is rational, so $P \in E(\mathbb{Q})$. The normalized conjugation gives $\epsilon P_K = \sigma P_K$. In particular $P_K \neq 0$ implies $\epsilon = \sigma$.

4.2 The local chain and its canonical module

The cyclic-three equation on $\mathcal{E}_a$, in reciprocal subgroup coordinate x, is

$$x^4 - 6x^2 - 4ax - 3 = 0.$$

Put $y = -3/x$. The quotient and normalized dual have parameters

$$a = \frac{x^3 - 6x + y}{4}, \qquad a' = \frac{y^3 - 6y + x}{4}, \qquad xy = -3.$$

These are the deformation coordinates of the cyclic isogeny, as follows either from the subgroup calculation in [Zhu14, Propositions 2.1, 2.3 and Corollary 3.2], or from the following normalized Vélu expression. For $r = 1/x$ and $F(t) = t^3 + at^2 + t$, set

$$\mathcal{V}_x(t) = t + \frac{2F'(r)}{t - r} + \frac{4F(r)}{(t - r)^2} - (-2r + 2/r).$$

The isogeny is

$$(t, v) \longmapsto (\mathcal{V}_x(t)/x^2,\ v\mathcal{V}_x'(t)/x^3).$$

Its differential multiplier is x, its dual multiplier is y, and their composition is $[-3]$. Substitution at $t = 1, -1$ gives $\mathcal{V}_x(1)/x^2 = 1$ and $\mathcal{V}_x(-1)/x^2 = -1$. Thus the chosen cyclic-four marking is transported without exchanging its two halving subgroups.

The completed fine depth-three ring is

$$\begin{aligned} B_0 &= R[[x, y]]/(xy + 3), \\ \mathcal{A} &= B_0[z, w]/(P_z, P_w), \\ P_z &= z^3 + xz^2 + (x^2 - 6)z - y, \\ P_w &= w^3 + yw^2 + (y^2 - 6)w - x. \end{aligned} \tag{4.3}$$

It is completed at $(3, x, y, z, w)$. Each monic cubic selects an end extension different from the reversing isogeny. Its rank over B_0 is nine. The linear terms in the equations for P_z, P_w eliminate y, x, and the maximal ideal is generated by z, w. The two-dimensional local ring is therefore regular. The generic chain interpretation identifies this normal ring with the actual fine modular normalization.

The following identities will be used repeatedly:

$$u = \frac{4-(x+z)^2}{1-x^2}, \qquad v = \frac{4-(y+w)^2}{1-y^2}, \qquad u^3 = 1-y^2, \qquad v^3 = 1-x^2, \tag{4.4}$$
$$z = \frac{(u-1)^2}{y}, \qquad w = \frac{(v-1)^2}{x}, \qquad P_z' = -xu^2(z-x), \qquad P_w' = -yv^2(w-y).$$

They are verified by replacing z^3, w^3, xy using (4.3); thus every denominator is specified before reduction. We fix the chain orientation explicitly. The original elliptic curve in the moduli problem is the right endpoint E_3; the three arrows $E_3 \to E_2 \to E_1 \to E_0$ have normalized multipliers $(-3/w, y, z)$. The first two deletions $\gamma(\tau) = 9\tau$ leave the left pair $(z, -3/z)$. The matrices satisfy

$$\beta \mathsf{T} \beta^{-1} = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} \in \Gamma_0(9M) \cap \Gamma(4), \qquad \mathsf{L} \in \Gamma_0(9M) \cap \Gamma(4), \qquad \beta(\tau) = 3\tau.$$

Thus the upper-triangular T fixes x, y, z and permutes the three right extensions w; the lower-triangular S fixes x, y, w and permutes the three left extensions z. Since each cubic has degree three, these actions are nontrivial. The quotient $\mathsf{S}(u)/u$ is a constant primitive cube root by $u^3 = 1 - y^2$ and $z = (u-1)^2/y$, and $\mathsf{S}(v) = v$. Reversal by w_{27} sends (x, y, z, w) to (y, x, w, z), hence exchanges u, v. This is reversal on the marked model: the displayed Vélu formulas preserve the two halving subgroups, and the reverse composition $[-3]$ is the identity on the four-marking. Conjugation therefore gives

$$\mathsf{S} : (u, v) \longmapsto (\zeta_3 u, v), \qquad \mathsf{T} : (u, v) \longmapsto (u, \zeta_3 v).$$

All subsequent choices of constants use this ζ_3.

With $k = z - x,\ l = w - y$, put

$$U_x = (1-x^2)(1-x^2-2xk), \quad U_y = (1-y^2)(1-y^2-2yl), \quad a = k + 4x, \quad b = l + 4y.$$

Then

$$yU_x = k^2 a, \qquad xU_y = l^2 b, \qquad -3 = \frac{k^2 l^2 ab}{U_x U_y}. \tag{4.5}$$

Modulo three, with $D_0 = 1 - k^2 l^2$,

$$x = \frac{l^2(l+k^3)}{D_0}, \quad y = \frac{k^2(k+l^3)}{D_0}, \quad a = \frac{k+l^3}{D_0}, \quad b = \frac{l+k^3}{D_0}.$$

The components are $C_0 = (b)$, $C_1 = (l)$, $C_2 = (k)$, $C_3 = (a)$, with infinity on C_3, and

$$F = \operatorname{div}(3) = C_0 + 2C_1 + 2C_2 + C_3.$$

The two adjacent contacts $C_0 \cap C_1, C_2 \cap C_3$ have length three at each supersingular point; every other contact has length one. This follows by setting respectively $l = 0, k = 0$ in the last two equations, and gives the full intersection matrix below.

The monic canonical generator is

$$\Omega_{\mathcal{A}} = \frac{dx/x}{P'_z P'_w}. \tag{4.6}$$

If $F_0 = k^2 l^2 (k + l^3)(l + k^3)$, its reduction is

$$\overline{\Omega}_{\mathcal{A}} = -D_0 \, \frac{dk}{(F_0)_l}.$$

To obtain the sign and unit, eliminate x, y in the two cubics. Their implicit determinant reduces to D_0, and the mixed equation $xy + 3$ reduces to F_0/D_0^2. Differentiating this equation in l, with k fixed, before restriction to a multiple component yields the formula.

4.3 The original Fourier lattice and a global old correction

Let w_B be the degree of the Hodge line on a fine tame component, and let κ_B be the degree of its cusp divisor. As in (3.1),

$$\deg \mathrm{SS} = 2w_B, \qquad 2g(B) - 2 = 2w_B - \kappa_B.$$

The contacts just computed, together with $F \cdot C_j = 0$, give, after deleting the infinity row and dividing by w_B,

$$\mathcal{M} = \begin{pmatrix} 18 & -6 & -2 \\ -6 & 6 & -2 \\ -2 & -2 & 6 \end{pmatrix}.$$

For a generic holomorphic differential integral at infinity let $-x_j$ be its order on C_j, $j < 3$. Adjunction and the effectiveness of its horizontal divisor give

$$\mathcal{M}(x_0, x_1, x_2)^{\mathsf{t}} \le (18, 6, 6)^{\mathsf{t}} + b_B (1, 1, 1)^{\mathsf{t}}, \qquad b_B = 2 - \kappa_B / w_B < 2.$$

The inverse matrix is nonnegative: if $\mathcal{M} z \ge 0$, choose a minimum of z_j / e_j, where $(e_0, e_1, e_2) = (1, 2, 2)$; the positive omitted infinity intersections rule out a negative minimum. Solving the first row of the inverse inequality gives $x_0 < 4$, hence $x_0 \le 3$. Inverting the remaining two rows with $x_0 \le 3$ gives $x_1 < 6$, hence $x_1 \le 5$. The last row then gives $x_2 < 4$, hence $x_2 \le 3$. Thus, for every g in the full Fourier lattice,

$$3g \in H^0(\omega(W - C_3)), \qquad W = 2C_0 + 3C_1 + C_2. \tag{4.7}$$

This argument applies to all g; no eigenform assumption occurs.

Here is the nonreduced polar calculation needed to turn this estimate into a global oldform. Put

$$A_0 = C_0 + 2C_1 + C_2, \quad Y = C_0 + C_1, \quad G = C_2 + C_3, \quad W = A_0 + Y, \quad F = A_0 + G, \quad Z = Y \cap G.$$

Write local equations a_0, y_0, g_0 for these divisors on the regular surface. Since y_0, g_0 have no common prime divisor,

$$(a_0 g_0) \cap (a_0 y_0) = (a_0 g_0 y_0), \qquad (a_0 y_0) : (a_0 g_0) = (y_0).$$

Consequently, for $\mathcal{L} = \omega(W - C_3)$ and $\mathcal{P} = \mathcal{L}|_W$,

$$\begin{aligned} 3\mathcal{O}_W &\simeq \mathcal{O}_Y(-F), & 0 &\longrightarrow \mathcal{O}_Z(-A_0) \longrightarrow \mathcal{O}_W/3\mathcal{O}_W \longrightarrow \mathcal{O}_{A_0} \longrightarrow 0, \\ 3\mathcal{P} &\simeq \omega_Y(-C_2 - 2C_3), & 0 &\longrightarrow \omega_Y(-C_3)|_Z \longrightarrow \mathcal{P}/3\mathcal{P} \longrightarrow \omega_{A_0}(Y - C_3) \longrightarrow 0. \end{aligned} \tag{4.8}$$

Also $9\mathcal{O}_W = 0$, and $\operatorname{length}(Z) = 8w_B$. In particular the embedded term in the second line cannot be deleted.

Let $\gamma : X_3 \to X_1$ delete the first two isogenies. The monic frames in (4.6), or composition of the two different formulas for one-step deletion, give

$$\operatorname{Diff}_{\gamma,\mathrm{vert}} = (0, 1, 3, 2), \qquad \gamma^*(\text{lower connected component}) = C_3.$$

For a lower Rosenlicht differential $(h, 0)$, lift it to κ_0 and form $\Theta_h = \gamma^*\kappa_0/3$. Its orders are at least $(-1, -1, 1, 2)$. Changing the lift by 3η changes this form by $\gamma^*\eta$, which has zero polar class on Y. It therefore induces an actual sheaf map

$$\gamma|_Y^*\omega_B \longrightarrow \omega_Y(-C_2 - 2C_3).$$

Both component maps $Y \to B$ are isomorphisms. The map is nonzero at their generic points, and on each branch its two line bundles have the same degree: the contact 3SS in adjunction is cancelled by SS + 2SS. Its determinant thus has no zero on either branch, including the contact subscheme. The map is an isomorphism on all of Y. Its global sections are diagonal, because the difference of the two restrictions lies in $H^0(B, \omega_B(-3\mathrm{SS})) = 0$. Every such h lifts globally by Lemma 2.3. Applying this construction to the polar class of $9g$ supplies

$$\zeta_{\mathrm{str}} = 9g - \gamma^*\kappa_0/3 \in H^0(\omega(-C_2 - 2C_3)).$$

Locally its reduction is $ka^2H_s\Omega_{\mathcal{A}}$.

At a wild marking point the fine marking action is

$$h(t) = \frac{t+3}{1-t} \quad (t = x, y, z, w).$$

On the lower one-step model its invariants are

$$n(x) = \frac{x(9-x^2)}{x^2-1}, \quad n(y) = \frac{y(9-y^2)}{y^2-1}, \qquad n(x)n(y) = -27.$$

Thus the actual coarse node is $R[[n(x), n(y)]]/(n(x)n(y) + 27)$, as also proved in Lemma 3.5. On its normalized branch, $n = t^3/(1-t^2)$, with different four. The invariant fine bottom differential therefore descends to a form h with at most simple supersingular poles and no cusp poles. Choose a form h' on the other lower component whose residues are the negatives at the paired nodes. It exists because the residues of h sum to zero. The pair (h, h') is a Rosenlicht differential. Its global lift κ defines

$$\zeta = 9g - \frac{\gamma^*\kappa}{3} = 9g - 3V_9\kappa, \qquad i^*\zeta = 9i^*g. \tag{4.9}$$

The factor $3V_9$ is the differential pullback $9V_9$ divided by three.

After pullback to the fine model, the difference from κ_0 reduces to $(0, h'_{\mathrm{fine}})$. A logarithmic coarse pole pulls back with order $-3 + 4 = 1$, so h'_{fine} is holomorphic. The divided pullback of

the difference has orders at least $(0,1,3,1)$. Hence ζ is canonical and has orders at least one on both C_2, C_3. Along every vertical prime the fine/coarse cover has ramification index one and separable residue extension of degree three; its different there is zero. It is free on the generic curve and on the cusp discs. Canonicality therefore descends at every height-one point and then by reflexivity to the normal coarse model. This descent uses no division by the order of its wild stabilizer.

The precise difference between the two corrections will matter:

$$\zeta = kaH\Omega_{\mathcal{A}} \pmod 3, \qquad H = aH_s + k^2 lJ. \tag{4.10}$$

Indeed their coefficient difference lies in $k^3 l\mathcal{A}$, and, in the reduced equation $F_0 = k^2 l^2 ab$, units being included in H_s, J,

$$(k^3 l, F_0) : (ka) = (k^2 l, kl^2 b).$$

For an explicit verification, cancel k; after subtracting a multiple of $kl^2 b$, the product $a(H - aH_s)$ is divisible by $k^2 l$. Since a is relatively prime to $k^2 l$, its coefficient is divisible by a, giving the first generator. The second generator annihilates ka modulo F_0. This proves (4.10) for the chosen correction.

We may first replace g by its three-depletion $g - V_3 U_3 g$. This is Fourier integral, its difference is an actual oldform, and it does not change $i^* g$. If α_j, β_j delete the last and first isogeny, respectively, the corrected source then also satisfies

$$\operatorname{Tr}_{\beta_3} \zeta = -\beta_2^* \kappa. \tag{4.11}$$

This follows by applying trace to (4.9): the degree-three trace of $\beta_3^* \beta_2^* \kappa/3$ is $\beta_2^* \kappa$, and the trace of the depleted g is zero.

4.4 The wild coarse order and its rational resolution

At a wild point introduce invariant functions

$$r_c = -yv/u, \quad s_c = -xu/v, \quad t_c = (y-x)/(uv), \quad d_c = s_c - r_c, \quad k_c = t_c - d_c, \quad q_c = k_c^2/3.$$

The subscript c is retained throughout for these coarse functions; R remains the coefficient ring, $K = \operatorname{Frac}(R)$, and S the full Fourier lattice. The identities in (4.4) give

$$r_c s_c = -3, \quad t_c^3 = d_c(d_c^2 - 9), \quad k_c^3 + 3d_c k_c^2 + 3d_c^2 k_c + 9d_c = 0.$$

The function q_c is integral: on each fine vertical branch $v(k_c) \geq 1$, whereas the multiplicity of three is at most two. The invariant order is

$$\mathcal{O} = B'\langle 1, k_c, q_c \rangle, \qquad B' = R[[r_c, s_c]]/(r_c s_c + 3), \tag{4.12}$$

with multiplication

$$\begin{aligned} &k_c^2 = 3q_c, && k_c q_c = -3d_c q_c - d_c^2 k_c - 3d_c, \\ &q_c^2 = 2d_c^2 q_c + (d_c^3 - d_c)k_c + 3d_c^2. \end{aligned} \tag{4.13}$$

These formulas prove closure and freeness over B'. They also prove normality. At a vertical generic point d_c is a unit, and the special equations are $k_c = 0$, $q_c(q_c + d_c^2) = 0$. At $q_c = 0$

the simple root gives a regular branch of multiplicity one. At $q_c = -d_c^2$, solving for q_c gives $3 = k_c^2/q_c$, a regular branch of multiplicity two. The characteristic-zero curve is smooth, as is seen from $t_c^3 = s_c^3 - r_c^3,\ r_c s_c = -3$. Thus the order is regular in codimension one and, being free over the Cohen–Macaulay ring B', satisfies S_2. It is normal. Its fraction field has degree three over that of B': the polynomial for k_c is Eisenstein at d_c after three is inverted. The actual degree-three marking action fixes it, so $\mathcal{O} = \mathcal{A}^{\langle h\rangle}$.

We compute its canonical lattice explicitly. Put

$$\mathfrak{b} = \frac{dr_c}{r_c t_c^2}, \qquad \mathfrak{e} = \frac{k_c}{3}\mathfrak{b}.$$

For $f(k_c) = k_c^3 + 3d_c k_c^2 + 3d_c^2 k_c + 9d_c$, one has $f' = 3t_c^2$. The monic trace-dual test, applied to $H = h_0 + h_1 k_c + h_2 k_c^2$, is

$$\operatorname{Tr}\!\left(\frac{H q_c}{f'}\right) = h_0/3 - d_c h_1 + 2d_c^2 h_2.$$

Thus the allowable numerators have $h_0 \in 3B'$, and

$$\omega_{\mathcal{O}} = \frac{(3, k_c, k_c^2)}{3t_c^2}\frac{dr_c}{r_c} = \mathcal{O}\mathfrak{b} + \mathcal{O}\mathfrak{e} = B'\langle \mathfrak{b}, \mathfrak{e}, q_c\mathfrak{b}\rangle. \tag{4.14}$$

For example $k_c\mathfrak{e} = q_c\mathfrak{b}$ and $q_c\mathfrak{e} = -d_c q_c\mathfrak{b} - d_c^2\mathfrak{e} - d_c\mathfrak{b}$; these identities verify the last equality without assuming that the ring is Gorenstein.

We include a resolution check because this canonical lattice must be the original Néron lattice. Blow up the maximal ideal of $\mathcal{O}$. Its exceptional support is $\mathbf{P}^1_{[r_c:s_c]}$. In the r_c-chart put $s_c = r_c(1+d), k_c = r_c k, q_c = r_c q$; then $3 = -r_c^2(1+d)$, and the strict transform has equations

$$\begin{aligned} k^2 + r_c(1+d)q &= 0,\\ kq + r_c d^2 k - r_c d(1+d) - r_c^2 d(1+d)q &= 0,\\ q^2 + dk - 2r_c d^2 q - r_c^2 d^3 k + r_c^2 d^2(1+d) &= 0. \end{aligned}$$

When d is a unit the third equation solves for k. Substitution in the other two leaves

$$q^3 - r_c d^2 q^2 + r_c^2 d^2\big((d+1)^2 + 1\big)q + r_c d^2(d+1) = 0. \tag{4.15}$$

If also $d+1$ is a unit, the coefficient of r_c is a unit, so $r_c = q^3$ times a unit and the central exceptional component has multiplicity six.

At $d = 0$ the tangent equations are

$$k^2 + r_c q, \quad kq - r_c d, \quad q^2 + dk.$$

Their projective curve is the twisted cubic $[r_c : k : q : d] = [s^3 : s^2 t : -st^2 : -t^3]$. Blowing up the point resolves it: on its r_c-chart the first and second equations solve the divided q, d with unit derivatives; on its d-chart they solve the divided k, r_c. The latter chart has $3 = \text{unit}\, d^2 q_1^6$. It supplies one multiplicity-two exceptional component. At $d = -1$, put $b = d + 1$ and blow up (r_c, b, q). On the r_c-chart, $b = r_c B, q = r_c Q$, the divided equation is

$$r_c Q^3 - r_c d^2 Q^2 + r_c d^2(b^2+1)Q + d^2 B = 0;$$

its B-derivative is a unit. Its fibre is a unit times $r_c^4 Q(Q+1)^2$. On the q-chart the equation becomes

$$q\{1 - rd^2 + r^2d^2(b^2+1)\} + rd^2b_1 = 0,$$

Near $r = 0$ the coefficient in braces is a unit, so $q = rb_1$ times a unit and $3 = \text{unit}\, r^5 b_1^4$. On the b-chart it is

$$bq_1^3 - brd^2q_1^2 + br^2d^2(b^2+1)q_1 + rd^2 = 0,$$

so $r = bq_1^3$ times a unit and $3 = \text{unit}\, b^5 q_1^6$. The locus with $r \neq 0$, including $r = -1$, is covered by the preceding r_c-chart. The s_c-chart gives the reflected calculation. These charts are regular and cover the exceptional locus.

The exceptional graph consequently has central component E_0, one arm F_0, and two chains L_1, L_2, R_1, R_2. Their multiplicities are $6; 2; 5, 4; 5, 4$. All are rational; their self-intersections are -2, except $F_0^2 = -3$. For the cycle

$$Z_0 = 3E_0 + F_0 + 2L_1 + L_2 + 2R_1 + R_2$$

the intersections are $(-1, 0, 0, 0, 0, 0)$. The addition order

$$E_0, F_0, L_1, L_2, R_1, R_2, E_0, L_1, R_1, E_0$$

builds Z_0 with every intersection of the preceding cycle with the added component at most one. In the exact sequence for adding a rational component C, the kernel is $\mathcal{O}_{\mathbf{P}^1}(-D \cdot C)$, whose H^1 vanishes when $D \cdot C \leq 1$. Induction proves $H^1(\mathcal{O}_{Z_0}) = 0$. Repeating the same addition order, using $Z_0 \cdot C \leq 0$, proves $H^1(\mathcal{O}_{nZ_0}) = 0$ for every n. The intersection form is negative definite (its successive principal minors have signs $-, +, -, +, -, +$ and absolute values $2, 5, 7, 9, 9, 9$). The cycles nZ_0 are cofinal among exceptional infinitesimal neighborhoods. Formal functions therefore gives $R^1 f_* \mathcal{O} = 0$. The singularity is rational. The other source charts are regular and the cusps are smooth. The canonical–Néron comparison (2.7) now gives equality on the actual global model.

The two branch orders of ζ give useful additional information. Write $\zeta = A\mathfrak{b} + B\mathfrak{e}$. The branches C_2, C_3 are respectively

$$(r_c, k_c, q_c) = (0, 0, -s_c^2), \qquad (r_c, k_c, q_c) = (0, 0, 0).$$

On C_2, $\mathfrak{b}$ has order one and $\mathfrak{e}$ has order zero, so $B(0, s_c, 0, -s_c^2) = 0$. On C_3 the multiplication table gives

$$k_c/3 = -d_c(q_c + 1)/(q_c + d_c^2), \qquad B(0, s_c, 0, 0) = s_c A(0, s_c, 0, 0).$$

The two evaluations of B differ by a multiple of s_c^2; hence $A(0) = B(0) = 0$. Thus

$$A, B \in \mathfrak{m}_{\mathcal{O}}. \tag{4.16}$$

In the B'-basis of (4.14), the same conditions give a unique expression

$$\zeta = a_\zeta r_c \mathfrak{b} + b_\zeta r_c \mathfrak{e} + c_\zeta q_c \mathfrak{b}, \qquad a_\zeta, b_\zeta, c_\zeta \in B'.$$

Indeed the middle branch makes the coefficient of $\mathfrak{e}$ divisible by r_c, and the outer branch then makes the coefficient of $\mathfrak{b}$ divisible by r_c. We next determine the relation between the two constant coefficients of this actual coarse section.

Differentiating $u^3 = 1 - y^2,\ v^3 = 1 - x^2$, with $xy = -3$, gives the exact identity

$$\mathfrak{b} = \frac{dx/x}{3uv} = -uvkl\,\Omega_{\mathcal{A}}.$$

On the fine outer branch $a = 0$, the reduced formulas are

$$k = -l^3, \quad x = l^3, \quad y = z = 0, \quad u = 1, \quad v = 1 - l^2, \quad s_c = -\frac{l^3}{1 - l^2}, \quad U_x = (1 - l^6)(1 + l^6).$$

Also $y/a = k^2/U_x$, and the multiplication table gives $k_c/3 = -1/s_c$ at its generic point. Consequently the first outer coefficients are

$$\frac{r_c\mathfrak{b}}{a\Omega_{\mathcal{A}}}\bigg|_{a=0} = -\frac{l^{10}(1 - l^2)^2}{U_x},$$
$$\frac{r_c\mathfrak{e}}{a\Omega_{\mathcal{A}}}\bigg|_{a=0} = \frac{q_c\mathfrak{b}}{a\Omega_{\mathcal{A}}}\bigg|_{a=0} = -\frac{l^7(1 - l^2)^3}{U_x}.$$

Here the second equality uses $q_c = 3(k_c/3)^2$ and $3/a = -xk^2/U_x$.

Equation (4.10), lifted to the fine regular ring and divided by $a\Omega_{\mathcal{A}}$, shows that the restriction of $\zeta/(a\Omega_{\mathcal{A}})$ to $a = 0$ is zero modulo l^9. The strong summand contains a; the weak summand k^3lJ has order at least ten; and the ambiguity from a multiple of three has order at least nine, since

$$(3/a)|_{a=0} = -l^9/U_x.$$

Every nonconstant B'-coefficient on this branch starts in order at least three because $r_c = 0$ and $s_c = -l^3/(1 - l^2)$. The coefficient of l^7 in the preceding display is therefore $-\overline{b_\zeta}(0) - \overline{c_\zeta}(0)$, and must vanish. Define

$$B_s = \overline{b_\zeta}(0) = -\overline{c_\zeta}(0). \tag{4.17}$$

The resulting constant coefficient contribution is

$$\overline{a_\zeta}(0)r_c\mathfrak{b} + B_s(r_c\mathfrak{e} - q_c\mathfrak{b}).$$

The $r_c\mathfrak{b}$ direction is removed by an actual old differential in the coarse period calculation below.

4.5 The two high discriminants

Let $\varpi_2^2 = -3$, $R' = R[\varpi_2]$, and $p_c = k_c/\varpi_2$. The normalization of the preceding coarse ring after this base change is

$$\mathcal{O}' = R'[[r_c, s_c, p_c]]/(r_cs_c - \varpi_2^2,\ p_c^3 - d_c^2p_c + \varpi_2 d_c(1 - p_c^2)). \tag{4.18}$$

It is finite and birational over the base-changed order. It is S_2; at each vertical generic point d_c is a unit and its reduced roots $0, d_c, -d_c$ are distinct, so it is regular there. Its generic curve is smooth. These checks prove normality. The derivative of the second equation with respect to p_c is

$$3p_c^2 - d_c^2 - 2\varpi_2 d_c p_c = -(d_c + \varpi_2 p_c)^2 = -t_c^2.$$

Thus $\mathfrak{b}$ generates its canonical module, and $\varpi_2\mathfrak{e} = -p_c\mathfrak{b}$.

We need the resolved canonical order here; the singularity in (4.18) is not rational. Blow up $(\varpi_2, r_c, s_c, p_c)$. The tangent equations $r_c s_c = \varpi_2^2,\ \varpi_2(s_c - r_c) = 0$ describe four ruling lines. In the p_c-chart put $r_c = p_c r, s_c = p_c s, \varpi_2 = p_c t$. The strict equations are

$$rs = t^2, \qquad p_c\{1 - (s-r)^2\} + t(s-r)(1-p_c^2) = 0.$$

The second solves p_c as a unit times $t(s-r)$. Thus ϖ_2 is a unit times $rs(s-r)$. Resolving $rs = t^2$, in the chart $s = rt_1^2, t = rt_1$, gives

$$\varpi_2 = \text{unit}\, r^3 t_1^2(t_1^2 - 1).$$

There is a multiplicity-three central curve and arms at $t_1 = 0, 1, -1, \infty$. In the first blowup's r_c-chart, $s_c = r_c t^2, p_c = r_c p, \varpi_2 = r_c t$, the equation is

$$r_c\{p^3 - (t^2-1)^2 p\} + t(t^2-1)(1 - r_c^2 p^2) = 0.$$

Its derivative at $r_c = 0, t = 0, \pm 1$ is a unit. At $t = 0$ it gives $\varpi_2 = \text{unit}\, r_c^2 p(p-1)(p+1)$. The symmetric chart gives the other end. Two arms have multiplicity two and meet the three corresponding outer branches; two have multiplicity one. The central curve has self-intersection -2, the arms have self-intersection -3, and the canonical discrepancy is

$$-(2C + A_1 + A_2 + A_3 + A_4).$$

This can also be read from adjunction: its intersections with the central curve and each arm are respectively zero and one, exactly those of the displayed discrepancy.

Set $\alpha = \varpi_2 \zeta = (\varpi_2 A - p_c B)\mathfrak{b}$. At the central divisor the valuations of $(r_c, s_c, \varpi_2, p_c)$ are $(3, 3, 3, 2)$; the original maximal ideal has minimum value three. By (4.16) the coefficient has value at least five there. At every arm it has value at least two. Subtracting the canonical discrepancy leaves orders at least two on the central component and at least one on each multiplicity-two arm. These are exactly the coefficients of $F - F_{\text{red}}$.

At a fine point the quadratic normalization is instead

$$r^2 = ab/(U_x U_y), \qquad \varpi_2 = klr, \qquad \Omega_{\text{new}} = kl\Omega_{\mathcal{A}}.$$

Equation (4.10) gives $\alpha = karH\Omega_{\text{new}}$. Resolving the ordinary double point gives exceptional multiplicity three and coefficient order at least three. At each end four reduced branches meet, so one more point blowup is required. In its l-chart $k = lK, r = lR$, one has $a = \text{unit}\, lR^2$ and $K \in (l, R)^2$. Thus kar has order at least eight; the canonical pullback adds one. In the k-chart, $l = kL, r = kR$, one has $a = \text{unit}\, k$, giving orders at least four and then five. Each new fibre multiplicity is $3 + 1 + 1 + 1 = 6$, so the required order five is obtained in both charts. The resulting reduced fibre is nodal.

We have proved on a regular proper model that $\bar{\alpha}$ belongs to $\omega_{F_{\text{red}}}$ inside ω_F. Its reduced Picard identity component is semiabelian. For $\delta = 9, 11$, the elliptic curve is still additive after $K(\varpi_2)/K$. Indeed its (-3)-twist has respectively discriminant valuation $3, 5$, and these become $6, 10 < 12$ after the extension, so those equations remain minimal and additive. The twist isomorphism gives

$$\omega_{E,K(\varpi_2)} = \varpi_2\, (\text{unit})\, \omega_E.$$

The map from the additive special elliptic identity component to the semiabelian quotient is zero: a homomorphism from $\mathbb{G}_a$ to a torus is zero, and one to an abelian variety is zero by properness and the extension from $\mathbf{A}^1$ to $\mathbf{P}^1$. Hence

$$i^*\alpha \in \varpi_2 R' \omega_{E,K(\varpi_2)}.$$

Write $i^*\zeta = a\omega_E$, with $a \in R$, using the canonical–Néron equality already proved. The last display says $a \in \varpi_2 R' \cap R = 3R$. Together with (4.9), this proves $3i^*S \subseteq R\omega_E$ for $\delta = 9, 11$.

For completeness the list of discriminants used here follows from the local algorithm and Ogg's conductor formula. At three a potentially multiplicative quadratic twist has conductor at most two. Types III, III* and I_0^* in odd characteristic have conductor two: their discriminant valuations are $3, 9, 6$, respectively. The remaining additive types II, IV, IV*, II* have respectively $1, 3, 7, 9$ geometric components, so $3 = \delta + 1 - \#\text{components}$ gives $3, 5, 9, 11$. The completed-square II and IV tests and the conductor formula are those of [Cre97, Chapter 3, pp. 62–68]. Quadratic twisting by -3 exchanges II with IV* and IV with II*, as is also checked by substituting the completed-square normal forms and removing the maximal powers of three.

4.6 Proper periods for the two low discriminants

Assume now $\delta = 3$ or 5. The correction ζ is in the original Néron lattice, so $i^*\zeta = a\omega_E$ with $a \in R$. If $3 \mid c_E$, then $c_E a \in 3R$, and (4.9) already proves the theorem. For the rest of the section $c_E \in \mathbb{Z}_3^\times$, with $\bar{c}_E \in \mathbb{F}_3^\times$. The residue assertion follows from $c_E \in \mathbb{Q}$. The coefficients of g may belong to the chosen unramified extension.

In a completed-square original minimal equation write

$$E: \quad Y^2 = X^3 + a_2X^2 + a_4X + a_6, \quad \omega_E = \frac{dX}{2Y}, \quad \eta_E = X\omega_E.$$

The proper de Rham representative of η_E has transition Y/X, since

$$\eta_E - d(Y/X) = (a_4/X + 2a_6/X^2)\omega_E.$$

Fix the cup orientation $\langle \omega_E, \eta_E \rangle = -1$. The coefficient of $c_E i^*\zeta$ is, up to this single fixed sign, $c_E\langle \zeta, \varphi^*\eta_E \rangle$. After a good-reduction extension write

$$X = s_E + u_E^2 X_g, \quad Y = u_E^3 Y_g, \quad \omega_g = u_E\omega_E, \quad \eta_E = u_E\eta_g + (s_E/u_E)\omega_g.$$

Here $\eta_g = X_g\omega_g$. We use the good-target formal parameter

$$\theta_E = -X_g/Y_g = -u_E(X - s_E)/Y, \qquad \theta_E(O) = 0, \tag{4.19}$$

and the same symbol for its reduction on the good special fibre. Global Hodge isotropy gives the exact identity

$$c_E\langle \zeta, \varphi^*\eta_E \rangle = c_E u_E \langle \zeta, \varphi^*\eta_g \rangle. \tag{4.20}$$

The translation s_E, the original target, and the original scalar have all been retained.

Translate the second-kind pole on the good elliptic curve away from the finitely many node images and images of constant components. This is a proper cohomological operation: if $P_* = (r, s)$ and $\lambda_* = (Y + s)/(X - r)$, the chord formulas give

$$X_\Delta = \lambda_*^2 - a_2 - X - r, \qquad d\lambda_* = (X - X_\Delta)\omega_E.$$

Thus the second-kind forms differ by $d\lambda_*$; their transitions differ by $\lambda_* + (Y_\Delta/X_\Delta) - (Y/X)$, which is regular at the two pole sections. The complete horizontal inverse divisor of the chosen good pole is finite flat. Relative residue there uses the canonical finite trace, including collisions and inseparable residue maps. In particular it is integral on integral canonical forms; there is no division by a geometric degree. This is the same-base pullback, trace, and cup duality of [Cai09, Propositions 4.5–4.6 and Theorem 5.11], applied to regular proper models, whose cohomological flatness is supplied by their smooth outer sections.

Every rational reduced component maps constantly to the good elliptic curve. Ordinary component normalizations are already defined over $K(\sqrt{-3})$: in (4.18) the ordinary roots $p_c = 0, \pm d_c$ are simple, and the fine multiplicities one and two become reduced over the same quadratic extension. Further inertia therefore fixes these ordinary function fields pointwise. It acts nontrivially on the good target, since good reduction over that quadratic field would make 2δ divisible by twelve, whereas $2\delta = 6, 10$. A nonidentity origin-preserving elliptic automorphism has finite fixed locus. Equivariance forces the good image of each connected ordinary component to be constant. Ordinary components will accordingly be treated outside, and separately from, the supersingular configurations below.

There is also an exact reason to discard the two *own* elliptic components of each supersingular configuration. For an actual old map $q : X \to Y$, newness says $\varphi_* q^* = 0$. On the abelian quotients of the special Picard identity components, the column associated with an old elliptic component is the sum of its actual component pullbacks. In the charts below only one source component dominates that old component, and its residue degree is three, equal to the generic degree of q. The valuation degree formula excludes any second component. Its column is therefore one isogeny, also when it is purely inseparable. Composition with φ_* is zero, so the good target map of the own component is constant.

4.7 All the positive-genus components

Fix $\iota^2 = -1$ in an unramified coefficient extension, and use ι also for its residue. Put

$$t^2 = 1 + 2\zeta_3, \quad t^4 = -3, \quad \alpha^3 = 2, \quad \Lambda = \alpha + 1, \quad L = \Lambda^2 - \Lambda + 1, \quad h_0 = \alpha^2, \quad \rho = \Lambda/t.$$

Then $\Lambda^3 = 3L,\ t = -\rho^3/L,\ \Lambda = -\rho^4/L$, and $v_\rho(3) = 12$. Good scaling on the original target has $v_\rho(u_E) = \delta$, even if a further good extension is needed.

