Counterexamples to Robust Strictly Positive Real Synthesis for High-Order Interval Polynomials

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Abstract

We show that robust Hurwitz stability does not guarantee equal-degree strictly positive real (SPR) synthesis for high-order interval polynomial families. Specifically, for every order $n \geq 10$, we construct a monic Hurwitz-stable interval polynomial family with rational endpoints and positive width in every nonleading coefficient such that every nonzero real numerator of degree at most n produces a strictly negative real part at some member and some positive frequency. At order ten, twelve fixed vertex–frequency evaluations on four Kharitonov vertices form a matrix of full column rank 11 admitting a strictly positive linear dependence, ruling out all nonzero real numerators of degree at most 10 (even weakly). An additional negative moment enables inductive low-end degree lifting $a(s) \mapsto sa(s) + \eta$ to propagate this finite obstruction to every order $n \geq 10$.

Keywords: strictly positive real synthesis; interval polynomials; Hurwitz stability; counterexamples; degree lifting

1 Introduction

Strictly positive real (SPR) transfer functions play a central role in passivity analysis, absolute stability, adaptive control, and output-error system identification [3, 8]. For a single Hurwitz-stable denominator $a(s)$, choosing $b(s) = a(s)$ trivially yields a transfer function with strictly positive real part on the imaginary axis. When the denominator varies over an uncertain family, however, robust SPR synthesis requires a *single* fixed real numerator $b(s)$ of prescribed degree such that $\mathrm{Re}[b(\mathrm{i}\omega)/a(\mathrm{i}\omega)] > 0$ holds simultaneously for every admissible denominator a and every real frequency ω.

For real interval polynomial families, in which each coefficient varies independently in a closed interval, Kharitonov's theorem [5] reduces robust Hurwitz stability to four vertex polynomials. For a *prescribed* numerator, checking the SPR condition across the entire family likewise reduces to these four vertices [1, 6]. Synthesis is fundamentally harder because the same numerator must satisfy all four vertex inequalities at all frequencies simultaneously. For low-order Hurwitz-stable interval families ($n \leq 4$), Anderson et al. [1], Betser and Zeheb [2], and Yu and Wang [7, 6] established the existence of a common SPR numerator. For convex combinations of two polynomials (segments), common-numerator synthesis is possible in arbitrary order [9]. Yet an interval box allows all coefficients to vary independently, so the segment theorem does not settle whether robust Hurwitz stability suffices for interval families of arbitrary order.

In this paper, we resolve the high-order question negatively for every order $n \geq 10$ (Theorem 2.2). At order ten, we construct an explicit rational monic Hurwitz-stable interval box with positive width in all ten nonleading coefficients and exhibit twelve vertex–frequency evaluation rows on four Kharitonov vertices that have full column rank 11 together with a strictly positive linear dependence. This finite linear-algebraic obstruction rules out every nonzero real numerator of degree at most 10, even under the weaker nonnegative real-part condition. Moreover, the same twelve weights satisfy an additional strict moment inequality, which allows us to propagate the finite obstruction inductively from degree d to $d + 1$ via low-end degree lifting $a(s) \mapsto s\,a(s) + \eta$.

The remainder of the paper is organized as follows. Section 2 formulates the interval SPR synthesis problem, recalls the four-vertex reduction, and states Theorem 2.2. Section 3 establishes the finite obstruction criterion, verifies the order-ten seed obstruction, and proves the inductive degree-lifting lemma to complete the proof of Theorem 2.2. Section 4 concludes with remarks on open intermediate orders, and Appendix A records the exact rational data for the order-ten certificate.

