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QUANTUM PERMUTATIONS AND THE STANDARD MODEL ALGEBRA

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ABSTRACT. The finite part of the Standard Model, in the Chamseddine–Connes approach, has algebra $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$, and its quantum isometry groups were computed by Bhowmick, D'Andrea and Dabrowski. We explain how the quantum automorphism group of the complex algebra $B_F = \mathbb{C} \oplus \mathbb{C} \oplus M_2(\mathbb{C}) \oplus M_3(\mathbb{C})$ generated by A_F, with its canonical trace, is a cocycle twist of the quantum permutation group S_{15}^+, and how the color parts of the quantum isometry groups are twists of S_9^+, and of a quantum group $G_3 \subset S_9^+$ preserving an oriented area tensor on $\mathbb{F}_3^2$. We prove that G_3 is infinite, with classical version of order 216, while the quantum symmetry group of the unoriented area tensor is classical, of order 432. With the trace coming from the fermions, the coaction preserves the weak and color summands separately, and the quantum automorphism group becomes a twist of $S_4^+ \hat{*} S_9^+$. We also discuss matrix models and maximality, the fermions and the Heisenberg group of order 27, the untwisting of color, which leads to color Lie algebras in the sense of Rittenberg, Wyler and Scheunert, the spectral action, and the mixing matrices.

CONTENTS

Introduction 1
1. Twisting finite quantum spaces 4
2. The Chamseddine-Connes algebra 6
3. Physical trace and weak sector 9
4. Real structure and area tensors 11
5. Matrix models and maximality 15
6. Fermions and the Heisenberg group 17
7. Untwisting and color Lie algebras 20
8. Spectral action and classicality 23
9. Mixing matrices and Hadamard 26
10. A look into Yang-Mills theory 30
11. Further questions and outlook 33
References 36

INTRODUCTION

This paper is the outline of a program, whose aim is to understand the Standard Model of particle physics in terms of quantum permutations, by using the noncommutative geometry approach of Chamseddine and Connes, and the twisting methods for quantum permutation groups.

The Standard Model of particle physics can be formulated, following Chamseddine and Connes, as the product of an ordinary spin manifold M with a finite noncommutative space F, whose algebra is $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$, see [22, 23, 24, 28, 34]. The gauge group, which is $(U_1 \times SU_2 \times SU_3)/\mathbb{Z}_6$, the Higgs field, which appears as an inner fluctuation of the Dirac operator, and the spectral action all emerge from this finite geometry.

It is natural to ask whether F has quantum symmetries, in the sense of Wang [87]. This question was answered by Bhowmick, D'Andrea and Dabrowski in [12], with the quantum isometry group of F being $C(U_1) * A_{aut}(M_3(\mathbb{C}))$ in their complex setting, and $C(U_1) * C(PU_3)$ in their real setting, where the coactions must extend to the complexification of the real algebra A_F. Here U_1 is a circle acting on the weak summand, and the color part is free in the complex setting, and classical in the real setting.

In this paper we revisit this question by using quantum permutations instead of quantum rotations. Our starting point is the twisting result $A_{aut}(\mathbb{C}_\sigma[G]) \simeq A_{aut}(\mathbb{C}[G])^\sigma$ from [8], which shows that the matrix algebra $M_N(\mathbb{C})$ is a twisted version of the algebra of functions on N^2 points, and that the quantum group PU_N^+ is a twist of $S_{N^2}^+$. This extends to arbitrary finite dimensional C^*-algebras, see [18]: any such algebra B, with its canonical trace, is a twisted version of $N = \dim B$ points, in the sense that its quantum automorphism group is a cocycle twist of S_N^+. In particular, the complex algebra $B_F = \mathbb{C} \oplus \mathbb{C} \oplus M_2(\mathbb{C}) \oplus M_3(\mathbb{C})$ generated by the Chamseddine–Connes algebra corresponds in this way to $15 = 1 + 1 + 4 + 9$ points. We will present a short proof of all this, which also covers the quantum subgroups defined by extra invariant maps, that we will need.

In this picture the weak sector corresponds to S_4^+, which is classical in disguise, untwisting into SO_3, and the color sector corresponds to S_9^+, which is free on both sides, untwisting into PU_3^+. However, all this uses the canonical trace of B_F, and with this trace the 15 points get mixed by S_{15}^+, and even the electroweak part alone, corresponding to $1+1+4=6$ points, has a free symmetry, coming from S_6^+. The physically relevant trace is rather the one coming from the fermions. This trace has unnormalized weights $8, 4, 4, 4$, per generation, on the minimal projections of the 4 summands, and, after normalization, the operator mm^* acts on these summands by the scalars $4, 8, 16, 24$, which are all distinct, and this forces the coaction to preserve each summand. To be more precise, we have the following result:

Theorem A. *With the trace coming from its action on the fermions, the quantum automorphism group of B_F is the twist of $S_4^+ \hat{*} S_9^+ \subset S_{15}^+$, the quantum subgroup preserving each of the 4 blocks of the partition $15 = 1 + 1 + 4 + 9$. Equivalently, its algebra of functions is the free product $C(SO_3) * C(PU_3^+)$.*

Thus, with the physical trace, the coaction preserves the weak and color summands separately, their coefficient algebras appearing in a free product, and freeness appears exactly at the level of quarks. The weak part, which is the twist of S_4^+, is then reduced by the Dirac operator to a circle, according to [12], acting on the off-diagonal matrix units of $M_2(\mathbb{C})$, while the color part survives intact.

Our main result concerns the real case, and the color part. The classical color group in the real case is characterized by the fact that it preserves the opposite product of $M_3(\mathbb{C})$ as well, and on the permutation side the opposite product becomes an oriented area tensor b on the 9 points of $\mathbb{F}_3^2$. We have the following result:

Theorem B. *The quantum subgroup $G_3 \subset S_9^+$ preserving the oriented area tensor on $\mathbb{F}_3^2$ is a twist of the real color group PU_3. It is infinite and noncommutative, and its classical version is the affine symplectic group $\mathbb{F}_3^2 \rtimes SL_2(3)$, of order 216. By contrast, the quantum symmetry group of the unoriented area tensor $Re(b)$ is classical, equal to $\mathbb{F}_3^2 \rtimes GL_2(3)$, of order 432.*

The first part works in fact for any finite abelian group H, with a twist of PU_N appearing inside $S_{N^2}^+$, where $N = |H|$, and the description of this twist by a single oriented triangle tensor is what we would like to emphasize here. As for the second part, the proof uses the fact that the third point on the line through 2 points a, c of $\mathbb{F}_3^2$ is $-a - c$, and the commutation of the coordinates comes from a general quasigroup argument.

We discuss as well a number of related topics. We construct explicit matrix models for the quantum groups G_H, coming from the Weyl operators, we determine exactly when they are commutative, we compute the Hopf image of any family of them, the classical version of G_H corresponding to the Clifford group, and we discuss the maximality of the classical version inside G_H, which holds precisely when H has prime exponent. Regarding the fermions, the color triplets cannot be untwisted one at a time, by a determinant argument, and the projective clock and shift cannot be lifted to commuting symmetries of the finite geometry, but the Heisenberg group of order 27, which is isomorphic to the flavor group $\Delta(27)$, acts on the fermion space of the Standard Model by symmetries of the whole finite geometry. Also, keeping the usual quark module, with the observed connected CKM mixing, at most 3 color points can act. We also discuss some classicality mechanisms, including one coming from the bosonic spectral action: under a coefficientwise reality condition, the natural transformation of 1-forms preserves the bosonic action on every tested geometry precisely for the classical finite quantum groups, while for the actual Standard Model geometry a 2-sided spatial implementer restores the bosonic invariance.

The question of untwisting is in fact the key question raised by the present work. Indeed, the above results show that the algebra of the finite geometry of the Standard Model is, in a precise sense, a twisted version of the algebra of functions on a finite set of points, and the natural problem is that of understanding what happens when the whole theory, including the fermions, the gauge fields and the action, is transported to the untwisted side, where the symmetries are given by quantum permutations. We discuss as well the untwisting of color, at the level of the various objects involved. The twist preserves the dimensions of the representations, and the dimensions of the irreducible representations of the color quantum group S_9^+ turn out to be the Fibonacci numbers $1, 8, 55, 377, \ldots$, those of G_3 being those of PU_3. On the other hand, for G_3, which is the twist of PU_3, the twist changes the symmetric braiding, the flip becoming a graded flip, with cube roots of unity as coefficients, which is formally reminiscent of the parastatistics of Greenberg [52], which was the origin of color. At the level of Lie algebras, we obtain:

Theorem C. *The Lie algebra gl_3 is the twist of the color Lie algebra $\mathbb{C}[\mathbb{Z}_3^2]$, with bracket $[e_g, e_h] = (1 - w^{\det(g,h)})e_{g+h}$, where $w = e^{2\pi i/3}$. Accordingly, in the Weyl basis, the 8 gluon components are labeled by the nonzero elements of $\mathbb{F}_3^2$, with the pairs of independent labels interacting via trigonometric structure constants.*

Here the color Lie algebras are those of Rittenberg, Wyler and Scheunert [77, 79], and the trigonometric structure constants are those of Patera–Zassenhaus and Fairlie–Fletcher–Zachos [44, 74]. As for the quarks, these correspond after untwisting to modules over the twisted group algebra of $\mathbb{Z}_3^2$, in formal analogy with the discrete torsion of Vafa and Witten [83, 84]. In the weak sector the same happens at $N = 2$, with the Pauli matrices instead of the clock and shift matrices, and with the cube roots of unity being replaced by signs.

Finally, we discuss the mixing matrices, where the unitary invariant of Connes [29] reduces in the commuting square case to complex Hadamard matrices, which are well-known to produce quantum permutation groups, and where the subgroups of the weak quantum group S_4^+ meet the classical lepton flavor groups A_4, S_4, A_5. We also show that the groups of order 216 and 432 found above act on the 4 mutually unbiased bases of $\mathbb{C}^3$ via A_4 and S_4, with the trimaximal matrix appearing as a transition matrix between these bases, while G_3 itself has no nontrivial action on $k \leq 8$ points.

The paper is organized as follows: 1 contains the general twisting results, 2 deals with the Chamseddine–Connes algebra, 3 deals with the physical trace and the weak sector, 4 deals with the real structure, 5 discusses matrix models and maximality, 6 discusses the fermions, 7

discusses the untwisting of color, 8 discusses the spectral action, 9 discusses the mixing matrices, 10 discusses the Yang–Mills theory, as a program for later, and 11 contains a list of questions.

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1. Twisting finite quantum spaces

We use Wang's quantum automorphism groups [87], in the tracial setting from [5]. Given a finite dimensional C^*-algebra A, with a faithful trace $tr : A \to \mathbb{C}$, we denote by $A_{aut}(A, tr)$ the universal Hopf algebra coacting on A via a unital $*$-algebra morphism $\alpha : A \to A \otimes A_{aut}(A, tr)$, such that $(tr \otimes id)\alpha = tr(.)1$.

More generally, given a family T of linear maps $T_i : A^{\otimes k_i} \to A^{\otimes l_i}$, we denote by $A_{aut}(A, tr, T)$ the quotient of $A_{aut}(A, tr)$ making all the maps T_i colinear. For $A = \mathbb{C}^N$ with the uniform measure we obtain the algebra $C(S_N^+)$, and for $A = M_N(\mathbb{C})$ with the normalized trace we obtain the algebra $C(PU_N^+)$, see [5, 87].

Assume that Γ is a finite abelian group, and that A is Γ-graded, with the trace supported in degree 0. That is, $A = \oplus_{g\in\Gamma} A_g$ with $A_g A_h \subset A_{g+h}$, $A_g^* = A_{-g}$, $1 \in A_0$, and $tr(A_g) = 0$ for $g \neq 0$, the condition $A_g^* = A_{-g}$ being needed for having a grading compatible with the involution. Given a bicharacter $\sigma : \Gamma \times \Gamma \to \mathbb{T}$, we can then define a twisted product on A, as follows:

$$a \cdot b = \sigma(g, h)ab \quad , \quad \forall a \in A_g, b \in A_h$$

We denote by A_σ the resulting algebra, and we keep the same trace on it. The grading of A is a coaction of the group algebra $\mathbb{C}[\Gamma]$, namely $a \to a \otimes g$ for $a \in A_g$, which preserves the trace, so we have a Hopf algebra map $\pi : A_{aut}(A, tr) \to \mathbb{C}[\Gamma]$. We can therefore twist $A_{aut}(A, tr)$ by using the cocycle σ, via the map π, as in [8, 41]. We denote the result by $A_{aut}(A, tr)^\sigma$, and we work at the level of the dense Hopf $*$-algebras, the C^*-algebras being then obtained by universal completion. The product of the twisted Hopf algebra is given by the following formula, in Sweedler notation:

$$[x][y] = \sum \sigma^{-1}(\pi(x_1), \pi(y_1))\sigma(\pi(x_3), \pi(y_3))[x_2y_2]$$

In a homogeneous basis of A, with indices standing for degrees, the coefficients of the coaction $\alpha(e_g) = \sum_a e_a \otimes u_{ag}$ are therefore multiplied according to the formula $[u_{ag}][u_{bh}] = \frac{\sigma(g,h)}{\sigma(a,b)}[u_{ag}u_{bh}]$, the factor $\sigma(a, b)$ coming from the product in the first tensor leg, and the factor $\sigma(g, h)$ coming from the product of the inputs, as in the proof of [8, Theorem 1.3].

For maps $T : A^{\otimes k} \to A^{\otimes l}$ which preserve the degree, we define $T^\sigma = \Omega_l^{-1} T \Omega_k$, where Ω_k multiplies $a_1 \otimes \ldots \otimes a_k$, with $a_i \in A_{g_i}$, by the scalar $\prod_{j=1}^{k-1} \sigma(g_1 + \ldots + g_j, g_{j+1})$. Observe that the multiplication of A_σ is $m^\sigma = m\Omega_2$. With these conventions, we have the following result, extending [8]:

Theorem 1.1. *Assume that (A, tr) is Γ-graded as above, with the trace supported in degree 0, and let T be a family of degree preserving maps. We have then*

$$A_{aut}(A_\sigma, tr, T^\sigma) \simeq A_{aut}(A, tr, T)^\sigma$$

at the level of the dense Hopf $*$-algebras, and of their universal C^*-completions, and in particular these 2 Hopf algebras are isomorphic as coalgebras, via an isomorphism mapping coefficients to coefficients, and preserving the Haar integrals.

Proof. Let us set $L = A_{aut}(A, tr, T)$, and let $\pi : L \to \mathbb{C}[\Gamma]$ be the Hopf algebra map coming from the grading. As explained in [41, 67], the twisting by the 2-cocycle $\sigma(\pi \otimes \pi)$ does not change the coalgebra, and produces a monoidal equivalence between the categories of comodules over L and over L^σ, which is the identity on the underlying spaces and on the morphisms, and whose tensor structure is given by the maps Ω above. In other words, a linear map $T : A^{\otimes k} \to A^{\otimes l}$ which preserves the degree is L-colinear if and only if $T^\sigma = \Omega_l^{-1} T \Omega_k$ is L^σ-colinear.

We apply this to the multiplication m, to the unit, to the trace, which is supported in degree 0, so that $tr^\sigma = tr$, and to the maps T_i. We conclude that the coaction of L on A, regarded as a linear map, is a coaction of L^σ on A_σ, which makes colinear the multiplication $m^\sigma = m\Omega_2$, the unit, the trace and the maps T_i^σ. Regarding the involutions, the one of A_σ is given by $a \to \overline{\sigma(g,-g)} a^*$ for $a \in A_g$, and the compatibility of the coaction with the involutions follows as in [8], the values of σ being in $\mathbb{T}$. Thus, by universality, we obtain a Hopf algebra map $\Phi : A_{aut}(A_\sigma, tr, T^\sigma) \to L^\sigma$, mapping coefficients to coefficients.

The algebra A_σ is graded by the same grading, with the trace supported in degree 0, and the maps T_i^σ preserve the degree, so by symmetry, using the cocycle σ^{-1}, we have as well a Hopf algebra map $\Psi : L \to A_{aut}(A_\sigma, tr, T^\sigma)^{\sigma^{-1}}$, mapping coefficients to coefficients. By universality, Φ commutes with the maps to $\mathbb{C}[\Gamma]$ coming from the grading, so it can be twisted by σ^{-1}, and $\Phi^{\sigma^{-1}}\Psi$ and $\Psi^\sigma\Phi$ are the identity on the coefficients, which generate the algebras. Thus Φ is an isomorphism. Finally, since the twisting does not change the coalgebra, the Haar integrals coincide on the common vector space, as explained in [8], the products of coefficients appearing in the moments being of course transported via the above formula. □

For related deformation results, in the general setting of δ-forms, see [18]. In what follows we will only need Theorem 1.1, and its particular cases.

