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VASCONCELOS' CONJECTURE ON THE MODULE OF DIFFERENTIALS

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Abstract. Let k be a field and let S be a local k-algebra essentially of finite type whose module of Kähler differentials $\Omega_{S/k}$ has finite projective dimension. A conjecture of Vasconcelos, in the form studied by Herzog and Briggs, asserts that S is then a complete intersection when $\operatorname{char} k = 0$. We give a complete proof, in particular one independent of the Eisenbud–Mazur conjecture, that S is a reduced complete intersection when $\operatorname{char} k = 0$, and, over an arbitrary field, when some minimal prime of S is a generic point at which S is a field separably generated over k; in the latter case $\operatorname{pd}_S \Omega_{S/k} \leq 1$ and S is essentially smooth at all of its generic points. In positive characteristic p, Vasconcelos' conjecture for reduced algebras holds for every such algebra exactly when $[k : k^p] \leq p$.

1. Introduction

Let k be a field and let S be a local ring essentially of finite type over k, that is, a localization of a finitely generated k-algebra at a prime ideal. Vasconcelos conjectured that the projective dimension $\operatorname{pd}_S \Omega_{S/k}$ of the module of Kähler differentials can only take the values 0, 1 and ∞, for affine algebras over an arbitrary field and leaving questions of separability aside [Vas78, §4, (C2)]. In the form studied by Herzog [Her81, §1, (C2)] and Briggs [Bri22], for $\operatorname{char} k = 0$, the conjecture asserts that

$$\text{if } \operatorname{pd}_S \Omega_{S/k} < \infty, \text{ then } S \text{ is a complete intersection.} \tag{C2}$$

See the introduction of [Bri22] for the history of this conjecture and of its companion [Vas78, §0, (C1)]: for an ideal I of finite projective dimension in a local ring R,

$$\text{if } \operatorname{pd}_{R/I}(I/I^2) < \infty, \text{ then } I \text{ is generated by a regular sequence.} \tag{C1}$$

In characteristic zero, the graded cases of both conjectures were settled by Avramov and Herzog, (C1) in [AH94, Thm. 3] and (C2) in [AH94, Thm. 1]. Briggs proved (C1) in full [Bri22, Thm. A]. For the module of differentials, Briggs proved the conjecture for algebras without non-trivial evolutions [Bri22, Thm. 3.4], so that in characteristic zero it would follow from the

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Eisenbud–Mazur conjecture on evolutions and symbolic squares. Since the Eisenbud–Mazur conjecture has been shown to fail [MX26], that reduction does not prove Herzog's (C2), and a different argument is needed.

In this paper we prove the conjecture for ordinary Kähler differentials in characteristic zero, and we determine exactly what survives in positive characteristic.

Throughout, $(R, \mathfrak{n}, \kappa)$ denotes a local ring R with maximal ideal $\mathfrak{n}$ and residue field κ, and $\kappa(\mathfrak{p})$ is the residue field of a ring at a prime ideal $\mathfrak{p}$. For a finite module M over R we write $\operatorname{pd}_R M$ for its projective dimension, $\mu_R(M)$ for its minimal number of generators and $\operatorname{Ass}_R M$ for its set of associated primes; $\operatorname{ht} I$ is the height of an ideal I, and $\operatorname{grade} I$ is the common length of the maximal R-regular sequences in a proper ideal I [Mat86, Thm. 16.7]. If R is Cohen–Macaulay, for instance regular, then [Mat86, Thms. 17.3 and 17.4]

$$\operatorname{grade} I = \operatorname{ht} I \quad \text{and} \quad \operatorname{depth} R_{\mathfrak{p}} = \operatorname{ht} \mathfrak{p} \quad \text{for every prime ideal } \mathfrak{p}.$$

A ring is *non-trivial* if it is not the zero ring. A *generic point* of a ring is a minimal prime ideal of it.

In positive characteristic p we write $\mathbf{F}_p$ for the prime field and k^p for the subfield of p-th powers of a field k.

A local k-algebra is *essentially smooth* if it is a localization of a smooth k-algebra, and a local k-algebra essentially of finite type is called a *complete intersection over k* if it is of the form $R/(g_1, \dots, g_c)$ with R a localization of a polynomial ring over k at a prime ideal and $g_1, \dots, g_c$ an R-regular sequence. Whether a given presentation $S = R/I$ with R regular has I generated by a regular sequence does not depend on the presentation [Mat86, Thm. 21.2(ii)].

Theorem 1.1. *Let k be a field of characteristic zero and let S be a non-trivial local k-algebra essentially of finite type. If $\operatorname{pd}_S \Omega_{S/k} < \infty$, then*

- *S is a reduced complete intersection over k, and*
- *$\operatorname{pd}_S \Omega_{S/k} \leq 1$.*

Reducedness is a conclusion here, not a hypothesis. In positive characteristic the conclusion can fail even for domains, and the correct statement involves the generic points of S.

Theorem 1.2. *Let k be any field and let S be a non-trivial local k-algebra essentially of finite type with $\operatorname{pd}_S \Omega_{S/k} < \infty$. Suppose that for some minimal prime η of S the local ring S_η is a field which is separably generated over k. Then*

- *S is a reduced complete intersection over k,*
- *the ring $S_{\eta'}$ is essentially smooth over k for every minimal prime η' of S, and*
- *$\operatorname{pd}_S \Omega_{S/k} \leq 1$.*

Corollary 1.3. *Let k be a perfect field and let S be a non-trivial local k-algebra essentially of finite type with $\operatorname{pd}_S \Omega_{S/k} < \infty$. If the local ring S_η is reduced for some minimal prime η of S, then*

- *S is a reduced complete intersection over k,*
- *the ring $S_{\eta'}$ is essentially smooth over k for every minimal prime η' of S, and*
- $\operatorname{pd}_S \Omega_{S/k} \leq 1$.

In particular every reduced such S is a complete intersection over k.

For reduced algebras the dividing line is the *imperfection degree* $[k : k^p]$ of the base field k.

Theorem 1.4. *Let k be a field of positive characteristic p. The following are equivalent.*

(1) $[k : k^p] \leq p$.
(2) *Every non-trivial reduced local k-algebra S essentially of finite type with $\operatorname{pd}_S \Omega_{S/k} < \infty$ is a complete intersection over k.*

Outline of the proof. The proof descends in three steps. Differentials occur only in the first step.

Step 1: *from differentials to a quotient of the conormal module.* Present S as R/I with R a localization of a polynomial ring in v variables over k. The conormal sequence

$$I/I^2 \xrightarrow{\delta} S \otimes_R \Omega_{R/k} \cong S^v \longrightarrow \Omega_{S/k} \longrightarrow 0$$

shows that the image $\delta(I/I^2)$ of δ, the module generated by the rows of the Jacobian matrix, is a quotient of I/I^2 with $\operatorname{pd}_S \delta(I/I^2) < \infty$ exactly when $\operatorname{pd}_S \Omega_{S/k} < \infty$. At a generic point of S whose local ring is a separably generated field, the Jacobian criterion says that $\delta(I/I^2)$ has rank at least $\operatorname{ht} I$; hence $\delta(I/I^2)$ needs at least $\operatorname{ht} I$ generators. The conjecture is thereby reduced to a statement in which differentials do not appear: a quotient of I/I^2 of finite projective dimension cannot have more than $\operatorname{ht} I$ generators, and if it has that many, then I is generated by a regular sequence.

To state it, let $(R, \mathfrak{n}, \kappa)$ be a local ring, $I \subsetneq R$ an ideal and $S = R/I$. Let

$$\sigma\colon I/I^2 \longrightarrow N$$

be an S-linear map into a finitely generated S-module, and for $g \in I$ write $[g]$ for the image of $\sigma(\bar{g})$ in $N/\mathfrak{n}N$, where $\bar{g}$ is the class of g in I/I^2. The statement does not require R to be regular: it holds whenever S has finite projective dimension over R, with the grade of I in place of its height.

Theorem 1.5. *Let $(R, \mathfrak{n}, \kappa)$ be a Noetherian local ring and $I \subsetneq R$ an ideal such that $S = R/I$ has finite projective dimension over R. Let $\sigma\colon I/I^2 \twoheadrightarrow N$ be a surjective S-linear map onto a finitely generated S-module with $\operatorname{pd}_S N < \infty$.*

(1) $\mu_S(N) \leq \operatorname{grade} I$.
(2) *If $\mu_S(N) \geq \operatorname{grade} I$, then equality holds and I is generated by an R-regular sequence whose images form a basis of $N/\mathfrak{n}N$.*

(3) *If* $\mu_S(N) = \operatorname{grade} I - 1$ *and* $\operatorname{depth} R_{\mathfrak{p}} = \operatorname{grade} I$ *for every associated prime* $\mathfrak{p}$ *of the* R*-module* S*, then* I *is generated by an* R*-regular sequence.*

If R is regular, then $\operatorname{pd}_R S < \infty$ by Serre's theorem [Stacks, Tag 00O7], $\operatorname{grade} I = \operatorname{ht}_R I$ and $\operatorname{depth} R_{\mathfrak{p}} = \operatorname{ht}_R \mathfrak{p}$; in parts (2) and (3) the ring S is then a complete intersection. Step 1 applies the theorem in this case to the image $N = \delta(I/I^2)$, with σ induced by δ. For $N = I/I^2$ and σ the identity one has

$$\mu_S(N) = \mu_R(I) \geq \operatorname{ht}_R I \geq \operatorname{grade} I,$$

so part (2) recovers the theorem of Briggs [Bri22, Thm. A], that is, (C1): if $\operatorname{pd}_R S < \infty$ and $\operatorname{pd}_S(I/I^2) < \infty$, then I is generated by a regular sequence. For regular R and a free module N of rank $\operatorname{ht}_R I$, part (2) is the classical criterion of Vasconcelos on free conormal summands [Vas68, §2, Prop.]. The Jacobian image is a priori neither free nor all of I/I^2, and this is why the general statement is needed.

