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# THE DEGREE PAIR $(56, 140)$ IN THE PLANE JACOBIAN PROBLEM: EXCLUSION OF THE SHAPE WITH STARTING CORNER $(7, 21)$ AND FINAL CORNER $(13/7, 3)$

DAVID NIEDBALA GIRAUDIN

Abstract. In the list of Guccione, Guccione, Horruitiner and Valqui of the possible shapes of a counterexample to the plane Jacobian conjecture with degrees at most 150, the family $F_{11}$ contributes one shape: the standard $(2, 5)$-pairs of degrees $(56, 140)$ with starting corner $(7, 21)$ and final corner $(13/7, 3)$. We show that no such pair exists, when it is normalised as in Corollary 5.21 of Guccione–Guccione–Valqui. The two sides of the Newton polygon are described by the results of Guccione–Guccione–Valqui; the left side is reduced as in the treatment of the corner $(7, 21)$ by the four authors for the degrees $(56, 84)$. In suitable polynomial coordinates the pair is then confined to an explicit finite system of polynomial equations, which has no solution: a chain of linear eliminations with rational coefficients leads to $840 = 0$. The eliminations are reproduced by an ancillary script, which also confirms the result by a Gröbner basis computation. All statements are proved relative to the cited results. Nothing in this note has been checked by a third party.

## 1. Statement

Throughout, $K$ is an algebraically closed field of characteristic zero, $L = K[x, y]$, $[P, Q] = P_x Q_y - P_y Q_x$ and $(m, n) = (2, 5)$. We use the notation of [1]: for a weight $(\rho, \sigma)$, $v_{\rho,\sigma}$ is the weighted degree, $\ell_{\rho,\sigma}$ the leading form, $\mathrm{st}_{\rho,\sigma}$ and $\mathrm{en}_{\rho,\sigma}$ its end points; $L^{(l)} = K[x^{\pm 1/l}, y]$. *Standard $(m, n)$-pairs*, their *starting triple* and their *regular corners* are defined in [1, Definitions 4.3, 6.2, 5.5]. Put $t := x^2 y^7$.

**Standing data 1.1.** $(P, Q)$ is a standard $(2, 5)$-pair in $L$ with $[P, Q] = J \in K^\times$, $\deg P = 56$, $\deg Q = 140$, starting triple $\big((7, 21), (1, 0), (7, -2)\big)$, and $(-1, 1) < \mathrm{Succ}_P(1, 0) < (-1, 0)$ as in [1, Corollary 5.21(4)]. Thus:

(D1) the support of $P$ lies in $\{(i, j) : 7i - 2j \le 14\}$, that of $Q$ in $\{7i - 2j \le 35\}$, and

$$\ell_{7,-2}(P) = \lambda_P \big(x\, h(t)\big)^2, \qquad \ell_{7,-2}(Q) = \lambda_Q \big(x\, h(t)\big)^5, \qquad \lambda_P, \lambda_Q \in K^\times,$$

with $h \in K[t]$ monic of degree 3, $h(0) \neq 0$ [1, Proposition 6.1, Theorem 7.6(2)].

**Theorem 1.2.** *There is no pair $(P, Q)$ as in Standing data 1.1.*

This is the shape of the family $F_{11}$ of [2, Section 5] for $j = 0$, the only member of that family with degrees at most 150 [2, Section 6]. The first arXiv version of [3] listed this family among those discarded, assuming that two expressions for the intersection number of a Jacobian pair coincide; the revised version (arXiv v2) establishes only an inequality between them and no longer contains that list. To our knowledge the shape has not been excluded elsewhere.