Let us discuss now the basic example. Let H be a finite abelian group, with $|H| = N$, and consider the group $\Gamma = \widehat{H} \times H$. We use the following bicharacters on Γ:

$$\sigma((\lambda, x), (\mu, y)) = \mu(x) \quad , \quad d((\lambda, x), (\mu, y)) = \lambda(y)\overline{\mu(x)}$$

Here d is the symplectic form on Γ, which is nondegenerate and alternating, and we have $\sigma(g,h)d(g,h) = \sigma(h,g)$. We use the group algebra $A = \mathbb{C}[\Gamma]$, with its standard basis $\{e_g\}_{g\in\Gamma}$, graded by $\deg(e_g) = g$, and with the trace $tr(e_g) = \delta_{g0}$. Via the Fourier transform, $\mathbb{C}[\Gamma] \simeq C(\widehat{\Gamma})$ is the algebra of functions on N^2 points, with the uniform measure. Observe also that the central extension of Γ by $\mathbb{T}$ defined by the cocycle σ is the Heisenberg group of H, and that d is its commutator form, $d(g,h) = \sigma(h,g)/\sigma(g,h)$. In the main case of interest, $H = \mathbb{Z}_n$, we have $\Gamma = \mathbb{Z}_n^2$, and with $w = e^{2\pi i/n}$ our bicharacters are given by $\sigma((i,j),(k,l)) = w^{kj}$ and $d((i,j),(k,l)) = w^{il-kj}$. On the other hand, the twisted algebra is a matrix algebra, coming from the clock and shift matrices, a construction which goes back to Weyl and Schwinger, and which is well-known in quantum information, where the corresponding operators are called generalized Pauli matrices. To be more precise, we have the following result:

Proposition 1.2. *We have an isomorphism $\mathbb{C}_\sigma[\Gamma] \simeq M_N(\mathbb{C})$, mapping the trace to the normalized trace, given by $e_{(\lambda,x)} \to M_\lambda T_x$, where M_λ is the multiplication by λ on $l^2(H)$, and T_x is the translation $(T_x f)(z) = f(z+x)$. For $H = \mathbb{Z}_n$, in terms of the matrix units $e_{ab} \in M_n(\mathbb{C})$, and with $w = e^{2\pi i/n}$, this isomorphism is given by:*

$$e_{(i,j)} \to \sum_k w^{ki} e_{k,k+j}$$

Proof. We have $T_x M_\mu T_x^{-1} = \mu(x) M_\mu$, so $M_\lambda T_x M_\mu T_y = \mu(x) M_{\lambda\mu} T_{x+y}$, which is the twisted product. The operators $M_\lambda T_x$ are linearly independent, and have trace 0 unless $(\lambda, x) = (1, 0)$,

so we obtain an isomorphism, mapping the trace to the normalized trace. The formula at $H = \mathbb{Z}_n$ is a reformulation of this. □

By combining the above results, we recover the following result from [8]:

$$A_{aut}(M_N(\mathbb{C}), tr) \simeq C(S^+_{N^2})^\sigma$$

At $N = 2$ the left term is $C(SO_3)$, see [5, 13], and we recover the fact that S^+_4 is a twist of SO_3, from [7]. At $N \geq 3$ both PU^+_N and $S^+_{N^2}$ are infinite and noncommutative. Observe that the twist changes the multiplication of the algebra, and this is unavoidable, due to the following fact:

Proposition 1.3. *For $N \geq 2$, any trace preserving coaction of $C(S^+_{N^2})$ on the unchanged algebra $M_N(\mathbb{C})$ is trivial.*

Proof. The coaction gives an N^2-dimensional corepresentation of $S^+_{N^2}$ having a fixed vector, and if nontrivial, this must be the magic representation $1 + r_1$, because the other irreducible representations have dimension $> N^2 - 1$. By restricting to the classical subgroup S_{N^2}, a transposition must act on $M_N(\mathbb{C})$ by an automorphism of trace $N^2 - 2$. But an involutive automorphism Ad_U has trace $|Tr(U)|^2 = (N - 2k)^2$, and $N^2 - 2$ is never of this form. □

Thus, the multiplication of the algebra must change, and the twisting from Theorem 1.1 is the natural way of changing it. Summarizing, we have obtained the following result:

Theorem 1.4. *For any finite abelian group H with $|H| = N \geq 2$, the quantum group PU^+_N is the twist of $S^+_{N^2}$ by the cocycle σ on $\widehat{H} \times H$, while $S^+_{N^2}$ itself has no nontrivial action on the unchanged algebra $M_N(\mathbb{C})$.*

Proof. This follows from Theorem 1.1, Proposition 1.2 and Proposition 1.3. □

Summarizing, the matrix algebra $M_N(\mathbb{C})$ is a twisted version of the algebra of functions on N^2 points, and its quantum symmetries are twisted quantum permutations, with the twisting being unavoidable. In the next section we extend this to arbitrary finite dimensional C^*-algebras, and in particular to the Chamseddine–Connes algebra, with its summands $M_2(\mathbb{C})$ and $M_3(\mathbb{C})$ corresponding to 4 and 9 points.

2. The Chamseddine-Connes algebra

Before getting into the Standard Model, let us extend the above to arbitrary finite dimensional C^*-algebras. Such an algebra is a direct sum of matrix algebras, $B = \oplus_i M_{n_i}(\mathbb{C})$, and its canonical trace, coming from the left regular representation, is given by $tr = \oplus_i \frac{n_i^2}{N} tr_i$, where $N = \dim B = \sum_i n_i^2$, and where tr_i are the normalized traces, see [5]. This is the unique trace on B satisfying the δ-form condition $mm^* = N \cdot id$, which is the natural condition for the quantum automorphism groups, as explained in [5], and as we will see in section 3 below. We have the following result, which is known, see [18]:

Theorem 2.1. *Any finite dimensional C^*-algebra $B = \oplus_i M_{n_i}(\mathbb{C})$, with its canonical trace, is a twisted version of $N = \sum_i n_i^2$ points, in the sense that*

$$A_{aut}(B, tr) \simeq C(S^+_N)^\sigma$$

with $\sigma = \prod_i \sigma_i$ being the product of the cocycles σ_i from section 1, on the group $\Gamma = \prod_i \Gamma_i$, with $\Gamma_i = \widehat{H_i} \times H_i$ and $|H_i| = n_i$, acting on the N points by translations on the blocks.

Proof. This is proved in [18, Theorem 3.1], with a construction using translations on the individual blocks and a product cocycle, and here is a short proof, in our setting. We grade the algebra $A = \oplus_i \mathbb{C}[\Gamma_i]$ by Γ, with the group algebras graded via the inclusions $\Gamma_i \subset \Gamma$. Via the Fourier transform we have $A \simeq \mathbb{C}^N$, and the canonical trace of A is the uniform measure, which is supported in degree 0. By Proposition 1.2, applied on each block, the twisted algebra is $A_\sigma = B$, with the trace becoming the canonical trace of B. Thus Theorem 1.1 applies, and gives the result, the map $\pi : C(S_N^+) \to \mathbb{C}[\Gamma]$ coming from the action of the dual group $\widehat{\Gamma} \simeq \Gamma$ on the N points. □

This is consistent with the results in [5, 39], where $A_{aut}(B, tr)$ was shown to have the fusion rules of SO_3, for $N \geq 4$, and to be monoidally equivalent to $C(S_N^+)$. Theorem 2.1 is a stronger statement, because a cocycle twist is a monoidal equivalence which preserves the dimensions of the corepresentations.

The finite part of the Standard Model is the real spectral triple $F = (A_F, H_F, D_F, \gamma_F, J_F)$, with algebra $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$, see [23, 24, 28, 34, 85]. For the general theory of finite spectral triples, see Krajewski [60] and Paschke–Sitarz [73], the latter discussing as well the symmetries coming from Hopf algebras, and for the relation with Morita equivalence, see [38]. Let us describe F explicitly, following [24, 85]. For one generation, the particles are the following 16 fermions, with the quarks coming in 3 colors, and with R, L standing for the right-handed and left-handed components:

leptons	ν_R	e_R	ν_L	e_L
quarks	u_R	d_R	u_L	d_L

Thus the space of particles is $H_f = \mathbb{C}^{16}$, with the quarks contributing $4 \times 3 = 12$ dimensions, and the leptons 4 dimensions. Together with the antiparticles, and with the 3 generations, we obtain the following Hilbert space, of dimension 96:

$$H_F = (H_f \oplus \overline{H_f}) \otimes \mathbb{C}^3$$

The algebra A_F acts on H_F as follows, with $a = (\lambda, q, m)$. On the particles, the quaternion q, viewed as a 2×2 complex matrix, acts on the left-handed doublets (ν_L, e_L) and (u_L, d_L), while λ acts on ν_R, u_R, and $\bar{\lambda}$ acts on e_R, d_R. On the antiparticles, λ acts on the antileptons, and m acts on the color index of the antiquarks, the leptons being regarded in this way as having a 4th color. The generation space $\mathbb{C}^3$ is not touched by the algebra.

The real structure is the antiunitary $J_F(\xi, \bar{\eta}) = (\eta, \bar{\xi})$, exchanging particles and antiparticles, and the grading γ_F is $+1$ on the left-handed particles and the right-handed antiparticles, and -1 on the other fermions. These satisfy the KO-dimension 6 relations, $J_F^2 = 1$, $J_F D_F = D_F J_F$, $J_F \gamma_F = -\gamma_F J_F$. As for the Dirac operator, this is of the following form:

$$D_F = \begin{pmatrix} S & T^* \\ T & \bar{S} \end{pmatrix}$$

Here S acts on the particles, connecting the left and right components, and is built from the Yukawa matrices $Y_\nu, Y_e, Y_u, Y_d \in M_3(\mathbb{C})$, acting on the generation space. Up to the normalization coming from the Higgs vacuum, the singular values of these matrices are the Dirac masses of the fermions, and the mixing matrices CKM and PMNS measure the relative position of the diagonalizing bases, for the up and down quarks, and for the neutrinos and electrons, with the light neutrino masses and the PMNS matrix coming in fact from the full neutral mass matrix, including the Majorana term below. As for T, this is the Majorana mass term, connecting ν_R with its antiparticle.

The algebra A_F is real, and the complex algebra that it generates on H_F is the following algebra, of dimension 15, see [12]:

$$B_F = \mathbb{C} \oplus \mathbb{C} \oplus M_2(\mathbb{C}) \oplus M_3(\mathbb{C})$$

Here the 2 copies of $\mathbb{C}$ appear because a complex number $\lambda \in \mathbb{C} \subset A_F$ acts on some fermions by λ, and on others by $\bar{\lambda}$, so that the complex span of these operators is $\mathbb{C} \oplus \mathbb{C}$, and $M_2(\mathbb{C})$ is the complexification of $\mathbb{H}$. Modulo many things, we have the following dictionary, for the summands of B_F:

summand	gauge group	interaction	points
$\mathbb{C} \oplus \mathbb{C}$	U_1	hypercharge	$1+1$
$M_2(\mathbb{C})$	SU_2	weak	4
$M_3(\mathbb{C})$	SU_3	strong	9

Observe that this dictionary is only a heuristic one, the gauge group of the Standard Model, which acts on the fermions as $(U_1 \times SU_2 \times SU_3)/\mathbb{Z}_6$, coming from the unimodular unitary group of A_F, and not of B_F, with the hypercharges being fixed by the unimodularity condition, see [24, 85]. This gauge group is not to be confused with the quantum symmetry groups discussed below. In what follows, we will be interested in the algebra B_F itself, and in its quantum symmetries. The canonical trace of B_F has weights $\frac{1}{15}, \frac{1}{15}, \frac{4}{15}, \frac{9}{15}$ on the 4 summands, and these are exactly the weights of the uniform measure on $1+1+4+9=15$ points. By applying Theorem 2.1, we obtain the following result:

Theorem 2.2. *Consider the group $\Gamma = \mathbb{Z}_2^2 \times \mathbb{Z}_3^2$, acting on $15 = 1+1+4+9$ points, with $\mathbb{Z}_2^2$ acting by translations on the 4 points, and $\mathbb{Z}_3^2$ acting by translations on the 9 points. We have then*

$$A_{aut}(B_F, tr) \simeq C(S_{15}^+)^\sigma$$

with $\sigma = \sigma_2 \times \sigma_3$ being the product of the cocycles from section 1, for $H = \mathbb{Z}_2$ and $H = \mathbb{Z}_3$.

Proof. This is the particular case of Theorem 2.1, for the algebra B_F. □

At the level of the summands, the weak part $M_2(\mathbb{C})$ corresponds to S_4^+, which untwists into the classical group $SO_3 = PU_2$, while the color part $M_3(\mathbb{C})$ corresponds to S_9^+, which untwists into the free quantum group PU_3^+. Thus, summand by summand, freeness appears exactly at the level of quarks. We will see in section 3 that this summand by summand picture is the correct one, when using the trace coming from the fermions.

Note also that the 9-dimensional magic representation of S_9^+ decomposes as $1+8$, and this corresponds, under the twist, to $M_3(\mathbb{C}) = \mathbb{C}1 \oplus sl_3(\mathbb{C})$, with 8 being the number of gluons. Similarly, $4 = 1+3$ at the weak level. However, the analogy with the gluons has its limits, due to the following fact:

Proposition 2.3. *The 8-dimensional representations r_1 of S_9^+ and $sl_3(\mathbb{C})$ of SU_3 have different cubic invariants, namely:*

$$\dim Hom_{S_9^+}(1, r_1^{\otimes 3}) = 1 \quad , \quad \dim Hom_{SU_3}(1, sl_3(\mathbb{C})^{\otimes 3}) = 2$$

In matrix terms, PU_3^+ preserves the trilinear form $Tr(ABC)$, but not $Tr(BAC)$.

Proof. For S_9^+ the fixed spaces of the tensor powers of $p = 1 + r_1$ have dimensions given by the noncrossing partitions, $|NC(k)| = 1, 1, 2, 5$ for $k = 0, 1, 2, 3$, and by inclusion-exclusion we obtain $5 - 3 \cdot 2 + 3 \cdot 1 - 1 = 1$. For SU_3 we have $\dim End_{SU_3}((\mathbb{C}^3)^{\otimes k}) = k!$ for $k \leq 3$, and we obtain $6 - 3 \cdot 2 + 3 \cdot 1 - 1 = 2$. The last assertion follows from the fact that PU_3^+ preserves the multiplication of $M_3(\mathbb{C})$ and the trace, but not the opposite multiplication. □

In familiar notation, the decompositions $8 \otimes 8 = 1 + 8 + 55$ for S_9^+, and $8 \otimes 8 = 1 + 8_s + 8_a + 10 + \overline{10} + 27$ for SU_3, both of dimension 64, are distinguished by their invariant tensors. In physics terms, the 2 invariant cubics $Tr(ABC)$ and $Tr(BAC)$ correspond to the symmetric and antisymmetric color tensors d_{abc} and f_{abc}, and the free theory only keeps the complex combination $Tr(T_aT_bT_c) = \frac{1}{4}(d_{abc} + if_{abc})$. The structure constants f_{abc} alone come back only with the opposite multiplication, which is related to the real structure, as discussed in section 4 below.

The quantum symmetries of the whole finite geometry F, taking into account H_F, D_F, γ_F and J_F, were computed by Bhowmick, D'Andrea and Dabrowski in [12], by using Goswami's quantum isometry groups [14, 50], the result being as follows:

Theorem 2.4 (Bhowmick, D'Andrea, Dabrowski)**.** *Under the assumptions on the mass matrices from* [12, section 3.1]*, the quantum group of orientation and real structure preserving isometries of F is*

$$QISO_J^+(D_F) = C(U_1) * A_{aut}(M_3(\mathbb{C}), tr)$$

with U_1 acting on $M_2(\mathbb{C})$ via $e_{12} \to e_{12} \otimes z$, $e_{21} \to e_{21} \otimes z^*$, and trivially on the rest of $\mathbb{C} \oplus \mathbb{C} \oplus M_2(\mathbb{C})$. When requiring the coaction to extend to the complexification of the real algebra A_F, compatibly with its real form, the result becomes $QISO_{\mathbb{R}}^+(D_F) = C(U_1) * C(PU_3)$.

In the regime from [12, section 3.1], which requires spectral separation and nonzero simple charged spectra, the effective answer does not depend on the remaining parameters, such as the mixing matrices, while at $D_F = 0$ the commutation with D_F imposes no constraint at all. Observe that the weak part SO_3 is reduced by D_F to a circle, and that the U_1 factor lives on the weak summand. We should mention here that the Higgs field appears in the Chamseddine–Connes approach as an inner fluctuation of the Dirac operator of $M \times F$, while the above concerns the stabilizer of a fixed finite operator D_F, so that a relation with the electroweak symmetry breaking would require a field, a vacuum, and the corresponding gauge action. Observe also that the quantum group in Theorem 2.4 is the effective quantum group, acting on the algebra, whose algebra of functions is the Woronowicz subalgebra generated by the coefficients of the adjoint coaction, inside the algebra of a larger, universal quantum group implementing the coaction on H_F, and that these 2 objects must not be identified. Also, the global quantum gauge group of [13] is different, its mixed weak and color coefficients not being reconstructible from the 2 factors. By combining with Theorem 1.4, we obtain:

Theorem 2.5. *In terms of quantum permutations, the result in* [12] *reads*

$$QISO_J^+(D_F) \simeq C(U_1) * C(S_9^+)^\sigma$$

in the complex setting, with σ being the Heisenberg cocycle on $\mathbb{Z}_3^2 \subset S_9$.

Proof. This follows indeed from Theorem 2.4 and Theorem 1.4, applied at $H = \mathbb{Z}_3$. □

Observe that the circle $U_1 \subset SO_3$ does not contain the Klein subgroup $\mathbb{Z}_2^2 \subset SO_3$ which supports the weak twist from section 1, so the factor $C(U_1)$ has no direct description in terms of S_4^+, and remains untouched by the twist.

3. Physical trace and weak sector

The results in section 2 use the canonical trace of B_F. However, in the spectral setting the algebra comes with its representation on H_F, and the natural trace is then the one coming from this representation. In order to discuss this question, let us first record a general fact. Given $B = \oplus_i M_{n_i}(\mathbb{C})$ with a faithful trace tr, we denote by t_i the trace of a minimal projection of $M_{n_i}(\mathbb{C})$, so that $\sum_i n_i t_i = 1$, and we use the scalar product $< a, b >= tr(ab^*)$.