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2 Problem Formulation and Main Result

Write polynomial coefficients in ascending order and consider the nth-order interval polynomial box

$$\mathcal{K} = \left\{ a(s) = \sum_{j=0}^{n} a_j s^j : l_j \le a_j \le u_j,\ 0 \le j \le n \right\}, \qquad 0 < l_j \le u_j < \infty. \tag{2.1}$$

A real polynomial $a \in \mathbb{R}[s]$ is *Hurwitz stable* if all of its complex roots lie in the open left half-plane $\{s \in \mathbb{C} : \operatorname{Re} s < 0\}$, and the box $\mathcal{K}$ is *Hurwitz* (or robustly Hurwitz stable) if every member $a \in \mathcal{K}$ is Hurwitz stable. Following the equal-degree convention in [8], a real polynomial $b \in \mathbb{R}[s]$ of exact degree n is a *common SPR numerator* for $\mathcal{K}$ if

$$\operatorname{Re} \frac{b(\mathrm{i}\omega)}{a(\mathrm{i}\omega)} > 0 \qquad (\forall\, a \in \mathcal{K},\ \forall\, \omega \in \mathbb{R}). \tag{2.2}$$

Setting $t = \omega^2 \ge 0$, decompose any real polynomial $a(s) = \sum_{j=0}^{n} a_j s^j$ into its even and odd parts along the imaginary axis:

$$a(\mathrm{i}\omega) = E_a(t) + \mathrm{i}\omega O_a(t), \qquad E_a(t) = \sum_{k \ge 0} a_{2k}(-t)^k, \qquad O_a(t) = \sum_{k \ge 0} a_{2k+1}(-t)^k. \tag{2.3}$$

For a candidate numerator $b(s) = \sum_{j=0}^{n} b_j s^j$, define the real frequency pairing polynomial

$$\mathcal{P}_{a,b}(t) \coloneqq E_a(t)E_b(t) + t\, O_a(t)O_b(t) = R(a,t) \cdot (b_0, \ldots, b_n), \tag{2.4}$$

where the evaluation row vector $R(a,t) = \big(R_0(a,t), \ldots, R_n(a,t)\big) \in \mathbb{R}^{n+1}$ has entries

$$R_{2k}(a,t) = (-t)^k E_a(t), \qquad R_{2k+1}(a,t) = (-t)^k t\, O_a(t). \tag{2.5}$$

Whenever a is Hurwitz stable, $|a(\mathrm{i}\omega)|^2 = E_a(\omega^2)^2 + \omega^2 O_a(\omega^2)^2 > 0$, so

$$\operatorname{Re} \frac{b(\mathrm{i}\omega)}{a(\mathrm{i}\omega)} = \frac{\mathcal{P}_{a,b}(\omega^2)}{|a(\mathrm{i}\omega)|^2}, \tag{2.6}$$

and the sign of $\operatorname{Re}[b(\mathrm{i}\omega)/a(\mathrm{i}\omega)]$ coincides with the sign of $\mathcal{P}_{a,b}(\omega^2)$.

Lemma 2.1 (Four-vertex reduction [1, 6, 8])**.** For each fixed $t \ge 0$, as a ranges over $\mathcal{K}$, the pair $\big(E_a(t), O_a(t)\big)$ sweeps a closed axis-aligned rectangle whose four corners are realized by four Kharitonov vertex polynomials in $\mathcal{K}$ independent of t. Consequently, for any fixed $b \in \mathbb{R}[s]$ and $t \ge 0$, $\mathcal{P}_{a,b}(t) > 0$ (respectively, ≥ 0) holds for all $a \in \mathcal{K}$ if and only if it holds at these four vertices.

We now state the main theorem of this paper.

Theorem 2.2 (Counterexamples for all $n \ge 10$)**.** For every integer $n \ge 10$, there exists a Hurwitz box $\mathcal{K}$ of the form (2.1) with $l_n = u_n = 1$ and rational endpoints $l_j < u_j$ for every $0 \le j < n$ such that, for every nonzero real polynomial $b \in \mathbb{R}[s]$ of degree at most n, there exist $a \in \mathcal{K}$ and $\omega > 0$ satisfying

$$\operatorname{Re} \frac{b(\mathrm{i}\omega)}{a(\mathrm{i}\omega)} < 0.$$

At order $n = 10$, twelve fixed tests (a_m, ω_m) on four Kharitonov vertices suffice for all such b.

