# RATIONAL POINTS ON INTERSECTIONS OF CUBIC AND QUADRIC HYPERSURFACES IN 28 VARIABLES

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Abstract. Let $X \subset \mathbb{P}^{n-1}$ be a non-singular complete intersection of a cubic and a quadric hypersurface defined over $\mathbb{Q}$. Browning, Dietmann and Heath-Brown established that $X$ satisfies the Hasse principle when $n \geq 29$. In this article we improve this result to $n \geq 28$. More precisely, we establish an asymptotic formula for a smoothly weighted count of the integral points on $X$ in an expanding box, and we deduce that $X$ satisfies weak approximation.

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## Contents

1. Introduction 1
2. Preliminaries 6
3. Exponential sums 10
4. The dissection 16
5. The major arcs 20
6. The nested $\delta$-symbol 22
7. Proof of the theorems 28
8. Quadratic forms of rank $n-1$ 30
9. Weak approximation 37
References 42

## 1. Introduction

Let $C, Q \in \mathbb{Q}[x_1, \dots, x_n]$ be a cubic form and a quadratic form whose common zero locus is a complete intersection $X \subset \mathbb{P}^{n-1}$. This paper is concerned with the Hasse principle for $X$. Thus we would like to show that $X(\mathbb{Q})$ is non-empty as soon as $X$ has points over $\mathbb{R}$ and over $\mathbb{Q}_p$ for every prime $p$, under the mildest possible assumption on the number of variables $n$.

For complete intersections cut out by forms of the same degree, a classical theorem of Birch [2] establishes the Hasse principle when the number of variables is sufficiently large in terms of the degree, the number of forms and the dimension of a certain singular locus. In Birch's theorem the number of variables has to grow quadratically with the number of forms. For systems of quadratic forms and of cubic forms that define a smooth complete intersection, Rydin Myerson [18, 19] has shown that linear growth suffices, and in [17] he obtains a result of the same kind for generic systems of forms of higher degree. Birch's method does not adapt easily to forms of differing degrees, because the process of differencing loses all trace of the forms of lower degree.

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For a cubic form and a quadratic form that are both diagonal, work of Wooley [21, 22] handles $n \geqslant 13$, under mild conditions on the coefficients. For general forms the first results are due to Browning, Dietmann and Heath-Brown [3]. Among other things they proved that if $X$ is non-singular and $n \geqslant 29$, then $X(\mathbb{Q}) \neq \emptyset$ if and only if $X(\mathbb{R}) \neq \emptyset$; in this range $X$ has points over $\mathbb{Q}_p$ for every $p$. Their proof is based on a version of the Hardy–Littlewood circle method in two variables. Browning and Heath-Brown [4] later extended Birch's method to systems of forms of differing degrees. For a non-singular intersection of a cubic and a quadric, their work gives an asymptotic formula with a power saving when $n \geqslant 37$ [4, Corollary 1.8], and hence the Hasse principle and weak approximation in this range. Over the function field $\mathbb{F}_q(t)$ with $\mathrm{char}(\mathbb{F}_q) > 3$, Glas [10] has obtained an asymptotic formula and weak approximation for $n \geqslant 26$, by means of a two-dimensional version of the $\delta$-method. Over $\mathbb{Q}$ a two-dimensional $\delta$-symbol has recently been developed by Li, Rydin Myerson and Vishe [15], who applied it to pairs of quadratic forms.

A different approach to systems of two equations was introduced by Munshi [16], who established an asymptotic formula for the number of integral points of bounded height on non-singular intersections of two quadrics in at least 11 variables. (For non-singular intersections of two quadrics the Hasse principle had been established by Colliot-Thélène, Sansuc and Swinnerton-Dyer [6] in at least 9 variables, by algebraic methods.) In his *nested $\delta$-method* the first equation is detected by the $\delta$-symbol of Duke, Friedlander and Iwaniec [9] and of Heath-Brown [12] with a modulus $q_1$; the second equation is then split into a congruence modulo $q_1$ and an equation of smaller size, and the latter is detected by a second $\delta$-symbol with a correspondingly smaller modulus $q_2$. In [16, §1, p. 1192] it is remarked that for the intersection of a quadric and a cubic this method should yield an asymptotic formula as soon as $n \geqslant 28$. We are not aware of a published proof. The purpose of this paper is to provide one, by a variant of the nested $\delta$-method.

**Theorem 1.1.** *Let $n \geqslant 28$ and let $X \subset \mathbb{P}^{n-1}$ be a non-singular complete intersection of a cubic hypersurface and a quadric hypersurface, both defined over $\mathbb{Q}$. Then $X(\mathbb{Q}) \neq \emptyset$ if and only if $X(\mathbb{R}) \neq \emptyset$.*

For $n \geqslant 29$ this is contained in [3, Theorem 1.3], so that the new case is $n = 28$. The quadric hypersurface containing $X$ is unique, and if $X$ is non-singular then the corresponding quadratic form has rank $n$ or $n-1$, by [3, Corollary 3.2]. Both cases are covered by Theorem 1.1. They require different arguments at one point of the proof, as we explain below. No conditions at the finite primes appear in Theorem 1.1, because $X(\mathbb{Q}_p) \neq \emptyset$ for every prime $p$ in this range (see Lemmas 7.1 and 8.16). For an arbitrary intersection of a cubic and a quadric in $n > 13$ variables, Zahid [23] has shown that $X(\mathbb{Q}_p) \neq \emptyset$ for every prime $p > 293$.

Theorem 1.1 is a consequence of an asymptotic formula. By [3, Corollary 3.2], after a linear change of variables over $\mathbb{Q}$, a variety $X$ as in Theorem 1.1 is given by $C = Q = 0$, where

$$\begin{aligned} &C \in \mathbb{Z}[x_1, \dots, x_n] \text{ is a non-singular cubic form,} \\ &Q(\mathbf{x}) = \sum_{i=1}^{n} d_i x_i^2, \qquad d_i \in \mathbb{Z}, \quad d_1 \cdots d_{n-1} \neq 0. \end{aligned} \tag{1.1}$$

Thus $Q$ has rank $n$ if $d_n \neq 0$, and rank $n-1$ if $d_n = 0$. For a weight function $w$ on $\mathbb{R}^n$ and $P \geqslant 1$ let

$$N_w(P) = \sum_{\substack{\mathbf{x}\in\mathbb{Z}^n \\ C(\mathbf{x})=Q(\mathbf{x})=0}} w(\mathbf{x}/P).$$

**Theorem 1.2.** *Let $n \geqslant 28$, let $C$ and $Q$ be as in (1.1), and suppose that $X : C = Q = 0$ is non-singular and $X(\mathbb{R}) \neq \emptyset$. Let $w$ be the weight function constructed in §2.2. Then for every $\varepsilon > 0$*

$$N_w(P) = \mathfrak{S}\,\mathfrak{J}\,P^{n-5} + O_\varepsilon\Big(P^{n-5-1/43+\varepsilon}\Big),$$

*where $\mathfrak{S}$ and $\mathfrak{J}$ are the singular series and the singular integral defined in §5. Both are positive.*

The exponent $1/43$ is dictated by the case $n = 28$. For larger $n$ it could be improved, but we have not pursued this.

The method also yields weak approximation. Recall that $X$ is said to satisfy weak approximation if the image of $X(\mathbb{Q})$ under the diagonal embedding is dense in $X(\mathbb{R}) \times \prod_p X(\mathbb{Q}_p)$, where the product is over all primes and carries the product topology.

**Theorem 1.3.** *Let $n \geqslant 28$ and let $X$ be as in Theorem 1.1. Then $X$ satisfies weak approximation.*

Since $X$ is non-singular, $X(\mathbb{R})$ is Zariski dense in $X$ as soon as it is non-empty, by [5, Lemme 3.1.2]. Theorem 1.3 therefore implies that $X(\mathbb{Q})$ is Zariski dense in $X$ whenever $X(\mathbb{R}) \neq \emptyset$.

Over $\mathbb{F}_q(t)$ the corresponding statement for $n \geqslant 26$ is due to Glas [10]. Over $\mathbb{Q}$ it is remarked in [3] that the arguments given there may be developed to establish weak approximation; the details are left to the reader. For $n \geqslant 37$ weak approximation follows from [4, Corollary 1.8]. Theorem 1.3 is deduced from a version of Theorem 1.2 in which $\mathbf{x}$ is restricted to a residue class $\mathbf{x} \equiv \mathbf{b} \pmod{H}$ (Theorem 9.1). We detect the congruence condition by additive characters. This leads to the sums $S(\alpha, \beta)$ with an additional factor $e(\boldsymbol{\theta} \cdot \mathbf{x})$, where $\boldsymbol{\theta} \in H^{-1}\mathbb{Z}^n$. In the argument outlined below, the only effect of this factor is that the dual variable $\mathbf{m}$ is replaced by $\mathbf{m} - q_1 q_2 \boldsymbol{\theta}$ in the oscillatory integrals. The complete exponential sums are the same as before. The necessary changes are described in §9.

**Outline of the proof.** We write $S(\alpha, \beta) = \sum_{\mathbf{x}} w(\mathbf{x}/P) e(\alpha C(\mathbf{x}) + \beta Q(\mathbf{x}))$, so that $N_w(P)$ is the integral of $S(\alpha, \beta)$ over the unit square. The argument of [3] combines a Weyl bound, which depends only on a rational approximation to $\alpha$, with a bound obtained from Poisson summation to a modulus that is a common denominator for $\alpha$ and $\beta$. Neither bound exploits cancellation in the sums over the numerators. Our starting point is different. Heath-Brown's $\delta$-symbol [12], applied to the equation $C(\mathbf{x}) = 0$ with level $P^{3/2}$, gives the exact formula

(1.2)
$$N_w(P) = \frac{c}{P^3} \sum_{q_1 \ll P^{3/2}} \sideset{}{^*}\sum_{a_1 \bmod q_1} \int_{\mathbb{R}} p_{r_1}(t)\, \mathcal{H}\Big(\frac{a_1}{q_1} + \frac{t}{P^3}\Big) dt, \qquad \mathcal{H}(\alpha) = \int_0^1 S(\alpha, \beta)\, d\beta,$$

in which $c = 1 + O(P^{-N})$, $r_1 = q_1 P^{-3/2}$, and $p_{r_1}(t)$ is bounded and is negligible unless $|t| \leqslant P^{3/2+\varepsilon}/q_1$ (Lemma 4.1). Put $\Theta = 1 + |t|$. We divide the pairs $(q_1, t)$ into three regions.