Proposition 3.1. *The operator mm^* acts on each summand $M_{n_i}(\mathbb{C})$ as the scalar n_i/t_i. Thus, any trace preserving coaction on B leaves invariant each of the subalgebras $B_\lambda = \oplus_{n_i/t_i=\lambda} M_{n_i}(\mathbb{C})$, with $\alpha(1_\lambda) = 1_\lambda \otimes 1$.*

Proof. The elements $f_{ab} = e_{ab}/\sqrt{t_i}$, with $e_{ab} \in M_{n_i}(\mathbb{C})$, form an orthonormal basis of B, and we have $m^*(f_{ac}) = \frac{1}{\sqrt{t_i}} \sum_b f_{ab} \otimes f_{bc}$, so $mm^*(f_{ac}) = \frac{n_i}{t_i} f_{ac}$. A trace preserving coaction is a unitary corepresentation on $(B, <, >)$, as explained in [5], so m^* is colinear, and so is mm^*, and its eigenspaces B_λ are therefore invariant. Finally, $\alpha(1_\lambda) \in B_\lambda \otimes A$ for any λ, and these elements sum up to $1 \otimes 1$, so $\alpha(1_\lambda) = 1_\lambda \otimes 1$. □

For the canonical trace we have $t_i = n_i/N$, and so $mm^* = N$, which is the δ-form condition from [5]. Regarding now the trace coming from a Hilbert space, the point is that this is invariant under the quantum groups that we are interested in, which are all of Kac type, as follows:

Proposition 3.2. *Assume that B acts on a finite dimensional Hilbert space H, via a representation π, and that a coaction of a Kac type quantum group Q on B is implemented by a unitary corepresentation U on H, in the sense that $(\pi \otimes id)\alpha(b) = U(\pi(b) \otimes 1)U^*$. Then α preserves the trace $Tr_H \circ \pi$.*

Proof. With $U = \sum_{ki} e_{ki} \otimes u_{ki}$ and $X = \pi(b)$, we have the following formula:

$$(Tr_H \otimes id)(U(X \otimes 1)U^*) = \sum_{ij} X_{ij} \sum_k u_{ki} u_{kj}^*$$

In the Kac case the conjugate corepresentation $\bar{U} = (u_{ki}^*)$ is unitary too, so $\sum_k u_{ki} u_{kj}^* = \delta_{ij}$, and we obtain $Tr_H(X)1$, as claimed. □

Let us compute now the trace Tr_{H_F} on B_F. With notations from section 2, the first copy of $\mathbb{C}$, where λ acts, corresponds to ν_R, u_R and to the 4 antileptons, the second copy, where $\bar{\lambda}$ acts, corresponds to e_R, d_R, while $M_2(\mathbb{C})$ acts with multiplicity 4 on the left-handed doublets, and $M_3(\mathbb{C})$ acts with multiplicity 4 on the antiquarks. Thus, for one generation, we have the following table:

summand	fermions	$Tr_{H_f \oplus \overline{H_f}}$(min. proj.)	n_i/t_i
$\mathbb{C}$ (λ)	$\nu_R, u_R, \bar{\nu}_R, \bar{e}_R, \bar{\nu}_L, \bar{e}_L$	8	4
$\mathbb{C}$ $(\bar{\lambda})$	e_R, d_R	4	8
$M_2(\mathbb{C})$	ν_L, e_L, u_L, d_L	4	16
$M_3(\mathbb{C})$	$\bar{u}_R, \bar{d}_R, \bar{u}_L, \bar{d}_L$	4	24

Here the quark entries count the 3 colors, the total is $8 + 4 + 2 \cdot 4 + 3 \cdot 4 = 32$, and the last column is computed with the normalized trace. With 3 generations all the traces get multiplied by 3, and the normalized trace stays the same. We have checked this table with the program `maradona.py`, which builds the representation of B_F on $\mathbb{C}^{32}$, and computes the operator mm^* for both traces.

The 4 values of n_i/t_i being distinct, Proposition 3.1 shows that the summands of B_F cannot be mixed by quantum symmetries preserving Tr_{H_F}. This is in contrast with the canonical trace, where even the electroweak part has genuine quantum symmetries:

Proposition 3.3. *The quantum automorphism group of the electroweak algebra $\mathbb{C} \oplus \mathbb{C} \oplus M_2(\mathbb{C})$, with its canonical trace, is the twist of S_6^+, and is infinite and noncommutative.*

Proof. The first assertion follows from Theorem 2.1, with $N = 6$. Under the twist, the coaction on the algebra corresponds to the magic representation $1 + r_1$ of S_6^+, with r_1 irreducible, so its commutant has dimension 2. On the other hand, the classical automorphism group $\mathbb{Z}_2 \times SO_3$

leaves invariant the subspaces spanned by $1_1 + 1_2$, by $1_1 - 1_2$, by 1_{M_2}, and $sl_2(\mathbb{C})$, so the commutant of its action has dimension at least 4. Thus our quantum group is not classical, and since its classical version is $\mathbb{Z}_2 \times SO_3$, it is infinite and noncommutative. □

Summarizing, the canonical trace leads to large free quantum groups, mixing all the summands, while the physical trace forces the coaction to preserve each summand. In this latter case, we have the following result, with $S_4^+ \hat{*} S_9^+ \subset S_{15}^+$ standing for the quantum subgroup preserving each of the 4 blocks of the partition $15 = 1+1+4+9$, whose algebra is $C(S_4^+) * C(S_9^+)$:

Theorem 3.4. *With the trace Tr_{H_F} coming from the fermions, we have*

$$A_{aut}(B_F, Tr_{H_F}) \simeq C(S_4^+ \hat{*} S_9^+)^\sigma \simeq C(SO_3) * C(PU_3^+)$$

with σ being the cocycle from Theorem 2.2.

Proof. By Proposition 3.1 and the above table, a coaction preserving Tr_{H_F} leaves invariant the 4 summands, and preserves the normalized trace on each of them. Conversely, such a coaction preserves the canonical trace too, and Tr_{H_F}. Thus our algebra is $A_{aut}(B_F, tr, T)$, where tr is the canonical trace, and T consists of the maps $\mathbb{C} \to B_F$ given by $1 \to 1_i$, which are of degree 0, so that $T^\sigma = T$. By Theorem 1.1 this is the twist of the quantum permutation algebra of the 15 points preserving the indicator functions of the 4 blocks, which is $C(S_4^+) * C(S_9^+)$. Finally, the universal property of the free product shows that our algebra is also the free product of the algebras A_{aut} of the summands, which are $\mathbb{C}, \mathbb{C}, C(SO_3), C(PU_3^+)$. □

This result is in agreement with [12], where the quantum symmetries of the whole finite geometry turn out to be $C(U_1) * A_{aut}(M_3(\mathbb{C}))$, with a free product between the circle acting on the weak summand and the color part, the weak part $C(SO_3) = C(S_4^+)^\sigma$ being reduced by the Dirac operator to $C(U_1)$. Regarding S_4^+ itself, this is the Pauli quantum group from [10], see also section 5. In permutation terms, the picture is that the free product $S_4^+ \hat{*} S_9^+$ gets reduced by D_F, with only a circle surviving from the weak part, and with the color quantum group surviving intact.

Let us describe as well the weak part in more detail. At $N = 2$ the isomorphism from Proposition 1.2 maps the 4 points of $\mathbb{Z}_2^2$, or rather the corresponding basis of $\mathbb{C}[\mathbb{Z}_2^2]$, to the matrices $1, X, Z, ZX$, where X, Z are the Pauli matrices, and so the magic representation $1 + r_1$ of S_4^+ corresponds to the decomposition $M_2(\mathbb{C}) = \mathbb{C}1 \oplus sl_2(\mathbb{C})$, with r_1 being the adjoint representation of SO_3. In physical terms, the 3 nontrivial labels of $\mathbb{Z}_2^2$, that is, the 3 nontrivial characters of the 4 points, correspond to the 3 weak directions, and the trivial label corresponds to the unit. The twisting is very mild here, because S_4^+ and SO_3 have the same fusion rules, and the same dimensions of the irreducible representations, namely $1, 3, 5, 7, \ldots$ However, the algebras $C(S_4^+)$ and $C(SO_3)$ are of course very different, the first one being noncommutative, as shown for instance by the Pauli model from section 5, and the passage from one to the other is precisely the twisting. We will come back to the weak sector, and to its untwisting, in section 7 below.

4. Real structure and area tensors

In the setting of [12], the real structure J_F is imposed in both cases from Theorem 2.4, and the difference between them comes from the requirement, in the real case, that the coaction extends to the complexification of A_F, compatibly with its real form. This complexification has dimension 24, and contains 2 copies of $M_3(\mathbb{C})$, the second one being the conjugate algebra $\overline{M_3(\mathbb{C})}$, which is isomorphic to the opposite algebra $M_3(\mathbb{C})^{op}$, via $\bar{a} \to a^*$. This suggests that the classical color group should be characterized by the fact that it preserves the opposite product too, and this is indeed the case, as follows:

Proposition 4.1. *The quotient of $A_{aut}(M_N(\mathbb{C}), tr)$ making the opposite product $m^{op}(a\otimes b) = ba$ colinear is $C(PU_N)$.*

Proof. Consider the element $F = \sum_{ab} e_{ab} \otimes e_{ba} \in M_N(\mathbb{C}) \otimes M_N(\mathbb{C})$, which is proportional to $m^*(1)$, and is therefore invariant. In the algebra $M_N(\mathbb{C}) \otimes M_N(\mathbb{C})$ the element F is the flip of $\mathbb{C}^N \otimes \mathbb{C}^N$, so $F^2 = 1$ and $F(x\otimes y)F = y\otimes x$. On the other hand, the left and right multiplications by F can be written as follows, with F being inserted in the middle:

$$L_F = (m^{op} \otimes m)(id \otimes F \otimes id) \quad , \quad R_F = (m \otimes m^{op})(id \otimes F \otimes id)$$

Thus, if both m and m^{op} are colinear, so is the flip $\Sigma = R_F L_F$ of $M_N(\mathbb{C}) \otimes M_N(\mathbb{C})$, and this forces the coefficients of the coaction to commute. Thus the quantum group is classical, and so is a closed subgroup of the group of automorphisms of $M_N(\mathbb{C})$, which is PU_N. Conversely, PU_N preserves m^{op}, and this gives the result. □

As an illustration, the program `caniggia.py` checks numerically at $N = 2, 3$ that the algebra generated by the maps built from m, m^{op} and their adjoints contains Σ. Let us transport now m^{op} to the permutation side. With notations from section 1, we define a product on $\mathbb{C}[\Gamma] = C(\widehat{\Gamma})$ by:

$$b(e_g \otimes e_h) = d(g, h)e_{g+h}$$

In the point basis of $\widehat{\Gamma} \simeq \Gamma$, identified via d, the map b is given by the following formula, valid for any finite abelian group H:

$$b(\delta_a \otimes \delta_c) = \frac{1}{|\Gamma|} \sum_r \overline{d(a - r, c - r)}\, \delta_r$$

Indeed, with $\chi_g(r) = d(g, r)$ we have $\delta_a = |\Gamma|^{-1} \sum_g \overline{d(g, a)}\chi_g$, and inserting this in the formula $b(\chi_g \otimes \chi_h) = d(g, h)\chi_{g+h}$, the summation over g forces $h = a - r$, which gives the result. At $H = \mathbb{Z}_3$, so that $\Gamma = \mathbb{Z}_3^2$ and $d(g, h) = w^{\det(g,h)}$ with $w = e^{2\pi i/3}$, we obtain a formula involving the oriented area of the triangle (r, a, c):

$$b(\delta_a \otimes \delta_c) = \frac{1}{9} \sum_r w^{-\det(a-r,c-r)} \delta_r$$

Since $\sigma(g, h)d(g, h) = \sigma(h, g)$, we have $b^\sigma = m_\sigma^{op}$, where m_σ is the product of $\mathbb{C}_\sigma[\Gamma] = M_N(\mathbb{C})$. In other words, the twist transports the pointwise product into the product of $M_N(\mathbb{C})$, and the area tensor b into the opposite product. This leads to:

Definition 4.2. Given a finite abelian group H with $|H| = N$, we let $G_H \subset S_{N^2}^+$ be the quantum subgroup making the area tensor b colinear, $b \in Hom(p^{\otimes 2}, p)$, where p is the magic representation. At $H = \mathbb{Z}_3$ we use the notation $G_3 \subset S_9^+$.

In other words, G_H is the quantum group of quantum permutations of the N^2 points which preserve the oriented areas. With this notion in hand, we have the following result:

Theorem 4.3. *For any finite abelian group H with $|H| = N \geq 2$, the quantum group $G_H \subset S_{N^2}^+$ is a twist of PU_N, and is infinite and noncommutative.*

Proof. The map b preserves the degree, so Theorem 1.1 applies, and combined with Proposition 4.1 this gives $C(G_H)^\sigma \simeq C(PU_N)$. The algebra $C(G_H)$ is infinite dimensional, because it is isomorphic to $C(PU_N)$ as a coalgebra, while its classical version is finite, being a subgroup of S_{N^2}. Thus G_H is infinite and noncommutative. □

Let us mention that embeddings of twists of classical groups into quantum permutation groups, supported on a finite abelian subgroup, go back to [9, Theorem 8.3], see also [27], and that the new point here is the description of the twist of PU_N by a single oriented area tensor.

Regarding now the classical version, this is something quite explicit, because a permutation of the N^2 points preserving the oriented areas must be affine. To be more precise, we have the following result:

Proposition 4.4. *The classical version of G_H is the affine symplectic group $\Gamma \rtimes Sp(\Gamma, d)$. At $N = 2$ we have $G_H = S_4^+$, and at $N = 3$ the classical version of G_3 is $\mathbb{F}_3^2 \rtimes SL_2(3)$, of order 216, so that $G_3 \neq S_9^+$.*

Proof. Identify the N^2 points with Γ, via d. A permutation f preserving b preserves the quantities $d(a-r, c-r)$, and up to a translation we can assume $f(0) = 0$. Then $d(f(a), f(c)) = d(a, c)$ for any a, c, and so $d(f(x+y) - f(x) - f(y), f(z)) = 1$ for any x, y, z. Since f is bijective and d is nondegenerate, f is additive, and so is a symplectic automorphism. Conversely, the affine symplectic maps preserve b. At $N = 2$ the group $PU_2 = SO_3$ is the twist of S_4^+, so $G_H = S_4^+$. At $N = 3$ we have $Sp(\Gamma, d) = SL_2(3)$, and we have also checked by computer, with the program `seedorf.py`, that exactly 216 of the 9! permutations preserve b. □

At $N = 3$ it is natural to allow orientation reversal too. The program `seedorf.py` finds that exactly 216 permutations of the 9 points send b to $\bar{b}$, so that exactly 432 permutations preserve the unoriented area $Re(b)$, forming the group $\mathbb{F}_3^2 \rtimes GL_2(3)$. We will see below that there are no further quantum symmetries of $Re(b)$.

Interestingly, the Heisenberg group of order 27, which will appear in section 6, is isomorphic to the group $\Delta(27)$, which is well-known as a flavor symmetry group, going back to [19], and whose automorphism group has order 432 as well, see [56], the generalized CP transformations of $\Delta(27)$ models being built from such automorphisms, see [26, 56]. However, the natural action of $Aut(\Delta(27))$ on the 9 one-dimensional characters of $\Delta(27)$ fixes the trivial character, so it is not the transitive affine action on the 9 points, and the equality of the orders does not identify these 2 permutation representations. Also, an orientation reversing permutation of the 9 points is not by itself a CP symmetry, which would require compatible transformations of all the multiplets, and of the spacetime and spinor variables, see [26].

Back to the Standard Model, we can now formulate the following result, which is the main result of this paper:

Theorem 4.5. *In the real setting, and under the assumptions from* [12, section 3.1]*, the color part of the quantum isometry group of the finite geometry of the Standard Model, which is PU_3 by* [12]*, is the twist of the quantum permutation group $G_3 \subset S_9^+$, preserving the oriented area tensor on $\mathbb{F}_3^2$. Thus, in the real setting, we have*

$$QISO_{\mathbb{R}}^+(D_F) \simeq C(U_1) * C(G_3)^\sigma$$

with G_3 being infinite and noncommutative, and with classical version of order 216.

Proof. This follows from Theorem 2.4, Theorem 4.3 and Proposition 4.4, applied at $H = \mathbb{Z}_3$. □

Thus, in the permutation picture, the color symmetry of the real Standard Model, which is classical in [12], corresponds to a genuinely quantum permutation group. The same phenomenon appears at $N = 2$, where the classical SO_3 untwists into S_4^+. For a related study of the twists of SU_3, see [16].

Geometrically, the 9 points of $\mathbb{F}_3^2$ on which G_3 acts can be thought of as the 9 inflection points of a plane cubic, the 12 affine lines being the lines joining them, and the classical version of G_3

being the Hessian group of order 216, see [2]. The oriented area tensor b is then an invariant of this configuration, and G_3 can be regarded as the quantum symmetry group of the Hesse configuration, equipped with its oriented areas. We will come back to this point of view in section 9, in connection with the mixing matrices.