Remark 2.3 (Mechanism of the obstruction). Theorem 2.2 rules out not only equal-degree SPR numerators satisfying (2.2), but every nonzero real numerator of degree at most n, even under the weaker nonnegative real-part requirement $\operatorname{Re}[b(\mathrm{i}\omega)/a(\mathrm{i}\omega)] \ge 0$. In the order-ten construction (Appendix A), several interior endpoint ratios take $l_j/u_j = 1 - 10^{-300}$ very close to 1; the obstruction arises because all frequency constraints must be met by a single numerator, and the twelve evaluation rows on four vertices have full column rank 11 together with a strictly positive linear dependence, forcing every nonzero numerator to take a strictly negative value on at least one row.

3 Order-Ten Obstruction and Inductive Degree Lifting

In this section we prove Theorem 2.2. We begin with a general finite-matrix criterion that certifies the nonexistence of any nonzero weak numerator (Lemma 3.1), apply it to the explicit order-ten box (Proposition 3.2 and Corollary 3.3), and then lift the obstruction inductively from order d to $d+1$ (Lemma 3.4).

3.1 Finite Obstruction Criterion and the Order-Ten Counterexample

Lemma 3.1 (Finite obstruction criterion)**.** Let $\mathcal{K}$ be a Hurwitz box of degree $d \geq 1$. Suppose there exist finitely many test pairs $(a_m, t_m) \in \mathcal{K} \times (0, \infty)$ $(1 \leq m \leq M)$ and strictly positive weights $\lambda_1, \ldots, \lambda_M > 0$ such that the $M \times (d+1)$ matrix R with rows $R(a_m, t_m)$ has full column rank $d+1$ and satisfies

$$\sum_{m=1}^{M} \lambda_m R(a_m, t_m) = 0. \tag{3.1}$$

Then for every nonzero real polynomial $b \in \mathbb{R}[s]$ of degree at most d, there exists an index $m \in \{1, \ldots, M\}$ such that

$$\operatorname{Re} \frac{b(\mathrm{i}\sqrt{t_m})}{a_m(\mathrm{i}\sqrt{t_m})} < 0.$$

Proof. Identify $b(s) = \sum_{j=0}^{d} b_j s^j$ with its nonzero coefficient vector $\mathbf{b} = (b_0, \ldots, b_d)^{\mathsf{T}} \in \mathbb{R}^{d+1}$. Since R has full column rank $d+1$, the vector $R\mathbf{b} \in \mathbb{R}^M$ is nonzero. On the other hand, (3.1) gives $\sum_{m=1}^{M} \lambda_m (R\mathbf{b})_m = 0$ with all $\lambda_m > 0$. Hence at least one entry $(R\mathbf{b})_m = \mathcal{P}_{a_m, b}(t_m)$ must be strictly negative. Because $a_m \in \mathcal{K}$ is Hurwitz stable, $|a_m(\mathrm{i}\sqrt{t_m})|^2 > 0$, and (2.6) yields $\operatorname{Re}[b(\mathrm{i}\sqrt{t_m})/a_m(\mathrm{i}\sqrt{t_m})] < 0$. □

Proposition 3.2 (Order-ten seed obstruction)**.** There exist a rational monic Hurwitz box $\mathcal{K}_{10}$ of degree ten with $l_j < u_j$ for all $0 \leq j \leq 9$, twelve test pairs (a_m, t_m) $(1 \leq m \leq 12)$ on four vertices of $\mathcal{K}_{10}$ with $t_m > 0$, and strictly positive weights $\lambda_1, \ldots, \lambda_{12} > 0$ such that the 12×11 matrix R with rows $R(a_m, t_m)$ has full column rank 11 and satisfies

$$\sum_{m=1}^{12} \lambda_m R(a_m, t_m) = 0, \qquad \sum_{m=1}^{12} \lambda_m \big[-O_{a_m}(t_m)\big] < 0. \tag{3.2}$$

Proof. The explicit rational endpoints of $\mathcal{K}_{10}$, the root-bracketing intervals, the twelve vertex–frequency pairs, and the rational determinant/linear-system checks are given in Appendix A.