First consider the coarse curve. Set

$$r_c = t^2 r, \quad s_c = t^2/r, \quad t_c = t^2 Y/r, \qquad Y^3 + r^6 = 1,$$

and

$$h = r^2 + Y,\ s = r^2 - Y,\ H = (h - h_0)/t^2,\ Z = s/t, \qquad U = H/h,\ V = Z/h.$$

Its degree-two quotient and covering equation are

$$V^2 = U^3 + t^2 U^2 - U, \qquad r^2 = \frac{h_0(1 + tV)}{2(1 - t^2 U)}. \tag{4.21}$$

For $z_C = -U/V$, the three poles of the covering function are $z_C = 0, \pm t$. Its three zeros are

$$z_j = (1 - h_0\zeta_3^j)/t, \quad V = -1/t, \quad U = (1 - h_0\zeta_3^j)/t^2.$$

The two triples have centers at distance of order ρ, and points in either triple are separated at order ρ^3. Blow up the base disc first at radius ρ, then at radius ρ^3 in each cluster. The base has

its good elliptic component, a central rational curve, and two rational leaves. The normalized double cover splits over the good elliptic component into two new elliptic curves $r = \pm\iota$. On the central curve its square class is a unit times $(w-c)/w$, so its normalization is rational. Each leaf is a double cover branched at its three marked points and its attaching node; its equation is $v^2 = p^3 - p$. These are the two own elliptic components. There are four central attachment nodes, all of ρ-thickness one. The three ordinary attachments to each own component have thickness three. The six displayed branch points exhaust the double-cover branch divisor.

For example the own r_c-chart is

$$r_c = t/v, \quad s_c = t^3 v, \quad k_c = t^3 p/v, \qquad \begin{aligned}\mathcal{F} = p^3 - (t^2v^2-1)^2 p + v^2(t^2v^2-1) \\ - t^2p^2(t^2v^2-1) = 0.\end{aligned} \tag{4.22}$$

Its reduction is $v^2 = p^3 - p$, and

$$\mathfrak{b} = t^{-2} v\, dv/\mathcal{F}_p, \quad \mathfrak{e} = -t^{-3} p\, dv/\mathcal{F}_p, \quad q_c\mathfrak{b} = -p^2 v^{-1} dv/\mathcal{F}_p, \qquad \mathcal{F}_p = -\{t^2(p+v^2)-1\}^2.$$

The action $t \mapsto \iota t$ is exactly $(t,p,v) \mapsto (\iota t, -p, \iota v)$, and fixes the original r_c, s_c, k_c.

We now construct the extra components on the fine curve. From (4.4),

$$r_c^3 = \frac{y(y^2-9)}{y^2-1}, \qquad \mathcal{K} = \frac{y-t^2}{y+t^2}, \qquad \mathcal{K}^3 = \frac{r^3+1}{r^3-1}.$$

This is the actual degree-three marking cover. Use the central parameter

$$q = \frac{r(h-h_0)}{\Lambda s - h + h_0}.$$

Writing $w_c = -(h-h_0)/(\Lambda s)$, the equation for its base is

$$-h_0^2 \Lambda w_c + h_0 \Lambda^2 s w_c^2 - L s^2 w_c^3 + h_0 s - \Lambda s^2 w_c = 0.$$

At $\Lambda = 0$, its relevant branch is $r = q^3$; its derivative in r is a nonzero unit times $q^6(q^2+1)^3$. With

$$\mathcal{K} = \frac{r+1}{r-1}(1 + \Lambda Z_c), \quad G(r) = \frac{-2r(r^2+1)}{(r+1)^2(r^2+r+1)}, \quad g(q) = \frac{q(q^2+1)}{(q^2-1)^2},$$

the complete equation is

$$L Z_c^3 + \Lambda^2 Z_c^2 + \Lambda Z_c = G(r).$$

Its first uncancelled residual after $Z_c = g(q)$ is $\Lambda/g(q)$. The differential of $1/g$ is

$$d(1/g) = \frac{(q^2-1)(q^4+1)}{q^2(q^2+1)^2}\, dq.$$

The six critical points are therefore

$$(q^2-1)(q^4+1) = 0.$$

To construct actual integral neighborhoods also at $q = \pm 1$, use the unit Kummer function $J = \mathcal{K}Y$, for which $J^3 = -(r^3+1)^2$; at $q = -1$ use $Y/\mathcal{K}$. The following approximation records

every mixed-characteristic carry needed:

$$j_0 = -(q^3+1)^2,$$
$$j_1 = \frac{q^{13}+q^{10}+q^9-q^8+q^7+q^5+q^4+q^3-q^2-1}{q(q^2+1)},$$
$$J_{\mathrm{app}} = j_0 + \Lambda j_1, \qquad \frac{-(r^3+1)^2}{J_{\mathrm{app}}^3} = 1 - \Lambda^4/g(q) + O(\Lambda^5).$$

These identities follow by substituting the base equation modulo Λ^5, using $\Lambda^3 = 3L$. On the disc at $q=-1$, make the explicit substitution

$$(r, q, \mathcal{K}, J) \longmapsto (-r, -q, \mathcal{K}^{-1}, Y/\mathcal{K});$$

its approximation is $j_0(-q) + \Lambda j_1(-q)$, whose reduction at that disc is a unit. The transformed cube is $-(r^3-1)^2$. In these respective charts the denominators are units and the zeros of the derivative are simple. Hensel lifting, then factoring $F(q) - F(a) = (q-a)^2 U(q)$ and taking a unit square root, gives an integral parameter T with

$$J^3/J_{\mathrm{app}}^3 = b^3 + \Lambda^4 T^2.$$

Set $\lambda = \zeta_3 - 1$, $T = \rho x$, and $J/J_{\mathrm{app}} = b(1+\lambda Z)$. The exact equation is

$$Z^3 + (3/\lambda)Z^2 + (3/\lambda^2)Z = C_0 x^2, \qquad C_0 = \Lambda^4\rho^2/(\lambda^3 b^3), \quad \bar{C}_0 = -1. \tag{4.23}$$

Its smooth proper reduction is $Z^3 - Z = -x^2$, of genus one. Adjoin the six required constants b and take a common finite extension. The equations above then give six disjoint actual genus-one affinoids; all valuations retain $v_\rho(3) = 12$.

There are four further, inherited, fine elliptic affinoids. On either own chart the preceding fourfold model has $t = a^3 z$ times a unit reducing to one, $z^2 = w^3 - w$. Its old map on this residue curve is

$$(w, z) \longmapsto \Big(w/z,\ (w^2+1)/(w^2-1) \Big);$$

the inverse is obtained from $V = w/z$ and $Z = (w^2+1)/(w^2-1)$ by $w = 2V^2/(Z-1)$ and $z = 2V/(Z-1)$. These identities follow from $V^2 = w/(w^2-1)$. After $a = \rho e$, its normalization has $e^3 z = -1$. Putting $V_f = -1/e$, $U_f = w - V_f^2$ gives

$$w = U_f^3, \quad z = V_f^3, \quad V_f^2 = U_f^3 - U_f.$$

The old map is the displayed isomorphism composed with Frobenius of degree three. This proves the single-column old-map assertion used above.

Over each new coarse component let $r_0 = \pm\iota$ denote its reduced value. In the equation for Z_c, put $Z_c = \rho T$. Its reduction is $T^3 = r_0 V$. With $U_f = U + T^2$, $V_f = r_0 T$, one obtains

$$U = U_f^3, \quad V = V_f^3, \quad V_f^2 = U_f^3 - U_f.$$

The Jacobian of these two equations has rank two: the first coarse equation has nonzero V-derivative, and the second has nonzero U-derivative. They are smooth normalized affinoids. Differentiating their actual equations gives

$$\frac{\mathrm{pullback}(\omega_{\mathrm{coarse}})}{\rho^2} = \omega_f \pmod{\rho}, \quad \rho^7 \mathfrak{b} = \omega_f, \quad \rho^{11}\Omega_{\mathcal{A}} = -\omega_f, \quad (k,l,a)/\rho^2 = (r_0, -r_0, r_0). \tag{4.24}$$

Here $G'(r_0) = 1$, $dr/\rho^3 = -r_0\omega_{\mathrm{coarse}}$, and $\Lambda/\rho^2 = -\rho^2 + \cdots$; these give the factor ρ^2. The same factor follows from the two nontrivial deck displacements of T, each of order one. It is unrelated to an incoming differential multiplier μ of a separate elliptic isogeny.

There are exactly ten positive-genus vertices in the fine supersingular domain: two own, two inherited new, and six central new. Here is a degree argument that also checks exhaustion. The original field is the connected degree-nine Kummer extension

$$x = -3/y, \quad u^3 = 1 - y^2, \quad v^3 = 1 - 9/y^2.$$

Valuations at $y = 1$ and $y = 3$ show independence of the two Kummer classes. The six branch points are $1, -1, 3, -3, 0, \infty$, each with inertia three. Riemann–Hurwitz gives

$$2g - 2 = -18 + 6 \cdot 3(3 - 1) = 18, \qquad g = 10.$$

Every constructed elliptic affinoid has $v(k), v(l) > 0$; they lie in the same strict supersingular domain and are pairwise disjoint. The semistable genus formula $g = \sum_v g_v + b_1$, and the fact that a positive-genus type-two point belongs to every semistable vertex set, [BPR13, Proposition 3.13 and Remarks 4.17–4.18], leave only rational additional vertices. The prime-to-three marking is étale on the strict local deformation and introduces no uncounted cover of these affinoids. The genus-ten calculation applies to this strict supersingular domain. The ordinary components were treated by inertia above.

4.8 The target equation and the discriminant-five periods

On the original additive identity domain put $w_E = X/Y$ and $z_E = 1/X$. If $a_i = 3A_i$, its smooth equation is

$$z_E = w_E^2(1 + a_2 z_E + a_4 z_E^2 + a_6 z_E^3).$$

It is a contraction in the coefficient ring for bounded w_E. Lagrange inversion supplies the integral convergent primitive

$$\mathcal{L}(w_E) = -\sum_{n \geq 0} \frac{[z^n](1 + a_2 z + a_4 z^2 + a_6 z^3)^n}{2n + 1} w_E^{2n+1}, \qquad d\mathcal{L} = \omega_E.$$

Every numerator term has valuation at least $\lceil n/3 \rceil$. For $n \geq 5$, $\lceil n/3 \rceil - v_3(2n+1) \geq 2$: when the latter valuation is $j \geq 3$, the smallest possible index is $(3^j - 1)/2$, and the inequality increases thereafter; $j = 0, 1, 2$ is immediate. At $n = 2, 3, 4$ the only possible exceptions are settled by

$$[z^4](1 + a_2 z + a_4 z^2 + a_6 z^3)^4 = a_2^4 + 12a_2^2 a_4 + 6a_4^2 + 12a_2 a_6.$$

The extra factors of three make their reduction zero. Consequently

$$\overline{\mathcal{L}} = -w_E - \bar{A}_2 w_E^3.$$

The Néron mapping property extends the original modular map on the smooth canonical ordinary component. Thus $W = w_E(\varphi)/c_E$ is a rational function on its proper tame normalization B, regular away from supersingular points, including the cusps. The primitive has its full three-depleted Fourier expansion. For $\theta = d \log j$, $D_\theta = d/\theta$, this proves

$$D_\theta^2 W = W + \bar{A}_2 \bar{c}_E^2 W^3. \tag{4.25}$$

Indeed $D_\theta^3 = D_\theta$, by its logarithmic definition or its full expansion $-q\,d/dq$, and the square of this operator fixes every exponent prime to three in the primitive. The two Weierstrass equations above have invariants

	c_4	Δ
$\mathcal{E}_a$	$16(a^2-3)$	$16(a^2-4)$
$\mathcal{E}'_a$	$16(a^2+12)$	$256(a^2-4)^2$.

Thus

$$j(\mathcal{E}_a) = \frac{256(a^2-3)^3}{a^2-4}, \qquad j(\mathcal{E}'_a) = \frac{16(a^2+12)^3}{(a^2-4)^2}.$$

The modular object is $\mathcal{E}'_a$. Reducing in its fine cyclic-four parameter $t_0 = a$ gives

$$j = j(\mathcal{E}'_a) = \frac{t_0^6}{(1-t_0^2)^2}, \qquad \theta = \frac{t_0}{1-t_0^2}\,dt_0. \tag{4.26}$$

If $\bar{A}_2 \neq 0$, a pole of W has order at most two: at a simple zero of θ, D_θ^2 increases a pole order by at most four, whereas W^3 triples it.

We record the complete target trace kernel. For a monic cubic P, $\mathrm{Tr}(z^j/P') = (0,0,1)$, $j = 0,1,2$, by the residue at infinity. Applying this to both cubics in (4.3), newness of $\beta = \varphi^*\omega_E$ gives exactly

$$\beta = (a + bz + cw + dzw)\Omega_{\mathcal{A}}, \qquad a,b,c,d \in B_0. \tag{4.27}$$

Here c is a coefficient function, not c_E. The two generic old squares with the unit-degree marking cover are Cartesian: odd-degree isogenies transport their two-torsion markings bijectively. Hence these are the actual global trace-zero conditions.

At the six central elliptic interiors, substitution in the Kummer equation gives

$$\frac{x}{\rho^6} = -q^{-3}, \quad \frac{y}{\rho^6} = -q^3, \quad \frac{k}{\rho^2} = -q, \quad \frac{l}{\rho^2} = -q^{-1}, \quad \rho^{11}\Omega_{\mathcal{A}} \text{ has order zero.}$$

Changing J/J_{app} by $1 + \lambda Z$ gives increments $\Delta u, \Delta v, \Delta k, \Delta l$ whose leading terms on the reduction are

$$\overline{\Delta u/\rho^6} = -q^3 Z, \quad \overline{\Delta v/\rho^6} = -q^{-3} Z, \quad \overline{\Delta k/\rho^4} = q^2 Z, \quad \overline{\Delta l/\rho^4} = q^{-2} Z.$$

Expand coefficients of (4.27) uniquely as a constant plus pure x- and y-series, using $xy = -3$. Good-target integrality gives $v_\rho(\beta) \geq -\delta$. The numerator orders $0, 2, 4$, with respective functions $1,\ b_0 q + c_0/q,\ d_0$, force $a_0, b_0, c_0, d_0 \in 3R$, since these functions are independent on the entire six-point critical scheme. For $\delta = 3$ the order-six functions $a_x q^{-3} + a_y q^3$ also force $a_x, a_y \in 3R$. A coefficient that vanishes modulo ρ is in $3R$, since it belongs to the original unramified ring; it gains twelve ρ-orders.

For $\delta = 5$, the completed-square type-IV normal form has $v_3(a_2) = 1,\ v_3(a_4) \geq 2,\ v_3(a_6) = 2$. Thus $\bar{A}_2 \neq 0$ in (4.25). On the canonical outer component

$$(k,l,x,y,z,w) = (-t_0^3, t_0, t_0^3, 0, 0, t_0), \qquad \Omega_{\mathcal{A}} = dt_0/t_0^9.$$

The unique order-six pole of β is $a_x t_0^{-6} dt_0$, whereas $\bar{\beta} = -\bar{c}_E dW$ has poles of order at most three. Therefore $a_x \in 3R$. The exact partial Fricke relation (4.2), including its translation on

the whole Néron model, transports this pole bound to the opposite outer component and gives $a_y \in 3R$ as well.

Equation (4.10) permits the strong source on all eight new fine interiors: its error has order at least -1, and its three-multiple error has order at least one. Both are positive after multiplication by u_E. Put $h = H_s(0,0)$ and $v = \overline{u_E/\rho^5}$. On a central component write its Morse inverse as $q(T) = q + \gamma T + \cdots$. With $U = -Z, V = x, \omega_0 = dU/(2V) = -dx$, one obtains

$$\overline{u_E\zeta} = \frac{v\gamma h q^4}{(q^2+1)^2}\omega_0, \qquad \overline{u_E\beta} = \frac{v\gamma(a_x q^{-2} + a_y q^4)}{(q^2+1)^2}\omega_0 = 0.$$

At $q = -1$ the symmetric Kummer chart changes both orientation factors; their fourth powers agree: $\gamma^4 = q^6(q^2+1)^6$. On the two inherited new interiors (4.24) gives respectively

$$\overline{u_E\zeta} = v r_0 h \omega_f, \qquad \overline{u_E\beta} = -v r_0 (a_x - a_y)\omega_f = 0.$$

The reduced source forms extend holomorphically to the proper elliptic curves. A morphism between supersingular elliptic curves with zero differential pulls back a second-kind proper class into the Hodge line. To see the assertion with its normalization, on $E_0 : V^2 = U^3 - U$ put $\eta_0 = U\omega_0$. The proper transition is V/U; Frobenius gives

$$(V/U)^3 \eta_0 = (U - U^{-1})\, dU/2,$$

whose residue period is one. Hence $F^*\eta_0 = \omega_0,\ F^*\omega_0 = 0$. Also $F^2 = [-3]$. Any homomorphism is defined over $\mathbb{F}_9$, since $F^2\lambda = \lambda^{(9)}F^2$; its Hodge-preserving matrix has opposite diagonal the cube of its Hodge coefficient, by $F\lambda = \lambda^{(3)}F$. The assertion follows, including further inseparability. Thus all eight complete fine proper periods vanish.

We supply the corresponding coarse estimate, including the only Hodge localization that is needed. The integral good Hodge frame on the degree-two quotient is

$$\omega = \frac{dh}{2ths} = \frac{dU}{2h_0 V}.$$

Indeed, differentiation of $Y^3 + r^6 = 1$ gives $dh = -2rsh\, dr/Y^2$, whereas $dU = h_0\, dh/(t^2h^2)$ and $V = s/(th)$. Consequently $\mathfrak{b} = -\omega/t^3$. The original wild inertia fixes t, h, s, and so fixes this particular frame exactly. Under $t \mapsto \iota t$, the original coordinates give $r \mapsto -r$, $Y \mapsto Y$; hence h, s are fixed and $\omega \mapsto \iota^3\omega$. On its two new elliptic interiors the exact expressions are

$$r_c\mathfrak{b} = -r\omega/t, \quad r_c\mathfrak{e} = ((h_0 - 1)/t^3)\omega + (H/t)\omega, \quad q_c\mathfrak{b} = \frac{(h_0 - 1 + t^2H)^2}{t^3r^2}\omega. \tag{4.28}$$

Their orders are respectively $-3, -5, -1$. Every nonconstant B'-coefficient gains six orders. There is an actual global integral Hodge form κ_s that equals ω modulo t^3 on this entire supersingular configuration and zero modulo t^3 on the others. Here is the lifting argument. The universal good elliptic differential pulls back to a canonical form on the actual modular curve. To check each ordinary overlap, use the own chart (4.22), where

$$r_c = t/v, \qquad \omega = -t^3\mathfrak{b} = -\frac{tv\, dv}{\mathcal{F}_p} = \frac{t^3\, dr_c}{r_c^3\mathcal{F}_p}.$$

On the overlap r_c is a unit, and $\mathcal{F}_p$ is a unit at all three finite attachments. Thus this form is divisible by t^3; the other endpoint gives the same calculation with s_c. Keeping this section on

one whole supersingular neighborhood and zero on the others defines a global section modulo t^3. Relative duality gives $H^1(\omega)$ free, so it lifts globally. Tame projection onto the character ι^3 is permitted because its order is four. The original wild inertia then satisfies

$$(\gamma-1)\kappa_s \in t^3 H^0(\omega), \qquad \tau\kappa_s = \iota^3\kappa_s.$$

If $i^*\kappa_s = D_s\omega_E$, same-base integral canonical trace over any further good extension gives

$$v_\rho(D_s) \geq \delta, \quad v_\rho((\gamma-1)D_s) \geq 9+\delta, \quad \tau D_s = \iota^3 D_s.$$

Write $D_s = b_0 + b_1\rho + b_2\rho^2$ over $R[t]$. The two nonconstant rows of the wild action, with $a = (1+t^2)/2$, $\zeta_3 - 1 = t^2 a$, and $\gamma\rho = \zeta_3\rho - ta$, are

$$-tab_1 + t^2a^2b_2 \in t^4R[t], \qquad t^2ab_1 - 2\zeta_3 tab_2 \in t^4R[t].$$

Elimination uses the unit $t^2a - 2\zeta_3$ and gives $b_1, b_2 \in t^3R[t]$. Hence $D_s = b_0 + \rho^{10}\mathcal{O}$. Its initial valuation is at least three, and its tame character permits only powers $t^{3 \bmod 4}$, so

$$v_\rho(D_s) \geq 9. \tag{4.29}$$

Replace the first Hodge term of $r_c\mathfrak{e}$ in (4.28) by this global κ_s. Its local error has order at least $-5+9 = 4$; its global original scalar has order at least $-5+9 = 4$ by (4.29). The remaining $H\omega/t$ has order -3, and $q_c\mathfrak{b}$ has order -1. Multiplication by the actual u_E, of order five, makes every complete coarse polar block positive. The own, ordinary, and rational components have constant good image, and the preceding genus calculation exhausts the other fine components. Thus (4.20) belongs to the maximal ideal of the good extension. Its coefficient lies in the original unramified ring, whose intersection with that ideal is $3R$. We have proved

$$c_E i^*\zeta \in 3R\omega_E \qquad (\delta = 5). \tag{4.30}$$

4.9 The remaining coarse period when $\delta = 3$

For the rest of the proof the original type-II equation is

$$Y^2 = X^3 + 3A_2X^2 + 3A_4X + 3A_6, \qquad A_4, A_6 \in R^\times. \tag{4.31}$$

It remains minimal over $R[t]$, $t^4 = -3$. Indeed a scaling by t would require the translated constant term to have t-valuation at least six, whereas its unique smallest term has valuation $0, 3, 4$ for a translation of valuation $0, 1, \geq 2$, respectively. A larger scaling is excluded by the discriminant valuation twelve. Thus ω_E is canonical over this fourfold extension.

We first check the original images at the own branches, since their original additive values enter the coarse period. In (4.22) tame inertia fixes the branch and acts as $(t, p, v) \mapsto (\iota t, -p, \iota v)$. An image at the singular point of the additive Weierstrass fibre would have $X = t^2x$, $Y = t^2(v_0 + t^2y)$, with $v_0^2 = -A_6 \neq 0$. This follows by comparing the three least terms in (4.31). But inertia sends Y/t^2 to $-Y/t^2$, and fixes its nonzero constant residue v_0, a contradiction. Its original image is therefore in the smooth additive identity domain.

Here is a trace calculation that controls the possible closed functions in that image. Write $\beta = a\mathfrak{b} + c\mathfrak{e} + d\,q_c\mathfrak{b}$ in (4.14). The two old deck transformations have weights

	r_c	s_c	t_c	$\mathfrak{b}$
S	ζ_3^2	ζ_3	ζ_3^2	ζ_3^2
T	ζ_3	ζ_3^2	ζ_3^2	ζ_3^2

and

$$\beta = (a - c\,d_c/3 + d\,d_c^2/3)\mathfrak{b} + (c - 2d\,d_c)t_c\mathfrak{b}/3 + d\,t_c^2\mathfrak{b}/3.$$

Use the unique constant-plus-pure-r_c, s_c expansions of its coefficients. Zero old traces give

$$d_0 = 0, \quad c_{s_c,1} = 6d_{s_c,2}, \quad c_{r_c,1} = -6d_{r_c,2}, \quad a_{r_c,2} + c_{r_c,3} = -3d_{r_c,4}.$$

For example the invariant $s_c t_c \mathfrak{b}$ coefficient is $(c_{s_c,1} - 2d_0 - 6d_{s_c,2})/3$; the last relation also uses $r_c^4 s_c^2 = 9r_c^2$, with $r_c s_c = -3$ retained throughout coefficient extraction.

Integrality on the own chart, at successive orders t^{-3}, t^{-2}, t^{-1}, forces $c_0, a_0, c_{r_c,1}, a_{r_c,1}, c_{r_c,2} \in 3R$. For instance the second order has coefficient $a_0 v - c_{r_c,1} p/v$; the relation $v^2 = p^3 - p$ makes its two coefficients independent. Substitution of the displayed exact trace relations then gives the complete reduced differential

$$\bar{\beta} = \bar{c}_{r_c,3}\, d\{1/(p^2 - 1)\}.$$

At each of the three finite attachments the original canonical form has pole order at most one: in a regular node $t = xy^m$ its frame on $x = 0$ is a unit times dy/y^m, and here $m = 1$. The last differential has poles of order three at $p = \pm 1$; hence it is zero.

At the generic point of this own component the actual old extension has degree three, residue Frobenius, and basis $1, v, v^2$, with v^3 in the old field. This follows also from its original coordinate $r_c = t/v$: the old map has coordinate $r_c^3 = t^3/v^3$. The monic algebra has residue a field of degree three and uniformizer t, so it is already the normalized discrete valuation ring. Its trace is three times coefficient projection. Thus $\mathrm{Tr}\,/3$ is integral on this particular old valuation ring. Apply it to $\mathcal{L}(w_E)$. Its trace has zero derivative, by the actual old trace of β, and is therefore constant. Since $\bar{\beta} = 0$, the function $\bar{w}_E$ is a cube in the source residue field. Projection is the identity on the Frobenius subfield. Reduction of the one constant trace gives

$$-\bar{w}_E - \bar{A}_2 \bar{w}_E^3 = \text{constant}.$$

A rational function algebraic over the constant field is constant. This proves original constancy, including the closed-cube ambiguity, on both own components.

On the two coarse new components, (4.28) and (4.29) leave, for $\delta = 3$, only the constant $r_c\mathfrak{e}$ coefficient multiplying $H\omega/t$. The $r_c\mathfrak{b}$ term is removed by an actual old Hodge form as follows. In the coarse coordinates the old quotient has

$$X = 3t_c/s_c, \qquad Y = -27/s_c^3.$$

The identities $t_c^3 = s_c^3 - r_c^3$ and $r_c s_c = -3$ give $Y^2 = X^3 - 27$ and, by differentiation, $dX/(2Y) = r_c\mathfrak{b}$. Both coordinates are fixed by the actual deck transformation T, so this is its degree-three old quotient. On this curve the change $X = 3 + t^6 X_g$, $Y = t^9 Y_g$ gives $Y_g^2 = X_g^3 + t^2 X_g^2 - X_g$, and $\omega_{\mathrm{old,g}} = t^3 \omega_{\mathrm{old}}$. Its actual old pullback is $r_c\mathfrak{b}$. On the ordinary overlap the integral good multiple is

$$t^3 r_c \mathfrak{b} = \frac{t^2\, dv}{\mathcal{F}_p} = -\frac{t^3\, dr_c}{r_c^2 \mathcal{F}_p}.$$

It is zero modulo t^3 there. The same $H^1(\omega)$-lifting localizes this Hodge form on one entire old supersingular configuration. Dividing its old pullback by t^3 removes precisely $r_c\mathfrak{b}$, with integral

canonical error. Its new elliptic pairing is exactly zero. All nonconstant coefficients gain six orders; all remaining $q_c\mathfrak{b}$ terms are positive.

The actual good target maps on these two new components are inseparable. Here is the stronger coefficient check needed to see this. Applying the own calculation at both ends gives

$$a_0, c_0, a_{r_c,1}, a_{s_c,1}, c_{r_c,1}, c_{s_c,1}, c_{r_c,2}, c_{s_c,2} \in 3R.$$

The two invariant first coefficients are

$$\begin{aligned} a_{r_c,1} + c_0/3 + c_{r_c,2} + 2d_{r_c,1} + 3d_{r_c,3} - d_{s_c,1} &= 0, \\ a_{s_c,1} - c_0/3 - c_{s_c,2} - d_{r_c,1} + 2d_{s_c,1} + 3d_{s_c,3} &= 0. \end{aligned}$$

Modulo three these give $d_{r_c,1} + d_{s_c,1} = 0$ and $c_0/3 = 0$. Thus $c_0 \in 9R$, while $d_0 = 0$. The orders of $\mathfrak{b}, \mathfrak{e}, q_c\mathfrak{b}$ at a new coarse component are $-9, -11, -1$, and those of r_c, s_c are six. Every term of β therefore has order at least three, so $\overline{u_E\beta} = 0$.

Normalize each good elliptic reduction to $E_0 : Z^2 = H^3 - H$, and set $\omega_0 = dH/(2Z)$, $\eta_0 = H\omega_0$. For any homomorphism $\psi : E_0 \to E_0$, put $z_C = -H/Z$ on the source and write the target formal parameter as

$$\theta_E\psi = Az_C + Jz_C^3 + O(z_C^5).$$

Its proper de Rham matrix in (ω_0, η_0) is

$$\begin{pmatrix} A & J^3 \\ 0 & A^3 \end{pmatrix}.$$

For a proof beyond the Frobenius calculation already given, write $A = m + n\iota \in \mathbb{F}_9$ and subtract the endomorphism $[m] + [n]I$, where $I(H, Z) = (-H, \iota Z)$. Its third formal coefficient is zero and its matrix is $\operatorname{diag}(A, A^3)$. The difference has zero differential and factors through Frobenius. The computation $F^*\eta_0 = \omega_0$ then supplies the off-diagonal J^3. This also covers the zero map. In particular an inseparable map with formal leading coefficient Jz_C^3 has $\overline{\langle \eta_0, \psi^*\eta_0 \rangle} = J^3$.

Let B_s be the residue-field coefficient defined in (4.17). Put $v = \overline{u_E/t}$. The remaining full proper coarse contribution is

$$\bar{c}_E v B_s\big(\langle \eta_0, \psi_+^*\eta_0 \rangle + \langle \eta_0, \psi_-^*\eta_0 \rangle\big).$$

Its endpoint normalization can be calculated without choosing unrelated node units. On the central component put $w_c = z_C/\rho$. The reduction of (4.21) gives $r^2 = -(1 + w_c^{-1})^3$. Choose $q^2 = 1 + w_c^{-1}$, $r = \iota q^3$; the new attachments are $q = \pm 1$, the own attachments are $q = 0, \infty$, and

$$pz_C = \rho, \qquad \bar{p} = q^2 - 1.$$

For $f = \theta_E\varphi$, the complete target coefficient bounds above give $v_\rho(\beta) \geq 4$ on the central curve and hence $v_\rho(df) \geq 7$. In fact $v_\rho(f) \geq 3$. If $0 < m < 3$ were its first order, its leading central function would be a cube, with poles at the two new nodes of order at most $\lfloor m \rfloor \leq 2$, and with zero values at the own nodes, where f has order at least three. A cube cannot have a pole of order one or two; it would be a constant vanishing at an own node, a contradiction. The proof applies to fractional orders after any further extension: rational chains contract to the same normal node and regular functions descend through that contraction.

Thus $\overline{f/u_E}$ is a closed cube with poles of order at most three at $q = \pm 1$. If the two inseparable formal leading coefficients are J_+, J_-, the exact node equation, with its factors $2q$, gives

$$\overline{f/u_E} = C + \frac{J_+}{v(q-1)^3} - \frac{J_-}{v(q+1)^3}.$$

At the own endpoints $f/u_E = -(X - s_E)/Y$ reduces to $-w_E$. Consequently

$$J_+ + J_- = v(w_{\text{own},0} - w_{\text{own},\infty}).$$

The good linear coefficient in (4.31) gives $v^4 = \bar{A}_4$, since $t^4 = -3$. The complete coarse contribution is therefore exactly

$$\bar{A}_4 \bar{c}_E^4 B_s (W_{\text{own},0} - W_{\text{own},\infty})^3. \tag{4.32}$$

This calculation includes both new components and their proper transitions. An incoming factor μ in an intermediate elliptic map multiplies its actual formal coefficients; it is never divided out in this formula.

4.10 The actual nodes of the eight fine new components

For a central component in (4.23), choose a constant d of ρ-valuation $16/3$ and put

$$z = Z/x, \qquad m = dT/(\lambda Z), \qquad \tau = d\rho/\lambda.$$

Then $mz = \tau$, with $v_\rho(\tau) = 1/3$. If $V = z^3 x$, its exact equation is

$$V^2 + \{(3/\lambda)z^2 - C_0\}V + (3/\lambda^2)z^4 = 0.$$

At $z = 0$, the root $V = C_0$ has unit derivative. It therefore defines $V(z)$ integrally, and

$$T = \frac{\lambda^3}{d^3\rho^2} V(z) m^3, \qquad \lambda Z = \frac{\lambda}{\tau^2} V(z) m^2.$$

The first coefficient is a unit and the second has valuation $16/3$. The inverse Morse series and J_{app} are integral; thus the original chart functions are integral in $\mathcal{O}[[m, z]]/(mz - \tau)$. Conversely m, z are the displayed rational functions, and recover the original functions on the generic annulus. This proves that this is the actual normal node.

The inherited new nodes require different rational parameters. Keep the exact $r_0^2 = h_0/2$, put

$$q = \frac{rU}{\rho V - U}, \quad p = -(r+q)/q, \quad z = (Z_c - g(q))/\rho, \quad m = \tau/z, \quad \tau^3 = \rho.$$

Let $r = r_0 + m^9 R,\ q = -r_0 + m^3 Q$, and

$$A_0 = r_0^2 - r^2, \qquad D_0 = r^2\Lambda - r_0^2 p.$$

The two original equations become

$$\begin{aligned} L A_0^2 - A_0 p^2 D_0 + \Lambda^2 A_0 D_0 + \Lambda D_0^2 = 0, \\ L Z_c^3 + \Lambda^2 Z_c^2 + \Lambda Z_c - G(r) = 0. \end{aligned}$$

Substitute $\rho = (mz)^3,\ \Lambda = -\rho^4/L,\ 3 = -\rho^{12}/L^4$. The equations are divisible by m^{18} and m^9. The second has unit R-derivative because $G'(r_0) = 1$. Modulo z^{12} its solution is exactly $r = q^3$: there $\Lambda = 3 = 0$, and $G(q^3) = g(q)^3$. The first divided equation is consequently divisible by z^{12} as a whole formal series. At $m = 0$ the two divided leading expressions are

$$-R^2 + RQ^3 + z^{12}Q^2, \qquad Q^3 + z^{12} - R.$$

Eliminating R and dividing by z^{12} gives $Q^2 - Q^3 - z^{12}$; its derivative at $Q = 1, z = 0$ is -1. Hensel's lemma solves Q, R in $\mathcal{O}[[m, z]]/(mz - \tau)$. The recovered U, V have poles at most z^{-6}, z^{-9}, so $t^2U, tV, \Lambda Z_c$ are integral and vanish at the node. The original x, y, u, v are then integral by their unit formulas. The converse rational formulas above prove birationality. This identifies the actual inherited node, also of ρ-length $1/3$.

An infinitesimal displacement of order ρ^ν on any of these eight elliptic components has pole order at most $\lfloor 3\nu \rfloor$ at its attachment. Equality at integral 3ν is excluded for a closed term whose central reduction is zero. The good image of the central rational components is indeed O: the original type-II Néron fibre is connected additive, its map to the good elliptic special fibre is zero, and the constants agree along the ordinary and own connections.

4.11 The fine period through the second and third jets

We now give the complete jet calculation. This is the only case in which negative source orders survive the good scaling. Choose a root s_E of the cubic in (4.31), and $u_E^4 = -f'(s_E)$. Then $v_\rho(s_E) = 4,\ v_\rho(u_E) = 3$, and the good equation is

$$Y_g^2 = X_g^3 + \frac{3s_E + 3A_2}{u_E^2} X_g^2 - X_g.$$

Its middle coefficient has order at least six. Use the target formal parameter θ_E of (4.19). The point P_K is a three-torsion point and reduces to O on the supersingular good reduction E_0, so this parameter is defined at P_K. Its formal group law is additive through degree three: the first coefficient from $a_4 = -1$ has total degree five, and the coefficient from a_2 has valuation at least six. Equation (4.23), after its unit constant normalization, is also constant to the required order: $3/\lambda$ has order six, $3/\lambda^2 = -1 + O(\rho^6)$, and $C_0 = -1 + O(\rho^4)$. Thus the standard E_0 formal coordinates apply through order three.