Let us discuss now the unoriented version of the area tensor. We consider the entrywise real part $Re(b)$ of the tensor b, written in the point basis as above, which is invariant under the orientation reversing affine maps too. Observe that taking adjoints in an intertwining relation reverses the order of the products of magic entries, while the entrywise conjugation in the point basis does not follow from such an operation, so preserving b does not imply preserving $Re(b)$. We will need the following general fact:

Proposition 4.6. *For a finite quasigroup $(X,*)$, the quantum subgroup of S_X^+ making colinear the map $L(v_a\otimes v_c)=v_{a*c}$ is classical, namely the group of automorphisms of $(X,*)$.*

Proof. For a magic unitary p, the colinearity of L reads $\sum_{x*y=k}p_{xi}p_{yj}=p_{k,i*j}$. Fix a,b,i,j, and set $k=a*b$, $P=p_{ai}$, $Q=p_{bj}$, $R=p_{k,i*j}$. By multiplying this relation on the left by P, and separately on the right by Q, and using the orthogonality on the columns of p, and the unique division in the quasigroup, we obtain $PQ=PR=RQ$. Thus $(PQ)^2=PRRQ=PRQ=PQ$, so PQ is a contractive idempotent, and so an orthogonal projection, and so $PQ=(PQ)^*=QP$. Thus the coordinates commute, and the classical points are the permutations preserving the operation $*$. □

For a related rigidity result, in the associative case, see [59]. Back now to $\mathbb{F}_3^2$, the point is that the passage from b to $Re(b)$ kills all the quantum symmetries, as follows:

Theorem 4.7. *The quantum subgroup of S_9^+ preserving the unoriented area tensor $Re(b)$ on $\mathbb{F}_3^2$ is the classical group $\mathbb{F}_3^2\rtimes GL_2(3)$, of order 432. In particular, this quantum group does not contain G_3, and its intersection with G_3 is the classical version of G_3.*

Proof. Set $v_a=\delta_a$ and $1=\sum_r v_r$, and let $C(v_a\otimes v_c)$ be the sum of the vectors v_r with $\det(a-r,c-r)=0$. Since the real parts of the nontrivial cube roots of unity are $-1/2$, we have $Re(b)(v_a\otimes v_c)=-\frac{1}{18}1+\frac{1}{6}C(v_a\otimes v_c)$. For $a\neq c$ the points r on the line through a,c are $a,c,-a-c$, and we obtain, for any a,c:

$$C(v_a\otimes v_c)=v_a+v_c+v_{-a-c}+\delta_{ac}(1-3v_a)$$

All the terms here, except for $L(v_a\otimes v_c)=v_{-a-c}$, are built from the unit, the summation functional, the pointwise multiplication and the identity, so they are intertwiners for any magic unitary, and preserving $Re(b)$ is equivalent to preserving L. Now $a*c=-a-c$ is a quasigroup operation, so by Proposition 4.6 our quantum group is classical. Now a permutation f preserving L and fixing 0 satisfies $f(-a)=-f(a)$, and then $f(a+b)=f(a)+f(b)$, so the permutations preserving L are the affine maps. Finally, the intersection with G_3 is classical, and so contained in the classical version of G_3, which is contained in $\mathbb{F}_3^2\rtimes GL_2(3)$. The program `gullit.py` checks the above formulae, finds that 432 permutations preserve L, and checks that the Weyl models from section 5 below intertwine b, but not $Re(b)$. □

In fact, the orientation of the areas, which is what makes room for quantum symmetries here, is invisible on the incidence structure of the 12 affine lines, and this structure has only classical symmetries. To be more precise, with the quantum automorphism group of a graph being defined by a magic unitary commuting with its adjacency matrix, we have:

Theorem 4.8. *The quantum automorphism group of the incidence graph of the Hesse configuration, having as vertices the 9 points of $\mathbb{F}_3^2$ and the 12 affine lines, is the classical group $\mathbb{F}_3^2\rtimes GL_2(3)$, of order 432.*

Proof. The degree vector, which is the image of the all-one vector by the adjacency matrix, is fixed, and since its entries are 4 on the points and 3 on the lines, the magic unitary is block diagonal, with blocks u on the points and v on the lines. With $I : \mathbb{C}^{12} \to \mathbb{C}^9$ being the incidence matrix, we have $uI = Iv$, and I^* intertwines too. Now let $T = Im_l(I^* \otimes I^*)$, with m_l being the pointwise product of the functions on the lines. Since there is a unique line through 2 distinct points, and the 4 lines through a point a cover the other points once, and a itself four times, we have:

$$T(v_a \otimes v_c) = v_a + v_c + v_{-a-c} + \delta_{ac}1$$

Thus, after subtracting universal intertwiners, we obtain the map L from the proof of Theorem 4.7, and by Proposition 4.6 the coefficients of u commute. Also, for $a \neq c$, the vector $m_l(I^*v_a \otimes I^*v_c)$ is the indicator of the line through a, c, and these vectors span the functions on the lines, so the coefficients of v lie in the algebra generated by those of u, and commute too. Finally, the classical automorphisms are the permutations of the points preserving the lines, which are the affine maps. The program `haaland.py` checks the above formula on all 81 pairs of points, and finds 432 permutations preserving the lines. □

5. Matrix models and maximality

The quantum groups G_H have explicit matrix models, coming from the Weyl operators. Let us denote by $W_g = M_\lambda T_x$, with $g = (\lambda, x) \in \Gamma$, the Weyl operators from Proposition 1.2, which form an orthonormal basis of $M_N(\mathbb{C})$, with respect to the scalar product $< a, b >= tr(ab^*)$. Given a unitary $x \in U_N$, we can consider the following matrix of rank one projections, acting on the Hilbert space $M_N(\mathbb{C})$:

$$P^x_{rs} = Proj(W_r x W_s^*)$$

Here the indices r, s range over Γ, identified with the N^2 points via d. We have the following result, in the spirit of the flat models from [11]:

Proposition 5.1. *The matrix P^x is magic, and the area tensor b intertwines it, so we have a representation $\pi_x : C(G_H) \to B(M_N(\mathbb{C}))$, given by $p_{rs} \to P^x_{rs}$.*

Proof. We have the following formulae, with the scalar product $< a, b >= tr(ab^*)$:

$$< W_r x W_s^*, W_r x W_t^* >= tr(W_s^* W_t) = \delta_{st} \quad , \quad < W_r x W_s^*, W_q x W_s^* >= tr(W_r W_q^*) = \delta_{rq}$$

Thus the rows and columns of P^x consist of orthonormal bases, and P^x is magic. Regarding b, consider the operators $V_{ab}(Y) = W_a Y W_b^*$, which form a basis of $B(M_N(\mathbb{C}))$, and the coefficients $t_{ab}(x) = tr(W_a^* x W_b x^*)$ of the conjugation action of PU_N on $M_N(\mathbb{C})$, in the Weyl basis. By expanding the rank one projections, and using $W_r W_a W_r^* = d(a, r)W_a$, we obtain, with $m = |\Gamma|$:

$$P^x_{rs} = \frac{1}{m} \sum_{a,b} d(a, r)\overline{d(b, s)}\, t_{ab}(x) V_{ab}$$

Since $V_{ab}V_{cd} = \sigma(a, c)\overline{\sigma(b, d)}V_{a+c,b+d}$, this shows that $p_{rs} \to P^x_{rs}$ is the evaluation at x of the twisted version of the algebra of functions on PU_N, which is $C(G_H)$, via the map $f \to \sum_{ab} f_{ab}(x)V_{ab}$, where f_{ab} are the components of f under the left and right translations by the Weyl group, as in [27, Theorem 4.11]. Thus π_x is well-defined, and b intertwines P^x. We have verified the above formula with the program `koeman.py`, for $H = \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_2^2$, and the intertwining property at $N = 3$ with the program `kempes.py`. □

Let us discuss now the commutativity of these models. The Weyl group $K \simeq \Gamma$ sits inside PU_N, via $g \to [W_g]$, and its normalizer C_H, which is the projective Clifford group, fits into an exact sequence

$$1 \to K \to C_H \to Sp(\Gamma, d) \to 1$$

with the quotient map being given by the conjugation action on the labels of the Weyl operators, see [49]. In particular C_H is finite, of order $|\Gamma| \cdot |Sp(\Gamma, d)|$, and we have:

Proposition 5.2. *The image of π_x is commutative if and only if $[x] \in C_H$. In particular, π_x has noncommutative image for any x outside a finite set of points of PU_N.*

Proof. The column $s = 0$ of P^x consists of the projections onto the orthonormal basis $\{W_r x\}_r$. If all the entries of P^x commute, each rank one projection P^x_{rs} commutes with these, so it is diagonal in this basis, and so xW_s^* must be proportional to some $W_r x$, for any s. Thus $[x]$ normalizes K. Conversely, if $[x]$ normalizes K, then each $W_r x W_s^*$ is proportional to some $W_t x$, so all the projections P^x_{rs} are diagonal in the basis $\{W_t x\}_t$, and so commute. The program `benzema.py` checks at $N = 3$ that C_3 has order 216, that π_x is commutative for $x \in C_3$, and that it is noncommutative for random x, and for small perturbations of elements of C_3. □

At $N = 2$ the Weyl operators are, up to phases, the Pauli matrices, and the models π_x with $x \in SU_2$ are the Pauli representation of $C(S_4^+)$, which was shown to be faithful in [10]. Thus Proposition 5.1 can be regarded as a generalization of the Pauli representation. In general, the integrated model $x \to \pi_x$, with $x \in PU_N$, is faithful on the dense Hopf $*$-algebra, as shown in [27, Theorem 4.11], where the Hopf image of the full Weyl model is identified with the twist of PU_N. More generally, we have the following result, where the Hopf image of a family of representations is the smallest Hopf algebra quotient through which they all factor:

Theorem 5.3. *Given a nonempty subset $X \subset PU_N$, the joint Hopf image of the models π_x with $x \in X$ is the twist of the algebra of functions on the closed subgroup $\overline{\langle K, X\rangle} \subset PU_N$.*

Proof. We work at the level of the dense Hopf $*$-algebras, and we identify the polynomial functions on the twist of PU_N with those on PU_N, as vector spaces. Decompose such a function under the left and right translations by K, as $f = \sum_{a,b} f_{ab}$. By the proof of Proposition 5.1, see also [27, Theorem 4.11], π_x maps f_{ab} to $f_{ab}(x)V_{ab}$, where $V_{ab}(Y) = W_a Y W_b^*$, and since these operators form a basis of $B(M_N(\mathbb{C}))$, finite Fourier inversion shows that $\pi_x(f) = 0$ precisely when f vanishes on KxK. Similarly, the iterated coproduct being given by evaluation at products, the tensor $(\pi_{x_1} \otimes \ldots \otimes \pi_{x_m})\Delta^{(m-1)}(f)$ vanishes precisely when f vanishes on $Kx_1K \ldots Kx_mK$. The union of these sets, over all words in X, is a semigroup, whose closure is a closed subsemigroup of a compact group, and so a closed subgroup, which contains X, and $K = (Kx)x^{-1}$ for $x \in X$, so it is $H = \overline{\langle K, X\rangle}$. Thus the largest Hopf ideal contained in the joint kernel of our models is the ideal of functions vanishing on H. Conversely, since $K \subset H$, the restriction to H is a Hopf algebra quotient of the twisted algebra, and every model factors through it. We have checked at $N = 3$, with the program `benzema.py`, that the kernels of $f \to \pi_x(f)$ and of $f \to f_{|KxK}$ coincide on the coefficients of the magic representation. □

In particular, with $X = C_H$, we obtain that the twist of $C(C_H)$ is a quotient of $C(G_H)$, which is commutative by Proposition 5.2, and has dimension $|C_H| = |\Gamma| \cdot |Sp(\Gamma, d)|$, so it is the algebra of functions on the classical version of G_H, computed in Proposition 4.4. Thus the classical version of G_H corresponds, under the twist, to the projective Clifford group C_H, which answers a question raised in [27, section 4].

Also, for Haar almost every $x \in PU_N$, the single model π_x has Hopf image G_H. Indeed, consider the closed subgroup $L = \overline{\langle K, x\rangle}$. The complexified Lie algebra of L is invariant under $Ad(K)$, so it is a sum of Weyl lines $\mathbb{C}W_g$, and there are finitely many such sums. A proper nonzero such sum has a proper closed stabilizer in PU_N, because the adjoint representation is irreducible, and so x must belong to a Haar null set. Also, if L is finite, then x is a torsion element, and these elements form a null set. Thus, outside a null set we have $L = PU_N$, and Theorem 5.3 gives the result.

The restriction to abelian groups H in all this matters. Indeed, consider a nice error basis $\{U_g\}$ indexed by a finite nonabelian group F, and restrict the above model to the parameters $x = U_k$. Each vector $U_i U_k U_j^*$ is proportional to another basis element, with label ikj^{-1}, so the projections are diagonal in the basis $\{U_g\}$, and so commute. Thus the model factors through evaluations at the permutations $i \to g^{-1}ik$, and the Hopf image is the classical group of left and right translations $(F \times F)/\{(z,z)|z \in Z(F)\}$, of order $|F|^2/|Z(F)|$, while a twist of $C(F)$ would have dimension $|F|$. For instance the 16 operators $Z^a X^b J^s \otimes Z^s X^t$ with $a, b, s, t \in \mathbb{F}_2$, where Z, X are the Pauli matrices and J is the Hadamard matrix of order 2, form such a basis of $M_4(\mathbb{C})$, with $|F| = 16$ and $|Z(F)| = 4$, so the Hopf image has order 64. We have checked all this with the program `thuram.py`, which verifies that these operators form a nice error basis, computes the group law of the labels and its center, and computes the group generated by the above permutations.

Let us discuss now maximality. Under the twist, by [27, Lemma 4.13], the closed quantum subgroups of G_H containing K correspond to the closed subgroups of PU_N containing K, and the classical version $\Gamma \rtimes Sp(\Gamma, d)$ corresponds to C_H, as explained above. We have the following result, in the spirit of the maximality questions for $S_N \subset S_N^+$, see [27]:

Theorem 5.4. *For a nontrivial finite abelian group H, the classical version $\Gamma \rtimes Sp(\Gamma, d)$ is maximal among the closed quantum subgroups of G_H precisely when H has prime exponent, that is, $H = \mathbb{Z}_p^r$ with p prime.*

Proof. By untwisting, the question is whether C_H is maximal in PU_N. For $H = \mathbb{Z}_p^r$, the conjugation by K has distinct 1-dimensional weight spaces $\mathbb{C}W_g$ with $g \neq 0$ on $sl_N(\mathbb{C})$, and the symplectic group acts transitively on the nonzero labels, so C_H acts irreducibly on $sl_N(\mathbb{C})$. Thus there is no proper closed overgroup of positive dimension, because the complexified Lie algebra of its connected component would be a nonzero invariant subspace, hence equal to $sl_N(\mathbb{C})$, and the overgroup would be PU_N itself. The finite overgroups are excluded by the results of Nebe, Rains and Sloane on the maximality of the complex Clifford groups associated to $\mathbb{Z}_p^r$, see [71, Theorems 6.5 and 7.3] and [72], via a finite lift: the preimage in SU_N of a finite overgroup of C_H is finite, and adjoining the finite group of scalar phases of their Clifford group, we obtain a finite overgroup of it. Conversely, if the exponent of H is composite, pick a prime p dividing it, and set $D = \Gamma[p]$, which is a proper nonzero characteristic subgroup. The algebra spanned by the operators W_g with $g \in D$ is a proper nonscalar subalgebra of $M_N(\mathbb{C})$, and its normalizer in PU_N contains C_H, because C_H acts on the labels by symplectic automorphisms, which preserve D. This normalizer is proper, because a subalgebra invariant under all the unitary conjugations must be trivial or the whole algebra, and has positive dimension, because it contains the conjugations by the unitaries of our nonscalar subalgebra. By twisting, we obtain an intermediate quantum group. □

For instance at $H = \mathbb{Z}_9$ the subgroup $D = \Gamma[3] \simeq \mathbb{Z}_3^2$ is isotropic for d, so the algebra spanned by the operators W_g with $g \in D$ is commutative, of dimension 9, and so is a maximal abelian subalgebra of $M_9(\mathbb{C})$, generated by the cubes X^3, Z^3 of the shift and clock matrices. Its normalizer in PU_9 is the corresponding monomial group, namely a torus extended by S_9, which gives an intermediate group. On the other hand, for $H = \mathbb{Z}_3^2$, which has the same number 81 of points, there are no intermediate quantum groups. Thus the maximality question depends on the group structure of H, and not only on its order.

6. Fermions and the Heisenberg group

The fermions of the Standard Model live in H_F, which is a module over A_F, and the quarks come in color triplets $\mathbb{C}^3$, which are irreducible modules over $M_3(\mathbb{C})$. A natural question is

whether the twisting from the previous sections can be performed at the level of the fermions too. The answer is negative, for single triplets:

Proposition 6.1. *The following hold:*

(1) *If $V = \mathbb{C}^3$ is a color triplet, with $End(V) = M_3(\mathbb{C})$ as algebra object, there is no tensor functor extending the inverse twist which sends $End(V)$ to $\mathbb{C}^9$, with its pointwise product.*
(2) *For $H = \mathbb{Z}_N$, if the projective Weyl translations of $\mathbb{C}^N$ lift to commuting unitaries of the form $M_\lambda \otimes A$, $T_x \otimes B$ on $\mathbb{C}^N \otimes E$, then $N|\dim E$.*

Proof. (1) A tensor functor preserves duals and contractions, so it maps the algebra $End(V) = V \otimes V^*$ to $End(F(V))$, whose center is 1-dimensional, while $\mathbb{C}^9$ has a 9-dimensional center.