For Hurwitz stability of every $a \in \mathcal{K}_{10}$, four disjoint intervals $[L_h, H_h]$ $(h = 1, 2, 3, 4)$ each bracket a strict sign change of $O_a(t)$ uniformly over $\mathcal{K}_{10}$, yielding four distinct positive roots of the quartic $O_a(t)$. At the boundary points L_h, H_h together with $t = 0$ and $t \to \infty$, the signs of $E_a(t)$ force one root of the quintic $E_a(t)$ in each of the five complementary open gaps $(0, L_1), (H_1, L_2), (H_2, L_3), (H_3, L_4), (H_4, \infty)$. Since $\deg E_a = 5$ and $\deg O_a = 4$, these roots are simple and strictly interlaced, and $E_a(0), O_a(0) > 0$. By the Hermite–Biehler theorem [4], every $a \in \mathcal{K}_{10}$ is Hurwitz stable.

For the matrix conditions (3.2), exact rational elimination shows that the first eleven rows $R(a_1, t_1)$, …, $R(a_{11}, t_{11})$ are linearly independent, so R has rank 11. Expressing $-R(a_{12}, t_{12})$ in the basis of the first eleven rows yields eleven strictly positive coordinates $\lambda_1, \ldots, \lambda_{11} > 0$. Setting $\lambda_{12} = 1$ gives $\sum_{m=1}^{12} \lambda_m R(a_m, t_m) = 0$, and direct evaluation of the same weights verifies $\sum_{m=1}^{12} \lambda_m [-O_{a_m}(t_m)] < 0$. □

Combining Lemma 3.1 with the first identity in Proposition 3.2 immediately establishes the order-ten case of Theorem 2.2; the second inequality in (3.2) supplies the extra sign needed for degree lifting.

Corollary 3.3 (Order-ten nonexistence)**.** For the rational monic Hurwitz box $\mathcal{K}_{10}$ and the twelve fixed vertex–frequency pairs (a_m, t_m) in Proposition 3.2, every nonzero real polynomial $b \in \mathbb{R}[s]$ of degree at most 10 satisfies $\operatorname{Re}[b(\mathrm{i}\sqrt{t_m})/a_m(\mathrm{i}\sqrt{t_m})] < 0$ for at least one $m \in \{1, \ldots, 12\}$.

3.2 Inductive Degree Lifting and Proof of Theorem 2.2

Lemma 3.4 (Low-end degree lifting)**.** Suppose a compact monic Hurwitz box $\mathcal{K}_d$ of degree $d \geq 1$ admits finitely many positive-frequency tests (a_m, t_m) $(1 \leq m \leq M)$ whose evaluation matrix has full column rank $d+1$ and whose positive weights $\lambda_m > 0$ satisfy (3.2) (with sum over $1 \leq m \leq M$). Then there exists a rational monic Hurwitz box $\mathcal{K}_{d+1}$ of degree $d+1$, with every nonleading coefficient interval of positive width, admitting $M+1$ positive-frequency tests with full column rank $d+2$ and positive weights satisfying (3.2).

Proof. We construct $\mathcal{K}_{d+1}$ in three steps.

Step 1: Hurwitz stability of the shifted family. For a parameter $\eta > 0$, replace each $a \in \mathcal{K}_d$ by the monic degree-$(d+1)$ polynomial

$$A(s) = s\, a(s) + \eta.$$

These polynomials form a compact monic box (initially with constant coefficient fixed at η). At $\eta = 0$, every $s\, a(s)$ has a simple root at $s = 0$ because $a_0 \geq l_0 > 0$, while its remaining d roots lie uniformly inside the

open left half-plane by compactness of $\mathcal{K}_d$. Differentiating $A(s(\eta)) = 0$ at $\eta = 0$ gives $s'(0) = -1/a_0 < 0$. By the implicit function theorem on the compact box $\mathcal{K}_d$, the continuation of the zero root admits a uniform expansion $-\eta/a_0 + O(\eta^2)$, which is strictly negative for all sufficiently small $\eta > 0$, while the other d roots remain in the open left half-plane. Thus the lifted box is Hurwitz for all sufficiently small $\eta > 0$.