## 2. The roots of $h$

**Lemma 2.1.** *In Standing data 1.1, $h = (t - c)^3$ with $c \in K^\times$.*

*Proof.* By [1, Definition 6.2], $((7, 21), (7, -2))$ is a regular corner of the $(2, 5)$-pair $(P, Q)$ in $L^{(1)}$, with $\frac{1}{2}\,\mathrm{st}_{7,-2}(P) = (1, 0)$. In $\tilde{z} := x^{2/7} y$ one has $\ell_{7,-2}(P) = x^2 p(\tilde{z})$ with $p(\tilde{z}) = \lambda_P h(\tilde{z}^7)^2$; the bracket of $\ell_{7,-2}(P)$ and $\ell_{7,-2}(Q)$ vanishes, both being powers of $x\, h(t)$ up to constants, $p$ has at least seven distinct roots and $v_{1,-1}(\mathrm{st}_{7,-2}(P)) = v_{1,-1}(2, 0) > 0$. By [1, Proposition 5.16]

*Date*: 20 November 2026.
ORCID: 0009-0009-1526-1178.

some root $\lambda \neq 0$ of $p$ has multiplicity at least $\frac{m}{l} \cdot \frac{a\rho+bl\sigma}{\rho+\sigma} = 2 \cdot \frac{49-42}{5} = \frac{14}{5}$ (the root can be taken different from 0 because $p(0) \neq 0$). A root of $h$ of multiplicity $k$ gives roots of $p$ of multiplicity $2k$; so some root $c$ of $h$ has multiplicity $k \geq 2$, and by [1, Proposition 5.18] some $(2,5)$-pair in $L^{(7)}$ has a regular corner at $(1,0)+k(\frac{2}{7},1)$. For $k=2$ this point is $(11/7,2)$. By [1, Proposition 5.19] it would be the first entry of a regular corner of Case I or of Case II (in the sense of [1, Section 5, before Remark 5.9]). Case II requires $\gcd(11,2) > 1$ [1, Proposition 5.14(4)]. In Case I, with direction $(\rho,\sigma)$, [1, Proposition 1.13, Corollary 5.7(2)] give $2(\frac{11}{7}\rho+2\sigma) = \frac{2}{7}(\rho+\sigma)$, that is $(\rho,\sigma) = (13,-10)$, $v(P) = \frac{6}{7}$, $v(Q) = \frac{15}{7}$; by [1, Proposition 5.13] one of $\frac{13}{7} \cdot 2a' = \frac{6}{7}$, $\frac{13}{7}k' = \frac{15}{7}$, $\frac{13}{7}k' = \frac{6}{7}$ would have a solution in positive integers, which is not the case. Hence $k=3$. □

## 3. The left side

**Lemma 3.1.** *There is $\gamma \in K$ such that*

$$\hat{P}(z,v) := P\big(zv^3+\gamma v^2,\, v^{-1}\big), \qquad \hat{Q}(z,v) := Q\big(zv^3+\gamma v^2,\, v^{-1}\big)$$

*are polynomials in $z, v$. Give $z$ the weight 1 and $v$ the weight 2. Then $\hat{P}$ has weighted degree 14 with top form $\lambda_P z^2(z^2-cv)^6$, $\hat{Q}$ has weighted degree 35 with top form $\lambda_Q z^5(z^2-cv)^{15}$, and*

