All-Order Robust Strictly Positive Real Synthesis for Wide-Interval Polynomial Families

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Abstract

We establish a uniform all-order existence theorem for robust strictly positive real (SPR) synthesis of Hurwitz-stable interval polynomial families. Specifically, for every order $n \geq 1$, we prove that a common real SPR numerator of exact degree n always exists whenever the lower endpoint of each interior coefficient interval is at most 10^{-5} times its upper endpoint, while the constant and leading coefficients vary over arbitrary compact positive intervals. The threshold 10^{-5} is independent of both the polynomial order n and the stability margin of the family. Under whole-box Hurwitz stability, such wide interior variation forces geometric separation between successive characteristic frequency scales, decoupling the synthesis constraints into n frequency bands; combining Brouwer's fixed-point theorem in small-margin bands with bandwise strict convexity yields a common exact-degree-n SPR numerator across the entire family.

Keywords: strictly positive real synthesis; interval polynomials; Hurwitz stability; common numerator; scale separation

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1 Introduction

Strictly positive real (SPR) transfer functions play a central role in passivity analysis, absolute stability, adaptive control, and output-error system identification [3, 8]. For a single Hurwitz-stable denominator $a(s)$, choosing $b(s) = a(s)$ trivially yields a transfer function with strictly positive real part on the imaginary axis. When the denominator varies over an uncertain family, however, robust SPR synthesis requires a *single* fixed real numerator $b(s)$ of prescribed degree such that $\mathrm{Re}[b(\mathrm{i}\omega)/a(\mathrm{i}\omega)] > 0$ holds simultaneously for every admissible denominator a and every real frequency ω.

For real interval polynomial families, in which each coefficient varies independently in a closed interval, Kharitonov's theorem [5] reduces robust Hurwitz stability to four vertex polynomials. For a *prescribed* numerator, checking the SPR condition across the entire family likewise reduces to these four vertices [1, 6]. Synthesis is fundamentally harder because the same numerator must satisfy all four vertex inequalities at all frequencies simultaneously. For low-order Hurwitz-stable interval families ($n \leq 4$), Anderson et al. [1], Betser and Zeheb [2], and Yu and Wang [7, 6] established the existence of a common SPR numerator. For convex combinations of two polynomials (segments), common-numerator synthesis is possible in arbitrary order [9]. Yet an interval box allows all coefficients to vary independently, so understanding structural regimes that guarantee common-numerator synthesis in arbitrary order remains a fundamental problem.

In this paper, we identify a uniform structural regime in which equal-degree robust SPR synthesis succeeds at every order $n \geq 1$ (Theorem 2.2). Specifically, whenever every interior coefficient interval $[l_j, u_j]$ $(1 \leq j < n)$ of a Hurwitz-stable interval box satisfies $l_j/u_j \leq 10^{-5}$—with arbitrary compact positive intervals for a_0 and a_n—there always exists a common real SPR numerator of exact degree n. At first glance, requiring wide coefficient intervals may seem counterintuitive; however, under the joint hypothesis that the *entire* box is Hurwitz stable, wide interior variation forces adjacent characteristic frequency scales apart geometrically. This separation decouples the positive frequency axis into n bands on each of which a single Kharitonov vertex is extremal, allowing a common numerator to be constructed via Brouwer's fixed-point theorem (for bands with arbitrarily small stability margins) and verified via bandwise strict convexity and quadratic lower bounds.

The remainder of the paper is organized as follows. Section 2 formulates the interval synthesis problem, recalls the four-vertex reduction, and states Theorem 2.2. Section 3 establishes the four-coefficient stability inequality, geometric scale separation, and stationary-point alignment in the small-margin regime. Section 4 develops the frequency band decomposition, tail decay estimates, and bandwise positivity to complete the proof of Theorem 2.2. Section 5 concludes with remarks on sharp thresholds and open questions.

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2 Problem Formulation and Main Result

Write polynomial coefficients in ascending order and consider the nth-order interval polynomial box

$$\mathcal{K} = \left\{ a(s) = \sum_{j=0}^{n} a_j s^j : l_j \le a_j \le u_j,\ 0 \le j \le n \right\}, \qquad 0 < l_j \le u_j < \infty. \tag{2.1}$$

A real polynomial $a \in \mathbb{R}[s]$ is *Hurwitz stable* if all of its complex roots lie in the open left half-plane $\{s \in \mathbb{C} : \operatorname{Re} s < 0\}$, and the box $\mathcal{K}$ is *Hurwitz* (or robustly Hurwitz stable) if every member $a \in \mathcal{K}$ is Hurwitz stable. Following the equal-degree convention in [8], a real polynomial $b \in \mathbb{R}[s]$ of exact degree n is a *common SPR numerator* for $\mathcal{K}$ if

$$\operatorname{Re} \frac{b(\mathrm{i}\omega)}{a(\mathrm{i}\omega)} > 0 \qquad (\forall\, a \in \mathcal{K},\ \forall\, \omega \in \mathbb{R}). \tag{2.2}$$