At $q = -1$ fix the symmetric chart by $Z \mapsto -Z,\ x \mapsto \iota x,\ T \mapsto \iota T$. The uniform Morse coefficient is then

$$\gamma^2 = -q^3(q^2 + 1)^3, \qquad q(T) = q + \gamma T + eT^2 + \cdots$$

on all six critical points. Let h, h_k, h_l be the constant and first coefficients of $\underline{H_s(k, l)}$, and let $b_x, c_x, b_y, c_y, d_x, d_y$ be the indicated original pure coefficients in (4.27). Set $v = \overline{u_E/\rho^3}$,

$$\begin{aligned} D_0 &= (q^2 + 1)^2, \qquad F = -q^4/D_0, \\ G &= (b_x q^{-1} + c_x q^{-3} + b_y q^5 + c_y q^3)/D_0, \\ H &= ((b_x + c_x)q^{-2} + (b_y + c_y)q^4)/D_0. \end{aligned}$$

On E_0, use $U = -Z, V = x, \omega = -dx, \eta = Z dx$. Write ψ for the reduced homomorphism and C for the constant formal translation in the actual map. The coefficients used in the relative jets

are

$$\begin{aligned} A &= -v\gamma G, & K &= -v(\gamma^2 G' + 2eG),\\ L_1 &= v\gamma H, & M_1 &= -v\gamma(d_x q^{-2} + d_y q^4)/D_0,\\ s_0 &= -v\gamma hF, & s_1 &= vh(\gamma^2 F' + 2eF),\\ s_\eta &= -v\gamma h q^3/(q^2+1), & s_H &= -v\gamma(h_k q^5 + h_l q^3)/D_0. \end{aligned} \tag{4.33}$$

The central cubic constants. Put $L_0 = \mathrm{Frac}(R)(t,\alpha)$, enlarging its maximal unramified subextension to contain the residue constants used below. Its valuation is normalized by $v_\rho(\rho) = 1$. Thus

$$\Lambda = -\rho^4/L, \qquad 3 = -\rho^{12}/L^4, \qquad L = \Lambda^2 - \Lambda + 1,$$

and ρ is a uniformizer of L_0. The polynomials $\Lambda^3 - 3\Lambda^2 + 3\Lambda - 3$ and $t^4 + 3$ are Eisenstein of coprime degrees three and four. Their compositum has ramification index twelve, and $v_3(\Lambda/t) = 1/12$, which proves the uniformizer assertion. The jet coefficients below belong to the residue field and are lifted to $\mathcal{O}_{L_0}$; a coefficient error divisible by three has ρ-valuation at least twelve. Equations for the central cubic constants retain their mixed-characteristic coefficients.

Let q range over the six critical residue points. The critical value of the cubic ratio defining the central cover is

$$b_q^3 = 1 - c(q)\Lambda^4 + O(\Lambda^5), \qquad c(q) = \frac{(q^2-1)^2}{q(q^2+1)}.$$

Here the actual critical point, the Morse parameter, and the coefficients of the inverse Morse series are defined over L_0. Indeed the critical point equation has simple derivative after its displayed first factor is removed, and the quadratic Morse coefficient is a unit; Hensel lifting and the extraction of a square root therefore take place over an unramified enlargement of L_0. Replacing the critical residue point by its actual lift changes its value only by $O(\Lambda^5)$ in this formula. For $q^4 = -1$, reduction gives

$$D_0 = -q^2, \qquad c(q) = \frac{q}{q^2+1}, \qquad c(q)^2 = -1, \qquad c(q)(q + q^{-1}) = 1. \tag{4.34}$$

For $q^2 = 1$, the value of c is zero.

Using the fixed ι, choose one of the two critical values having $c = \iota$, and choose a cube root b_0 of that value. Write

$$\delta_b = b_0 - 1, \qquad \varpi_b = \delta_b/\rho^5, \qquad \epsilon_b = \delta_b/\rho^4 = \rho\varpi_b.$$

The equation

$$\varpi_b^3 + \frac{3}{\rho^5}\varpi_b^2 + \frac{3}{\rho^{10}}\varpi_b + \frac{1 - b_0^3}{\rho^{15}} = 0$$

is Eisenstein: the valuations of its last three coefficients are $7, 2, 1$. Consequently

$$v_\rho(\varpi_b) = 1/3, \qquad v_\rho(\delta_b) = 16/3, \qquad v_\rho(\epsilon_b) = 4/3.$$

The symbol ϵ_b denotes this entire algebraic constant.

Every unit in $1 + \rho^{20}\mathcal{O}_{L_0}$ has a cube root in $1 + \rho^8\mathcal{O}_{L_0}$. To see this, for e of valuation at least twenty solve

$$u = e/3 - u^2 - u^3/3$$

on $\rho^8\mathcal{O}_{L_0}$. The right side preserves that ball. Its difference at u, u' is divisible by

$$(u-u')\left(u+u'+\frac{u^2+uu'+u'^2}{3}\right),$$

and the parenthesis has valuation at least four. Iteration converges to the required solution of $(1+u)^3 = 1+e$. It follows that the four roots may be chosen compatibly as

$$b_q = b_0^{\epsilon_q} h_q, \qquad \epsilon_q = c(q)/\iota \in \{1,-1\}, \qquad h_q \in 1+\rho^8\mathcal{O}_{L_0}.$$

For these choices, and as identities of whole constants,

$$\delta_{b,q} = b_q - 1, \qquad \epsilon_{b,q} := \frac{\delta_{b,q}}{\rho^4} = \epsilon_q\epsilon_b + O(\rho^4). \tag{4.35}$$

For example, when $\epsilon_q = -1$, use $b_0^{-1} - 1 = -\delta_b + \delta_b^2/(1+\delta_b)$, where $v_\rho(\delta_b^2) > 8$. For $q = 1, -1$ one may instead choose $b_q \in 1+\rho^8\mathcal{O}_{L_0}$, so that $v_\rho(\epsilon_{b,q}) \geq 4$. These choices fix the cube-root ambiguity in the coordinates $J/J_{\text{app}} = b_q(1+\lambda Z)$.

The original functions and their differential. On the four components with $q^4 = -1$ all denominators in

$$y = t^2\frac{Y+J}{Y-J}, \qquad x = t^2\frac{Y-J}{Y+J}, \qquad u = \frac{2\sqrt{-J}}{Y-J}, \qquad v = -\frac{2r\sqrt{-J}}{Y+J}$$

are units after their indicated fixed powers of ρ have been removed. In these rational formulas and in $d\log x$ below, x, y, u, v denote the original chain functions. In $T = \rho x$ and the resulting elliptic expansions, x is the affine elliptic coordinate. The target unit $v = \overline{u_E/\rho^3}$ is restored in the relative-jet paragraph. At $\Lambda = 0$ the residues are

$$r = q^3, \quad Y = 1-q^6, \quad J = -(q^3+1)^2, \quad \sqrt{-J} = q^3+1, \quad \frac{y}{\rho^6} = -q^3, \quad \frac{x}{\rho^6} = -q^{-3}, \quad u = v = 1.$$

Write $Y_0 = 1-q^6$ and $j_0 = -(q^3+1)^2$ in the following coefficient calculation. The first Λ-coefficients of r, Y, obtained by substitution in the exact base equation, are

$$r_1 = \frac{(q^4+1)(q^6-1)}{q(q^2+1)}, \qquad Y_1 = \frac{q^{12}+q^8+q^6}{q^2+1}.$$

With the displayed j_1 in $J_{\text{app}} = j_0 + \Lambda j_1$, logarithmic differentiation of the last two unit formulas gives

$$u_\Lambda = \frac{j_1}{2j_0} - \frac{Y_1-j_1}{Y_0-j_0} = q^2, \qquad v_\Lambda = \frac{r_1}{q^3} + \frac{j_1}{2j_0} - \frac{Y_1+j_1}{Y_0+j_0} = q^{-2}.$$

Multiplying J by $1+\delta_{b,q}$, while fixing the base point, gives

$$u_{\delta_b} = \frac{1}{2} + \frac{j_0}{Y_0-j_0} = q^3, \qquad v_{\delta_b} = \frac{1}{2} - \frac{j_0}{Y_0+j_0} = q^{-3}.$$

Thus $u-1 = \Lambda q^2 + \delta_{b,q}q^3 + \cdots$ and $v-1 = \Lambda q^{-2} + \delta_{b,q}q^{-3} + \cdots$. Inserting these in $k = (u-1)^2/y - x$ and $l = (v-1)^2/x - y$ gives the new terms. The factor λZ in J has the same first variation as

$\delta_{b,q}$, and $\lambda/\rho^4 = -\rho^2 + O(\rho^6)$. Writing the inverse Morse series as $q(T) = q + \gamma T + eT^2 + \cdots$, with $T = \rho x$, therefore gives the full relative expansions

$$\begin{aligned} \frac{k}{\rho^2} &= -q - \rho\gamma x - \epsilon_{b,q} q^2 + \rho^2(q^2 Z - e x^2) + O_{>2}, \\ \frac{l}{\rho^2} &= -q^{-1} + \rho\gamma q^{-2} x - \epsilon_{b,q} q^{-2} + \rho^2\{q^{-2} Z + (eq^{-2} - \gamma^2 q^{-3}) x^2\} + O_{>2}. \end{aligned} \tag{4.36}$$

Here and below $O_{>r}$ means terms of valuation strictly greater than r in the indicated normalized expansion. Products $\rho\epsilon_{b,q}$ and $\epsilon_{b,q}^2$ have valuations $7/3$ and $8/3$, respectively, and belong to $O_{>2}$.

For the monic canonical frame one must also check the numerator. The coefficient of $\delta_{b,q}$ in $\log x$ is

$$-\frac{j_0}{Y_0 - j_0} - \frac{j_0}{Y_0 + j_0} = q^{-3} - q^3.$$

Its derivative in characteristic three is zero. The leading coefficient of $d\log x$ has valuation five, coming from $\Lambda\, dq$; the derivative of its normalized Λ-coefficient is $-q^3$ on $q^4 = -1$. The next contributions involving $\delta_{b,q}$ have relative valuation at least $v_\rho(\delta_{b,q}) = 16/3$: a Λ-term has the additional factor $\Lambda\delta_{b,q}\, dq$, and a dZ-term the factor $\lambda\delta_{b,q}$. The characteristic-zero derivative of $q^{-3} - q^3$ is divisible by three. None of these terms affects the relative jet through order two. Consequently the relative $\epsilon_{b,q}$-correction to the monic frame is $-p_q\epsilon_{b,q}$, where $p_q = q + q^{-1}$. This follows from $\Omega_{\mathcal{A}} = -d\log x/(3u^2v^2kl)$ and $\Delta k/k = q\epsilon_{b,q}$, $\Delta l/l = q^{-1}\epsilon_{b,q}$. For completeness, the coefficient before the critical-point specialization is

$$W = -\frac{d}{dq}\left(\frac{Y_1 - j_1}{Y_0 - j_0} - \frac{Y_1 + j_1}{Y_0 + j_0}\right) = \frac{q^{12} - q^{10} + q^8 - q^6 + q^4 - q^2 + 1}{q^9 - q^7 + q^5}.$$

With $F = -q^4/(q^2+1)^2$, reduction modulo $q^4 + 1$ gives

$$W = -F/q^3, \qquad W'' = 0, \qquad \gamma^2 W' + q^{-3} - q^3 = -\gamma^2 F'/q^3.$$

These identities follow by putting the rational functions over their displayed denominators and reducing their numerators. Before this reduction, the numerator and denominator expansions give

$$\rho^{11}\Omega_{\mathcal{A}} = \Big\{q'(T)W(q(T)) + \rho(q^{-3} - q^3)x + \rho^2\gamma W(q)p_q Z - \epsilon_{b,q}\gamma W(q)p_q\Big\}\, dx + O_{>2}.$$

Here $dZ = -x\, dx$. In $q'(T)W(q(T))$, the two coefficients involving eW' in degree two add to $3e\gamma W' = 0$, while the remaining degree-two coefficient is $\gamma^3 W''/2 = 0$. Thus the full monic frame is

$$\rho^{11}\Omega_{\mathcal{A}} = \left\{-\frac{\gamma F}{q^3} - \rho\frac{\gamma^2 F' + 2eF}{q^3}\, x + \rho^2\frac{\gamma}{q^2+1} Z + \epsilon_{b,q}\frac{\gamma F}{q^3}(q + q^{-1})\right\} dx + O_{>2}. \tag{4.37}$$

The complete relative jet through order two. Return to the elliptic coordinates $U = -Z, V = x$, $\omega = -dx$, and to $v = \overline{u_E/\rho^3}$. Retain the coefficient table (4.33). The strong numerator ka^2H_s has the same $\epsilon_{b,q}$-jet as k^3h, since $a - k$ has valuation six, hence relative valuation four, and the variation of H_s has valuation greater than two. Its linear $\epsilon_{b,q}$-coefficient is zero in characteristic three. The additional scaled source term is therefore

$$-\rho^{-2}\epsilon_{b,q}p_q s_0\omega.$$

For the target, the leading numerator divided by ρ^8 is

$$N = b_x q^{-2} + c_x q^{-4} + b_y q^4 + c_y q^2.$$

The change from k, l, followed by the change of frame, is

$$\begin{aligned}(b_x q^{-1} + c_x q^{-5} + b_y q^5 + c_y q) - (q + q^{-1})N \\ = -(b_x + c_x)q^{-3} - (b_y + c_y)q^3.\end{aligned}$$

Multiplication by the leading frame coefficient $-v\gamma q/D_0$ gives $+L_1\omega$. Since $d(-L_1 x) = L_1\omega$, the additional displacement is $-\epsilon_{b,q} L_1 x$. Thus the two complete jets are

$$\begin{aligned}h_{\mathrm{map}} &= \rho K Z - \epsilon_{b,q} L_1 x + \rho^2 (L_1 Z x + M_1 x) + O_{>2}, \\ u_E \zeta &= \rho^{-2} s_0 \omega + \rho^{-1} s_1 x\, dx - \rho^{-2} \epsilon_{b,q} p_q s_0 \omega + s_\eta \eta + s_H \omega + O_{>0}.\end{aligned} \tag{4.38}$$

The constant formal translation is retained separately as C.

These expansions remain complete after adjoining the good target coordinates or any further finite extension. Indeed

$$(u_E/\rho^3)^4 = L^{-4}(A_4 + 2A_2 s_E + s_E^2), \qquad v_\rho(s_E) = 4,$$

so u_E/ρ^3 differs from a unit in an unramified extension of L_0 first in order four. The good target equation differs from E_0 first in order six. The two actual affine source equations and the coefficients of their differentials through order two belong to L_0, apart from the displayed whole constant $\epsilon_{b,q}$. An additional first nonconstant map coefficient of order $0 < \nu \leq 2$ would consequently have zero differential. It is a cube of a rational function on E_0. A nonconstant rational function with its only pole at O has pole order at least two, so such a cube has pole order at least six. The actual node bound is $\lfloor 3\nu \rfloor$. It excludes $\nu < 2$. At $\nu = 2$, the only possible pole-six principal part has nonzero order-zero central reduction, whereas the actual central map reduces to O; the listed terms have pole at most five and cannot cancel it. Thus it is absent as well. This argument applies to every fractional value of ν in every finite extension. Constant coefficients are absorbed in C.

For $q = 1, -1$, the compatible roots lie in L_0 and their relative corrections begin in order four. The symmetric chart at -1 gives the same conclusion. On the two inherited components the affine coordinate changes displayed in (4.44) are defined over L_0 through order two. The same node argument excludes additional fractional nonconstant coefficients on these four components.

The integer coefficients follow from the original coordinate increments after (4.27) and the frame above. The identities

$$F''/2 = 0, \qquad \gamma^2 G''/2 + H = 0$$

in the critical algebra check respectively the absent second Taylor term and the fact that the second-order target differential is $L_1 d(Zx) + M_1 dx$, where $d(Zx) = (Z - x^2)dx$. They can be checked by differentiating each of the four numerator monomials in G, then reducing by $(q^2 - 1)(q^4 + 1)$.

Use the good pole $P_* = (0, 0)$. It avoids all constant and node images, and

$$\eta_{P_*} = -\omega/U, \qquad \eta - \eta_{P_*} = d(V/U).$$

Put $H_0 = -1/U$, $D_E = d/\omega$. If $\theta_E\psi = Az + Jz^3 + \cdots$, the integer-order polar contribution, including the constant translation, is

$$\begin{aligned}\mathcal{B}_q^{(0)} = & -s_0A^3/\rho^2 + (s_1A^2 + s_0KA)D_EH_0(C)/\rho + s_\eta J^3 - s_HA^3 \\ & + (s_1KA - s_0L_1A - s_0K^2/2)J - s_0M_1A^2 \quad (\text{mod positive valuation}).\end{aligned} \tag{4.39}$$

Here is a residue derivation specifying the attachment terms. For the proper map $\psi + C$, the Hodge source has period $-A^3$. The source η has period J^3; its extra attachment term is positive since η_{P_*} vanishes to order two at O. The exact source $x\,dx = -dZ$ contributes $\mathrm{Res}_O[-Z(\psi + C)^*\eta_{P_*}] = A^2D_EH_0(C)$. For displacement $h = \rho KZ + \rho^2(L_1Zx + M_1x)$, the change of the whole moving-pole residue is minus the residue at O of the source times

$$hH_0(\psi + C) + (h^2/2)D_EH_0(\psi + C).$$

Only division by two occurs. The formal expansions

$$U = z^{-2} + z^2 + \cdots,\ V = -z^{-3} - z + \cdots,\quad \omega = (1 + z^4 + \cdots)dz$$

give

integrand	Res_O
$\omega ZxH_0(\psi)$	AJ
$\omega xH_0(\psi)$	A^2
$\omega Z^2D_EH_0(\psi)/2$	$J/2$
$x\,dx\,ZH_0(\psi)$	$-AJ$
$\omega Z^3H_0(\psi)$	0
$\omega ZH_0(\psi + C)$	$-AD_EH_0(C)$

because $H_0(\psi) = -A^2z^2 + AJz^4 + \cdots$ and $D_EH_0(\psi) = Az + Jz^3 + \cdots$. The unit residues may be evaluated at $C = 0$, because translating by a formal C changes them by terms of positive valuation. The displayed negative translation term retains its full C-dependence. These identities prove (4.39) for the integer-order contribution to the finite flat polar trace, also when poles collide.

The proper fractional block. The pole $P_* = (0, 0)$ and the function $H_0 = -1/U$ are as in the proper polar calculation. One has

$$D_EH_0 = 2V/U^2, \qquad D_E^2H_0 = 1.$$

For every formal constant C,

$$\mathrm{Res}_O\big(\omega xH_0(\psi + C)\big) = A^2. \tag{4.40}$$

Indeed $x = -z^{-3} - z + \cdots$, and the coefficient of z^2 in $H_0(\psi(z) + C)$ is $A^2D_E^2H_0(C)/2 = -A^2$. The Hodge proper period is translation invariant and equals $-A^3$. Hence the new source term contributes $\rho^{-2}\epsilon_{b,q}p_qs_0A^3$, and moving the entire finite flat polar divisor by $-\epsilon_{b,q}L_1x$ contributes $\rho^{-2}\epsilon_{b,q}s_0L_1A^2$. The additional whole proper block is exactly

$$\rho^{-2}\epsilon_{b,q}C_q, \qquad C_q = s_0A^2\big((q + q^{-1})A + L_1\big),\quad (\mathrm{mod}\ v_\rho > 0). \tag{4.41}$$

Its products with the order-one jet have valuation at least $1/3$, and its quadratic products have valuation at least $2/3$. The source terms and map terms omitted in (4.38) have positive

valuation in the full polar trace: its pole is away from the attachment and its finite flat trace is integral, also for colliding poles and inseparable reduced maps.

Thus the complete central polar contribution is

$$\mathcal{B}_q^{(0)} + \rho^{-2}\epsilon_{b,q}C_q \pmod{v_\rho > 0}.$$

The selected diagonal supplies further equations that cannot be imposed on an arbitrary local trace kernel. Its actions on the central components are

$$\delta_{\mathsf{S}}q/\rho^2 = -1/[q(q^2+1)], \quad \delta_{\mathsf{T}}q/\rho^2 = q^5/(q^2+1), \quad \delta_{\mathsf{S}}Z = 1/[q(q^2+1)], \quad \delta_{\mathsf{T}}Z = q^3/(q^2+1).$$

The last two coefficients belong to $\mathbb{F}_3^\times$ on the critical scheme. Thus same eliminates the $q = \pm 1$ components, and opposite eliminates the four $q^4 = -1$ components: a nonzero AS translation has rational quotient, so an invariant map to E_0 is constant. On the two inherited new components both generators act as $(U_f, V_f) \mapsto (U_f - 1, V_f)$. Same eliminates them, whereas opposite allows them. The own components were already eliminated by the actual old columns.

Vanishing of the target Hodge coefficient on the eliminated components first gives

$$c_y = -b_x,\ b_y = -c_x \quad \text{(same)}, \qquad c_y = b_x,\ b_y = -c_x \quad \text{(opposite)}.$$

Write $Q_{\mathsf{S}}, Q_{\mathsf{T}}$ for the coefficients of $\delta_{\mathsf{S}}q/\rho^2, \delta_{\mathsf{T}}q/\rho^2$ displayed above. Put $Q = Q_{\mathsf{S}} + Q_{\mathsf{T}}$ or $Q_{\mathsf{S}} - Q_{\mathsf{T}}$ on an allowed central component. Pointwise $Q = -1/q$, but differentiate its full rational expression before using that equality. The actual source formal displacement is

$$\rho d + \rho^2 e_1 x + \cdots, \qquad d = -Q/\gamma, \quad e_1 = -Q' + 2eQ/\gamma^2.$$

The fractional displacement changes under this diagonal first in order $4/3 + 1 > 2$. The nonconstant order-two coefficient of $\varphi\mathsf{K} - \varphi = P_K$ is therefore $Ae_1 + Kd = 0$. After (4.33) this is the division-free equation $(GQ)' = 0$. Its numerator modulo $q^4 + 1$ is $2c_x(1 + q^2)$ in the same case and modulo $q^2 - 1$ is c_x in the opposite case. The denominators are units. We obtain

$$c_x = b_y = 0, \qquad c_y = -\sigma b_x, \qquad \overline{\theta_E(P_K)/\rho} = v b_x. \tag{4.42}$$

The last coefficient is one global value, including on the two allowed inherited components. Write $b = b_x$. In particular $P_K = 0$ implies $b = 0$.

After these relations $FG' - F'G = 0$ and $s_0 H = s_\eta G$ in the relevant critical algebra. They follow by inserting $G = b(q^{-1} - \sigma q^3)/D_0$. Thus the coefficient of $D_E H_0(C)$ in (4.39) is zero, for every fractional constant translation. Also $s_1 A = -s_0 K$, and $1 + 1/2 = 0$ in characteristic three cancels the K^2 terms. The unit block is

$$s_\eta(J^3 + A^2 J) - s_H A^3 - s_0 M_1 A^2. \tag{4.43}$$

On an inherited new component, set $a = r_0 = \pm\iota$. The actual affine coordinates through order two are

$$V = Y^3 - \rho + \rho^2 Y + O(\rho^3), \qquad S^3 - S - Y^2 = \rho Y^3 + \rho^2(1 - Y^4) + O(\rho^3).$$

Thus $X = S + \rho Y^3 + \rho^2(1 - Y^4)$ gives $Y^2 = X^3 - X \pmod{\rho^3}$. The actual node parameter constructed above has expansion

$$z = -aY/X^2 - \rho a/X^3 + \rho^2 aY(X^{-5} + X^{-3} - X) + O(\rho^3). \tag{4.44}$$

For an explicit derivation, the first three Hasse Taylor coefficients of $g(-a+\Delta)$ are $1, a, -1$; insert

$$q = -r/(1-\rho V/U), \quad r = a + a\rho^3 Y^3 + O(\rho^4), \quad U = X^3 - \rho Y^3 - \rho^2(1-Y^4)$$

in $z = (Z_c - g(q))/\rho$. All integer-three terms start in order twelve and $r_0 - a$ starts in order four. This gives the whole rational coefficients in (4.44). Since $D_E(-aY/X^2) = a$, its displacement terms are

$$-X^{-3}, \qquad Y(X^{-5} + X^{-3} - X).$$

The first is regular at O; the second has precisely the principal part of $-XY$, of pole order five. In particular there is no unaccounted simple attachment pole.

After (4.42), on this component

$$\begin{aligned} u_E\beta &= A\omega - \rho^2 va(d_x - d_y)\omega + \text{positive}, && A = -vb, \\ u_E\zeta &= \rho^{-2}s_0\omega - s_0Y^2\omega - v(h_k - h_l)\omega + \text{positive}, && s_0 = vah. \end{aligned}$$

The node bound excludes a closed nonconstant term of order one. At order two its only possible cube is X^3, of pole six, which has nonzero central order-zero reduction and cannot be cancelled by the pole-five attachment above. Thus, retaining the constant translation C, the actual map displacement is $\rho^2 M_a Y$, where $M_a = va(d_x - d_y)$. Since

$$-Y^2\omega = \eta + d(XY),$$

the whole inherited polar block is

$$-s_0A^3/\rho^2 + s_0(J_a^3 - A^2J_a) + v(h_k - h_l)A^3 - s_0M_aA^2. \tag{4.45}$$

Indeed the proper η term gives J_a^3, the exact XY term gives $-A^2J_a$, and moving the pole by $\rho^2 M_a Y$ gives $-s_0M_aA^2$. There is no negative constant-translation term, because the leading Hodge proper cup is translation invariant. These observations justify every term of (4.45) with the actual node parameter.

The negative ρ^{-2} terms cancel on the whole six-point algebra: each coefficient of s_0A^3 is an odd function of q. On the inherited pair their coefficients are opposite and their A's are equal.

After (4.42), in the same case and on $q^4 = -1$,

$$F = q^2, \quad G = -bq, \quad H = b(1+q^2), \quad \gamma^4 = 1, \quad A = v\gamma bq, \quad s_0 = -v\gamma hq^2,$$

so that

$$C_q = -v^4hb^3(1+q^2), \qquad c(q)C_q = -v^4hb^3q.$$

In the opposite case these four reduced maps are constant, so $A = 0$ and $C_q = 0$. Adding the four components with the compatible roots of (4.35) gives a single term

$$\rho^{-1}\varpi_b\,\widetilde{C} + O_{>0}, \qquad \widetilde{C} \in \mathcal{O}_{L_0}, \qquad \overline{\widetilde{C}} = \sum_{q^4=-1} \frac{c(q)}{\iota} C_q = 0. \tag{4.46}$$

All coefficients may be lifted in L_0; the target units, critical lifts, and coefficients differing first in order four give positive errors after multiplication by $\rho^{-1}\varpi_b$. The last equality follows from $\sum_{q^4=-1} q = 0$. Since $v_\rho(\widetilde{C}) \geq 1$, the valuation of the entire term is at least $1/3$.

The global third coefficient. The third jet is controlled by the same global point P_K. For the central coordinate, the original normalizers have, through order four, the rational Möbius expression

$$q \longmapsto r\,\frac{N+\ell a}{D-\ell a}, \quad N=\frac{qs}{r+q}, \quad D=\frac{rs}{r+q}, \quad \ell=\lambda/\Lambda, \quad a=1 \text{ or } 2r^2.$$

Here $\ell=\rho^2+O(\rho^6)$, and changes of r and of the remaining Λ-coefficients start in order six. Expanding the displayed fraction gives the first vector field $a(r+q)^2/(rs)$ and half its product with its derivative as the second coefficient. The two fields commute. The selected diagonal therefore has the exact required expansion

$$\Delta q=\rho^2 Q+\rho^4 QQ'/2+O(\rho^6).$$

At the affine two-torsion point $x=Z=0$ this gives its invariant formal displacement

$$\rho d+\rho^3 d_3+O(\rho^4), \qquad d=-Q/\gamma, \quad d_3=-QQ'/(2\gamma)+eQ^2/\gamma^3.$$

The exact translation formula from the two-torsion point is $V(P+h)=V(h)/U(h)^2$, whose invariant coordinate is $-x+O(x^5)$; thus no order-three term has been lost. Use the selected diagonal on its allowed central components. The exact normalizer formula involves only r,s,Λ,λ, so its displacement coefficients are unchanged by the choice of b_q. The additional map term $-\epsilon_{b,q}L_1x$ has difference $+\rho\epsilon_{b,q}L_1d+O_{>3}$, because the invariant coordinate there is $-x+O(x^5)$. The complete target identity through valuation three is therefore

$$\theta_E(P_K)=\rho Ad+\rho\epsilon_{b,q}L_1d+\rho^3(Jd^3+Ad_3+Kd^2-M_1d)+O_{>3}. \tag{4.47}$$

The cancellation of the other cubic terms follows directly from

$$Ad_3+Kd^2=vQ(GQ'/2-G'Q)-3veGQ^2/\gamma^2=0:$$

use $(GQ)'=0$, then $1+1/2=0$ in characteristic three. Terms of differential order $\nu>2$ acquire a further order under the selected diagonal, and thus do not enter this equation. A closed nonconstant term is a cube; its difference under the diagonal acquires at least three orders. This also controls terms from any larger good-reduction field.

For the same diagonal on $q^4=-1$, one has

$$d=1/(q\gamma), \qquad Ad=vb, \qquad L_1d=vb(q+q^{-1}), \qquad \epsilon_{b,q}L_1d=\epsilon_b\,vb/\iota+O_{>2}.$$

Choose lifts of the residue constants in the unramified coefficient field. The common coefficient T_3 is defined by

$$T_3=\begin{cases}\overline{\rho^{-3}\left(\theta_E(P_K)-\rho\widetilde{vb}-\rho\epsilon_b\,\widetilde{vb}/\iota\right)}, & \text{same},\\ \overline{\rho^{-3}\left(\theta_E(P_K)-\rho\widetilde{vb}\right)}, & \text{opposite}.\end{cases} \tag{4.48}$$

The compatible lifts just used have error first in order four: the original coefficients differ from their unramified lifts by a multiple of three, the critical lifts and Morse coefficients first change in order four, and $u_E/\rho^3=\widetilde{v}+O(\rho^4)$. Thus the quantities inside the displayed residues have valuation at least zero. The value is common to all allowed components because the left side of (4.47) is the single point P_K, and its whole term of valuation $7/3$ has just been subtracted. In

the opposite case both central allowed components and both inherited components have the expansion without that term. Consequently

$$T_3 = Jd^3 - M_1 d, \qquad J = \gamma^3\{q^3 T_3 - v(d_x + d_y q^6)/D_0\}. \tag{4.49}$$

These equations also hold when $A = 0$ or $b = 0$.

On either inherited allowed component, the opposite diagonal fixes the original y and sends r to $\zeta_3 r$. Its exact action is

$$Z_c' = \frac{(1 + \Lambda Z_c)K_0(r)/K_0(\zeta_3 r) - 1}{\Lambda}, \qquad K_0(r) = (r+1)/(r-1).$$

Since $Y = aZ_c/\rho$, expansion gives $Y' = Y + \rho + O(\rho^4)$. The coefficients of orders two and three vanish: λ/Λ has its next term in order six and $r - r_0$ starts in order three on this chart. The constant affine E_0 identification can be continued through order three with Y fixed, because its X-derivative is a unit. Its formal displacement at $X = Y = 0$ is $-\rho + O(\rho^4)$. It follows that the very same T_3 satisfies

$$J_a = M_a - T_3. \tag{4.50}$$

Substitute (4.49) into (4.43), and (4.50) into (4.45). The results, after summing the indicated whole residue schemes, are

components	sum of unit periods divided by v^4
$q^4 = -1$ (same)	$b^3 h_l + h d_y^3$
$q^2 = 1$ (opposite)	$-b^3 h_k - b^3 h_l + h d_x^3 + h d_y^3$
$a = \iota, -\iota$ (opposite)	$-h(d_x^3 - d_y^3) + hb^2(d_x - d_y) + b^3(h_k - h_l)$.

The following reduction rules compute these sums for arbitrary coefficients. In the first row reduce $q^4 = -1$, $D_0 = -q^2$, and every even power of γ by $\gamma^2 = -q^3(q^2+1)^3$. The trace of q^n is zero unless $4 \mid n$, when it is $(-1)^{n/4}$. For the second row reduce $q^2 = 1, D_0 = 1$; the trace of an odd power is zero and that of an even power is -1. Use $(a+b)^3 = a^3 + b^3$ before either reduction. For the last row use $a^2 = -1$, $A = -vb$, $s_0 = vah$, $M_a = va(d_x - d_y)$; all odd powers of a sum to zero and the sum of a^2 is one. More explicitly, applying these rules to the four summands $s_\eta J^3,\ s_\eta A^2 J,\ -s_H A^3,\ -s_0 M_1 A^2$, in that order and dividing by v^4, gives

$q^4 = -1$	$h d_y^3$	$-hb^2 d_x$	$b^3 h_l$	$hb^2 d_x$
$q^2 = 1$	$h(d_x^3 + d_y^3)$	$hb^2(d_x + d_y)$	$-b^3(h_k + h_l)$	$-hb^2(d_x + d_y)$.

In each entry every coefficient of T_3^3, T_3 is zero. The sums of the four entries are the first two rows above. Adding the last two rows, using $2 = -1$, gives

$$\begin{aligned} &\text{same:} && \bar{A}_4(b^3 h_l + h d_y^3), \\ &\text{opposite:} && \bar{A}_4\{b^3 h_l - h d_y^3 + hb^2(d_x - d_y)\}. \end{aligned} \tag{4.51}$$

We used $v^4 = \bar{A}_4$, which follows from $u_E^4 = -f'(s_E)$ and $3/\rho^{12} = -1/L^4$. The whole fractional contribution has positive valuation by (4.46). Thus (4.51) gives the complete reduced fine periods before the external factor $\bar{c}_E$ in (4.20).

4.12 Global old traces, cusp gluing, and endpoint transport

We next identify the source coefficients in (4.51) intrinsically. The map α_3 traces z, and β_3 traces w, with the chain orientation fixed above. On the old curve obtained by tracing z, the relevant ordinary component has $y = 0,\ x = w^3$. Modulo three,

$$P_z = z(z-x)^2, \quad \frac{ka^2 H_s}{P_z} = \frac{zH_s(z-x,w)}{z-x}.$$

The complete monic finite-root residue is $xH_s(0,w)$, including the double root. The old plane equation is

$$F_{\mathrm{old}} = yw^3 + y^2w^2 + (y^3 - 6y)w + 3, \qquad \Omega_{\mathrm{old}} = dw/(F_{\mathrm{old}})_y.$$

On $y = 0$ its frame is dw/w^3. Consequently the actual global canonical trace has ordinary restriction

$$\nu = H_s(0,t_0)\,dt_0. \tag{4.52}$$

The weak correction divided by P_z is $(z-x)wJ$, with no finite pole, so its trace is zero there. Three-multiple errors likewise reduce to zero. Thus $\nu_0 = h, \nu_1 = h_l$ for the original ζ.

The differential ν is regular at every cusp: on each normalized cusp chart it is the canonical finite trace of the regular source differential, and finite duality preserves regular dualizing forms. Formula (4.52) gives regularity at every fine supersingular point. At a wild coarse point the marking quotient has ramification index three and different four. A pole of order m downstairs would pull back to order $4 - 3m$; regularity upstairs therefore permits only $m \le 1$. These wild points are the only possible poles.

The exact old trace compatibility is

$$C(\nu) = \kappa_e, \tag{4.53}$$

where (κ_e, κ_c) is the actual lower Rosenlicht reduction, and C is the relative Cartier trace. The generic square made by $\alpha_3, \beta_3, \alpha_2, \beta_2$ is Cartesian: two length-two chains with their common length-one chain concatenate to the cyclic length-three chain. Trace and pullback thus give

$$\mathrm{Tr}_{\beta_2}\,\mathrm{Tr}_{\alpha_3}\,\zeta = -\,\mathrm{Tr}_{\alpha_2}\,\beta_2^*\kappa$$

by (4.11). For x a unit and $y = -3/x$, $P_z \bmod 3 = z(z-x)^2$ and $P_{z,x} = z(z-x)$. The trace of the lower étale form $f(z)dz$ is the finite residue of $-f(z)P_{z,x}dz/P_z = -f(z)dz/(z-x)$, hence $-f(x)dx$. The connected coordinate is

$$e = -3/z = x(z^2 + xz + x^2 - 6), \quad \bar{e} = x(z-x)^2, \quad \bar{e}_x = z(z-x), \quad \bar{e}_z = -x(z-x).$$

For $g(e)de$ its multiplication trace vanishes at the two reduced roots, and its differential trace is the residue of $xg(e)dz$, also zero. Thus the right-hand old restriction is κ_e. The sole component over the left-hand canonical target has relative Frobenius degree three, exhausting the generic degree; its canonical trace is Cartier, locally $t^{3n+2}dt \mapsto s^n ds$, $s = t^3$. This proves the identity with its relative coefficient convention.