$$\hat{P}_z\hat{Q}_v - \hat{P}_v\hat{Q}_z = -J\,v.$$

*Proof.* This is the reduction of [4, Section 4, Case $(7,21)$], stated there for $(m,n) = (2,3)$; we give the argument in detail. Apply the flip $\psi_1$ of [1, Section 8], so that the weight $(7,-2)$ becomes $(\rho_0,\sigma_0) = (-2,7)$, with $\mathrm{st}_{\rho_0,\sigma_0} = (42,14)$ and $\mathrm{en}_{\rho_0,\sigma_0} = (0,2)$ for the flipped $P$, and $\frac{1}{m}\,\mathrm{st} = \frac{1}{n}\,\mathrm{st} = (21,7)$. Let $F$ be the element of [1, Theorem 2.6] for $(\rho_0,\sigma_0)$; then $v_{-2,7}(F) = 5$. If $\mathrm{st}(F) = (1,1)$ then $\mathrm{en}(F) \neq (1,1)$, since $F$ is not a monomial [1, Remark 2.5], so $\mathrm{en}(F) \sim \mathrm{en}(P) \sim (0,1)$ and $\mathrm{en}(F) = (0,5/7)$, which is impossible; hence $\mathrm{st}(F) \sim (21,7)$, so $\mathrm{st}(F) = (15,5) = \frac{5}{7}(21,7)$ and $q=7$ in [1, Corollary 7.4]. That corollary applies to the flipped pair (its hypotheses hold with $(a/l,b) = (21,7)$, and $\rho_0+\sigma_0 > 0$): for every weight between $(1,-3)$ and $(\rho_0,\sigma_0)$ at which the pair has an edge, the edge form of $P$ is a power $R^{14}$. In the original variables: for a weight $(-s,1)$ with $s<3$ carrying an edge with starting point $(14,42)$, the form is $(xy^3-\gamma y^{3-s})^{14}$ up to a constant, so $s \in \mathbb{Z}$, and $s \geq 2$ by the condition on $\mathrm{Succ}_P(1,0)$; thus $s=2$. Here $(14,42) = \mathrm{en}_{7,-2}(P)$ is the starting point of the edge at $\mathrm{Succ}_P(1,0)$, because $\deg P = 56 = v_{1,1}(14,42)$. The translation $x \mapsto x+\gamma y^{-2}$ removes this edge [1, Proposition 3.9] (take $\gamma=0$ if there is no such edge). By [1, Proposition 3.9] the leading forms of the translated pair at the weights between $(7,-2)$ and $(-2,1)$ are unchanged, and its $(-2,1)$-form is the monomial $x^{14}y^{42}$; so an edge with starting point $(14,42)$ and $s<3$ could only occur for $2<s<3$. The same corollary applies to the translated pair: it lies in $L^{(1)}$, the form $\ell_{\rho_0,\sigma_0}$ and the degrees $v_{1,1}$, $v_{0,1}$ of the flipped pair are unchanged, and so is $F$ (the element $G_0$ of [1, Theorem 2.6] can be kept, and $(0,5/7)$ is excluded in $L^{(1)}$ as well). Hence the form of such an edge would again be a power $R^{14}$, with $R$ a $(-s,1)$-homogeneous element of $K[x,y^{\pm1}]$ whose only possible monomials are $xy^3$ and $y^{3-s}$, which forces $s \in \mathbb{Z}$. So no edge with $s<3$ and starting point $(14,42)$ remains. Since the weight $(-3,1)$ lies within a half-turn of $(7,-2)$, the point $(14,42)$ is then still leading for $(-3,1)$ [1, Remark 2.10, Proposition 3.7]; hence every monomial $x^iy^j$ of the translated $P$ satisfies $j \leq 3i$, that is, $x^iy^j = z^iv^{3i-j}$ is a monomial in $z = xy^3$, $v = y^{-1}$. For every weight $(-s,1)$ with $s<3$ the values of $P$ and $Q$ are positive, since $(14,42)$ and $(35,105)$ remain in the supports and $v_{-s,1}(14,42) = 14(3-s)$, $v_{-s,1}(35,105) = 35(3-s)$. At the weight $(-2,1)$, if $P$ has an edge there, [1, Proposition 7.3] with $T_0 = Q$ gives $[\ell(Q),\ell(P)] = 0$ and proportional values, so $\ell(Q) \sim R^{35}$ and the same translation applies. For the translated pair and $2 \leq s < 3$, the leading forms commute by [1, Proposition 1.13] (the sum of the values exceeds $1-s$), so $\ell_{-s,1}(Q)$ is proportional to a power of the monomial $\ell_{-s,1}(P) = x^{14}y^{42}$ [1, Proposition 2.1(2)], hence a monomial at a point $(i,3i)$; by (D1) such a point of the support has $i \leq 35$, and $(35,105)$ is in the support, so $\ell_{-s,1}(Q)$ is the monomial at $(35,105)$ and $j-si \leq 35(3-s)$ on the support of the translated $Q$. Letting $s$ tend to 3 gives $j \leq 3i$ there.

The translation replaces $x^iy^j$ by monomials $x^{i-r}y^{j-2r}$ of $(7,-2)$-weight $7i-2j-3r$, so (D1) holds for the translated pair with the same leading forms. Since $x^iy^j = z^iv^{3i-j}$ has $7i-2j = i+2(3i-j)$, the bound $7i-2j \le 14$ is the weighted degree bound, and for $x = zv^3$, $y = v^{-1}$ one has $t = z^2v^{-1}$ and, by Lemma 2.1, $x^2(t-c)^6 = z^2(z^2-cv)^6$, which gives the top form; likewise for $Q$. The Jacobian follows from $dx \wedge dy = -v\,dz \wedge dv$. □

## 4. The right side

Write $X := x^{1/7}$, so that $L^{(7)} = K[X^{\pm1}, y]$.