Setting $t = \omega^2 \ge 0$, decompose any real polynomial $a(s) = \sum_{j=0}^{n} a_j s^j$ into its even and odd parts along the imaginary axis:

$$a(\mathrm{i}\omega) = E_a(t) + \mathrm{i}\omega O_a(t), \qquad E_a(t) = \sum_{k \ge 0} a_{2k}(-t)^k, \qquad O_a(t) = \sum_{k \ge 0} a_{2k+1}(-t)^k. \tag{2.3}$$

For a candidate numerator $b(s) = \sum_{j=0}^{n} b_j s^j$, define the real frequency pairing polynomial

$$\mathcal{P}_{a,b}(t) \coloneqq E_a(t)E_b(t) + t\, O_a(t)O_b(t). \tag{2.4}$$

Whenever a is Hurwitz stable, $|a(\mathrm{i}\omega)|^2 = E_a(\omega^2)^2 + \omega^2 O_a(\omega^2)^2 > 0$, so

$$\operatorname{Re} \frac{b(\mathrm{i}\omega)}{a(\mathrm{i}\omega)} = \frac{\mathcal{P}_{a,b}(\omega^2)}{|a(\mathrm{i}\omega)|^2}, \tag{2.5}$$

and the sign of $\operatorname{Re}[b(\mathrm{i}\omega)/a(\mathrm{i}\omega)]$ coincides with the sign of $\mathcal{P}_{a,b}(\omega^2)$.

Lemma 2.1 (Four-vertex reduction [1, 6, 8])**.** For each fixed $t \ge 0$, as a ranges over $\mathcal{K}$, the pair $\big(E_a(t), O_a(t)\big)$ sweeps a closed axis-aligned rectangle whose four corners are realized by four Kharitonov vertex polynomials in $\mathcal{K}$ independent of t. Consequently, for any fixed $b \in \mathbb{R}[s]$ and $t \ge 0$, $\mathcal{P}_{a,b}(t) > 0$ (respectively, ≥ 0) holds for all $a \in \mathcal{K}$ if and only if it holds at these four vertices.

We now state the main theorem of this paper.

Theorem 2.2 (All-order synthesis under wide interior intervals)**.** Let $n \ge 1$, and let $\mathcal{K}$ be a Hurwitz box of the form (2.1). If the interior coefficient intervals satisfy

$$\frac{l_j}{u_j} \le 10^{-5} \qquad (1 \le j < n), \tag{2.6}$$

then there exists a real polynomial $b \in \mathbb{R}[s]$ of exact degree n satisfying (2.2).

Remark 2.3 (Uniformity and direction of the width condition). The threshold (2.6) in Theorem 2.2 is uniform in both the order n and the stability margin of $\mathcal{K}$, and imposes no width restriction on the constant coefficient a_0 or the leading coefficient a_n. Note the direction of the inequality: (2.6) requires each interior coefficient to vary across a factor of at least 10^5. Under the joint hypothesis that the *entire* box is Hurwitz stable, such wide variation forces adjacent characteristic frequency scales apart by a factor of at least 10^{10}, decoupling the synthesis constraints across frequency bands.

3 Scale Separation and Stationary-Point Alignment

Since the cases $n = 1$ and $n = 2$ of Theorem 2.2 are elementary (handled at the end of Section 4.2), we assume throughout Sections 3 and 4.1 that $n \ge 3$ and that $\mathcal{K}$ is a Hurwitz box satisfying (2.6). Define the coefficient ratios, maximum interior ratio, and four-coefficient stability ratios by

$$\xi_j \coloneqq \frac{l_j}{u_j} \quad (0 \le j \le n), \qquad \epsilon \coloneqq \max_{1 \le j < n} \xi_j \le 10^{-5}, \qquad D_j \coloneqq \frac{l_j l_{j+1}}{u_{j-1} u_{j+2}} \quad (1 \le j \le n-2). \tag{3.1}$$

A key structural feature of the proof is the distinction between the *small-margin regime* $1 < D_j \le 4$ (where the stability margin can be arbitrarily small, so crude approximations can lose positivity) and the *large-margin regime* $D_j > 4$ (where direct coefficient dominance suffices).

3.1 Four-Coefficient Ratios and Baseline Scale Separation

Lemma 3.1 (Four-coefficient stability inequality)**.** Every Hurwitz-stable polynomial $a(s) = \sum_{j=0}^{n} a_j s^j$ satisfies $a_j a_{j+1} > a_{j-1} a_{j+2}$ for all $1 \le j \le n-2$. Consequently, for the Hurwitz box $\mathcal{K}$, every four-coefficient ratio satisfies $D_j > 1$ $(1 \le j \le n-2)$.