Step 2: Continuation of the positive dependence and an added test near $t = 0$. From (2.3), the even and odd parts of $A(s) = s\,a(s) + \eta$ are

$$E_A(t) = \eta - t\,O_a(t), \qquad O_A(t) = E_a(t). \tag{3.3}$$

Comparing (2.5) at degree $d+1$ and degree d, when $\eta = 0$ the lifted row $R^{(d+1)}(A_m, t_m) \in \mathbb{R}^{d+2}$ satisfies

$$R_0^{(d+1)}(A_m, t_m) = -t_m O_{a_m}(t_m), \qquad R_{j+1}^{(d+1)}(A_m, t_m) = t_m R_j^{(d)}(a_m, t_m) \quad (0 \le j \le d).$$

Consider the rescaled weights $\tilde{\lambda}_m(0) = \lambda_m/t_m > 0$. At $\eta = 0$, they annihilate columns $1, \dots, d+1$ of the lifted matrix and yield a strictly negative sum in column 0:

$$S(0) \coloneqq \sum_{m=1}^{M} \tilde{\lambda}_m(0)\big[-t_m O_{a_m}(t_m)\big] = \sum_{m=1}^{M} \lambda_m\big[-O_{a_m}(t_m)\big] < 0.$$

Since columns $1, \dots, d+1$ at $\eta = 0$ have rank $d+1$, choose $d+1$ linearly independent rows of this submatrix. Keeping the remaining $M - (d+1)$ weights fixed and solving the $(d+1) \times (d+1)$ linear system on this nonsingular minor continues $\tilde{\lambda}_m(0)$ continuously to strictly positive weights $\tilde{\lambda}_m(\eta) > 0$ for small $\eta > 0$ that still annihilate columns $1, \dots, d+1$, while their column-0 sum satisfies $S(\eta) \to S(0) < 0$.

Now pick any member $a_* \in \mathcal{K}_d$ and add an $(M+1)$-th test on $A_*(s) = s\,a_*(s) + \eta$, temporarily at $t = 0$. By (3.3), its evaluation row is $(\eta, 0, \dots, 0)$ and its extra moment is $-O_{A_*}(0) = -E_{a_*}(0) = -a_*(0) < 0$. Assign it the positive weight $\mu(\eta) = -S(\eta)/\eta > 0$. Then:

- all $d+2$ columns of the $(M+1) \times (d+2)$ matrix have weighted sum zero;
- the matrix has full column rank $d+2$ (the new row is nonzero only in column 0, while the first M rows have rank $d+1$ on columns $1, \dots, d+1$);
- as $\eta \downarrow 0$, the extra moment from the first M tests remains bounded, whereas the new test contributes $-\mu(\eta)a_*(0) = \frac{S(\eta)}{\eta} a_*(0) \to -\infty$.

Fix a sufficiently small $\eta > 0$ so that the lifted box is Hurwitz and the total extra moment is strictly negative. Perturbing the $(M+1)$-th node from $t = 0$ to a sufficiently small $t_{M+1} > 0$ and solving continuously on the enlarged $(d+2) \times (d+2)$ nonsingular minor preserves full column rank $d+2$, positivity of all $M+1$ weights, and strict negativity of the extra moment.