**Lemma 4.1.** *There are $\lambda \in K^\times$ with $\lambda^7 = c$ and $\lambda_3, \lambda_4 \in K$ such that, with*

$$Y := \eta + \lambda X^{-2} + \lambda_3 X^{-3} + \lambda_4 X^{-4}, \qquad P_1(X,\eta) := P\big(X^7, Y\big), \qquad Q_1(X,\eta) := Q\big(X^7, Y\big),$$

*and with the weight $W(X^i\eta^j) := 5i-21j$, every monomial of $P_1$ has $W \le 4$ and every monomial of $Q_1$ has $W \le 10$. The monomials of $P_1$ with $W = 4$ lie between $X^5\eta$ and $X^{26}\eta^6$, both with non-zero coefficient; those of $Q_1$ with $W = 10$ lie between $X^2$ and $X^{65}\eta^{15}$, both with non-zero coefficient.*

*Proof.* By Lemma 2.1 and its proof, every root $\lambda$ of $\tilde z^7 = c$ has multiplicity $6 \ge \frac{14}{5}$ in $p$, and by [1, Proposition 5.18] the automorphism $\varphi(y) = y + \lambda x^{-2/7}$ gives a $(2,5)$-pair $(\varphi P, \varphi Q)$ in $L^{(7)}$ with a regular corner at $A_1 := (1,0) + 3(2/7,1) = (13/7, 3)$, of direction $(\rho',\sigma') < (7,-2)$. We follow this corner as in the proof of [1, Proposition 5.19]. Case II is impossible, because [1, Proposition 5.14(4)] requires $\gcd(13,3) > 1$; this holds at every step below, since it depends only on the point $A_1$. In Case III, [1, Proposition 5.17] gives $\rho' \mid 7$, hence $\rho' = 7$ (no direction $(1,\sigma')$ is smaller than $(7,-2)$ in the interval of directions) and $\sigma'/\rho' = -j/7$ with $3 \le j \le 6$, and the translation $y \mapsto y + \lambda' x^{-j/7}$ produces a $(2,5)$-pair in $L^{(7)}$ with a regular corner at the same point $A_1$ and a smaller direction. After finitely many steps Case I occurs, for a pair $(P_1,Q_1)$ and a direction $(\rho,\sigma)$. There $v_{\rho,\sigma}(P_1) + v_{\rho,\sigma}(Q_1) = \rho+\sigma$ [1, Proposition 1.13], while $\mathrm{en}(P_1) = 2A_1 = \frac{2}{5}\,\mathrm{en}(Q_1)$ [1, Corollary 5.7(2)]. Hence $2(\frac{13}{7}\rho + 3\sigma) = \frac{2}{7}(\rho+\sigma)$, that is $(\rho,\sigma) = (5,-3)$, $v(P_1) = \frac{4}{7}$, $v(Q_1) = \frac{10}{7}$. By [1, Proposition 5.13] either $\frac{1}{2}\,\mathrm{st}(P_1) = (a'/7, 0)$ with $a' \in \mathbb{N}$, which would give $\frac{10}{7}a' = \frac{4}{7}$; or $\{\mathrm{st}(P_1), \mathrm{st}(Q_1)\} = \{(k'/7,0), (1-k'/7,1)\}$ with $k' \in \mathbb{N}$. If $\mathrm{st}(P_1) = (k'/7,0)$ then $5k' = 4$; so $\mathrm{st}(Q_1) = (k'/7,0)$ with $5k' = 10$, that is $\mathrm{st}(Q_1) = (2/7,0)$ and $\mathrm{st}(P_1) = (5/7,1)$. The directions of the Case III steps lie strictly between $(5,-3)$ and $(7,-2)$, so $j \in \{3,4\}$, each at most once. Finally $7\,v_{5,-3}(x^{i/7}y^j) = 5i-21j$, $\mathrm{en}(P_1) = (26/7, 6)$ and $\mathrm{en}(Q_1) = (65/7, 15)$. □