At a wild point the actual old trace satisfies

$$\begin{aligned}
&\alpha_3^*\,\mathrm{Tr}_{\alpha_3}(r_c\mathfrak{b}) = 0,\\
&\alpha_3^*\,\mathrm{Tr}_{\alpha_3}(r_c\mathfrak{e}) = r_c(t_c + r_c)\mathfrak{b},\\
&\alpha_3^*\,\mathrm{Tr}_{\alpha_3}(q_c\mathfrak{b}) = (t_c + r_c)^2\mathfrak{b}.
\end{aligned}$$

These formulas follow by expanding $k_c = t_c - s_c + r_c$ and selecting the invariant monomials under S. On the canonical branch $r_c = 0, t_c = s_c$, $\mathfrak{b} = -ds_c/s_c^3$. Expanding each B'-coefficient as a constant plus pure r_c- and s_c-series shows that every nonconstant coefficient has zero residue there: positive r_c-powers vanish or introduce a factor three through $r_c s_c = -3$, and positive s_c-powers leave a regular differential. Consequently (4.17) gives

$$\operatorname{Res}_s \nu = -\overline{c_\zeta}(0) = B_s, \qquad \sum_{s \text{ wild}} \operatorname{Tr} B_s = 0. \tag{4.54}$$

The last equality is the proper residue theorem. For the fine parameter t_0, the coarse parameter is $n = t_0^3/(1 - t_0^2)$. In characteristic three,

$$\frac{dn}{n} = -\frac{t_0}{1 - t_0^2}\, dt_0.$$

A regular coarse remainder pulls back with order at least four. Consequently the fine jets at a wild point are $h = 0$ and $h_l = -B_s$.

We also need whole-parameter transport at the two ends. The Vélu formulas above preserve the same cyclic-four subgroup. The source parameter of the chain on C_3 reduces to l, and its target parameter on C_0 reduces to k:

$$\frac{(-3/w)^3 - 6(-3/w) + w}{4} \longmapsto l, \qquad \frac{(-3/z)^3 - 6(-3/z) + z}{4} \longmapsto k.$$

These are identities of whole reduced rational functions, so there is no second-jet change of coordinate. The two supersingular labels differ by Frobenius cubed. On E_0, $F^2 = [-3]$; every cyclic prime-to-three subgroup, and the auxiliary four-marking since $-3 = 1 \bmod 4$, is fixed by F^2. Thus the labels and the target coefficients are in $\mathbb{F}_9$. Only on this finite set does the paired end have cubic coefficient conjugation.

The exact Kummer character coefficients of (4.27), after multiplication by $9(1 - x^2)^2(1 - y^2)^2/(dx/x)$, are

$$\begin{aligned} C_{11} &= 4a - 2(x + y)(b + c) + (x + y)^2 d, \\ C_{21} &= 2a + 2xb - (x + y)c - x(x + y)d, \\ C_{12} &= 2a - (x + y)b + 2yc - y(x + y)d, \\ C_{22} &= a + xb + yc - 3d. \end{aligned}$$

They follow from the exact inverse-different identities

$$\frac{1}{P'_z} = -\frac{u^2 + 2u}{3(1 - x^2)(1 - y^2)}, \qquad \frac{z}{P'_z} = \frac{(x + y)u - xu^2}{3(1 - x^2)(1 - y^2)}$$

and their w, v counterparts. Same invariance sets $C_{11} = C_{22} = 0$; opposite sets $C_{21} = C_{12} = 0$. Reducing gives $a = -yc, b = -yd$ on $x = 0$, and $a = -xb, c = -xd$ on $y = 0$. Together with (4.42) these yield the actual nonclosed outer principal parts

$$W_{C_3} = \frac{b\, l^{-2} - d_x l^{-1}}{\bar{c}_E} + \text{closed terms}, \qquad W_{C_0} = \frac{\sigma b\, k^{-2} + d_y k^{-1}}{\bar{c}_E} + \text{closed terms}.$$

Using (4.2) in the natural tame moduli parameters and the finite $\mathbb{F}_9$ conjugation therefore gives

$$\sigma b = \epsilon b^3, \qquad d_y = -\epsilon d_x^3, \quad d_y^3 = -\epsilon d_x. \tag{4.55}$$

Arbitrary source coefficients are not cubed in this transport; their traces always use relative Frobenius.

For the coarse endpoints let s denote their common target tame label. With $p = w_E(P)/\bar{c}_E \in \mathbb{F}_3$, the actual values are

$$W_{\mathrm{own},0} = W(s)^3, \qquad W_{\mathrm{own},\infty} = \epsilon W(s) + p. \tag{4.56}$$

Their difference cubed is $W(s) - \epsilon W(s)^3 - p$. This uses Frobenius on the finite supersingular set, not an inverse Frobenius on an ordinary function field.

4.13 Wild values and the three possible coefficients A_2

The values in (4.56) are finite even when $\bar{A}_2 = 0$. We give the required node argument. For a normal node $\mathcal{R}[[x,y]]/(xy - \varpi^e)$ at an ordinary–own attachment, the good parameter $f = -X_g/Y_g$ is regular and both original generic branch images lie in the smooth additive identity domain. There

$$f/u_E = -w_E + s_E w_E z_E.$$

It is integral at the two vertical primes; at horizontal primes u_E is a unit. Normality therefore makes f/u_E integral at the node. Since $1/Y_g = -f^3 U(f)$, with U a unit,

$$w_E = -f/u_E - s_E (f/u_E)^3 U(f).$$

Thus w_E is integral and has the same finite value on both branches. Rational connecting chains contract to this normal node; functions descend because the proper birational contraction has pushforward $\mathcal{O} = \mathcal{O}$. The hypotheses on the ordinary branch follow from the Néron mapping property; those on the own branch were proved above. This establishes the actual endpoint values for all $\bar{A}_2$.

At a full-wild point represent the supersingular object by

$$E_0 : y^2 = x^3 - x, \qquad u(x,y) = (x+1, y).$$

The group subtraction formula gives $u - u^2 = (x^3, -y^3) = -F_3$. A cyclic prime-to-three subgroup stable under u is consequently Frobenius stable. The full two-torsion is also rational over $\mathbb{F}_3$. Every full-wild point of the oriented tame curve is therefore $\mathbb{F}_3$-rational. Since W descends from the original rational map,

$$W(s)^3 = W(s).$$

The actual quotient parameter is $n = t_0^3/(1 - t_0^2)$, and $\theta = -dn/n$. At a regular function's value $n = 0$, the derivation $D_\theta = -n\, d/dn$ and its square vanish. Equation (4.25) gives

$$(1 + \bar{A}_2) W(s) = 0. \tag{4.57}$$

The selected translation has the following exact arithmetic constraint:

$$P_K \neq 0 \implies \bar{A}_2 = \bar{A}_6 = -1, \quad b^2 = 1, \quad \epsilon = \sigma, \quad \mathrm{SS}_{\mathrm{wild}} = \varnothing. \tag{4.58}$$

Indeed nontrivial conjugation over $\mathbb{Q}(\zeta_3)$ sends K to K^{-1}, so it sends P_K to $-P_K$. Its X-coordinate is rational and $Y \in \sqrt{-3}\mathbb{Q}_3$. The complete three-division polynomial, divided by three, is

$$X^4 + 4A_2X^3 + 6A_4X^2 + 12A_6X + 12A_2A_6 - 3A_4^2.$$

A negative rational valuation of X would make X^4 uniquely smallest; a unit X would force a unit Y, impossible in $\sqrt{-3}\mathbb{Q}_3$. Thus $X \in 3\mathbb{Z}_3$. Writing $Y = \sqrt{-3}y_0$ in the curve equation gives y_0 a unit and $\bar{A}_6 = -1$. The constant term of the division polynomial must be in $9\mathbb{Z}_3$, giving $\bar{A}_2\bar{A}_6 = \bar{A}_4^2 = 1$. The good formal coordinate of this point has order one, by $v_\rho(s_E) = 4,\ v_\rho(u_E) = 3,\ v_\rho(Y) = 6$. Equation (4.42) therefore makes its coefficient b one nonzero global constant. Comparing the double-pole terms in (4.25) gives $\bar{A}_2 b^2 = -1$, hence $b = \pm 1$. The Fricke equality was proved after (4.2). Finally a double pole cannot descend through a quotient of ramification index three, whereas W does descend. Thus no full-wild point occurs.

If $\bar{A}_2 = 1$, then $P_K = 0$, $b = 0$, and the complete principal part at each fine point is $-d_x/(\bar{c}_E t_0)$. Equations (4.51) and (4.55) identify its fine contribution, before the external c_E, with

$$\sigma\epsilon\bar{A}_4\bar{c}_E \sum_{\text{fine}} \operatorname{Tr}\operatorname{Res}(\nu W).$$

At a wild point $W = 0$ by (4.57), and ν has at most a logarithmic pole. There are no cusp residues. The proper residue theorem makes this sum zero. The complete coarse sum is, by (4.32) and (4.56), $-\bar{A}_4\bar{c}_E^4 p^3 \sum_{\text{wild}} \operatorname{Tr} B_s = 0$. This proves the $\bar{A}_2 = 1$ branch.

4.14 The closed third poles when $\bar{A}_2 = 0$

Here $P_K = 0,\ b = 0$, and $D_\theta^2 W = W$. The nonclosed principal part is a simple pole. Differentiating it twice can produce at most a third pole, so the equation excludes every closed pole of order greater than three. Write the complete local data as

$$W_{\text{pp}} = A_s t_0^{-3} + B_s^{\text{pp}} t_0^{-1}, \qquad B_s^{\text{pp}} = -d_x/\bar{c}_E.$$

If u_s is the regular linear coefficient, the equation gives $u_s = -A_s - B_s^{\text{pp}}$, and

$$(D_\theta W)_{\text{pp}} = -B_s^{\text{pp}} t_0^{-3} - A_s t_0^{-1}.$$

In particular the closed third pole cannot be omitted from $\operatorname{Res}(\nu W)$.

Let S_w be the full-wild set on the actual proper tame curve B, and $I = \mathcal{O}_B(-S_w)$. The simple principal parts $A_s^{1/3}/t_0$ at non-wild supersingular points, together with value zero at S_w, define

$$\alpha \in H^1(B, I).$$

There are no prescribed cusp poles. This class is intrinsic: changing a parameter changes the closed cubic term by a regular cube and transforms its root principal part by the same parameter change. The simple nonclosed term cannot contribute to a cubic pole. Equation (4.57) gives $W = 0$ at S_w.

We prove $\alpha = 0$, including its relative boundary. First the actual wild quotient has a smooth lift over the original unramified ring. In a coordinate reducing to t_0 its action and invariant are

$$h(t_0) = \frac{t_0 + 3}{1 - t_0}, \qquad n = \frac{t_0^3 - 9t_0}{1 - t_0^2}, \qquad t_0^3 + nt_0^2 - 9t_0 - n = 0.$$

The last monic equation gives a free rank-three algebra over $R[[n]]$. Its generic fixed field has degree three, and normality of $R[[n]]$ then proves $R[[t_0]]^{\langle h\rangle} = R[[n]]$. The tame marking deforms étale; elsewhere the effective stabilizers have order prime to three, and the cusp marking action is free. Hence B has a smooth proper good model over R.

In characteristic three its ramified trace table is

$$\mathrm{Tr}(1) = 0, \quad \mathrm{Tr}(t_0) = -n, \quad \mathrm{Tr}(t_0^2) = n^2, \quad \mathrm{Tr}(t_0^{-1}) = 0. \tag{4.59}$$

The first three are Newton identities for $t_0^3 + nt_0^2 - n = 0$; the last follows from $t_0^{-1} = t_0 + t_0^2/n$. Thus

$$\mathrm{Tr}(\mathcal{O}_{t_0}) \subset (n), \qquad \mathrm{Tr}(t_0^{-1}\mathcal{O}_{t_0}) \subset (n),$$

including every regular remainder.

For $\ell \nmid 6M$, take actual pullback and function trace on the ℓ-isogeny correspondence. At fine unramified charts they commute with extracting the cubic-root principal part, since $\mathrm{Tr}(f^3) = \mathrm{Tr}(f)^3$. At a ramified branch above S_w, the preceding trace inclusions put even its regular remainder in I. At an unramified branch a stable kernel takes a wild endpoint to another wild endpoint, so its prescribed zero value is preserved. This constructs the Hecke action on $H^1(B, I)$ with its full relative condition.

The original modular-map identity gives $T_\ell W = a_\ell W$. On the actual correspondence the sum of φ is $a_\ell\varphi$; its possible constant is zero at the marked infinity. For the auxiliary two-marking choose representatives $\left(\begin{smallmatrix} 1 & 2j \\ 0 & \ell \end{smallmatrix}\right)$, so all preserve that cusp marking. Reducing in the original additive identity group gives the stated function identity. Its Fourier normalization is

$$T_\ell^{\mathrm{fun}} = V_\ell + \ell U_\ell, \qquad W(q) = -\sum_{3\nmid n} a_n q^n/n$$

in this linear branch. The equality $a_{\ell n} + \ell a_{n/\ell} = a_\ell a_n$ checks every coefficient. Extracting the root does not change $a_\ell \bmod 3 \in \mathbb{F}_3$, so

$$T_\ell \alpha = a_\ell \alpha. \tag{4.60}$$

The original residual inertia excludes its image in $H^1(B, \mathcal{O}_B)$. Indeed for $A_2 \in 3R$, the division polynomial has Newton polygon joining $(0,1)$ to $(4,0)$. Put $\varpi^8 = 3,\ X = \varpi^2 x,\ Y = \varpi^3 y$. The scaled equation reduces to $y^2 = x^3$; the divided torsion polynomial reduces to $x^4 - \bar{A}_4^2 = x^4 - 1$. Its eight nonzero points are distinct in the smooth locus and lift by Hensel's lemma. The additive coordinate $\overline{\varpi X/Y}$ identifies the original $E[3]$ with $(\mathbb{F}_9, +)$. If inertia sends ϖ to $\zeta_8\varpi$, it acts on this coordinate by ζ_8^{-1}. Thus its two characters are

$$\omega_2^7,\ \omega_2^5. \tag{4.61}$$

A nonzero $H^1(\mathcal{O}_B)$ eigenclass with (4.60) lifts through the finite torsion-free weight-two Hecke algebra to a characteristic-zero tame-level eigensystem. Serre duality gives the same eigenvalues, since the transpose correspondence differs only by scalar action on the cyclic subgroup and the two-orientation, both trivial here. The corresponding Jacobian has good reduction at three, and its residual constituent is a subquotient of a finite flat group scheme over the unramified ring. Raynaud's bound [Ray74, Theorem 3.4.3 and Corollary 3.4.4] makes its fundamental-character digits zero or one. With determinant cyclotomic, the possible pairs are $1, \chi_3$ or ω_2, ω_2^3; the

two-dimensional formulation and eigenvalue lifting are also stated in [Ser87, §2.8, Proposition 4, and §§3.1.6–3.1.7]. The digits of $5 = (12)_3$, $7 = (21)_3$ contradict this bound. Equality of away-Hecke traces identifies the semisimple residual representations by Chebotarev and Brauer–Nesbitt. Hence the image of α in $H^1(\mathcal{O}_B)$ is zero.

The remaining boundary lies in $H^0(\mathcal{O}_{S_w})/H^0(\mathcal{O}_B)$. For every $\ell = 2 \bmod 3$, $\ell \nmid 6M$, the polynomial $X^2 + X + 1$ is irreducible on $E_0[\ell]$; no cyclic ℓ-line is u-stable. Every branch above a wild point is therefore a ramified orbit of three, whose value trace is zero by (4.59). Thus $T_\ell = 0$ on this boundary. If $\alpha \neq 0$, (4.60) would give $a_\ell = 0 \bmod 3$ at every such inert prime. It follows that $E[3]^{\mathrm{ss}} \simeq E[3]^{\mathrm{ss}} \otimes \chi_3$, by comparing traces also at the split primes. But twisting (4.61) gives ω_2^3, ω_2, a different pair. This contradiction proves $\alpha = 0$.

There is therefore a global rational function R_A with the prescribed simple principal parts, no other poles, and zero values at S_w. Apply the proper residue theorem to $R_A \kappa_e$. Its wild residues are zero by the prescribed values, and it has no cusp residues. Cartier residue adjunction and (4.53) give

$$\sum_{\text{fine}} \operatorname{Tr} A_s \nu_2(s) = 0.$$

After identifying the relative Frobenius twist over a perfect residue field, Cartier is $C(\sum a_n t^n dt) = \sum a_{3n+2}^{1/3} t^n dt$. Thus the preceding sum is the cube of the zero sum of residues $R_A C(\nu)$, with the residue field traces transported by Frobenius. The proper residue theorem for νW, whose wild residues are zero, separately gives

$$\sum_{\text{fine}} \operatorname{Tr}(A_s \nu_2 + B_s^{\mathrm{pp}} \nu_0) = 0.$$

Together these annihilate the complete fine functional (4.51). The coarse sum vanishes as in the preceding branch because $W = 0$ at S_w. This proves the $\bar{A}_2 = 0$ case, including its closed third poles and its relative boundary.

4.15 The oriented Igusa class when $\bar{A}_2 = -1$

Put $w = \bar{c}_E W$, keeping the original rational unit. Then

$$D_\theta^2 w = w - w^3.$$

For $P_K \neq 0$, the common coefficient $b = \pm 1$ and (4.58) exclude all wild points. In the same case $\epsilon = \sigma = 1$, and (4.51) is exactly $\bar{A}_4 \bar{c}_E \sum \operatorname{Res}(\nu W)$ before the external factor. This is zero by the proper residue theorem. In the opposite case $\epsilon = \sigma = -1$; subtracting that same global residue leaves exactly

$$\bar{c}_E \bar{A}_4 \sum_{\text{fine}} \operatorname{Tr}\{\nu_0(s)(d_s - d_s^3)\}, \qquad d_s = d_x(s). \tag{4.62}$$

We construct a global primitive for these simple principal parts, including every cusp condition.

The complete Laurent expansion forced by the original differential equation is

$$w = b/t_0^2 - d_s/t_0 + u_s + e_s t_0 + \cdots, \qquad e_s = d_s - d_s^3, \quad e_s^3 = -e_s,$$

and

$$(D_\theta w)_{\mathrm{pp}} = b/t_0^4 + d_s/t_0^3 - b/t_0^2 - d_s^3/t_0.$$

The coefficients are at the actual $\mathbb{F}_9$ marked supersingular points. Define the actual rational function on B

$$R_0 = b/j - w^3 - D_\theta w + w. \tag{4.63}$$

Since $D_\theta(1/j) = -1/j$, the equation gives $D_\theta R_0 = -R_0$. Using $1/j = t_0^{-6} + t_0^{-4} + t_0^{-2}$ and $b^3 = b$ gives its entire principal part

$$(R_0)_{\mathrm{pp}} = -e_s(t_0^{-3} + t_0^{-1}).$$

It is the Artin–Schreier image of e_s/t_0. There are no other poles. In particular it is regular at every cusp; there θ has logarithmic residue prime to three, so $D_\theta R_0 = -R_0$ gives cusp value zero. Thus $Y^3 - Y = R_0$ is unramified on the proper curve, with the specified local regularizations.

With the orientation determined by the isogeny calculation below, the degree-three part of the characteristic-three level-nine Igusa character is

$$X^3 - X = -1/j. \tag{4.64}$$

Precisely, it is the ordinary rank-one étale three-adic Tate character modulo nine, followed by quotient by $\{\pm 1\}$. This is the explicit equation of [Kea88, §5, pp. 261–265, and §6, pp. 266–268]. The equation can also be checked from its order-nine torsion calculation: for $y^2 = x^3 + j^2x^2 - j^5$, the coordinate z on the twice-Frobenius twist satisfies

$$z^3 - j^{17}z^2 + (j^{33} - j^{32})z - j^{47} - j^{45} = 0.$$

Substitution $z = j^{16}/X - j^{16} + j^{15}$ gives, after multiplication by X^3/j^{47}, $j(1 - jX + jX^3)$, which is precisely (4.64).

A prime-to-three isogeny induces an isomorphism of the rank-one Tate modules. Conjugating a scalar character by a scalar does nothing, and a quadratic twist disappears in the quotient by $\{\pm 1\}$. Fixing one identification of that quotient with the additive $\mathbb{F}_3$, the same oriented character therefore occurs at both ends of every cyclic d-isogeny:

$$[1/j_d] = [1/j] \quad \text{in } k(B)/(F-1)k(B). \tag{4.65}$$

There is no factor d. This holds over the actual coarse function field: the generic automorphism group $\{\pm 1\}$ fixes every cyclic subgroup, so the subgroup descends on a chosen generic elliptic model; equivalently the projected character is already twist independent. It follows that a rational function A_d exists on the actual coarse $X_0(d)$ with

$$A_d^3 - A_d = 1/j_d - 1/j. \tag{4.66}$$

As an orientation check on $X_0(2)$, its parameters $j = (z+1)^3/z^2$, $j_2 = (z+1)^3/z$ give exactly

$$1/j_2 - 1/j = (1/(z+1))^3 - 1/(z+1).$$

Here is a self-contained derivation of the function Hecke identity used below. Trace (4.66) for $d = \ell \neq 3$ on the level-one correspondence:

$$T_\ell(1/j) - (\ell+1)/j = H^3 - H, \qquad H = \mathrm{Tr}(A_\ell).$$

The left side has at most a simple pole at $j = 0$ and no other finite poles. To check also the wild branches, choose fine invariant differentials with isogeny multiplier a, a unit. Verschiebung

compatibility gives $\mathrm{Ha}_t = a^{-2}\mathrm{Ha}_s$; since $j = \mathrm{Ha}^6/\Delta$ and the discriminants are units in a good supersingular deformation, j_s, j_t have the same order. Thus j_s/j_t is regular, and

$$jT_\ell(1/j) = \mathrm{Tr}(j_s/j_t)$$

is integral at $j = 0$. At a cusp every isogenous curve is a Tate curve, so the trace vanishes at infinity. A pole of H would give a pole of order divisible by three in $H^3 - H$, contradicting the simple-pole bound. Hence H is constant on the j-line and the left side is zero:

$$T_\ell(1/j) = (1 + \ell)/j. \tag{4.67}$$

For $P_K \neq 0$ the global point of character χ_3 gives $E[3]^{\mathrm{ss}} = 1 + \chi_3$; hence $a_\ell = 1 + \ell \bmod 3$ at good primes. At $\ell \parallel M$, multiplicative reduction has semisimplification $\eta_\ell + \eta_\ell\chi_3$, with $\eta_\ell(\mathrm{Frob}_\ell) = a_\ell = \pm 1$. Comparison gives $a_\ell \in \{1, \ell\} \bmod 3$. At $\ell^2 \mid M$ the local newform Euler factor is one. At three it is also one. Define, over $\mathbb{F}_3$,

$$P(V) = \prod_{\ell \parallel M} (1 - (\ell/a_\ell)V_\ell) \prod_{\ell^2 \mid M} (1 - V_\ell)(1 - \ell V_\ell) = \sum_{d \mid M} v_d V_d. \tag{4.68}$$

Every occurring d is an actual divisor of M.

The identity of full Fourier series is

$$\bar{f}(q) = \sum_{d \mid M} v_d \Delta(q^d). \tag{4.69}$$

We verify it at every Euler factor. Since $E_4 = 1 \bmod 3$, the full expansion of $1/j$ is Δ. The geometric function operator is $V_\ell + \ell U_\ell = \ell T_\ell^{(12)} \bmod 3$. Equation (4.67) therefore gives $\tau(\ell) = 1 + \ell \bmod 3$, including $\ell = 2$ on the level-one correspondence. The one-dimensional space of weight-twelve level-one cusp forms gives its Hecke recursion

$$\tau(\ell^{r+1}) = \tau(\ell)\tau(\ell^r) - \ell^{11}\tau(\ell^{r-1});$$

the relation follows equally by composing the cyclic correspondences and separating their one backtracking term. Modulo three this is the Euler denominator $(1-X)(1-\ell X)$. Multiplicativity accounts for all primes, and $\Delta = q\prod(1 - q^{3m})^8 \bmod 3$ accounts for three-depletion. For $\ell \parallel M$,

$$(1 - X)(1 - \ell X) = (1 - a_\ell X)(1 - (\ell/a_\ell)X);$$

for $\ell^2 \mid M$ both factors are removed. These exact formal identities prove (4.69) to every order.

At the original infinity cusp $\bar{\beta} = \bar{c}_E \bar{f}\, dq/q = -dw$ and $\theta = -dq/q$. Therefore $D_\theta w = \bar{c}_E \bar{f}$. On the auxiliary cusp $q = q_{\mathrm{aux}}^2$, both differentials acquire the same factor two. Equality of the complete rational-function expansions in (4.69) gives

$$D_\theta w = \bar{c}_E \sum_{d \mid M} v_d / j_d. \tag{4.70}$$

This is an equality of functions. If written as modular forms it has the Hasse factors $\mathrm{Ha}^5 f = \Delta$, or $f/\mathrm{Ha} = \Delta/\mathrm{Ha}^6 = 1/j$; no identification of weights two and twelve without those factors is involved.

Equations (4.65) and (4.70) now give

$$[R_0] = (b - \bar{c}_E P(1))[1/j].$$

At a fine supersingular point,

$$1/j - \{(t_0^{-2})^3 - t_0^{-2}\} = t_0^{-4} - t_0^{-2}.$$

Its leading pole order four is prime to three; no further AS coboundary can remove it, because a pole of order m in a function produces leading order $3m$ in its AS image. Thus every nonzero $\mathbb{F}_3$-multiple of $[1/j]$ is ramified, while $[R_0]$ is unramified. We conclude $b = \bar{c}_E P(1)$. Put

$$S_0 = \sum v_d A_d, \qquad G_0 = -w - \bar{c}_E S_0.$$

Then $G_0^3 - G_0 = R_0$ exactly. Near a fine point, the AS image of $G_0 - e_s/t_0$ is regular; a pole would produce a pole in that image. Hence G_0 has exactly the prescribed simple principal part e_s/t_0. It is regular at every other point, including every cusp, by the same argument. There are no wild points in this branch, so ν is holomorphic. The residue theorem for $G_0\nu$ annihilates (4.62). This completes both nonzero-P_K diagonals.

4.16 The relative wild boundary when $P_K = 0$

It remains to treat $\bar{A}_2 = -1,\ P_K = 0$. Here $b = 0$, and the complete fine principal parts of W are simple. Using (4.51), (4.55), (4.54), and (4.56), the full fine-plus-coarse scalar, including its external c_E, is

$$\bar{A}_4(1 - \epsilon - \sigma\epsilon) \sum_{s \in S_w} \mathrm{Tr}\{B_s W(s)\}. \tag{4.71}$$

Indeed the fine part is $-\sigma\epsilon\bar{A}_4\bar{c}_E^2 \sum B_s W(s)$; the coarse part is $\bar{A}_4\bar{c}_E^4 \sum B_s(W(s)^3 - \epsilon W(s) - p)^3$. Use $W(s)^3 = W(s)$, $\bar{c}_E^2 = \bar{c}_E^4 = 1$, and $\sum B_s = 0$. The four possible coefficients of the sum are $-1, 0, 1, 1$. They do not cancel uniformly. We prove the value pairing itself is zero.

The entire wild value module has the action

$$T_\ell = (1 + \ell)\,\mathrm{id}, \qquad \ell \nmid 6M. \tag{4.72}$$

For $\ell = 2 \bmod 3$, every cyclic kernel is in a free orbit of three under u, and its value trace vanishes by (4.59). This includes simple fine poles and all their regular remainders. For $\ell = 1 \bmod 3$, the Euclidean ring $\mathbb{Z}[u]$ has two primes $\pi_\ell, \bar{\pi}_\ell$ of norm ℓ. Their kernels are the two u-stable cyclic lines. Their quotient maps identify with these actual endomorphisms of E_0. Each is an invertible polynomial in u on the cyclic M-subgroup, so preserves it. Modulo two it is a nonzero scalar of $\mathbb{F}_4 = \mathbb{Z}[u]/2$, and preserves the A_3 orientation. Both branches return to the same marked wild point, each with coefficient one. All other branches have zero trace as above. This proves (4.72).

The original identity $T_\ell W = a_\ell W$ therefore shows that either all wild W-values are zero, or $a_\ell = 1 + \ell \bmod 3$ for every $\ell \nmid 6M$. In the latter case Chebotarev, determinant cyclotomic, and semisimple comparison give $E[3]^{\mathrm{ss}} = 1 + \chi_3$. A subline is not needed in this zero-translation case. All Euler arguments proving (4.68)–(4.70) are valid in this subcase. The AS function (4.63) with $b = 0$ is regular and zero at the wild points: $w(s)^3 = w(s)$, and $D_\theta w(s) = 0$. Its AS class is unramified everywhere. Thus $P(1) = 0$ by the same ramification test; at a wild point one can use $1/j = n^{-2}$ directly, whose pole order two is prime to three.

Existence of a wild point forces every prime dividing M to be $1 \bmod 3$: a stable cyclic ℓ^e-subgroup contains a stable order-ℓ line, which is impossible for an inert ℓ. If $M = 1$, then $P(1) = 1$, a contradiction, so all wild values are already zero. Assume $M > 1$. We determine the relative values as follows.

Normalize the functions in (4.66) on the actual coarse $X_0(d)$ by

$$A_d(\infty) = 0, \qquad A_1 = 0.$$

The right side of the AS equation is regular and zero at infinity, so A_d is regular there and has a value in $\mathbb{F}_3$; subtracting it gives the unique normalization. On the actual cyclic chain of degree de, write $\phi_d(E, C_{de}) = (E/C_d, C_{de}/C_d)$. The AS image of $A_{de} - A_d - A_e\phi_d$ is zero, so it is an $\mathbb{F}_3$-constant. Every map sends the distinguished chain $\mathrm{Tate}(q) \to \mathrm{Tate}(q^d) \to \mathrm{Tate}(q^{de})$ to the distinguished infinity. Its constant is zero:

$$A_{de} = A_d + A_e\phi_d. \tag{4.73}$$

This also applies to repeated primes, provided the total kernel is cyclic.

At a wild point both j, j_d have order two on the coarse normalization: on the fine deformation their orders are six and the common order-three stabilizer has ramification three. A pole of A_d would therefore create a pole of order at least three in $A_d^3 - A_d$, whereas the right side has order at most two. Thus every A_d is regular there and (4.73) can be evaluated.

An $\mathbb{Z}[u]$-stable cyclic ℓ^e-subgroup lies in exactly one of the two lifted ℓ-adic eigenlines. Indeed the idempotents split it into two parts, and a direct sum of two nonzero ℓ-primary groups cannot be cyclic. It is consequently $\ker(\pi_\ell^e)$ for one of the two primes. Quotient by π_ℓ leaves the same chosen direction, preserves all other prime-power subgroups, and preserves the two-orientation. Each later elementary ℓ-isogeny is the same coarse $X_0(\ell)$-point. Iterating (4.73) gives

$$A_d(s) = \sum_{\ell \mid M} \mathrm{ord}_\ell(d) A_\ell(s).$$

All these are values of actual cyclic isogenies, not of a correspondence with backtracking.

Every local factor of P is now $1 - X$ if $\ell \parallel M$, or $(1 - X)^2$ if $\ell^2 \mid M$. For $S_0 = \sum v_d A_d$, the product rule gives

$$S_0(s) = \sum_{\ell \mid M} A_\ell(s) P_\ell'(1) \prod_{q \mid M,\ q \neq \ell} P_q(1). \tag{4.74}$$

Each $P_q(1)$ is zero. Thus $S_0(s) = 0$ at every wild point if M has two distinct prime divisors, or if it is a prime power of exponent at least two, since then $P_\ell'(1) = 0$.

For these level shapes, compare with the original function without changing its additive origin. For the actual d-isogeny, $\theta_d = d \log j_d = d^{-1}\theta$; the equality follows either from Hasse and Kodaira–Spencer compatibility, or from their full Tate expansions, where $d^{-1} = d$ in $\mathbb{F}_3^\times$. Here every $d \mid M$ is $1 \bmod 3$. Differentiating (4.70) gives $D_\theta^2 w = -D_\theta w$. Its original equation therefore gives $w^3 - w = D_\theta w$. Since $P(1) = 0$, equations (4.66) and (4.70) also give

$$(\bar{c}_E S_0)^3 - \bar{c}_E S_0 = D_\theta w.$$

It follows on the connected curve that $w - \bar{c}_E S_0$ is one $\mathbb{F}_3$-constant. By (4.74), $W(s)$ has the same value at every wild point. Equation (4.54) annihilates their weighted sum.

There is one remaining level shape, $M = \ell$ prime and $\ell = 1 \bmod 3$. The two stable lines $\ker \pi_\ell, \ker \bar{\pi}_\ell$ are exchanged by the actual automorphism

$$(x, y) \longmapsto (-x, \iota y),$$

which conjugates u to u^{-1}. They represent one geometric point of the original coarse $X_0(\ell)$ after forgetting the two-marking. The particular W used here descends to that original tame curve: it is the additive coordinate of the original modular map on its naturally normalized outer component. Hence its finite values at all wild points above that point are equal. Again $\sum B_s = 0$ gives zero. This proves the vanishing of (4.71) for every sign pair and completes $\bar{A}_2 = -1$.

Completion of Theorem 4.1. For $\delta = 3$, the complete good-pole calculation has only the fine functional (4.51) and the coarse functional (4.32) at nonpositive order. Every other component and every omitted higher coefficient has strictly positive contribution, or has constant good image. The three cases $\bar{A}_2 = 0, 1, -1$ just proved make the full residue-field sum zero. Thus the coefficient of $c_E i^* \zeta$ lies in the maximal ideal of the chosen good extension. It belongs to the original unramified ring, so it lies in $3R$. Equation (4.30) gives the same conclusion for $\delta = 5$, and the case $3 \mid c_E$ was proved before any unit reduction. Using the actual old identity $i^*\zeta = 9i^*g$ yields $3c_E i^* g \in R\omega_E$ for every original Fourier-integral g. The unit degrees of the auxiliary markings cancel by their norm–pullback identity, and the inclusion descends through the unramified extension. The high-discriminant argument gave the stronger inclusion for $\delta = 9, 11$. Finally (2.4) proves the asserted ARS inequality, with the original m_E, r_E, c_E. □

5 The prime three at conductor depth four

Throughout this section, $N = 81M$, $3 \nmid M$, and E is the original optimal quotient of $J_0(N)$. We retain the differential, embedding, and Fourier-integral lattice of Section 2. Valuations on algebraic extensions of K are normalized by $v(3) = 1$. If a differential is considered on a component of a semistable model, its valuation means the valuation of its coefficient in the relative dualizing lattice at the generic point of that component.

Theorem 5.1. *If $v_3(N) = 4$, then*

$$9i^*S \subseteq R\omega_E.$$

Consequently $v_3(c_E r_E/m_E) \le 2$, and in particular $v_3(r_E/m_E) \le 2$.

The proof requires both the ordinary and the supersingular parts of the stable reduction. We first construct one actual old differential which removes the relevant poles of every element of $9S$. We then identify the components on which the original elliptic quotient can be nonconstant, and compute the trace of the remaining differential on those components.

5.1 The cyclic model and the first pole reduction

We use the regular cyclic models with auxiliary prime-to-three level. Write X_d for the model at depth d, and number its components so that the original cusp ∞ lies on C_d. Auxiliary level four can be used to obtain a fine moduli scheme. Passage from level four to level two, retaining the determinant components and dividing by the ineffective central sign, has degree eight and admits integral averaging at three. On the level-two presentation, the central automorphism

-1 acts ineffectively on the moduli parameter. Canonical differentials and all the weight-two quantities below descend through this ineffective action.

Let B denote the smooth tame base component, let g_B be its genus, and let $\kappa > 0$ be its number of cusps, counted with their geometric multiplicities. Put

$$w = \frac{2g_B - 2 + \kappa}{2}.$$

On a fine cover this is the degree of the Hodge bundle. On a level-two quotient it is simply the displayed number; the formulas below concern canonical divisors and remain valid. The supersingular divisor has degree $2w$.