The pairs with $q_1$ and $\Theta$ at most a small power of $P$ form the major arcs for the cubic. They are treated classically, and produce the main term (§5).

If $q_1^{n/8-2}\Theta^{n/8-1} \geqslant P^{2+\eta_0}$, where $\eta_0 > 0$ is a small parameter, we bound $\mathcal{H}(\alpha)$ by the supremum of $|S(\alpha,\beta)|$ over $\beta$, and use the Weyl bound $S(\alpha,\beta) \ll P^{n+\varepsilon}(q_1\Theta)^{-n/8}$, which holds uniformly in $\beta$. We give a complete proof of it in §4, by Davenport's form of Weyl differencing, as in [3, Lemma 4.1]. This *Weyl region* contributes $O(P^{n-5-\eta_0+\varepsilon})$ (§4).

In the remaining region we detect the equation $Q(\mathbf{x}) = 0$ in the manner of Munshi. We write it as the congruence $q_1 \mid Q(\mathbf{x})$ together with the equation $Q(\mathbf{x})/q_1 = 0$, and apply the $\delta$-symbol to the latter, with the level

$$R = \frac{P}{(q_1\Theta)^{1/2}}.$$

This level depends on $t$ as well as on $q_1$. The reason is that we may insert a factor $B(\Theta Q(\mathbf{x}/P))$, with a smooth function $B$ such that $B(0) = 1$, at no cost: it oscillates no faster than the factor $e(tC(\mathbf{x}/P))$ which is already present. After this the integer $Q(\mathbf{x})/q_1$ has size $R^2$. The sum over $\mathbf{x}$ now has the modulus $q_1q_2$, where $q_2 \ll R$ is the modulus of the second $\delta$-symbol, and we apply Poisson summation. The result is an expression of the shape

$$\sum_{a_1 \bmod q_1}^{*} \mathcal{H}\Big(\frac{a_1}{q_1} + \frac{t}{P^3}\Big) \approx \frac{P^n}{R^2} \sum_{q_2 \ll R} \frac{1}{(q_1q_2)^n} \sum_{\mathbf{m}\in\mathbb{Z}^n} S_{q_1,q_2}(\mathbf{m})\, I_{q_1,q_2}(t;\mathbf{m})$$

(Lemma 6.1), in which $S_{q_1,q_2}(\mathbf{m})$ is a complete exponential sum modulo $q_1q_2$ and $I_{q_1,q_2}(t;\mathbf{m})$ is an oscillatory integral. Outside the Weyl region the conductor $P(q_1\Theta)^{1/2}$ of this sum is at most $P^{5/3+\eta_0}$.

The sum $S_{q_1,q_2}(\mathbf{m})$ runs over $a_1$ modulo $q_1$, over the numerator $a_2$ of the second $\delta$-symbol modulo $q_2$, and over $\mathbf{k}$ modulo $q_1q_2$ with $q_1 \mid Q(\mathbf{k})$. For the sum over $\mathbf{k}$ we use the estimates of [3, §6] for complete sums, which rest on Deligne's theorem and give square-root cancellation except for moduli with a large cube-full part. The sum over $a_1$ is estimated trivially. The sum over $a_2$, on the other hand, can be evaluated: for the primes $p \mid q_2$ that do not divide $q_1$ the sum over $\mathbf{k}$ is a product of quadratic Gauss sums, and what remains of the sum over $a_2$ is a Ramanujan sum if $n$ is even, as in the critical case $n = 28$, and a Gauss sum if $n$ is odd. In both cases we save a factor $q_2^{1/2}$ (Lemma 3.7). The primes that divide both moduli require a separate argument, based on stationary phase modulo prime powers (Lemma 3.9). These bounds are combined in the mean value estimate of Proposition 3.12, which is the arithmetic heart of the paper. The integrals $I_{q_1,q_2}(t;\mathbf{m})$ are treated in §6, where the final bookkeeping is carried out in Lemmas 6.8–6.12.

The description in the last paragraph applies when $Q$ has rank $n$. Suppose now that $d_n = 0$, so that the variable $x_n$ does not occur in $Q$. Then the sum over $k_n$ in $S_{1,e}(\mathbf{m})$ is not a Gauss sum, and for an individual $\mathbf{m}$ the sum over $a_2$ saves nothing. On the other hand, a shift of $k_n$ by multiples of $q_1$ shows that

$$S_{q_1,q_2}(\mathbf{m}) = 0 \qquad \text{unless } q_2 \mid m_n$$

(Lemma 8.2). So only one value of $m_n$ in $q_2$ contributes, and on average over $\mathbf{m}$ this makes up for the saving that is lost. The mean value estimate that replaces Proposition 3.12 is Proposition 8.7. Its proof needs only the estimates of [3, §6], which are available for forms $Q$ of rank $n-1$. The case $d_n = 0$ is treated in §8, and it leads to the same exponent in the error term.

**The savings.** It may be helpful to see, at a heuristic level, how much is saved at each step. Consider the pairs $(q_1, t)$ with $q_1 \asymp Q_1$ and $\Theta \asymp 1$. By (1.2) their contribution to $N_w(P)$ is about

$$\frac{1}{P^3} \sum_{q_1 \asymp Q_1} \sideset{}{^*}\sum_{a_1 \bmod q_1} \mathcal{H}\Big(\frac{a_1}{q_1} + \frac{t}{P^3}\Big).$$

The trivial bound $\mathcal{H}(\alpha) \ll P^{n-2}$ gives $P^{n-5}Q_1^2$, so that a factor $Q_1^2$ has to be saved; this is $P^{8/3}$ for $Q_1 = P^{4/3}$, and $P^3$ for the largest moduli. After the second $\delta$-symbol and Poisson summation the dual variable $\mathbf{m}$ has length $q_1q_2/P$ in each coordinate. The number of $\mathbf{m}$ is therefore $(q_1q_2/P)^n$, which cancels the factor $P^n(q_1q_2)^{-n}$, and the contribution is about

$$\frac{1}{P^3} \cdot Q_1 \cdot \frac{1}{R^2} \cdot R \cdot \max |S_{q_1,q_2}(\mathbf{m})| = \frac{Q_1}{P^3 R} \max |S_{q_1,q_2}(\mathbf{m})|, \qquad R = PQ_1^{-1/2}, \tag{1.3}$$

the maximum being over $q_1 \asymp Q_1$, $q_2 \asymp R$ and the relevant $\mathbf{m}$. Square-root cancellation in the sum over $\mathbf{k}$ gives $|S_{q_1,q_2}(\mathbf{m})| \ll q_1q_2(q_1q_2)^{n/2}$, and the evaluation of the sum over $a_2$ improves this to $q_1q_2^{1/2}(q_1q_2)^{n/2}$. Inserting the latter into (1.3), with $q_1q_2 \asymp PQ_1^{1/2}$, we obtain

$$\frac{Q_1^2}{P^3R^{1/2}} (PQ_1^{1/2})^{n/2} = P^{\frac{n}{2}-\frac{7}{2}} Q_1^{\frac{n+9}{4}},$$

which is the first term in Proposition 6.5. This increases with $Q_1$, and it is needed only outside the Weyl region, that is, for $Q_1 \leqslant P^{8(2+\eta_0)/(n-16)}$. For $n = 28$ and $\eta_0 \to 0$ the largest such modulus is $Q_1 = P^{4/3}$, with $R = P^{1/3}$ and $q_1q_2 \asymp P^{5/3}$. In this critical case the main term has size $P^{23}$, and the three estimates for $S_{q_1,q_2}(\mathbf{m})$ lead to the following bounds.

| Estimate used for $S_{q_1,q_2}(\mathbf{m})$ | Bound for $\lvert S_{q_1,q_2}(\mathbf{m})\rvert$ | Contribution |
|---|---|---|
| Trivial | $(q_1q_2)^{n+1}$ | $P^{139/3}$ |
| Square-root cancellation in the sum over $\mathbf{k}$ | $q_1q_2\,(q_1q_2)^{n/2}$ | $P^{23}$ |
| In addition, evaluation of the sum over $a_2$ | $q_1q_2^{1/2}(q_1q_2)^{n/2}$ | $P^{23-1/6}$ |

Thus the bounds of [3] alone bring us exactly to the size of the main term when $n = 28$, and the saving from the sum over $a_2$ is what makes this case accessible. If $Q$ has rank $n-1$, the last line of the table has to be read differently. The bound for an individual sum is then $q_1q_2^{3/2}(q_1q_2)^{n/2}$, which is larger than the bound in the second line. But only the $\mathbf{m}$ with $q_2 \mid m_n$ contribute, and since $q_1q_2/P > q_2$ in the critical case, these are one in $q_2$ of all the relevant $\mathbf{m}$. The contribution is again $P^{23-1/6}$. The Weyl bound saves $P^{\eta_0}$ in its region, while moving the boundary of the Weyl region by $P^{\eta_0}$ costs $P^{(37/6)\eta_0}$ in the critical case (Lemma 6.8); the relation $\eta_0 + (37/6)\eta_0 = 1/6$ gives $\eta_0 = 1/43$. There is a second critical case, $q_1 \asymp \Theta \asymp P^{1/2}$ and $q_2 \asymp 1$, in which the moduli $q_1$ with a large cube-full part dominate; it also gives $P^{23-1/6}$ (Lemma 6.12). The same bookkeeping shows that the method does not reach $n = 27$, where the corresponding bound exceeds the main term by a factor of about $P^{1.1}$.

**Notation.** Throughout, $n$ is the number of variables. In the theorems, in Proposition 5.3, and in §§6–9 we have $n \geqslant 28$; some intermediate results hold for smaller $n$, as stated. For $\mathbf{x} \in \mathbb{R}^n$ we write $|\mathbf{x}| = \max_i |x_i|$. We put $e(x) = e^{2\pi i x}$ and $e_q(x) = e(x/q)$. A sum $\sum^*_{a \bmod q}$ runs over the residues $a$ modulo $q$ with $(a, q) = 1$. The letter $\varepsilon$ denotes an arbitrarily small positive number, not necessarily the same at each occurrence, and

$N$ an arbitrarily large positive number. Implied constants depend at most on $n$, $C$, $Q$, the weight $w$, $\varepsilon$ and $N$. For a positive integer $q$, $\omega(q)$ is the number of distinct prime factors of $q$, $\tau(q)$ the number of divisors, and $v_p(q)$ the exponent of the prime $p$ in $q$. We put

$$D = d_1 \cdots d_n, \qquad \Delta = 2D, \qquad \widetilde{Q}(\mathbf{m}) = \sum_{i=1}^{n} m_i^2 \prod_{j \neq i} d_j,$$

and for a positive integer $q$

$$q^\flat = \prod_{p \mid \Delta} p^{v_p(q)}, \qquad q^\sharp = q/q^\flat, \qquad \mathrm{cf}(q) = \prod_{p^e \| q,\ e \geqslant 3} p^e, \qquad \kappa(q) = q^\flat\ \mathrm{cf}(q^\sharp).$$

(In §8, where $d_n = 0$, the definition of $D$ is changed to $D = d_1 \cdots d_{n-1}$.) We write $\beta \mid \Delta^\infty$ if every prime factor of $\beta$ divides $\Delta$. We shall use that the number of $\beta \mid \Delta^\infty$ with $\beta \leqslant x$ is $O((\log 3x)^{\omega(\Delta)})$, and that the number of cube-full integers up to $x$ is $O(x^{1/3})$.