(2) From $T_x M_\lambda = w M_\lambda T_x$ with w a primitive root of unity, the commutation of $M_\lambda \otimes A$ and $T_x \otimes B$ gives $AB = wBA$, and taking determinants gives $w^{\dim E} = 1$. □

Thus the quarks, taken one triplet at a time, do not survive the passage to 9 points. However, the obstruction in (2) disappears in 2 natural ways. First, the Heisenberg group of order 27, which is the central extension of $\mathbb{Z}_3^2$ generated by clock and shift, acts genuinely on $\mathbb{C}^3$. Second, the color space can be paired with a second 3-dimensional space, as in $\mathbb{C}^3 \otimes \overline{\mathbb{C}}^3$, on which the Weyl translations commute, with all 9 characters as weights. In the Standard Model, a natural candidate for this second space would be the space of the 3 generations, but this is excluded, at least under natural requirements, as follows:

Proposition 6.2. *Assume that D_F^2, restricted to the antiquarks of a given chirality, has an eigenspace which is a single color triplet, as is the case for nondegenerate quark masses. Then there are no commuting unitaries U, V on H_F, commuting with D_F, γ_F, and implementing on $M_3(\mathbb{C}) \subset A_F$ the adjoint actions of the clock and shift matrices.*

Proof. Such unitaries preserve the support of $M_3(\mathbb{C})$, which is the space of antiquarks, and commute with γ_F and D_F^2, so they preserve the above eigenspace $E \simeq \mathbb{C}^3$. Since $M_3(\mathbb{C})$ acts irreducibly on E, by Schur's lemma the restrictions of U, V to E are scalar multiples of the clock and shift matrices. But these do not commute, their commutator being a nontrivial cube root of unity. □

Thus the commuting lifts are excluded, even when using the 3 generations, and the genuinely nonabelian Heisenberg group is the right object.

On the other hand, there are real spectral triples with genuine S_N^+ symmetry, provided that the fermions are changed. With $L = l^2(X)$, $|X| = N \geq 4$, and P the projection onto the constant functions, one can take $H = L \otimes L \otimes \mathbb{C}^2$, with $C(X)$ acting on the first leg, $D = (P \otimes 1 + 1 \otimes P) \otimes \sigma_x$, $\gamma = 1 \otimes 1 \otimes \sigma_z$, and J the flip of the 2 copies of L, composed with σ_x and complex conjugation. This is a real, even spectral triple of KO-dimension 6, satisfying the first order condition, the commutators $[D, a]$ acting on the first leg, and the opposite algebra acting on the second leg. The quantum group S_N^+ acts on it via $U = u_{13} u_{23}$, which commutes with D because the magic unitary u fixes the constant functions, and which implements the coaction on the first leg. As for the real structure, the flip of the 2 legs, combined with the adjoint on the coefficients, leaves U invariant, because $(u_{ij} u_{kl})^* = u_{kl} u_{ij}$, which is a reality condition of coefficientwise type, in the sense of section 8. We have checked all this numerically at $N = 4$, with the Pauli model of S_4^+ from section 5, with the program `baresi.py`. Thus the obstruction to quantum symmetry comes from the fermion content of the Standard Model, and not from the axioms themselves.

For the Standard Model itself, the color structure of H_F makes the Heisenberg group act in a very natural way, and we have the following result, which shows that this group is the right object:

Theorem 6.3. *The Heisenberg group of order* 27*, which is isomorphic to the flavor group* $\Delta(27)$*, acts on* H_F *by unitaries commuting with* D_F, γ_F, J_F*, implementing on* $M_3(\mathbb{C}) \subset A_F$ *the adjoint action of the clock and shift matrices, and acting trivially on* $\mathbb{C} \oplus \mathbb{H}$*.*

Proof. In the Standard Model, the color index of each quark or antiquark is carried by a tensor factor $\mathbb{C}^3$, the leptons being colorless, and D_F and γ_F act trivially on this factor, because the Yukawa couplings and the Majorana mass term are color-blind. We let the Heisenberg group act via clock and shift on the color factor of the antiquarks, on which $M_3(\mathbb{C})$ acts, via the complex conjugate matrices on the color factor of the quarks, and trivially on the leptons. These unitaries commute with D_F, γ_F, and with the antiunitary J_F, which exchanges particles and antiparticles, and they implement the adjoint action of clock and shift on $M_3(\mathbb{C})$. □

This action exists for any choice of the Yukawa couplings and of the mixing matrices, and puts no constraint on them. Observe also that, in the flavor models, the group $\Delta(27)$ acts on the generations, and not on color.

Theorem 6.3 suggests performing the twisting at the level of F by a deformation of isospectral type, in the spirit of Connes and Landi [33], with the Heisenberg group instead of a torus. However, this cannot work in the usual monoidal framework: the color triplet is an object of the representation category of the Heisenberg group, its endomorphism algebra is an algebra object of this category, and any tensor functor maps it to a full matrix algebra, as in the proof of Proposition 6.1 (1), so the color algebra cannot become $\mathbb{C}^9$ in this way. Thus, the passage to 9 points at the level of the fermions requires changing the module data. In fact, keeping the usual quark module, such a change cannot produce many color points, as follows:

Proposition 6.4. *Keep* H_F*, the weak action,* J_F*,* γ_F *and* D_F*, with invertible quark mass matrices having simple singular spectra, and replace the action of* $M_3(\mathbb{C})$ *by a faithful representation* ρ *of* $\mathbb{C}^m$ *on the antiquarks, with the same support, and commuting with* γ_F*. If the order zero and first order conditions hold, then* $m \leq 3c$*, where* c *is the number of connected components of the bipartite support graph of the CKM matrix, having the up and down mass eigenstates as its 2 vertex sets. In particular, for connected CKM mixing,* $m \leq 3$*.*

Proof. For a minimal projection $f \in \mathbb{C}^m$, set $Q = J_F \rho(f) J_F^{-1}$, acting on the quarks. By order zero, Q commutes with the weak algebra, so on the left quarks we have $Q_L = 1_2 \otimes T$, with T acting on color and generations. With $M = diag(1_3 \otimes Y_u, 1_3 \otimes Y_d)$ being the mass arrow from the right quarks to the left quarks, the first order condition $[[D_F, e_w], Q] = 0$, with e_w being the unit of $\mathbb{H}$, gives $Q_L M = M Q_R$, and so $[Q_L, MM^*] = 0$. Thus T commutes with $1_3 \otimes Y_u Y_u^*$ and $1_3 \otimes Y_d Y_d^*$, and the simple spectra show that $T \in M_3(\mathbb{C}) \otimes \mathbb{C}^c$, because a matrix which is diagonal in both eigenbases has diagonal entries satisfying $\lambda_i = \mu_j$ when $V_{ij} \neq 0$, where V is the CKM matrix. Since M is invertible, each T is nonzero, and since these T are mutually orthogonal projections, there are at most $3c$ of them. The program `aguero.py` checks the commutant computation, for connected, block and monomial CKM matrices. □

Since the observed CKM matrix is connected, the usual quark module allows at most 3 color points, the bound being attained by the diagonal color algebra, and so 9 color points require a substantial change of the module data.

Let us record as well the decomposition of H_F under this action. The center $\mathbb{Z}_3$ of the Heisenberg group acts by mutually conjugate primitive cube roots of unity on the quarks and on the antiquarks, and trivially on the leptons, and this is the action of the center of SU_3, also known as triality. Since the 3-dimensional irreducible representations of the Heisenberg group are determined by their central character, we obtain, for the 3 generations:

$$H_F = 24 \cdot 1 \oplus 12 \cdot \mathbf{3} \oplus 12 \cdot \bar{\mathbf{3}}$$

Here the trivial part corresponds to the leptons, and $\mathbf{3}, \bar{\mathbf{3}}$ are the 3-dimensional representations given by clock and shift, and by their conjugates. Observe that the color singlets, such as those appearing in qqq and $q\bar{q}$, have trivial central character, the converse being false, since for instance $\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{1} \oplus \mathbf{8}$ for SU_3, with both components having trivial central character. We have checked this decomposition, for one generation, with the program `maradona.py`.

7. Untwisting and color Lie algebras

We discuss here the untwisting of the color sector, from $M_3(\mathbb{C})$ to the 9 points of $\mathbb{F}_3^2$. According to Theorem 1.1, the twist preserves the coalgebra structure, and in particular the dimensions of the representations, the fusion rules and the Haar integrals. Thus, all the quantities which can be computed from the representation theory alone are the same on both sides.

As a first illustration, let us compute the dimensions of the irreducible representations of S_N^+. For $N \geq 4$ these are indexed by $k \in \mathbb{N}$, with fusion rules $r_1 \otimes r_k = r_{k-1} + r_k + r_{k+1}$, and $\dim r_1 = N - 1$, see [5]. Thus the dimensions are given by:

$$d_0 = 1 \quad , \quad d_1 = N - 1 \quad , \quad d_{k+1} = (N-2)d_k - d_{k-1}$$

These dimensions should not be confused with the Catalan numbers $C_k = |NC(k)|$, which count the fixed points of the tensor powers of the magic representation, as in Proposition 2.3, and which do not depend on N. Here are the first values of these dimensions, computed with the program `rijkaard.py`:

N	d_1	d_2	d_3	d_4	d_5
4	3	5	7	9	11
5	4	11	29	76	199
6	5	19	71	265	989
7	6	29	139	666	3191
8	7	41	239	1393	8119
9	8	55	377	2584	17711

At $N = 4$ we obtain $d_k = 2k+1$, as for SO_3, and at $N \geq 5$ the growth is exponential, $d_k \sim cx^k$, with $x + x^{-1} = N - 2$. At the 2 special values $N = 5$ and $N = 9$, the latter corresponding to the color quantum group, we have the following result:

Proposition 7.1. *The dimensions of the irreducible representations of S_5^+ are the odd Lucas numbers, $d_k = L_{2k+1}$, namely $1, 4, 11, 29, 76, \ldots$ The dimensions of the irreducible representations of S_9^+, and of its twist PU_3^+, are the Fibonacci numbers $d_k = F_{4k+2}$, namely $1, 8, 55, 377, 2584, \ldots$*

Proof. The Lucas numbers satisfy $L_{n+2} = 3L_n - L_{n-2}$, with $L_1 = 1$, $L_3 = 4$, which is the above recurrence at $N = 5$. The Fibonacci numbers satisfy $F_{n+4} = 7F_n - F_{n-4}$, with $F_2 = 1$, $F_6 = 8$, which is the above recurrence at $N = 9$. The assertion about PU_3^+ follows from Theorem 1.4, because a twist preserves the dimensions of the corepresentations. The program `rijkaard.py` checks this as well. □

Thus, the 8-dimensional representation from section 2 sits inside a Fibonacci sequence, for S_9^+ and PU_3^+, with the next representation having dimension 55, and so on, while for G_3, which is a twist of PU_3, the dimensions are those of PU_3, namely $1, 8, 10, 10, 27, \ldots$ On the other hand, the twist changes the symmetric structure of the category, and more precisely the flip. Note here that the free pair S_9^+, PU_3^+ and the pair G_3, PU_3 have different tensor categories, and that the graded flip below concerns the latter pair only, the ordinary flip being not colinear for the free color quantum group. To be more precise, we have:

Proposition 7.2. *Under the correspondence $T \to T^\sigma$ from Theorem 1.1, the flip Σ of $M_N(\mathbb{C}) \otimes M_N(\mathbb{C})$, which is colinear for PU_N, corresponds to the following map of $\mathbb{C}[\Gamma] \otimes \mathbb{C}[\Gamma]$, which is colinear for G_H:*

$$T(e_g \otimes e_h) = d(g,h)e_h \otimes e_g$$

This map satisfies $T^2 = 1$, so it is a symmetric braiding, and at $N = 3$ its coefficients are cube roots of unity.

Proof. We must find T with $\Omega_2^{-1} T \Omega_2 = \Sigma$. With $T = \Omega_2 \Sigma \Omega_2^{-1}$ we obtain $T(e_g \otimes e_h) = \sigma(g,h)^{-1}\sigma(h,g)e_h \otimes e_g = d(g,h)e_h \otimes e_g$, as claimed. Since $d(g,h)d(h,g) = 1$ we have $T^2 = 1$, and since d is a bicharacter, T satisfies the hexagon relations, so it is a symmetric braiding. At $N = 3$ the values of d are the cube roots of unity. Finally, T is colinear for G_H by Theorem 1.1, because Σ is colinear for PU_N. □

This is quite interesting, historically. Color was introduced by Greenberg in 1964 [52] as a parastatistics of order 3 for the quarks, before being reformulated as an SU_3 gauge symmetry, and Proposition 7.2 shows that, after untwisting, a $\mathbb{Z}_3^2$-graded symmetric braiding comes back. We should mention however that this is a statement about the symmetric monoidal structure: the ordinary flip is not colinear for the free color quantum group, the pointwise product on $\mathbb{C}^9$ still commutes, and no quark exchange operator, Fock representation or parafermion relation is obtained in this way. At the level of Lie algebras, Proposition 7.2 leads to the color Lie algebras of Rittenberg and Wyler [77], also called ε-Lie algebras, studied by Scheunert [79]. These are Γ-graded vector spaces L with a bilinear bracket satisfying, for $x \in L_g, y \in L_h, z \in L_k$:

$$[y,x] = -\varepsilon(h,g)[x,y] \quad , \quad \varepsilon(k,g)[x,[y,z]] + \varepsilon(g,h)[y,[z,x]] + \varepsilon(h,k)[z,[x,y]] = 0$$

Here $\varepsilon : \Gamma \times \Gamma \to \mathbb{T}$ is a commutation factor, namely a bicharacter satisfying $\varepsilon(g,h)\varepsilon(h,g) = 1$. The basic example is that of the Lie superalgebras, where $\Gamma = \mathbb{Z}_2$ and $\varepsilon(g,h) = (-1)^{gh}$, and the general theory, including the Poincaré–Birkhoff–Witt theorem and the relation with the ordinary Lie superalgebras via cocycle twists, was developed by Scheunert in [79]. In our situation the grading group is $\Gamma = \widehat{H} \times H$, and the commutation factor is the symplectic form d, which takes its values in the N-th roots of unity. The point is that d is indeed a commutation factor, because it is a bicharacter satisfying $d(g,h)d(h,g) = 1$, and we have:

Proposition 7.3. *The group algebra $\mathbb{C}[\Gamma]$, with the following bracket, is a color Lie algebra, with commutation factor d:*

$$[e_g, e_h] = e_g e_h - d(g,h)e_h e_g = (1 - d(g,h))e_{g+h}$$

The Lie algebra gl_N is the twist of this color Lie algebra, with sl_N corresponding to the span of the elements e_g with $g \neq 0$, and with the center corresponding to $\mathbb{C}e_0$.

Proof. The bracket is $m - m \circ T$, with T being the braiding from Proposition 7.2, so under the correspondence from Theorem 1.1 it becomes $m_\sigma - m_\sigma \Sigma$, which is the commutator of $M_N(\mathbb{C})$. Explicitly, in terms of the Weyl operators, we have the following formula:

$$[W_g, W_h] = \sigma(g,h)(1 - d(g,h))W_{g+h}$$

The identities of a color Lie algebra are transported from the usual identities for the commutator, as in [79]. Finally, $d(0,h) = 1$, so e_0 is central, and $tr(W_g) = 0$ for $g \neq 0$. We have checked all this at $N = 3$ with the program `hagi.py`, including the ε-Jacobi identity for all the 729 triples of basis elements. □

As for the quarks, Proposition 6.1 shows that a color triplet has no image as a module over the 9 points. However, $\mathbb{C}^3$ is the unique irreducible projective representation of $\mathbb{Z}_3^2$ with cocycle σ, that is, the unique irreducible module over the twisted group algebra, and the class

$[\sigma] \in H^2(\mathbb{Z}_3^2, \mathbb{T}) = \mathbb{Z}_3$ is formally analogous to discrete torsion. This notion was introduced by Vafa [83], see also [84], and Douglas showed in [42] that it is implemented in D-brane gauge theories by using projective representations of the orbifold group. Making this analogy precise would require a module category, or a field theory, and we will not do this here.

Let us discuss now the gauge fields. A gl_3-valued gauge field on a manifold M can be decomposed, locally or on a trivial bundle, in the Weyl basis, which is orthonormal with respect to $< a, b >= tr(ab^*)$, and this decomposition is related to the finest grading of sl_N studied by Patera and Zassenhaus [74]. We have:

Theorem 7.4. *Write a gl_3-valued gauge field as $A = \sum_g A_g W_g$, with A_g being 1-forms on M, and with $g \in \mathbb{Z}_3^2$. The curvature $F = dA + A \wedge A$ has then components*

$$F_g = dA_g + \frac{1}{2} \sum_{h+k=g} \sigma(h,k)(1 - d(h,k)) A_h \wedge A_k$$

and the Yang–Mills density is $\sum_g |F_g|^2$, up to the coupling constant, with respect to $< a, b >= tr(ab^*)$ and to the metric of M. The U_1 part A_0 decouples, and the components of the 8 gluon fields are labeled by the nonzero elements of $\mathbb{F}_3^2$, with exactly 48 ordered pairs of labels having nonzero brackets, namely the pairs of independent labels.