Granting Theorem 1.5, part (2) shows that S is a complete intersection in the situation of Theorem 1.2. Only afterwards do we obtain injectivity of the conormal map, the bound $\operatorname{pd}_S \Omega_{S/k} \leq 1$, and reducedness together with smoothness at the generic points, all from one computation at the associated primes of S. There the module of differentials, having finite projective dimension over a ring of depth zero, is free of the same rank as at the given generic point. Hence the Jacobian matrix of the regular generators of I has an invertible maximal minor, S is smooth over k there, and a regular local ring of depth zero is a field. In characteristic zero the hypothesis on the generic point holds automatically: the module of differentials is free at every generic point, and in characteristic zero this already makes the local ring regular. Thus Theorem 1.1 is a corollary of Theorem 1.2. Over an imperfect field with $[k : k^p] \leq p$ the rank of N at a generic point may drop by one, and part (3) takes over. Over every field with $[k : k^p] > p$ a one-dimensional local domain with free differentials which is not a complete intersection is obtained from the monomial curve (t^3, t^4, t^5) by a purely inseparable change of coordinates (Proposition 7.2).

Step 2: *from a quotient of the conormal module to a change of rings.* To prove Theorem 1.5 one chooses an R-regular sequence $g_1, \ldots, g_\ell$ in I such that $[g_1], \ldots, [g_\ell]$ are linearly independent and which cannot be prolonged by an element of I with these two properties, and passes to

$$Q = R/(g_1, \ldots, g_\ell), \qquad J = I/(g_1, \ldots, g_\ell), \qquad Q/J = S.$$

The *residual ideal* J has grade

$$\operatorname{grade} J = \operatorname{grade} I - \ell.$$

If S has finite projective dimension over Q, two elementary facts finish the proof. First, an ideal of grade zero with a cyclic quotient of finite projective dimension is zero; so J contains a nonzerodivisor unless $J = 0$, and prime avoidance shows that $[g_1], \ldots, [g_\ell]$ span $N/\mathfrak{n}N$, for otherwise the sequence

could be prolonged. Second, an ideal of grade one with such a quotient is principal if Q has depth one at every associated prime of the quotient. The second fact rests on the factorization of ideals with finite free resolutions due to Buchsbaum and Eisenbud [BE74, Cor. 5.2]; this is the only result beyond the textbooks which is quoted in this paper, and it is needed only for part (3), that is, only for Theorem 1.4. Everything therefore depends on transferring finite projective dimension from R to Q.

Theorem 1.6 (change of rings along a regular sequence)**.** *Let $(R, \mathfrak{n}, \kappa)$ be a Noetherian local ring and $I \subsetneq R$ an ideal such that $S = R/I$ has finite projective dimension over R; this holds for every I when R is regular. Let $\sigma\colon I/I^2 \to N$ be an S-linear map into a finitely generated S-module with $\operatorname{pd}_S N < \infty$. If $g_1, \dots, g_t \in I$ is an R-regular sequence such that $[g_1], \dots, [g_t]$ are linearly independent over κ, where $[g]$ denotes the image of $\sigma(\bar{g})$ in $N/\mathfrak{n}N$, then*

$$\operatorname{pd}_{R/(g_1,\dots,g_t)} S \;\leq\; \operatorname{pd}_R S - t \;<\; \infty.$$

The regular sequence in this theorem is chosen in advance and is not extracted from homological data, and neither freeness of N nor injectivity of σ is assumed. This is the main technical result of the paper.

Step 3: *the change of rings.* Theorem 1.6 is proved by removing one regular element at a time, in the residue algebra $B = \kappa \otimes_R \mathcal{R}$ of a free DG algebra resolution $\mathcal{R}$ of S over R which has a variable e_j of degree one for each generator g_j of I. A finite free resolution of N, of length r, yields for every S-linear functional $\lambda\colon N \to \kappa$ a derivation D_λ of B of degree -1 with $D_\lambda(e_j) = \lambda\sigma(\bar{g}_j)$ whose commutator with the differential takes values in the ideal generated by the variables of degree less than r. This alone makes the commutator nilpotent in each degree, within a range depending only on r (Proposition 3.1); the construction originates in the proofs of [Bri22, Lem. 2.6 and Thm. 2.7]. A derivation D of this kind with $D(e_1) = 1$ forces the top group $\operatorname{Tor}^{R/(g_1)}_{\operatorname{pd}_R S}(S, \kappa)$ to vanish, so the projective dimension drops by one (Lemma 4.1). The derivations dual to the selected elements descend to the successive quotients, and the step can be repeated t times without changing N. The argument is self-contained; in particular, the Eisenbud–Mazur conjecture is not used.

Organization. The sections follow the three steps in reverse order, so that every result is proved before it is used.

Section 2 prepares Step 3. It fixes the DG algebra resolution $\mathcal{R}$ of S and its residue algebra B, and shows that a free DG module with triangular differential over $\mathcal{R}$ resolves its reduction (Lemma 2.1).

Section 3 carries out the first half of Step 3. It constructs the derivation D_λ of B from a finite free resolution of N and proves that its commutator with the differential is nilpotent within a range depending only on r (Proposition 3.1).

Section 4 completes Step 3. One regular element is removed in Lemma 4.1, and repeating this along the prescribed sequence proves Theorem 1.6.

Section 5 carries out Step 2. It treats the residual ideal in grades zero and one (Lemmas 5.1 and 5.2) and proves Theorem 1.5 with a maximal regular sequence whose images in $N/\mathfrak{n}N$ are linearly independent.

Section 6 carries out Step 1. It computes the rank of the Jacobian image at a generic point (Lemma 6.1) and proves Theorem 1.2, Theorem 1.1 and Corollary 1.3.

Section 7 carries out Step 1 in positive characteristic. Theorem 1.5(3) gives Corollary 7.1 and the implication (1)⇒(2) of Theorem 1.4; Proposition 7.2 gives the converse.

Declaration of generative AI: As the outcome of about three weeks of intensive discussion between the second author and a ChatGPT model, a proof was proposed which rests on two deep results, Briggs' theorem on radical elements of the homotopy Lie algebra [Bri22, Thm. 2.7] and a theorem of Briggs, Grifo and Pollitz on cohomological support varieties [BGP24]. A careful study of their proofs showed which parts of them the argument actually needs, and this led to the comparatively elementary proof given here, which uses neither result. The authors reviewed and edited the content as needed and take full responsibility for the content of the article.

2. Conventions and resolutions

This section fixes conventions and constructs the resolution of S in which the computations of Sections 3 and 4 take place. All rings are commutative and Noetherian. In a minimal free resolution of a finite module M over a local ring $(T,\mathfrak{m},\kappa)$ all differentials vanish after tensoring with κ; therefore, if $\operatorname{Tor}_q^T(M,\kappa)=0$ for some $q\geq 1$, then $\operatorname{pd}_T M\leq q-1$.

2.1. **DG algebras and modules.** Let $S=R/I$ be a quotient of a local ring $(R,\mathfrak{n},\kappa)$ and let $g\in I$. Finite projective dimension of S over the quotient $R/(g)$ will be detected on $\operatorname{Tor}^{R/(g)}(S,\kappa)$. To compare it with $\operatorname{Tor}^R(S,\kappa)$ we resolve S over R by an algebra in which g is the differential of a variable e; dividing out g and e then yields a resolution over $R/(g)$. Resolutions of this kind were introduced by Tate [Tat57, §2]. The multiplication is what makes derivations available, and a derivation of a free algebra is determined by its values on the variables.

We use homological grading, with differentials of degree -1. Every differential is denoted by ∂, the degree of a homogeneous element a by $|a|$, and the component of degree k of a graded object M by M_k; $H(M)$ is the homology of M in all degrees together. A DG algebra is strictly graded-commutative and concentrated in nonnegative degrees. A DG module P over a DG algebra $\mathcal{T}$ satisfies

$$\partial(ap)=\partial(a)p+(-1)^{|a|}a\,\partial(p),$$

and we put

$$pa=(-1)^{|a||p|}ap.$$

A *derivation of degree* -1 from $\mathcal{T}$ to P is a map β, linear over the ring $\mathcal{T}_0$, which lowers degrees by one and satisfies

$$\beta(ab) = \beta(a)b + (-1)^{|a|}a\,\beta(b).$$

For a ring T and a set Y of variables of positive degrees, $T[Y]$ denotes the free strictly graded-commutative T-algebra on Y: polynomial in the variables of even degree and exterior in those of odd degree. A *monomial* is a product of variables; the monomials form a T-basis of $T[Y]$. A derivation of degree -1 from $T[Y]$ to a graded module is determined by its values on Y, and these may be prescribed arbitrarily in the right degrees.

2.2. **A free resolution by a DG algebra.** Let $(R, \mathfrak{n}, \kappa)$ be a local ring, $I \subsetneq R$ an ideal with generators $g_1, \dots, g_n$, not necessarily minimal, and $S = R/I$. Let $\mathcal{R} = R[X]$ be a free DG R-algebra on variables $X = X_1 \sqcup X_2 \sqcup \cdots$, with X_k in degree k, such that

$$\begin{aligned} X_1 &= \{e_1, \dots, e_n\}, & \partial e_j &= g_j \quad (1 \le j \le n), \\ H_0(\mathcal{R}) &= S, & H_k(\mathcal{R}) &= 0 \quad (k \ge 1). \end{aligned} \tag{2.1}$$

Such an algebra exists [Tat57, Thm. 1]: start from the Koszul complex $R[X_1]$ and, for $k = 1, 2, \dots$ in turn, adjoin variables of degree $k+1$ whose differentials are cycles representing generators of the R-module H_k of the algebra built so far. The elements of degree at most k are unchanged by this step, so the homology in degrees below k is unchanged and H_k becomes zero. Each $\mathcal{R}_k$ is a free R-module and $\mathcal{R}_0 = R$, so $\mathcal{R}$ is a free resolution of S over R. We put

$$B = \kappa \otimes_R \mathcal{R} = \kappa[X], \qquad H_j(B) = \operatorname{Tor}_j^R(S, \kappa).$$

Then $B_0 = \kappa$, and every e_j is a cycle of B because $g_j \in \mathfrak{n}$. For $g \in I$ we write $\bar{g}$ for its class in I/I^2. When an S-linear map $\sigma\colon I/I^2 \to N$ into a finitely generated S-module is given, $[g]$ denotes the image of $\sigma(\bar{g})$ in $N/\mathfrak{n}N$, as in the Introduction.