Proof. Set $P(x) = E_a(-x)$ and $Q(x) = O_a(-x)$. By Hermite–Biehler strict interlacing [4], the roots of P and xQ are simple, nonpositive, and strictly alternating, so both Q/P and $P/(xQ)$ admit partial-fraction expansions of the form $c + \sum w/(x+\alpha)$ with $w > 0$ and $\alpha \ge 0$ (in $P/(xQ)$, the residue at $x = 0$ is $P(0)/Q(0) = a_0/a_1 > 0$). For either quotient A/B and each pole $-\alpha$, set $H = B/(x+\alpha)$ (taking out-of-range coefficients to be zero). Comparing coefficients in $A = cB + \sum wH$ yields

$$A_k B_{k+1} - B_k A_{k+1} = \sum w\big(H_k^2 - H_{k-1} H_{k+1}\big).$$

Since all roots of H are real and nonpositive, Newton's inequalities give $H_k^2 - H_{k-1}H_{k+1} \ge 0$, with strict inequality whenever $H_k > 0$. For Q/P, every H has positive coefficients throughout its degree range; for $P/(xQ)$, the pole at zero alone contributes $H = Q$. Reading these two sets of inequalities in alternating index order gives $a_j a_{j+1} > a_{j-1} a_{j+2}$ for every $1 \le j \le n-2$. Choosing $a_j = l_j$, $a_{j+1} = l_{j+1}$, $a_{j-1} = u_{j-1}$, and $a_{j+2} = u_{j+2}$ in $\mathcal{K}$ yields $D_j > 1$. □

Lemma 3.2 (Baseline scales and geometric separation)**.** There exist positive baseline coefficients $\beta_0, \dots, \beta_n > 0$ satisfying

$$\frac{\beta_j}{\beta_{j+1}} \le \frac{l_{j+1}}{u_{j+2}} \quad (0 \le j \le n-2), \qquad \frac{\beta_{j+1}}{\beta_j} \le \frac{l_j}{u_{j-1}} \quad (1 \le j \le n-1). \tag{3.2}$$

Writing $W_j \coloneqq u_j/u_{j+2}$ $(0 \le j \le n-2)$ and $\tau_j \coloneqq \beta_{j-1}/\beta_{j+1}$ $(1 \le j \le n-1)$, we have the geometric scale separation

$$\frac{W_j}{W_{j+1}} \le \epsilon^2 \quad (0 \le j \le n-3), \qquad \frac{\tau_j}{\tau_{j+1}} \le \epsilon^2 \quad (1 \le j \le n-2). \tag{3.3}$$

Proof. Fix $\delta = 1/16$, set $\beta_1 = 1$ and $\beta_0 = \delta l_1/u_2$, and define the interior ratios inductively by

$$\frac{\beta_{i+1}}{\beta_i} = \begin{cases} \dfrac{u_{i+2}}{l_{i+1}}, & 1 < D_i \le 4, \\[2ex] \sqrt{\dfrac{u_{i+2} l_i}{l_{i+1} u_{i-1}}}, & D_i > 4, \end{cases} \qquad (1 \le i \le n-2), \tag{3.4}$$

and finally $\beta_n = \delta l_{n-1}\beta_{n-1}/u_{n-2}$. For each $1 \le i \le n-2$, introduce the dimensionless quantities

$$A_i \coloneqq \frac{u_{i+2}\beta_i}{l_{i+1}\beta_{i+1}}, \qquad B_i \coloneqq \frac{u_{i-1}\beta_{i+1}}{l_i \beta_i}. \tag{3.5}$$

By (3.4) and Lemma 3.1, $(A_i, B_i) = (1, 1/D_i)$ when $1 < D_i \le 4$, and $A_i = B_i = 1/\sqrt{D_i} < 1/2$ when $D_i > 4$. Together with $\beta_0/\beta_1 = \delta l_1/u_2 < l_1/u_2$ and $\beta_n/\beta_{n-1} = \delta l_{n-1}/u_{n-2} < l_{n-1}/u_{n-2}$, this proves (3.2). Finally,

$$\frac{W_j}{W_{j+1}} = \frac{u_j u_{j+3}}{u_{j+1}u_{j+2}} = \frac{\xi_{j+1}\xi_{j+2}}{D_{j+1}} \le \epsilon^2,$$

and multiplying $\beta_{j-1}/\beta_j \le l_j/u_{j+1}$ with $\beta_{j+2}/\beta_{j+1} \le l_{j+1}/u_j$ (from the second inequality in (3.2) at index $j+1$) gives $\tau_j/\tau_{j+1} = (\beta_{j-1}/\beta_j)(\beta_{j+2}/\beta_{j+1}) \le (l_j/u_{j+1})(l_{j+1}/u_j) = \xi_j\xi_{j+1} \le \epsilon^2$, establishing (3.3). □

3.2 Stationary-Point Alignment in the Small-Margin Regime

For each $0 \le i < n$, let $a^{(i)} \in \mathcal{K}$ denote the Kharitonov vertex taking lower endpoints l_j at indices $j \equiv i, i+1 \pmod 4$ and upper endpoints u_j at all other indices. As shown in Lemma 4.1 below, $a^{(i)}$ will be the minimizing vertex of $\mathcal{P}_{a,b}(t)$ on the ith frequency band.