Proof. We have $A \wedge A = \sum_{hk} \sigma(h,k) A_h \wedge A_k W_{h+k}$, and by antisymmetrizing, using $\sigma(k,h) = \sigma(h,k) d(h,k)$, we obtain the formula of F_g. The formula of the Yang–Mills density follows from the orthonormality of the Weyl operators. Finally, $d(h,k) = w^{\det(h,k)}$, so $1 - d(h,k) = 0$ precisely when h, k are dependent. In particular, $F_0 = dA_0$, because $h + k = 0$ implies that h, k are dependent. For each of the 8 nonzero labels h, there are 6 nonzero labels k independent from h, which gives 48 pairs. □

With the normalization $J_g = w^{2g_1 g_2} W_g$, where $g = (g_1, g_2)$, we obtain the following formula, as checked by `hagi.py`:

$$[J_g, J_h] = 2i \sin\left(\frac{2\pi \det(g,h)}{3}\right) J_{g+h}$$

These are the trigonometric structure constants of Fairlie, Fletcher and Zachos [44]. Regarding reality, write $A = \sum_g B_g J_g$, with new coefficients B_g. For an su_3 connection, in the anti-Hermitian convention, we have $B_0 = 0$, and since $J_g^* = J_{-g}$, the condition $A^* = -A$ reads $B_{-g} = -\overline{B_g}$, while in the Weyl basis, where $W_g^* = w^{g_1 g_2} W_{-g}$, it reads $A_{-g} = -w^{g_1 g_2} \overline{A_g}$. Thus the 8 labels pair into 4 complex 1-forms, corresponding to the 8 real color directions. Observe that the labels g index the Weyl basis, which is a Fourier basis with respect to the 9 points, so the points themselves do not label individual gluons. After untwisting, by Proposition 7.3 the structure constants become simply $1 - d(h,k)$, the ordinary spacetime 1-forms and the statistics of the quantized fields being unchanged. Making sense of the corresponding untwisted Yang–Mills theory is an interesting question. Let us compare this with the other twists of the Standard Model geometry which appeared in the literature. In the twisted spectral triples of Connes and Moscovici [35], the algebra is unchanged, and the commutators with the Dirac operator are replaced by twisted commutators $Da - \rho(a)D$, with ρ being an automorphism of the algebra. These were applied to the Standard Model by Devastato and Martinetti [40], the breaking of the grand symmetry down to the Standard Model becoming a dynamical process, and were further studied in [45, 63, 69], notably in connection with the Lorentzian signature. A different notion, the twisted reality condition, where the real structure is twisted instead, was introduced by Brzeziński, Ciccoli, Dabrowski and Sitarz [20].

These twists are not our cocycle twist, which changes the product of the algebra, and leaves the representation theory of the quantum symmetries unchanged. There is however a formal

analogy: for an element a which is an eigenvector of ρ, with eigenvalue χ, the twisted commutator is $Da-\chi aD$, which has the same shape as the bracket in Proposition 7.3. In our setting the Dirac operator D_F is color-blind, by Theorem 6.3, so it has degree 0, and, whenever the transport to the untwisted side exists, its commutators untwist into usual commutators. Thus a link with [40, 63] would rather involve a Dirac operator, or a real structure, of nonzero degree.

Let us discuss now the weak case. At $N = 2$ we have $\Gamma = \mathbb{Z}_2^2$, the Weyl operators are $1, X, Z, ZX$, with X, Z being Pauli matrices, and $d(g, h) = (-1)^{\det(g,h)}$. Thus, in the weak sector, the graded flip is given by signs, instead of cube roots of unity, and we are led to the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded color Lie algebras considered by Rittenberg and Wyler in [78]. To be more precise, we have the following result:

Theorem 7.5. *The Lie algebra gl_2 is the twist of the color Lie algebra $\mathbb{C}[\mathbb{Z}_2^2]$, with commutation factor $d(g, h) = (-1)^{\det(g,h)}$, and with bracket given by*

$$[e_g, e_h] = 2e_{g+h} \quad , \quad \forall g \neq h,\ g, h \neq 0$$

and $[e_0, e_h] = [e_g, e_g] = 0$. This bracket is symmetric on the nonzero labels, and accordingly, in the Pauli basis, the 3 components of the weak gauge field are labeled by the nonzero elements of $\mathbb{F}_2^2$, with all the 6 ordered pairs of distinct labels having nonzero brackets.

Proof. This is the case $N = 2$ of Proposition 7.2, Proposition 7.3 and Theorem 7.4, with $d(g, h) = -1$ precisely when g, h are distinct and nonzero, so that $1 - d(g, h) = 2$, and $[e_h, e_g] = -d(h, g)[e_g, e_h] = [e_g, e_h]$. We have checked all this with the program `kluivert.py`, which also checks that the models P^x from section 5 are magic and noncommutative, for random $x \in SU_2$. □

Summarizing, each scalar summand corresponds to a single point, where nothing happens, the electroweak algebra corresponds to $1 + 1 + 4$ points, with S_4^+ acting on the 4 points and untwisting into SO_3, with a graded flip given by signs, and the color sector corresponds to 9 points, with the free pair S_9^+, PU_3^+, and, on restricting to G_3, PU_3, with a graded flip given by cube roots of unity. Observe however that electromagnetism, with charge $Q = T_3 + Y$, mixes the weak and hypercharge sectors, so it is not modeled by a single scalar summand. The electroweak sector, which is the simplest one, certainly deserves a separate study.

8. Spectral action and classicality

We discuss here some general mechanisms forcing quantum symmetries to be classical, which are relevant to the passage from the finite geometry F to the product $M \times F$. The first one is the flip mechanism, used in the proof of Proposition 4.1: if the flip Σ intertwines $u^{\otimes 2}$, for a faithful corepresentation u, then the coefficients of u commute. The second one concerns tensor products:

Proposition 8.1. *Let A_1, A_2 be unital C^*-algebras with coactions $\alpha_i : A_i \to A_i \otimes C(G)$, and assume that a coaction α on the spatial tensor product $A_1 \otimes A_2$ satisfies $\alpha(a \otimes 1) = \alpha_1(a)_{13}$ and $\alpha(1 \otimes b) = \alpha_2(b)_{23}$. Then the coefficients of α_1 commute with those of α_2. In particular, if both coefficient algebras generate $C(G)$, then G is classical.*

Proof. The elements $a \otimes 1$ and $1 \otimes b$ commute, so their images commute too, and by applying linear forms on the A_1 and A_2 legs, we obtain the commutation of the coefficients. □

This does not apply to the global actions on $M \times F$, which are trivial on the first leg, but shows that local quantum symmetries on products of independent algebras must be handled with care. Recall also that, by [51], a compact connected smooth manifold has no genuine compact quantum group symmetries, in the sense of faithful smooth actions, so for a product $M \times F$, for

an action preserving the manifold factor, the effective action on this factor is classical, although this alone does not classify the actions on the product.

More generally, the passage from F to the product $M\times F$, with algebra $C^\infty(M)\otimes A_F$, Hilbert space $L^2(M,S)\otimes H_F$ and Dirac operator $D_M\otimes 1+\gamma_5\otimes D_F$, raises problems at 3 levels. First, the global symmetries are no problem, because the quantum group acts via $id\otimes$coaction, and the cocycle is constant along M, so the twisting passes. Second, the local symmetries, or gauge transformations, are maps $M\to G$, and for quantum groups one needs quantum families of maps [80], or $C(M)$-linear coactions, as in [13], which is a hard problem. Observe however that $H^2(\mathbb{Z}_3^2,\mathbb{T})=\mathbb{Z}_3$ is discrete, so on a connected manifold M a locally defined twist class is forced to be constant, but this does not produce a global Weyl frame on a nontrivial bundle, so twisting fiberwise is fine only locally, or on trivial bundles. Third, there is the question of the fermions, discussed in section 6, which is where the real obstruction lies. Observe also that, at the level of the gauge fields, the untwisting can be performed locally, or on trivial bundles, as in Theorem 7.4, with the color Lie algebra from Proposition 7.3 replacing gl_3.

The third mechanism concerns the spectral action [23]. Consider a real spectral triple (A,H,D,J), and a finite quantum group Q, with $C(Q)$ acting on $H_Q=L^2(C(Q),h)$ by left multiplication, and with $j(a)=a^*$, and set $K=J\otimes j$. Consider a coaction implemented by a unitary U on $H\otimes H_Q$, which commutes with $D\otimes 1$. There are 2 natural reality conditions for U, namely the full commutation $KU=UK$, and the following coefficientwise condition, used in [12, Definition 2.12]:

$$KU(\xi\otimes 1)=U(J\xi\otimes 1)\quad,\quad\forall\xi\in H$$

Given a classical self-adjoint 1-form $A_1=\sum a_i[D,b_i]$, we have the fluctuated operator $D_{A_1}=D+A_1+\varepsilon' JA_1J^{-1}$, and its transformed version $D'=D\otimes 1+A_1'+\varepsilon' KA_1'K^{-1}$, with $A_1'=U(A_1\otimes 1)U^*$. The natural invariance requirement for the bosonic spectral action, with cutoff function f and scale Λ, is then the following equality of operators on H_Q:

$$(Tr_H\otimes id)f(D'/\Lambda)=Tr_Hf(D_{A_1}/\Lambda)1$$

Under full commutation we have $D'=U(D_{A_1}\otimes 1)U^*$, and the invariance follows by functional calculus, and the Kac type partial trace identity from the proof of Proposition 3.2. Under the coefficientwise condition this fails, because KL_aK^{-1} involves a right multiplication by a^*, and we have the following result, which shows that the invariance then detects all the nonclassical finite quantum groups:

Theorem 8.2. *For a finite quantum group Q, the following are equivalent:*

1. *$C(Q)$ is commutative.*
2. *The above bosonic invariance holds for all real product geometries, coefficientwise real actions, classical 1-forms and cutoffs.*
3. *The bosonic invariance holds for the regular coaction of Q, with 1-forms built from the counit block projection, on a fixed 2-point external triple, or alternatively on the flat torus $\mathbb{T}^4$, each of these 2 geometries being sufficient.*

Proof. Write $C(Q)=\oplus_\beta M_{n_\beta}(\mathbb{C})$, with central projections z_β, and $d=\sum_\beta n_\beta^2$. Let e be the counit block projection, $\Pi=\Delta(e)$, and $P=L_\Pi$, $R=R_\Pi$, acting on $H_Q\otimes H_Q$. For the regular coaction we have $U(L_e\otimes 1)U^*=P$ and $KPK^{-1}=R$. In the block $\alpha\otimes\beta$, the element Π projects onto the invariant vectors of $\pi_\alpha\otimes\pi_\beta$, which by Schur's lemma form the line spanned by a unit maximally entangled vector when β is the dual of α, and are zero otherwise. In the Hilbert–Schmidt realization the map $X\to\Pi X\Pi$ is then a rank one projection, whose vector has n_β^2 equal Schmidt coefficients $1/n_\beta$, and with E being the ordinary partial trace over the

first leg, we obtain:

$$E(PR) = \sum_\beta n_\beta^{-2} L_{z_\beta} \quad , \quad E(P) = E(R) = 1 \quad , \quad E(1) = d$$

For the 2-point witness, represent $\mathbb{C}^2$ on $\mathbb{C}^3$, with basis $0, L, R$ carrying the bimodule labels $(1,1), (2,1), (1,2)$, let J exchange L, R and conjugate the coefficients, and set $B = |0\rangle\langle L| + |L\rangle\langle 0|$, $C = JBJ^{-1}$, $D = B + C$ and $\gamma = diag(1, -1, -1)$. This is a real even spectral triple satisfying the first order condition, and with $p = diag(0, 1, 0)$ we have $B = (1-2p)[D, p]$. Take its product with $(C(Q), H_Q, 0, j)$, with the regular coaction, which satisfies the coefficientwise condition, and the 1-form $-B \otimes L_e$. Since e is central, the fluctuated operator is $(B+C) \otimes (1 - L_e)$, whose kernel has dimension $d + 2$. After transformation we obtain $B \otimes (1 - P) + C \otimes (1 - R)$, and on a joint eigenspace of P, R the kernel has dimension 1, unless $P = R = 1$ there, where it has dimension 3. With a cutoff selecting the kernel, the transformed action is therefore $d + 2E(PR)$, which equals $d + 2/n_\beta^2$ on the block β, so the invariance fails as soon as some $n_\beta > 1$. The same works with the heat cutoff $f(D) = e^{-sD^2}$, at any scale $s > 0$, the transformed action minus the original one being $2(1 - e^{-s})^2(n_\beta^{-2} - 1)$ on the block β.

On the 2π-periodic flat torus $\mathbb{T}^4$, with periodic spin structure and $D = -i \sum \gamma^\mu \partial_\mu$, use instead the 1-form $t\gamma^1 \otimes L_e$, with $0 < t < 1/2$, coming from $u^*[D, u] = \gamma^1$ with $u = e^{ix_1}$. The original shifts cancel, while after transformation the first momentum is shifted by $t(P - R)$, and only the zero momentum with $P = R$ contributes to the kernel, which gives $4(d - 2 + 2E(PR))$ instead of $4d$. Conversely, in the classical case the action is fiberwise given by unitaries commuting with D and J, and the invariance holds. We have checked all this with the programs `beckham.py` and `zlatan.py`, the latter building the transformed operators, and computing the heat cutoff version, for the duals of S_3 and Q_8. □

The torus witness shows in fact more than the failure of invariance. Indeed, the 1-forms $A_1 = 0$ and $A_1 = t\gamma^1 \otimes L_e$ give the same fluctuated operator D_{A_1}, but different transformed operators, so the prescribed transformation of the 1-forms does not descend to the fluctuated Dirac operators.

For the Standard Model geometry itself, a natural question is whether the invariance can be restored by using a second, 2-sided implementing unitary W, commuting with K and with the Dirac operator. Observe that if U has coefficients acting by left multiplications, and $KU = UK$, then these coefficients lie in $L(C(Q)) \cap R(C(Q))$, which is the center of $C(Q)$, because $L_x = R_y$ gives $x = y$ by evaluating at 1, and then $xz = zx$ for any z. Thus such a W must be of a different type. For the Standard Model, such a W exists, the point being that the weak algebra acts on the particle doublets, while the color algebra acts on the antiquarks, so that the weak phase and the color action can be placed on opposite legs of $L^2(Q)$. To be more precise, we have the following result:

Proposition 8.3. *For the quantum groups Q from Theorem 2.4, acting on F, there is a unitary W on $H_F \otimes L^2(Q)$, spatial in the Haar representation, which commutes with $D_F \otimes 1$, $\gamma_F \otimes 1$ and K, implements the effective coaction on the represented complex algebra B_F, and satisfies $(Tr_{H_F} \otimes id)(W(X \otimes 1)W^*) = Tr(X)1$. Thus, for classical 1-forms, the bosonic invariance holds for F, and for its products with spin manifolds, with the global action, trivial on the manifold, and with cutoffs giving trace-class operators.*

Proof. Let $\mathcal{M} = L^\infty(Q)$, which is a finite von Neumann algebra, Q being of Kac type, and write L_a, R_a for the left and right multiplications on $L^2(Q)$, so that $jL_aj = R_{a^*}$. The color coaction is a trace preserving morphism $\beta : M_3(\mathbb{C}) \to M_3(\mathbb{C}) \otimes \mathcal{M}$, and the projections $\beta(e_{11})$ and $e_{11} \otimes 1$ both have center-valued trace $1/3$, so there is a partial isometry s with $s^*s = e_{11} \otimes 1$

and $ss^* = \beta(e_{11})$. Then $v = \sum_i \beta(e_{i1})s(e_{1i} \otimes 1)$ is a unitary with $\beta(b) = v(b \otimes 1)v^*$, and trace preservation gives $\sum_i v_{ij}v_{ik}^* = \delta_{jk}1$. With z being the generator of the weak circle, set $h_\nu = h_u = 1$ and $h_e = h_d = z^*$, and let W act on the particles of species s by $L_{h_s}V_R$, and on the antiparticles of species s by $R_{h_s^*}V_L$, where $V_L = (L_{v_{ij}})$ and $V_R = (R_{v_{ij}^*})$ act on the color indices of the quarks, and are trivial on the leptons. Then W is unitary, and commutes with K, because $jL_aj = R_{a^*}$. On the weak doublets V_R cancels, and we obtain $e_{12} \to e_{12} \otimes z$, while on the antiquarks the right phases commute with the left color action, and we obtain β, so W implements the coaction. Since W acts in the same way on both ends of each mass arrow, and trivially on the generations and on the Majorana arrow, it commutes with D_F and γ_F. The partial trace formula follows from $\sum_i w_{ij}w_{ik}^* = \delta_{jk}1$ for the color blocks w of W, which comes from the above relation for v, and from $R_aR_b = R_{ba}$. Finally, W commutes with K and with the Dirac operator, so $D' = W(D_{A_1} \otimes 1)W^*$, and the partial trace formula gives the invariance. We have checked all this with the program `vanbasten.py`, in a model with $\mathcal{M} = \mathbb{C}[S_3]$, 3 generations, generation mixing and a Majorana term. □

Observe that W is not claimed to be a corepresentation, and that the compatibility with the real form of A_F, which selects $QISO_{\mathbb{R}}^+$ in Theorem 2.4, is an additional requirement, not used here.

This is in agreement with [12, section 4], where the invariance of the spectral action under the quantum isometries of F is established by using the conjugacy identity [12, (4.4)], which follows from the full commutation with K, but not from the coefficientwise condition alone, as the above regular examples show. Thus such statements depend on the precise prescription for transforming the fluctuations, on the reality condition imposed on the implementer, and on the geometry itself, the obstruction from Theorem 8.2 coming from the regular coactions.

9. Mixing matrices and Hadamard

The Standard Model has 2 mixing matrices, namely the CKM matrix for the quarks, relating the mass eigenstates of the up and down quarks, and the PMNS matrix for the leptons, relating the mass and flavor eigenstates of the neutrinos. Both are unitaries $C \in U_3$, defined up to phases on rows and columns, at least as far as the oscillations are concerned, 2 Majorana phases of the PMNS matrix being physical in the Majorana case, but invisible in the oscillations, and in the Chamseddine–Connes framework they appear inside the Dirac operator D_F, via the Yukawa couplings, see [24, 28].

In [29], Connes interpreted such a matrix as the relative position of 2 maximal abelian subalgebras of $M_n(\mathbb{C})$, namely the diagonal algebra D and its conjugate UDU^*, and extended this into a unitary invariant of Riemannian manifolds, which together with the spectrum of the Dirac operator is a complete invariant, see also [30]. A key example in [29], in the section on the CKM matrix and the Fourier transform, is the Fourier matrix $F_3 = (w^{ij})$, with $w = e^{2\pi i/3}$, which is said to fall in the class of commuting squares of Popa [76].