A DG module as in the next lemma is built by adjoining free generators one degree at a time, and such a module cannot distinguish quasi-isomorphic algebras: this is the DG form of the fact that tensoring with a bounded below complex of free modules preserves quasi-isomorphisms. We use the lemma twice: to lift a finite resolution over S to a DG module over $\mathcal{R}$, and to compute Tor over a quotient of R by a regular element.

Lemma 2.1. *Let $\mathcal{T} \to U$ be a surjective quasi-isomorphism of DG algebras. Let M be a DG $\mathcal{T}$-module which is graded free on homogeneous basis elements of nonnegative degrees, such that the differential of every basis element is a $\mathcal{T}$-linear combination of basis elements of smaller degree. Then $M \to U \otimes_{\mathcal{T}} M$ is a surjective quasi-isomorphism.*

Proof. Let $\mathfrak{k} = \ker(\mathcal{T} \to U)$, an acyclic DG ideal. The kernel of $M \to U \otimes_{\mathcal{T}} M$ is the direct sum of the $\mathfrak{k}v$ over the basis elements v, and it suffices to show that it is acyclic. Let z be a homogeneous cycle in it, let q be the largest

degree of a basis element occurring in z, and write the corresponding part of z as $\sum_{|v|=q} a_v v$ with $a_v \in \mathfrak{k}$. Since ∂v involves only basis elements of degree less than q, the coefficient of v in $\partial z = 0$ is ∂a_v; hence $\partial a_v = 0$. Choose $b_v \in \mathfrak{k}$ with $\partial b_v = a_v$. Then $z - \partial\big(\sum_{|v|=q} b_v v\big)$ is a cycle in the kernel which involves only basis elements of degree less than q. As z involves finitely many basis elements, of nonnegative degrees, repeating this step exhibits z as a boundary. □

3. Derivations from a finite free resolution

This section constructs, from a finite free resolution of N, the derivations with which a regular element will be divided out in Section 4.

3.1. **Motivation.** Keep the notation of §2.2, let $\sigma\colon I/I^2 \to N$ be an S-linear map into a finitely generated S-module with $\operatorname{pd}_S N < \infty$, and suppose that $g = g_1$ is a nonzerodivisor of R with $[g] \neq 0$. Put $Q = R/(g)$. We want to show that $\operatorname{pd}_Q S < \infty$.

The prototype is the argument of Vasconcelos for a free module N [Vas68, §2, Prop.]. The homomorphism $S \to I/gI$, $1 \mapsto g$, is well defined. Since g is a nonzerodivisor on R and on I, a finite free resolution of I over R reduces modulo g to a free resolution of I/gI over Q, so

$$\operatorname{pd}_Q(I/gI) \leq \operatorname{pd}_R I = \operatorname{pd}_R S - 1.$$

If N is free, there is an S-linear map $N \to S$ taking $\sigma(\bar{g})$ to 1, and composing it with $I/gI \to I/I^2 \xrightarrow{\sigma} N$ splits the map $S \to I/gI$. Thus S is a direct summand of I/gI over Q, and $\operatorname{pd}_Q S \leq \operatorname{pd}_R S - 1$. When N is not free there is no such splitting: the only functionals at our disposal are the maps $\lambda\colon N \to \kappa$ with $\lambda\sigma(\bar{g}) = 1$. The construction of this section is what remains of the splitting over the residue field.

In terms of resolutions, we want to compare the homology of B, which is $\operatorname{Tor}^R(S, \kappa)$ and is bounded when $\operatorname{pd}_R S < \infty$, with the homology of $C = B/(e_1)$, which will turn out to be $\operatorname{Tor}^Q(S, \kappa)$ (Lemma 4.1). As a graded algebra B is the tensor product over κ of C and the exterior algebra on e_1, and the simplest way to pass from B to C is to contract against e_1: one looks for a derivation D of B of degree -1 with $D(e_1) = 1$. If D commuted with the differential in the graded sense, that is, if

$$\theta = \partial D + D\partial$$

vanished, then B would split as a complex into C and a shifted copy of C, and boundedness would pass from $H(B)$ to $H(C)$ at once. This is the chain-level form of the splitting above, and it is what happens when N is free. In general $\theta \neq 0$. The map which θ induces on $H(C)$ lowers degrees by two; it plays the role, on Tor, of the operators of degree two attached to the hypersurface $R/(g)$ by Gulliksen [Gul74, Thm. 3.1] and Eisenbud [Eis80, §1], for which finite projective dimension over $R/(g)$ is a nilpotence property. We

use this comparison only as a guide and do not need to make it precise. The task is to choose D in such a way that θ is visibly nilpotent in each degree.

The values $D(e_j)$ are forced on us only through a functional on the conormal module. Indeed, a variable x of degree two has $\partial x = \sum_{j=1}^n a_j e_j$ with $\sum_{j=1}^n a_j g_j = 0$, and for $\theta(x)$ to have no component in degree zero one needs

$$\sum_{j=1}^{n} \bar{a}_j D(e_j) = 0,$$

where $\bar{a}_j$ is the image of a_j in κ; so $g_j \mapsto D(e_j) \in \kappa$ should come from an S-linear map $I/I^2 \to \kappa$. Extending D to the higher variables of $\mathcal{R}$ means solving one equation per variable, and over κ these equations have no reason to be solvable. The idea, which goes back to the proofs of [Bri22, Lem. 2.6 and Thm. 2.7], is to solve them one level higher. If the functional factors as $\lambda\sigma$ through the module N, with a finite free resolution G of length r, one first lifts σ to a derivation β from $\mathcal{R}$ into a DG module P lifting G; there the equations are solvable because P is a resolution of N. One then composes with the projection of P onto κ given by λ. The only step which does not commute with differentials is this last projection, and it fails to commute only through basis elements of G of degree at least two, with defects of positive degree at most $r-1$. Thus θ takes values in the ideal generated by the variables of degree less than r, and on each of these variables a bounded number of applications of θ reaches degree zero. This is all that is needed for nilpotence.

3.2. **The construction.**

Proposition 3.1. *Let $(R, \mathfrak{n}, \kappa)$, $I = (g_1, \dots, g_n)$, $S = R/I$, $\mathcal{R}$ and B be as in §2.2. Let $\sigma\colon I/I^2 \to N$ be an S-linear map into an S-module N which has a resolution $G \to N$ by finitely generated free S-modules $G_0, \dots, G_r$. For every S-linear map $\lambda\colon N \to \kappa$ there is a derivation $D_\lambda\colon B \to B$ of degree -1 such that*

$$D_\lambda(e_j) = \lambda\sigma(\bar{g}_j) \qquad (1 \le j \le n) \tag{3.1}$$

and such that the commutator

$$\theta_\lambda = [\partial, D_\lambda] = \partial D_\lambda + D_\lambda \partial$$

satisfies

$$\theta_\lambda^u(B_{k+2u}) = 0 \tag{3.2}$$

for all integers $k \ge 0$ and $u \ge 1$ with

$$u > k\lfloor r/2 \rfloor.$$

Moreover θ_λ is a chain map and a derivation of B of degree -2, its image is contained in the ideal of B generated by the variables of degrees $1, \dots, r-1$, and $\theta_\lambda = 0$ if $r \le 1$.

Proof. Write $\rho\colon G_0 \to N$ for the augmentation and choose bases of the G_k.

Step 1: lifting the resolution. We construct a DG $\mathcal{R}$-module P, graded free on a finite basis $\Pi = \Pi_0 \sqcup \cdots \sqcup \Pi_r$ with Π_k in degree k and in bijection with the chosen basis of G_k, such that the differential of each element of Π_k involves only $\Pi_0, \dots, \Pi_{k-1}$ and such that $S \otimes_{\mathcal{R}} P = G$. Put $\partial p = 0$ for $p \in \Pi_0$. Suppose that the differential has been defined on $\Pi_0, \dots, \Pi_{k-1}$, so that the free module $P^{<k}$ on these elements is a DG module with $S \otimes_{\mathcal{R}} P^{<k} = G_{<k}$, the part of G in degrees less than k. By Lemma 2.1, applied to the augmentation $\mathcal{R} \to S$, the projection $P^{<k} \to G_{<k}$ is a surjective quasi-isomorphism. For $p \in \Pi_k$ with image $\bar{p} \in G_k$, the element $\partial\bar{p} \in G_{k-1}$ is a cycle. Lift it to an element of $P^{<k}$ of degree $k-1$; the differential of this lift lies in the acyclic kernel of the projection and is a cycle there, so after subtracting a primitive in the kernel we obtain a cycle $z_p \in P^{<k}$ lifting $\partial\bar{p}$. Put $\partial p = z_p$. This defines P. By Lemma 2.1 again, $\varphi\colon P \to G$ is a surjective quasi-isomorphism, hence so is

$$\varepsilon = \rho\varphi\colon P \longrightarrow N.$$

Only the basis of P is finite; P itself is not bounded.