Step 3: Rational full-width enlargement. By openness of the Hurwitz property and compactness of the lifted box, we may slightly enlarge each nonleading coefficient interval (including the constant coefficient $[\eta, \eta]$) to a rational closed interval of positive width while keeping the entire box Hurwitz stable. Every tested polynomial A_m remains inside the enlarged rational box $\mathcal{K}_{d+1}$, completing the induction step. □

Proof of Theorem 2.2. For $n = 10$, the assertion is Corollary 3.3. For any $n > 10$, applying Lemma 3.4 inductively $n - 10$ times starting from Proposition 3.2 produces a rational monic Hurwitz box $\mathcal{K}_n$ with positive width in all nonleading coefficients and a finite positive-frequency test set satisfying the hypotheses of Lemma 3.1, which yields the stated inequality for every nonzero $b \in \mathbb{R}[s]$ of degree at most n. □

4 Conclusion

We have shown that robust Hurwitz stability of an nth-order interval polynomial family does not guarantee equal-degree SPR synthesis in general: for every $n \ge 10$, there exist full-width rational Hurwitz boxes that exclude every nonzero real numerator of degree at most n, even under the weaker nonnegative real-part condition. Because the order-ten obstruction is certified by a finite rank-11 matrix with a positive linear dependence and a negative extra moment, low-end degree lifting propagates it to every higher order. Whether equal-degree interval SPR synthesis holds for the intermediate orders $n \in \{7, 8, 9\}$ remains an open question.

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A Rational Data for the Order-Ten Certificate

This appendix specifies the exact rational endpoints, root-bracketing tests, and twelve evaluation rows used in Proposition 3.2. All operations are carried out over $\mathbb{Q}$ in the squared frequency variable $t = \omega^2$.

A.1 Interval Endpoints and Root-Bracketing Signs

Fix the rational parameters

$$z = 10^{-24}, \qquad e = z^{10}, \qquad \gamma = 10^{-6}, \qquad \nu = 10^{-8}, \qquad q = 1 - 10^{-300}.$$

For $y \in \{q, 1\}$, define the eleven endpoint ratios

$$(\xi_0, \dots, \xi_{10}) = (e,\ \gamma/y,\ y,\ y,\ e,\ \gamma/y,\ y,\ y,\ e,\ \gamma/y,\ 1).$$

Set $d_h = f_h = (1+\gamma)^{-1}$ for $h \in \{1, 3, 4\}$, $(d_2, f_2) = (1 - \gamma\nu, \nu)$, and $(g_{2h-1}, g_{2h}) = (1 - z^{h+1})(d_h, f_h)$ for $h = 1, 2, 3, 4$. Starting from $U_0 = U_1 = U_2 = 1$, compute recursively

$$U_{i+2} = g_i \xi_i \xi_{i+1} \frac{U_i U_{i+1}}{U_{i-1}} \quad (1 \le i \le 8), \qquad u_j(y) = \frac{U_j}{U_{10}}, \qquad l_j(y) = \xi_j u_j(y) \quad (0 \le j \le 10). \tag{A.1}$$

Below, unadorned symbols (l_j, u_j) denote the actual box endpoints ($y = q$, where $l_j < u_j$ for all $j \le 9$ and $l_{10} = u_{10} = 1$), whereas barred symbols $(\bar{l}_j, \bar{u}_j, \bar{\xi}_j)$ denote the auxiliary values at $y = 1$.

For $h = 1, 2, 3, 4$, define the reference frequencies and bracket endpoints

$$r_h = \frac{\bar{u}_{2h-1}}{\bar{u}_{2h+1}}, \qquad L_h = r_h \bar{\xi}_{2h-1}(1 - 10^{-155}), \qquad H_h = \frac{r_h(1 + 10^{-155})}{\bar{\xi}_{2h+1}},$$

which satisfy $0 < L_1 < H_1 < L_2 < H_2 < L_3 < H_3 < L_4 < H_4$. Put $s_h = (-1)^{h-1}$. For each h, let the superscripts $+$ and $-$ denote the vertex choices maximizing and minimizing $s_h E_a(t)$ and $s_h O_a(t)$, respectively: specifically, maximizing takes u_j when $s_h(-1)^{\lfloor j/2 \rfloor} > 0$ and l_j otherwise on the corresponding parity, and minimizing reverses the endpoints. For example, a_h^{+-} maximizes $s_h E_a$ and minimizes $s_h O_a$. Exact rational evaluation (via Horner's scheme: initialize $E = a_{10}$, $O = a_9$, and for $k = 4, 3, 2, 1, 0$ update $E \leftarrow a_{2k} - tE$ and, when $k < 4$, $O \leftarrow a_{2k+1} - tO$) verifies the sixteen strict inequalities

