The ordered 2×2 corner pattern: a removal exponent of at least $9/2$, and other partial results

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Abstract

Let $P = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. An ordered induced copy of P in a binary matrix is given by rows $i < j$ and columns $c < d$ with entries $(1,0)$ in row i and $(1,1)$ in row j. Whether P admits a polynomial removal lemma is, to our knowledge, open; it is the last open 2×2 case. This note does not settle it and records partial results. Main result: for every $q \geq 2$ we construct a square binary matrix of side $n = 6q^3$, with exactly $n/6$ copies of P, each on a pair of adjacent rows, these pairs being disjoint, whose distance to P-freeness is exactly $(n/6)^{4/3} + n/6$ entry changes; consequently any removal exponent for P is at least $9/2$. Further results: (1) if the columns are split into m consecutive intervals on each of which the matrix is P-free, and any two rows can be made P-free by deleting at most K columns, then the $n \times n$ matrix is within $6m\,n^{3/2}\sqrt{K} + 3mn$ entry changes of a P-free matrix; (2) two explicit families show that no bound of the form CK per row, or CKn in total, holds with an absolute constant, even for two intervals; (3) for matrices made of two interleaved "staircase" classes we give a repair with at most $4K$ changes per row; (4) the block-repair lemma used by Lupini for the 2×2 identity has no direct analogue for P; (5) for matrices in which every copy uses two adjacent rows, we give two safe repair moves and a matrix with a single copy at linear distance from P-freeness. Statements are labelled [**proved**], [**computed**] or [**open**]. The results are established facts whose novelty has not been verified. The note was prepared with an AI model; the proofs were examined by independent AI reviews and by computation, and nothing has been checked by a third party.

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1 Setting

The general polynomial removal question for ordered binary matrices is surveyed in [2]; [1] proves it for finite families closed under row permutations, and a weak form (a small set of entries meeting all copies) for every single pattern.

Matrices are binary; rows are numbered from top to bottom and columns from left to right. A row is identified with the set of its columns containing a 1. For a matrix M, $D(M)$ is the minimum number of entry changes (in either direction) producing a P-free matrix.

Lemma 1. [**proved**] *A matrix is P-free if and only if, for all rows $x < y$, the set $x \cap y$ is a final segment of y (every column of y to the right of a common column is common). Equivalently: for every column c, the rows containing c, read from top to bottom and restricted to the columns to the right of c, are decreasing for inclusion.*

Proof. A copy is a pair of rows $x < y$ containing c together with a column $d > c$ in y and not in x. □

Definition 2. For rows $x < y$, $d(x, y)$ is the minimum number of columns whose deletion makes the pair P-free.

Lemma 3 (cut formula)**. [proved]** *For rows $x < y$ with a common columns,*

$$d(x,y) = \min_{\Pi}\Big(\#\{\text{common columns} \le \Pi\} + \#\{\text{columns of } y \text{ not in } x, \text{ to the right of } \Pi\}\Big),$$

the minimum being over cut positions Π. Moreover the set of common columns differs from the set of the last a ones of y in at most $2d(x,y)$ columns.

Proof. The copies of the pair are the edges of the bipartite graph joining a common column c to a column $d > c$ of $y \setminus x$; $d(x,y)$ is its minimum vertex cover. Given a cover, let Π be just to the left of the leftmost common column not in the cover: the cover contains all common columns left of Π and all columns of $y \setminus x$ right of Π. For an optimal cut, let e and h be the two numbers. To the right of Π, y has $a-e$ common columns and h others. If $h \le e$, the last a ones of y are these $a-e+h$ columns and $e-h$ more, and the symmetric difference has at most $2e$ elements. If $h > e$, the last a ones of y all lie to the right of Π and the difference has at most $2h$ elements. □

[computed] Both assertions were checked on 300 000 random pairs of rows.

2 Gluing consecutive intervals

Theorem 4 (gluing m intervals)**. [proved]** *Let M have n rows and W columns, the columns being partitioned into m consecutive intervals $I_1, \dots, I_m$ such that each restriction $M|_{I_j}$ is P-free. Suppose $d(x,y) \le K$ for all pairs of rows, with $K \ge 1$, and put $s = \lceil 2\sqrt{KW}\,\rceil$. Then M can be made P-free by deleting fewer than $3ms$ ones in each row. In particular*

$$D(M) < 3mns \le 6m\,n\sqrt{KW} + 3mn.$$

Proof. If $K > W$ then $s > 2W$ and deleting all ones costs at most $W < s$ per row; so assume $K \le W$. For a finite set E of columns let $\mathrm{tr}(E)$ be E without its s largest elements. Say that rows $x < y$ are *linked through* I_i if $\mathrm{tr}(x \cap I_i) \cap \mathrm{tr}(y \cap I_i) \neq \emptyset$. Since $M|_{I_i}$ is P-free, the common columns in I_i form a final segment of $y \cap I_i$; a common column that is not among the last s of $y \cap I_i$ therefore forces more than s common columns in I_i. Let $\mathrm{Anc}_{<j}(y)$ be the set of rows reached from y by going up through links of indices $< j$, step by step. Define

$$Q_y^{(j)} = (y \cap I_j) \cap \bigcap_{z \in \mathrm{Anc}_{<j}(y)} (z \cap I_j), \qquad N_y \cap I_j = \mathrm{tr}\big(Q_y^{(j)}\big).$$

N is P-free. Inside I_j: for z above y, $(y \cap I_j) \setminus z$ is an initial segment of $y \cap I_j$, so $Q_y^{(j)}$ is $y \cap I_j$ minus an initial segment. Removing initial segments keeps P-freeness: if $d < d'$ are kept in y and d is kept in x, then $d' \in x$ (final segment) and d' is to the right of d, hence kept in x. Removing the last s ones of every row also keeps P-freeness: if $c < d$ with c common, d kept in y and removed from x, then y has at least s ones after d, all in x, so d is not among the last s ones of x. Across intervals: if $x < y$ share a column of I_i in N, they are linked through I_i, because $\mathrm{tr}(Q) \subseteq \mathrm{tr}(E)$ whenever $Q \subseteq E$. For $j > i$ this gives $\mathrm{Anc}_{<j}(y) \supseteq \{x\} \cup \mathrm{Anc}_{<j}(x)$, hence $Q_y^{(j)} \subseteq Q_x^{(j)}$ and $\mathrm{tr}(Q_y^{(j)}) \subseteq \mathrm{tr}(Q_x^{(j)})$ (an element with s successors in $Q_y^{(j)}$ has s successors in $Q_x^{(j)}$). So no copy has its common column in I_i and its other column in I_j.