Step 2: lifting the conormal map. We construct a derivation $\beta\colon \mathcal{R} \to P$ of degree -1 with

$$\partial\beta = -\beta\partial, \qquad \varepsilon\beta(e_j) = \sigma(\bar{g}_j) \quad (1 \le j \le n). \tag{3.3}$$

Choose $\beta(e_j) \in P_0$ with $\varepsilon\beta(e_j) = \sigma(\bar{g}_j)$. The first identity holds on e_j because $\partial P_0 = 0$ and β vanishes on $\mathcal{R}_0 = R$. Let $x \in X_k$ with $k \ge 2$, and suppose that β has been defined on the variables of degree less than k and satisfies $\partial\beta = -\beta\partial$ on the subalgebra they generate, which contains ∂x. Then $\partial\beta(\partial x) = -\beta(\partial^2 x) = 0$, so $\beta(\partial x)$ is a cycle of P of degree $k-2$. If $k \ge 3$ it is a boundary because $H_{k-2}(P) = 0$. If $k = 2$, write $\partial x = \sum_{j=1}^{n} a_j e_j$ with $a_j \in R$; then $\sum_{j=1}^{n} a_j g_j = 0$ and

$$\varepsilon\beta(\partial x) = \sum_{j=1}^{n} a_j \sigma(\bar{g}_j) = \sigma\Big(\overline{\textstyle\sum_{j=1}^{n} a_j g_j}\Big) = 0,$$

so $\beta(\partial x)$ is again a boundary, since ε induces $H_0(P) \cong N$. Choose $\beta(x)$ with $\partial\beta(x) = -\beta(\partial x)$. Since $\partial\beta + \beta\partial$ is additive, R-linear and satisfies the product rule $ab \mapsto (\partial\beta + \beta\partial)(a)\, b + a\, (\partial\beta + \beta\partial)(b)$, it vanishes on $\mathcal{R}$ once it vanishes on the variables. The map β does not depend on λ.

Step 3: projecting to κ. Put $\bar{P} = \kappa \otimes_R P$, a graded free B-module on Π, and $\bar{\beta} = \kappa \otimes_R \beta$. Let $\gamma_\lambda\colon \bar{P} \to B$ be the B-linear map of degree zero with

$$\gamma_\lambda(p) = \begin{cases} \lambda\rho(\bar{p}) & \text{if } p \in \Pi_0, \\ 0 & \text{if } p \in \Pi_k,\ k \ge 1, \end{cases} \qquad \text{and put} \qquad D_\lambda = \gamma_\lambda\bar{\beta}.$$

Then D_λ is a derivation of degree -1, and (3.1) follows from (3.3). By (3.3),

$$\theta_\lambda = \partial\gamma_\lambda\bar{\beta} + \gamma_\lambda\bar{\beta}\partial = \alpha_\lambda\bar{\beta}, \qquad \alpha_\lambda = \partial\gamma_\lambda - \gamma_\lambda\partial. \tag{3.4}$$

The *defect* α_λ satisfies $\alpha_\lambda(ap) = (-1)^{|a|} a\,\alpha_\lambda(p)$ for $a \in B$. It vanishes on Π_0. On $p \in \Pi_1$ its value is

$$\alpha_\lambda(p) = -\gamma_\lambda(\partial p) = -\lambda\rho(\partial\bar{p}) = 0,$$

because $\rho\partial = 0$ on G_1. On $p \in \Pi_j$ with $j \geq 2$ its value is an element $\alpha_\lambda(p) \in B_{j-1}$, of positive degree at most $r-1$. Let $\mathfrak{J} \subseteq B$ be the ideal generated by the variables of degrees $1, \dots, r-1$. A monomial of positive degree j contains a variable of degree at most j, so every homogeneous element of B of positive degree at most $r-1$ lies in $\mathfrak{J}$. Hence

$$\theta_\lambda(B) \subseteq \mathfrak{J}, \tag{3.5}$$

and $\theta_\lambda = 0$ if $r \leq 1$, in which case $\mathfrak{J} = 0$. As the commutator of two derivations of odd degree, θ_λ is a derivation of degree -2, and

$$\partial\theta_\lambda = \partial D_\lambda \partial = \theta_\lambda \partial.$$

Step 4: powers of θ_λ. If $r \leq 1$, then $\theta_\lambda = 0$ and (3.2) holds. Let $r \geq 2$ and put $h = \lfloor r/2 \rfloor \geq 1$. For a variable y of degree at most $r-1$, the element $\theta_\lambda^h(y)$ lies in $\mathfrak{J}$ by (3.5) and has degree

$$|y| - 2h \leq r - 1 - 2h \leq 0.$$

As $\mathfrak{J}$ has no nonzero elements of degree zero, $\theta_\lambda^h(y) = 0$. We claim that

$$\theta_\lambda^{hn+1}(B) \subseteq \mathfrak{J}^{\,n+1} \qquad (n \geq 0). \tag{3.6}$$

For $n = 0$ this is (3.5). Let $n \geq 1$ and $b \in B$, and write $\theta_\lambda(b) = \sum_y a_y y$, a finite sum over variables y of degrees $1, \dots, r-1$ with $a_y \in B$. Since θ_λ is a derivation of even degree,

$$\theta_\lambda^{hn}(a_y y) = \sum_{i=0}^{hn} \binom{hn}{i} \theta_\lambda^{hn-i}(a_y)\,\theta_\lambda^i(y).$$

The terms with $i \geq h$ vanish. For $i < h$ the factor $\theta_\lambda^i(y)$ lies in $\mathfrak{J}$, by (3.5) if $i \geq 1$, and

$$\theta_\lambda^{hn-i}(a_y) = \theta_\lambda^{h(n-1)+1}(\theta_\lambda^{h-1-i}(a_y)) \in \mathfrak{J}^{\,n}$$

by induction. Hence $\theta_\lambda^{hn+1}(b) \in \mathfrak{J}^{\,n+1}$, which proves (3.6). Finally, $\mathfrak{J}$ is generated in positive degrees, so $\mathfrak{J}^{\,k+1}$ has no nonzero elements of degree k. If $u > hk$, then $u \geq hk+1$, and (3.6) gives

$$\theta_\lambda^u(B_{k+2u}) \subseteq \left(\mathfrak{J}^{\,k+1}\right)_k = 0,$$

which proves (3.2). □

4. Change of rings

4.1. **One regular element.** The next lemma isolates what a derivation of degree -1 with nilpotent commutator gives. Its proof is the chain-level form of the argument by the change-of-rings exact sequence below. Let S be a quotient of a local ring $(T, \mathfrak{m}, \kappa)$ with $\operatorname{pd}_T S < \infty$, and let f be a nonzerodivisor of T which maps to zero in S. For $U = T/(f)$ there is an exact sequence

$$\cdots \to \operatorname{Tor}^T_k(S,\kappa) \to \operatorname{Tor}^U_k(S,\kappa) \xrightarrow{\chi} \operatorname{Tor}^U_{k-2}(S,\kappa) \to \operatorname{Tor}^T_{k-1}(S,\kappa) \to \cdots,$$

so χ is surjective onto $\operatorname{Tor}^U_k(S,\kappa)$ for $k \geq \operatorname{pd}_T S$, and the group $\operatorname{Tor}^U_{\operatorname{pd}_T S}(S,\kappa)$ vanishes as soon as some power of χ kills $\operatorname{Tor}^U_{\operatorname{pd}_T S+2u}(S,\kappa)$. This general sequence is not used below. In the proof its role is played by the short exact sequence of complexes given by multiplication by e, and the connecting map of that sequence is the map induced by $[\partial, D]$ for any derivation D with $D(e) = 1$.

Lemma 4.1. *Let $(T, \mathfrak{m}, \kappa)$ be a local ring, let S be a non-trivial quotient of T with $\operatorname{pd}_T S < \infty$, and let $f \in \ker(T \to S)$ be a nonzerodivisor of T. Let $\mathcal{T} = T[Y]$ be a free DG T-algebra with $H_0(\mathcal{T}) = S$ and $H_k(\mathcal{T}) = 0$ for $k \geq 1$, having a variable e of degree one with $\partial e = f$, and put $B = \kappa \otimes_T \mathcal{T}$.*

1. *The DG algebra $\mathcal{T}/(f,e)$ is a free DG algebra over $U = T/(f)$ on the variables $Y \setminus \{e\}$, and it is a free resolution of S over U. In particular $H(B/(e)) = \operatorname{Tor}^U(S,\kappa)$.*
2. *Suppose that D is a derivation of B of degree -1 with $D(e) = 1$ and that $\theta = [\partial, D]$ satisfies $\theta^u(B_{\operatorname{pd}_T S+2u}) = 0$ for some integer $u \geq 1$. Then*

$$\operatorname{pd}_{T/(f)} S \;\leq\; \operatorname{pd}_T S - 1.$$

Proof. (1) Let $\mathcal{K} = T[e] \subseteq \mathcal{T}$, the Koszul complex on f. Since f is a nonzerodivisor, $\mathcal{K} \to U$ is a surjective quasi-isomorphism. As a graded $\mathcal{K}$-module, $\mathcal{T}$ is free on the monomials in the variables other than e, and the differential of such a monomial of degree q is a $\mathcal{K}$-linear combination of monomials of degree less than q. By Lemma 2.1,

$$\mathcal{T} \longrightarrow U \otimes_{\mathcal{K}} \mathcal{T} = \mathcal{T}/(f,e)$$

is a quasi-isomorphism. The target is $U[Y \setminus \{e\}]$ as a graded algebra, so it is a complex of free U-modules resolving S, and tensoring it with κ gives $C := B/(e)$.

(2) We have $\operatorname{pd}_T S \geq 1$: the nonzerodivisor f annihilates the non-trivial ring S, so S is not free over T. By (1) and the remark at the beginning of Section 2 it suffices to prove $H_{\operatorname{pd}_T S}(C) = 0$. Recall that $H_k(B) = \operatorname{Tor}^T_k(S,\kappa) = 0$ for $k > \operatorname{pd}_T S$.