Let us recall that 2 subalgebras $D_1, D_2 \subset M_n(\mathbb{C})$, together with a common subalgebra $D_0 \subset D_1 \cap D_2$, form a commuting square with respect to the trace when $E_{D_1}E_{D_2} = E_{D_0}$, the maps E being the conditional expectations, see [76]. In the case of maximal abelian subalgebras, with $D_0 = \mathbb{C}1$, which corresponds to the setting of Connes, who works with labeled minimal idempotents, we have the following well-known result, see [54, 76, 81]:

Proposition 9.1. *For a unitary $U \in U_n$, we have $E_DE_{UDU^*} = E_{\mathbb{C}1}$ if and only if $|U_{ij}|^2 = \frac{1}{n}$ for any i, j, that is, if and only if $\sqrt{n}U$ is a complex Hadamard matrix. In this case we have as well $E_{UDU^*}E_D = E_{\mathbb{C}1}$, and $D \cap UDU^* = \mathbb{C}1$.*

Proof. Let $e_i \in D$ be the minimal projections, and $f_j = Ue_jU^*$. Since $E_{\mathbb{C}1} = E_{\mathbb{C}1}E_{UDU^*}$, the equality to be proved holds if and only if it holds on each f_j. But we have $E_D(f_j) = \sum_i n\,tr(e_if_j)e_i = \sum_i |U_{ij}|^2 e_i$, while $E_{\mathbb{C}1}(f_j) = \frac{1}{n}1$, which gives the result. The other assertions follow by taking adjoints, and from $E_D(x) = x$ for $x \in D \cap UDU^*$. □

When formulating instead the commuting square condition over $D \cap UDU^*$, the identity matrix gives a trivial example. In general, the bipartite graph having as edges the pairs (i, j) with $U_{ij} \neq 0$ decomposes U, after permuting the rows and columns, into square blocks, the intersection $D \cap UDU^*$ being scalar on each block, and the commuting square condition holds precisely when each block is, up to normalization, a complex Hadamard matrix of its own size.

Thus, the commuting square case of the Connes invariant is precisely the complex Hadamard case. Now recall that any complex Hadamard matrix $H \in M_N(\mathbb{C})$ produces a quantum permutation group $G \subset S_N^+$, as follows, see [6]: the vectors $\xi_{ij} = H_i/H_j$, obtained by dividing the rows of H entrywise, form a magic basis of $\mathbb{C}^N$, the magic matrix $P_{ij} = Proj(\xi_{ij})$ defines a representation $\pi : C(S_N^+) \to M_N(\mathbb{C})$, and G is given by the Hopf image of π, that is, $C(G)$ is the smallest Hopf algebra quotient of $C(S_N^+)$ through which π factors.

Regarding now the physics, the quark mixing matrix CKM is close to the identity, which is the opposite of the Hadamard regime. However, the lepton mixing matrix PMNS is much closer to the Hadamard regime. In fact, the Fourier matrix $F_3/\sqrt{3}$ itself was proposed in 1978 by Cabibbo and by Wolfenstein as a lepton mixing matrix, under the name trimaximal mixing [21, 92]. The more recent tribimaximal ansatz of Harrison, Perkins and Scott [55] is as follows, up to signs and phases:

$$U_{TB} = \begin{pmatrix} \sqrt{2/3} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \\ -1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \end{pmatrix}$$

The middle column here is flat, as for a Hadamard matrix. The matrix $F_3/\sqrt{3}$, which predicts $|U_{e3}|^2 = 1/3$, was ruled out by the data long ago, and the measurement of a nonzero reactor angle θ_{13} by Daya Bay and others [3] ruled out U_{TB} too. As for the trimaximal scheme TM2, keeping the flat middle column of U_{TB}, this predicts $\sin^2\theta_{12} = 1/(3\cos^2\theta_{13}) \simeq 0.341$, with $\sin^2\theta_{13} \simeq 0.022$, and the first precision measurement of θ_{12} by JUNO [58] gives a value close to 0.31, with an uncertainty below 0.01, which puts exact TM2 under strong tension, leaving only approximate textures, with explicit deviations. Thus the lepton mixing matrix is much closer to the Hadamard regime than the quark mixing matrix, and we will make this more precise at the end of this section.

Most models producing tribimaximal or related mixing use a finite flavor group, acting on the 3 generations via a 3-dimensional representation. The classic examples are the tetrahedral group A_4, introduced by Ma and Rajasekaran [66] and used by Altarelli and Feruglio [1] for tribimaximal mixing, the octahedral group S_4, and the icosahedral group A_5, which produces the golden ratio mixing [43]. These, together with the cyclic and dihedral groups, are the finite subgroups of SO_3, acting on $\mathbb{R}^3$.

Now recall from section 3 that SO_3 is the twist of the weak quantum group S_4^+. The quantum subgroups of S_4^+ were classified in [7], by using the isomorphism $S_4^+ \simeq SO_3^{-1}$, and include, besides S_4 and its subgroups, the twists S_4^τ and A_5^τ of the octahedral and icosahedral groups, which are not classical, and are subject to a McKay type correspondence. Thus, the classical flavor groups have quantum permutation counterparts, acting on 4 points, and the integration theory over S_4^+, developed in [10], could be useful here.

There is also a link with the fermions, discussed in section 6. The group $\Delta(27)$ acts on the 3 generations in many flavor models [19, 56], and its 3-dimensional representations are generated

by the clock and shift matrices, the Fourier matrix F_3 interchanging their eigenbases, which are therefore mutually unbiased. Thus, the same matrix F_3 appears in the color twist from section 1, in the Connes example from [29], and in the trimaximal mixing from [21, 92]. Summarizing, we have the following result:

Theorem 9.2. *The Connes invariant of a pair of maximal abelian subalgebras $D, UDU^* \subset M_n(\mathbb{C})$ corresponds to a commuting square over $\mathbb{C}1$ precisely when $\sqrt{n}U$ is a complex Hadamard matrix, and in this case it produces a quantum permutation group $G \subset S_n^+$. For the trimaximal matrix $F_3/\sqrt{3}$ this quantum permutation group is the group $\mathbb{Z}_3$.*

Proof. The first assertion is Proposition 9.1. Regarding the second assertion, for a Fourier matrix F_G we have $\xi_{ij} = \xi_{i-j,0}$, so the magic matrix is circulant, and its coefficients generate the algebra $C(G)$, as explained in [6]. At $G = \mathbb{Z}_3$ we obtain the result. □

Let us discuss now the color case in more detail. At $N = 3$, the 8 nonzero labels of the Weyl operators W_g split into 4 pairs $\{g, 2g\}$, corresponding to the 4 lines through 0 in $\mathbb{F}_3^2$, and we have:

Proposition 9.3. *For each of the 4 lines L through 0 in $\mathbb{F}_3^2$, the operators W_g with $g \in L$ have a common eigenbasis B_L of $\mathbb{C}^3$, unique up to phases and permutations. These 4 bases are mutually unbiased, and the transition matrix between any 2 of them, multiplied by $\sqrt{3}$, is a complex Hadamard matrix, equivalent to F_3.*

Proof. For $g \neq 0$ we have $W_{2g} \sim W_g^2$, and W_g has the 3 cube roots of unity as eigenvalues, up to a common phase, because $tr(W_g) = tr(W_g^2) = 0$. This gives the bases B_L. If g, h are independent, $W_h W_g \sim W_g W_h$ with a nontrivial cube root of unity as scalar, so W_h permutes cyclically the eigenvectors e_i of W_g, up to phases. Thus for an eigenvector f of W_h the quantities $|< e_i, f >|$ do not depend on i, and so are all equal to $1/\sqrt{3}$. Finally, any 3×3 complex Hadamard matrix is equivalent to F_3, see [54, 81]. □

These are the mutually unbiased bases of Ivanović and Wootters–Fields [57, 91]. Explicitly, consider the Fourier matrix and the phase matrix, given by:

$$F_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & w & w^2 \\ 1 & w^2 & w \end{pmatrix} \quad , \quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & w \end{pmatrix}$$

Then, up to phases and to the order of the vectors, the 4 bases B_L with L spanned by $(1,0), (0,1), (1,2), (1,1)$ are given by the columns of the following unitary matrices:

$$1_3 \quad , \quad \frac{F_3}{\sqrt{3}} \quad , \quad \frac{DF_3}{\sqrt{3}} \quad , \quad \frac{D^2F_3}{\sqrt{3}}$$

Thus the transition matrices between the standard basis and the other 3 bases are the trimaximal matrix $F_3/\sqrt{3}$ and its versions twisted by the phases of D, and the transition matrices between 2 nonstandard bases are of the form $F_3^* D^k F_3/3$ with $k = 1, 2$, which are complex Hadamard matrices too, after multiplication by $\sqrt{3}$, as predicted by Proposition 9.3. Observe also that the 9 points of $\mathbb{F}_3^2$, together with the 12 affine lines, form the Hesse configuration of the 9 inflection points of a plane cubic, the Heisenberg group of order 27 acting by clock and shift on the Hesse pencil $x^3 + y^3 + z^3 + txyz = 0$, and the group of order 216 from Proposition 4.4 being the Hessian group, see [2]. In connection with the mixing matrices, observe that the bases B_L are well-defined only as sets of rays, the ordered eigenbases being unique up to phases and permutations, so the natural objects here are the 4 sets of rays, corresponding to the 4 parallel classes of lines of $\mathbb{F}_3^2$. The point is that the Clifford group permutes these 4 sets of rays, and that the flavor groups A_4 and S_4 appear in this way, as follows:

Theorem 9.4. *The projective Clifford group $C_3 \subset PU_3$, which is the twisted counterpart of the classical version of G_3, of order 216, permutes the 4 mutually unbiased bases B_L, and its action factors as*

$$C_3 \to C_3/K \simeq SL_2(3) \to PSL_2(3) \simeq A_4$$

via the action of $SL_2(3)$ on the 4 lines through 0. When adding complex conjugation, which corresponds to orientation reversal, we obtain a group of order 432, acting on the 4 bases via $GL_2(3) \to PGL_2(3) \simeq S_4$.

Proof. An element $U \in C_3$ satisfies $UW_gU^* \sim W_{Ag}$, where $A \in SL_2(3)$ is its image in the exact sequence from section 5, so U maps B_L to B_{AL}. The Weyl group K acts trivially, as does -1, and the kernel of the action of $SL_2(3)$ on the 4 points of the projective line over $\mathbb{F}_3$ is $\{\pm 1\}$, with image $PSL_2(3) \simeq A_4$. Complex conjugation maps $W_{(i,j)}$ to $W_{(-i,j)}$, which corresponds to a matrix of determinant -1, and we obtain in this way $GL_2(3)$, acting via $PGL_2(3) \simeq S_4$. We have checked all this with the program `cantona.py`, which constructs the 4 bases, verifies that the Fourier matrix and the phase gate $diag(1, 1, w)$ normalize the Weyl group and induce generators of $SL_2(3)$, and computes the images of $SL_2(3)$ and $GL_2(3)$ in the symmetric group on the 4 bases, with element orders matching those of A_4 and S_4. □

This is quite interesting, physically. In the flavor models, the mixing matrix appears as the transition matrix between the eigenbases of the residual symmetries preserved by the charged leptons and by the neutrinos, and it was shown by Lam [62] that tribimaximal mixing, for all values of the Yukawa couplings, requires the horizontal group S_4, or a group containing it, while A_4 naturally leads to trimaximal mixing, with the qualification, explained by Grimus, Lavoura and Ludl [53], that the symmetries of the effective mass terms need not be symmetries of the underlying Lagrangian. In the above picture, if the residual symmetries are the cyclic subgroups of the Weyl group, the trimaximal matrix $F_3/\sqrt{3}$ is the transition matrix between the eigenbases of clock and shift, and A_4 and S_4 are the permutation groups of the 4 mutually unbiased bases, induced by the classical versions of the quantum groups from section 4. Let us stress however that this provides no mechanism fixing the eigenbases of the physical Yukawa couplings: the CKM and PMNS matrices can vary continuously, with the color construction being unchanged, so the above is only a conditional statement, about mass matrices having specified residual Weyl symmetries.

At the quantum level, the 4 bases B_L correspond to the 4 parallel classes of lines of $\mathbb{F}_3^2$, and a natural question is whether the color quantum group G_3 acts on these 4 classes, which would give a link with the weak quantum group S_4^+. The answer is negative, and in fact we have the following result:

Proposition 9.5. *The quantum group G_3 has no nontrivial coaction on a set of $k \leq 8$ points. In particular, the action of its classical version on the 4 mutually unbiased bases, via A_4, does not extend to G_3.*

Proof. A coaction of G_3 on k points gives a k-dimensional magic corepresentation, having the constant vector as fixed vector. Since G_3 is a twist of PU_3, its irreducible corepresentations have the same dimensions as those of PU_3, namely the dimensions $(a+1)(b+1)(a+b+2)/2$ of the irreducible representations of SU_3 having highest weight (a, b) with $a = b$ modulo 3. The smallest nontrivial such dimension is 8, attained by the adjoint representation, so for $k \leq 8$ our corepresentation is a sum of trivial ones, and so the coaction is trivial. The program `gullit.py` lists these dimensions, namely $1, 8, 10, 27, 28, 35, \ldots$ □

Thus, the passage to the quantum symmetry loses here a classical quotient action, and the bound $k \leq 8$ is sharp, as shown by the action on the 9 points. As a consequence, a link between

the color and weak sectors, via mixing, must be searched elsewhere, for instance at the level of the classical versions.

Finally, as a classical baseline for the deviation from the Hadamard regime, for a mixing matrix $U \in U_n$ we can use the following rephasing invariant quantity, which vanishes precisely in the Hadamard case, and equals $n-1$ for a permutation matrix:

$$\delta(U) = \sum_{ij} \left(|U_{ij}|^2 - \frac{1}{n} \right)^2 = \sum_{ij} |U_{ij}|^4 - 1$$

Observe that $\delta(U) = ||E_D E_{UDU^*} - E_{\mathbb{C}1}||^2$, with the Hilbert–Schmidt norm of linear maps on $M_n(\mathbb{C})$, so δ measures the failure of the commuting square condition from Proposition 9.1. At $n = 3$ we have $\delta = 2$ for the identity, $\delta = 1/3$ for U_{TB}, and $\delta = 0$ for the transition matrices between the mutually unbiased bases, as checked with the program `overmars.py`, which also checks Proposition 9.1, the above Hilbert–Schmidt formula, and the unitarity of U_{TB}. For the CKM matrix, which is close to the identity, δ is close to its maximal value 2, while for the PMNS matrix, which is close to U_{TB}, it is close to $1/3$, a precise comparison requiring the experimental values, with their uncertainties. Observe that δ is unchanged by complex conjugation, so it loses the sign of the Jarlskog invariant J, and that it compresses even the information carried by the moduli, which determine $|J|$. Summarizing, the question of understanding the mixing matrices in terms of quantum permutations is natural, with the Hadamard case being well understood, and with the deviation from the Hadamard regime remaining to be measured by a suitable quantum group invariant, refining δ.

10. A look into Yang-Mills theory

The Yang–Mills theory with gauge quantum group S_N^+, or its twists, should normally come only after the untwisting of the whole Standard Model, discussed in sections 3 and 7, and after the passage from F to $M \times F$. Indeed, there is no point in formulating a gauge theory for objects which are still twisted. We gather here, as a program, a few observations which make sense independently of this.

In 2 dimensions, Yang–Mills theory on a closed surface Σ_g of genus g and area A can be solved exactly, and the Migdal–Witten formula [70, 89] gives the partition function as a sum over the irreducible representations, with c being the Casimir:

$$Z(\Sigma_g, A) = \sum_\lambda (\dim \lambda)^{2-2g} e^{-A\, c(\lambda)}$$

As explained by Witten in [90], this sum encodes the intersection pairings on the moduli spaces of flat connections, and at $A = 0$ it becomes the Witten zeta function $\sum_\lambda (\dim \lambda)^{-s}$, evaluated at $s = 2g-2$. Since this formula only uses the representation theory, one can formally write it for S_N^+, and the result is invariant under twisting. With the dimensions d_k from section 7, and with the Casimir $c(k) = k(k+1)$ of the SO_3 case, we are led to the following formal quantities:

$$Z_N(\Sigma_g, A) = \sum_k d_k^{2-2g} e^{-Ak(k+1)} \quad , \quad \zeta_N(s) = \sum_k d_k^{-s}$$

The program `rijkaard.py` computes these quantities numerically, and here are the values of ζ_N at $s = 2, 4$, corresponding to the genus $g = 2, 3$, with 4 decimals:

N	4	5	6	7	8	9
$\zeta_N(2)$	1.2337	1.0722	1.0430	1.0290	1.0210	1.0160
$\zeta_N(4)$	1.0147	1.0040	1.0016	1.0008	1.0004	1.0002

At $N = 4$ these are the values $\pi^2/8$ and $\pi^4/96$, coming from the SO_3 theory, and at $N \geq 5$ the values approach 1 quickly, the main correction being $(N-1)^{-s}$, coming from the representation r_1. In general, we have the following result:

Theorem 10.1. *The formal S_N^+ partition function $Z_N(\Sigma_g, A)$, with $N \geq 4$, is well-defined for any $A > 0$ and any genus g, and as $N \to \infty$, with $A > 0$ fixed, it diverges for $g = 0$, is independent of N for $g = 1$, and tends to 1 for $g \geq 2$. At $A = 0$ the series $\zeta_N(2g-2)$ converges for $g \geq 2$, with exponential speed when $N \geq 5$, at $N = 4$ we have $\zeta_4(s) = (1 - 2^{-s})\zeta(s)$ for $s > 1$, as for SO_3, and $\zeta_N(s) \to 1$ as $N \to \infty$, for any $s > 0$.*

Proof. For $A > 0$ the Gaussian factor $e^{-Ak(k+1)}$ dominates the exponential growth of d_k^{2-2g}. Since $d_{k-1} \leq d_k$, the recurrence gives $d_{k+1} \geq (N-3)d_k$, and so $d_k \geq (N-1)(N-3)^{k-1}$ for $k \geq 1$. Thus for $N \geq 5$ and $s > 0$ the series $\sum_{k \geq 1} d_k^{-s}$ converges exponentially fast, and is bounded by $(N-1)^{-s} \sum_{k \geq 0} (N-3)^{-sk}$, which tends to 0 as $N \to \infty$. At $N = 4$ we have $\sum_k (2k+1)^{-s} = (1 - 2^{-s})\zeta(s)$. Finally, for $g = 0$ we have $Z_N \geq (N-1)^2 e^{-2A}$, for $g = 1$ we have $Z_N = \sum_k e^{-Ak(k+1)}$, and for $g \geq 2$ we have $1 \leq Z_N \leq \zeta_N(2g-2)$, which tends to 1. □

This is to be compared with the large N results for U_N and SU_N, where the limit of the partition function was computed by Lemoine [64], in terms of the Witten zeta function, the Jacobi theta function and the Euler function, and where the Wilson loops were shown by Dahlqvist and Lemoine [36, 37] to converge to the planar master field. Understanding the analogues of these results for S_N^+, where the large N limit of the partition function is trivial in genus $g \geq 2$, but where the Wilson loops should remain interesting, is a natural question.