Cost in I_j. At most s for tr, plus $\max_z |(y \cap I_j) \setminus z|$ over $z \in \mathrm{Anc}_{<j}(y)$ (the sets are nested initial segments). Let $a_{<j}(x,y)$ be the number of common columns to the left of I_j and $p_j(x,y)$ the number of columns of $y \cap I_j$ not in x. Every column left of I_j precedes every column of I_j, so $\min(a_{<j}, p_j) \le d \le K$. Fix a path of links $y = z_0 > z_1 > \cdots > z_H = z$. Extract $u_0 = z_0$ and, while $u_i = z_t$ with $t < H$, let $u_{i+1} = z_{t'}$ where $t' > t$ is the largest index with $a_{<j}(z_{t'}, z_t) > K$; it exists since $a_{<j}(z_{t+1}, z_t) > s \ge K$. We reach $u_h = z$, and each hop has $p_j \le K$, so $|(y \cap I_j) \setminus z| \le hK$.

For $0 \le i \le h-2$ let S_i be the set of the last $s+1$ ones of u_{i+1} in the interval of its next link on the path. If $i' \ge i+2$ then $u_{i'+1}$ is higher on the path than u_{i+2}, so $a_{<j}(u_{i'+1}, u_{i+1}) \le K$ by the

choice of u_{i+2}, and $|S_i \cap S_{i'}| \le K$. Put $q = \lfloor s/K \rfloor$; since $W \ge K$ we have $s \ge 2K$ and $q \ge s/(2K)$. If $\lfloor h/2 \rfloor \ge q$, the sets $S_0, S_2, \dots, S_{2q-2}$ have a union of size at least

$$\sum_{t=0}^{q-1} (s + 1 - tK) > qs - \frac{Kq^2}{2} \ge \frac{qs}{2} \ge \frac{s^2}{4K} \ge W,$$

which is impossible. Hence $h \le 2q-1$ and $hK < 2s$. Altogether fewer than $ms + 2(m-1)s < 3ms$ deletions per row. □

[computed] The construction was programmed. On 741 891 random matrices with 2 to 5 P-free intervals the output is always P-free; the parameter s being drawn at random, in the 17 492 cases with $s \ge 2K$ and $s^2 \ge 4KW$ the cost bounds hold.

Remark 5. The counting step uses neither P-freeness nor the shape of the rows. **[proved]** In any matrix with W columns, if $z_0 > \cdots > z_H$ are rows such that consecutive ones share more than s columns, with $s \ge 2K$ and $s^2 \ge 4KW$, one can go from z_0 to z_H in fewer than $2s/K$ hops along the sequence, each hop joining two rows that share more than K columns.

Corollary 6. **[proved]** *Call a matrix a* staircase matrix *if its nonzero columns are partitioned into classes and every row, restricted to each class, is a final segment of that class. If the matrix has n rows and W columns, the classes form R maximal runs of consecutive columns and $d \le K$ for all pairs of rows, with $K \ge 1$, then $D < 3Rn\lceil 2\sqrt{KW}\,\rceil$.*

3 Two families limiting the gluing bounds

Proposition 7 (chain family)**. [proved]** *Let $n_0 \ge 2$, $h \ge 1$. Columns, from left to right: blocks $Q_{n_0}, \dots, Q_1, Q_0$ of h columns each (part A), then columns $e_{n_0}, \dots, e_2$ (part B). The rows are $r_1, \dots, r_{n_0}$ from top to bottom, and r_i consists of Q_i, Q_{i-1} and the columns e_j with $j \le i$. Then $M|_A$ and $M|_B$ are P-free, the matrix is a staircase matrix with $n_0 + 2$ classes, $d \le 1$ for all pairs, and every P-free matrix differs from it in at least $\min(h/2, (n_0 - 1)/2)$ entries of some row.*

Proof. Rows r_i, r_j with $j \ge i + 2$ share no column of A, and $r_i \cap B$ is a final segment of $r_j \cap B$. For $j = i + 1$ the common columns in A are those of Q_i, a final segment of $r_{i+1} \cap A$, and the only later column of r_{i+1} missing from r_i is e_{i+1}. Suppose a P-free matrix changes at most t entries per row with $2t < h$. The modified rows r'_i, r'_{i+1} keep a common column in Q_i, to the left of all of B, so every column of B in r'_{i+1} is in r'_i. Hence the columns of B in r'_{n_0}, at least $n_0 - 1 - t$ of them, are in r'_1, which has at most t: $t \ge (n_0 - 1)/2$. □

[computed] Exact values for $(n_0, h) = (3, 2), (5, 4), (7, 6), (9, 8)$: the minimum number of changes in the most modified row is $1, 2, 3, 4$, and $D = 2, 6, 12, 18$. Consequently (take $h \ge n_0 - 1$, so that $K = 1$ and $m = n_0 + 2$) a per-row bound of the form $g(m)K$ for staircase matrices with m classes requires $g(m) \ge (m - 3)/2$, and the per-row bound of order $\sqrt{KW}$ in Theorem 4 cannot be replaced by CK.