The regularity theorem [KM85, Theorem 6.6.1] applies to this representable tame moduli problem. In the unrestricted characteristic-p crossing formula (3.2), put $p = 3$ and $n = 4$. The three internal factors are

$$(u - v^9)^2(u^3 - v^3)^2(u^9 - v)^2 = (u - v^9)^2(u - v)^6(u^9 - v)^2.$$

Together with the two external factors this gives

$$\operatorname{div}(3) = F = C_0 + 2C_1 + 6C_2 + 2C_3 + C_4 \tag{5.1}$$

and the intersection matrix wI, where

$$I = \begin{pmatrix} -54 & 18 & 2 & 2 & 2 \\ 18 & -18 & 2 & 2 & 2 \\ 2 & 2 & -2 & 2 & 2 \\ 2 & 2 & 2 & -18 & 18 \\ 2 & 2 & 2 & 18 & -54 \end{pmatrix}. \tag{5.2}$$

Here all reduced components are copies of the appropriate Frobenius twist of B. The local inputs to this matrix are the five branches

$$v - u^{81}, \quad v - u^9, \quad v - u, \quad u - v^9, \quad u - v^{81}$$

in the characteristic-three cyclic deformation, with multiplicities $(1, 2, 6, 2, 1)$. Their contacts are 9SS for the first and last adjacent pairs, and SS for every other pair. These give the off-diagonal entries in (5.2); intersection with the principal divisor F gives the diagonal entries. These are the cyclic deformation formula of [KM85, Theorem 13.4.7]. The different formulas used below are those of [CNS24, §§5.9–5.12].

Here is a uniform pole bound, including the integer rounding. For $g \in S$, put $x_j = -\operatorname{ord}_{C_j}(g)$, $0 \le j \le 3$. The expansion at the original cusp gives $\operatorname{ord}_{C_4}(g) \ge 0$. The horizontal part of the divisor of g is effective. Adjunction therefore gives

$$Mx \le \operatorname{diag}(M) + (2 - \kappa/w)\mathbf{1}, \qquad M = -I_{\{0,1,2,3\},\{0,1,2,3\}}.$$

Since the left side is integral and $\kappa > 0$, we may replace the right side by $(55, 19, 3, 19)^t$. The inverse of M is entrywise nonnegative. Indeed, for $e = (1, 2, 6, 2)^t$ one has $Me = (2, 2, 2, 18)^t > 0$, and the off-diagonal entries of M are nonpositive. If $t = \min_j(y_j/e_j) < 0$, attained at j, then

$(My)_j \le t(Me)_j < 0$. Thus $My \ge 0$ implies $y \ge 0$; the same argument proves nonsingularity, so the inverse is nonnegative. Direct multiplication gives

$$M^{-1}(55, 19, 3, 19)^t = (85/18, 76/9, 58/3, 14/3)^t.$$

Initially $x \le (4, 8, 19, 4)$. The second row then gives $18x_1 \le 137$, hence $x_1 \le 7$; the third gives $2x_2 \le 33$, hence $x_2 \le 16$. We have proved

$$g \in H^0(X_4, \omega_{X_4/R}(4C_0 + 7C_1 + 16C_2 + 4C_3)). \tag{5.3}$$

Let $\beta_d : X_d \to X_{d-1}$ delete the first isogeny of the cyclic chain. Set $U = \mathrm{Tr}_{\beta_4}$ and $V = \beta_4^*/3$, where trace and pullback are operations on differentials. At the original cusp,

$$U\left(\sum a_n q^n \frac{dq}{q}\right) = \sum a_{3n} q^n \frac{dq}{q}, \qquad V\left(h(q)\frac{dq}{q}\right) = h(q^3)\frac{dq}{q}, \qquad UV = 1.$$

Consequently

$$g_0 = (1 - VU)g \in S, \qquad Ug_0 = 0, \qquad i^* g_0 = i^* g. \tag{5.4}$$

The last equality uses an actual lower-level oldform, not a projection on a space of reduced components. This construction takes place at the original coarse level. Pullback to the auxiliary cover preserves the trace-zero equality because the prime-to-three marking is transported by the isogeny.

We record why the trace-zero condition improves the central pole. For a finite generically separable extension of discrete valuation rings $A \subset B$, with ramification index e, the dualizing trace satisfies

$$\mathrm{Tr}(\mathfrak{m}_B^n \omega_B) = \mathfrak{m}_A^{\lfloor n/e \rfloor} \omega_A \quad (n \in \mathbb{Z}). \tag{5.5}$$

Indeed, let $\mathfrak{C}$ be the inverse different. For the B-ideal $J = \mathfrak{m}_B^n \mathfrak{C}$, the inclusion $\mathrm{Tr}(J) \subseteq \mathfrak{m}_A^r$ is equivalent to $J \subseteq \mathfrak{m}_A^r \mathfrak{C}$: multiplication by every element of B preserves J, so the defining trace condition for the inverse different applies to every element of J. The latter inclusion is equivalent to $n \ge er$. This proves (5.5). If the residue degree is one and $n \equiv -1 \pmod e$, the resulting map between the two one-dimensional graded pieces is nonzero and hence is an isomorphism.

For β_4, the complete table at the ordinary generic points is

component	C_0	C_1	C_2	C_3	C_4
image	c_0	c_0	c_1	c_2	c_3
e	1	2	3	1	1
f	1	1	1	3	3

$$R_{\beta_4,\mathrm{vert}} = (0, 1, 6, 2, 1).$$

In particular, C_2 is the unique component over c_1, and its residual map is an isomorphism. The inseparable residual maps occur on C_3, C_4. Put $\eta = 9g_0$. Equation (5.3) first gives poles bounded by $(2, 3, 4, 0, -2)$. An exact pole of order four at C_2 would, by (5.5) with $e = 3, n = -4$, have nonzero trace in the pole-two graded piece at c_1. There is no other component which can cancel this contribution. Since $U\eta = 0$, its pole is at most three. The restriction of the resulting line bundle to C_3 has degree

$$(2g_B - 2) + (2C_0 + 3C_1 + 3C_2 - C_3 - 2C_4) \cdot C_3 = -\kappa.$$

Its section vanishes there. Thus

$$\eta \in H^0(X_4, L), \qquad L = \omega_{X_4/R}(D), \qquad D = 2C_0 + 3C_1 + 3C_2 - C_3 - 2C_4. \tag{5.6}$$

5.2 An old differential on the entire nonreduced cycle

Let

$$A = C_0 + 2C_1 + 3C_2, \qquad \gamma = \beta_2\beta_3\beta_4 : X_4 \longrightarrow X_1.$$

Write D_0, D_1 for the two components of X_1, with $\infty \in D_1$. The actual pullbacks and different are

$$\begin{aligned} \gamma^* D_0 &= C_0 + 2C_1 + 6C_2 + 2C_3, \qquad \gamma^* D_1 = C_4, \\ R_{\gamma,\mathrm{vert}} &= (0,1,9,5,3). \end{aligned} \tag{5.7}$$

These formulas also follow successively from $\beta_2^* D_0 = c_0 + 2c_1$, then $\beta_3^*\beta_2^* D_0 = c_0 + 2c_1 + 2c_2$. Since $A \le \gamma^* D_0$, restriction of γ factors scheme-theoretically as an actual morphism $\phi : A \to B = D_0$. Its restrictions to C_0, C_1, C_2 are isomorphisms; the residual degree nine on $C_3 \to D_0$ is irrelevant to this assertion.

For $h \in H^0(B, \omega_B)$, lift the Rosenlicht differential $(h, 0)$ to a global canonical differential η_h on X_1, using Lemma 2.3. Define

$$\Theta_h = \gamma^* \eta_h / 9. \tag{5.8}$$

Its component orders are at least

$$(0,1,9,5,3) + (0,0,0,0,1) - 2(1,2,6,2,1) = (-2,-3,-3,1,2) = -D.$$

Pullback has its actual effective horizontal different, so this construction introduces no horizontal poles. Replacing η_h by $\eta_h + 3\zeta$ changes Θ_h by $\gamma^*\zeta/3$, whose orders are at least

$$(-1,-1,3,3,2) \ge -D + A = (-1,-1,0,1,2).$$

The restriction of Θ_h to A is therefore independent of the lift.

This restriction defines a morphism of line bundles

$$\phi^* \omega_B \longrightarrow L|_A. \tag{5.9}$$

Here is the local linearity needed for this statement. Extend a local function f on B to the lower nodal fibre by taking on the other branch its value at the node, and lift this pair to a function $\widetilde{f}$ on X_1. Two canonical lifts of $(fh, 0)$, one of which is $\widetilde{f}\eta_h$, differ by $3\omega_{X_1/R}$. Their divided pullbacks have zero restriction to A by the preceding order estimate. Moreover $\gamma^*\widetilde{f}|_A = \phi^* f$. This proves $\mathcal{O}_A$-linearity, including at the supersingular intersections.

At the generic points of C_0, C_1, C_2, the map (5.9) is nonzero: the residual maps are isomorphisms and the orders in (5.7) are exactly the orders used in the divided pullback. Adjunction and (5.2) give

$$\deg(L|_{C_j}) = 2g_B - 2 \qquad (j = 0,1,2).$$

The source has the same degree. A nonzero map of line bundles of equal degree on a smooth proper curve is an isomorphism. Hence (5.9) is an isomorphism on every reduced component; its local multiplier is a unit modulo the nilradical and therefore a unit on A. Thus it is an isomorphism on the entire nonreduced cycle.

We also need its global sections. If $A_{\mathrm{red}} = C_0 + C_1 + C_2$, the kernel of restriction to A_{red} is

$$L(-A_{\mathrm{red}})|_{C_1 + 2C_2}.$$

The degree on each reduced component is $-\kappa$. A section therefore vanishes on $C_1 + C_2$; the line left on C_2 is $L(-C_0 - 2C_1 - 2C_2)|_{C_2}$, again of degree $-\kappa$. It has no section. Restriction to the reduced cycle is injective. Under (5.9), a reduced section is a triple (h_0, h_1, h_2) of regular differentials on the same actual B. Agreement on the intersections forces $h_i - h_j$ to vanish on SS. Since

$$\deg \omega_B(-\mathrm{SS}) = -\kappa < 0,$$

the three differentials are equal. Conversely a single h pulls back to a section on all of A. We have proved

$$H^0(A, L|_A) = H^0(B, \omega_B), \quad \text{through the actual divided pullback (5.8).} \tag{5.10}$$

For $\eta = 9g_0$, choose the unique h corresponding to its restriction. Then

$$\xi_f = 9g_0 - \Theta_h \in H^0\big(X_4, \omega_{X_4/R}(C_0 + C_1 - C_3 - 2C_4)\big). \tag{5.11}$$

The differential used for the original elliptic quotient must itself be an original coarse oldform. The construction of h is equivariant for the tame marking group, so h is invariant. It descends to a rational differential h_0 on the original tame component. At a wild marking stabilizer the ramification and different are $(3, 4)$ or $(6, 7)$. Regularity upstairs therefore allows at most simple poles downstairs. They occur only at supersingular points. Their residues sum to zero.

The original depth-one model is nodal by Lemma 3.5. The residue exact sequence gives a differential h_1 on its other component with opposite residues at every node. Thus (h_0, h_1) is a Rosenlicht differential. Lift it on the original model and pull it back by γ, divided by nine; denote the resulting original oldform by Θ. Its pullback to the fine model differs from Θ_h by a differential Δ with

$$\mathrm{ord}_{C_j}(\Delta) \geq R_\gamma + \gamma^* D_0 - 2F = (-1, -1, 3, 3, 1). \tag{5.12}$$

Indeed the two lower lifts have the same restriction to D_0, so their difference vanishes on that component. Set

$$\xi = 9g_0 - \Theta = \xi_f - \Delta. \tag{5.13}$$

Then

$$\xi \in H^0\big(X_4, \omega_{X_4/R}(C_0 + C_1 - C_3 - C_4)\big), \qquad i^*\xi = 9i^*g. \tag{5.14}$$

Every oldform in these equalities is a characteristic-zero global pullback from an actual lower-level curve. In particular, no division by the degree six of the full-two marking cover occurs.

5.3 The actual middle projection and its integral normalization

Let λ be the Legendre parameter of the middle elliptic curve in the cyclic chain of length four. If r_j is a local Cartier equation for C_j, the middle projection has vertical different

$$(2, 5, 21, 5, 2). \tag{5.15}$$

The target-projection different at cyclic type (a, b) is

$$d_{a,b} = \begin{cases} b, & a = 0, \\ 3^{\min(a,b)-1}(2b - 1), & a, b > 0, \\ 0, & b = 0. \end{cases}$$

This is [CNS24, Proposition 5.12], with the inseparable residual degrees included. Subtracting the pulled-back lower different gives $R_{\beta_3} = (0, 1, 2, 1)$ and $R_{\beta_2} = (0, 1, 1)$, as well as the value of R_{β_4} above. The two first steps have combined different $(0, 1, 9, 4, 2)$. The source-projection different $(2, 1, 0)$ at depth two pulls back first to $(2, 4, 1, 0)$ and then to $(2, 4, 12, 1, 0)$. Adding proves (5.15). Thus, up to a local unit,

$$\Omega = \frac{d\lambda}{r_0^2 r_1^5 r_2^{21} r_3^5 r_4^2} \tag{5.16}$$

is the canonical frame on the actual regular cyclic surface at a supersingular point.

We next determine the values of the r_j on the new affinoids. This is a calculation of functions through a finite map; it does not identify canonical lattices across a wild quotient. Take a full Drinfeld-nine basis P, Q on the middle elliptic curve E_2. The actual cyclic object is

$$E_2/\langle P\rangle \longrightarrow E_2 \longrightarrow E_2/\langle Q\rangle,$$

where the first arrow is the dual of the quotient by P. Its generic kernel is cyclic of order 81. The closed cyclic level condition extends it over the flat Drinfeld deformation ring, and the prime-to-three marking is transported by these actual isogenies. This gives a finite map to the cyclic deformation ring: the middle map is proper, and its fibres are the finite sets of incoming and outgoing subgroup configurations.

The full level ring has the twelve branches indexed by $\mathbf{P}^1(\mathbb{Z}/9\mathbb{Z})$. A branch parameter is $T([a]P + [b]Q)$, for a primitive vector (a, b), with the usual identification by a unit of $\mathbb{Z}/9\mathbb{Z}$. The regularity and divisor statement for full level structures gives

$$\operatorname{div}(\zeta_9 - 1) = \sum_{L \in \mathbf{P}^1(\mathbb{Z}/9\mathbb{Z})} L, \qquad \operatorname{div}(3) = 6 \sum_{L} L;$$

see [Lur18, Proposition 6.7, Corollary 6.9, and Lemma 6.11]. The partitions above $C_0, \dots, C_4$ have respectively $(1, 2, 6, 2, 1)$ members. If $e_j = (1, 2, 6, 2, 1)_j$, then

$$\operatorname{div}(r_j) = \frac{6}{e_j} \sum_{L \mapsto C_j} L. \tag{5.17}$$

This can also be checked from the generic group: the full group has order 3888, the cyclic middle cover has degree 108, and its diagonal stabilizer has order 36. After fixing the Weil pairing, the groups are $\mathrm{SL}_2(\mathbb{Z}/9\mathbb{Z})$, of order 648, and its diagonal subgroup of order six. Quotient by the ineffective central sign leaves a group of order three. The resulting cyclotomic normalization need not be the original regular cyclic ring; formulas (5.16) and (5.17) refer to the original ring and its actual finite cover, respectively.

In the region $v(\lambda + 1) \geq 1$, the height-two formal group has $[3](T)$ with linear coefficient 3, unit coefficient at T^9, and intermediate coefficients in $(3, \mathrm{Ha})$. After an unramified extension of the constants, rescale the formal coordinate so that the reduction of its T^9 coefficient is one; this requires only an eighth root of a unit. Its Newton polygon gives $v(T) = 1/8$ for every nonzero three-torsion point, and $v(T) = 1/72$ for every primitive nine-torsion point. Every branch parameter in (5.17) therefore has value $1/72$. Since the total exponent in that formula is six, we obtain

$$v(r_j) = 1/12 \qquad (0 \leq j \leq 4). \tag{5.18}$$

All local units have value zero at these points, since they reduce to nonzero constants at the supersingular closed point.

5.4 Four Hermitian affinoids and the canonical scale

We give the finite-level normalization explicitly. This will determine both the canonical scale and the actual points to be used in the support argument below. On the coarse curve $X_0(81)$, use [McM08, §3 and Proposition 3.5] in the following form:

$$\begin{aligned} x^3 &= z^6 + 9z^3 + 27, \qquad w_{81}(x,z) = (3x/z^2, 3/z), \\ f &= 2 + \frac{3(x-z)}{z^2 - x + 3}, \\ g &= \frac{x^2 + 10z - 9 - z(z-2)^3}{(2/3)(z^2 - x + 3)^2}, \qquad g^2 = f^3 + \tfrac{1}{4}. \end{aligned} \tag{5.19}$$

The double cover of the (f,g)-curve is obtained by adjoining the square root of

$$\mathcal{D} = (8f^2 - 80f + 44)g + 4f^4 - 28f^3 - 36f^2 + 40f - 22.$$

Choose $\beta^4 = 3$, $s = \beta^2$, $\theta^3 = 2 + s$, and $\gamma_0 = \theta^2 + 2\theta + 3$. Then $v(\gamma_0 + 1) = 1/3$. Above the disk $v(f - \gamma_0) \geq 2/3$, the coordinates

$$F_0 = \frac{f - \gamma_0}{(\gamma_0 + 1)^2}, \qquad Y = \frac{\sqrt{\mathcal{D}}}{\beta^3(\gamma_0 + 1)}$$

have smooth reduction

$$Y^2 = F_0^3 + F_0. \tag{5.20}$$

The conjugate disk supplies the other coarse new affinoid.

Put $h = x/z$, $S_0 = z + 3/z$. The middle j-invariant is given by the following exact identities, which follow by substitution in (5.19):

$$\begin{aligned} h &= \frac{f^2 - f + g - 1/2}{f + g - 1/2}, \qquad h^3 = S_0^3 - 9S_0 + 9, \\ J &= h^3 - 6 - 27/h^3, \qquad j_c = J^3, \\ j_c - 1728 &= \frac{(h^6 - 18h^3 - 27)(h^6 + 27)^2}{h^9}, \\ h^6 - 18h^3 - 27 &= (S_0^2 - 12)(S_0^2 - 3)^2, \qquad S_0^2 - 12 = (z - 3/z)^2. \end{aligned} \tag{5.21}$$

For example, the intermediate depth-two parameter is $t = z^9/x^3$, and its usual j-expression $(t + 27)(t + 3)^3/t$ becomes J^3. This also fixes the middle projection, as opposed to an endpoint projection.

We need the reductions of both j_c and its derivative, one half-grade apart. We include a finite arithmetic certificate and the bound on its omitted tail. In $\mathbb{Q}(\theta)$, put $b = \theta + 1$; its polynomial is

$$b^6 - 6b^5 + 15b^4 - 24b^3 + 27b^2 - 18b + 6.$$

It is Eisenstein at three. Accordingly the value of a reduced polynomial $\sum_{i=0}^{5} a_i b^i$ is exactly $\min_i(v_3(a_i) + i/6)$; different fractional parts prevent cancellation. The chosen value of g at $f = \gamma_0$ is

$$g_{\gamma_0} = \frac{2\gamma_0^4 - 14\gamma_0^3 - 18\gamma_0^2 + 20\gamma_0 - 11}{-2(2\gamma_0^2 - 20\gamma_0 + 11)}.$$

Writing $d = f - \gamma_0$, $g = \sum_{n\ge 0} g_n d^n$, and $g_0 = g_{\gamma_0}$, all coefficients are fixed by

$$2g_0 g_n = [d^n](\gamma_0 + d)^3 - \sum_{j=1}^{n-1} g_j g_{n-j} \qquad (n \ge 1). \tag{5.22}$$

Substitute $d = (\gamma_0 + 1)^2 F_0$, divide the two series in the first line of (5.21), and then form $J^3 - 1728$. For $c_n = [F_0^n](j_c - 1728)$, reduction modulo the displayed degree-six polynomial gives

n	0	1	2	3	4	5	6	7	8	9	10
$v(c_n)$	∞	7	13/2	6	13/2	41/6	13/2	7	22/3	6	15/2

and

n	11	12	13	14	15	16	17	18	19	20	21
$v(c_n)$	47/6	15/2	8	25/3	8	17/2	53/6	15/2	9	28/3	9.

The nonzero residues needed here are

$$\overline{c_3/3^6} = \overline{c_9/3^6} = -1, \qquad \overline{2c_2/(3^6 s)} = \overline{4c_4/(3^6 s)} = -1.$$

These identities are obtained by the rational operations in (5.22); together with the Eisenstein polynomial they specify the arithmetic completely, without a floating-point approximation or a choice of a formal solution.

To control all higher coefficients, work on the larger open disk $v(F_0) > -1/6$, so $v(d) > 1/2$. There

$$g = g_{\gamma_0}\sqrt{1+Q}, \qquad Q = \frac{3\gamma_0^2 d + 3\gamma_0 d^2 + d^3}{g_{\gamma_0}^2}, \qquad v(Q) > 1/6.$$

The binomial series converges and gives $v(g - g_{\gamma_0}) > 5/6$. The denominator defining h has value $1/3$, its numerator has value $5/6$, and hence $v(h) = 1/2$, $v(J) = 1$. Thus $v(j_c - 1728) \ge 3$ on this larger disk. The Gauss coefficient inequality, with the radii tending to its boundary, gives $v(c_n) \ge 3 + n/6$. For $n \ge 22$ this is greater than $13/2$. Differentiation multiplies by an integer and cannot lower that bound. We have therefore proved the identities of reductions

$$\overline{\frac{j_c - 1728}{3^6}} = -(F_0^9 + F_0^3) = -Y^6, \qquad \overline{\frac{dj_c/dF_0}{3^6 s}} = -(F_0^3 + F_0) = -Y^2. \tag{5.23}$$

Adjoining the actual full-two structure means adjoining a solution of

$$256(\lambda^2 - \lambda + 1)^3 - j_c \lambda^2 (\lambda - 1)^2 = 0.$$

The useful exact factorization is

$$j(\lambda) - 1728 = \frac{64(\lambda+1)^2(\lambda-2)^2(2\lambda-1)^2}{\lambda^2(\lambda-1)^2}.$$

The last two lines of (5.21) split the quadratic resolvent on the new disk after extending constants: the only remaining square-root factor is $h/h(\gamma_0)$, which is one plus a function of positive norm. For either choice of $r = \sqrt{j_c - 1728}/216$, set $\lambda = -1 + 3T$. The actual cubic normalization is

$$T(T-1)(2T-1) - r(3T-1)(3T-2) = 0. \tag{5.24}$$

Its leading coefficient is a unit and its reduced derivative is -1. By (5.23) its reduction is

$$T^3 - T = \alpha Y^3, \qquad \alpha^2 = -1.$$

Choose a with $a^3 = \alpha$, $a^2 = -1$, and put

$$W = T - aY, \qquad Z = -F_0 - W^2.$$

Substitution gives

$$W^3 - W = aY, \qquad Z^3 + Z = W^4. \tag{5.25}$$

This is the smooth affine part of a genus-three Hermitian curve. In particular the cubic cover is connected. Each of the two coarse disks has two such components, giving four actual affinoid points on the full-two cyclic curve.

The cubic equation (5.24) is finite étale on these affine integral models, so its vertical different is zero. Differentiating (5.25) gives $dF_0/Y = -a\,dW$. Differentiating the Legendre identity and using both formulas in (5.23) now gives

$$\overline{d\lambda/(3s)} = -dW, \qquad v(d\lambda) = 3/2. \tag{5.26}$$

The quotients of differential expressions used here extend over $Y = 0$, as follows either from the smooth cubic equation or from the nonzero reduced derivative in (5.24). This is a finite-level canonical calculation on the actual normalization. It is not a consequence of a purely inseparable comparison of abstract curves.

Prime-to-three markings deform uniquely in the strict henselian supersingular deformation. Their stabilizer is a subgroup $H \subseteq \mathrm{Aut}(E_{\mathrm{ss}})/\{\pm 1\} = S_3$. Up to conjugacy $H = 1, C_2, C_3, S_3$. On the preceding cubic cover, C_3 translates T, and a transposition exchanges its two branches. Thus over each coarse new disk the four cases give, respectively, two Hermitian components, one Hermitian component, two elliptic quotients, and one elliptic quotient. The degree-three quotient is finite étale at the vertical generic points, by (5.24).

Combining (5.18), (5.16), and (5.26) gives

$$v(\Omega) = \frac{3}{2} - \frac{35}{12} = -\frac{17}{12}.$$

The coefficient of ξ_f in (5.11) contains $r_3 r_4^2/(r_0 r_1)$, and the correction in (5.12) has total exponent five. Consequently

$$v(\xi_f) \geq -4/3, \qquad v(\Delta) \geq -1, \qquad v(\xi) \geq -4/3. \tag{5.27}$$

There is also a useful description of the boundary coefficient. Choose $\rho^{72} = 3$. The formal-group equation for every primitive nine-torsion parameter V_0, at the present affinoids, gives $\overline{(V_0/\rho)^{72}} = -1$. In a reduced characteristic-three field this implies $\overline{(V_0/\rho)^8} = -1$. The latter polynomial is separable and has constant roots. Hence all normalized leading branch parameters are constants, as are the leading values of units in the original complete local ring. From (5.26), the coefficient of ξ at value $-4/3$, when nonzero, is a constant multiple of dW.

Wild arithmetic inertia on the coarse new disk is $F_0 \mapsto F_0 + c$, $Y \mapsto Y$, where $c \neq 0$ and $c^3 + c = 0$; this is the explicit action in [McM08, §4.1]. The middle Legendre parameter is

defined over the original unramified field, so T is fixed. Consequently the same inertia element τ acts on (5.25) by

$$W \mapsto W, \qquad Z \mapsto Z - c.$$

The pole orders of W, Z at the unique point at infinity are three and four, and $\operatorname{div}(dW) = 4\infty$. Thus $Z\,dW$ is globally regular, and

$$dW = -c^{-1}(\tau - 1)(Z\,dW). \tag{5.28}$$

For an actual component map to a good elliptic target, dualizing trace commutes with inertia and sends global regular differentials to global regular differentials. An element of order three acts trivially on the one-dimensional regular-differential space in characteristic three. Equation (5.28) therefore kills the boundary trace. Wild inertia also acts trivially on the scalar associated graded of a valuation, so the assertion applies at the negative grade $-4/3$. It survives a marking quotient of order two by averaging, whose denominator is a unit.

5.5 The coefficient at a wild marking stabilizer

For $H = C_3$ or S_3, the regular primitive $Z\,dW$ in (5.28) need not descend. We instead use invariance of the original differential ξ.

The following actual endpoint chart fixes the orientation of the wild action. On the marked three-isogeny curve put

$$E_a : v^2 = t^3 + at^2 + t, \qquad a(x) = \frac{x^3 - 6x - 3/x}{4}, \qquad h(x) = \frac{x+3}{1-x}.$$

The three-division polynomial is $3t^4 + 4at^3 + 6t^2 - 1$, and its substitution at $t = 1/x$ is zero. The points with $t = 1$ give the cyclic-four marking; quotient by $(0, 0)$ gives the full-two Legendre structure, with $\lambda = (a-2)/(a+2)$. The identities

$$\begin{aligned} a(h(x)) &= 2(a(x) - 6)/(a(x) + 2), \\ \lambda(h(x)) &= (\lambda(x) - 1)/\lambda(x), \qquad h^3(x) = x, \\ h(x)h(-3/x) &= -3 \end{aligned} \tag{5.29}$$

show that h is the actual three-cycle of the full-two labels; the marked isogeny presentation is also the one in [Zhu14, Proposition 2.1].

Retain the first and last three-isogenies of the length-four chain, and pull back their outward parameters to x, y. Their divisors are C_0, C_4; these components meet transversely, so x, y are regular parameters of the cyclic local ring R_4. A single permutation of the full-two labels commutes with both endpoint maps, because the labels are transported through the same isogeny chain. Thus the action on both x, y is the same h, and not opposite generators. Since (5.1) gives $3 \in \mathfrak{m}_{R_4}^{12}$,

$$h(x) = x + x^2 \pmod{\mathfrak{m}^3}, \qquad h(y) = y + y^2 \pmod{\mathfrak{m}^3}. \tag{5.30}$$

We explicitly include the constraint unit in the canonical action. The regular ring has a presentation

$$R_4 = W[[x, y]]/(3 - F(x, y)), \qquad F \in (x, y)^{12},$$

where W is the unramified coefficient ring. To obtain it, lift an expression of $3 \in \mathfrak{m}^{12}$ to $W[[x,y]]$. The resulting quotient is regular of dimension two and surjects onto R_4, so it is R_4. Lift h to the ambient power series ring and write $h(3-F) = U(3-F)$. For $\mathfrak{n} = (3,x,y)$,

$$(U-1)(3-F) = F - h(F) \in \mathfrak{n}^{12}.$$

The associated graded ring is a polynomial domain and $3-F$ has order one, whence $U-1 \in \mathfrak{n}^{11}$. For a relative canonical frame Ω, its multiplier is the Jacobian divided by U. Equation (5.30) gives

$$h(\Omega)/\Omega = 1 + 2(x+y) \pmod{\mathfrak{m}^2}.$$

The branches C_1, C_3 have tangent lines $x=0, y=0$. Choose their equations r_1, r_3 with those initial terms. Comparing degree-two terms in $h(r_j) = U_j r_j$ gives

$$h(r_1)/r_1 = 1 + x \pmod{\mathfrak{m}^2}, \qquad h(r_3)/r_3 = 1 + y \pmod{\mathfrak{m}^2}.$$

Thus a frame for the line in (5.14) is

$$\nu = \frac{r_3 y}{x r_1}\Omega, \qquad h(\nu)/\nu = 1 + y \pmod{\mathfrak{m}^2}.$$

Write $\xi = K\nu$, with $K \in R_4$. Since ξ is an original coarse differential, $h(K)h(\nu)/\nu = K$. Its degree-one part first forces the constant term of K to be zero. If $K = ax + by \pmod{\mathfrak{m}^2}$, the degree-two part is

$$ax^2 + axy + 2by^2 = 0.$$

The associated graded ring is $k[x,y]$, so $a = b = 0$. Hence

$$K \in \mathfrak{m}^2. \tag{5.31}$$

We will also use one further consequence. If its quadratic part is $ax^2 + bxy + dy^2$, the cubic part of the invariance equation is

$$2ax^3 + (a+b)x^2 y + 2bxy^2 = 0.$$

Thus $a = b = 0$, and the quadratic part is a multiple of y^2.

On a new affinoid $v(x) = v(y) = 1/12$, and the frame ν has value $-17/12$. Equation (5.31) therefore gives

$$v(\xi) \geq -5/4. \tag{5.32}$$

The normalized cubic quotient is finite étale at its vertical generic point, so this is the same canonical bound on the actual elliptic quotient component.

5.6 The original quotient and the support of its specialization

We next prove that the estimates just obtained concern precisely the supersingular components which can map nontrivially to the original E. Rational cohomology will only be used for this constancy assertion; it supplies no scalar to an integral differential comparison.

Fix $\ell \neq 3$, with coefficients in $\overline{\mathbb{Q}}_\ell$, and retain the prime-to-three auxiliary marking. Let X_m be the proper modular curve at full level 3^m on the middle elliptic curve, and let U_m be its supersingular tube. The actual open immersion defines

$$j_{m,!} : H^1_c(U_m, \overline{\mathbb{Q}}_\ell) \longrightarrow H^1(X_m, \overline{\mathbb{Q}}_\ell). \tag{5.33}$$

This is extension by zero, and it commutes with level change, with the away-from-three Hecke operators, and with the $\mathrm{GL}_2(\mathbb{Q}_3)$-action on the full tower. Cusps belong to the ordinary complement, so compactification does not change this map. The ordinary/supersingular decomposition and the uniformization of the tubes are described geometrically in [Cho15, §§2.1–2.3 and 6.1]; these geometric constructions apply to $\overline{\mathbb{Q}}_\ell$-cohomology.

Let $D_{\mathbb{Q}}$ be the quaternion algebra ramified at three and at infinity, and let $D_3 = D_{\mathbb{Q}} \otimes \mathbb{Q}_3$. The uniformization, including the prime-to-three marking set, gives

$$H^1_c(U_\infty) = \Big(\mathcal{A}(K^{(3)}) \otimes H^1_c(\mathrm{LT}_\infty)\Big)^{D_3^\times}. \tag{5.34}$$

Here $\mathcal{A}(K^{(3)})$ is the locally constant automorphic function space on the definite quaternionic double quotient at the given away-from-three level. Equivalently one can form the full contracted product of the Lubin–Tate tower and the adelic marking set before taking cohomology. Formula (5.34) retains the stabilizers and all global marking multiplicities.

For odd residual characteristic, at a fixed central character χ_0, [Wei16, Theorem 3.1.1] gives the equivariant decomposition

$$H^1_c(\mathrm{LT}_\infty)[\chi_0] = \bigoplus_{\Pi \text{ discrete series}} \Pi \otimes \mathrm{JL}(\Pi^\vee) \otimes \mathcal{H}(\Pi)'. \tag{5.35}$$

For a supercuspidal Π, the last factor is its two-dimensional Weil representation with the normalization of that theorem. Inserting this into (5.34) retains, for each Π, the full factor

$$\Big(\mathcal{A}(K^{(3)}) \otimes \mathrm{JL}(\Pi^\vee)\Big)^{D_3^\times}.$$

Taking invariants preserves the component inclusions used below.

On a semistable model, normalization and the vanishing-cycle sequence identify the middle-weight term as

$$\mathrm{Gr}^W_1 H^1(X_m) = \bigoplus_C H^1(C^\nu). \tag{5.36}$$

For $H^1_c(U_m)$, the corresponding sum runs over components in that tube. The map induced by (5.33) is the inclusion of these summands. This description follows by applying normalization to a semistable affinoid covering: components give the displayed terms, and nodes and ends give the terms in the other weights. The polarization pairing on (5.36) is the sum of the nondegenerate pairings of the component curves.

If the local representation of E is principal series, its dual is principal series also. Equations (5.34)–(5.35) imply that its global factor pairs to zero with the image of (5.33). After a field extension where E has good reduction, $H^1(E)$ has weight one. Functoriality for the actual modular map therefore makes

$$H^1(E_k) \longrightarrow H^1(C^\nu)$$

zero for every supersingular component C. A nonconstant map of smooth proper curves to E_k is finite and has nonzero pullback on $\overline{\mathbb{Q}}_\ell$-cohomology: its purely inseparable part is a universal homeomorphism, and for its separable part trace composed with pullback is its positive degree. Thus every such component is contracted. This applies to every semistable refinement, without an assumed list of its components.

We now treat the supercuspidal case and identify its original newvector. Write $K_0 = \mathrm{GL}_2(\mathbb{Z}_3)$ and $K_r = 1 + 3^r M_2(\mathbb{Z}_3)$. A conductor-four supercuspidal representation with trivial central character is twist-minimal: [HNS19, Lemma 2.1] applies because its central-character conductor is less than half its conductor. Proposition 2.9 of that paper gives an unramified quadratic torus T and

$$\Pi = c\mathrm{Ind}_{ZT(\mathbb{Z}_3)K_1}^{\mathrm{GL}_2(\mathbb{Q}_3)} \chi, \qquad \rho = \mathrm{Ind}_{T(\mathbb{Z}_3)K_1}^{K_0} \chi, \qquad \dim \rho = 6.$$

On $K_1/K_2 = M_2(\mathbb{F}_3)$, the characters of ρ, each with multiplicity one, are $1 + 3X \mapsto \psi(\mathrm{tr}(BX))$, with B in the trace-zero, determinant-one elliptic orbit. Triviality of the central character forces trace zero, and $T^2 + 1$ is the only irreducible trace-zero quadratic over $\mathbb{F}_3$. Explicitly the six matrices are

$$\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}, \quad \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}. \tag{5.37}$$

Both off-diagonal entries of every matrix are nonzero.