## 5. Common coordinates

Let $\mathcal{A}$ be the set of formal sums $\sum c_{ij}X^i\eta^j$ ($i \in \mathbb{Z}$, $j \ge 0$) such that, for every $r$, only finitely many terms have $W = 5i-21j \ge r$. It is a ring containing $K[X^{\pm1},\eta]$; for $f \ne 0$ let $W(f)$ be the largest weight of its terms and $\ell(f)$ the sum of the terms of that weight. Then $\ell(fg) = \ell(f)\ell(g)$ and $W(fg) = W(f) + W(g)$, and $1+\varepsilon$ is invertible in $\mathcal{A}$ when $W(\varepsilon) < 0$.

**Lemma 5.1.** *Let $\gamma$, $\lambda, \lambda_3, \lambda_4$ be as in Lemmas 3.1 and 4.1, and put*

$$\zeta := z^2 - c\,v - \kappa_1 z - \kappa_2, \qquad \kappa_1 := 7\lambda^2\lambda_3, \quad \kappa_2 := 7\lambda^4(\lambda\lambda_4 - \lambda_3^2).$$

*Then $K[z,v] = K[z,\zeta]$. Write $\hat P = \sum p_{ab}z^a\zeta^b$ and $\hat Q = \sum q_{ab}z^a\zeta^b$. Then $p_{ab} \ne 0$ implies $5a-b \le 4$ and $a+2b \le 14$; $q_{ab} \ne 0$ implies $5a-b \le 10$ and $a+2b \le 35$. Moreover $p_{1,1}$, $p_{2,6}$, $q_{2,0}$, $q_{5,15}$ are non-zero, and*

$$\hat P_z\hat Q_\zeta - \hat P_\zeta\hat Q_z = \frac{J}{c^2}\,\big(z^2 - \kappa_1 z - \kappa_2 - \zeta\big).$$

*Proof.* Since $c \ne 0$, $v = (z^2 - \kappa_1 z - \kappa_2 - \zeta)/c$, so $K[z,v] = K[z,\zeta]$; giving $\zeta$ the weight 2, this substitution does not increase the weighted degree of Lemma 3.1, which gives $a+2b \le 14$ and $a+2b \le 35$, and the top form of $\hat P$ becomes $\lambda_P z^2\zeta^6$, that of $\hat Q$ becomes $\lambda_Q z^5\zeta^{15}$. In $\mathcal{A}$ the element $Y$ of Lemma 4.1 is $\lambda X^{-2}(1+\varepsilon)$ with $W(\varepsilon) < 0$; it is invertible. The assignment $v \mapsto Y^{-1}$, $z \mapsto X^7Y^3 - \gamma Y$ defines a ring morphism $K[z,v] \to \mathcal{A}$ which sends $\hat P$ to $P_1$ and

$\hat{Q}$ to $Q_1$, because $P(x,y) = \hat{P}\big((x-\gamma y^{-2})y^3, y^{-1}\big)$ in $K[x,y^{\pm 1}]$. Expanding (ancillary script), the image of $z$ has $W = 5$ and leading term $\lambda^3 X$, and the image of $\zeta$ has $W = -1$ and leading term $7\lambda^5\eta X^4$: the terms of weights 10, 5, 0 cancel by the choice of $c = \lambda^7$, $\kappa_1$, $\kappa_2$. Hence the image of $z^a\zeta^b$ has leading term a non-zero multiple of $\eta^b X^{a+4b}$, of weight $5a-b$, and these monomials are distinct for distinct $(a,b)$. Therefore, for a non-zero polynomial, $W\big(\sum c_{ab} z^a\zeta^b\big) = \max\{5a-b : c_{ab} \neq 0\}$, and its leading part is the image of the corresponding terms. Lemma 4.1 gives the values 4 and 10 for the images of $\hat{P}$ and $\hat{Q}$; the monomials $X^5\eta$, $X^{26}\eta^6$, $X^2$, $X^{65}\eta^{15}$ correspond to $z\zeta$, $z^2\zeta^6$, $z^2$, $z^5\zeta^{15}$. Finally $dz \wedge d\zeta = -c\, dz \wedge dv$, so the bracket in $(z,\zeta)$ is $-\frac{1}{c}$ times the bracket in $(z,v)$, which is $-Jv$ by Lemma 3.1. □

## 6. The finite system

Writing again $z$ for $z - \kappa_1/2$ (this replaces $\kappa_1$ by 0, changes $\kappa_2$, and preserves all the statements of Lemma 5.1, the leading terms of the images and the weighted degrees being unchanged), a pair as in the standing data yields a solution of the following system, in which $s$ stands for $\zeta$.