To handle indices with $1 < D_i \le 4$, we perturb the baseline coefficients β_j by positive multipliers z_j ($b_j = \beta_j z_j$) with ratios $x_i = z_{i+1}/z_i$ chosen via Brouwer's fixed-point theorem so that the normalized pairing on band i has a stationary point precisely at a zero of the rotated real part of $a^{(i)}(\mathrm{i}\omega)$.

Proposition 3.3. There exist positive coefficients $b_j = \beta_j z_j$ $(0 \le j \le n)$ with $z_0 = z_1 = 1$ and $z_n = z_{n-1}$ such that the interior ratios $x_i \coloneqq z_{i+1}/z_i$ $(1 \le i \le n-2)$ satisfy $|x_i - 1| \le 4000\epsilon$, with $x_i = 1$ whenever $D_i > 4$. Moreover, for each $1 \le i \le n-2$ with $D_i \le 4$, setting $T_i \coloneqq l_i/u_{i+2}$, the normalized pairing

$$H_i(v) \coloneqq \frac{\mathcal{P}_{a^{(i)},b}(T_i v)}{b_i l_i (T_i v)^i} \tag{3.6}$$

has a stationary point $v_i \in [3/4, 5/4]$ with $H_i(v_i) > 0$.

Proof. Let $x = (x_1, \dots, x_{n-2})$ range over the cube $[7/8, 9/8]^{n-2}$ and recover $z_0, \dots, z_n$ from $z_0 = z_1 = 1$, $z_{i+1} = x_i z_i$, and $z_n = z_{n-1}$. Adjacent z_j differ by at most a factor of $8/7$ in either direction.

Fix an index i with $1 < D_i \le 4$, and set $t = T_i v = \omega^2$ ($\omega > 0$) for $v \in [1/2, 2]$. Divide both $a^{(i)}(\mathrm{i}\omega)$ and $b(\mathrm{i}\omega)$ by $(\mathrm{i}\omega)^i$, and express their real and imaginary parts in units of $(l_i, \omega l_{i+1})$ for $a^{(i)}$ and $(b_i, \omega\beta_{i+1} z_i)$ for b. Using $a_i^{(i)} = l_i$, $a_{i+1}^{(i)} = l_{i+1}$, $a_{i+2}^{(i)} = u_{i+2}$, $a_{i-1}^{(i)} = u_{i-1}$, and $T_i = l_i/u_{i+2}$, these four normalized components take the form

$$R_a(v) = 1 - v + e_1(v), \quad I_a(v) = 1 - (D_i v)^{-1} + e_2(v), \qquad R_b(v) = 1 + e_3(v), \quad I_b(v) = x_i + e_4(v). \tag{3.7}$$

Define the C^1 norm $\|f\|_1 \coloneqq \sup_{v\in[1/2,2]} |f(v)| + \sup_{v\in[1/2,2]} |f'(v)|$. We claim that $\|e_1\|_1, \|e_2\|_1 < 64\epsilon^2$ and $\|e_3\|_1, \|e_4\|_1 < 64\epsilon$. Indeed:

- In R_a, the nearest omitted lower term at $v = 1$ has size at most $u_{i-2}u_{i+2}/l_i^2 = \xi_{i-1}\xi_{i+1}/(D_{i-1}D_i) \le \epsilon^2$, and the nearest upper term has size at most $T_i/W_{i+2} \le \epsilon^2$ (since $T_i = \xi_i W_i$ and $W_i/W_{i+2} \le \epsilon^4$).
- In I_a, the nearest omitted lower term at $v = 1$ (present when $i \ge 3$) has magnitude $(\xi_{i-3}/D_i)\, W_{i-3}/T_i$, which is bounded above by $W_{i-3}/T_i \le \epsilon^2$ (using $W_{i-1}/T_i = \xi_{i+1}/D_i$ and (3.3)), and the nearest omitted upper term has magnitude at most $\xi_i\xi_{i+2}/D_{i+1} \le \epsilon^2$.
- By (3.3), each subsequent step outwards in R_a or I_a introduces an additional factor of at most ϵ^2 (out-of-range terms are absent).
- In R_b and I_b, (3.2) and the first branch of (3.4) give $\tau_i/T_i \le \epsilon$ and $T_i/\tau_{i+1} \le \epsilon$. Accounting for the factor $(8/7)^2$ from adjacent z-ratios, the nearest omitted terms at $v = 1$ cost at most 3ϵ, and each subsequent parity step costs at most $2\epsilon^2$.