**Declaration on the use of AI.** The strategy of the paper, namely to detect the quadratic equation by a nested $\delta$-symbol in the manner of Munshi inside a circle-method treatment of the cubic equation, was chosen by the author, and the author directed the work throughout. In developing the detailed arguments and in preparing the manuscript, the author made use of the AI system Claude (Anthropic). The author has checked all the arguments in the paper and takes full responsibility for its content.

## 2. Preliminaries

We begin by recording the normalisation of the forms $C$ and $Q$ and the choice of the weight function. We then collect the facts about the $\delta$-symbol and about oscillatory integrals that will be needed later.

### 2.1. Normalisation.

Let $X$ be as in Theorem 1.1. By [3, Corollary 3.2], which rests on a result of Aznar [1, §2], there are a non-singular cubic form $C \in \mathbb{Z}[\mathbf{x}]$ and a diagonal quadratic form $Q \in \mathbb{Z}[\mathbf{x}]$ of rank at least $n-1$ such that, after a linear change of variables over $\mathbb{Q}$, $X$ is given by $C = Q = 0$. After a permutation of the variables (1.1) holds. Since the statement of Theorem 1.1 is invariant under linear changes of variables over $\mathbb{Q}$, it suffices to prove Theorem 1.2. From now on $C$ and $Q$ are as in (1.1), $X$ is non-singular, and $X(\mathbb{R}) \neq \emptyset$.

*In §§2–7 we assume in addition that $d_n \neq 0$.* The case $d_n = 0$ is treated in §8, where we indicate which of the results of §§2–7 carry over, and replace the others.

*Remark* 2.1. The diagonal shape of $Q$ in (1.1) is a normalisation, and not a restriction in Theorem 1.1. Every quadratic form over $\mathbb{Q}$ is diagonalised by a linear change of variables over $\mathbb{Q}$. Such a change maps the rational points and the real points of $X$ bijectively to those of the new variety, it preserves the non-singularity of $X$ and the rank of the quadratic form, and it transforms $C$ into another cubic form, whose coefficients become integers after a scaling. We use the diagonal shape for three reasons. The bounds of [3, §6] for the complete exponential sums, which we quote in (3.1) and Lemmas 3.1–3.3, are stated for diagonal $Q$. The Gauss sums in Lemma 3.7 factorise over the coordinates. And the gradient of $Q$ is $(2d_iz_i)_i$ with $|d_i| \geqslant 1$, which is used in Lemma 2.10. The lemmas of §3 that are proved here, Lemmas 3.7, 3.8 and 3.9, hold in the same way for $Q(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x}$ with a symmetric integer matrix $\mathbf{A}$ and primes $p \nmid 2 \det \mathbf{A}$, with the adjoint form of $\mathbf{A}$ in place of $\widetilde{Q}$.

Theorem 1.2, on the other hand, is a statement about the normalised forms. Under the change of variables the integral points of the original variety correspond to the points of a lattice in the new coordinates, and not to all of $\mathbb{Z}^n$. An asymptotic formula for the integral points in the original coordinates, for a quadratic form that is not diagonal, would require a lattice condition on $\mathbf{x}$. Such a condition is a union of congruence conditions of the kind considered in §9, but we have not pursued this.

2.2. **The weight function.** Every point of $X(\mathbb{R})$ is non-singular. By homogeneity there is $\mathbf{z}_0 \in \mathbb{R}^n$ with $0 < |\mathbf{z}_0| \leqslant 1/4$, $C(\mathbf{z}_0) = Q(\mathbf{z}_0) = 0$ and $\nabla C(\mathbf{z}_0)$, $\nabla Q(\mathbf{z}_0)$ linearly independent. We fix indices $i_1 \neq i_2$ such that the $2 \times 2$ minor of the matrix with rows $\nabla C(\mathbf{z}_0)$, $\nabla Q(\mathbf{z}_0)$ formed by the columns $i_1, i_2$ is non-zero. By continuity and compactness we can fix $\rho_0 > 0$ and $c_0 > 0$ such that the closed ball $\mathcal{U} = \{\mathbf{z} : |\mathbf{z} - \mathbf{z}_0| \leqslant 2\rho_0\}$ has the following properties:

(U1) $\mathcal{U} \subset (-1/2, 1/2)^n$ and $|\mathbf{z}| \geqslant |\mathbf{z}_0|/2$ for $\mathbf{z} \in \mathcal{U}$;
(U2) $|s_3 \nabla C(\mathbf{z}) + s_2 \nabla Q(\mathbf{z})| \geqslant c_0(|s_3| + |s_2|)$ for all $\mathbf{z} \in \mathcal{U}$ and $(s_3, s_2) \in \mathbb{R}^2$;
(U3) the minor above does not vanish on $\mathcal{U}$, and $\mathbf{z} \mapsto (C(\mathbf{z}), Q(\mathbf{z}), (z_i)_{i \neq i_1, i_2})$ is a diffeomorphism from a neighbourhood of $\mathcal{U}$ onto its image.

Let $w_0 \in C^\infty(\mathbb{R}^n)$ be non-negative, supported in $\{|\mathbf{y}| \leqslant 1\}$, with $w_0(\mathbf{0}) > 0$, and put

$$w(\mathbf{z}) = w_0\Big((\mathbf{z} - \mathbf{z}_0)/\rho_0\Big).$$

Then $w$ is smooth, non-negative, supported in $\{|\mathbf{z} - \mathbf{z}_0| \leqslant \rho_0\} \subset \mathcal{U}$, and $w(\mathbf{z}_0) > 0$. We put

$$S(\alpha, \beta) = \sum_{\mathbf{x} \in \mathbb{Z}^n} w(\mathbf{x}/P)\, e\Big(\alpha C(\mathbf{x}) + \beta Q(\mathbf{x})\Big),$$

$$\mathcal{H}(\alpha) = \sum_{\substack{\mathbf{x} \in \mathbb{Z}^n \\ Q(\mathbf{x}) = 0}} w(\mathbf{x}/P)\, e\Big(\alpha C(\mathbf{x})\Big) = \int_0^1 S(\alpha, \beta)\, d\beta.$$

2.3. **The $\delta$-symbol.** Let $\varpi \in C^\infty(\mathbb{R})$ be the non-negative function of [12, §3], supported in $[1/2, 1]$ with $\int \varpi = 1$, and for $x > 0$ and $y \in \mathbb{R}$ let

$$h(x, y) = \sum_{j=1}^{\infty} \frac{1}{xj} \Big\{ \varpi(xj) - \varpi\Big(\frac{|y|}{xj}\Big) \Big\}.$$

**Lemma 2.2.** *For every $R > 1$ there is a constant $c_R$ with $c_R = 1 + O_N(R^{-N})$ such that, for every $k \in \mathbb{Z}$,*

$$\delta(k) = \frac{c_R}{R^2} \sum_{q=1}^{\infty} \sideset{}{^*}\sum_{a \bmod q} e_q(ak)\, h\Big(\frac{q}{R}, \frac{k}{R^2}\Big),$$

*where $\delta(0) = 1$ and $\delta(k) = 0$ for $k \neq 0$. The function $h$ is smooth on $(0, \infty) \times \mathbb{R}$, and $h(x, y) = 0$ unless $x \leqslant \max\{1, 2|y|\}$. Moreover, for all integers $j, N \geqslant 0$,*

$$\frac{\partial^j h}{\partial y^j}(x, y) \ll_{j,N} x^{-1-j} \Big( x^N + \min\Big\{1, \Big(\frac{x}{|y|}\Big)^N\Big\}\Big), \tag{2.1}$$

*and the term $x^N$ may be omitted if $j \geqslant 1$.*

*Proof.* This is [12, Theorem 1 and Lemma 5]. □

We need three consequences. Put $c_C = \sup_{\mathbf{z}\in\mathcal{U}} |C(\mathbf{z})|$ and $r_0 = \max\{1, 2c_C\}$. Fix $\psi \in C_c^\infty(\mathbb{R})$ with $\psi = 1$ on $[-c_C, c_C]$, and a smooth even function $B$ with $B = 1$ on $[-1, 1]$ and $\operatorname{supp} B \subset [-2, 2]$. For $0 < r \leqslant r_0$, respectively $0 < r \leqslant 4$, define

$$p_r(t) = \int_{\mathbb{R}} h(r, y)\psi(y)e(-ty)\, dy,$$
$$g_r(y) = h(r, y)B(y), \qquad \widehat{g}_r(\tau) = \int_{\mathbb{R}} g_r(y)e(-\tau y)\, dy.$$

**Lemma 2.3.** *Let $f \in C_c^\infty(\mathbb{R})$, $x > 0$ and $N \geqslant 2$. Then*

$$\int_{\mathbb{R}} f(y)h(x, y)\, dy = f(0) + O_N\Big(x^{N-1}\big(\|f\|_{L^1} + \|f^{(N)}\|_{L^1}\big)\Big).$$

*Proof.* For fixed $x$ the sum defining $h(x, y)$ is finite, uniformly for $y$ in a compact set. For $s \in \mathbb{R}$ put

$$\Omega(s) = \varpi(s)/s, \qquad G(s) = \int_0^\infty \big\{f(sv) + f(-sv)\big\}\varpi(v)\, dv.$$

Then $\Omega \in C_c^\infty((0, \infty))$, and $G$ is smooth, even and compactly supported, with $G(0) = 2f(0)$. The substitution $y = \pm xjv$ gives $\int f(y)\varpi(|y|/(xj))\, dy = xj\, G(xj)$. Hence

$$\int_{\mathbb{R}} f(y)h(x, y)\, dy = \Big(\int f\Big)\sum_{j\geqslant 1}\Omega(xj) - \sum_{j\geqslant 1} G(xj) = \Big(\int f\Big)\sum_{j\in\mathbb{Z}}\Omega(xj) - \frac{1}{2}\sum_{j\in\mathbb{Z}} G(xj) + f(0).$$