We include the Mackey calculation at principal level two. For the Cartan representatives $g_r = \mathrm{diag}(3^r, 1)$, $r \geq 0$, the term $r = 0$ contributes ρ. If $r \geq 1$, the group $g_r K_2 g_r^{-1} \cap K_0$ contains

$$U_1^- = \{1 + 3tE_{21} : t \in \mathbb{F}_3\}.$$

Indeed the preimage of this element has its lower-left entry divisible by 3^{r+1}, hence by nine. Every character in (5.37) is nontrivial on U_1^-. Thus the corresponding fixed space is zero in every other Mackey term, and

$$\Pi^{K_2} = \rho. \tag{5.38}$$

Let D denote the full diagonal group of order 36 in $\mathrm{GL}_2(\mathbb{Z}/9\mathbb{Z})$. It is the group which forgets the generators of the two actual cyclic-nine lines in the middle construction. With $a = \mathrm{diag}(1, 9)$,

$$a^{-1} K_0(81) a = D K_2.$$

The diagonal principal units kill all but the two weight lines with zero diagonal in (5.37); the residual diagonal ratio -1 interchanges these two lines. Its square is central and acts trivially. Hence

$$\dim \rho^D = 1. \tag{5.39}$$

Casselman's newvector theorem [Cas73, Theorem 1] also gives dimension one for the conductor-four $K_0(81)$ space. Equations (5.38) and (5.39), together with the actual middle isogeny correspondence, identify its translate. The target is still the original E.

At full-nine level an ordinary component is indexed by its connected-kernel line in $\mathbf{P}^1(\mathbb{Z}/9\mathbb{Z})$ and by its pairing label. For the line $\langle e_1 \rangle$, the upper unipotent group acts trivially on its reduced Igusa component: the connected marked point is zero on a reduced characteristic-three base, so replacing the other marked point by its sum with any multiple of the connected one changes nothing. The identity on the ordinary open extends to its compactification. Ordinary component cohomology is therefore induced from Borel modules on which U_1^+ acts trivially. By (5.37), $\rho^{U_1^+} = 0$. Frobenius reciprocity for every conjugate Borel and every pairing label shows that ρ occurs in none of these component spaces. The actual specialization of the original new line therefore contracts every ordinary component in the supercuspidal case. Components over cusps are rational and are contracted as well.

For principal series $\Pi = \mathrm{Ind}(\chi, \chi^{-1})$ with $a(\chi) = 2$, the analogous finite type is regular split. Its K_1/K_2-weights form the twelve-element orbit of $\mathrm{diag}(b, -b)$, $b \neq 0$. The U_1^+-fixed weights are the six matrices

$$\begin{pmatrix} b & t \\ 0 & -b \end{pmatrix}, \quad \begin{pmatrix} -b & t \\ 0 & b \end{pmatrix} \qquad (t \in \mathbb{F}_3).$$

The upper unipotent quotient acts transitively on each set of three. Its full invariant, or Jacquet, space is therefore the sum of the two diagonal characters (χ, χ^{-1}) and their interchange. The five D-orbits on $\mathbf{P}^1(\mathbb{Z}/9\mathbb{Z})$ have sizes $1, 2, 6, 2, 1$. On an axis the stabilizer is D, and on either near-axis orbit it contains independent wild diagonal ratios. The conductor-two character is nontrivial on each of these stabilizers. On the mixed orbit the stabilizer is scalar and the two characters cancel. Thus the original D-invariant new line has ordinary support only on the central ordinary pair. Together with the preceding extension-by-zero argument, this proves that these are the only possible nonconstant components for a principal-series target.

For later use we identify all supersingular support in the supercuspidal case. The mixed-characteristic affinoids of [IMT11, §§3.1–3.5] apply to $\mathbb{Q}_3$. At height zero, each of the six fixed-determinant components of full Drinfeld-nine space contains twenty-four actual Hermitian tails. Their labels satisfy

$$x_0^8 = y_0^8 = -1, \qquad x_0^3 y_0 - x_0 y_0^3 = \zeta, \tag{5.40}$$

and their reductions have the form $A^3 + A = \zeta z^4 - c_0$. All lie strictly in the region where the middle Hasse height exceeds $3/4$.

Let $\mathcal{W}$ be the direct sum of H^1 of these 144 proper residual curves, with its actual specialization map. [IMT11, Proposition 7.3 and Corollary 7.4] give

$$\mathcal{W} = \bigoplus_w \pi_w^\vee \otimes \tau_w \otimes (w \circ a_L), \qquad \dim \pi_w = 6, \quad \dim \tau_w = 3,$$

for strongly primitive characters w of the units in the unramified quadratic ring modulo nine. The acting groups are $\mathrm{GL}_2(\mathbb{Z}/9\mathbb{Z})$, $\mathcal{O}_{D_3}^\times$, and inertia. A fixed conductor-four Π has the two Galois-conjugate characters w, w^σ, and their combined contribution is $6(3 + 3) = 36$ at height zero. Modulo central scalar $3^{\mathbb{Z}}$, the tower has two framing heights. A division-algebra uniformizer moves these actual components to the other height. The division-algebra representation in the same propositions is the induction from $L^\times U_{D_3}^1$, and has dimension six. Thus on both heights the contribution is

$$6 \cdot 6 \cdot 2 = 72.$$

By (5.38) and (5.35), this is the entire contribution of Π^{K_2}. Equality is inside the same equivariant component specialization map. It follows that its supersingular component support is contained in these tails. Tensoring with $\mathcal{A}(K^{(3)})$ and taking the invariants in (5.34) preserves that inclusion.

5.7 Identification of the same four points

We now prove that the images of the tails just described are the four points for which (5.27) was computed. A genus bound on an actual proper curve avoids any assumption that abstractly isomorphic residual curves determine the same points.

Fix the Weil pairing and the full-two marking. The corresponding proper full-nine curve is $X(18)$. Its projective index and number of cusps are

$$\mu = \frac{18^3}{2}(1-2^{-2})(1-3^{-2}) = 1944, \qquad c = \frac{18^2}{2}(1-2^{-2})(1-3^{-2}) = 108.$$

There are no elliptic fixed points. Hence $g(X(18)) = 1 + \mu/12 - c/2 = 109$. Its twelve ordinary normalizations are the full-two Igusa-nine curves modulo the central sign. We compute their genus directly, and will use the same equation in the trace argument.

In characteristic three, write the Legendre curve as $y^2 = x^3 + Ax^2 + Bx$, where $A = -(\lambda+1)$, $B = \lambda$. The division-polynomial identity is

$$x([3]Q) = \frac{x(Q)^3\{x(Q)^6 + ABx(Q)^3 + A^2B^2\}}{\{Ax(Q)^3 - B^2\}^2}.$$

Thus Verschiebung on x-coordinates is

$$V(t) = \frac{t^3 + ABt^2 + A^2B^2t}{A^2t^2 + AB^2t + B^4},$$

and the nonzero kernel-V coordinate is B^2/A. For a primitive kernel-V^2 point, modulo sign, its coordinate therefore satisfies

$$Ax^3 + (A^4B^3 - A^6B^2)x^2 + (A^7B^6 - A^3B^8)x - B^{14} = 0. \tag{5.41}$$

Divide by A, denote the coefficients by a_0, b_0, c_0, and substitute $x = y + b_0/a_0$, $z = 1/y$. The result is

$$Cz^3 + a_0z + 1 = 0, \quad a_0 = \lambda^2(\lambda-1)^2(\lambda+1)^3, \quad C = -\frac{\lambda^{10}(\lambda-1)^{10}}{(\lambda+1)^3}.$$

Putting $z = (\lambda+1)^3Y/[\lambda^4(\lambda-1)^4]$ then gives

$$Y^3 - Y = \frac{\lambda^2(\lambda-1)^2}{(\lambda+1)^6}.$$

Finally, with $u = 1/(\lambda+1)$ and $Z = Y - u^2$,

$$Z^3 - Z = u^4 - u^2. \tag{5.42}$$

The right side has one reduced pole, of order four. Its different is $2(4+1) = 10$, so Riemann–Hurwitz gives genus three. The ordinary components consequently account for genus 36.

Let $\mathcal{Z}$ be the supersingular tube with this fixed determinant and framing height, after quotient by the central sign. The full-two Legendre line has a single supersingular point. In any semistable refinement of $X(18)$ containing the twelve ordinary components and any prescribed finite set of positive-genus points in $\mathcal{Z}$, the genus formula is the sum of the component genera and the first Betti number of the dual graph. All terms are nonnegative. Therefore

$$\sum_{z\in\mathcal{Z}} g(z) \le 109 - 36 = 73, \tag{5.43}$$

where the assertion applies to every finite subset and hence to all positive-genus points, including points in further refinements.

Here is a set of actual points reaching this bound. The central $m = 0$ affinoid has the Deligne–Lusztig reduction $XY^3 - X^3Y = 1$, of genus three. Its central-sign quotient, with $t = X/Y$, $v = 1/Y^2$, has equation $v^2 = t - t^3$ and genus one. Its middle Hasse height is $3/4$.

There are four neighboring $m = 0$ points. To construct them over $\mathbb{Q}_3$, take the four actual lattice-neighbor translates of Weinstein's $m = 0$ affinoid at infinite level. The matrices

$$g_a = \begin{pmatrix} 3 & a \\ 0 & 1 \end{pmatrix} \quad (a = 0, 1, 2), \qquad g_\infty = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$$

have determinant three; their columns generate the four index-three sublattices of $\mathbb{Z}_3^2$. Pair each matrix with a division-algebra uniformizer of reduced norm three. This restores framing height zero and preserves the fixed determinant. The division-algebra action changes the framing, not the underlying formal elliptic curve or its Hasse height. For each neighbor matrix g,

$$g^{-1} K_2 g \subset K_1.$$

Its residual action on the $m = 0$ Deligne–Lusztig curve is therefore trivial. The finite-level quotient statement [Wei16, Theorem 5.1.2 and Lemma 6.4.2] gives a nonconstant residual map from each actual finite-level component to that curve. After the central-sign quotient there is still a nonconstant map to its genus-one quotient, so each point has genus at least one. Positive residual genus prevents its removal in semistable pruning.

Their Hasse height and distinctness can also be checked on the actual isogeny. At the central point, every nonzero three-torsion parameter has value $1/8$; the determinant relation makes the leading residues of distinct lines distinct. If C is one of the four lines and $P \notin C$, the quotient isogeny satisfies

$$v(f_C(P)) = \sum_{c \in C} v(P -_F c) = 3/8.$$

This is a nonzero point of the dual kernel. The target Newton polygon has canonical three-torsion value $(1 - h')/2 = 3/8$, so $h' = 1/4$. At that height the canonical line is unique; its full-level marking records the chosen neighbor. Thus the four points are distinct. Equivalently their adjacent maximal orders are distinct and are individually fixed, not identified, by K_2.

The remaining points come from the mixed-characteristic calculations [IMT11, §§3.1–3.5]. The $Z_{1,1}$ affinoids have four directions and sixteen genus-one tails in each; the central sign acts freely on their sixteen labels, by $y_0 \mapsto -y_0$. They give thirty-two genus-one points in $\mathcal{Z}$, all of Hasse height $1/2$. The twenty-four Hermitian tails in (5.40) are also freely paired by the central sign and give twelve genus-three points, all of Hasse height greater than $3/4$. The different heights distinguish the four families, and the residue labels or the neighboring kernel lines distinguish their members. Their genus sum is

$$1 + 4 \cdot 1 + 32 \cdot 1 + 12 \cdot 3 = 73. \tag{5.44}$$

Together with (5.43), this proves that each neighbor has genus exactly one and that there are no additional positive-genus points. In particular every such point of Hasse height greater than $3/4$ is one of the twelve Hermitian points. The construction of the four neighbors used the unramified local field itself; no assertion requiring the underlying local field to have absolute ramification at least two or three has been applied here.

It remains to take the actual diagonal quotient. On the fixed determinant, central-sign quotient, its remaining group is C_3, represented by $\mathrm{diag}(1+3s, 1-3s)$. [IMT11, Lemma 5.1 and Proposition 5.3] give its action on the Hermitian tail as

$$(A, z) \longmapsto (A - s(x_0 y_0^3 + x_0^3 y_0)^3, z). \tag{5.45}$$

In absolute ramification one the correction term G_0 in that proposition is zero: its two coefficients $f_{1,0}$, $f_{0,1}$ are zero. Thus (5.45) is the actual action here. Writing $r = x_0/y_0$, the six possible ratios belong to $\mathbb{F}_9 \setminus \mathbb{F}_3$; the translation is zero precisely when $r^3 + r = 0$. There are two such ratios. Among the twelve central-sign orbits of labels, exactly four therefore have trivial residual action. Their quotients have genus three; the other eight have rational fixed field $k(z)$.

Wild residue-field inseparability does not alter this genus count. For a finite invariant ring $A^G \subset A$, the reduction of A^G injects into the invariants of the reduction. If an element v of a residual field is fixed by a group of order three, lift it to $a \in A$. The orbit polynomial $\prod_{g \in G}(T - ga)$ reduces to $(T-v)^3 = T^3 - v^3$. Hence v^3 lies in the residual field of the quotient. If a prime-to-three stabilizer is also present, the same argument with order $3n$ has coefficient $-nv^3$, and n is a unit. The remaining extension between the quotient residue field and the residual invariant field is thus purely inseparable. A purely inseparable extension between function fields of smooth curves over the perfect residue field preserves the genus. This proves the asserted genus three or zero for the actual quotient points.

Let $b_1, \dots, b_4$ be the four distinct points constructed from (5.24). Their middle parameter is $\lambda = -1 + 3T$, with $v(T) \geq 0$, so their middle Hasse height is at least one. Under the same finite modular map, any point above b_j has a finite extension of its residual function field. Since b_j has genus three, that point has positive genus: remove the purely inseparable part and apply Riemann–Hurwitz to the remaining finite separable map. Its Hasse height is still at least one, since the middle Hasse function is pulled back from the same elliptic curve. By (5.44) it is one of the twelve tails. We have just proved that the tails have exactly four positive-genus images under the diagonal quotient. The four b_j are therefore precisely these images.

Combining this identification with the equivariant support calculation proves the following assertion about the original modular map. In the supercuspidal case every nonconstant specialization component lies on the four affinoids already bounded in (5.27), or on their actual marking quotients. If a quotient component had a nonconstant map, any component above it dominating it would also have a nonconstant map, so the assertion descends through every $H = 1, C_2, C_3, S_3$. In the principal-series case the only possible nonconstant components are the central ordinary pair.

5.8 The discriminant and the central Fricke sign

Put $\delta = v_3(\Delta_{\min}(E))$. The local classification [DFV24, Proposition 5.2.2] gives, at conductor exponent four, potentially good reduction with semistability defect three or six. The local representation has either the principal-series form $\mathrm{Ind}(\chi, \chi^{-1})$, with $a(\chi) = 2$, or the supercuspidal form used above. We recall how the possible minimal discriminants follow, so the conductor and discriminant exponents remain distinct. Potentially multiplicative types are excluded. Type I_0^* acquires good reduction over a tame quadratic extension, and types III and III* over a tame quartic extension; these assertions follow from [DD13, Theorem 3]. None has the nontrivial wild inertia present here. The remaining types are $\mathrm{II}, \mathrm{IV}, \mathrm{IV}^*, \mathrm{II}^*$, with respectively $1, 3, 7, 9$

geometric components. Ogg's conductor formula $4 = \delta + 1 - \#\{\text{components}\}$ gives

$$\delta \in \{4, 6, 10, 12\}. \tag{5.46}$$

In the principal-series case with $\delta = 4$, the inertia character has order three. Here is a differential check of the tame part. If the semistability defect were six, pass to the minimal good-reduction extension over the maximal unramified field. A tame involution acts by negation on a uniformizer. In the change to a good minimal equation, write

$$\omega_E = a^{-1}\omega_{\text{good}}, \qquad v(a) = \delta/12.$$

Its differential character under that involution is $(-1)^{\delta/2}$, because the ramification index is six. For $\delta = 4$ this is $+1$. But the faithful inertia action on the good elliptic curve would give a nontrivial origin-preserving involution; in characteristic three its only such involution is $[-1]$, whose differential is -1. This contradiction proves that the tame involution is absent. Thus $\chi(-1) = 1$. The local newform formula [Sch02, Proposition 3.1.1 and §3.2] gives for the actual partial Fricke involution the eigenvalue

$$\varepsilon(\chi \oplus \chi^{-1}) = \chi(-1) = 1.$$

In particular $w_{81}i = i$ for this original quotient.

We describe the same involution on the central model. The marked three-isogeny chart of (5.29) has consecutive cyclic-isogeny relation

$$\mathcal{F}(A, B) = A^3 - A^2B^3 + 6A^2B + 3AB^2 - 9B = 0. \tag{5.47}$$

It follows by substituting the marked quotient parameter $-3/x$ into the equation $a(x)$ above; the factor corresponding to the immediately dual isogeny is removed. To fix the endpoint orientation, a four-isogeny chain and its reversed dual chain have parameters

$$(-3/z, x_{\text{int}}, w_{\text{int}}, B), \qquad (-3/B, -3/w_{\text{int}}, y_{\text{int}}, z).$$

The two outward parameters are therefore interchanged: $w_{81}(x, y) = (y, x)$. The middle elliptic object is fixed. Transport of a cyclic prime-to-three subgroup around the half-chain differs by [9], which preserves that subgroup. Thus this is the actual partial Fricke on the original $\Gamma_0(81M)$-curve, including its tame marking quotient.

On the ordinary branch let $Y(t)$ be the solution of $\mathcal{F}(Y(t), t) = 0$ with $Y(t) \equiv t^3 \pmod 3$. Eliminating the two internal parameters of the chain gives

$$3 = -\frac{Y(x)Y(y)(Y(x) - Y(y))^2}{(1 - Y(x)Y(y))^2}. \tag{5.48}$$

This identity can also be obtained by substituting the two relations $\mathcal{F}(Y(x), x) = \mathcal{F}(Y(y), y) = 0$ and the two dual parameters into (5.47); the remaining factor is its displayed symmetric numerator. Set $T_0 = (x + y)/2$, $r = x - y$. Its defining equation is even in r, and a dualizing generator dT_0/F_r is odd under Fricke. The central DVR has uniformizer r, and $\operatorname{ord}_{C_2}(3) = 6$. A differential with nonnegative central order consequently has plus average of central order at least one: modulo r, Fricke fixes its coefficient and negates the canonical frame.

For $\delta = 4$ in the principal-series case we may therefore replace ξ by the actual global differential

$$\xi^{+} = \frac{\xi + w_{81}^{*}\xi}{2}, \qquad i^{*}\xi^{+} = i^{*}\xi = 9i^{*}g. \tag{5.49}$$

The division by two is integral. The canonical cost at the central ordinary normalization is $9/6$. Indeed over the ordinary base the ramified constant extension is generated by ζ_9; its different has valuation $v(\Phi_9'(\zeta_9)) = 3/2 = 9/6$, and the residual Igusa cover is separable and finite étale on the ordinary open, where its vertical different is zero. The extra central zero just proved lowers the cost for ξ^{+} to $8/6$.

5.9 The surviving central coefficient and its poles

Put

$$\eta = \zeta_9 - \zeta_9^{-1}, \qquad \eta^6 + 6\eta^4 + 9\eta^2 + 3 = 0, \qquad v(\eta) = 1/6.$$

For $\sigma : \zeta_9 \mapsto \zeta_9^4$, the cubic expression for $\sigma(\eta)$, or direct use of the displayed equation, gives $\sigma(\eta)/\eta \equiv 1 \pmod{\eta^2}$. For $\delta = 4$, let

$$\beta = \overline{\eta^{8}\xi^{+}} \tag{5.50}$$

on a proper central ordinary normalization. This is a global meromorphic differential arising from an actual global differential. It is fixed by arithmetic σ, and by the stabilizer of the prime-to-three marking. On the ordinary open, the separable Igusa C_3-cover therefore expresses β as the pullback of a rational differential on the tame base. We identify this base and its supersingular parameter explicitly. On the mixed ordinary orbit the reduced chain is

$$E_0 \xrightarrow{F} E_0^{(3)} \xrightarrow{F} E_0^{(9)} \xrightarrow{V} E_0^{(3)} \xrightarrow{V} E_0.$$

The map γ retains the last Verschiebung arrow. Its component D_0, identified with C_2, is parametrized by the target E_0. Transport around $FV = [3]$ preserves the cyclic tame subgroup and the full-two labels. By [Lur18, Proposition 6.7(3)], the corresponding ordinary full-nine branch parametrizes a primitive generator of $\ker(V^2 : E_0^{(9)} \to E_0)$. Thus λ in (5.41)–(5.42) is the Legendre parameter of E_0; the middle curve has parameter λ^9 on the reduced fibre.

We prove the required pole bounds at its missing supersingular points. On the original regular model choose $t = r_4$ to be the last outward isogeny parameter. The target parameter of that arrow is $-3/t$. In the endpoint chart (5.29) its elliptic parameter is

$$a(-3/t) = \frac{-27/t^3 + 18/t + t}{4}.$$

At the generic point of the ordinary D_0 branch, where t is invertible, reduction gives $a(-3/t) = t$. As rational functions on its smooth compactification,

$$\lambda = \frac{t+1}{t-1}, \qquad \lambda + 1 = \frac{t}{1-t}, \qquad u = \frac{1}{t} - 1. \tag{5.51}$$

Hence $\lambda + 1$ has a simple zero at the supersingular point. The identity $h(t)h(-3/t) = -3$ in (5.29) transports the same label three-cycle to the target parameter and gives $h(u) = u - 1$.

Let $r = r_2$, and choose equations for the other components in the form

$$r_j = ta_j(t, r) + rb_j(t, r), \qquad a_j(t, 0) \in k[[t]]^{\times} \quad (j = 0, 1, 3).$$

These expressions use the transverse intersection of C_2 with each of the other branches. They do not replace Cartier equations by characteristic-three initial forms. The different of dt is $(0,1,9,5,4)$; the last entry is one greater than that of the divided lower canonical pullback, because t is the endpoint parameter. Thus the fine remainder has, up to a regular unit absorbed into k, the expression

$$\xi_f = \frac{k\,dt}{r_0 r_1^2 r^9 r_3^4 t^2}, \qquad k \in R_4. \tag{5.52}$$

On the ordinary normalization $\epsilon = \eta/(rt)$ is a unit. Multiplying (5.52) by η^9 gives

$$\eta^9 \xi_f = k\epsilon^9 \big((r_0/t)(r_1/t)^2 (r_3/t)^4\big)^{-1} dt.$$

Substitute $r = \eta/(t\epsilon)$. The first coefficient in η of r_j/t has at most a double pole in t, whereas that of a regular function k has at most a simple pole. The zeroth coefficients of the r_j/t are units. In characteristic three the ninth power ϵ^9 has no first-order variation. Its zeroth coefficient is on the base: its sixth power is a base unit, and an element of a separable cubic extension whose sixth power is in the base has separable degree dividing two, hence belongs to the base. It follows that the zeroth base differential is regular and the first base coefficient has pole order at most two.

Taking the plus average removes the zeroth coefficient. Fricke changes the endpoint base parameter by a first-order term with at most a simple Hasse pole, since $x - y = r$ and $r/\eta = 1/(t\epsilon)$. If that change is the vector field $V(t)\partial_t$, the first change of a regular differential β_0 is $d(V(t)\beta_0/dt)$, which has at most a double pole. We have proved that the base differential whose pullback is β has pole order at most two at every fine supersingular point.

At a wild marking stabilizer the bound improves to regularity on that fine base. Use the original weak frame and coefficient from (5.31):

$$\xi = K\nu, \qquad \nu = \frac{r_3 y}{x r_1}\Omega, \qquad K \in (x,y)^2.$$

The cubic invariance computation already proved shows that the quadratic part of K is dy^2. Since r_1, r_3 have initial terms x, y, and Fricke negates Ω, its contribution to the plus average is

$$\frac{d}{2}\left(\frac{y^4}{x^2} - \frac{x^4}{y^2}\right)\Omega = \frac{d(y^2 - x^2)^3}{2x^2y^2}\Omega \quad \text{in characteristic three.}$$

It has a third-order zero in the central normal direction $x - y$, and contributes nothing to the first η-grade. Changing a frame by a unit leaves this quadratic calculation unchanged, because K has no terms of degree zero or one. All errors in replacing the branch ratios by their initial terms have total coefficient degree at least three.

Such higher terms give a regular first base coefficient. Relative to (5.52), the weak frame has one extra factor $1/t$. For $K \in (x,y)^3$, its restriction K_0 is divisible by t^3, and its normal derivative is divisible by t^2. Substitution $r = \eta/(t\epsilon)$ therefore gives a first coefficient of K/t divisible by t^0. A unit variation has a pole of order at most two, but it is multiplied by K_0/t, which is divisible by t^2. It is regular as well. The zeroth base differential vanishes to order at least two; the Fricke vector field has pole order at most one, so its first Lie derivative is again regular. The correction Δ in (5.12) has central order at least three, and so does its Fricke

transform. Neither changes the zeroth or first coefficient. This proves the bounds

marking stabilizer	pole order of the fine base differential for β
1 or C_2	≤ 2
C_3 or S_3	≤ 0.

(5.53)

For $\delta = 6$, use the boundary differential of ξ, namely $\beta_0 = \overline{\eta^9 \xi}$. The same zeroth-coefficient calculation shows that its fine base differential is regular. At a wild marking point it vanishes at least once, since $K \in (x, y)^2$ and the weak frame has just the extra factor $1/t$. This separate zeroth-grade assertion will be used below; it does not follow by applying a first-grade argument to an arbitrarily chosen lift.

5.10 A global primitive on the original Igusa quotient

We construct a single global rational primitive for the arithmetic difference operator and then control its poles. In (5.42), arithmetic inertia is $\sigma Z = Z + 1$, $\sigma u = u$. Here λ is the Legendre parameter of the target E_0 identified in (5.51). In particular, the pole orders in (5.53) are measured on this same tame base.

For the three-cycle of the full-two labels, (5.29) gives $h(u) = u - 1$. Substituting in (5.42) gives

$$h(Z) = Z - u + c, \qquad c \in \mathbb{F}_3.$$

The constant c is global on each connected curve: $h(Z) - Z + u$ has zero Artin–Schreier image, so it is one of the three constants, not an independently chosen local function. Summing its three transforms gives

$$\operatorname{Tr}_h Z = 1, \qquad 2 \operatorname{Tr}_h Z^2 = Z + u^2 + (1 + c)u + (2 + 2c + c^2).$$

Consequently the global rational function

$$T = Z + u^2 + (1 + c)u \tag{5.54}$$

is h-invariant and satisfies $(\sigma - 1)T = 1$. On the Igusa curve the pole orders of u, Z at a supersingular point are three and four; hence T has pole order at most six and no other poles.

For the coefficient β in (5.50), put $P = T\beta$. It is a global rational differential, is invariant under the actual wild marking stabilizer, and satisfies

$$(\sigma - 1)P = \beta. \tag{5.55}$$

It descends through the separable marking function-field quotient. If the other central branch has the opposite arithmetic orientation, take the negative primitive there. Equivalently, for the equation $Z^3 - Z = \varepsilon(u^4 - u^2)$ and $\sigma Z = Z + 1$, use

$$T = Z + \varepsilon u^2 + (\varepsilon + c)u.$$

A marking involution commutes with arithmetic σ; averaging over it therefore preserves (5.55). This operation uses only $1/2$, and works whether the involution preserves a central branch or exchanges the pair.

We verify the poles after the actual marking quotient. The Igusa cover over the fine tame base has ramification three and different ten at a supersingular point. A base pole of order two

pulls back to order at least $10-6=4$; after multiplication by T the order is at least -2. Thus at $H=1$, P has pole order at most two. At a point fixed by C_2, the tame quotient has ramification two and different one. If P_{down} is its descended differential, then $\mathrm{ord}(P)=2\,\mathrm{ord}(P_{\mathrm{down}})+1$; the bound -2 therefore gives $\mathrm{ord}(P_{\mathrm{down}})\geq -1$. If the involution exchanges points, the original bound of two applies at each image.

At $H=C_3$, (5.53) gives regularity on the fine base, so upstairs $\mathrm{ord}(P)\geq 10-6=4$. The marking action has lower jump one: $q=u/Z$ is a uniformizer and the displayed formula for h gives $h(q)=q+q^2+\cdots$. Its ramification index is three and its different is $2(1+1)=4$. Thus the invariant P descends regularly. For $H=S_3$ first take this wild quotient and then the tame order-two quotient; a regular invariant differential descends regularly in that tame step. We have proved, on the original central components,

H	maximum pole order of P
1	2
C_2 at a ramified point	1
$C_3,\ S_3$	0.

(5.56)

In the unramified C_2 case the first row applies. There are no other poles: u, Z, T are regular away from the supersingular set, and the coefficient comes from the original global cuspidal differential.

For $\delta=6$, apply the same global construction to $\beta_0=\overline{\eta^9\xi}$. Its fine base differential is regular everywhere, so $T\beta_0$ has order at least four at every fine supersingular point. At a wild marking point its additional base zero gives an even larger order. The different computations just made show that this primitive is regular after every original marking quotient. Equation (5.55) holds with β_0 in place of β.

5.11 The actual central trace and integral rounding

Let $f: C\to E_k$ be one of the nonconstant specialization maps from an original central component to the good elliptic curve. Its local representation is principal series, as proved above. Each inducing character has conductor two and is nontrivial on $(1+3\mathbb{Z}_3)/(1+9\mathbb{Z}_3)$. Under local reciprocity, the cyclotomic generator $\zeta_9\mapsto\zeta_9^4$ generates this quotient, up to inversion. Thus the chosen arithmetic σ acts nontrivially on the good elliptic target. Every supersingular point of C is fixed by arithmetic σ. Its image is therefore the unique fixed point O of the nontrivial order-three automorphism γ of E_k. Over the algebraically closed residue field we can write

$$E_k: y^2=x^3-x,\qquad \gamma(x,y)=(x+1,y).$$

At O, a uniformizer $-x/y$ is sent to itself plus its cube to leading order; the lower jump is two. On the Igusa curve, $q=u/Z$ satisfies $\sigma(q)-q=-u/Z^2+\cdots$, of order five, so the arithmetic lower jump at its supersingular point is four.

At a point of the first row of (5.56), the local degree $e_s=\mathrm{ord}_s f^*t_O$ is at least two. Indeed if $e_s=1$, formal inversion would identify the two completed local DVRs equivariantly, forcing their lower jumps to agree. They are four and two. If the map has an inseparable factor, e_s is already divisible by three. At a ramified C_2 marking point the possible pole is only simple, so no lower bound beyond $e_s\geq 1$ is needed; the wild marking cases have no pole.

By the finite duality estimate (2.11), a pole of order n_s can give a pole of order at most $\lceil n_s/e_s\rceil$ in $\mathrm{Tr}_f P$. This estimate follows directly from $n_s s\leq f^*(\lceil n_s/e_s\rceil O)$, the integral duality

trace and its projection formula; it is valid for inseparable maps also. Equation (5.56) and the preceding local-degree argument show that $\operatorname{Tr}_f P$ has at most one simple pole, at O. The residue theorem on the proper target forces its residue to vanish, so it is regular. Since γ acts trivially on $H^0(E_k, \omega_{E_k})$, equivariance and (5.55) give

$$\operatorname{Tr}_f \beta = (\gamma - 1) \operatorname{Tr}_f P = 0. \tag{5.57}$$

For $\delta = 6$ the primitive of β_0 is already regular, and the same argument gives $\operatorname{Tr}_f \beta_0 = 0$.

We finish by relating these component computations to the original characteristic-zero scalar.

Proof of Theorem 5.1. Given $g \in S$, construct ξ by (5.13); its pullback by the original embedding is $9i^*g$. Pass to a finite extension where the source is semistable and E has good reduction, and extend the actual modular map on a regular semistable refinement. Remove from the target the finitely many special points which are images of contracted components. Near the generic point of its special fibre the resulting proper map is quasi-finite and hence finite. The dualizing trace there is the same trace as on the generic fibre and commutes with special-fibre base change. Thus the sum of the actual component traces computed above is exactly the leading coefficient of $i^*\xi$. This remains true when the residual component maps are inseparable.

In the supercuspidal case the support theorem places every nonconstant component on the four bounded new affinoids or their actual marking quotients. For $H = 1, C_2$, (5.27) gives value at least $-4/3$ and (5.28) kills its boundary trace. At $\delta = 4$ its trace value is therefore strictly greater than $-4/3$. For $H = C_3, S_3$, (5.32) gives the already strict bound $-5/4$. With the original good-reduction scale $v(a) = \delta/12$, the corresponding scalar bounds for $\delta = 4$ are

$$> -\frac{4}{3} + \frac{4}{12} = -1, \qquad \geq -\frac{5}{4} + \frac{4}{12} = -\frac{11}{12}.$$

For $\delta = 6, 10, 12$, even the unrefined value $-4/3$ gives respectively $-5/6, -1/2, -1/3$, all greater than -1.

In the principal-series case only the central ordinary pair can contribute. For $\delta = 4$ use the plus average in (5.49), which preserves the original functional. Its central value is at least $-8/6$; (5.57) kills the boundary coefficient, so its actual trace value is strictly greater than $-8/6$. For $\delta = 6$ the unaveraged value is at least $-9/6$, and the separate zeroth-grade trace vanishing makes it strictly greater. For $\delta = 10, 12$ the unaveraged value suffices. After adding $\delta/12$, these four original scalar bounds are, respectively,

$$> -1, \qquad > -1, \qquad \geq -2/3, \qquad \geq -1/2.$$

Write $9i^*g = \mu\omega_E$, where $\mu \in K$ is the original scalar. The scale $\omega_E = a^{-1}\omega_{\mathrm{good}}$ gives $v(\mu) = v(i^*\xi/\omega_{\mathrm{good}}) + \delta/12$; in the $\delta = 4$ principal-series case one uses ξ^+ with the same scalar. All the inequalities just proved say $v(\mu) > -1$. Because K is the original unramified field, $v(\mu)$ is an integer. Hence $\mu \in R$. Faithful flat descent gives the assertion over the original R, and Lemma 2.1 gives the two claimed inequalities. The Manin constant throughout is the original c_E. □

6 The prime two at depth two

Throughout this section R is an unramified discrete valuation ring over $\mathbb{Z}_2$, enlarged when necessary to split the supersingular points and tame markings. Its residue field is denoted by k.

Write $N = 4M$, with M odd. We retain the original optimal elliptic curve E, the embedding $i : E \hookrightarrow J_0(N)$, and the lattice S of Section 2.

Theorem 6.1. *If $v_2(N) = 2$, then $2i^*S \subseteq R\omega_E$. In particular $v_2(c_E r_E/m_E) \leq 1$.*

6.1 The full level-two model and its highest residue

The integral conjugation isomorphism [CNS24, Example 6.3, equation (6.3.1)] identifies the normal model of $X_0(4M)$ with the normal model for $\Gamma_0(M) \cap \Gamma(2)$. Thus a full Drinfeld level-two structure is available without a ramified extension: $\zeta_2 = -1$ already belongs to R. Add a fine auxiliary odd marking and denote the resulting regular model by $\mathcal{X}_f$. We work separately on every connected tame component and remove the auxiliary marking only by descent of the specified sections.

Let B be the smooth compactified fine tame curve, let $\mathcal{L}$ be its Hodge line bundle, and let D_{cusp} be its reduced cusp divisor. Put

$$\deg \mathcal{L} = \deg \mathrm{SS}, \qquad \kappa = \deg D_{\mathrm{cusp}} > 0.$$

The Hasse invariant Ha has weight one and simple zero divisor SS, and Kodaira–Spencer gives

$$\Omega^1_B = \mathcal{L}^{\otimes 2}(-D_{\mathrm{cusp}}), \qquad 2g(B) - 2 = 2\deg \mathrm{SS} - \kappa. \tag{6.1}$$

The fine special fibre has three reduced components I_K, indexed by $K \in \mathbf{P}^1(\mathbb{F}_2)$, whose normalizations are copies of B. At each supersingular point its regular completed local ring has an equation

$$u^2 v - u v^2 + G(u, v) + 2F_1(u, v) = 0, \qquad G \in (u, v)^4, \quad F_1 \in R[[u, v]]^\times. \tag{6.2}$$

These are the full-level local model and equation of Katz–Mazur, in the form stated for an arbitrary prime in [EP21, §2.1, Theorem 2.3]. The three reduced branches have distinct tangents $u = 0, v = 0, v = u$, so their pairwise intersection numbers are $\deg \mathrm{SS}$. Since the total fibre is principal, each self-intersection is $-2\deg \mathrm{SS}$.