Proposition 8 (sliding windows)**. [proved]** *Let $L \ge 16$ be even and $n \ge 5L^{3/2}$. Rows are numbered $0, \dots, n-1$. Part A has $n - 1 + L$ columns and row i contains the L consecutive columns $n - 1 - i, \dots, n - 2 - i + L$. Part B, to the right of A, consists of columns $e_T, \dots, e_1$ in this order, with $T = \lfloor (n-1)/(L-1) \rfloor$, and row i contains e_t for $t \le \lfloor i/(L-1) \rfloor$. Then $M|_A$ and $M|_B$ are P-free, $d \le 1$ for all pairs, and $D \ge n\sqrt{L}/80$.*

Proof. For $i < j$ the common columns in A are the last $L - (j - i)$ columns of the window of j (none if $j - i \ge L$) and $r_i \cap B$ is a final segment of $r_j \cap B$; a copy has its common column in A and its other column among the at most one e_t of $r_j \setminus r_i$ when $j - i \le L - 1$. Let N be P-free. If two rows of N share a column in the zone of A, every column of B of the lower one is in the upper one. Call a row *bad* if N changes at least $L/4$ of its entries in the zone of A. Two good rows at

distance at most $L/2$ keep a common column in A. Fix a block of $S = 4\lceil L^{3/2} \rceil$ consecutive rows. If it contains $L/2$ consecutive bad rows, or more than $S/2$ bad rows, it carries at least $L^2/8$ changes. Otherwise consecutive good rows are at distance at most $L/2$, the B-parts of the $r \geq S/2 \geq 2L^{3/2}$ good rows decrease from top to bottom in N, and for an index t whose first row $t(L-1)$ lies in the block the column e_t needs at least $\min(u_t, v_t)$ changes, where u_t, v_t count the good rows before and from that row. Between the good rows of ranks $\lceil r/4 \rceil$ and $\lfloor 3r/4 \rfloor$ there are at least $r/2 - 2$ rows, hence at least $\frac{r/2-2}{L-1} - 1 \geq \frac{r}{4L}$ indices t, each with $\min(u_t, v_t) \geq r/4$: at least $r^2/(16L) \geq L^2/4$ changes. Each block thus carries at least $L^2/8$ changes and there are at least $n/(2S) \geq n/(10L^{3/2})$ disjoint blocks. □

So no bound $D \leq CKN$ (N the larger side) holds for two P-free parts, even for intervals. When $n = 5L^{3/2}$ this family has $d \leq 1$, fewer than $3n$ columns and $D \geq n^{4/3}/137$, against an upper bound of order $n^{3/2}$ from Theorem 4.

4 Staircase matrices

For a staircase matrix with classes $S_1, \ldots, S_m$, row x is described by its starts $x_j \in S_j \cup \{\infty\}$: it contains the columns of S_j at or after x_j.

Lemma 9 (threshold criterion)**. [proved]** *For a row x and classes $i \neq j$ with $x_i < x_j$, let Δ_x^{ij} be the last column of S_j strictly between x_i and x_j if there is one, and $-\infty$ otherwise; put $\Delta_x^{ij} = -\infty$ when $x_i \geq x_j$. The matrix is P-free if and only if, for every row y and every pair of classes $\{i, j\}$, $\max(y_i, y_j)$ exceeds Δ_x^{ij} and Δ_x^{ji} for all rows $x < y$.*

Proof. A copy uses two columns of different classes, with common column $c \geq \max(x_i, y_i)$ in S_i and other column $d \in S_j$, $y_j \leq d < x_j$, $c < d$. If $x_i < x_j$ and $\max(y_i, y_j) \leq \Delta_x^{ij}$, then $c = \max(x_i, y_i)$ and $d = \Delta_x^{ij}$ form a copy. Conversely a copy gives $x_i \leq c < d < x_j$, so $d \leq \Delta_x^{ij}$, and $y_i \leq c < d$, $y_j \leq d$. □

[computed] Compared with the direct test on 60 000 staircase matrices with 2 to 5 classes: no disagreement.

Theorem 10 (two classes)**. [proved]** *Let M be a staircase matrix with two classes and $d \leq K$ for all pairs of rows, with $K \geq 1$. Then M can be made P-free by changing at most $4K$ entries in each row.*

Proof. An empty row forms no copy and is left unchanged. For a nonempty row x let $e_x < \lambda_x$ be its starts (with $\lambda_x = \infty$ if x meets only one class), E_x, L_x their classes, G_x the set of columns of L_x strictly between e_x and λ_x, and $\Delta_x = \max G_x$. By Lemma 9 the matrix is P-free iff $\lambda_y > \max_{x<y} \Delta_x$ for all y.

Step 1 (each row separately). If $|G_x| \leq K$, lower λ_x to the first column of L_x after e_x. Otherwise lower λ_x to the K-th largest column of G_x and let θ_x be the $(K+1)$-th largest; if $[e_x, \theta_x)$ contains at most K columns of E_x, raise e_x to the first column of E_x after θ_x. Cost at most $2K$. Afterwards a row either has no column between its starts, or is *heavy*: it has filled its K top gap columns, its level is θ_x, and $[e_x, \theta_x)$ contains more than K columns of E_x.

Fact 1. Let x be heavy and y an original row below x whose start in L_x is at most θ_x. The column θ_x and the K filled columns belong to y and not to the original row x, and every common column of E_x before θ_x is to their left; so there are at most K such common columns. Hence y starts after e_x in E_x and has at most K columns of E_x before θ_x.

Step 2 (top to bottom). Let Θ be the largest Δ among the final rows above y, attained by a row x, and $(i, j) = (E_x, L_x)$. If both starts of y after Step 1 are at most Θ: if the original start of y in class j is at most Θ, raise the start of y in class i to the first column of that class after Θ; otherwise raise the start in class j to the first column of that class after Θ.

Fact 2. A row modified in Step 2 has one start at most Θ and the other equal to the first column of its class after Θ; its gap columns are at most Θ, so it does not raise the level. By induction the level is always attained by a heavy row not modified in Step 2, and Fact 1 applies to it and to original rows.

Cost of Step 2. Step 1 lowers a start by at most K columns of its class. In the first case Fact 1 gives at most K columns of class i between the original start of y and Θ, so at most $2K$ deletions. In the second case the start in class j was above Θ originally and was lowered by at most K columns: at most K deletions. In the final matrix every row has a start above the level of the rows above it. □

[computed] On 201 318 two-class matrices the output is always P-free and no row exceeds $4K$ (maximum observed $2K$). **[open]** The same statement for $m \geq 3$ interleaved classes with a bound $g(m)K$ per row; by Proposition 7, $g(m) \geq (m-3)/2$.