By Poisson summation $\sum_{j\in\mathbb{Z}} F(xj) = x^{-1}\sum_{k\in\mathbb{Z}}\widehat{F}(k/x)$ for $F = \Omega$ and $F = G$. Now $\widehat{\Omega}(0) = \int \varpi(s)s^{-1}ds$ and

$$\widehat{G}(0) = \int_0^\infty \varpi(v)\int_{\mathbb{R}}\big\{f(sv) + f(-sv)\big\}ds\, dv = 2\Big(\int f\Big)\int_0^\infty \frac{\varpi(v)}{v}dv,$$

so the terms with $k = 0$ cancel. For $k \neq 0$ we have $\widehat{\Omega}(k/x) \ll_N (x/|k|)^N$ and $|\widehat{G}(k/x)| \leqslant (2\pi|k|/x)^{-N}\|G^{(N)}\|_{L^1}$, and $\|G^{(N)}\|_{L^1} \leqslant 2\|f^{(N)}\|_{L^1}\int \varpi(v)v^{N-1}dv \leqslant 2\|f^{(N)}\|_{L^1}$. Summing over $k \neq 0$ gives the claim. □

**Lemma 2.4.** *For $0 < r \leqslant r_0$ and $|y| \leqslant c_C$ we have $h(r, y) = \int_{\mathbb{R}} p_r(t)e(ty)\, dt$. Moreover*

$$p_r(t) \ll_N \min\{1, (r|t|)^{-N}\}, \qquad p_r(t) = 1 + O_N\big(r^{N-1}(1 + |t|)^N\big).$$

*Proof.* The function $y \mapsto h(r, y)\psi(y)$ is smooth with compact support, and it equals $h(r, y)$ for $|y| \leqslant c_C$; the first statement is Fourier inversion. By (2.1) with $N = 2$, $\int |h(r, y)\psi(y)|\, dy \ll 1$ and $\int |\partial_y^j(h(r, y)\psi(y))|\, dy \ll_j r^{-j}$ for $j \geqslant 1$, uniformly for $0 < r \leqslant r_0$. This gives the first bound, on integrating by parts $N$ times. The second follows from Lemma 2.3 with $f(y) = \psi(y)e(-ty)$. □

**Lemma 2.5.** *For $0 < r \leqslant 4$ we have $g_r(y) = \int_{\mathbb{R}} \widehat{g}_r(\tau)e(\tau y)\, d\tau$ and $\widehat{g}_r(\tau) \ll_N \min\{1, (r|\tau|)^{-N}\}$.*

*Proof.* Since $B$ has compact support, the function $g_r$ is smooth with compact support, and the first statement is Fourier inversion. By (2.1) with $N = 2$, $\int |g_r(y)|\, dy \ll 1$ and $\int |g_r^{(j)}(y)|\, dy \ll_j r^{-j}$ for $j \geqslant 1$, uniformly for $0 < r \leqslant 4$. The bound follows on integrating by parts $N$ times, as in the proof of Lemma 2.4. □

2.4. **Oscillatory integrals.** For $\boldsymbol{\gamma} = (\gamma_3, \gamma_2) \in \mathbb{R}^2$ and $\mathbf{u} \in \mathbb{R}^n$ let

$$f_\gamma(\mathbf{z}) = \gamma_3 C(\mathbf{z}) + \gamma_2 Q(\mathbf{z}), \qquad \|\boldsymbol{\gamma}\| = 1 + |\gamma_3| + |\gamma_2|,$$
$$J(\boldsymbol{\gamma}; \mathbf{u}) = \int_{\mathbb{R}^n} w(\mathbf{z})\, e\Big(f_\gamma(\mathbf{z}) - \mathbf{u}\cdot\mathbf{z}\Big)\, d\mathbf{z}.$$

This is the integral $I(\boldsymbol{\gamma}; \mathbf{u})$ of [3, §6]. We begin with the standard principle of non-stationary phase.

**Lemma 2.6.** *Let $N \geqslant 1$ and $A \geqslant 1$. Let $v \in C_c^\infty(\mathbb{R}^n)$ be supported in $\{|\mathbf{y}| \leqslant 2\}$ with $|\partial^{\mathbf{a}} v| \leqslant A$ for all multi-indices with $|\mathbf{a}| \leqslant N$. Let $F$ be a real smooth function on a neighbourhood of $\operatorname{supp} v$, and let $\lambda \geqslant 1$ be such that, on $\operatorname{supp} v$, $|\nabla F| \geqslant \lambda$ and $|\partial^{\mathbf{a}} F| \leqslant A\lambda$ for $2 \leqslant |\mathbf{a}| \leqslant N+1$. Then*

$$\int_{\mathbb{R}^n} v(\mathbf{y}) e(F(\mathbf{y}))\, d\mathbf{y} \ll_{n,N,A} \lambda^{-N}.$$

*Proof.* Let $\mathbf{b} = \nabla F/|\nabla F|_2^2$, where $|\cdot|_2$ is the Euclidean norm, and let $L^* v = -(2\pi i)^{-1} \nabla \cdot (v\,\mathbf{b})$. Since $(2\pi i)^{-1} \mathbf{b}\cdot\nabla e(F) = e(F)$, integration by parts gives $\int v e(F) = \int (L^*)^N v \cdot e(F)$. The hypotheses imply that each component of $\mathbf{b}$, and each of its partial derivatives of order at most $N$, is $O(\lambda^{-1})$ on $\operatorname{supp} v$: this is clear for $\mathbf{b}$ itself, and each differentiation either falls on a factor $\partial_i F$, producing a derivative of order at least two, which is $O(\lambda)$, or on a power of $|\nabla F|_2^2$, producing a factor $O(\lambda \cdot |\nabla F|_2)/|\nabla F|_2^2 = O(1)$. Hence $(L^*)^N v \ll \lambda^{-N}$. □

Let $c_4 = 2 \sup_{\mathbf{z}\in\mathcal{U}} (|\nabla C(\mathbf{z})| + |\nabla Q(\mathbf{z})|)$. By (U1), there are constants $0 < \kappa_1 \leqslant \kappa_2$ with $\kappa_1 \leqslant |(d_i z_i)_i| \leqslant \kappa_2$ for $\mathbf{z} \in \mathcal{U}$.

In the next four lemmas $\boldsymbol{\gamma} \in \mathbb{R}^2$ and $\mathbf{u} \in \mathbb{R}^n$ are arbitrary, and we put $F(\mathbf{z}) = f_\gamma(\mathbf{z}) - \mathbf{u}\cdot\mathbf{z}$. All partial derivatives of $F$ of order at least two are $O(\|\boldsymbol{\gamma}\|)$ on $\mathcal{U}$, and those of order at least four vanish. We shall also use the trivial bound

$$|J(\boldsymbol{\gamma}; \mathbf{u})| \leqslant \int_{\mathbb{R}^n} w(\mathbf{z})\, d\mathbf{z}. \tag{2.2}$$

**Lemma 2.7.** *If $|\mathbf{u}| \geqslant c_4 \|\boldsymbol{\gamma}\|$, then $J(\boldsymbol{\gamma}; \mathbf{u}) \ll_N |\mathbf{u}|^{-N}$.*

*Proof.* Here $|\nabla F| \geqslant |\mathbf{u}| - \frac{1}{2} c_4 \|\boldsymbol{\gamma}\| \geqslant |\mathbf{u}|/2$ on $\mathcal{U}$, and we may suppose that $|\mathbf{u}| \geqslant 2$. Lemma 2.6 applies with $\lambda = |\mathbf{u}|/2$, after a translation and dilation that carries $\operatorname{supp} w$ into $\{|\mathbf{y}| \leqslant 2\}$. □

**Lemma 2.8.** *We have $J(\boldsymbol{\gamma}; \mathbf{0}) \ll_N \|\boldsymbol{\gamma}\|^{-N}$.*

*Proof.* By (U2), $|\nabla f_\gamma| \geqslant c_0(|\gamma_3| + |\gamma_2|)$ on $\mathcal{U}$. If $|\gamma_3| + |\gamma_2| \geqslant 1/c_0$ we apply Lemma 2.6 with $\lambda = c_0(|\gamma_3| + |\gamma_2|)$; otherwise we use (2.2). □

**Lemma 2.9.** *For every $P \geqslant 1$,*

$$J(\boldsymbol{\gamma}; \mathbf{u}) \ll_{N,\varepsilon} P^{-N} + \operatorname{meas}\Big\{\mathbf{z} \in \mathcal{U} :\ |\nabla f_\gamma(\mathbf{z}) - \mathbf{u}| \leqslant P^\varepsilon \|\boldsymbol{\gamma}\|^{1/2}\Big\}.$$

*Proof.* Put $\theta = \|\boldsymbol{\gamma}\|$ and $\delta = \theta^{-1/2} \leqslant 1$. Let $\chi \in C_c^\infty(\mathbb{R}^n)$ be supported in $[-1,1]^n$ with $\sum_{\mathbf{j}\in\mathbb{Z}^n} \chi(\mathbf{y} - \mathbf{j}) = 1$, and write $w = \sum_{\mathbf{j}} w_{\mathbf{j}}$ with $w_{\mathbf{j}}(\mathbf{z}) = w(\mathbf{z})\chi(\mathbf{z}/\delta - \mathbf{j})$. Each $w_{\mathbf{j}}$ is supported in a cube of side $2\delta$, the supports have bounded overlap, and $O(\delta^{-n})$ of them are non-zero. Fix $\mathbf{j}$ with $w_{\mathbf{j}} \neq 0$. Suppose first that $|\nabla F(\mathbf{z})| \geqslant P^\varepsilon \theta^{1/2}$ for all $\mathbf{z} \in \operatorname{supp} w_{\mathbf{j}}$. With $\mathbf{z} = \delta(\mathbf{j} + \mathbf{y})$ we have $\int w_{\mathbf{j}} e(F) = \delta^n \int v(\mathbf{y}) e(\widetilde{F}(\mathbf{y}))\, d\mathbf{y}$, where $v(\mathbf{y}) = w(\delta(\mathbf{j}+\mathbf{y}))\chi(\mathbf{y})$ and $\widetilde{F}(\mathbf{y}) = F(\delta(\mathbf{j}+\mathbf{y}))$. Now $|\nabla \widetilde{F}| \geqslant \delta P^\varepsilon \theta^{1/2} = P^\varepsilon$, and the derivatives of $\widetilde{F}$ of order $k \geqslant 2$ are $O(\delta^k \theta) = O(1)$. Lemma 2.6 with $\lambda = P^\varepsilon$ shows that $\int w_{\mathbf{j}} e(F) \ll_N \delta^n P^{-N}$. Otherwise there is $\mathbf{z}^* \in \operatorname{supp} w_{\mathbf{j}}$ with $|\nabla F(\mathbf{z}^*)| < P^\varepsilon \theta^{1/2}$.