Let I_L be an infinity component. The pullback of an element $g \in S$ is integral along every such component. If its orders on the other two components are $-a, -b$, effectiveness of its horizontal divisor, adjunction, and (6.1) give

$$2a - b < 4, \qquad 2b - a < 4.$$

For example, the first left side is the vertical intersection with the first component, divided by $\deg \mathrm{SS}$; its upper bound is $(2g - 2 + 2\deg \mathrm{SS})/\deg \mathrm{SS} = 4 - \kappa/\deg \mathrm{SS}$, after dropping the nonnegative infinity contribution. Integrality of the orders gives $2a - b \leq 3$ and $2b - a \leq 3$, whence $a, b \leq 3$. Consequently $8g$ is canonical on the fine model and vanishes at least three times along I_L.

The restrictions of $8g$ on the other two normalizations are divisible by Ha. We identify these restrictions through the modular maps. Let ρ be the actual map sending the marked elliptic curve to its quotient by L. On both components different from I_L, its restriction identifies the normalization separably with the same tame curve B. In (6.2), choose u to cut out I_L. For a canonical frame Ω_f one may write

$$8g = u^3 H \Omega_f.$$

On each other branch, Ω_f has ordinary normalization pole order two. Thus its restriction is $\mathrm{Ha} w_K$, where $w_K \in H^0(B, \mathcal{L}(-D_{\mathrm{cusp}}))$.

The transvection fixing L interchanges the other two components, fixes ρ, and has first jets

$$u \longmapsto u, \qquad v \longmapsto v + u.$$

Its multiplier on Ω_f has constant term one: the tangent determinant is one, and the unit coefficient of 2 in (6.2) is preserved. It follows that the difference of $u^3 H\Omega_f$ and its transform belongs to $(u, v)^4 \omega_{\mathcal{X}_f/R}$. Comparison through the same map ρ shows that $w_K - w_{K'}$ vanishes at every supersingular point. But

$$H^0(B, \mathcal{L}(-D_{\mathrm{cusp}} - \mathrm{SS})) = H^0(B, \mathcal{O}_B(-D_{\mathrm{cusp}})) = 0.$$

Hence both restrictions are the same differential $\mathrm{Ha} w$. The construction is equivariant for changes of the auxiliary marking, so w is a Katz weight-one cusp form for $\Gamma_0(M)$. The argument has used a common leading jet; it has imposed no invariance condition on g under the transvection.

6.2 Two particular lifts at odd level

Lemma 6.2. *Let M be odd and let w be a Katz weight-one cusp form of level $\Gamma_0(M)$ over k. The two weight-two forms* $\mathrm{Ha} w$ *and the Frobenius-linearized square of w are reductions of characteristic-zero weight-two cusp forms at the same level.*

Proof. For $M = 1$ the weight-one cusp space is zero: on the fine curve $X(3)_k \simeq \mathbf{P}^1_k$, the bundle $\mathcal{L}(-D_{\mathrm{cusp}})$ has negative degree. Assume $M > 1$. The effective automorphism group of a supersingular elliptic curve in characteristic two is A_4. An effective stabilizer containing V_4 would lift to a subgroup containing Q_8 in the elliptic automorphism group. Choose an odd prime dividing M. On a preserved cyclic subgroup of that prime order all commutators act trivially, whereas the commutator -1 of Q_8 acts nontrivially. This is impossible. Thus an effective wild stabilizer is C_2; the other possibilities are trivial or the tame group C_3.

The good genus-zero curve $X(3)_k$ has a faithful A_4 deck action. An element of order two has, at its fixed point, a parameter in which it acts as

$$t \longmapsto \frac{t}{1 + ct}, \qquad c \in k^{\times}.$$

Its lower jump is one and its different exponent is two. Odd-order cyclic markings are étale deformation data, so this is the completed action for every wild point at the level under consideration.

The quotient w/Ha is an invariant rational function at such a point. Its order upstairs is at least -1 and, because it is invariant, is divisible by the ramification index two. Its order is therefore nonnegative, and w has order at least one. The Kodaira–Spencer differentials belonging to $\mathrm{Ha} w$ and w^2 have order at least two upstairs. For a coarse differential θ the exact order formula is

$$\mathrm{ord}_{\mathrm{up}}(h^*\theta) = 2\,\mathrm{ord}_{\mathrm{down}}(\theta) + 2. \tag{6.3}$$

Both differentials consequently descend holomorphically. At tame stabilizers, holomorphic invariant differentials descend holomorphically by the tame different formula. At cusps the odd widths and cusp vanishing give the same extension.

The coarse curve $X_0(M)_R$ is smooth and proper. Its special-fibre holomorphic differentials lift by cohomology and relative duality. Their pullbacks to the modular stack give the required weight-two lifts. If $k \neq \mathbb{F}_2$, use the square after the inverse coefficient-Frobenius twist to obtain the k-linear operation whose expansion is $w(q^2)$. The same valuation argument applies to that twist. □

Here and below, this lemma concerns these two specified forms; it is not a surjectivity assertion for all Katz weight-two forms at arbitrary odd level.

6.3 The actual old differential

Put $\mathcal{Y} = X_0(M)_R$ and $\mathcal{Z} = X_0(2M)_R$; these are the coarse modular curves. Write $\mathcal{Y}^{\mathrm{st}}$ and $\mathcal{Z}^{\mathrm{st}}$ for their modular stacks, and let $c : \mathcal{Z}^{\mathrm{st}} \to \mathcal{Z}$ be the coarse map. The special components of $\mathcal{Z}$ are C_μ, C_{et}; their stack counterparts are denoted by $\mathcal{C}_\mu, \mathcal{C}_{\mathrm{et}}$. Identify the multiplicative component with the odd-level curve by the forgetful map α, and the étale component by the quotient map γ. The same letters denote the corresponding maps of stacks. The line bundle $\mathcal{L}_{\mathrm{st}}$ below is the Hodge bundle on $\mathcal{Z}^{\mathrm{st}}$.

We first record precisely the comparison with the coarse curve. The source [Ces18a, Theorem A.4(b) and Remark A.6] gives

$$\omega_{\mathcal{Z}/R} \simeq c_* \omega_{\mathcal{Z}^{\mathrm{st}}/R}, \tag{6.4}$$

compatibly with pullback of differentials on the generic fibre. Here is the codimension-one argument, including the assertion about orders that we will need. Each vertical generic point is ordinary. On the elliptic curve locus with $j \neq 0, 1728$, the automorphism group is the constant group $\{1, -1\}$, and the coarse map is its rigidification; it is étale [Ces17, proof of Theorem 6.7]. In characteristic two this constant group is still étale. Equivalently, add any full auxiliary level $\ell \geq 3$, with ℓ odd and prime to M. Its scheme of markings is étale over the elliptic curve stack. After removing the ineffective central action, the resulting fine cover is étale over the coarse curve at each vertical generic point. Relative canonical modules are compatible with these étale maps. Thus, for an actual rational differential η,

$$\mathrm{ord}_{C_j}(\eta) = \mathrm{ord}_{\mathcal{C}_j}(c_K^* \eta), \qquad j \in \{\mu, \mathrm{et}\}. \tag{6.5}$$

The horizontal part of the comparison uses characteristic zero. After removing ineffective inertia, a completed local quotient there has the form $t = s^e$. A holomorphic invariant differential upstairs has the form $s^{e-1} a(s^e)\, ds = e^{-1} a(t)\, dt$, so it descends holomorphically, including at elliptic points and cusps. Finally, a normal noetherian surface is Cohen–Macaulay, and its canonical module is torsion-free, of rank one, and satisfies (S_2); it is therefore reflexive. A rational canonical section is consequently regular if and only if it is regular at every codimension-one point. This extends the comparison across all closed points of the special fibre, including the supersingular points with wild effective automorphisms. It also shows directly that a holomorphic differential on $\mathcal{Z}_K$ with nonnegative orders in (6.5) belongs to $H^0(\mathcal{Z}, \omega_{\mathcal{Z}/R})$.

We now construct the specified section on the stack. Let Vw denote the Frobenius-linearized square of w, so that $(Vw)(q) = w(q^2)$. Lemma 6.2 supplies odd-level weight-two cusp forms G, H whose reductions are $\mathrm{Ha}w, Vw$, respectively. In particular, their corresponding differentials on $\mathcal{Y}_K$ are holomorphic. Distinguish pullback of a weight-two form, using the homomorphism of Hodge bundles induced by the universal isogeny, from pullback of a differential by writing the former as α^{wt} and γ^{wt}. Set

$$\Phi = \alpha^{\mathrm{wt}} G - \gamma^{\mathrm{wt}} H \in H^0\big(\mathcal{Z}^{\mathrm{st}}, \mathcal{L}_{\mathrm{st}}^{\otimes 2}(-D_{\mathrm{cusp}})\big).$$

The restrictions of the Hodge pullbacks are those of [Ces18a, Lemma 3.10]. On $\mathcal{C}_\mu$, α and its Hodge pullback are isomorphisms, whereas γ^{wt} is zero because the quotient isogeny is relative Frobenius. On $\mathcal{C}_{\mathrm{et}}$, γ is an isomorphism of modular curves and its Hodge homomorphism vanishes simply at the supersingular divisor: under this identification it is the Verschiebung homomorphism, whose multiplier is Ha. Its weight-two multiplier is therefore Ha^2. The restriction of $\alpha^{\mathrm{wt}}G$ to $\mathcal{C}_{\mathrm{et}}$ is the Frobenius pullback of $\mathrm{Ha}w$, namely Ha^2Vw under the same Hodge identification. Consequently

$$\Phi|_{\mathcal{C}_\mu} = \mathrm{Ha}w, \qquad \Phi|_{\mathcal{C}_{\mathrm{et}}} = 0. \tag{6.6}$$

These statements include the cusps: the odd-level reduced cusp divisor is étale over R, and the component isomorphisms identify the respective cusp divisors, as in the proof of the cited lemma. The construction in [Ces18a, proof of Proposition 3.11] uses exactly the two particular lifts supplied above.

The Kodaira–Spencer comparison is on the modular stack:

$$\omega_{\mathcal{Z}^{\mathrm{st}}/R} \simeq \mathcal{L}_{\mathrm{st}}^{\otimes 2}(-D_{\mathrm{cusp}} + \mathcal{C}_{\mathrm{et}}). \tag{6.7}$$

This is [Ces18a, Lemma 3.12(b)], applied with $n = 2M$ and $\Gamma = \Gamma_0(n)$. Its hypothesis $v_2(n) = 1$ permits every odd M, including $M = 1, 3$, with arbitrary supersingular stabilizers. To recall the vertical factor explicitly, the forgetful map is étale at the multiplicative generic point. At the étale generic point, strict henselization gives a finite degree-two extension $A \subset B$ of DVRs with uniformizer 2 in both rings and purely inseparable residue extension of degree two. Its different has exponent $d \geq 1$. If $d \geq 2$, then $1/4$ would belong to the inverse different, although $\mathrm{Tr}_{\mathrm{Frac}(B)/\mathrm{Frac}(A)}(1/4) = 1/2 \notin A$. Thus $d = 1$, giving the single $\mathcal{C}_{\mathrm{et}}$ in (6.7).

Normalize (6.7) by the Tate differential dq/q at the multiplicative cusp. This is an integral normalization: the forgetful map is an isomorphism on its formal Tate parameter there, and the comparison is the usual odd-level Kodaira–Spencer comparison at that cusp. Let $\gamma_G = G(q)\,dq/q$ and $\gamma_H = H(q)\,dq/q$ on $\mathcal{Y}_K$, and define the actual characteristic-zero differential

$$f_0 = \alpha_K^*\gamma_G - 2\gamma_K^*\gamma_H \quad \text{on } \mathcal{Z}_K. \tag{6.8}$$

Indeed, at this cusp the quotient isogeny is $\mathbb{G}_m \to \mathbb{G}_m$, $z \mapsto z^2$, so its pullback on invariant elliptic differentials is multiplication by 2. Hence $\gamma^{\mathrm{wt}}H$ has expansion $4H(q^2)$, whereas $\gamma_K^*\gamma_H = 2H(q^2)\,dq/q$. Thus $c_K^*f_0$ is exactly the section associated to Φ by the normalized comparison, and f_0 is an actual old differential. Formula (6.8) also proves its holomorphicity everywhere on the characteristic-zero coarse curve.

By (6.6), Φ vanishes on the Cartier divisor $\mathcal{C}_{\mathrm{et}}$. There is one further zero when it is viewed as a canonical section in (6.7). More explicitly, if t is a local equation of $\mathcal{C}_{\mathrm{et}}$, λ is a frame of $\mathcal{L}_{\mathrm{st}}^{\otimes 2}(-D_{\mathrm{cusp}})$, and Ω_{st} is the canonical frame corresponding to λ/t, then

$$\Phi = ta\lambda \quad \Longleftrightarrow \quad c_K^*f_0 = t^2a\Omega_{\mathrm{st}}.$$

Thus $c_K^*f_0$ is canonical and has order at least two on $\mathcal{C}_{\mathrm{et}}$. The comparison and order equality above give the precise coarse assertion

$$f_0 \in H^0(\mathcal{Z}, \omega_{\mathcal{Z}/R}), \qquad \mathrm{ord}_{C_{\mathrm{et}}}(f_0) \geq 2, \qquad f_0|_{C_\mu} = \mathrm{Ha}w. \tag{6.9}$$

The last equality is read as a differential after pullback to the specified fine tame normalization. It follows from (6.6) and the usual Kodaira–Spencer comparison on the multiplicative component.

Let w_2 be the integral Atkin–Lehner involution of $\mathcal{Z}$, and put

$$h = \frac{w_2^* f_0}{4}.$$

It exchanges C_μ and C_{et}. The identities $\alpha w_2 = \gamma$ and $\gamma w_2 = \alpha$ show that its map on the tame normalizations is the identity under the stated forgetful and quotient identifications. Consequently h is integral on C_μ, has pole at most two on C_{et}, and $(4h)|_{C_{\mathrm{et}}} = \mathrm{Ha}w$, again on the fine normalization. The same identities give the useful normalization check

$$h = \frac{G(q^2) - H(q)}{2}\,\frac{dq}{q} \quad \text{at infinity.}$$

Its expansion is integral since $\overline{G}(q^2) = w(q^2) = \overline{H}(q)$.

The actual degeneracy map $\beta : X_0(4M) \to X_0(2M)$ sends (A, C_4) to $(A/C_2, C_4/C_2)$. On the full level-two description it retains L. It is Frobenius of degree two from I_L to C_μ, and is the same separable isomorphism from each other normalization to C_{et}, when followed by the quotient identification. At the original infinity cusp $q_{\mathcal{Z}} = q^2$. Thus, for differentials,

$$V_2 h = \frac{\beta^* h}{2}, \qquad 8V_2 h = \beta^*(4h). \tag{6.10}$$

The right side has precisely the two highest restrictions of $8g$. It vanishes on I_L, as also follows from its integral expansion there. Hence the fine calculation gives

$$\xi = g - V_2 h, \qquad 4\xi \in H^0(X_0(4M)_R, \omega). \tag{6.11}$$

Here the stated coarse membership is justified as follows. At each ordinary vertical generic point, removal of the ineffective central action makes the auxiliary odd marking cover étale, so the nonnegative fine canonical order is the coarse canonical order. The rational differential 4ξ is holomorphic at every horizontal point because both its summands are characteristic-zero cusp differentials on the coarse curve. Reflexivity of the canonical module on the normal coarse surface then extends it across all remaining closed points. This is the same argument as (6.5), with descent given by equality of orders in the étale local rings. The correction is an actual old differential, so $i^*\xi = i^* g$.

6.4 The last factor of two

The restriction of 4ξ to every fine normalization branch is holomorphic. Locally this follows because its factor u^2 cancels the normalization pole of order two of the dualizing generator. Equation (6.3) allows at worst a simple pole on each coarse normalization branch. All other points, including the cusps, are tame. The coarse special fibre is reduced: locally

$$A^H/(2) \hookrightarrow A/(2),$$

since $2b$ invariant in a torsion-free ring implies that b is invariant. The differential is already canonical, so its residues satisfy the Rosenlicht conditions.

For a reduced proper curve C, the canonical differentials with only simple poles on its normalization are exactly the image of $H^0(C^{\mathrm{sn}}, \omega)$, where C^{sn} is the seminormalization. Indeed the residue test against a local function then depends only on its constant value on the branches;

it is precisely the condition that the sum of the branch residues is zero. The normalization sequence shows that $\mathrm{Pic}^0(C^{\mathrm{sn}})$ is semiabelian.

By [CNS24, Theorem 6.12(c) and Proposition 6.2], the actual $X_0(4M)_R$ has rational singularities and its Picard identity model is the Néron identity model. The special identity component of E is $\mathbb{G}_a$, because $v_2(N) = 2$. Every homomorphism from $\mathbb{G}_a$ to a semiabelian variety is zero. For the affine torus part this follows from the absence of nonconstant invertible polynomials on $\mathbf{A}^1$; for the abelian quotient it follows because a map $\mathbf{A}^1 \to A$ extends to $\mathbf{P}^1$, and an abelian variety contains no rational curve. The reduction of 4ξ, being a cotangent vector from that semiabelian quotient, therefore annihilates the image of $\mathbb{G}_a$. Consequently

$$i^*(4\xi) \in 2R\omega_E.$$

Since $i^*\xi = i^*g$, this proves Theorem 6.1. The case $M = 1$ is vacuous, since $X_0(4)$ has genus zero.

7 The prime two at depth three

We now put $N = 8M$, with M odd.

Theorem 7.1. *If $v_2(N) = 3$, then $2i^*S \subseteq R\omega_E$. In particular $v_2(c_E r_E/m_E) \leq 1$.*

The assertion to be proved is

$$2i^*S \subseteq R\omega_E. \tag{7.1}$$

Here the fine special fibre has four components, and its positive-jet Picard group is larger. We first remove a Cartier-zero highest term, then construct an old differential by a residue lift on $X_0(2M)$. The wild comparison will use an explicit model at level 40.

7.1 The actual middle cover

Set

$$\mathsf{s} = \begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix}, \qquad \mathsf{t} = \begin{pmatrix} 1 & 0 \\ -4M & 1 \end{pmatrix}, \qquad \sigma = \mathsf{st}.$$

Conjugation by $\mathrm{diag}(2,1)$ identifies $\Gamma_0(8M)$ with the subgroup of $\Gamma_0(2M)$ consisting of matrices $\left(\begin{smallmatrix} a & b \\ 2Mc & d \end{smallmatrix}\right)$ for which b, c are even. The map

$$\Gamma_0(2M) \longrightarrow \mathbb{F}_2^2, \qquad \begin{pmatrix} a & b \\ 2Mc & d \end{pmatrix} \longmapsto (b, c)$$

is a surjective homomorphism: a, d are odd, and direct multiplication gives additivity of the two indicated entries modulo two. Thus

$$X_0(8M) \longrightarrow X_0(2M), \qquad \tau \longmapsto 2\tau \tag{7.2}$$

is a $V_4 = \langle \mathsf{s}, \mathsf{t} \rangle$ cover. Let Y_{8M} be its intermediate quotient by σ, and write $\rho : X_0(8M) \to Y_{8M}$. These maps extend to the normal integral modular models by normalization in the corresponding generic function fields.

Choose the auxiliary odd fine marking on the middle elliptic curve of $X_0(2M)$, pull it back along (7.2), and normalize. Denote this fine model by $\mathcal{X}_f$, and the original coarse model

$X_0(8M)_R$ by $\mathcal{X}$. The two deck transformations fix the middle marking. Their involution and commutation relations therefore hold on this same fine model.

For the newform of E, every even Fourier coefficient is zero. Hence $\mathsf{s}i = -i$. The full Fricke involution conjugates s to t and preserves the new line, so $\mathsf{t}i = -i$, and $\sigma i = i$. The operator

$$g \longmapsto \frac{g - \mathsf{s}^* g}{2}$$

simply deletes even powers of q. It preserves S and i^*g. We may therefore assume henceforth that g is odd in q.

7.2 Integral coordinates and the four branches

On a fine supersingular deformation, the family

$$E_a : \quad y^2 + axy + y = x^3$$

has an order-two subgroup parameter u satisfying $u^3 - au - 2 = 0$. The quotient parameter is $a' = a^2 + 3u - au^2$; these explicit isogeny formulas are given in [Rez08, §3]. Setting $v = a - u^2$ gives the identities

$$uv = -2, \qquad a = u^2 + v, \qquad a' = u + v^2.$$

For the next isogeny the subgroup polynomial factors as

$$(u_{\text{next}} - v)(u_{\text{next}}^2 + v u_{\text{next}} - u).$$

The first factor is the dual, backtracking isogeny. The quadratic factor gives exactly the two nonbacktracking choices.

Applying this on the two ends of the middle isogeny in (7.2) gives

$$\begin{aligned} B_0 &= R[[x,y]]/(xy+2), \\ A &= B_0[z,w]/(z^2 + xz - y,\ w^2 + yw - x), \\ D = A^{\mathsf{s}} &= R[[y,w]]/(2 + yw(w+y)). \end{aligned} \tag{7.3}$$

The algebra A is free over B_0 with basis $1, z, w, zw$. Its two equations eliminate x, y as integral formal series in z, w, by the invertible Jacobian at the origin. The remaining equation is $2 +$ a series in $(z,w)^2 = 0$. Thus A is regular. These monic nonbacktracking equations parametrize the cyclic eight-isogenies in characteristic zero, and normality identifies them with the integral normalization of the actual modular cover.

The actions and a canonical frame are

$$\begin{aligned} &\mathsf{s}(z) = -x - z, && \mathsf{s}(w) = w, \\ &\mathsf{t}(z) = z, && \mathsf{t}(w) = -y - w, \\ &\Omega_A = \frac{dx}{x(2z+x)(2w+y)}. \end{aligned} \tag{7.4}$$

Put $q_0 = 1 - zw$. Elimination gives the exact identities

$$x = \frac{w(w+z^2)}{q_0}, \quad y = \frac{z(z+w^2)}{q_0}, \quad 2 = -\frac{zw(w+z^2)(z+w^2)}{q_0^2}. \tag{7.5}$$

The special fibre is therefore reduced. Its normalized branches, with the last containing infinity, are

$$\begin{array}{c|cccc} & A_1 & A_2 & A_3 & A_4 \\ \hline (z,w) & (0,s) & (s^2,s) & (t,t^2) & (t,0). \end{array} \tag{7.6}$$

The pairs A_1, A_2 and A_3, A_4 have contact order two, and every cross pair has contact order one. In consistent normalization frames, Ω_A restricts to $ds/s^4, ds/s^4, dt/t^4, dt/t^4$. For example, with

$$P_0 = 2(1-zw)^2 + zw(w+z^2)(z+w^2),$$

one has $\Omega_A = -q_0\, dz/(\partial P_0/\partial w)$. On A_3, its numerator $1+t^3$ cancels the same factor in the derivative, leaving dt/t^4. The other three substitutions give the displayed frames.

The Hodge line and supersingular divisor on every fine tame normalization again satisfy (6.1). The self-intersection of each of the four components is $-4\deg \mathrm{SS}$. Anti-invariance under s makes the two pole orders on A_1, A_2 equal; denote them by a, and denote the pole order on A_3 by c. As in the preceding section, effectiveness of the horizontal divisor gives

$$2a - c < 6, \qquad 2c - a < 3. \tag{7.7}$$

The first inequality uses the intersection with A_1, whose canonical degree is $6\deg \mathrm{SS} - \kappa$; the second uses A_3 and divides its inequality by $2\deg \mathrm{SS}$. The nonnegative infinity order has been dropped. The integral inequalities imply $a \le 4, c \le 3$.

Since Ω_A changes sign under s, its coefficient in g is s-invariant. The divisors of y and $w+y$ in (7.5), together with normality of D, give

$$g = \frac{F}{y^4(w+y)^3}\Omega_A, \qquad F \in D. \tag{7.8}$$

This describes the allowed fractional ideal of an actual global input.

7.3 The highest Cartier term vanishes

The two actual endpoint maps give leading restrictions

$$\begin{aligned} \beta(g) &= (16g)|_{A_1} = (16g)|_{A_2} = sF(0,s)\, ds = \mathrm{Ha}\, w(g), \\ r(g) &= (8g)|_{A_3} = F(t^2,t^2)\, dt. \end{aligned} \tag{7.9}$$

In the first line the expression means the coefficient in the highest vertical grade; the restrictions become ordinary canonical restrictions after that grade is cleared. The maps identify the branches with the tame normalization using the endpoint elliptic curves. They are therefore independent of a choice of local formal parameter. The frame calculation following (7.6) proves both formulas. It also shows that

$$w(g) \in S_1(\Gamma_0(M);k)_{\mathrm{Katz}}, \qquad r(g) \in H^0(B,\Omega^1_B) \cap \ker \mathrm{Cartier}.$$

For the first assertion the leading differential has a simple zero at SS, and division by Ha gives the weight-one bundle in (6.1). For the second, the series $F(t^2,t^2)$ has only even powers, so its Cartier operator is zero; the cancellations in (7.5) show it is holomorphic at the supersingular points.

If $w(g) = 0$, then $F \in (y)$, since $D/(y) = k[[w]]$. Therefore $r(g)$ vanishes twice at each supersingular point. But

$$H^0(B, \Omega^1_B(-2\mathrm{SS})) = H^0(B, \mathcal{O}_B(-D_{\mathrm{cusp}})) = 0.$$

Consequently $\ker w \subseteq \ker r$: the image $V = \operatorname{im} r$ is a quotient of the image $W = \operatorname{im} w$.

To apply this statement with its precise Hecke and descent properties, pull the original g first to the *coarse* level $X_0(8L)$, where $L = 15M$, by the identity-coordinate degeneracy. Perform (7.9) on that original coarse input space. Full odd level is used only for the local coordinates. The endpoint maps, Hasse invariant and Kodaira–Spencer identifications commute with change of that marking, so the specified sections descend to the $\Gamma_0(L)$ stack. Define W, V as the images of these maps on the coarse input space. Then $W \twoheadrightarrow V$ follows directly from the kernel inclusion; no exactness of an invariant functor in characteristic two is needed.

For primes $\ell \nmid 2L$, the actual Hecke correspondences commute with both endpoints and with the two leading restrictions. On the elliptic locus this follows from the uniqueness of prime-to-two subgroup deformation; at the cusps the widths are odd. Multiplication by Ha also commutes with these Hecke operators, since the two weight normalizations differ by an odd scalar. Thus V is a quotient of W as an away-from-$2L$ Hecke module.

The weight-one doubling map [Wie14, §2, equations (2.3)–(2.4)] is injective and satisfies

$$\begin{aligned} \Psi : S_1 \oplus S_1 \longrightarrow S_2, \qquad (f, h) \longmapsto \mathrm{Ha} f + F_{\mathrm{lin}}(h), \\ T_2\Psi = \Psi \begin{pmatrix} T_2^{(1)} & 1 \\ -\langle 2 \rangle & 0 \end{pmatrix}. \end{aligned} \tag{7.10}$$

Here $F_{\mathrm{lin}}(h)(q) = h(q^2)$. Restrict the identity to trivial diamond character; then $\langle 2 \rangle = 1$, and the displayed operator is invertible. Since $3 \mid L$, no effective wild stabilizer occurs on $X_0(L)_k$: an order-four elliptic automorphism has no invariant line on its three-torsion, as $X^2 + 1$ is irreducible over $\mathbb{F}_3$. The doubled forms therefore descend holomorphically to the good coarse curve and lift to its characteristic-zero differential lattice. The same holds for V, on which T_2 is zero, since on weight-two differentials it is coefficient Frobenius composed with Cartier.

For weight-two forms of arbitrary odd level and trivial character, the ordinary residual local representation at two is reducible, whereas the nonordinary one is irreducible [KM19, §3.4, Theorem 11]. Every simple Hecke constituent of the doubled space lifts to an ordinary characteristic-zero eigenform, and every simple constituent of V would lift to a nonordinary one. Here lifting eigenvalues is applied to the finite Hecke order on the free characteristic-zero weight-two lattice. A common away-from-$2L$ system would give isomorphic semisimple global residual representations by Chebotarev and their equal determinants. The nonordinary representation is globally irreducible because its local restriction is irreducible; the identified ordinary representation would then have the same local restriction, a contradiction. Thus their simple supports are disjoint. Since V is a quotient of W, and W embeds in the doubled space, $V = 0$.

The pullback from tame level M to L is generically separable on the relevant normalization component. Vanishing there reflects vanishing before pullback. We have proved $r(g) = 0$ at the original level, without dividing by the covering degree. In (7.8) this forces $F \in (w + y)$, because substitution $y = w = t^2$ is injective on $k[[w]]$. The improved bound $c \leq 2$ in (7.7) then forces

$a \le 3$, hence a further factor y. We conclude that

$$\begin{aligned} F &= y(w+y)F_2, \quad F_2 \in D, \\ \gamma = 8g &= -xw^2F_2\Omega_A \in H^0(\mathcal{X}_f, \omega_{\mathcal{X}_f/R}), \\ \gamma|_{(\mathcal{X}_f)^\nu_k} &= (\beta,\beta,0,0), \qquad \beta = -F_2(0,s)\,ds. \end{aligned} \tag{7.11}$$

The ordinary generic points of the auxiliary marking cover are étale, so γ is also canonical on the original normal coarse model by reflexivity.

7.4 An integral comparison at level forty

We shall need the Picard–Néron comparison for both the original curve and its diagonal quotient. We establish it by explicit models, retaining the distinction between a convenient model and the normal coarse modular model.

Put

$$F = \eta(2\tau)^2\eta(10\tau)^2, \qquad V = F(2\tau), \qquad G = f_{40}.$$

Here F is the normalized newform of level 20, and G is the normalized newform of level 40 attached to

$$E_{40}: \quad Y_E^2 = X^3 - 7X - 6. \tag{7.12}$$

Their coefficient fields are $\mathbb{Q}$, and their Fricke eigenvalues at their respective new levels are -1; the level-40 identification and signs are recorded in [Cre97, Tables 1 and 3, entries 40A1 and 40A]. The eta product for F can also be checked directly: its two eta congruence sums are 24, its character is trivial since $2^2 10^2$ is a square, and its cusp orders are positive. Since $X_0(20)$ has genus one, it is the normalized generator; eta inversion gives its Fricke eigenvalue -1. The genus formula gives $\dim S_2(\Gamma_0(40)) = 3$, with basis F, V, G. The identities used below can be verified entirely by integer arithmetic. The finite certificate is given in Subsection 7.9; the bounds are 24 and 96. They are

$$G^2 = F^2 + 4V^2, \tag{7.13}$$

$$16A_f^2\big((D_qA_f)V - A_f(D_qV)\big)^2 = A_f^8 + 8A_f^6(2V)^2 - 2A_f^4(2V)^4 + 8A_f^2(2V)^6 + (2V)^8, \tag{7.14}$$

where $A_f = G - F$ and $D_q = q\,d/dq$. The second identity is an identity of holomorphic weight-sixteen forms: the derivative bracket is half the first Rankin–Cohen bracket of two weight-two forms, hence is holomorphic of weight six. The index of $\Gamma_0(40)$ is 72, so the stated bounds follow from the valence bound, equivalently the characteristic-zero form of Sturm's criterion [Stu87].

Define

$$x = \frac{G-F}{2V}, \qquad \eta_0 = \frac{F+G}{2}\frac{dq}{q}, \qquad Y = \frac{dx}{\eta_0}.$$

Equation (7.13) gives

$$F\frac{dq}{q} = (1-x^2)\eta_0, \quad V\frac{dq}{q} = x\eta_0, \quad G\frac{dq}{q} = (1+x^2)\eta_0. \tag{7.15}$$

Since

$$Y = \frac{A_f\big((D_qA_f)V - A_f(D_qV)\big)}{4V^4},$$

equation (7.14) is precisely

$$Y^2 = f(x) := x^8 + 8x^6 - 2x^4 + 8x^2 + 1. \tag{7.16}$$

The polynomial f is square-free: its Euclidean gcd with f' is 1. Its smooth projective double cover of $\mathbf{P}^1$ therefore has genus three. The nonconstant map from $X_0(40)_{\mathbb{Q}}$, supplied by x, Y, has degree one by Riemann–Hurwitz, since both curves have genus three. Thus (7.16) is an equation of $X_0(40)_{\mathbb{Q}}$ and $\eta_0 = dx/Y$. The integral equation

$$y_h^2 - (x^4+1)y_h = 2x^6 - x^4 + 2x^2, \qquad Y = 2y_h - x^4 - 1 \tag{7.17}$$

will supply its two endpoint charts.

Let S_{40} be the full lattice integral at infinity. Reducing (7.13) modulo two shows $G - F \in 2R[[q]]$, because $k[[q]]$ has no nilpotents. The initial coefficients

$$F = q - 2q^3 - q^5 + 2q^7 + q^9 + \cdots, \qquad G = q + q^5 - 4q^7 - 3q^9 + \cdots$$

show that $\eta_0, x\eta_0, x^2\eta_0$ have initial q-orders $1, 2, 3$, each with coefficient one. These first three coefficients recover their coordinates by a triangular matrix in $\mathrm{GL}_3(R)$. Consequently

$$S_{40} = R\eta_0 \oplus Rx\eta_0 \oplus Rx^2\eta_0. \tag{7.18}$$

For differentials the actual Fricke operator satisfies

$$W_{40}F = -2V, \qquad W_{40}V = -F/2, \qquad W_{40}G = -G.$$

Using (7.15), this gives

$$W_{40}x = \frac{1-x}{1+x}, \quad W_{40}Y = \frac{4Y}{(1+x)^4}, \quad W_{40}\eta_0 = -\frac{(1+x)^2}{2}\eta_0.$$

For $P(x) = a + bx + cx^2$, its transformed coefficient is

$$-\frac{(a+b+c) + (2a-2c)x + (a-b+c)x^2}{2}.$$

Thus the two-endpoint lattice is

$$\begin{aligned} L_{40} := S_{40} \cap W_{40}S_{40} &= \{(a+bx+cx^2)\eta_0 : a,b,c \in R,\ a-b+c \in 2R\} \\ &= R(2\eta_0) \oplus R((x+1)\eta_0) \oplus R((x^2-1)\eta_0). \end{aligned} \tag{7.19}$$

The canonical lattice of the actual coarse modular model is contained in L_{40}: its two endpoint cusps are smooth, and Fricke extends to that model.

We next compute a Néron lattice without making an identification of integral models. Blow up $\mathbf{P}^1_R$ at $(2, x+1)$. At the crossing set

$$u = x + 1, \qquad v = 2/u, \qquad uv = 2, \qquad T = \frac{Y/u^2 - u^2 - v^2}{2}.$$

The finite normalization of this chart has equation

$$T^2 + (u^2+v^2)T = 45 - 26(u+v) + 9(u^2+v^2) - 2(u^3+v^3) =: Q(u,v). \tag{7.20}$$

Indeed, modulo $uv-2$, the exact polynomial identity is

$$u^4(2T+u^2+v^2)^2 - f(u-1) = 4u^4\{T^2+(u^2+v^2)T - Q(u,v)\},$$

and the integral coordinate in (7.17) is

$$y_h = u^2T + u^4 - 2u^3 + 3u^2 - 2u + 3. \tag{7.21}$$

On the chart where u is a unit this is (7.17). On the chart where v is a unit the same formula applies to the Fricke coordinate $x' = v-1$, because $Y' = v^2(Y/u^2)$. Modulo two, (7.20) becomes

$$(T+1)(T+1+u^2+v^2) = 0, \qquad uv = 0.$$

Its derivative in T is a unit off $u = v = 0$. The endpoint charts, including the usual chart at infinity of the double cover, are smooth away from $x = 1$. The only possible singularity is $u = v = 0, T = 1$.