Bounding v^k and its derivative on $[1/2, 2]$ yields the geometric majorants

$$\|e_1\|_1, \|e_2\|_1 \le 8\epsilon^2 \sum_{h\ge0} (5 + 2h)(2\epsilon^2)^h < 64\epsilon^2, \qquad \|e_3\|_1, \|e_4\|_1 \le 12\epsilon \sum_{h\ge0} (3 + 2h)(4\epsilon^2)^h < 64\epsilon.$$

Since $R_a(v) = 1 - v + e_1(v)$ with $\|e_1\|_1 < 64\epsilon^2$, R_a has a unique simple zero $v_i \in [3/4, 5/4]$ with $R_a'(v_i) < 0$; note that v_i depends only on the vertex $a^{(i)}$ and not on x. At this zero, $I_a(v_i) > 0$: along the positive imaginary axis, the argument of any Hurwitz polynomial is strictly increasing (each root $-p + \mathrm{i}q$ with $p > 0$ contributes $p/[p^2 + (\omega - q)^2] > 0$ to $\frac{\mathrm{d}}{\mathrm{d}\omega} \arg a(\mathrm{i}\omega)$), and dividing by $(\mathrm{i}\omega)^i$ merely subtracts a constant; hence at any crossing where the real part decreases through zero, the imaginary part must be strictly positive.

Using $\beta_{i+1}/\beta_i = u_{i+2}/l_{i+1} = l_i/(l_{i+1}T_i)$ from (3.4), the normalized pairing (3.6) equals

$$H_i(v) = R_a(v)R_b(v) + v\, I_a(v)I_b(v).$$

Its principal part $(1 - v) + (v - D_i^{-1})x_i$ has derivative $x_i - 1$. Using the submultiplicativity of $\|\cdot\|_1$ (with bounds $2, 5, 1, 2$ for the four principal components and 3 for v on $[1/2, 2]$), we obtain

$$\left|H_i'(v_i) - (x_i - 1)\right| \le 4000\epsilon < \frac{1}{8}.$$

Define the continuous map $F : [7/8, 9/8]^{n-2} \to \mathbb{R}^{n-2}$ by setting $F_i(x) = H_i'(v_i)$ when $D_i \le 4$, and $F_i(x) = x_i - 1$ when $D_i > 4$. Because $|(x_i - F_i(x)) - 1| \le 4000\epsilon < 1/8$, the map $x \mapsto x - F(x)$ maps the compact convex cube $[7/8, 9/8]^{n-2}$ continuously into itself. By Brouwer's fixed-point theorem, there exists a fixed point x^* at which $F(x^*) = 0$. This gives $x_i^* = 1$ for $D_i > 4$, $|x_i^* - 1| \le 4000\epsilon$ for all i, and $H_i'(v_i) = 0$ for all $D_i \le 4$. Finally, at v_i, $R_a(v_i) = 0$ and $I_b(v_i) \ge 7/8 - 64\epsilon > 0$, so $H_i(v_i) = v_i I_a(v_i) I_b(v_i) > 0$, using the true sign of $I_a(v_i)$ regardless of how small $D_i - 1$ may be. □

4 Frequency Band Decomposition and Proof of Theorem 2.2

4.1 Frequency Band Decomposition and Envelope Decay

Fix the numerator $b(s) = \sum_{j=0}^n b_j s^j$ constructed in Proposition 3.3. Its alternating parity zeros partition $[0, \infty)$ into n frequency bands on each of which a single vertex $a^{(i)}$ minimizes $\mathcal{P}_{a,b}(t)$.

Lemma 4.1 (Frequency band decomposition)**.** There exist strictly increasing nodes $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = \infty$ such that, for each $0 \le i < n$, the vertex $a^{(i)}$ minimizes $\mathcal{P}_{a,b}(t)$ over $\mathcal{K}$ at every finite $t \in [t_i, t_{i+1}]$. Putting $\sigma_j \coloneqq b_{j-1}/b_{j+1}$ $(1 \le j < n)$, we have

$$\frac{\sigma_j}{2} < t_j < 2\sigma_j \qquad (1 \le j < n),$$

and whenever $1 \le i \le n - 2$ satisfies $D_i \le 4$, the stationary point satisfies $T_i v_i \in (t_i, t_{i+1})$.

Proof. Since adjacent z-ratios differ by at most $(8/7)^2$, (3.3) implies $\sigma_j/\sigma_{j+1} \le 2\epsilon^2$. Setting $t = \sigma_j v$ ($v \in [1/2, 2]$), the two terms of degrees $j-1$ and $j+1$ in the corresponding parity component (E_b if j is odd, O_b if j is even), normalized by the degree-$(j-1)$ term, give $1 - v$. At $v = 1$, the nearest omitted term on either side has magnitude at most $(2\epsilon^2)^2$, and each subsequent parity step introduces a factor of at most $2\epsilon^2$. The same geometric series bound as in Proposition 3.3 keeps the C^1 error below $128\epsilon^2$, yielding a unique simple zero $t_j \in (\sigma_j/2, 2\sigma_j)$. Meanwhile, the opposite parity component is dominated by its degree-j term (neighboring terms cost at most $2\epsilon^2$ at $v = 1$). Because these $n-1$ intervals are disjoint, they exhaust all zeros of E_b and O_b on $(0, \infty)$.