Theorem 11 (pivot reduction)**. [proved]** *Let M be an $n \times n$ matrix with $d(x, y) \leq k$ for all pairs of rows and let $s \geq 1$ be an integer. There is a staircase matrix N with $m \leq n/s$ classes such that $|M \triangle N| \leq n^2 k/s + ns$ and $d_N(x, y) \leq (2m+1)k$ for all pairs of rows.*

Proof. Process the rows from bottom to top, keeping a set of classified columns. For the row w, let S be the set of its ones in unclassified columns. If $|S| \geq s$, w becomes a *pivot* and S a new class: every row x above w is replaced, on the columns of S, by the nearest final segment of S. Otherwise the ones of w in S are erased. When w becomes a pivot, the entries of the columns of S in the rows above w are still those of M; since w is identically 1 on S, turning x into a final segment of S costs exactly the number of columns of S to delete to make the pair (x, w) P-free on S, at most k. There are at most n/s pivots, hence at most $n^2 k/s$ changes, and the erasures cost less than s per row. A row below a pivot was processed before it, so its ones in the then unclassified columns were classified into another class or erased: it is zero on that class, and a column never classified is zero. For the distances, let $x < y$; delete the columns of a minimum set for the pair in M and those where x or y was modified by a rounding: at most $k + 2mk$ columns. On the remaining columns N differs from M only by erasures. An erasure in y creates no copy of the pair. An erasure of the entry (x, c) happens when c is unclassified at the time of x; then the row y, processed earlier, is zero at c in N. So every copy of the pair in N on these columns is a copy in M. □

Corollary 12. [proved, conditional] *Suppose there are constants $C > 0$ and $a \geq 0$ such that every $n \times n$ staircase matrix with m classes and $d \leq K$ for all pairs satisfies $D \leq C\, m^a\, n^{3/2}\sqrt{K}$. Then every $n \times n$ matrix with $d \leq \kappa n$ for all pairs of rows, $0 < \kappa \leq 1$, satisfies $D \leq C' \kappa^{1/(2a+3)} n^2$.*

Proof. If $\kappa n < 1$ the matrix is P-free. Otherwise let $m = \lceil \kappa^{-1/(2a+3)} \rceil \leq n$ and apply Theorem 11 with $s = \lceil n/m \rceil$; the matrix obtained has at most m classes, cost at most $\kappa m n^2 + 2n^2/m$ and pair distances at most $(2m+1)\kappa n$. The hypothesis adds $C\, m^a n^{3/2} \sqrt{(2m+1)\kappa n} \leq 2C\, m^{a+1/2} \sqrt{\kappa}\, n^2$. □

By Theorem 4 the hypothesis of the corollary holds, with $a = 1$, when the classes are intervals of columns. **[open]** It is not known for interleaved classes. The step from the conclusion of the corollary to a polynomial removal lemma for P is not treated in this note.

5 The row-block method does not transfer directly

Lupini [3] proved polynomial removal for the ordered 2×2 identity. His short-span estimate keeps blocks of consecutive rows separated by zeroed rows, observes that a column pair is *mixed* in at most one block, and repairs a block of m rows with Q mixed pairs at cost at most $2m\sqrt{2Q}$ by rows that are lattice polynomials of the rows of the block, which preserves compatibility between blocks. Write $x\, R\, y$ if the row x placed above the row y forms no copy of P.

Lemma 13. **[proved]** *For fixed y, $\{x : x\,R\,y\}$ is closed under coordinatewise minimum and maximum. For fixed x, $\{y : x\,R\,y\}$ is closed under coordinatewise minimum and under passing to a smaller row, but not under maximum.*

Proof. For fixed y the condition is $x_c \le x_d$ for all $c < d$ with $y_c = y_d = 1$. For fixed x it is "y_c and y_d are not both 1" for all $c < d$ with $(x_c, x_d) = (1, 0)$; with $x = (1, 0)$ the rows $(1, 0)$ and $(0, 1)$ qualify and their maximum does not. □

Say that a column pair $c < d$ is *mixed* in a block if some row of the block has entries $(1, 0)$ and another has $(1, 1)$ there. **[proved]** If every copy of P has row span at most h and the retained blocks are separated by at least h rows, a column pair is mixed in at most one block, since a row $(1, 0)$ of a block and a row $(1, 1)$ of a lower block form a copy of span larger than h.

Proposition 14 (lock family)**.** **[proved]** *Let m be even, $k \ge 1$, $r \ge k$. Columns, from left to right: k columns $b^i_1, \dots, b^i_k$ for each $i \le m/2$, k columns $a^j_1, \dots, a^j_k$ for each $j \le m/2$, then c, d, $e_1, \dots, e_k$. Block B: rows $x_i = \{b^i_1, \dots, b^i_k, c\}$, $i \le m/2$, above rows $y_j = \{a^j_1, \dots, a^j_k, c, d, e_1, \dots, e_k\}$, $j \le m/2$. Below B: for each i, r rows equal to $\{b^i_1, \dots, b^i_k, c\}$, and for each j, r rows equal to $\{a^j_1, \dots, a^j_k, c\}$. Then every copy of P has both rows in B; B has exactly $k + 1$ mixed column pairs and the rows below B have none; and $D \ge m(k + 1)/4$.*

Proof. Two rows not both in B share either all the columns of the lower one or only c, which is the last column of the lower one. In B the mixed pairs are (c, d) and (c, e_t). Let N be P-free; for $i \le m/2$ let $\gamma(i)$ be the number of changes in x_i and in its r rows below, and $\gamma'(j)$ likewise for y_j. If x_i loses c, either one of its r rows below is untouched, and then x_i must lose its k columns b, or all of them changed: $\gamma(i) \ge k + 1$. Likewise if y_j loses c. If x_i and y_j both keep c, each of the $k + 1$ columns $d, e_1, \dots, e_k$ must be added to x_i or removed from y_j: $\gamma(i) + \gamma'(j) \ge k + 1$. So for every pair (i, j) one of the two is at least $(k + 1)/2$; this holds for all i or for all j. □

[computed] Exact distance for $(m, k, r) = (4, 2, 2), (4, 3, 3), (6, 3, 3), (6, 4, 4), (4, 6, 6)$: $6, 8, 12, 15, 14$, that is $m(k + 1)/2$ each time, against $m\sqrt{k + 1}$ for the quantity $m\sqrt{Q}$. The family does not contradict a short-span estimate for P (its distance is of the order of its side); it shows, as k grows, that the cost of a block is not controlled by its mixed pairs alone; the lower bound comes from the compatibility of B with the rows below it.