Here is the complete resolution at that point. Write $w = T - 1$. In the regular three-dimensional ambient local ring $R[[u,v,w]]/(uv-2)$, with regular parameters u, v, w, the surface equation is

$$G_0 = w^2 + (u^2+v^2+2)w - 44 + 26(u+v) - 8(u^2+v^2) + 2(u^3+v^3) = 0. \tag{7.22}$$

Its initial form is w^2. Blow up the closed point in the ambient threefold and take the strict transform. The w-chart has no exceptional point on the surface, since its exceptional equation would be $1 = 0$. On the u-chart put $v = uV_1, w = uW_1$, so $2 = u^2V_1$. Dividing G_0 by u^2 gives exactly

$$\begin{aligned} G_1 = {} & W_1^2 + u(1+V_1+V_1^2)W_1 + 13uV_1(1+V_1) \\ & - 11u^2V_1^2 + u^3V_1(1+V_1^3) - u^6V_1^3(1+V_1^2). \end{aligned} \tag{7.23}$$

The quotient by this monic polynomial is free over the ambient two-dimensional chart; u is a non-zero-divisor there. Thus division by u^2 has already saturated the equation and has not retained an embedded component. The v-chart is symmetric. The reduced exceptional curve is a projective line, given on this chart by $u = W_1 = 0$.

At its point $V_1 = a$, the coefficient of u modulo two is $a(1+a)$. Therefore only $a = 0, 1, \infty$ can be singular. The tangent quadrics at these three points, in the indicated regular ambient parameters, are

$V_1 = 0$	$W_1^2 + uW_1 + uV_1$
$V_1 = 1,\ Z_1 = V_1 - 1$	$W_1^2 + uW_1 + uZ_1 + u^2$
$V_1 = \infty$	the first quadratic on the symmetric chart.

Each projective conic is smooth even in characteristic two. The derivative with respect to V_1, or Z_1, forces $u = 0$; the other derivative identifies the remaining coordinates; the quadratic then forces both to be zero. There is no projective singular point. Blowing up each of these three points therefore resolves the strict transform: on every chart the exceptional equation is a smooth affine piece of this projective tangent cone, and the surface was already regular off the centers. These are all four blowups.

The original surface is a complete intersection, regular in codimension one, hence normal by Serre's criterion. This proves that (7.20) is the asserted normalization. Together with its endpoint charts it gives a proper normal flat model $\mathcal{C}_{40}$ of $X_0(40)$. All four blowups are crepant. At each center the ambient threefold is regular and the surface has multiplicity two; the ambient contribution $2E$ is canceled by the multiplicity $2E$ in adjunction. This computation also works directly for relative dualizing sheaves in mixed characteristic. On the first u-chart the old ambient generator $du \wedge dw/u$ pulls back to $du \wedge dW_1$, whereas the new ambient generator is $du \wedge dW_1/u^2$. Its factor u^2 cancels the factor in $G_0 = u^2 G_1$. The same regular-center adjunction applies to the subsequent three blowups. If $\pi : \widetilde{\mathcal{C}}_{40} \to \mathcal{C}_{40}$ is the resulting regular resolution, then

$$\omega_{\widetilde{\mathcal{C}}_{40}/R} = \pi^* \omega_{\mathcal{C}_{40}/R}, \qquad H^0(\widetilde{\mathcal{C}}_{40}, \omega) = H^0(\mathcal{C}_{40}, \omega). \tag{7.24}$$

The second equality follows from normality, $\pi_* \mathcal{O} = \mathcal{O}$, and the projection formula.

The dualizing generator on (7.20) is

$$\Omega = \frac{du}{u(2T + u^2 + v^2)}.$$

Here $\eta_0 = \Omega/u$. Consequently

$$(a + bx + cx^2)\eta_0 = \left(\frac{a - b + c}{u} + (b - 2c) + cu \right) \Omega. \tag{7.25}$$

If $a - b + c \in 2R$, the first term is integral, since $2/u = v$. The other terms are integral as well. On the two smooth endpoint charts, the polynomial canonical bases in (7.17), and their Fricke transforms, show that every element of L_{40} is canonical. Conversely, the two smooth cusp disks impose membership in both lattices in (7.19). Therefore

$$H^0(\mathcal{C}_{40}, \omega) = L_{40}.$$

The regular model $\widetilde{\mathcal{C}}_{40}$ has a smooth special point, for instance the cusp with $x = 0$. By (7.24) and [CNS24, Proposition 6.2], its canonical lattice, hence the Néron differential lattice of $J_0(40)$, is L_{40}. The same proposition supplies the first inclusion in

$$L_{40} = H^0(\mathcal{J}_0(40), \Omega^1) \ \subseteq \ H^0(X_0(40)_R, \omega) \ \subseteq \ L_{40}.$$

The last inclusion was obtained from the two actual modular cusps. Equality and that proposition prove that the actual coarse $X_0(40)_R$ has rational singularities. Its Proposition 6.10(v), including the local stabilizer argument, now gives rational singularities of the actual $X_0(8M)_R$ for every odd $M > 1$. For $M = 1$ the curve has genus zero. In particular the connected Picard scheme of the actual model is the identity component of the Néron model.

7.5 The diagonal quotient and its completed pairs

Let $\mathcal{Y}_{8M}$ be the normal integral quotient of the actual middle cover by $\sigma = \mathsf{st}$. It has a smooth cusp with parameter q. Indeed,

$$\sigma(\infty) = \frac{2M - 1}{4M}.$$

The denominator has 2-adic valuation 2, whereas infinity has valuation 3. By the ordinary component description [CNS24, Lemma 5.11], their reductions lie on different components, and

both are away from the supersingular locus. The two cusp disks are disjoint copies of $R[[q]]$, exchanged by σ. The invariant subring of their product is one copy of $R[[q]]$. This argument uses a free orbit, so it requires no division by two. The same freeness holds at every cusp. At a reduced cusp a/c of $\Gamma_0(2M)$, put $w_c = 2M/\gcd(c^2, 2M)$. A parabolic inertia generator is

$$\begin{pmatrix} 1 - acw_c & a^2 w_c \\ -c^2 w_c & 1 + acw_c \end{pmatrix}.$$

Its image under the middle deck map is $(1, 0)$ if c is even: then a, w_c are odd and $-c^2 w_c/(2M)$ is even. If c is odd, then $v_2(w_c) = 1$, and its image is $(0, 1)$. Thus cusp inertia never contains $\sigma = (1, 1)$. An odd auxiliary marking multiplies these widths by odd numbers and preserves the conclusion. After unramified splitting of the cusp fields, the involution exchanges disjoint smooth cusp disks, and their quotient is smooth.

We first describe the fine quotient. In the completed ring A of (7.3), put

$$U = (1 - yz)(1 - xw) = 1 - yz - xw + xyzw.$$

Then

$$C = A^\sigma = B_0[U]/(U^2 - (1 + x^3)(1 + y^3)), \qquad B_0 = R[[x, y]]/(xy + 2). \tag{7.26}$$

Here is an integral check of the invariants, which is useful because the group order is not invertible. Write an element of A in the free B_0-basis $1, z, w, zw$ as $a + bz + cw + ezw$. Invariance under σ gives $2b = ey,\ 2c = ex$. Since $2 = -xy$, these say $e = -xb = -yc$. The ideals (x) and (y) in the regular ring B_0 are prime and have intersection (xy). Thus $e = xyh,\ b = -yh,\ c = -xh$, proving $A^\sigma = B_0 \oplus B_0(U - 1)$. The quadratic relation in (7.26) follows by substitution in the two equations of A. As an invariant subring of the normal ring A, C is normal; the displayed equation also makes it Gorenstein.

We give its fine resolution as well. With $Z = U - 1$, its equation in the regular threefold $R[[x, y, Z]]/(xy + 2)$ is

$$Z^2 - xyZ - x^3 - y^3 - x^3y^3 = 0. \tag{7.27}$$

The initial form is Z^2. After the blowup of the closed point, the Z-chart contains no exceptional point. On the x-chart put $y = xv, Z = xw$. The exact strict equation is

$$w^2 - x(1 + v^3 + vw) - x^4v^3 = 0, \qquad 2 + x^2v = 0. \tag{7.28}$$

It is monic in w, and x is a non-zero-divisor in the quotient; this is the saturated strict transform. On the exceptional line, only the points $v = a$, $a^3 = 1$, are singular. At such a point put $v = a + h$; the tangent quadratic is

$$w^2 + axw + a^2xh.$$

Its projective conic is smooth: the h-derivative gives $x = 0$, the equation gives $w = 0$, and the x-derivative then gives $h = 0$. The symmetric y-chart has no additional singularity. Blowing up these three points resolves the surface. As before, every center is a multiplicity-two hypersurface point in a regular threefold, so all four blowups have zero discrepancy. This is a mixed-characteristic calculation, not an appeal to the classification of quotient singularities in characteristic zero.

The fine global quotient is regular off these points: on the ordinary locus the involution exchanges distinct normalized branches, and at the cusps it exchanges the disjoint disks just described. Glue the local resolutions to obtain a proper regular resolution of the fine quotient. Crepancy and normality give equality of its two global canonical lattices, exactly as in (7.24). The smooth cusp permits the application of [CNS24, Proposition 6.2]; hence the fine quotient has rational singularities.

We next treat the wild coarse quotient at the universal tame level $M = 5$. In the modular coordinates above, the characters of the two actual deck involutions give

$$\sigma^* F = -F, \qquad \sigma^* V = -V, \qquad \sigma^* G = G.$$

Thus $\sigma(x) = -1/x$ and $\sigma(Y) = Y/x^4$. The invariant functions

$$t = x - 1/x, \qquad V_0 = Y/x^2$$

satisfy $V_0^2 = t^4 + 12t^2 + 16$. Set

$$u_0 = (V_0 + t^2)/2, \qquad X = u_0 + 1, \qquad Y_E = t(u_0 + 3).$$

Direct substitution and differentiation give

$$\begin{aligned} Y_E^2 &= X^3 - 7X - 6, \\ \frac{dX}{2Y_E} &= \frac{dt}{V_0} = (1 + x^2)\frac{dx}{Y} = G\frac{dq}{q}. \end{aligned} \tag{7.29}$$

Conversely $t = Y_E/(X + 2)$ recovers the quotient function field. The discriminant of (7.12) is 6400, of 2-adic valuation $8 < 12$. This integral Weierstrass equation is therefore minimal at two, and $\omega_{40} = dX/(2Y_E)$ is its Néron differential. The identity (7.29) establishes the differential normalization on this specific curve.

The smooth cusp on the actual $\mathcal{Y}_{40}$ has the same q-parameter as its source cusp. Since G is integral with first coefficient one, its cusp-integral lattice is $R\omega_{40}$. Proposition 6.2 of [CNS24] now gives

$$R\omega_{40} \subseteq H^0(\mathcal{Y}_{40}, \omega) \subseteq R\omega_{40}.$$

It follows that the actual $\mathcal{Y}_{40}$ has rational singularities. The comparison uses its quotient cusp and the minimal elliptic equation (7.12).

For later use we record why the comparison propagates as a statement about completed pairs, including their maps. Put the odd cyclic marking on the middle elliptic curve. An automorphism of this marked deformation acts on the two two-element sets of non-backtracking choices. Exchanging the elements of a two-element set is its unique nonidentity permutation; hence the automorphism commutes with both exchanges s, t. If H is the effective stabilizer of the odd marking, the completed source–quotient pair is therefore

$$(A^H, C^H), \tag{7.30}$$

with its inclusion $C^H \subset A^H$. The odd marking deforms uniquely over the 2-primary deformation, by the étaleness of prime-to-two torsion. Thus the pair is independent of the odd level once its stabilizer action is specified.

For $M > 1$ the effective stabilizer is cyclic; this is the faithful action on an odd cyclic marking used in [CNS24, Lemma 6.9 and Proposition 6.10]. Inside $\mathrm{Aut}(E_{\mathrm{ss}})/\{\pm 1\} \simeq A_4$, it is $1, C_2$, or C_3. All subgroups C_2 are conjugate. Their inverse images are cyclic of order four, and they are diagonalizable on $E_{\mathrm{ss}}[5]$, since $\mathbb{F}_5$ contains fourth roots of unity. An eigenline provides a cyclic 5-marking realizing each of them. At the supersingular point the connected order-two middle subgroup is unique, and both end choices specialize to their unique connected subgroups. This realization therefore applies to the entire pair (7.30), including the quotient map.

The level-40 comparison proves rational singularities for all the rings C^{C_2} that occur. The fine resolution proves the assertion for $H = 1$; the tame case $H = C_3$ follows from [CNS24, Lemma 6.8]. Away from supersingular points the ordinary and cusp descriptions above apply. Hence every actual $\mathcal{Y}_{8M}$, $M > 1$ odd, has rational singularities. For $M = 1$ the generic quotient has genus zero and the same smooth cusp, so both differential lattices vanish and Proposition 6.2 gives the assertion. We have proved, uniformly in M, the identifications

$$\mathrm{Pic}^0_{X_0(8M)_R/R} \simeq \mathcal{J}_0(8M)^0, \qquad \mathrm{Pic}^0_{\mathcal{Y}_{8M}/R} \simeq \mathcal{J}(\mathcal{Y}_{8M})^0. \tag{7.31}$$

7.6 A global residue lift at the actual old level

Return to $\gamma = 8g$ in (7.11). The fine restriction β is holomorphic. At a tame stabilizer of order e, the differential order formula $\mathrm{ord}_{\mathrm{up}} = e\,\mathrm{ord}_{\mathrm{down}} + e - 1$ implies that its descended restriction is holomorphic. At a wild C_2 stabilizer, the different exponent is two, by (6.3). Thus

$$\mathrm{ord}_{\mathrm{up}}(\beta) = 2\,\mathrm{ord}_{\mathrm{down}}(\beta) + 2.$$

The descended β has at most a simple pole at such a point. It is holomorphic everywhere else, including at cusps. Conversely, a simple pole at one of these coarse wild points pulls back to a holomorphic fine differential.

Put $\mathcal{Z} = X_0(2M)_R$. Its special fibre is reduced and nodal, with two smooth normalized tame components and the supersingular points paired between them. This assertion at depth one includes the prime two; see [BLR90, §9.7, p. 288]. We include the local wild check. The fine node is $R[[u, v]]/(uv + 2)$. Its completed quotient by the effective C_2 is independent of the odd marking, and each such subgroup is realized by a cyclic 5-marking, as in (7.30). The curve $X_0(10)$ has genus zero, two normalized $\mathbf{P}^1$ components, and one supersingular gluing point. To see the last assertion, the six cyclic 5-subgroups form a transitive A_4-orbit: an order-four elliptic automorphism has two eigenlines on $E_{\mathrm{ss}}[5]$, giving effective stabilizer C_2 and orbit size $12/2 = 6$. The normalization genus formula then shows that the local normalization quotient has length one. A reduced two-branch curve singularity with this value is a node: its subring in the two normalization power-series rings already lies in the codimension-one ring of pairs with equal constant term, and equality of codimensions makes the two rings equal. Thus the wild quotient is nodal for every M. A tame branch-preserving quotient of a node is again nodal, as one sees by taking separate invariant branch parameters. The other points are smooth.

Keep β on the normalized component reached separably by A_1, A_2. Its residues sum to zero by the residue theorem. On the other normalized component choose a meromorphic differential β', with at most simple poles at the paired supersingular points, whose residues are the negatives of the corresponding residues of β. Such a differential exists: the residue sequence for a smooth proper curve maps $H^0(\Omega^1(D))$ onto the vectors of residues with sum zero. Its only obstruction is precisely the sum just computed. At a nonwild point the required residue is zero, so β' is

holomorphic there. The pair (β, β') satisfies the opposite-residue condition at every node, hence is a Rosenlicht differential on $\mathcal{Z}_k$.

By Lemma 2.3, it lifts to

$$\kappa \in H^0(\mathcal{Z}, \omega_{\mathcal{Z}/R}). \tag{7.32}$$

The lifting mechanism uses cohomological flatness and relative duality. The special curve is reduced, proper, and connected, with a smooth cusp; its only global functions are constants. Consequently $H^1(\mathcal{Z}, \mathcal{O})$ is torsion-free and relative duality gives $H^1(\mathcal{Z}, \omega) = R$. The reduction sequence for ω then makes $H^0(\mathcal{Z}, \omega) \to H^0(\mathcal{Z}_k, \omega)$ surjective.

Use the actual degeneracy map

$$b_4 : X_0(8M) \longrightarrow X_0(2M), \qquad \tau \longmapsto 4\tau.$$

In the fine coordinates its old parameters are

$$u = w, \qquad v = y(w + y), \qquad uv = -2. \tag{7.33}$$

On A_1, A_2 it maps separably with degree one to the first old component. On A_3, the surviving coordinate is $u = t^2$, and on A_4 it is $v = t^4$; these are relative Frobenius and its square. Thus, as rational differentials on the normalized special fibre,

$$b_4^*\kappa = (\beta, \beta, 0, 0). \tag{7.34}$$

The zero restrictions do not depend on the holomorphic part chosen in β'. The equalities hold also on the coarse normalizations, since they hold at their generic ordinary points with the actual modular maps.

Both $8g$ and $b_4^*\kappa$ are canonical on the original normal coarse model $\mathcal{X}$. The special fibre is reduced: its fine completed equation is the product of the four distinct factors in (7.5), and its generic component multiplicities are all one. Normality makes the total surface Cohen–Macaulay, so the special fibre has no embedded associated points. Alternatively, for a stabilizer quotient, $A^H/(2)$ injects into $A/(2)$, because invariance of $2a$ implies invariance of a in a torsion-free ring. This also proves reducedness for the diagonal quotient.

Equations (7.11) and (7.34) now show that

$$\eta = \frac{8g - b_4^*\kappa}{2} \in H^0(\mathcal{X}, \omega_{\mathcal{X}/R}). \tag{7.35}$$

Indeed, their difference has order at least one along each multiplicity-one special component; after division by two it is regular at every height-one point. Reflexivity of the canonical sheaf gives global regularity. Write $\kappa = h(q)dq/q$ at the old infinity cusp. The differential pullback, including its scalar, is

$$b_4^*\kappa = 4h(q^4)\frac{dq}{q}.$$

Hence

$$\eta = 4g - 2V_4\kappa, \qquad i^*\eta = 4i^*g. \tag{7.36}$$

The second identity follows from newness, since $i^*b_4^*\kappa = 0$. All coefficients of the q-expansion of η belong to $2R$. Its reduction therefore vanishes identically on the normalized infinity component A_4.

We will also need its precise fine poles at the other branches. Pull κ back to the fine old node and write

$$\kappa = K(u,v)\Omega_0, \qquad \Omega_0 = du/u.$$

Both normalized special restrictions are holomorphic there: at a wild point this follows from the different exponent two and the simple-pole bound above. Consequently $K \in (u,v)$. Choose L_0, M_0 in the old node ring with $K = uL_0 + vM_0$, and evaluate them below using (7.33). Matching the first restriction to $-F_2(0,s)ds$ gives the exact relation

$$F_2 + L_0 = yH_0, \qquad H_0 \in D,$$

because $D/(y) = k[[w]]$. Differentiating the two actual nonbacktracking equations gives

$$b_4^*\Omega_0 = (w+2y)(2z+x)\Omega_A, \qquad y(2z+x) = 2(yz-1).$$

Substitution in (7.35), without dropping the second old component, yields the integral identity

$$\frac{\eta}{\Omega_A} = w^2H_0 - zw^2L_0 - wyL_0(2z+x) - (w+y)M_0(w+2y)(yz-1). \tag{7.37}$$

One may verify it without division in the local ring: use $F_2 = yH_0 - L_0$, $x = w(w+y)$, and $y = z(z+x)$, multiply by two, and subtract the numerator in (7.35). The resulting polynomial is

$$(2+xy)(H_0w^2 + M_0w^2 + 3M_0wy + 2M_0y^2),$$

which vanishes. Modulo two, (7.37) simplifies to

$$\eta/\Omega_A = w^2(H_0 - zL_0) + xM_0(1-yz). \tag{7.38}$$

On A_1, A_2 both w^2 and x have order at least two. On A_3 we have $w^2 = t^4, x = 0$; on A_4 we have $w = x = 0$. Since the normalization pole of Ω_A has order four, this proves

A_1, A_2	poles of order at most two
A_3	holomorphic
A_4	identically zero.

(7.39)

In particular the term involving M_0, coming from the second old branch through a divided Frobenius pullback, has been retained explicitly.

7.7 The cotangent annihilator at nonwild points

For a proper reduced curve, the normalization units sequence is

$$H^0(\mathcal{O}_{\mathcal{X}_k^\nu}^\times) \longrightarrow \prod_P (\widetilde{\mathcal{O}}_P^\times/\mathcal{O}_P^\times) \longrightarrow \mathrm{Pic}^0(\mathcal{X}_k) \longrightarrow \prod_j \mathrm{Pic}^0(\mathcal{X}_{k,j}^\nu) \longrightarrow 0.$$

The units on each normalized proper component are constants. They affect only the constant-unit torus. They impose no relation on the positive-jet unipotent groups. Thus the unipotent radical is the product

$$U_{\mathcal{X}} = \prod_P U_{\mathcal{X},P}. \tag{7.40}$$

The same is true for $\mathcal{Y}$. A homomorphism $\mathbb{G}_a \to \mathrm{Pic}^0$ has zero image in the abelian quotient and in the torus quotient, and hence lands in this product. We will apply this to the actual special homomorphism from $\mathcal{E}_k^0 = \mathbb{G}_a$, since E has additive reduction at conductor depth three.

At a point with trivial effective stabilizer, the completed special ring is

$$\overline{A} = k[[z, w]]/(zw(w + z^2)(z + w^2)).$$

Its conductor in the four normalization rings has order four on every branch: for instance the product of the other three branch equations restricts to order $1 + 2 + 1 = 4$ on A_1, and similarly on the others. Modulo this conductor the normalization has dimension 16. The image of $\overline{A}$ has the following basis:

$$1, z, w, z^2, zw, w^2, z^3, w^3. \tag{7.41}$$

To check the assertion, substitute the four parametrizations (7.6) and reduce modulo the fourth powers of the branch parameters. These eight vectors are independent. Every monomial not displayed either has order at least four on each branch, or has the same four truncated restrictions as a combination of the displayed ones. In particular z^2w and zw^2 have zero truncated restrictions, and mixed monomials of higher degree do also. Thus the normalization quotient has dimension $16 - 8 = 8$. After removal of the three-dimensional constant torus, the local unipotent group has dimension five.

Multiplying by units from $\overline{A}$ gives a unique representative for its class of the form

$$(1,\ 1 + as + ps^2 + cs^3,\ 1 + bt + qt^2,\ 1). \tag{7.42}$$

Here is an explicit elimination procedure, also explaining uniqueness. First remove the series on A_1 and A_4 by units in w and z; their truncated restrictions at those two branches form triangular bases. The remaining degree-three coefficient on A_3 is removed with the unit in zw. The five coefficients left in (7.42) are complementary to the eight-dimensional space in (7.41), after constants are removed, so no further identification is possible.

Multiplying the four branch series and performing the same elimination gives

$$\begin{aligned} a'' &= a + a', & b'' &= b + b', \\ p'' &= p + p' + aa', & q'' &= q + q' + bb', \\ c'' &= c + c' + ap' + pa' + bq' + qb'. \end{aligned}$$

For the last line, the product on A_3 first has cubic term $bq' + qb'$; the unit in zw removes that term and adds it to the cubic term on A_2. In particular

$$[2](a, b, p, q, c) = (0, 0, a^2, b^2, 0). \tag{7.43}$$

The reduced subgroup killed by two is therefore $a = b = 0$. The image of every homomorphism from $\mathbb{G}_a$ lies in that reduced subgroup, since $\mathbb{G}_a$ is reduced and is itself killed by two. Its tangent vectors have normalization representatives

$$(0,\ ps^2 + cs^3,\ qt^2,\ 0).$$

The canonical cotangent pairing is the sum of the branch residues of their products with a differential. The bounds (7.39) make each such residue zero. Thus $\overline{\eta}$ annihilates the tangent image of any $\mathbb{G}_a$ at a point with trivial stabilizer.

For an effective stabilizer C_3, pull back to the fine local cover. Pullback followed by norm is multiplication by three, and the residue pairing upstairs is three times the pairing downstairs. The pulled-back additive image is still contained in the reduced two-torsion described above. The fine residue pairing vanishes, and three is a unit in k. This proves the same annihilator statement at every nonwild point.

7.8 The cotangent annihilator at wild points

If M has a prime factor congruent to 3 (mod 4), all effective stabilizers are odd [CNS24, Lemma 6.9], so the preceding subsection already applies everywhere. We may therefore assume that $M > 1$ and every prime factor of M is congruent to 1 (mod 4). We first obtain a factorization on the original elliptic curve. Choose an odd integer a with $a^2 \equiv -1 \pmod{M}$, and put

$$b = -\frac{a^2+1}{2M}, \qquad h = \begin{pmatrix} a & b \\ 2M & -a \end{pmatrix}.$$

The integer b is odd, because $a^2 + 1 \equiv 2 \pmod 8$. The matrix h belongs to $\Gamma_0(2M)$, has determinant one and trace zero, and represents the diagonal coset in the quotient $\mathbb{F}_2^2$ of (7.2). Its fixed point in the upper half-plane gives a fixed point of σ on the generic source. Thus the degree-two map $\rho : X_0(8M) \to Y_{8M}$ is ramified.

It follows that $\rho^* : J(Y_{8M}) \to J_0(8M)$ is injective. Indeed, a nontrivial element of its kernel would be a nontrivial line bundle of order two whose associated étale double cover of Y_{8M} becomes trivial on $X_0(8M)$. The degree-two map would then factor through that étale double cover, and the equal degrees would make the factor an isomorphism, contrary to ramification. Since $\sigma i = i$, the connected image of i lies in the identity invariant abelian subvariety, which is the image of ρ^*. Hence there is an actual homomorphism

$$j : E \longrightarrow J(Y_{8M}), \qquad i = \rho^* j. \tag{7.44}$$

Néron functoriality extends this factorization integrally, and (7.31) identifies its special fibre with the actual Picard pullback. Equation (7.40) puts $j_k(\mathbb{G}_a)$ in the product of the local unipotent groups $U_{\mathcal{Y},P}$.

Lemma 7.2. *For an effective C_2 completed pair (A^{C_2}, C^{C_2}) as in* (7.30)*, a canonical special differential on the source that vanishes on one normalized branch annihilates*

$$\mathrm{Lie}(\rho^* : U_{\mathcal{Y},P} \to U_{\mathcal{X},P}).$$

Proof. We first work on the actual coarse level-40 curves. The tame curve $X_0(5)_k$ is $\mathbf{P}^1$, and the transitive action on its six supersingular cyclic markings was proved in Subsection 7.6. Hence the actual special fibre $X_0(40)_k$ has four normalized $\mathbf{P}^1$ components, all meeting over one singular point, and no other singular point. Each component has multiplicity one. Its genus is three, so the normalization units sequence has local constant torus $\mathbb{G}_m^3$ and local unipotent dimension three. The four global component constants kill exactly that torus. Therefore

$$\mathrm{Pic}^0(X_0(40)_k) = U_{\mathcal{X},P}.$$

Similarly, $\mathcal{Y}_{40,k}$ has two normalized $\mathbf{P}^1$ components and a single singular point. Its genus is one, so its connected Picard group is its one-dimensional local unipotent group, identified with the additive reduction of (7.12) by (7.31).

Duality identifies the full dual of $\operatorname{Lie} U_{\mathcal{X},P}$ with $H^0(X_0(40)_k, \omega)$. The local expression of the pairing is

$$([f_j], \alpha) \longmapsto \sum_j \operatorname{Res}(f_j \alpha_j). \tag{7.45}$$

The tangent space of $U_{\mathcal{X},P}$ is the positive-jet quotient of the normalization. Two local canonical differentials give the same functional on this tangent space if and only if their difference has at most simple poles on every branch. To check both implications, test (7.45) on each individual positive monomial t_j^n, $n \geq 1$. The resulting residues recover precisely the coefficients of $t_j^{-n-1} dt_j$, that is, all principal parts of order at least two.

Let α be a local canonical differential vanishing on a chosen branch. There is a unique global canonical α_0 inducing the same functional on $\operatorname{Lie} U_{\mathcal{X},P}$. Their difference has at most simple poles. On the chosen branch, α_0 therefore has at most a simple pole at its one singular point. The residue theorem on this normalized $\mathbf{P}^1$ makes that residue zero. The restriction is holomorphic, and hence zero.

Suppose first that the chosen branch is the infinity branch. Lift α_0 to a global canonical differential on $X_0(40)_R$. Such a lift exists by relative duality and base change: the reduced connected special fibre has $H^0(\mathcal{O}) = k$, so $H^1(\omega) = R$ and reduction on $H^0(\omega)$ is surjective. Its canonical lattice is L_{40} by Subsection 7.4. Since the special restriction at infinity is zero, this lift has q-expansion whose coefficients all belong to $2R$, and hence belongs to $2S_{40}$.

Let $i_0 = \rho_{40}^* : E_{40} \to J_0(40)$. The exact quotient (7.29) and the trace–pullback identity give

$$i_0^*(F\, dq/q) = i_0^*(V\, dq/q) = 0, \qquad i_0^*(G\, dq/q) = 2\omega_{40}.$$

The first two equalities also follow from the anti-invariance of their differential lines under σ; the third is the degree-two trace of the pulled-back differential ω_{40}. By (7.15),

$$i_0^*\eta_0 = \omega_{40}, \qquad i_0^*(x\eta_0) = 0, \qquad i_0^*(x^2\eta_0) = \omega_{40}.$$

Consequently

$$i_0^*(2S_{40}) \subseteq 2R\omega_{40}.$$

The reduction of the lifted α_0 therefore annihilates the special tangent image of ρ_{40}^*; so does α, since their functionals are the same.

The assertion holds for any chosen branch. The actual deck transformations exchange the two members of each paired set, and Fricke exchanges the two paired sets. Fricke conjugates s to t, hence normalizes σ. These transformations descend to the diagonal quotient and commute with its Picard pullback in the corresponding square. They act transitively on the four branches and transport the proved assertion to every one of them.

Finally, (7.30) identifies the whole completed source–target pair for every effective wild point with the level-40 pair, after an unramified extension and a conjugacy of the C_2 stabilizer. The proof there established the map as well as the two rings. Normalization, the positive-jet quotient, and residue pairing are functorial under this identification. The just-proved annihilator therefore transports to every odd M, proving the lemma. □

Proof of Theorem 7.1. The case $M = 1$ has $X_0(8)$ of genus zero, so there is no elliptic quotient to consider. For $M > 1$, start with any $g \in S$ and make the odd projection defined at the beginning of this section. It preserves i^*g. The highest-term argument gives the canonical $8g$, and the global residue lift constructs the canonical η satisfying (7.36). Via (7.31), the reduction

of i is an actual homomorphism from $\mathcal{E}_k^0 = \mathbb{G}_a$ to the special Picard group. Its image lies in the local unipotent product (7.40).

At nonwild points, Subsection 7.7 shows that $\overline{\eta}$ annihilates the relevant tangent image. If wild points occur, the factorization (7.44) reduces their contribution to the local images of $U_{\mathcal{Y},P}$. The differential $\overline{\eta}$ vanishes on the normalized infinity component by (7.36). That component passes through every supersingular gluing point. Lemma 7.2 therefore annihilates each wild contribution. The component constants contribute only to a torus, whose tangent is absent from the image of $\mathbb{G}_a$; no further term remains.

Thus $i^*\eta$ has zero reduction as an invariant differential on $\mathcal{E}^0$:

$$i^*\eta \in 2R\omega_E.$$

By (7.36), this says $4i^*g \in 2R\omega_E$, or $2i^*g \in R\omega_E$. Faithfully flat unramified descent gives the assertion over $\mathbb{Z}_2$. The lattice identity of Lemma 2.1 gives the stated valuation bound. □

7.9 An exact coefficient certificate

We specify all arithmetic needed for (7.13) and (7.14), so that the modular equations used in the proof can be reproduced without a modular-forms database. Expand

$$F = q\prod_{n\ge1}(1-q^{2n})^2(1-q^{10n})^2, \qquad V(q) = F(q^2), \qquad G = \sum_{n\ge1} a_n q^n.$$

The coefficients of G are determined by $a_1 = 1$, multiplicativity for relatively prime arguments, $a_2 = 0,\ a_5 = 1$, and

$$a_{\ell^e} = \begin{cases} a_\ell a_{\ell^{e-1}} - \ell a_{\ell^{e-2}}, & \ell \nmid 10,\ e \ge 2, \\ a_\ell^e, & \ell \mid 10. \end{cases}$$

For a good odd prime, counting points on (7.12) gives

$$a_\ell = -\sum_{x\in\mathbb{F}_\ell}\left(\frac{x^3-7x-6}{\ell}\right).$$

The value $a_5 = 1$ follows also from split multiplicative reduction: the cubic modulo five has double root 3, simple root 4, and square tangent coefficient $3-4=4$. Additive reduction gives $a_2 = 0$. Here are all required prime coefficients through 100:

ℓ	2	3	5	7	11	13	17	19	23
a_ℓ	0	0	1	−4	4	−2	2	4	4
ℓ	29	31	37	41	43	47	53	59	61
a_ℓ	−2	−8	6	−6	−8	4	6	−4	−2
ℓ	67	71	73	79	83	89	97		
a_ℓ	8	0	−6	0	−16	−6	−14		

The following integer-arithmetic program performs the two finite comparisons. Lists are coefficient vectors modulo q^{101}. The program computes the prime coefficients from the cubic. The two printed lists of nonzero residual coefficients are both empty, in fact through degree 100, exceeding the necessary bounds.

```
B = 100
one = [1] + [0]*B
def lin(*terms):
    return [sum(c*a[n] for c,a in terms)
            for n in range(B+1)]
def mul(a,b):
    c = [0]*(B+1)
    for i,ai in enumerate(a):
        if ai:
            for j in range(B+1-i):
                c[i+j] += ai*b[j]
    return c
def power(a,e):
    r = one[:]
    while e:
        if e % 2:
            r = mul(r,a)
        a = mul(a,a)
        e //= 2
    return r
def deriv(a):
    return [n*a[n] for n in range(B+1)]

F = [0,1] + [0]*(B-1)
for step in (2,10):
    for n in range(step,B+1,step):
        h = one[:]
        h[n] = -1
        F = mul(F,mul(h,h))
V = [F[n//2] if n % 2 == 0 else 0
     for n in range(B+1)]
primes = [p for p in range(2,B+1)
          if all(p % d for d in range(2,p)
                 if d*d <= p)]
ap = {}
for p in primes:
    if p == 2:
        ap[p] = 0
    elif p == 5:
        ap[p] = 1
    else:
        def chi(v):
            r = pow(v % p,(p-1)//2,p)
            return -1 if r == p-1 else r
        ap[p] = -sum(chi(x**3-7*x-6)
                     for x in range(p))
expected = [0,0,1,-4,4,-2,2,4,4,-2,-8,6,
            -6,-8,4,6,-4,-2,8,0,-6,0,-16,-6,-14]
assert [ap[p] for p in primes] == expected

def coefficient(n):
    result = 1
```

```
    for p in primes:
        e = 0
        while n % p == 0:
            n //= p
            e += 1
        if not e:
            continue
        if p in (2,5):
            result *= ap[p]**e
        else:
            prev,cur = 1,ap[p]
            for j in range(2,e+1):
                prev,cur = cur,ap[p]*cur-p*prev
            result *= cur
    assert n == 1
    return result
G = [0] + [coefficient(n) for n in range(1,B+1)]
A = lin((1,G),(-1,F))
W = lin((2,V))
bracket = lin((1,mul(deriv(A),V)),
              (-1,mul(A,deriv(V))))
R4 = lin((1,power(G,2)),(-1,power(F,2)),
         (-4,power(V,2)))
R16 = lin((16,mul(power(A,2),power(bracket,2))),
          (-1,power(A,8)),
          (-8,mul(power(A,6),power(W,2))),
          (2,mul(power(A,4),power(W,4))),
          (-8,mul(power(A,2),power(W,6))),
          (-1,power(W,8)))
assert all(c == 0 for c in R4[:25])
assert all(c == 0 for c in R16[:97])
print([(n,c) for n,c in enumerate(R4) if c])
print([(n,c) for n,c in enumerate(R16) if c])
```

All factors beyond the displayed truncation begin too late to affect these coefficients. The genus and holomorphic weight computations in Subsection 7.4, together with the bounds $4 \cdot 72/12 = 24$ and $16 \cdot 72/12 = 96$, turn these finite coefficient identities into identities of modular forms. Thus the certificate proves the two equations used in the integral model computation.

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