(S1) $\hat{P} = \sum p_{ab} z^a s^b$ with $a,b \geq 0$, $a+2b \leq 14$, $5a-b \leq 4$; $\hat{Q} = \sum q_{ab} z^a s^b$ with $a,b \geq 0$, $a+2b \leq 35$, $5a-b \leq 10$; and $p_{1,1}\, p_{2,6}\, q_{5,15} \neq 0$.

(S2) $\hat{P}_z \hat{Q}_s - \hat{P}_s \hat{Q}_z = J'\,(z^2 - \kappa - s)$ with $J' \in K^\times$, $\kappa \in K$.

In (S1), $\hat{P}$ has 15 possible monomials and $\hat{Q}$ has 72. The system is deliberately weaker than what is known: the top forms of Lemma 3.1 and the fact that $P$ and $Q$ are polynomials in $y$ give further restrictions, which are not needed.

**Lemma 6.1.** *If* (S1)–(S2) *has a solution, it has one with* $p_{1,1} = p_{2,6} = q_{5,15} = 1$ *and* $p_{0,0} = q_{0,0} = q_{4,12} = q_{2,6} = 0$. *For such a solution* $q_{2,0} = \frac{5}{16}$, $q_{3,5} = \frac{15}{8}$, $q_{4,10} = \frac{5}{2}$ *and* $J' = -\frac{5}{8}$.

*Proof.* The system is preserved by $(\hat{P},\hat{Q}) \mapsto (\alpha\hat{P}(\mu z, \mu^2 s), \beta\hat{Q}(\mu z, \mu^2 s))$, with $\kappa$ replaced by $\kappa/\mu^2$ and $J'$ by $\alpha\beta\mu^5 J'$; it multiplies $p_{1,1}$ by $\alpha\mu^3$, $p_{2,6}$ by $\alpha\mu^{14}$ and $q_{5,15}$ by $\beta\mu^{35}$, so we may assume that these three coefficients are 1. The system is also preserved by adding constants to $\hat{P}$ and $\hat{Q}$, and by $\hat{Q} \mapsto \hat{Q} + \alpha'\hat{P}^2 + \beta'\hat{P}$, since the monomials of $\hat{P}^2$ satisfy $a+2b \leq 28$ and $5a-b \leq 8$; the coefficient of $z^4 s^{12}$ in $\hat{P}^2$ is $p_{2,6}^2 = 1$ and $\hat{P}$ has no such monomial, and the coefficient of $z^2 s^6$ in $\hat{P}$ is 1; so $\alpha'$ and then $\beta'$ can be chosen to obtain $q_{4,12} = q_{2,6} = 0$. These operations do not change the terms with $5a-b = 10$ of $\hat{Q}$. The terms of (S2) with $5a-b = 10$ read $\{zs + z^2 s^6,\ q_{2,0} z^2 + q_{3,5} z^3 s^5 + q_{4,10} z^4 s^{10} + z^5 s^{15}\} = J' z^2$, where $\{f,g\} = f_z g_s - f_s g_z$, because the bracket of $z^a s^b$ and $z^{a'} s^{b'}$ is a multiple of $z^{a+a'-1} s^{b+b'-1}$, on which $5a-b$ takes the value of the sum minus 4. This equation has the unique solution stated (ancillary script). □

**Proposition 6.2.** *The system* (S1)–(S2), *normalised as in Lemma 6.1, has no solution.*

*Proof.* Call the terms with $5a-b = 4-k$ the *layer* $k$ of $\hat{P}$ and those with $5a-b = 10-k$ the layer $k$ of $\hat{Q}$; put $J'$ in layer 0 and $\kappa$ in layer 10. The bracket of a term of layer $i$ of $\hat{P}$ with a term of layer $j$ of $\hat{Q}$ is a monomial $z^a s^b$ with $5a-b = 10-(i+j)$, and $z^2$, 1, $s$ have $5a-b = 10, 0, -1$. So the coefficients of the monomials with $5a-b = 10-k$ in (S2) form, once the layers $< k$ are known, a system of linear equations in the unknowns of layer $k$, with a matrix of rational numbers (the brackets with the normalised layers of order 0).