6 Matrices of row span one

The row span of a copy is, up to the factor n, the *height* of a 2×2 submatrix considered by Alon and Ben-Eliezer [1]; their question about copies of large height concerns a pattern over four symbols and is different from the statements below. For the 2×2 identity, Lupini [3] shows that any removal exponent is at least 3; for P the family of the next section gives $9/2$.

Say that a matrix has *row span at most h* if every copy of P uses two rows at distance at most h. Rows are written $x_1, \dots, x_n$ from top to bottom, F_Π is the set of columns of index larger than Π and I_Π its complement. We write $x\,R\,y$ when a row x placed above a row y forms no copy with it, that is, when $x \cap y$ is a final segment of y.

Lemma 15 (counting)**.** **[proved]** *In a matrix of row span one, every pair of columns carries at most one copy; an $n \times n$ matrix of row span one has at most $\binom{n}{2}$ copies. For row span at most h, every pair of columns carries at most h^2 copies.*

Proof. If rows $(i, i + 1)$ and $(j, j + 1)$, $j > i$, both carry a copy on columns $c < d$, then $j = i + 1$ is impossible (row $i + 1$ would both contain and miss d), and $j \ge i + 2$ gives a copy between rows i and $j + 1$. For span h: let i_0 be the first row with pattern $(1, 0)$ on (c, d) lying above a row with pattern $(1, 1)$, and j_1 the last row with pattern $(1, 1)$ lying below a row with pattern $(1, 0)$; every copy has its top row in $[j_1 - h, j_1)$ and its bottom row in $(i_0, i_0 + h]$. □

Corollary 16. **[proved]** *If P satisfies "$D \geq \varepsilon n^2$ implies at least $c\varepsilon^C n^4$ copies", then every $n \times n$ matrix of row span one satisfies $D \leq C' n^{2-2/C}$. Conversely, a family of matrices of row span one with $D \geq n^{2-o(1)}$ would rule out a polynomial removal lemma for P.*

Proposition 17 (a single copy at linear distance)**.** **[proved]** *Let $m \geq 0$. Take four blocks b, b'', a, a'' of $m+1$ columns, followed by three columns $c < t < d$. Rows, from top to bottom: $m+1$ copies of $\{t\}$; $x = b \cup b'' \cup \{c, t\}$; $y = a \cup a'' \cup \{c, d\}$; then $m+1$ copies of each of $b \cup \{c\}$, $b'' \cup \{t\}$, $a \cup \{c\}$, $a'' \cup \{d\}$. This matrix, with $5m+7$ rows and $4m+7$ columns, contains exactly one copy of P, and $D \geq m+1$. Consequently any exponent C in a polynomial removal lemma for P satisfies $C \geq 4$.*

Proof. Two rows with disjoint supports, or the upper one containing the lower one, or whose only common column is the last column of the lower row, form no copy; this covers all pairs except (x, y), whose only common column is c and where y has only d above c. Suppose at most m entries are changed and the result is P-free. In each of the five groups of $m+1$ equal rows one row is untouched, and in each block one column $\beta, \beta'', \alpha, \alpha''$ is untouched in all rows. Let x', y' be the modified rows. Since $\beta \in x'$ and an untouched row $b \cup \{c\}$ lies below, $c \in x'$; likewise $t \in x'$, $c \in y'$, $d \in y'$. The pair (x', y') has the common column c and $d \in y'$, hence $d \in x'$. Then an untouched row $\{t\}$ above x' gives a copy on (t, d), a contradiction. Appending m all-zero columns gives a square matrix of side $n = 5m+7$ with the same single copy and the same distance (a zero column belongs to no copy, and a submatrix of a P-free matrix is P-free). With one copy and $\varepsilon \geq 1/(7n)$, the inequality $1 \geq c\varepsilon^C n^4$ for all such n forces $C \geq 4$. □

[computed] For blocks and repetitions of size $1, \ldots, 5$: one copy, and exact distance $2, \ldots, 6$. The proposition is consistent with a bound of the form $D \leq Cn\sqrt{\tau}$, where τ is the least number of entries meeting all copies (here $\tau = 1$, and $D \leq n$ trivially).

Lemma 18 (splice)**.** **[proved]** *Let u, x, y be sets of columns and Π a cut such that $x \cap I_\Pi$ is empty or has its last element in y. Put $s = (x \cap I_\Pi) \cup (y \cap F_\Pi)$. If $u\,R\,x$ and $u\,R\,y$, then $u\,R\,s$.*

Proof. If u misses $x \cap I_\Pi$, then $u \cap s = u \cap y \cap F_\Pi$, and the elements of s above its least element belong to y, hence to u. Otherwise let t be the least element of $u \cap x \cap I_\Pi$. All of x above t lies in u, in particular the last element c^* of $x \cap I_\Pi$, which is in y; so all of y above c^* lies in u, and $y \cap F_\Pi \subseteq u$. Thus all of s above t lies in u. □

Theorem 19 (two safe moves, bottom-up)**.** **[proved]** *Let M have row span one. Say that the state $\mathcal{J}(a)$ holds if every pair of rows (t, s) with $s \geq t+2$ satisfies $x_t\,R\,x_s$, and every adjacent pair $(t, t+1)$ with $t \geq a+1$ does too. In the state $\mathcal{J}(a)$ put $U = x_a$, $W = x_{a+1}$ and consider:*

(G1) for a row $r \in \{a, a+1\}$ and a column y, delete y from every row of the vertical closure $\mathrm{Cl}(r, y)$*, the smallest set $S \ni r$ such that $t \in S$, $s > t$, $y \in x_s$ and $x_t \cap x_s \cap I_{y-1} \neq \varnothing$ imply $s \in S$;*

(G2) for a cut Π such that $U \cap F_\Pi \subseteq W$ and $U \cap I_\Pi$ is empty or has its last element in W, replace U by $U \cup (W \cap F_\Pi)$.