At $t = 0$, both $E_b(0) = b_0 > 0$ and $O_b(0) = b_1 > 0$. As t crosses $t_1, t_2, \ldots, t_{n-1}$ in turn, the zeros alternate between E_b and O_b, starting with E_b at t_1. Hence on each band $[t_i, t_{i+1}]$ (for $\omega > 0$), both the real and imaginary parts of $b(\mathrm{i}\omega)/(\mathrm{i}\omega)^i$ are nonnegative. Minimizing the pairing $\mathcal{P}_{a,b}(t)$ in these rotated axes selects the lower endpoints at indices $j \equiv i, i+1 \pmod 4$ and the upper endpoints elsewhere, which is precisely $a^{(i)}$. Continuity includes the band endpoints and $t = 0$. Finally, for $D_i \le 4$, the bounds $\tau_i/T_i, T_i/\tau_{i+1} \le \epsilon$ from Proposition 3.3 give $\sigma_i/T_i, T_i/\sigma_{i+1} \le 2\epsilon$, placing $T_i v_i \in [\frac{3}{4}T_i, \frac{5}{4}T_i]$ strictly inside (t_i, t_{i+1}). □

For any fixed vertex $a \in \mathcal{K}$, expand its pairing with b as $\mathcal{P}_{a,b}(t) = \sum_{k=0}^{n} c_k t^k$.

Lemma 4.2 (Coefficient bounds and tail decay)**.** For every vertex a of $\mathcal{K}$ and every $0 \le k \le n$:

(i) $|c_k| \le 4 b_k u_k$, and if $a_k = u_k$, then $c_k \ge (1 - 4\epsilon) b_k u_k$;

(ii) on band i ($t \in [t_i, t_{i+1}]$), successive upper envelopes $b_j u_j t^j$ decrease by a factor of at most $q_0 := 8\epsilon^3$ when moving upwards from $j \ge i+2$, or downwards from $j \le i-1$.

Proof. Expanding (2.4) yields

$$c_k = \sum_h (-1)^h b_{k-h} a_{k+h},$$

summed over all integers h for which both $0 \le k - h \le n$ and $0 \le k + h \le n$. By (3.2) and $z_{k\pm1}/z_k \le 8/7$, each distance-one term ($h = \pm 1$) has magnitude at most $(8/7) b_k l_k$. Putting $r_j := u_{j+1}/u_j$, we have $r_{p+1}/r_{p-1} = \xi_p \xi_{p+1}/D_p \le \epsilon^2$, so $r_{k+h}/r_{k-h} \le \epsilon^{2h}$ for $h \ge 1$. Hence for $h \ge 1$, the ratio of successive envelopes $b_{k-h} u_{k+h}$ satisfies

$$\frac{b_{k-h-1} u_{k+h+1}}{b_{k-h} u_{k+h}} \le \frac{8}{7} \xi_{k-h} \frac{r_{k+h}}{r_{k-h}} \le 2\epsilon^{2h}.$$

Reversing coefficient order gives the same bound for $h \le -1$. Summing over $|h| \ge 2$, all terms at distance at least two together cost at most $8\epsilon^2 b_k l_k$. Combining the central term $b_k a_k$ with the distance-one and tail bounds yields (i) (with $c_0 = b_0 a_0$ and $c_n = b_n a_n$ exact).

For (ii), if a high tail exists on band i ($i \le n-2$), Lemma 4.1 gives $t \le t_{i+1} < 2b_i/b_{i+2}$. For any $j \ge i+2$, applying (3.2) and $W_{j-1} \ge W_{i+1}$ (from (3.3)) yields

$$t \frac{b_{j+1} u_{j+1}}{b_j u_j} \le 2 \left(\frac{8}{7}\right)^3 \frac{l_{i+1} l_{i+2}}{u_{i+2} u_{i+3}} \xi_j \frac{u_{j+1}}{u_{j-1}} \le 8\epsilon^3 = q_0.$$

The low-tail estimate ($j \le i-1$) follows symmetrically by reversing all coefficient indices and replacing t with $1/t$. □

4.2 Bandwise Positivity and Proof of Theorem 2.2

We verify strict positivity of $\mathcal{P}_{a^{(i)},b}(t)$ separately on the interior bands and on the two boundary bands, and then assemble the proof of Theorem 2.2.

Lemma 4.3 (Positivity on interior bands)**.** For every interior index $1 \le i \le n-2$ and every $t \in [t_i, t_{i+1}]$, we have $\mathcal{P}_{a^{(i)},b}(t) > 0$.