As a graded algebra B is free over the exterior algebra on the cycle e, the projection $B \to C$ has kernel eB, and $\partial(eb) = -e\,\partial b$. Hence left

multiplication by e gives a short exact sequence of complexes

$$0 \longrightarrow C[1] \xrightarrow{e} B \longrightarrow C \longrightarrow 0,$$

where $C[1]_k = C_{k-1}$ with differential $-\partial$. Its long exact homology sequence [Stacks, Tag 0111] contains the exact sequences

$$H_{k+2}(C) \xrightarrow{\tau} H_k(C) \longrightarrow H_{k+1}(B),$$

so τ is surjective for $k \geq \operatorname{pd}_T S$. Now

$$\theta(e) = \partial(1) + D(\partial e) = 0,$$

so the derivation θ preserves the ideal eB and induces a chain map $\bar{\theta}$ on C. To compute τ, let $\bar{c} \in C_{k+2}$ be a cycle, lift it to $c \in B_{k+2}$ and write $\partial c = eb$. Then $\tau[\bar{c}] = [\bar{b}]$, while $D(e) = 1$ gives

$$\theta c = \partial(Dc) + D(eb) = \partial(Dc) + b - eD(b),$$

that is,

$$\bar{b} = \bar{\theta}\,\bar{c} - \partial\,\overline{Dc} \qquad \text{in } C.$$

Thus τ is the map $H(\bar{\theta})$ induced by $\bar{\theta}$. Consequently

$$\tau^u = H(\bar{\theta}^{\,u})\colon H_{\operatorname{pd}_T S+2u}(C) \longrightarrow H_{\operatorname{pd}_T S}(C)$$

is surjective, as a composite of surjections. It is also zero, because $\bar{\theta}^{\,u}(C_{\operatorname{pd}_T S+2u})$ is the image of $\theta^u(B_{\operatorname{pd}_T S+2u}) = 0$. Hence $H_{\operatorname{pd}_T S}(C) = 0$. □

Note that D is applied only to elements of B; it need not induce a map on $C = B/(e)$. The lemma holds in every characteristic and involves no completeness hypothesis.

4.2. **Proof of Theorem 1.6.** If $t = 0$ there is nothing to prove. Let $t \geq 1$ and extend the given sequence to a finite generating list $g_1, \dots, g_n$ of I. Form $\mathcal{R}$ and B as in §2.2. Since $[g_1], \dots, [g_t]$ are linearly independent in $N/\mathfrak{n}N$, there are S-linear maps $\lambda_i\colon N \to N/\mathfrak{n}N \to \kappa$ with

$$\lambda_i \sigma(\bar{g}_j) = \delta_{ij} \qquad (1 \leq i, j \leq t).$$

Let r be the length of a finite free resolution of N over S. Proposition 3.1 gives derivations $D_i = D_{\lambda_i}$ of B with $D_i(e_j) = \delta_{ij}$ for $i, j \leq t$, whose commutators $\theta_i = [\partial, D_i]$ satisfy $\theta_i^u(B_{k+2u}) = 0$ whenever $u > k\lfloor r/2 \rfloor$.

For $0 \leq j \leq t$ put

$$Q_j = R/(g_1, \dots, g_j), \qquad \mathcal{R}^{(j)} = \mathcal{R}/(g_1, \dots, g_j, e_1, \dots, e_j).$$

We prove by induction on j that $\mathcal{R}^{(j)}$ is a free DG Q_j-algebra on the variables $X \setminus \{e_1, \dots, e_j\}$ which resolves S, and that

$$\operatorname{pd}_{Q_j} S \leq \operatorname{pd}_R S - j. \tag{4.1}$$

For $j = 0$ this is the choice of $\mathcal{R}$. Let $1 \leq j \leq t$ and assume the statement for $j - 1$. The image of g_j in Q_{j-1} is a nonzerodivisor which maps to zero in S, and it is the differential of the variable e_j of $\mathcal{R}^{(j-1)}$. Lemma 4.1(1), applied to $T = Q_{j-1}$, $\mathcal{T} = \mathcal{R}^{(j-1)}$ and this element, gives the assertion about

$\mathcal{R}^{(j)}$. The residue algebra of $\mathcal{R}^{(j-1)}$ is $B/(e_1,\dots,e_{j-1})$. The derivation D_j maps e_ib to $-e_iD_j(b)$ for $i<j$, since $D_j(e_i)=0$; hence it induces a derivation $\bar{D}_j$ of $B/(e_1,\dots,e_{j-1})$, still with $\bar{D}_j(e_j)=1$. Its commutator with the differential is the map induced by θ_j, so its u-th power vanishes in degree $k+2u$ whenever $u>k\lfloor r/2\rfloor$. Take $k=\operatorname{pd}_{Q_{j-1}}S$, which is finite by the induction hypothesis, and any u with $u>k\lfloor r/2\rfloor$. Lemma 4.1(2) gives

$$\operatorname{pd}_{Q_j}S\le\operatorname{pd}_{Q_{j-1}}S-1\le\operatorname{pd}_R S-j.$$

This completes the induction, and (4.1) for $j=t$ is the theorem. □

The module N and its resolution are never replaced by quotients in this induction; what changes is only the algebra on which the fixed derivations act. This is essential. An induction on t which replaces R, I and N by $R/(g_1)$, $I/(g_1)$ and $N/S\sigma(\bar{g}_1)$ does not work, because the last module may have infinite projective dimension. For instance, let

$$R=\kappa[x,y,z]_{(x,y,z)},\qquad I=(x^2,y^2),\qquad N=S/(z)\oplus S/(z+x),$$

with $\sigma\colon I/I^2\cong S^2\twoheadrightarrow N$ the coordinatewise quotient map. Then $\operatorname{pd}_S N=1$, since z and $z+x$ are nonzerodivisors on S, and for $g_1=x^2+y^2$ one finds $N/S\sigma(\bar{g}_1)\cong S/(z,x)$. This module has infinite projective dimension over S: because the annihilator of x in S is xS, the minimal free resolution of $S/(x)$ is infinite, and z is a nonzerodivisor on $S/(x)$.

5. Quotients of finite projective dimension

5.1. **The residual ideal.** Theorem 1.6 is used as follows. In the situation of Theorem 1.5, choose a regular sequence $g_1,\dots,g_\ell$ in I with independent images in $N/\mathfrak{n}N$, as long as possible, and pass to

$$Q=R/(g_1,\dots,g_\ell),\qquad J=I/(g_1,\dots,g_\ell),\qquad Q/J=S.$$

Then $\operatorname{pd}_Q(Q/J)<\infty$, and the grade of the residual ideal J is $\operatorname{grade}I-\ell$. The question what I looks like has become the question which ideals of small grade in a local ring have a cyclic quotient of finite projective dimension. In grade zero, that is, for ideals consisting of zerodivisors, the answer is that only the zero ideal does: an ideal with a finite free resolution has a well-defined rank, namely the alternating sum of the ranks of the free modules, and this rank is seen at every associated prime. In grade one the answer is that the ideal is principal, provided the ring has depth one at every associated prime of the quotient; this follows from the factorization theorem of Buchsbaum and Eisenbud [BE74, Cor. 5.2]. We state the grade-zero fact for modules, because the module form is used again in Section 6, at the associated primes of a complete intersection.

Lemma 5.1. *Let T be a local ring and M a finitely generated T-module with $\operatorname{pd}_T M<\infty$.*

(1) *The modules $M_\mathfrak{p}$ which are free over $T_\mathfrak{p}$, for prime ideals $\mathfrak{p}$ of T, all have the same rank, and $M_\mathfrak{p}$ is free for every $\mathfrak{p}\in\operatorname{Ass}T$.*

(2) *If $M_{\mathfrak{p}_0}$ is free of rank s for some prime ideal $\mathfrak{p}_0$ of T, then every surjection $\pi\colon T^s \twoheadrightarrow M$ is an isomorphism.*
(3) *If $M = T/J$ for an ideal $J \subsetneq T$ which is contained in an associated prime of T, then $J = 0$.*

Proof. (1) Let E be the alternating sum of the ranks of the modules in a finite free resolution of M. For a prime $\mathfrak{p}$ such that $M_{\mathfrak{p}}$ is free, the localized resolution is a split exact sequence of free modules, so $M_{\mathfrak{p}}$ has rank E. For $\mathfrak{p} \in \operatorname{Ass} T$ the ring $T_{\mathfrak{p}}$ has depth zero, so $M_{\mathfrak{p}}$ is free by the Auslander–Buchsbaum formula [Stacks, Tag 090V].

(2) The case $s = 0$ is trivial. By (1), $M_{\mathfrak{p}}$ is free of rank s for every $\mathfrak{p} \in \operatorname{Ass} T$, so $\pi_{\mathfrak{p}}$ is a surjection between free modules of the same finite rank over a local ring, and therefore an isomorphism. The kernel of π is a submodule of T^s which vanishes at every associated prime of T^s. Its associated primes are among those of T^s [Stacks, Tag 02M3], so it has none, and it is zero [Stacks, Tag 0587].

(3) If $J \subseteq \mathfrak{p}_0 \in \operatorname{Ass} T$, then $(T/J)_{\mathfrak{p}_0}$ is nonzero and cyclic, and it is free by (1), hence free of rank one. Part (2), applied to $T \twoheadrightarrow T/J$, gives $J = 0$. □

Equivalently, a nonzero proper ideal with a cyclic quotient of finite projective dimension contains a nonzerodivisor; this ideal form is classical.