On the lattice, in the sense of Wilson [88], one can try to use magic unitaries on the links. By Proposition 8.1, commuting link algebras, each generating the whole coefficient algebra, would force the quantum group to be classical, which suggests using free products of link algebras. The Haar state then integrates the link variables via the Weingarten formula, with noncrossing partitions [10]. For S_4^+ the full Pauli model is available, while at 9 points the Weyl models from section 5 factor through G_3, whose Haar measure differs from that of S_9^+, already at the level of the third moment of the magic character, which is 6 for G_3, and 5 for S_9^+, so that integrating over unrestricted S_9^+ links requires its own Weingarten formula. However, a lattice theory would also need edge reversal, vertex gauge transformations, an action or a measure, and a prescription for the Wilson loops. For a general framework for lattice gauge fields in noncommutative geometry, see the gauge networks of Marcolli and van Suijlekom [68].

Finally, regarding large N, recall that in the limit of 't Hooft [82] the planar diagrams dominate, and that the corresponding $1/N$ expansions of matrix models are now well understood, see for instance [17]. Planar diagrams are noncrossing, and so are governed by free probability, and in 2 dimensions, for the master field on the plane, the holonomy along a simple loop of area t is distributed as the free unitary Brownian motion of Biane [15] at time t, see [4, 65]. On the other hand, the representation theory of $S_{N^2}^+$ is governed by the noncrossing partitions at any N, and the Weingarten formula for $S_{N^2}^+$ computes integrals as sums over noncrossing partitions with exact N-dependence, while the corresponding formulae for SU_N only become noncrossing as $N \to \infty$. However, the noncrossing partitions appearing in the intertwiner spaces and the planar ribbon graphs of the gauge theory expansions count different objects, as already shown at finite rank by the different cubic multiplicities in Proposition 2.3, and relating them, if possible, is part of the program.

Let us also record what happens in the 2 cases of interest for the Standard Model. In the weak case $N = 4$ the dimensions are $d_k = 2k + 1$, and the S_4^+ partition function coincides with the partition function of the SO_3 theory, as it should, because S_4^+ is a twist of SO_3. In the color case $N = 9$ the dimensions are the Fibonacci numbers F_{4k+2}, from Proposition 7.1, this

concerning S_9^+ and its twist PU_3^+, while the color group PU_3 of the real case, and its twist G_3, have the representation theory of PU_3. The zeta function becomes then a Fibonacci zeta function, as follows:

$$\zeta_9(s)=\sum_{k\geq 0}\frac{1}{F_{4k+2}^s}=1+\frac{1}{8^s}+\frac{1}{55^s}+\frac{1}{377^s}+\ldots$$

Since $F_{4k+2}\sim\varphi^{4k+2}/\sqrt{5}$, with φ being the golden ratio, this series converges for any $s>0$, and is dominated by its first terms. In particular, in genus $g\geq 2$ the color partition function at $A=0$ is very close to 1, the correction coming from the 8-dimensional representation, of order 8^{2-2g}. This is in contrast with the SU_3 theory, where the dimensions grow only polynomially, the Witten zeta function being of number theoretic nature.

Let us point out, however, that the above Casimir weights, coming from the SO_3 case, do not come from a positive heat kernel on S_N^+ at $N\geq 5$, and so cannot be used as such for the Yang–Mills measure. To be more precise, we have the following result:

Theorem 10.2. *For $N\geq 5$, the linear functionals φ_t on the dense Hopf $*$-algebra of $C(S_N^+)$ given by $\varphi_t(u_{ij}^{(k)})=\delta_{ij}e^{-tk(k+1)}$, where $u^{(k)}$ are the irreducible corepresentations, are not positive, for $t>0$ small.*

Proof. Let χ be the character of r_1, which is self-adjoint, with $\chi^2=1+\chi+\chi_2$, and set $d=N-1$, so that $\dim r_2=d^2-d-1$. If φ_t is a state, the variance of χ must be ≥ 0, and this variance is given by:

$$V(t)=1+de^{-2t}+(d^2-d-1)e^{-6t}-d^2e^{-4t}$$

But we have $V(0)=0$ and $V'(0)=-2d-6(d^2-d-1)+4d^2=-2N(N-4)$, so $V(t)<0$ for $t>0$ small, when $N\geq 5$. We have checked this with the program `stoichkov.py`, which also computes the large N behavior from Theorem 10.1. □

The problem is the quadratic growth of the Casimir weights. In fact, the positive central heat laws on S_N^+ have rates growing at most linearly, as follows, with D_k being the polynomial in N giving d_k, so that $D_0=1$, $D_1=N-1$ and $D_{k+1}=(N-2)D_k-D_{k-1}$:

Theorem 10.3. *For $N\geq 5$, the rates of a central convolution semigroup of states on $C(S_N^+)$, given by $\varphi_t(u_{ij}^{(k)})=\delta_{ij}e^{-tc_k}$, satisfy $0\leq c_k\leq(N-1)c_1D_k'(N)/D_k(N)$. Thus, if $c_1>0$, the sphere sum $\sum_k d_k^2e^{-Ac_k}$ diverges for $0<A\leq 2\eta\sqrt{N(N-4)}/((N-1)c_1)$, where $\eta=arccosh((N-2)/2)$.*

Proof. Let X be the character of the magic representation, which is a sum of N projections, so that $0\leq X\leq N$, and $\chi_k=D_k(X)$, by the fusion rules. With T_j being the Chebyshev polynomials, we have $D_k(x)=1+2\sum_{j=1}^kT_j((x-2)/2)$, and since $|T_j'|\leq j^2=T_j'(1)$ on $[-1,1]$, and T_j' is positive and increasing on $[1,\infty)$, we obtain $|D_k'(x)|\leq D_k'(N)$ and $D_k(x)\leq D_k(N)$ for $x\in[0,N]$. By functional calculus, we obtain:

$$0\leq 1-\frac{D_k(X)}{D_k(N)}\leq(N-X)\frac{D_k'(N)}{D_k(N)}$$

Now by applying φ_t, and using $\varphi_t(D_k(X))=d_ke^{-tc_k}$ and $\varphi_t(N-X)=(N-1)(1-e^{-tc_1})$, then dividing by t, and letting $t\to 0$, we obtain the estimate. Finally, we have $D_k(N)=\sinh((k+1/2)\eta)/\sinh(\eta/2)$, so $\log D_k(N)=k\eta+O(1)$ and $D_k'(N)/D_k(N)=k/\sqrt{N(N-4)}+O(1)$, which gives the divergence. We have checked all this with the program `ribery.py`. □

According to Franz, Kula and Skalski [47, Theorem 10.10], the following rates, which attain the above bound with $c_1 = 2$, come from a positive central convolution semigroup:

$$c_k = 2(N-1)\frac{D_k'(N)}{D_k(N)}$$

At $N = 4$ these rates are exactly $k(k+1)$, as checked by `ribery.py`. Thus a positive replacement of the Casimir weights exists, the price to pay being the divergence of the sphere sum at small area, the surface amplitudes and their gluing remaining to be defined. For these rates, logarithmic differentiation of the formula of $D_k(N)$ gives, with $\Delta = \sqrt{N(N-4)}$:

$$\Delta\frac{D_k'(N)}{D_k(N)} = \left(k+\frac{1}{2}\right)\coth\left(\left(k+\frac{1}{2}\right)\eta\right) - \frac{1}{2}\coth\left(\frac{\eta}{2}\right)$$

Thus, at the threshold $A_* = 2\eta\Delta/((N-1)c_1)$, the summands of the sphere sum tend to $\exp(\eta\sqrt{N/(N-4)})/(N-4)$, while for $A > A_*$ they decay exponentially, so the threshold in Theorem 10.3 is sharp, as checked with exact integer arithmetic by `ribery.py`. For related work on these processes, see Freslon, Teyssier and Wang [48].

11. Further questions and outlook

We have seen in this paper that the complex algebra B_F generated by the Chamseddine–Connes algebra is a twisted version of 15 points, and that the color symmetry of its finite geometry is a twist of a quantum permutation group on 9 points, namely S_9^+ in the complex setting, and the quantum group G_3 preserving the oriented areas in the real setting. With the trace coming from the fermions, the coaction preserves the weak and color summands separately, and the untwisting turns the matrix algebras into algebras of functions on points, the flip into a graded flip, for G_3 in the color case, and the Lie algebras gl_2, gl_3 into color Lie algebras.

All this raises a number of questions, and we first list 3 of them, which are quite concrete, and can be investigated with the methods of the present paper:

(1) Can one construct a real spectral triple of KO-dimension 6, satisfying the first order condition, with algebra $\mathbb{C}\oplus\mathbb{H}\oplus\mathbb{C}^9$, by changing the module data of F, for instance via an independent representation, or a module category construction, and how would it be related to F? Observe that the monoidal transport of the Heisenberg action and the commuting lifts are excluded, as explained in section 6, and that, by Proposition 6.4, the usual quark module with connected CKM mixing allows at most 3 color points.

(2) What are the intermediate quantum groups in Theorem 5.4, in the composite exponent case? Also, since the incidence graph of the Hesse configuration has only classical symmetries, by Theorem 4.8, and adding intertwiners can only shrink this group, which replacements for the ordinary incidence, such as phase-valued tensors on the 9 points, lead to genuinely quantum groups, see [46] for the quantum symmetries of hypergraphs? And since G_3 cannot act on the 4 parallel classes of lines, by Proposition 9.5, is there a weaker structure, such as a quantum space of rays, on which the color and weak quantum groups can meet?

(3) What happens in the extensions of the Standard Model with a larger algebra, such as the Pati–Salam model [75], with algebra $\mathbb{H}\oplus\mathbb{H}\oplus M_4(\mathbb{C})$ in the spectral version [25], or the algebra $M_2(\mathbb{H})\oplus M_4(\mathbb{C})$ from [22]? Note that at $N = 4$ there are 2 choices for H, namely $\mathbb{Z}_4$ and $\mathbb{Z}_2^2$, leading to different 16-point geometries, distinguished by maximality, by Theorem 5.4, and a selection criterion between them should come from the fermions, the real structure, or the Dirac operator. Also, how does this relate to the twisted spectral triple approach to the breaking of the grand symmetry from [40]?

Regarding the first question, Proposition 6.4 shows that, as long as the usual quark module, the first order condition and the observed connected CKM mixing are kept, at most 3 color points can act, the bound being attained by the diagonal color algebra. Thus a faithful 9-point replacement must change at least one of the module, support, mass or first order hypotheses of Proposition 6.4, and with the measured masses and mixing fixed, a change in the module data or in the axioms, such as the twisted first order condition from [40], is the natural question, the necessary dimension of a replacement module being open. Regarding the second question, Theorems 4.7 and 4.8 show that the quantum symmetries of the 9 points are carried by the orientation of the areas, so the natural candidates are the replacements of the ordinary incidence by structures with phases. As for the third question, the case $N = 4$ is the first one where 2 different groups H appear, and is a natural test case for the role of the fermions in the choice of H.

Besides the above concrete questions, there are a number of more advanced questions, regarding the passage to the spacetime manifold, the gauge theory, the renormalization and the mixing matrices, as follows:

(1) The passage from the finite geometry to the full Standard Model, on $M \times F$, raises the question of local quantum symmetries, discussed in [13], which is open. By [51], for a smooth action preserving the manifold factor, the action on this factor is classical, although this alone does not classify the actions on the product. Related questions are those of the spectral action prescription from section 8, beyond the classical 1-forms, and of making sense of the untwisted Yang–Mills theory, with the color Lie algebra from Proposition 7.3, and the graded flip from Proposition 7.2.

(2) Develop the Yang–Mills program from section 10: the positive replacements of the Casimir weights, from [47], and their surface amplitudes, the S_N^+ Wilson loops in 2 dimensions and their large N behavior, the lattice theory with magic unitaries on the links, and the relation between the noncrossing combinatorics of $S_{N^2}^+$ and the planar limit of 't Hooft.

(3) Regarding renormalization, recall that this is governed by the Hopf algebra of Feynman graphs of Connes and Kreimer [31, 32], with the Ward–Takahashi identities of QED being implemented as relations in this Hopf algebra, as shown by van Suijlekom [86], and with the underlying universal symmetry being the cosmic Galois group $U^* = U \rtimes \mathbb{G}_m$ of Connes and Marcolli [34], whose normal part U is pro-unipotent, and so of Lie type. A quantum version of U, which would provide a link with the present framework, would require a theory of quantum groups beyond the locally compact setting of Kustermans and Vaes [61].

(4) Regarding the mixing matrices, what is the quantum group invariant measuring the deviation from the Hadamard regime, refining the classical quantity δ from section 9? Also, what is the role of the quantum subgroups of S_4^+, discussed in section 9, in connection with the flavor groups?

Let us also state clearly what the present paper does not do. The algebra A_F, its fermion representation, and the mass and mixing data are inputs here, and rewriting them in terms of quantum permutations cannot by itself predict the number of colors or generations, the coupling constants, or the mixing angles. Also, we have no action principle for untwisted quarks, and no new observable or phenomenological fit. What we have is a dictionary, which transports the algebraic structures of the finite geometry to the side of the quantum permutations, and which identifies precisely where this transport works, as for the algebra, its traces and the gauge fields, and where it stops, as for the individual quark triplets, the unoriented areas, or the spectral action with coefficientwise reality.

The main new results here are Theorems 3.4, 4.3, 4.5, 4.7, 4.8, 5.3, 5.4, 8.2, 10.2, 10.3 and Propositions 4.6, 5.2, 6.2, 6.4, 8.3, 9.5, the other statements being either known, as indicated in the text, or of a more heuristic nature.

From a mathematical point of view, the main phenomenon found in this paper is the contrast between the oriented and unoriented area tensors. On the same 9 points, the complex oriented area tensor has an infinite quantum symmetry group, with 216 classical points, while its real part has only classical symmetries, forming a group of order 432, the mechanism being that the adjoint of an intertwining relation reverses the order of the products, while the entrywise conjugation does not. We believe that this phenomenon, where a finite geometry decorated with phases has genuinely quantum symmetries, which disappear when the phases are forgotten, should appear in many other situations. On the physics side, the physical trace and the spectral action provide constraints which are invisible at the level of the fusion rules, the former depending on the fermion representation, and the latter on the reality condition, and we hope that such constraints will help in understanding the untwisting of the whole finite geometry.

These questions are of course not independent, and all depend on a good understanding of the untwisting procedure. As for the order in which all this should be done, QED comes before QCD, and the weak sector, which is classical in disguise, should be understood first, with the Pauli quantum group [10] and its quantum subgroups [7] as main tools, the color sector, which is genuinely free, coming next. Also, there is no point in formulating a gauge theory for objects which are still twisted, so the Yang–Mills theory and the renormalization should come at the end, while the mixing matrices, which provide a direct contact with the experimental data, should be investigated in parallel. Also, the questions from the first list above, which are finite dimensional, can be investigated by computer, with the methods of the present paper, while those from the second list will certainly require new tools. Thus, a natural order for this program is as follows:

Step 1. The electroweak sector, where the algebra is $\mathbb{C}\oplus\mathbb{C}\oplus M_2(\mathbb{C})$, corresponding to $1+1+4$ points, and where the untwisting is governed by S_4^+, the Pauli matrices and a graded flip given by signs. This is the simplest case, and the subgroups of S_4^+, which are explicitly known, should allow a detailed analysis, including the role of the circle surviving from the weak symmetry in the computation of Bhowmick, D'Andrea and Dabrowski, and of the Higgs field, as an inner fluctuation.

Step 2. The color sector, with the untwisting of $M_3(\mathbb{C})$ into 9 points, the quarks becoming modules over the twisted group algebra of $\mathbb{Z}_3^2$, and with the color Lie algebra from section 7 replacing gl_3. Here the main problem is that of formulating the fermionic part of the theory, compatibly with the real structure and with the first order condition, which requires changing the module data, the monoidal transport of the Heisenberg action and the commuting lifts being both excluded, as explained in section 6.

Step 3. The passage from the finite geometry F to the full product $M\times F$, with the inner fluctuations, the spectral action, and the gauge fields written in terms of the untwisted objects, and with the question of local quantum symmetries.

Step 4. The Yang–Mills theory, with positive replacements of the Casimir weights, and the renormalization, as discussed in section 10 and in the above questions, which should normally come at the end, once the whole Standard Model is untwisted.

Summarizing, the present paper is rather an outline of a program, and we intend to develop it in a series of forthcoming papers.

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