$$\begin{aligned} s_h E_{a_h^{+-}}(L_h) < 0, \quad s_h O_{a_h^{+-}}(L_h) > 0, \\ s_h E_{a_h^{++}}(H_h) < 0, \quad s_h O_{a_h^{++}}(H_h) < 0 \end{aligned} \qquad (h = 1, 2, 3, 4).$$

By extremality of the four vertices (Lemma 2.1), these inequalities hold for every $a \in \mathcal{K}_{10}$ and yield the root-bracketing and Hermite–Biehler interlacing used in Proposition 3.2.

A.2 The Twelve Test Rows and Rational Verification

Set $\alpha = 10^{-15}$. The first eleven tests (a_m, t_m) take $t_m = r_h v$ and the actual vertex of $\mathcal{K}_{10}$ listed in Table 1 (using the oriented $+/-$ convention for $s_h E_a, s_h O_a$ defined above).

Table 1: The first eleven squared-frequency tests (a_m, t_m) with $t_m = r_h v$.

m	h	v	vertex	m	h	v	vertex
1	1	$\gamma(1-\alpha)$	$+-$	7	3	$\gamma(1-\alpha)$	$+-$
2	1	$1+\alpha$	$++$	8	3	$1-\alpha$	$++$
3	1	10^{-18}	$--$	9	3	$1/4$	$+-$
4	1	4	$-+$	10	4	$1+\alpha$	$+-$
5	2	$1+\alpha$	$+-$	11	4	$(1-\alpha)/\gamma$	$++$
6	2	4	$++$				

To specify the twelfth test, set $p_1 = 1$, compute recursively for $h = 1, 2, 3, 4$:

$$p_{2h+1} = \frac{p_{2h-1}}{r_h}, \qquad p_{2h} = p_{2h+1} \frac{\bar{u}_{2h-1}}{\bar{l}_{2h}},$$

and put $p_0 = p_1 \bar{l}_1 / \bar{u}_2$ and $p_{10} = p_9 \bar{l}_9 / (e \bar{u}_8)$. Reuse the vertex $a_{12} = a_* := a_1^{--}$ of row 3 at the squared frequency

$$t_{12} = 10^{12} \left(1 + \sum_{j=0}^{9} (a_*)_j\right) \left(1 + \sum_{j=0}^{9} \frac{p_j}{p_{10}}\right).$$

Form the 12×11 rational matrix R with rows $R(a_m, t_m)$ from (2.5). Let $R_{\widehat{m}}$ denote the 11×11 matrix obtained by deleting row m, and set $d_m = (-1)^m \det R_{\widehat{m}}$ $(1 \le m \le 12)$. Then (3.2) is verified without any choice of normalization by the rational inequalities

$$d_{12} \neq 0, \qquad \frac{d_m}{d_{12}} > 0 \quad (1 \le m \le 11), \qquad \sum_{m=1}^{12} \frac{d_m}{d_{12}} \, O_{a_m}(t_m) > 0. \tag{A.2}$$

Equivalently, solving the single 11×11 linear system $R_{\widehat{12}}^{\mathsf{T}} w = -R_{12}^{\mathsf{T}}$ over $\mathbb{Q}$, setting $w_{12} = 1$, checking $w_m > 0$ $(1 \le m \le 11)$, and verifying $\sum_{m=1}^{12} w_m O_{a_m}(t_m) > 0$ establishes both assertions in (3.2). Positive row and column scalings may be applied during elimination provided any row scaling is applied to the $O_{a_m}(t_m)$ column as well.

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