Both moves preserve $\mathcal{J}(a)$. Hence, starting from M and treating the adjacent pairs from bottom to top, applying moves (G1)*,* (G2) *to the pair $(a, a+1)$ until it forms no copy (which* (G1) *alone always achieves), one obtains a P-free matrix; D is at most the number of entries changed.*

Proof. (G1) creates no copy in any matrix: a new copy on rows $t < s$ and columns $z < y'$ would have its zero entry (t, y') among the deleted ones, so $y' = y$ and $t \in S$, with $y \in x_s$ kept and $z \in x_t \cap x_s$, $z < y$; then $s \in S$ and y was deleted from x_s. (G2) changes row a only. For $s \geq a+2$, both U and W are R-related to x_s by $\mathcal{J}(a)$, every final segment F_Π is, and the set of rows x with $x\,R\,x_s$ is closed under union and intersection (Lemma 13); so $U \cup (W \cap F_\Pi)\,R\,x_s$. For $t \leq a-2$, $x_t\,R\,U$ and $x_t\,R\,W$, and the hypothesis on Π gives $U \cup (W \cap F_\Pi) = (U \cap I_\Pi) \cup (W \cap F_\Pi)$; Lemma 18 applies. The pairs $(a-1, a)$ and $(a, a+1)$ are not part of $\mathcal{J}(a)$. Once the pair $(a, a+1)$ forms no copy, $\mathcal{J}(a-1)$ holds. Finally, let c be the least column of $U \cap W$ having a column of $W \setminus U$ to its right; (G1) applied to $r = a+1$ and $y = c$ decreases the number of copies of the pair. □

[computed] Random moves on 222 026 matrices of row span one (sides 4 to 10; 584 501 moves (G1) and 97 260 moves (G2)): the state $\mathcal{J}$ is never violated and the output is always P-free. The file `verif_portee1.py` repeats this test and checks Lemmas 15 and 18 and Proposition 17. **[open]** A bound $D \le Cn^{2-\alpha}$ for all matrices of row span one. By Corollary 16 it is necessary for a polynomial removal lemma for P.

7 A span-one family at distance $n^{4/3}$

Throughout this section a *conflict* is a pair of adjacent rows carrying a copy of P. Fix integers $q \ge 2$ and $s, p \ge 1$, and let i, j, l range over $\{0, \dots, q-1\}$.

Columns, from left to right: groups $\xi_{i,l}$ (p columns each); then groups $\lambda_{j,l}$ (p columns each); then, for $i = 0, \dots, q-1$: the groups $\mu_{i,l}$ (p columns each, $l = 0, \dots, q-1$), followed, for $j = 0, \dots, q-1$, by the block Bl_{ij} made of the single columns $c_{ij0}, \dots, c_{ij,q-1}$, a group B_{ij} of p columns, and the single columns $e_{ij0}, \dots, e_{ij,q-1}$. Let Q_{ij} be the set of all columns to the right of Bl_{ij}, and F_i the set of all columns to the right of $\mathrm{Bl}_{i,q-1}$.

Rows, from top to bottom. For $i = 0, \dots, q-1$: for $j = 0, \dots, q-1$, first s rows $V_{ij} = B_{ij} \cup Q_{ij}$, then for $l = 0, \dots, q-1$ the two rows

$$U_{ijl} = \xi_{i,l} \cup \{c_{ij'l} : j' \le j\} \cup B_{ij} \cup Q_{ij},$$
$$W_{ijl} = \lambda_{j,l} \cup \{c_{i'jl} : i' < i\} \cup \mu_{i,l} \cup \{e_{ij'l} : j' < j\} \cup \{c_{ijl}, e_{ijl}\} \cup Q_{ij};$$

once this is done for all j (the rows attached to a pair (i,j) form the *bouquet* (i,j); bouquets are ordered lexicographically), for each l, s rows $K_{i,l} = \xi_{i,l} \cup \{c_{ijl}\}_j \cup F_i$. At the bottom: for each (j,l), s rows $J_{j,l} = \lambda_{j,l} \cup \{c_{ijl}\}_i$, then for each (i,l), s rows $R_{i,l} = \mu_{i,l} \cup \{e_{ijl}\}_j$. Call this matrix $M(q,s,p)$; it has $2q^3 + 4q^2 s$ rows and $2q^3 + 4q^2 p$ columns.

Proposition 20. [proved] *In $M(q,s,p)$ every pair of rows is P-free except the q^3 adjacent pairs (U_{ijl}, W_{ijl}), each of which carries exactly one copy, namely on columns (c_{ijl}, e_{ijl}). Hence $M(q,s,p)$ has row span one and exactly q^3 copies of P.*

Proof. By Lemma 1, a row x above a row y forms a P-free pair if and only if $x \cap y$ is a final segment of y. If $x = H \cup F$ with F a final segment of the column set and H to the left of F, then $y \cap F$ is a final segment of y, so the pair is P-free as soon as $H \cap y = \varnothing$, or x contains every element of y to the right of $\min(H \cap y)$. The rows V_{ij}, U_{ijl}, W_{ijl} have this form with $F = Q_{ij}$, and $K_{i,l}$ with $F = F_i$.

(A) *x in bouquet (i,j), y in a later bouquet (a,b).* The head B_{ij} of V_{ij} meets no such y. The head of $U_{ijl'}$ meets U_{abl} only if $a = i$, $l' = l$, and then $U_{ijl} \supseteq U_{ibl}$; it meets W_{abl} only in c_{ibl}, when $l' = l$, $a > i$, $b \le j$, and the elements of W_{abl} to the right of c_{ibl} lie in $F_i \subseteq Q_{ij}$. The head of $W_{ijl'}$ meets U_{abl} only in c_{ijl}, when $a = i$, $l' = l$, and the elements of U_{ibl} to the right of c_{ijl} lie in Q_{ij}; it meets W_{abl} in two cases: if $b = j$, $l' = l$, then $W_{ijl} \supseteq W_{ajl}$; if $a = i$, $l' = l$, the least common element lies in $\mu_{i,l}$ and all elements of W_{ibl} to its right ($\mu_{i,l}$, $e_{ib'l}$ for $b' < b$, c_{ibl}, e_{ibl}, Q_{ib}) belong to W_{ijl}.