Since $|\nabla F(\mathbf{z}) - \nabla F(\mathbf{z}^*)| \ll \theta\delta = \theta^{1/2}$ on $\operatorname{supp} w_{\mathbf{j}}$, the support of $w_{\mathbf{j}}$ is contained in the set $\{\mathbf{z} \in \mathcal{U} : |\nabla F(\mathbf{z})| \leqslant P^{2\varepsilon}\theta^{1/2}\}$ if $P$ is large, and we bound $\int w_{\mathbf{j}} e(F)$ by $\|w\|_\infty \operatorname{meas}(\operatorname{supp} w_{\mathbf{j}})$. Summing over $\mathbf{j}$, and replacing $\varepsilon$ by $\varepsilon/2$, we obtain the lemma. □

**Lemma 2.10.** *There are constants $K_0 \geqslant 1$ and $0 < c_1 < c_2$, depending only on $C$, $Q$ and $\mathcal{U}$, with the following property. Suppose that $|\gamma_2| \geqslant K_0(1 + |\gamma_3|)$. If $|\mathbf{u}| < c_1|\gamma_2|$ or $|\mathbf{u}| > c_2|\gamma_2|$, then $J(\boldsymbol{\gamma}; \mathbf{u}) \ll_N (|\gamma_2| + |\mathbf{u}|)^{-N}$. For all $\mathbf{u}$ and every $P \geqslant 1$,*

$$J(\boldsymbol{\gamma}; \mathbf{u}) \ll_{N,\varepsilon} P^{-N} + P^{\varepsilon}|\gamma_2|^{-n/2}.$$

*Proof.* We have $\nabla f_{\boldsymbol{\gamma}}(\mathbf{z}) = 2\gamma_2(d_i z_i)_i + \gamma_3 \nabla C(\mathbf{z})$. If $K_0$ is large enough, then for $|\gamma_2| \geqslant K_0(1 + |\gamma_3|)$ and $\mathbf{z}, \mathbf{z}' \in \mathcal{U}$,

$$\kappa_1|\gamma_2| \leqslant |\nabla f_{\boldsymbol{\gamma}}(\mathbf{z})| \leqslant 3\kappa_2|\gamma_2|, \qquad |\nabla f_{\boldsymbol{\gamma}}(\mathbf{z}) - \nabla f_{\boldsymbol{\gamma}}(\mathbf{z}')| \geqslant |\gamma_2|\,|\mathbf{z} - \mathbf{z}'|, \tag{2.3}$$

because $|d_i| \geqslant 1$ and $\nabla C$ is Lipschitz on $\mathcal{U}$. Let $c_1 = \kappa_1/2$ and $c_2 = 6\kappa_2$. If $|\mathbf{u}| < c_1|\gamma_2|$ or $|\mathbf{u}| > c_2|\gamma_2|$, then $|\nabla F| \gg |\gamma_2| + |\mathbf{u}|$ on $\mathcal{U}$, while the derivatives of $F$ of order at least two are $O(|\gamma_2|)$; Lemma 2.6 gives the first assertion. For the second we use Lemma 2.9. By the second inequality in (2.3), the set $\{\mathbf{z} \in \mathcal{U} : |\nabla f_{\boldsymbol{\gamma}}(\mathbf{z}) - \mathbf{u}| \leqslant P^{\varepsilon}\|\boldsymbol{\gamma}\|^{1/2}\}$ has diameter at most $2P^{\varepsilon}\|\boldsymbol{\gamma}\|^{1/2}/|\gamma_2| \ll P^{\varepsilon}|\gamma_2|^{-1/2}$, so its measure is $O(P^{n\varepsilon}|\gamma_2|^{-n/2})$. □

## 3. Exponential sums

This section is devoted to the complete exponential sums. We first recall the estimates of Browning, Dietmann and Heath-Brown for the sums that arise from a single modulus. We then introduce the sums $S_{q_1,q_2}(\mathbf{m})$ attached to the nested $\delta$-symbol, and estimate them according to the way in which the primes are distributed between the two moduli. The outcome is the mean value estimate of Proposition 3.12.

### 3.1. The complete sums of Browning, Dietmann and Heath-Brown.

For $q \in \mathbb{N}$, $\mathbf{a} = (a_3, a_2) \in \mathbb{Z}^2$ and $\mathbf{m} \in \mathbb{Z}^n$ let

$$S(\mathbf{a}, q; \mathbf{m}) = \sum_{\mathbf{y} \bmod q} e_q\Big(a_3 C(\mathbf{y}) + a_2 Q(\mathbf{y}) + \mathbf{m} \cdot \mathbf{y}\Big).$$

For a prime $p$ let $\nu_p$ be the largest integer $\nu$ such that $p^\nu$ divides one of $2d_1, \dots, 2d_{n-1}$, as in [3, §6]; thus $\nu_p = 0$ for $p \nmid \Delta$. Put $C_\Delta = \prod_{p|\Delta} p^{\nu_p}$. If $\gcd(q, a_3, a_2) = 1$ we factorise $q = q^{(0)}q^{(1)}q^{(2)}$, where

$$q^{(0)} = \prod_{\substack{p^e \| q \\ p^{1+\nu_p} | a_3}} p^e, \qquad q^{(2)} = \prod_{\substack{p^e \| q,\ e \geqslant 3 \\ p^{1+\nu_p} \nmid a_3}} p^e,$$

so that $q^{(1)}$ is cube-free and $q^{(2)}$ is cube-full.

We shall use four results of Browning, Dietmann and Heath-Brown, which are stated in [3, §6] for a non-singular cubic form $C$ and a diagonal form $Q$ with $d_1 \cdots d_{n-1} \neq 0$. In all of them $q \in \mathbb{N}$ and $\mathbf{a} \in \mathbb{Z}^2$ with $\gcd(q, a_3, a_2) = 1$. We record the hypotheses precisely. In [3, §6] the form $C$ is non-singular and $Q = \sum_i d_i x_i^2$ with $d_1 \cdots d_{n-1} \neq 0$, by [3, (6.1)]. The coefficient $d_n$ may vanish, and $\nu_p$ is defined in terms of $2d_1, \dots, 2d_{n-1}$ only, so these results apply verbatim when $d_n = 0$, as in §8. The numerators are assumed in [3, (6.2)] to satisfy $\gcd(q, \mathbf{a}) = 1$ and $1 \leqslant a_3, a_2 \leqslant q$. The second condition is no restriction, since $S(\mathbf{a}, q; \mathbf{m})$ depends only on $\mathbf{a}$ modulo $q$. Lemmas 3.1 and 3.2 are stated in [3] for every $\mathbf{m} \in \mathbb{Z}^n$, and Lemma 3.3 for every $\mathbf{m}_0 \in \mathbb{R}^n$ and $V \geqslant 1$. In all three the implied constants depend only on $n$, $C$, $Q$ and $\varepsilon$, and not on $\mathbf{a}$ or $\mathbf{m}$. This can be read off from the proofs. In [3, Lemma 6.2] the constant comes from the number $p^{\nu_p} \ll 1$ of choices for each of the variables $y_i$ there. In [3, Lemma 6.3] it comes from

Deligne's bound for moduli $p$ and Heath-Brown's bound for moduli $p^2$, applied to the polynomial $a_3C(\mathbf{x}) + a_2Q(\mathbf{x}) + \mathbf{m}\cdot\mathbf{x}$, whose leading form $a_3C$ is non-singular modulo $p$; the finitely many primes that divide $\Delta$, or modulo which $C$ is singular, are treated trivially. In [3, Lemma 6.4] the only condition on $\mathbf{a}$ is that $\gcd(q, a_3)$ be bounded, and here this takes the form $\gcd(q, a_3) \leqslant C_\Delta$. The first is the multiplicativity relation [3, (6.6)]: if $q = q'q''$ with $(q', q'') = 1$, then for all $\mathbf{m} \in \mathbb{Z}^n$

(3.1)
$$S(\mathbf{a}, q; \mathbf{m}) = S(\mathbf{a}', q'; \mathbf{m})\, S(\mathbf{a}'', q''; \mathbf{m}), \qquad \mathbf{a}' = (q''^2 a_3, q'' a_2), \quad \mathbf{a}'' = (q'^2 a_3, q' a_2).$$

The other three are the following lemmas.

**Lemma 3.1** ([3, Lemma 6.2])**.** *If $q = p^e$ and $p^{1+\nu_p} \mid a_3$, then $S(\mathbf{a}, q; \mathbf{m}) \ll q^{(n+1)/2}$ for all $\mathbf{m} \in \mathbb{Z}^n$.*

**Lemma 3.2** ([3, Lemma 6.3])**.** *If $q$ is cube-free and $p^{1+\nu_p} \nmid a_3$ for every prime $p \mid q$, then $S(\mathbf{a}, q; \mathbf{m}) \ll q^{n/2+\varepsilon}$ for all $\mathbf{m} \in \mathbb{Z}^n$.*

**Lemma 3.3** ([3, Lemma 6.4])**.** *If $q$ is cube-full and $\gcd(q, a_3) \leqslant C_\Delta$, then for all $\mathbf{m}_0 \in \mathbb{R}^n$ and $V \geqslant 1$*

$$\sum_{\substack{\mathbf{m}\in\mathbb{Z}^n \\ |\mathbf{m}-\mathbf{m}_0|\leqslant V}} |S(\mathbf{a}, q; \mathbf{m})| \ll q^{n/2+\varepsilon}\Big(V^n + q^{n/3}\Big).$$

**Corollary 3.4.** *Let $q \in \mathbb{N}$ and $\mathbf{a} \in \mathbb{Z}^2$ with $\gcd(q, a_3, a_2) = 1$. For all $\mathbf{m}_0 \in \mathbb{R}^n$ and $V \geqslant 1$,*

$$\sum_{|\mathbf{m}-\mathbf{m}_0|\leqslant V} |S(\mathbf{a}, q; \mathbf{m})| \ll q^{n/2+\varepsilon}\Big(q^{(0)}\Big)^{1/2}\Big(V^n + \Big(q^{(2)}\Big)^{n/3}\Big).$$