Lemma 5.2. *Let Q be a local ring and J an ideal with $\operatorname{pd}_Q(Q/J) < \infty$. If $\operatorname{depth} Q_{\mathfrak{p}} = 1$ for every associated prime $\mathfrak{p}$ of Q/J, then $J = aQ$ for a nonzerodivisor a of Q.*

Proof. The ideal J is nonzero, since otherwise the associated primes of Q, at which Q has depth zero, would be associated to Q/J. It has a finite free resolution, so by [BE74, Cor. 5.2] we have $J = aJ'$ for a nonzerodivisor a and an ideal J' which is either Q or contains a Q-regular sequence of length two. Multiplication by a is an isomorphism of Q/J' onto $aQ/aJ' = aQ/J \subseteq Q/J$, hence

$$\operatorname{Ass}(Q/J') \subseteq \operatorname{Ass}(Q/J).$$

If $J' \neq Q$, choose $\mathfrak{p} \in \operatorname{Ass}(Q/J')$. Then $\mathfrak{p} \supseteq J'$ contains a Q-regular sequence of length two, which is also $Q_{\mathfrak{p}}$-regular, so $\operatorname{depth} Q_{\mathfrak{p}} \geq 2$, while $\operatorname{depth} Q_{\mathfrak{p}} = 1$ because $\mathfrak{p} \in \operatorname{Ass}(Q/J)$. This contradiction shows $J' = Q$ and $J = aQ$. □

If Q is Cohen–Macaulay, the hypothesis of Lemma 5.2 says that every associated prime of Q/J has height one.

5.2. **Proof of Theorem 1.5.** Choose an R-regular sequence $g_1, \dots, g_\ell$ in I such that $[g_1], \dots, [g_\ell]$ are linearly independent in $N/\mathfrak{n}N$ and which is maximal with these two properties. It exists, since the empty sequence has them and linear independence bounds the length by $\mu_S(N)$. Let

$$Q = R/(g_1, \dots, g_\ell), \qquad J = I/(g_1, \dots, g_\ell).$$

Then $Q/J = S$ is non-trivial, and Theorem 1.6 gives $\operatorname{pd}_Q S < \infty$. If $J \neq 0$, then J is contained in no associated prime of Q by Lemma 5.1(3); by prime

avoidance [Stacks, Tag 00DS] it contains a nonzerodivisor of Q, whose lift to I prolongs $g_1, \dots, g_\ell$ to an R-regular sequence in I. Hence

$$\ell < \operatorname{grade} I \qquad \text{if } J \neq 0. \tag{5.1}$$

The images span. We show that $[g_1], \dots, [g_\ell]$ span $N/\mathfrak{n}N$. If $J = 0$, then $g_1, \dots, g_\ell$ generate I, and their images span because σ is surjective. Let $J \neq 0$, and suppose that the span W of $[g_1], \dots, [g_\ell]$ is a proper subspace. The set

$$I'' = \{g \in I : \ [g] \in W\}$$

is an ideal of R, and $I'' \neq I$ because $I \to N/\mathfrak{n}N$ is surjective. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ be the inverse images in R of the associated primes of Q; none of them contains I, since J is contained in no associated prime of Q. By prime avoidance [Stacks, Tag 00DS] there is $g_{\ell+1} \in I$ outside $I'' \cup \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_s$. Then $g_1, \dots, g_{\ell+1}$ is R-regular and $[g_1], \dots, [g_{\ell+1}]$ are linearly independent, contradicting maximality. Thus

$$\ell = \mu_S(N).$$

Parts (1) and (2). A regular sequence in I has length at most $\operatorname{grade} I$ [Mat86, Thm. 16.7], so $\mu_S(N) = \ell \leq \operatorname{grade} I$, which is (1). If $\mu_S(N) \geq \operatorname{grade} I$, then $\ell = \operatorname{grade} I$, and (5.1) gives $J = 0$. Hence $I = (g_1, \dots, g_\ell)$, and the ℓ independent elements $[g_i]$ form a basis of $N/\mathfrak{n}N$. This proves (2).

Part (3). Suppose $\mu_S(N) = \operatorname{grade} I - 1$ and $\operatorname{depth} R_\mathfrak{p} = \operatorname{grade} I$ for every $\mathfrak{p} \in \operatorname{Ass}_R S$. Then $\ell = \mu_S(N) = \operatorname{grade} I - 1$. The associated primes of the Q-module $Q/J = S$ are the ideals $\bar{\mathfrak{p}} = \mathfrak{p}/(g_1, \dots, g_\ell)$ with $\mathfrak{p} \in \operatorname{Ass}_R S$. For such $\mathfrak{p}$ the sequence $g_1, \dots, g_\ell$ is $R_\mathfrak{p}$-regular and $Q_{\bar{\mathfrak{p}}} = R_\mathfrak{p}/(g_1, \dots, g_\ell)$, so [Stacks, Tag 090R]

$$\operatorname{depth} Q_{\bar{\mathfrak{p}}} = \operatorname{depth} R_\mathfrak{p} - \ell = 1.$$

Lemma 5.2 gives $J = aQ$ with a a nonzerodivisor of Q. For a lift $g_{\ell+1} \in I$ of a, the sequence $g_1, \dots, g_\ell, g_{\ell+1}$ is R-regular and generates I. □

Only one regular sequence is chosen in this proof. The three parts correspond to three possibilities for the residual ideal: its grade cannot be negative, it vanishes in grade zero, and it is principal in grade one under the depth condition of part (3). When J has grade two or more there is no such conclusion: in Proposition 7.2 the ring R is regular, the residual ideal has height two and $N = 0$.

6. Differentials

6.1. **The Jacobian image.** Let k be a field and S a non-trivial local k-algebra essentially of finite type. Fix a polynomial ring $k[x_1, \dots, x_v]$, an ideal $\mathfrak{a}$ of it contained in a prime ideal $\mathfrak{q}$, and put $\Lambda = k[x_1, \dots, x_v]/\mathfrak{a}$, such that $S = \Lambda_{\mathfrak{q}/\mathfrak{a}}$. Let

$$R = k[x_1, \dots, x_v]_\mathfrak{q}, \qquad I = \mathfrak{a}R, \qquad S = R/I.$$

Then $(R, \mathfrak{n}, \kappa)$ is a regular local ring, so that Theorem 1.5 applies with $\operatorname{grade} I = \operatorname{ht}_R I$, and $\Omega_{R/k}$ is free on $\mathrm{d}x_1, \dots, \mathrm{d}x_v$. The conormal sequence of $k \to R \to S$ reads

$$I/I^2 \xrightarrow{\delta} S \otimes_R \Omega_{R/k} \cong S^v \longrightarrow \Omega_{S/k} \longrightarrow 0, \qquad \delta(\bar{f}) = \sum_{i=1}^{v} \frac{\partial f}{\partial x_i}\, \mathrm{d}x_i.$$

We call the image $\delta(I/I^2)$ of δ the *Jacobian image*, and we write σ for δ regarded as a map onto it; it plays the role of N in Theorem 1.5. Thus

$$\sigma\colon I/I^2 \twoheadrightarrow \delta(I/I^2), \qquad 0 \longrightarrow \delta(I/I^2) \longrightarrow S^v \longrightarrow \Omega_{S/k} \longrightarrow 0. \tag{6.1}$$

Since S^v is free, $\operatorname{pd}_S \delta(I/I^2) < \infty$ if and only if $\operatorname{pd}_S \Omega_{S/k} < \infty$. In that case $\delta(I/I^2)$ is a quotient of the conormal module with finite projective dimension, and Theorem 1.5 applies. The number $\mu_S(\delta(I/I^2))$ is the minimal number of generators of the module generated by the rows of the Jacobian matrix of a generating set of $\mathfrak{a}$; it is at least the rank of that matrix at the closed point. Theorem 1.5(1) says that

$$\mu_S(\delta(I/I^2)) \le \operatorname{ht} I,$$

and what has to be shown is that $\mu_S(\delta(I/I^2))$ is not smaller. Since the number of generators of a module can only drop under localization, it suffices to bound the rank of $\delta(I/I^2)$ from below at a generic point, where it is computed by the classical Jacobian criterion.

For a homomorphism of rings $A \to B$ we write $H_1(L_{B/A})$ for the first homology of the naive cotangent complex of B over A [Stacks, Tag 00S0]. It vanishes when B is a localization of a polynomial ring over A [Stacks, Tag 08Q1, Tag 00S7], it is $\mathfrak{b}/\mathfrak{b}^2$ when $A \to B$ is surjective with kernel $\mathfrak{b}$ [Stacks, Tag 07BP], and for a field extension K/k it vanishes if and only if K is separable over k [Stacks, Tag 0322]. For homomorphisms $A \to B \to C$ such that $\Omega_{B/A}$ is a free B-module there is the exact *Jacobi–Zariski sequence* [Stacks, Tag 00S2]

$$\begin{aligned} H_1(L_{B/A}) \otimes_B C \longrightarrow H_1(L_{C/A}) \longrightarrow H_1(L_{C/B}) \\ \longrightarrow \Omega_{B/A} \otimes_B C \longrightarrow \Omega_{C/A} \longrightarrow \Omega_{C/B} \longrightarrow 0. \end{aligned} \tag{6.2}$$

Lemma 6.1. *Let η be a minimal prime of S such that $K = S_\eta$ is a field, let $\mathfrak{p} \subseteq R$ be its inverse image. Then*

$$\dim_K \delta(I/I^2)_\eta = \operatorname{ht}_R \mathfrak{p} - \dim_K H_1(L_{K/k}),$$

and $H_1(L_{K/k}) = 0$ if K is separably generated over k. Consequently

$$\mu_S(\delta(I/I^2)) \;\ge\; \operatorname{ht}_R \mathfrak{p} - \dim_K H_1(L_{K/k}) \;\ge\; \operatorname{ht}_R I - \dim_K H_1(L_{K/k}).$$