*Step 1.* Solving these linear systems for $k = 0, 1, \ldots, 11$ with exact rational arithmetic (ancillary script; they are all solvable, and leave the four coefficients $q_{4,14}$, $q_{4,15}$, $q_{3,13}$, $q_{3,15}$ free) one finds, independently of the free coefficients,

$$p_{0,7} = -4, \qquad q_{3,16} = -10.$$

The term $-J's$ of (S2) enters at $k = 11$.

*Step 2.* The only monomials of $\hat{P}$ with $a+2b = 14$ are $z^2 s^6$ and $s^7$, and those of $\hat{Q}$ with $a+2b = 35$ are $z^5 s^{15}$, $z^3 s^{16}$, $z s^{17}$. Hence the coefficients of $z^2 s^{22}$ and of $s^{23}$ in (S2) are

$$-21\, p_{0,7}\, q_{3,16} + 28\, q_{1,17} = 0, \qquad -7\, p_{0,7}\, q_{1,17} = 0.$$

With the values of Step 1 the first gives $q_{1,17} = 30$, and the second becomes $840 = 0$. □

As an independent confirmation, the 104 equations of the normalised system (in 82 unknowns: the remaining 12 coefficients of $\hat{P}$, the remaining 68 coefficients of $\hat{Q}$, $\kappa$ and $J'$) generate the unit ideal of the polynomial ring over $\mathbb{Q}$: the reduced Gröbner basis computed by Singular [5] is $\{1\}$ (optional step of the ancillary script, which writes the input file).

*Proof of Theorem 1.2.* A pair as in the standing data gives a solution of (S1)–(S2) by Lemmas 2.1–5.1, hence a normalised one by Lemma 6.1, contradicting Proposition 6.2. □

## 7. Scope

- **Proved here, relative to the cited results:** the reduction of Theorem 1.2 to Proposition 6.2 (Lemmas 2.1–6.1). The inputs are (D1), which rests on [1, Proposition 6.1, Theorem 7.6(2), Remark 3.1], and [1, Definition 6.2, Theorem 2.6, Remarks 2.5, 2.10, Corollaries 5.7, 7.4, Propositions 1.13, 2.1, 3.7, 3.9, 5.13, 5.14, 5.16, 5.17, 5.18, 5.19, 7.3], as indicated. The condition on $\mathrm{Succ}_P(1,0)$ in the standing data is the one of [1, Corollary 5.21(4)], which is established there for the standard pairs attached to a minimal counterexample; the theorem is stated under this condition.
- **Verified by exact computation:** Step 1 of Proposition 6.2 (linear systems over $\mathbb{Q}$, SymPy) and its confirmation by a Gröbner basis (Singular); the equation of order 0 in Lemma 6.1; the arithmetic of Lemmas 2.1 and 4.1; the leading terms in Lemma 5.1 (ancillary file `verify_56_140.py` in the folder `anc`). The contradiction uses the term $s$ on the right-hand side of (S2), that is, the Jacobian $-Jv$ of Lemma 3.1.
- **Not claimed:** a statement on the plane Jacobian conjecture itself; on the other members of the family $F_{11}$ or on the families $F_9$, $F_{10}$ with the same starting corner; a new bound on the degree of a counterexample (see [4] for the one obtained by the four authors). The same shape with $P$ and $Q$ exchanged is covered by exchanging them. The completeness of the list of [2] is not re-examined.
- [1, Proposition 7.3 and Corollary 7.4], on which Lemma 3.1 rests, are stated in [1] with the indication that their proofs mimic those of Proposition 7.1 and Corollary 7.2; they are applied here to a Laurent translate, as in [1, Section 8] and [4]. The references [2] and [4] are arXiv preprints.
- The argument has been prepared by the author with AI assistance (Claude, Anthropic). Nothing in this note has been checked by a third party; it has not been validated by professionals.

## References

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Independent researcher, France