*Proof.* By (3.1), $S(\mathbf{a}, q; \mathbf{m})$ is the product of three sums with moduli $q^{(0)}$, $q^{(1)}$, $q^{(2)}$ and the same $\mathbf{m}$. Their numerators are obtained from $\mathbf{a}$ on multiplying $a_3$ and $a_2$ by integers coprime to the modulus, so the conditions of Lemmas 3.1, 3.2 and 3.3 hold for them. We apply (3.1) and Lemma 3.1 to the prime powers dividing $q^{(0)}$; the product of the implied constants is $O(q^\varepsilon)$. Then we apply Lemma 3.2 to $q^{(1)}$, and Lemma 3.3 to $q^{(2)}$ if $q^{(2)} > 1$. □

3.2. **The nested sums.** For $q_1, q_2 \in \mathbb{N}$ and $\mathbf{m} \in \mathbb{Z}^n$ let

$$S_{q_1,q_2}(\mathbf{m}) = \sideset{}{^*}\sum_{a_1 \bmod q_1} \sideset{}{^*}\sum_{a_2 \bmod q_2} \sum_{\substack{\mathbf{k} \bmod q_1q_2 \\ q_1 \mid Q(\mathbf{k})}} e_{q_1q_2}\Big(a_1q_2C(\mathbf{k}) + a_2Q(\mathbf{k}) + \mathbf{k}\cdot\mathbf{m}\Big).$$

The summand is well defined: if $q_1 \mid Q(\mathbf{k})$, then $a_2Q(\mathbf{k})/(q_1q_2)$ modulo 1 depends only on $a_2$ modulo $q_2$.

**Lemma 3.5.** *If $q_1 = q_1'q_1''$ and $q_2 = q_2'q_2''$ with $(q_1'q_2', q_1''q_2'') = 1$, then*

$$S_{q_1,q_2}(\mathbf{m}) = S_{q_1',q_2'}(\mathbf{m})\, S_{q_1'',q_2''}(\mathbf{m}).$$

*Proof.* Put $q' = q_1'q_2'$ and $q'' = q_1''q_2''$, and write $\mathbf{k} = q''\mathbf{k}' + q'\mathbf{k}''$ with $\mathbf{k}'$ modulo $q'$ and $\mathbf{k}''$ modulo $q''$. Then $q_1 \mid Q(\mathbf{k})$ if and only if $q_1' \mid Q(\mathbf{k}')$ and $q_1'' \mid Q(\mathbf{k}'')$. Let $\overline{q''}$ be an inverse of $q''$ modulo $q'$ and $\overline{q'}$ an inverse of $q'$ modulo $q''$, so that $e_{q'q''}(M) = e_{q'}(\overline{q''}M)e_{q''}(\overline{q'}M)$ for $M \in \mathbb{Z}$. Since $C(\mathbf{k}) \equiv q''^3C(\mathbf{k}')$, $Q(\mathbf{k}) \equiv q''^2Q(\mathbf{k}')$ and $\mathbf{k} \equiv q''\mathbf{k}'$ modulo $q'$,

$$e_{q'}\Big(\overline{q''}(a_1q_2C(\mathbf{k}) + a_2Q(\mathbf{k}) + \mathbf{k}\cdot\mathbf{m})\Big) = e_{q_1'q_2'}\Big(a_1'q_2'C(\mathbf{k}') + a_2'Q(\mathbf{k}') + \mathbf{k}'\cdot\mathbf{m}\Big),$$

with $a_1' = a_1q_2''q''^2$ and $a_2' = a_2q''$, and similarly for the other factor. As $a_1$ runs over the reduced residues modulo $q_1$, the pair $(a_1' \bmod q_1', a_1'' \bmod q_1'')$ runs over the pairs of reduced residues, and likewise for $a_2$. □

**Lemma 3.6.** *For all $q_1, q_2 \in \mathbb{N}$ and $\mathbf{m} \in \mathbb{Z}^n$ we have*

$$S_{q_1,q_2}(\mathbf{m}) = \frac{1}{q_1} \sum_{b \bmod q_1} \sideset{}{^*}\sum_{a_1 \bmod q_1} \sideset{}{^*}\sum_{a_2 \bmod q_2} S\Big((a_1q_2,\, a_2 + bq_2),\, q_1q_2;\, \mathbf{m}\Big),$$

*and $\gcd(q_1q_2, a_1q_2, a_2 + bq_2) = 1$ for all $b$, $a_1$, $a_2$ in the sum.*

*Proof.* We detect the condition $q_1 \mid Q(\mathbf{k})$ by $q_1^{-1}\sum_{b \bmod q_1} e_{q_1}(bQ(\mathbf{k}))$ and note that $e_{q_1}(bQ(\mathbf{k})) = e_{q_1q_2}(bq_2Q(\mathbf{k}))$. If a prime $p$ divides $q_2$, then $p \nmid a_2 + bq_2$. If $p$ divides $q_1$ but not $q_2$, then $p \nmid a_1q_2$. □

3.3. **The second modulus alone.**

**Lemma 3.7.** *There is a constant $A \geqslant 1$ such that, for all $e \in \mathbb{N}$ and $\mathbf{m} \in \mathbb{Z}^n$,*

$$|S_{1,e}(\mathbf{m})| \leqslant A^{\omega(e)}\, e^{(n+1)/2} \left(e^{\flat}\right)^{1/2} \left(e^{\sharp}, \widetilde{Q}(\mathbf{m})\right)^{1/2}.$$

*Here $(e^{\sharp}, 0) = e^{\sharp}$.*

*Proof.* By Lemma 3.5 it suffices to take $e = p^s$. We have

$$S_{1,p^s}(\mathbf{m}) = \sideset{}{^*}\sum_{a \bmod p^s} \prod_{i=1}^{n} G(ad_i, m_i), \qquad G(u,v) = \sum_{k \bmod p^s} e_{p^s}(uk^2 + vk).$$

Squaring out, $|G(u,v)|^2 \leqslant p^s\, \#\{h \bmod p^s : 2uh \equiv 0\} = p^s(2u, p^s)$. If $p \mid \Delta$ this gives $|S_{1,p^s}(\mathbf{m})| \leqslant p^s \cdot (2^n|D|)^{1/2}p^{sn/2}$, which is the claim with $A = (2^n|D|)^{1/2}$. Let $p \nmid \Delta$. Then, for $p \nmid u$, completing the square gives $G(u,v) = e_{p^s}(-\overline{4u}v^2)\left(\frac{u}{p}\right)^s G(1,0)$, where $\left(\frac{\cdot}{p}\right)$ is the Legendre symbol and $|G(1,0)| = p^{s/2}$. Hence

$$|S_{1,p^s}(\mathbf{m})| = p^{sn/2}\Big| \sideset{}{^*}\sum_{a \bmod p^s} \left(\frac{a}{p}\right)^{sn} e_{p^s}(aM)\Big|, \qquad M = -\overline{4D}\,\widetilde{Q}(\mathbf{m}),$$

on replacing $a$ by its inverse. If $sn$ is even, the sum over $a$ is a Ramanujan sum, of modulus at most $(p^s, M) \leqslant p^{s/2}(p^s, M)^{1/2}$. If $sn$ is odd, it is a Gauss sum for the character $\left(\frac{\cdot}{p}\right)$ modulo $p^s$: it vanishes unless $p^{s-1} \parallel M$, in which case its modulus is $p^{s-1/2} = p^{s/2}(p^s, M)^{1/2}$. Since $(p^s, M) = (p^s, \widetilde{Q}(\mathbf{m}))$, this proves the lemma. □

3.4. **Moduli with a common prime factor.** We first record a form of the method of stationary phase for sums modulo prime powers.

**Lemma 3.8.** *Let $p$ be an odd prime, $t \geqslant 1$, and*

$$g(\mathbf{z}) = \mathbf{z}^T\mathbf{B}\mathbf{z} + \boldsymbol{\ell}\cdot\mathbf{z} + c + p\,h(\mathbf{z}),$$

*where $\mathbf{B}$ is a symmetric $n \times n$ integer matrix with $p \nmid \det\mathbf{B}$, $\boldsymbol{\ell} \in \mathbb{Z}^n$, $c \in \mathbb{Z}$ and $h \in \mathbb{Z}[z_1, \dots, z_n]$. Then*

$$\Big|\sum_{\mathbf{z} \bmod p^t} e_{p^t}\Big(g(\mathbf{z})\Big)\Big| = p^{nt/2}.$$

*Proof.* For $t = 1$ the sum is a Gauss sum for the non-singular quadratic form $\mathbf{z}^T\mathbf{B}\mathbf{z}$ over $\mathbb{F}_p$. Let $t \geqslant 2$, $j = \lfloor t/2 \rfloor$ and $k = t - j$, and write $\mathbf{z} = \mathbf{x} + p^k\mathbf{v}$ with $\mathbf{x}$ modulo $p^k$ and $\mathbf{v}$ modulo $p^j$. The coefficients of the Taylor expansion of a polynomial with integer coefficients are polynomials with integer coefficients, and $2k \geqslant t$; hence $g(\mathbf{x} + p^k\mathbf{v}) \equiv g(\mathbf{x}) + p^k\nabla g(\mathbf{x})\cdot\mathbf{v} \pmod{p^t}$. Summing over $\mathbf{v}$,

$$\sum_{\mathbf{z} \bmod p^t} e_{p^t}(g(\mathbf{z})) = p^{nj} \sum_{\substack{\mathbf{x} \bmod p^k \\ \nabla g(\mathbf{x}) \equiv \mathbf{0} \bmod p^j}} e_{p^t}(g(\mathbf{x})).$$