Proof. Let $a = a^{(i)}$ and write $\mathcal{P}_{a,b}(t) = \sum_{k=0}^{n} c_k t^k$. By definition of $a^{(i)}$, we have $a_{i-1} = u_{i-1}$ and $a_{i+2} = u_{i+2}$, so Lemma 4.2(i) gives $c_{i-1} \ge (1 - 4\epsilon) b_{i-1} u_{i-1} > 0$ and $c_{i+2} \ge (1 - 4\epsilon) b_{i+2} u_{i+2} > 0$. In the scaled function $g_i(t) := t^{-i} \mathcal{P}_{a,b}(t)$, these two terms carry powers t^{-1} and t^2, whose second derivatives are strictly positive, while the central terms c_i and $c_{i+1} t$ vanish upon differentiating twice. By Lemma 4.2(i)–(ii), relative to the nearer of the two positive second derivatives ($2c_{i-1} t^{-3}$ on the low side and $2c_{i+2}$ on the high side), the sum of the absolute values of all more distant second derivatives on the same side is bounded by

$$\frac{4}{1 - 4\epsilon} \sum_{h \ge 1} \frac{(h+1)(h+2)}{2} q_0^h = \frac{4}{1 - 4\epsilon} \left[(1 - q_0)^{-3} - 1\right] < \frac{1}{2}. \tag{4.1}$$

Hence $g_i''(t) > 0$ throughout (t_i, t_{i+1}), so g_i is strictly convex on $[t_i, t_{i+1}]$.

- If $1 < D_i \le 4$, then $g_i(T_i v) = b_i l_i H_i(v)$ attains a strictly positive stationary value at $T_i v_i \in (t_i, t_{i+1})$ by Proposition 3.3 and Lemma 4.1. Strict convexity implies $g_i(t) > 0$ on all of $[t_i, t_{i+1}]$.
- If $D_i > 4$, then $x_i = 1$ and $A_i = B_i = 1/\sqrt{D_i} < 1/2$. In c_i, the neighbor $b_{i+1}u_{i-1}$ costs $B_i < 1/2$ relative to the central term $b_i l_i$, while the other neighbor $b_{i-1}l_{i+1}$ uses the lower endpoint l_{i+1} and costs at most $(8/7)\epsilon$. Symmetrically, in c_{i+1}, the neighbor $b_i u_{i+2}$ costs $A_i < 1/2$ relative to $b_{i+1}l_{i+1}$, and the other neighbor costs at most $(8/7)\epsilon$. Combining with the distance-≥ 2 bound $8\epsilon^2$ from Lemma 4.2, both central coefficients satisfy
$$\frac{c_i}{b_i l_i},\ \frac{c_{i+1}}{b_{i+1}l_{i+1}} \ge 1 - \frac{1}{2} - \frac{8}{7}\epsilon - 8\epsilon^2 > 0.$$
Meanwhile, the estimate (4.1) (without the derivative factors) shows that the remaining tails ($k \le i-2$ and $k \ge i+3$) are bounded in magnitude by less than half of the two positive outer terms $c_{i-1}t^{i-1}$ and $c_{i+2}t^{i+2}$. Therefore $\mathcal{P}_{a,b}(t) > 0$ on $[t_i, t_{i+1}]$. □

Lemma 4.4 (Positivity on the first and last bands)**.** We have $\mathcal{P}_{a^{(0)},b}(t) > 0$ for all $t \in [0, t_1]$ and $\mathcal{P}_{a^{(n-1)},b}(t) > 0$ for all $t \in [t_{n-1}, \infty)$.

Proof. On band 0, $a = a^{(0)}$ has $a_0 = l_0$, $a_1 = l_1$, and $a_2 = u_2$. The constant and linear coefficients of $\mathcal{P}_{a,b}(t)$ are $c_0 = b_0 l_0 > 0$ and $c_1 = S - b_0 u_2 - M$, where $S \coloneqq b_1 l_1$ and $M \coloneqq b_2 l_0$. Subtracting the high tail ($k \ge 3$) from $c_2 t^2$ via Lemma 4.2 leaves at least $\frac{1}{2}b_2 u_2 t^2$ (since $1 - 4\epsilon - 4q_0/(1-q_0) > 1/2$). By construction of β_0, $b_0 u_2 = \delta S$ with $\delta = 1/16$, while (3.2) and Proposition 3.3 give $M/S \le 1 + 8000\epsilon$. If $M \le (1-2\delta)S$, then $c_1 > 0$ and the quadratic $c_0 + c_1 t + \frac{1}{2}b_2 u_2 t^2$ is strictly positive on $[0, \infty)$. Otherwise $M > (1-2\delta)S$, and the negative part of c_1 satisfies $(-c_1)_+ \le (\delta + 8000\epsilon)S$; dividing its square by $c_0 \cdot (\frac{1}{2}b_2 u_2) = \frac{1}{2}b_0 u_2 M > \frac{1}{2}\delta(1-2\delta)S^2$ yields

$$\frac{(-c_1)_+^2}{\frac{1}{2}c_0 b_2 u_2} < \frac{2(\delta + 8000\epsilon)^2}{\delta(1-2\delta)} < 4.$$