Proof. Since $R_\mathfrak{p}/IR_\mathfrak{p} = S_\eta$ is a field, $IR_\mathfrak{p} = \mathfrak{p}R_\mathfrak{p}$, and the localization of δ at η is the map

$$\mathfrak{p}R_\mathfrak{p}/(\mathfrak{p}R_\mathfrak{p})^2 \longrightarrow \Omega_{R_\mathfrak{p}/k} \otimes_{R_\mathfrak{p}} K, \qquad f \longmapsto \mathrm{d}f, \tag{6.3}$$

with image $\delta(I/I^2)_\eta$. In the Jacobi–Zariski sequence (6.2) of $k \to R_\mathfrak{p} \to K$ the first term vanishes, because $R_\mathfrak{p}$ is a localization of a polynomial ring

over k, and $H_1(L_{K/R_\mathfrak{p}}) = \mathfrak{p}R_\mathfrak{p}/(\mathfrak{p}R_\mathfrak{p})^2$ maps to $\Omega_{R_\mathfrak{p}/k} \otimes_{R_\mathfrak{p}} K$ by (6.3). Hence $H_1(L_{K/k})$ is the kernel of (6.3). The ring $R_\mathfrak{p}$ is a regular local ring [Stacks, Tag 0AFS] of dimension $\operatorname{ht}_R \mathfrak{p}$, so the source of (6.3) has dimension $\operatorname{ht}_R \mathfrak{p}$, and $\dim_K \delta(I/I^2)_\eta = \operatorname{ht}_R \mathfrak{p} - \dim_K H_1(L_{K/k})$. If K is separably generated over k, then it is separable over k [Stacks, Tag 030X], so $H_1(L_{K/k}) = 0$. Finally $\mu_S(\delta(I/I^2)) \geq \dim_K \delta(I/I^2)_\eta$, and $\operatorname{ht}_R \mathfrak{p} \geq \operatorname{ht}_R I$ because $I \subseteq \mathfrak{p}$. □

The vector space $H_1(L_{K/k})$ depends only on the field extension K/k, and it vanishes exactly when K is separable over k [Stacks, Tag 0322]. Its dimension is what the Jacobian matrix fails to see in positive characteristic. Localizing (6.1) at η gives the first of the equalities

$$\dim_K \delta(I/I^2)_\eta = v - \dim_K \Omega_{K/k}, \qquad \operatorname{ht}_R \mathfrak{p} = v - \operatorname{trdeg}_k K,$$

and the second is the dimension formula for $k \to k[x_1, \dots, x_v]$, which is an equality because a field is universally catenary [Stacks, Tag 02IJ]. Comparing with Lemma 6.1,

$$\dim_K H_1(L_{K/k}) = \dim_K \Omega_{K/k} - \operatorname{trdeg}_k K,$$

which is Cartier's equality [Mat86, Thm. 26.10]; there it is stated over a perfect base field, here the prime field k_0 of k, with the kernel of $K \otimes_k \Omega_{k/k_0} \to \Omega_{K/k_0}$ in place of $H_1(L_{K/k})$; the two agree by the Jacobi–Zariski sequence of $k_0 \to k \to K$, since $H_1(L_{K/k_0}) = 0$. It is not used in this paper.

6.2. **Proof of Theorem 1.2.** Theorem 1.5 needs no injectivity of the conormal map, so it is applied first. Everything else then follows from one computation at the associated primes of S. Let η be the minimal prime of the theorem, let $K = S_\eta$ and $\mathfrak{p}$ be as in Lemma 6.1.

Complete intersection. By Lemma 6.1, where $H_1(L_{K/k}) = 0$ because K is separably generated,

$$\mu_S(\delta(I/I^2)) \geq \operatorname{ht}_R \mathfrak{p} \geq \operatorname{ht}_R I.$$

Theorem 1.5(2) gives $\mu_S(\delta(I/I^2)) = \operatorname{ht}_R I$ and shows that I is generated by an R-regular sequence of length $\operatorname{ht}_R I$; then $\operatorname{ht}_R \mathfrak{p} = \operatorname{ht}_R I$. Thus S is a complete intersection over k. Clearing denominators, which are units of R, we may take the generators $f_1, \dots, f_{\operatorname{ht}_R I}$ in $\mathfrak{a}$. Let

$$\Phi\colon S^{\operatorname{ht}_R I} \longrightarrow S^v$$

be the S-linear map sending the i-th basis vector to $\delta(\bar{f}_i)$. It is the composite of the surjection $S^{\operatorname{ht}_R I} \twoheadrightarrow I/I^2$ given by the $\bar{f}_i$ with δ, so its image is $\delta(I/I^2)$ and its cokernel is $\Omega_{S/k}$ by (6.1).

At the associated primes. By Lemma 6.1, $\dim_K \delta(I/I^2)_\eta = \operatorname{ht}_R \mathfrak{p} = \operatorname{ht}_R I$, so $(\Omega_{S/k})_\eta$ is free of rank $v - \operatorname{ht}_R I$ over K. Let $\mathfrak{r} \in \operatorname{Ass} S$. By Lemma 5.1(1), $(\Omega_{S/k})_\mathfrak{r}$ is free of rank $v - \operatorname{ht}_R I$. Hence the projection $S^v_\mathfrak{r} \to (\Omega_{S/k})_\mathfrak{r}$ splits, and its kernel $\delta(I/I^2)_\mathfrak{r}$ is free of rank $\operatorname{ht}_R I$. The map $\Phi_\mathfrak{r}$ maps $S^{\operatorname{ht}_R I}_\mathfrak{r}$ onto this kernel, so it is an isomorphism onto it, and $\Phi_\mathfrak{r}$ is a split injection. Its

reduction modulo $\mathfrak{r}S_{\mathfrak{r}}$ is therefore injective, so some maximal minor Δ of the Jacobian matrix $(\partial f_i/\partial x_j)$ is a unit in $S_{\mathfrak{r}}$ (take $\Delta = 1$ if $I = 0$). Let $\tilde{\mathfrak{r}} \subseteq \mathfrak{q}$ be the inverse image of $\mathfrak{r}$ in $k[x_1, \dots, x_v]$. After renumbering the variables, the k-algebra $k[x_1, \dots, x_v]_\Delta/(f_1, \dots, f_{\operatorname{ht}_R I})$ is standard smooth [Stacks, Tag 00T8], hence smooth [Stacks, Tag 00T7], and its local ring at $\tilde{\mathfrak{r}}$ is regular [Stacks, Tag 00TT]. This local ring is $S_{\mathfrak{r}}$, because $f_1, \dots, f_{\operatorname{ht}_R I}$ generate $\mathfrak{a}$ after localization at $\mathfrak{q}$, hence also after localization at $\tilde{\mathfrak{r}}$. A regular local ring is a domain [Stacks, Tag 00NP], and $S_{\mathfrak{r}}$ has depth zero, so $S_{\mathfrak{r}}$ is a field which is essentially smooth over k.

Conclusion. The kernel of Φ is a submodule of $S^{\operatorname{ht}_R I}$, and the nilradical of S is an ideal of S. Both vanish at every associated prime of S, the first because $\Phi_{\mathfrak{r}}$ is injective and the second because $S_{\mathfrak{r}}$ is a field. As in the proof of Lemma 5.1(2), both are zero [Stacks, Tag 02M3, Tag 0587]. Thus S is reduced, every associated prime of S is minimal, which gives the assertion on the minimal primes, and Φ is injective. Since Φ factors through the surjection $S^{\operatorname{ht}_R I} \twoheadrightarrow I/I^2$, both maps are isomorphisms, and (6.1) becomes the exact sequence

$$0 \longrightarrow I/I^2 \cong S^{\operatorname{ht}_R I} \xrightarrow{\delta} S^v \longrightarrow \Omega_{S/k} \longrightarrow 0. \tag{6.4}$$

Hence $\operatorname{pd}_S \Omega_{S/k} \leq 1$. No separability is used after the first step. □

6.3. **Characteristic zero and perfect fields.** In characteristic zero there is no hypothesis on generic points in Theorem 1.1, so one must show that the local rings of S at its generic points are fields. They have depth zero, so their modules of differentials are free, and in characteristic zero freeness of the module of differentials characterizes regularity.

Proof of Theorem 1.1. Let η be a minimal prime of S and $\tilde{\eta}$ its inverse image in Λ, so that $S_\eta = \Lambda_{\tilde{\eta}}$. The ring S_η has depth zero, so $(\Omega_{\Lambda/k})_{\tilde{\eta}} = (\Omega_{S/k})_\eta$ is free by the Auslander–Buchsbaum formula. Since $\operatorname{char} k = 0$, the local ring $\Lambda_{\tilde{\eta}}$ is therefore regular [Stacks, Tag 00TX], hence a domain [Stacks, Tag 00NP]. Being of dimension zero, it is a field. This field is finitely generated over k, hence separably generated since $\operatorname{char} k = 0$ [Stacks, Tag 0322, Tag 030O]. Apply Theorem 1.2. □

Proof of Corollary 1.3. Let η be a minimal prime with S_η reduced. A reduced local ring of dimension zero is a field, and S_η is finitely generated over k as a field. Every extension of a perfect field is separable [Stacks, Tag 030Y, Tag 030Z], and a finitely generated separable extension is separably generated [Stacks, Tag 030O]. Apply Theorem 1.2. □

Remark 6.2. The hypotheses of Theorem 1.2 and Corollary 1.3 cannot be dropped. For every field k of positive characteristic p the ring

$$S = \left(k[x, y]/(x^{2p}, x^p y^p, y^{2p})\right)_{(x,y)}$$

has $\Omega_{S/k} \cong S^2$, because the differentials of the three equations vanish, and it is not a complete intersection, because its defining ideal has height two and

three minimal generators. For $k = \mathbf{F}_p$ the residue field of its unique minimal prime η is separable over k; so the hypothesis of Theorem 1.2 concerns the local ring S_η and not only its residue field, and reducedness cannot be dropped in Corollary 1.3.

7. The imperfection-degree threshold

Theorems 1.1 and 1.2 settle the conjecture when some generic point is separably generated. It remains to determine what happens for reduced algebras over an imperfect field, where the Jacobian image can be too small.