Now $\nabla g(\mathbf{x}) = 2\mathbf{B}\mathbf{x} + \boldsymbol{\ell} + p\nabla h(\mathbf{x})$. If $\nabla g(\mathbf{x}) \equiv \nabla g(\mathbf{x}') \pmod{p^j}$ and $\mathbf{x} \equiv \mathbf{x}' \pmod{p^i}$ for some $i < j$, then $2\mathbf{B}(\mathbf{x}-\mathbf{x}') \equiv p(\nabla h(\mathbf{x}') - \nabla h(\mathbf{x})) \equiv \mathbf{0} \pmod{p^{i+1}}$, hence $\mathbf{x} \equiv \mathbf{x}' \pmod{p^{i+1}}$. By induction on $i$, the map $\mathbf{x} \mapsto \nabla g(\mathbf{x})$ is injective, hence bijective, on $(\mathbb{Z}/p^j\mathbb{Z})^n$. So the condition $\nabla g(\mathbf{x}) \equiv \mathbf{0} \pmod{p^j}$ holds for exactly one class $\mathbf{x}_0$ modulo $p^j$. If $t$ is even, then $k = j$, a single term remains, and the modulus of the sum is $p^{nj} = p^{nt/2}$. If $t$ is odd, then $k = j+1$ and $\mathbf{x} = \mathbf{x}_0 + p^j\mathbf{v}'$ with $\mathbf{v}'$ modulo $p$. Write $\nabla g(\mathbf{x}_0) = p^j\boldsymbol{\nu}$. The terms of order two in the Taylor expansion of $g$ at $\mathbf{x}_0$ are $\mathbf{v}'^T\mathbf{B}\mathbf{v}'$ plus $p$ times an integral quadratic form, and the terms of order at least three come from $ph$ alone. Since $3j \geqslant 2j+1$,

$$g(\mathbf{x}_0 + p^j\mathbf{v}') \equiv g(\mathbf{x}_0) + p^{2j}\Big(\mathbf{v}'^T\mathbf{B}\mathbf{v}' + \boldsymbol{\nu}\cdot\mathbf{v}'\Big) \pmod{p^{2j+1}}.$$

The sum over $\mathbf{v}'$ is again a Gauss sum of modulus $p^{n/2}$, and $p^{nj+n/2} = p^{nt/2}$. □

**Lemma 3.9.** *Let $p \nmid \Delta$ and $r, s \geqslant 1$. Then for all $\mathbf{m} \in \mathbb{Z}^n$*

$$|S_{p^r,p^s}(\mathbf{m})| \leqslant 2\,p^{(r+s)(n/2+1)}.$$

*Proof.* The sum over $a_1$ is a Ramanujan sum: $\sideset{}{^*}\sum_{a_1 \bmod p^r} e_{p^r}(a_1 C(\mathbf{k})) = p^r\mathbf{1}[p^r \mid C(\mathbf{k})] - p^{r-1}\mathbf{1}[p^{r-1} \mid C(\mathbf{k})]$. Hence

$$S_{p^r,p^s}(\mathbf{m}) = p^r\mathcal{N}^{(r)}(\mathbf{m}) - p^{r-1}\mathcal{N}^{(r-1)}(\mathbf{m}),$$

where

$$\mathcal{N}^{(j)}(\mathbf{m}) = \sideset{}{^*}\sum_{a \bmod p^s} \sum_{\substack{\mathbf{k} \bmod p^{r+s} \\ p^r \mid Q(\mathbf{k}),\ p^j \mid C(\mathbf{k})}} e_{p^{r+s}}\Big(aQ(\mathbf{k}) + \mathbf{k}\cdot\mathbf{m}\Big),$$

and it suffices to show that $|\mathcal{N}^{(j)}(\mathbf{m})| \leqslant p^{s+n(r+s)/2}$ for $j \in \{r-1, r\}$. Let $\mathbf{A} = \operatorname{diag}(d_1, \dots, d_n)$, so that $Q(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x}$ and $p \nmid 2\det\mathbf{A}$. We write $\mathbf{k} = \mathbf{u} + p^r\mathbf{v}$ with $\mathbf{u}$ modulo $p^r$ and $\mathbf{v}$ modulo $p^s$. The conditions $p^r \mid Q(\mathbf{k})$ and $p^j \mid C(\mathbf{k})$ depend only on $\mathbf{u}$, and

$$aQ(\mathbf{k}) + \mathbf{k}\cdot\mathbf{m} = aQ(\mathbf{u}) + \mathbf{u}\cdot\mathbf{m} + p^r(2a\mathbf{A}\mathbf{u} + \mathbf{m})\cdot\mathbf{v} + ap^{2r}Q(\mathbf{v}). \tag{3.2}$$

*The case $r < s$.* Write $\mathbf{v} = \mathbf{v}_0 + p^{s-r}\mathbf{v}_1$ with $\mathbf{v}_0$ modulo $p^{s-r}$ and $\mathbf{v}_1$ modulo $p^r$. Then $p^rQ(\mathbf{v}) \equiv p^rQ(\mathbf{v}_0) \pmod{p^s}$, and the sum over $\mathbf{v}_1$ is $p^{nr}$ if $2a\mathbf{A}\mathbf{u}+\mathbf{m} \equiv \mathbf{0} \pmod{p^r}$ and vanishes otherwise. For each $a$ this congruence holds for exactly one $\mathbf{u}$ modulo $p^r$; for it, writing $2a\mathbf{A}\mathbf{u}+\mathbf{m} = p^r\mathbf{w}$, the remaining sum over $\mathbf{v}_0$ is $\sum_{\mathbf{v}_0 \bmod p^{s-r}} e_{p^{s-r}}(aQ(\mathbf{v}_0)+\mathbf{w}\cdot\mathbf{v}_0)$, of modulus $p^{n(s-r)/2}$. Hence $|\mathcal{N}^{(j)}(\mathbf{m})| \leqslant p^s \cdot p^{nr} \cdot p^{n(s-r)/2}$.

*The case $r \geqslant s$.* Now $2r \geqslant r+s$, so by (3.2) the sum over $\mathbf{v}$ is $p^{ns}$ if $2a\mathbf{A}\mathbf{u} + \mathbf{m} \equiv \mathbf{0} \pmod{p^s}$ and vanishes otherwise. Moreover, if this congruence holds, then by (3.2) (with $\mathbf{u}$, $p^r\mathbf{y}$ in place of $\mathbf{u}$, $p^r\mathbf{v}$) the value of $aQ(\mathbf{u}) + \mathbf{u}\cdot\mathbf{m}$ modulo $p^{r+s}$ depends only on $\mathbf{u}$ modulo $p^r$. For each $a$ fix an integer $\bar{a}$ with $a\bar{a} \equiv 1 \pmod{p^{r+s}}$, and substitute $\mathbf{u} \equiv \bar{a}\mathbf{w}$. Then $aQ(\mathbf{u}) + \mathbf{u}\cdot\mathbf{m} \equiv \bar{a}(Q(\mathbf{w}) + \mathbf{w}\cdot\mathbf{m})$, and the conditions become $p^r \mid Q(\mathbf{w})$, $p^j \mid C(\mathbf{w})$ and $\mathbf{w} \equiv \mathbf{y}_0 \pmod{p^s}$, where $\mathbf{y}_0$ is the solution of $2\mathbf{A}\mathbf{y}_0 + \mathbf{m} \equiv \mathbf{0} \pmod{p^s}$. Thus

$$\mathcal{N}^{(j)}(\mathbf{m}) = p^{ns} \sideset{}{^*}\sum_{a \bmod p^s} \sum_{\substack{\mathbf{w} \bmod p^r,\ \mathbf{w}\equiv\mathbf{y}_0 \bmod p^s \\ p^r \mid Q(\mathbf{w}),\ p^j \mid C(\mathbf{w})}} e_{p^{r+s}}\Big(\bar{a}(Q(\mathbf{w}) + \mathbf{w}\cdot\mathbf{m})\Big).$$

Put $t = r - s$. If $t = 0$ the inner sum has at most one term, and $|\mathcal{N}^{(j)}(\mathbf{m})| \leqslant p^{ns+s}$, as required. Let $t \geqslant 1$; then $j \geqslant s$, and the inner sum is empty unless $p^s$ divides $Q(\mathbf{y}_0)$ and $C(\mathbf{y}_0)$, which we now assume. Write $\mathbf{w} = \mathbf{y}_0 + p^s\mathbf{z}$ with $\mathbf{z}$ modulo $p^t$, and $2\mathbf{A}\mathbf{y}_0 + \mathbf{m} = p^s\boldsymbol{\mu}$. Then

$$\mathcal{Q}(\mathbf{z}) := p^{-s}Q(\mathbf{y}_0 + p^s\mathbf{z}) = p^{-s}Q(\mathbf{y}_0) + 2\mathbf{y}_0^T\mathbf{A}\mathbf{z} + p^sQ(\mathbf{z}), \qquad \mathcal{C}(\mathbf{z}) := p^{-s}C(\mathbf{y}_0 + p^s\mathbf{z})$$

are polynomials with integer coefficients, the terms of degree at least two in $\mathcal{C}$ are divisible by $p^s$, and

$$\bar{a}\Big(Q(\mathbf{w})+\mathbf{w}\cdot\mathbf{m}\Big)=\bar{a}\Big(Q(\mathbf{y}_0)+\mathbf{y}_0\cdot\mathbf{m}\Big)+p^{2s}\bar{a}\Big(Q(\mathbf{z})+\boldsymbol{\mu}\cdot\mathbf{z}\Big).$$

The conditions on $\mathbf{w}$ are $p^t \mid \mathcal{Q}(\mathbf{z})$ and $p^{t'} \mid \mathcal{C}(\mathbf{z})$, where $t'=j-s\in\{t-1,t\}$. Detecting them by additive characters, we get

$$|\mathcal{N}^{(j)}(\mathbf{m})|\leqslant\frac{p^{ns}}{p^{t+t'}}\sideset{}{^*}\sum_{a \bmod p^s}\sum_{b_2 \bmod p^t}\sum_{b_1 \bmod p^{t'}}\Big|\sum_{\mathbf{z} \bmod p^t}e_{p^t}\Big(g(\mathbf{z})\Big)\Big|,$$

where $g(\mathbf{z})=\bar{a}(Q(\mathbf{z})+\boldsymbol{\mu}\cdot\mathbf{z})+b_2\mathcal{Q}(\mathbf{z})+p^{t-t'}b_1\mathcal{C}(\mathbf{z})$. Since $s\geqslant 1$, this polynomial has the form $\bar{a}Q(\mathbf{z})+\boldsymbol{\ell}\cdot\mathbf{z}+c+p\,h(\mathbf{z})$ of Lemma 3.8, with $\mathbf{B}=\bar{a}\mathbf{A}$. So the sum over $\mathbf{z}$ has modulus $p^{nt/2}$, and $|\mathcal{N}^{(j)}(\mathbf{m})|\leqslant p^{ns}\cdot p^s\cdot p^{nt/2}=p^{s+n(r+s)/2}$. □

3.5. **The remaining part.** In the next two lemmas $\varrho_1,\varrho_2\in\mathbb{N}$ are such that every prime factor of $\varrho_2$ divides both $\Delta$ and $\varrho_1$.