(B) *Same bouquet.* $V_{ij} \cap U_{ijl} = B_{ij} \cup Q_{ij}$, $V_{ij} \cap W_{ijl} = Q_{ij}$; for $l < l'$, $U_{ijl} \cap U_{ijl'} = B_{ij} \cup Q_{ij}$ and the three other intersections equal Q_{ij}: all final segments. Finally $U_{ijl} \cap W_{ijl} = \{c_{ijl}\} \cup Q_{ij}$ and the only element of $W_{ijl} \setminus U_{ijl}$ to the right of c_{ijl} is e_{ijl}: exactly one copy.

(C) *x in bouquet (i,j), $y = K_{a,l}$, $a \ge i$.* If $a > i$, $x \cap y = y \setminus \xi_{a,l}$. If $a = i$, $x \cap y$ is $\{c_{ibl}\}_{b>j} \cup F_i$, enlarged to y if $x = U_{ijl}$ and by c_{ijl} if $x = W_{ijl}$.

(D) *$x = K_{i,l}$, y below.* For y in a bouquet (a,b) with $a > i$, the head $\xi_{i,l} \cup \{c_{ibl}\}_b$ meets only W_{abl}, in c_{ibl}, and the elements of W_{abl} to its right lie in F_i. Moreover $K_{i,l} \cap K_{a,l'} = K_{a,l'} \setminus \xi_{a,l'}$ for $a > i$ and $K_{i,l} \cap K_{i,l'} = F_i$ for $l \ne l'$.

(E) $y = J_{b,l}$. For x in a bouquet or $x = K_{i,l'}$, $x \cap y$ is y itself (if $x = W_{ibl}$) or of the form $\{c_{abl} : a \geq a_0\}$. (F) $y = R_{a,l}$. $x \cap y$ is y itself, or $\{e_{abl} : b > j\}$, or empty. Rows of type J and R are pairwise disjoint or equal. □

Lemma 21 (rectangles)**. [proved]** *Let $R_1, \dots, R_m$ be rectangles $R_t = Z_t \times \Lambda_t$ in a grid, and let ε_t be the number of $t' < t$ with $R_{t'} \cap R_t \neq \varnothing$. Then $\left|\bigcup_t R_t\right| \geq \sum_t (\min(|Z_t|, |\Lambda_t|) - \varepsilon_t)^+$.*

Proof. Choose successively $D_t \subseteq R_t$ consisting of $(\min(|Z_t|, |\Lambda_t|) - \varepsilon_t)^+$ cells, no two in a common row or column, disjoint from the earlier $D_{t'}$. This is possible: the cells already chosen inside R_t come from at most ε_t rectangles, each contributing a matching; in the complete bipartite graph $Z_t \times \Lambda_t$ with ε_t matchings removed, every vertex of the smaller side keeps at least $\max(|Z_t|, |\Lambda_t|) - \varepsilon_t \geq \min(|Z_t|, |\Lambda_t|) - \varepsilon_t$ neighbours; so any $\min(|Z_t|, |\Lambda_t|) - \varepsilon_t$ vertices of the smaller side satisfy Hall's condition and can be matched. □

Theorem 22. [proved] *Every P-free matrix differs from $M(q, s, p)$ in at least $q^3\,(1 + \min(s, p))$ entries.*

Proof. Let N be P-free and Δ the set of entries where $N \neq M$. For each conflict $k = (i, j, l)$, with rows $U = U_{ijl}$, $W = W_{ijl}$ and columns $c = c_{ijl} < e = e_{ijl}$, one of the four entries of the copy has changed; fix one. This gives k a *type*, and with it a *class* of q conflicts, a group of s blocking rows, a target column $y \in \{c, e\}$, a row $x \in \{U, W\}$, a group G of p link columns and, for each earlier conflict k'' of the class, a *bonus entry*:

type	changed entry	class	blocking rows	y	G	bonus entry for k''
A	$N(W, c) = 0$	i varies	$J_{j,l}$	c	$\lambda_{j,l}$	$(W, c_{i''jl})$, $i'' < i$
K	$N(U, c) = 0$	j varies	$K_{i,l}$	c	$\xi_{i,l}$	$(U, c_{ij''l})$, $j'' < j$
R	$N(W, e) = 0$	j varies	$R_{i,l}$	e	$\mu_{i,l}$	$(W, e_{ij''l})$, $j'' < j$
V	$N(U, e) = 1$	l varies	V_{ij}	e	B_{ij}	$(U_{ijl''}, e)$, $l'' < l$

Let Z_k be the set of blocking rows z of k with $(z, y) \notin \Delta$, $a_k = s - |Z_k|$, Λ_k the set of $g \in G$ with $(x, g) \notin \Delta$, $b_k = p - |\Lambda_k|$, and $R_k = Z_k \times \Lambda_k$, a rectangle of the *block* (blocking rows) $\times G$.

(i) $R_k \subseteq \Delta$. For $z \in Z_k$, $g \in \Lambda_k$, if $(z, g) \notin \Delta$ then rows x, z and columns $g < y$ carry a copy in N: $(W, z; g, c)$ for type A, $(U, z; g, c)$ for K, $(W, z; g, e)$ for R, $(z, U; g, e)$ for V. (In M the blocking rows and the row x contain G, which lies to the left of y; the blocking rows contain y for types A, K, R and do not contain it for type V.)

(ii) Bonus. Let k'' precede k in the same class and type, with $R_{k''} \cap R_k \neq \varnothing$. Then the bonus entry of k for k'' lies in Δ. For types A, K, R there is $z \in Z_{k''} \cap Z_k$, so $N(z, y'') = N(z, y) = 1$; the column y'' is to the left of y and $M(x, y'') = 1$; if $N(x, y'') = 1$ then $(x, z; y'', y)$ is a copy. For type V there is $g \in \Lambda_{k''} \cap \Lambda_k$, so $N(U_{ijl''}, g) = N(U, g) = 1$ and $N(U, e) = 1$; if $N(U_{ijl''}, e) = 0$ then $(U_{ijl''}, U; g, e)$ is a copy.