Hence if $c_1 \ge 0$, the quadratic $c_0 + c_1 t + \frac{1}{2}b_2 u_2 t^2$ is strictly positive on $[0, \infty)$ directly; if $c_1 < 0$, then $(-c_1)_+^2 = c_1^2$, so the bound above shows that its discriminant is strictly negative. In either case, $\mathcal{P}_{a^{(0)},b}(t) > 0$ for all $t \in [0, t_1]$.

On the last band $i = n-1$, reversing the coefficient order and substituting $t \mapsto 1/t$ yields the exact same quadratic estimate and sign-case split, since $b_n u_{n-2} = \delta b_{n-1} l_{n-1}$ and the reversed multiplier ratio is bounded by $1/(1 - 4000\epsilon) \le 1 + 8000\epsilon$. □

Proof of Theorem 2.2. For $n = 1$, $b(s) = 1 + s$ gives $\mathcal{P}_{a,b}(t) = a_0 + a_1 t > 0$ for all $t \ge 0$. For $n = 2$, pick $b_1 = 1$ and choose $b_0, b_2 > 0$ small enough that $b_0 u_2 + b_2 u_0 < l_1$; then $\mathcal{P}_{a,b}(t) = b_2 a_2 t^2 + (a_1 - b_0 a_2 - b_2 a_0)t + b_0 a_0$ has all three coefficients strictly positive across $\mathcal{K}$.

For $n \ge 3$, take the degree-n polynomial b constructed in Proposition 3.3 (whose leading coefficient $b_n > 0$). By Lemma 4.1, the bands $[t_i, t_{i+1}]$ ($0 \le i < n$) cover $[0, \infty)$ with minimizing vertex $a^{(i)}$. Lemmas 4.3 and 4.4 show that $\mathcal{P}_{a^{(i)},b}(t) > 0$ on each band, so $\mathcal{P}_{a,b}(t) > 0$ for all $a \in \mathcal{K}$ and $t \ge 0$ by Lemma 2.1. Equation (2.5) and conjugate symmetry then yield (2.2) for every $a \in \mathcal{K}$ and every $\omega \in \mathbb{R}$. □

5 Conclusion

We have established a uniform all-order sufficient condition for equal-degree robust SPR synthesis of Hurwitz-stable interval polynomial families: whenever every interior coefficient interval satisfies $l_j/u_j \le 10^{-5}$ ($1 \le j < n$), whole-box Hurwitz stability forces geometric scale separation across frequency bands and guarantees the existence of a common exact-degree-n SPR numerator in every order $n \ge 1$. Determining the sharp threshold for the interior endpoint ratio l_j/u_j and relaxing the uniform requirement across all interior indices remain natural directions for future work.

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References

[1] B. D. O. Anderson, S. Dasgupta, P. Khargonekar, F. J. Kraus, and M. Mansour. Robust strict positive realness: characterization and construction. *IEEE Transactions on Circuits and Systems*, 37(7):869–876, 1990.

[2] A. Betser and E. Zeheb. Design of robust strictly positive real transfer functions. *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, 40(9):573–580, 1993.

[3] S. Dasgupta and A. Bhagwat. Conditions for designing strictly positive real transfer functions for adaptive output error identification. *IEEE Transactions on Circuits and Systems*, 34(7):731–736, 1987.

[4] Olga Holtz. Hermite–Biehler, Routh–Hurwitz, and total positivity. *Linear Algebra and its Applications*, 372:105–110, 2003.

[5] V. L. Kharitonov. The asymptotic stability of the equilibrium state of a family of systems of linear differential equations. *Differentsial'nye Uravneniya*, 14(11):2086–2088, 1978.

[6] Long Wang and Wensheng Yu. Robust SPR synthesis for low-order polynomial segments and interval polynomials. In *Proceedings of the 2001 American Control Conference*, volume 5, pages 3612–3617. IEEE, 2001.

[7] Wensheng Yu and Long Wang. Anderson's claim on fourth-order SPR synthesis is true. *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, 48(4):506–509, 2001.

[8] Wensheng Yu and Long Wang. Design of robust strictly positive real transfer functions. In Fei-Yue Wang and Derong Liu, editors, *Networked Control Systems: Theory and Applications*, pages 293–341. Springer, London, 2008.

[9] Wensheng Yu, Long Wang, and Jürgen Ackermann. Robust strictly positive real synthesis for polynomial families of arbitrary order. *Science in China Series F: Information Sciences*, 47(4):475–489, 2004.