7.1. **One element short.** When no generic point is separably generated, Lemma 6.1 still bounds $\mu_S(\delta(I/I^2))$ from below by $\operatorname{ht}_R I - \dim_K H_1(L_{K/k})$. If this dimension is at most one, the regular sequence of Theorem 1.5 is at most one element short, and Theorem 1.5(3) applies.

Corollary 7.1. *Let k be a field and let S be a non-trivial reduced local k-algebra essentially of finite type with $\operatorname{pd}_S \Omega_{S/k} < \infty$. If $\dim_{S_\eta} H_1(L_{S_\eta/k}) \le 1$ for every minimal prime η of S, then S is a complete intersection over k.*

Proof. Use the notation of §6.1. Since S is reduced, every S_η is a field and $\operatorname{Ass}_R S$ consists of the inverse images $\mathfrak{p}_\eta$ of the minimal primes η of S. Then $\operatorname{ht}_R \mathfrak{p}_\eta \ge \operatorname{ht}_R I$. Lemma 6.1 gives

$$\mu_S(\delta(I/I^2)) \;\ge\; \operatorname{ht}_R \mathfrak{p}_\eta - \dim_{S_\eta} H_1(L_{S_\eta/k}) \;\ge\; \operatorname{ht}_R I - 1 \qquad \text{for every } \eta.$$

If $\mu_S(\delta(I/I^2)) \ge \operatorname{ht}_R I$, Theorem 1.5(2) shows that S is a complete intersection. Otherwise $\mu_S(\delta(I/I^2)) = \operatorname{ht}_R I - 1$ and

$$\operatorname{ht}_R \mathfrak{p}_\eta \le \mu_S(\delta(I/I^2)) + 1 \le \operatorname{ht}_R I,$$

so $\operatorname{ht}_R \mathfrak{p}_\eta = \operatorname{ht}_R I$ for every η. As R is regular, this says that $\operatorname{depth} R_\mathfrak{p} = \operatorname{grade} I$ for every associated prime $\mathfrak{p}$ of the R-module S, and Theorem 1.5(3) applies. □

In positive characteristic p the dimension of $H_1(L_{K/k})$ is bounded by the size of a p-basis of k: the Jacobi–Zariski sequence of $\mathbf{F}_p \to k \to K$ embeds $H_1(L_{K/k})$ into $K \otimes_k \Omega_{k/\mathbf{F}_p}$, and $\Omega_{k/\mathbf{F}_p}$ has a basis indexed by a p-basis of k. This explains the form of Theorem 1.4: $H_1(L_{K/k})$ has dimension at most one for all extensions K/k exactly when $[k : k^p] \le p$.

Proof of Theorem 1.4, (1)⇒(2). Write $[k : k^p] = p^j$ with $j \le 1$, so that $\dim_k \Omega_{k/\mathbf{F}_p} = j$ by the description of differentials through p-bases [Stacks, Tag 07P2]. Let K/k be a field extension. Every extension of the perfect field $\mathbf{F}_p$ is separable [Stacks, Tag 030Y, Tag 030Z], so $H_1(L_{K/\mathbf{F}_p}) = 0$, and the Jacobi–Zariski sequence (6.2) of $\mathbf{F}_p \to k \to K$ gives an injection

$$H_1(L_{K/k}) \;\hookrightarrow\; K \otimes_k \Omega_{k/\mathbf{F}_p}.$$

Hence

$$\dim_K H_1(L_{K/k}) \le j \le 1.$$

Thus every reduced S as in (2) satisfies the hypothesis of Corollary 7.1, which proves (2). □

7.2. **A domain with free differentials.** For the converse we need, over every field with $[k : k^p] > p$, a reduced algebra with differentials of finite projective dimension which is not a complete intersection. The construction rests on one observation: in positive characteristic p a polynomial in p-th powers has zero differential, so equations in p-th powers do not affect the module of differentials. Over a perfect field such equations produce nilpotents, as in Remark 6.2. Over an imperfect field one can shift p-th powers by constants which are not p-th powers and obtain domains. Two p-independent constants suffice to transport the monomial curve (t^3, t^4, t^5), which is not a complete intersection; with only one, $H_1(L_{K/k})$ has dimension at most one and Corollary 7.1 applies.

Proposition 7.2. *Let k be a field of positive characteristic p containing two p-independent elements a, b, for instance $k = \mathbf{F}_p(a, b)$. In the polynomial ring $k[x, y, z]$ put*

$$\begin{aligned} &U = x^p - a, \qquad V = y^p - b, \qquad W = z^p - a, \\ &I = (V^2 - UW,\ VW - U^3,\ W^2 - U^2V), \end{aligned} \tag{7.1}$$

let $\mathfrak{m} = (U, V, z - x)$ and $R = k[x, y, z]_{\mathfrak{m}}$, $S = R/IR$. Then S is a one-dimensional local domain over k, it is not a complete intersection, and

$$\Omega_{S/k} \cong S^3, \qquad \operatorname{pd}_S \Omega_{S/k} = 0. \tag{7.2}$$

Proof. The monomial curve. For independent variables U, V, W the homomorphism

$$\begin{aligned} k[U, V, W]/(V^2 - UW,\ VW - U^3,\ W^2 - U^2V) &\longrightarrow k[t], \\ (U, V, W) &\longmapsto (t^3, t^4, t^5), \end{aligned} \tag{7.3}$$

is an isomorphism onto $\Gamma := k[t^3, t^4, t^5]$: the three relations reduce every monomial to a $k[U]$-linear combination of $1, V, W$, and the images of these normal forms have exponents in distinct residue classes modulo three, so they are linearly independent. This holds for every p, including $p = 2, 3$.

Primality and dimension. Since $k[U, V, W] = k[x^p, y^p, z^p]$, the ring $k[x, y, z]$ is free of rank p^3 over $k[U, V, W]$, and by (7.3)

$$k[x, y, z]/I \ \cong\ \Gamma[x, y, z]/(x^p - a - t^3,\ y^p - b - t^4,\ z^p - a - t^5),$$

which is free of rank p^3 over Γ because the three equations are monic in separate variables. In particular $\dim k[x, y, z]/I = 1$. Let $F = k(t) = \operatorname{Frac}\Gamma$. The generic fibre $F[x, y, z]/(x^p - (a + t^3),\ y^p - (b + t^4),\ z^p - (a + t^5))$ is a field if and only if $a + t^3$, $b + t^4$, $a + t^5$ are p-independent over F^p, that is, if and only if their differentials are linearly independent in $\Omega_{F/\mathbf{F}_p}$ [Stacks, Tag 07P2]. Since $k \to F$ is a purely transcendental, hence separable, extension, the map $F \otimes_k \Omega_{k/\mathbf{F}_p} \to \Omega_{F/\mathbf{F}_p}$ is injective and $\Omega_{F/\mathbf{F}_p} = (F \otimes_k \Omega_{k/\mathbf{F}_p}) \oplus F\,\mathrm{d}t$, so

$\mathrm{d}a, \mathrm{d}b, \mathrm{d}t$ are linearly independent. With respect to these three elements the differentials of $a+t^3$, $b+t^4$, $a+t^5$ have coefficient matrix

$$\begin{pmatrix} 1 & 0 & 3t^2 \\ 0 & 1 & 4t^3 \\ 1 & 0 & 5t^4 \end{pmatrix}.$$

Its determinant $5t^4 - 3t^2 = t^2(5t^2-3)$ is a nonzero polynomial in the transcendental element t in every characteristic (t^4+t^2 for $p=2$, $2t^4$ for $p=3$). Hence the generic fibre is a field of degree p^3 over F, and the free Γ-algebra $k[x,y,z]/I$ embeds into it, so I is a prime ideal of height two.

The maximal ideal. The residue ring $k[x,y,z]/(U,V,z-x)$ is the field $k(a^{1/p}, b^{1/p})$, so $\mathfrak{m}$ is maximal with $\operatorname{ht}\mathfrak{m} = 3$, and $U, V, \tau := z-x$ is a regular system of parameters of the regular local ring R. One has $W = (x+\tau)^p - a = U + \tau^p$, so $I \subseteq \mathfrak{m}^2$ and $S = R/IR$ is a one-dimensional local domain with $\operatorname{ht}_R IR = 2$.

Not a complete intersection. Let $\kappa = R/\mathfrak{m}R$ and write u^* for the initial form of $u \in R$ in the associated graded ring $\operatorname{gr}_{\mathfrak{m}} R = \kappa[U^*, V^*, \tau^*]$. The initial forms of the three generators are $V^{*2} - U^{*2}$, U^*V^* and U^{*2} (note $W^* = U^*$ since $\tau^p \in \mathfrak{m}^2$), which are linearly independent in $\mathfrak{m}^2/\mathfrak{m}^3$. Since $\mathfrak{m}IR \subseteq \mathfrak{m}^3$, the three generators are linearly independent modulo $\mathfrak{m}IR$, so $\mu_R(IR) = 3 > 2 = \operatorname{ht}_R IR$. An ideal of R generated by a regular sequence of length c has height c and is minimally generated by c elements, so IR is not generated by an R-regular sequence, and S is not a complete intersection [Mat86, Thm. 21.2(ii)].

Free differentials. Each generator of I is a polynomial in U, V, W, and $\mathrm{d}U = \mathrm{d}V = \mathrm{d}W = 0$ in $\Omega_{k[x,y,z]/k}$. Hence the conormal map $I/I^2 \to S \otimes_R \Omega_{R/k} = S^3$ is zero and $\Omega_{S/k} \cong S^3$, proving (7.2). □

Proof of Theorem 1.4, (2)⇒(1). If $[k : k^p] > p$, then k contains two p-independent elements a, b, and the ring S of Proposition 7.2 is a reduced local k-algebra essentially of finite type with free module of differentials which is not a complete intersection. Hence (2) fails. This includes fields of infinite imperfection degree. □

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