(iii) Distinct entries. Assign to k: the changed entry of its copy; the a_k entries (z, y), $z \notin Z_k$; the b_k entries (x, g), $g \notin \Lambda_k$; and its bonus entries lying in Δ. No entry is assigned to two conflicts: entries in a row U or W are assigned to the conflict of that row only, except the bonus entries of type V, which lie in the column e of the conflict receiving them and outside the columns used by the conflict of their row; entries in blocking rows lie in the column c or e of the conflict receiving them. The blocks are pairwise disjoint and disjoint from all these entries.

(iv) Count. Fix a type and a class and list its conflicts of that type in order. By (ii), k receives at least ε_k bonus entries, ε_k being the number of earlier rectangles meeting R_k. By (i) and Lemma 21, the block contains at least $\sum_k (\min(s - a_k, p - b_k) - \varepsilon_k)^+$ entries of Δ. The number of entries of Δ counted is thus at least

$$\sum_k \Big(1 + a_k + b_k + \varepsilon_k + (\min(s - a_k, p - b_k) - \varepsilon_k)^+\Big) \;\geq\; \sum_k (1 + \min(s + b_k, p + a_k)) \;\geq\; \sum_k (1 + \min(s, p)).$$

Summing over types and classes gives $|\Delta| \geq q^3(1 + \min(s, p))$. □

Proposition 23. **[proved]** *For all $q \geq 2$ and $s, p \geq 1$, replacing every row U_{ijl} by $U'_{ijl} = (U_{ijl} \setminus B_{ij}) \cup \{e_{ijl}\}$ yields a P-free matrix; hence the distance of $M(q,s,p)$ to P-freeness is at most $q^3(p+1)$. With Theorem 22: for $p \leq s$ this distance equals $q^3(p+1)$; in particular $M(q,q,q)$, a square matrix of side $n = 6q^3$ with $n/6$ copies of P on disjoint pairs of adjacent rows, is at distance exactly $q^4 + q^3 = (n/6)^{4/3} + n/6$.*

Proof. $U'_{ijl} = H' \cup Q_{ij}$ with $H' = \xi_{i,l} \cup \{c_{ij'l}\}_{j' \leq j} \cup \{e_{ijl}\}$, and we use the principle and the cases of the proof of Proposition 20. Pairs without a row of type U are unchanged. In the same bouquet, $U'_{ijl} \cap W_{ijl} = \{c_{ijl}, e_{ijl}\} \cup Q_{ij}$ is a final segment of W_{ijl}, $V_{ij} \cap U'_{ijl} = Q_{ij}$, and for $l \neq l'$ the intersections of U'_{ijl} with $U'_{ijl'}$ and $W_{ijl'}$ equal Q_{ij}. If x lies in an earlier bouquet and $y = U'_{abl}$, then $x \supseteq Q_{ij} \ni e_{abl}$ and the head of x meets y as in case (A). If $x = U'_{ijl}$ and y lies in a later bouquet (a,b), then H' meets $U'_{abl'}$ and $W_{abl'}$ as in case (A), and in addition meets W_{ibl} $(b > j)$ in e_{ijl}; this is then the least common element, and the elements of W_{ibl} to its right lie in Q_{ij}. For $y = K_{a,l'}$ or $J_{b,l'}$ nothing changes, since e_{ijl} belongs to none of these rows outside F_i. For $y = R_{a,l'}$, $x \cap y$ is y if $a > i$, empty if $a < i$, and $\{e_{ibl'} : b > j\}$, together with e_{ijl} when $l' = l$, if $a = i$. Finally $K_{i,l'} \cap U'_{abl} = U'_{abl} \setminus \xi_{a,l}$ for $a > i$. The cost is $p+1$ per conflict. □

Corollary 24. **[proved]** *For matrices of row span one, the distance to P-freeness is not bounded by $C(S+n)$, where S is the sum over adjacent pairs of rows of the distance of the pair: for $M(q,q,q)$, $S = n/6$. With $n/6$ copies at distance $(n/6)^{4/3} + n/6$, any removal exponent for P is at least $9/2$.*

[computed] Exact distances obtained with a constraint solver (optimality proved) agree with the formula for $(q,s,p) = (2,1,1), (2,2,2), (2,3,3), (3,1,1), (3,2,2), (3,3,3), (4,1,1), (2,3,1), (2,3,2), (3,3,1)$; for $p > s$ the two bounds differ and the solver gives $25, 17, 62, 17$ for $(2,2,3), (2,1,3), (3,1,2), (2,1,4)$. The ancillary script `verif_cube.py` (plain Python) rebuilds $M(q,s,s)$ for six sizes up to side 882 and checks Proposition 20, the entries used in the proof of Theorem 22, and the repair of Proposition 23. The same construction can be carried out for more general incidence patterns between conflicts and blocks (rows of bouquets in which the classes of type K are confined to one row and the classes of type J meet each row at most once); this extension is not written out here. For 135 421 random patterns of this kind the matrix has row span one and the analogous repair is P-free, of cost (number of conflicts)$\times(p+1)$. **[open]** Whether $D \leq C\, n\sqrt{m}$ holds for span-one matrices with m conflicts carrying one copy each; the family above gives $n\, m^{1/3}/6$.

8 Open questions

1. **[open]** Does P admit a polynomial removal lemma? A literature search on 20 November 2026 (arXiv, Zenodo, public repositories and the public race board for this problem) found the obstructions of [4, 6, 5] for larger patterns and the positive result [3] for the identity, and nothing on P.

2. **[open]** Extend Theorem 4 to two interleaved P-free parts, or to m interleaved staircase classes with a bound polynomial in m.

3. **[open]** Extend Theorem 10 to three classes.

4. **[open]** Prove $D \leq Cn^{2-\alpha}$ for matrices of row span one.

5. **[open]** Close the gap between the orders $n^{4/3}$ and $n^{3/2}$ for two P-free intervals with $K = 1$.

Ancillary files. Three plain Python scripts, with fixed parameters and no external module: `verif_corner_note.py` (Lemmas 3 and 9, Theorems 4 and 10, and the families of Propositions 7, 8 and 14, on random and small instances), `verif_portee1.py` (the section on row span one) and `verif_cube.py` (the family $M(q,s,p)$). The first two print their messages in French. They test the statements on finitely many instances and are not part of the proofs.

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