**Lemma 3.10.** *Let $\mathbf{m}_0\in\mathbb{R}^n$ and $V\geqslant 1$. Then*

$$\sum_{|\mathbf{m}-\mathbf{m}_0|\leqslant V}|S_{\varrho_1,\varrho_2}(\mathbf{m})|\ll(\varrho_1\varrho_2)^{n/2+1+\varepsilon}\varrho_2^{1/2}\Big\{\Big(\varrho_1^{\flat}\Big)^{1/2}V^n+\kappa(\varrho_1)^{n/3}\Big\}.$$

*The implied constant does not depend on $\mathbf{m}_0$.*

*Proof.* Put $q=\varrho_1\varrho_2$. By Lemma 3.6, $|S_{\varrho_1,\varrho_2}(\mathbf{m})|\leqslant\varrho_1^{-1}\sum_{b,a_1,a_2}|S(\mathbf{a},q;\mathbf{m})|$ with $\mathbf{a}=(a_1\varrho_2,a_2+b\varrho_2)$ and $\gcd(q,a_1\varrho_2,a_2+b\varrho_2)=1$; there are at most $\varrho_1^2\varrho_2$ triples $(b,a_1,a_2)$. Since $(a_1,\varrho_1)=1$ and every prime factor of $q$ divides $\varrho_1$, we have $v_p(a_1\varrho_2)=v_p(\varrho_2)$ for $p\mid q$. So the factorisation $q=q^{(0)}q^{(1)}q^{(2)}$ attached to $\mathbf{a}$ is the same for all triples. Let $\mathcal{A}$ be the set of primes $p\mid\Delta$ with $v_p(\varrho_2)\geqslant 1+\nu_p$. Then $q^{(0)}$ is the product of the $p^{v_p(q)}$ with $p\in\mathcal{A}$, and $q^{(2)}$ is the product of the $p^{v_p(q)}$ with $p\notin\mathcal{A}$ and $v_p(q)\geqslant 3$. For $p\nmid\Delta$ we have $v_p(q)=v_p(\varrho_1)$, and for $p\mid\Delta$ with $p\notin\mathcal{A}$ we have $v_p(q)\leqslant v_p(\varrho_1)+\nu_p$. Hence

$$q^{(0)}\mid\varrho_1^{\flat}\varrho_2,\qquad q^{(2)}\leqslant C_\Delta\,\kappa(\varrho_1),\qquad\Big(q^{(0)}\Big)^{1/2}\Big(q^{(2)}\Big)^{n/3}\leqslant C_\Delta^{n/3}\,\varrho_2^{1/2}\,\kappa(\varrho_1)^{n/3}.\tag{3.3}$$

For the last inequality we note that a prime $p\in\mathcal{A}$ contributes $p^{(v_p(\varrho_1)+v_p(\varrho_2))/2}\leqslant p^{v_p(\varrho_2)/2}p^{v_p(\varrho_1)n/3}$ to the left-hand side, a prime $p\mid\Delta$ with $p\notin\mathcal{A}$ contributes at most $p^{(v_p(\varrho_1)+\nu_p)n/3}$, and the primes $p\nmid\Delta$ contribute $\mathrm{cf}(\varrho_1^{\sharp})^{n/3}$. The lemma now follows from Corollary 3.4 and (3.3), on summing over the triples $(b,a_1,a_2)$. □

**Lemma 3.11.** *Let $\mathbf{m}_0,\mathbf{m}_1\in\mathbb{R}^n$ and $V\geqslant 1$. Then*

$$\sum_{\substack{|\mathbf{m}-\mathbf{m}_0|\leqslant V\\ \widetilde{Q}(\mathbf{m}-\mathbf{m}_1)=0}}|S_{\varrho_1,\varrho_2}(\mathbf{m})|\ll(\varrho_1\varrho_2)^{n/2+1+\varepsilon}\varrho_2^{1/2}\Big\{\Big(\varrho_1^{\flat}\Big)^{1/2}V^{n-1}\min\{V,\kappa(\varrho_1)\}+\kappa(\varrho_1)^{n/3}\Big\}.$$

*The implied constant does not depend on $\mathbf{m}_0$ and $\mathbf{m}_1$.*

*Proof.* We use the notation of the proof of Lemma 3.10. Let $\kappa'=q^{(2)}$. If $\kappa'>V$, then $\min\{V,\kappa(\varrho_1)\}\geqslant V/C_\Delta$ by (3.3), and the claim follows from Lemma 3.10. Let $\kappa'\leqslant V$. By (3.1) and Lemmas 3.1 and 3.2,

$$|S(\mathbf{a},q;\mathbf{m})|\ll(q/\kappa')^{n/2+\varepsilon}\Big(q^{(0)}\Big)^{1/2}|S(\mathbf{a}'',\kappa';\mathbf{m})|$$

for a suitable $\mathbf{a}''$, and the last factor depends only on $\mathbf{m}$ modulo $\kappa'$. For given $m_1,\dots,m_{n-1}$ there are at most two $m_n$ with $\widetilde{Q}(\mathbf{m}-\mathbf{m}_1)=0$, because this is a quadratic equation for $m_n$ in which the coefficient of $m_n^2$ is non-zero. Hence each

residue class modulo $\kappa'$ contains at most $2(2V/\kappa'+1)^{n-1} \ll (V/\kappa')^{n-1}$ vectors $\mathbf{m}$ with $|\mathbf{m}-\mathbf{m}_0| \leqslant V$ and $\widetilde{Q}(\mathbf{m}-\mathbf{m}_1)=0$. By Lemma 3.3, applied to a box of side $\kappa'$, the sum of $|S(\mathbf{a}'',\kappa';\boldsymbol{\nu})|$ over $\boldsymbol{\nu}$ modulo $\kappa'$ is $O(\kappa'^{3n/2+\varepsilon})$. Therefore

$$\sum_{\substack{|\mathbf{m}-\mathbf{m}_0|\leqslant V\\ \widetilde{Q}(\mathbf{m}-\mathbf{m}_1)=0}} |S(\mathbf{a},q;\mathbf{m})| \ll (q/\kappa')^{n/2+\varepsilon}\Big(q^{(0)}\Big)^{1/2}\Big(\frac{V}{\kappa'}\Big)^{n-1}\kappa'^{3n/2+\varepsilon} \ll q^{n/2+\varepsilon}\Big(q^{(0)}\Big)^{1/2}V^{n-1}\kappa',$$

and $\kappa' \leqslant \min\{V, C_\Delta\kappa(\varrho_1)\}$. By (3.3), summing over the triples $(b,a_1,a_2)$ gives the lemma. □

3.6. **The mean value.** The following proposition is the arithmetic input for §6. The factor $Q_2^{3/2}$, in place of the trivial $Q_2^2$, is the saving from the sum over $a_2$.

**Proposition 3.12.** *Let $Q_1, Q_2 \geqslant 1/2$ and $V \geqslant 1$, and for each $q_1$ let $\mathbf{m}_0(q_1) \in \mathbb{R}^n$ with $|\mathbf{m}_0(q_1)| \leqslant V_0$. Then*

$$\sum_{Q_1<q_1\leqslant 2Q_1}\ \sum_{Q_2<q_2\leqslant 2Q_2}\ \sum_{\substack{\mathbf{m}\in\mathbb{Z}^n\\ |\mathbf{m}-\mathbf{m}_0(q_1)|\leqslant V}} |S_{q_1,q_2}(\mathbf{m})|$$
$$\ll (Q_1Q_2VV_0)^{\varepsilon}\,(Q_1Q_2)^{n/2}Q_1^2Q_2^{3/2}\Big\{V^n\Big(1+Q_2^{1/2}V^{-1/2}\Big)+Q_1^{n/3-2/3}Q_2^{1/2}\Big\}.$$

*Proof.* Fix $q_1 \in (Q_1, 2Q_1]$, and write $\mathbf{m}_0 = \mathbf{m}_0(q_1)$. For $q_2 \in (Q_2, 2Q_2]$ put

$$g_2 = \prod_{\substack{p\nmid\Delta\\ p\mid q_1}} p^{v_p(q_2)}, \qquad \varrho_2 = \prod_{\substack{p\mid\Delta\\ p\mid q_1}} p^{v_p(q_2)}, \qquad e = \frac{q_2}{g_2\varrho_2}, \qquad g_1 = \prod_{p\mid g_2} p^{v_p(q_1)}, \qquad \varrho_1 = \frac{q_1}{g_1}.$$

Then $(e,q_1)=1$, the integers $g_1g_2$ and $\varrho_1\varrho_2 e$ are coprime, $g_1$ and $g_2$ have the same prime factors, none of which divides $\Delta$, and $(\varrho_1,\varrho_2)$ is as in Lemmas 3.10 and 3.11. By Lemma 3.5, applied twice, and Lemmas 3.7 and 3.9,

$$|S_{q_1,q_2}(\mathbf{m})| = |S_{\varrho_1,\varrho_2}(\mathbf{m})|\,|S_{1,e}(\mathbf{m})|\,|S_{g_1,g_2}(\mathbf{m})|$$
$$\leqslant |S_{\varrho_1,\varrho_2}(\mathbf{m})|\cdot A^{\omega(e)}e^{(n+1)/2}(e^\flat)^{1/2}\Big(e^\sharp,\widetilde{Q}(\mathbf{m})\Big)^{1/2}\cdot 2^{\omega(g_1)}(g_1g_2)^{n/2+1}.$$

The pair $(g_2,\varrho_2)$ determines $g_1$ and $\varrho_1$, and $e$ runs over integers in $(E/2,E]$, where $E = 2Q_2/(g_2\varrho_2)$. For $M \neq 0$ we have $\sum_{e'\leqslant x}(e',M)^{1/2} \leqslant \sum_{d\mid M} d^{1/2}\,x/d \leqslant \tau(M)x$. Splitting $e = e^\flat e^\sharp$ we obtain

$$\sum_{e\leqslant E} A^{\omega(e)}e^{(n+1)/2}(e^\flat)^{1/2}\Big(e^\sharp,\widetilde{Q}(\mathbf{m})\Big)^{1/2} \ll E^{(n+3)/2+\varepsilon}\times\begin{cases}\tau(|\widetilde{Q}(\mathbf{m})|), & \widetilde{Q}(\mathbf{m})\neq 0,\\ E^{1/2}, & \widetilde{Q}(\mathbf{m})=0.\end{cases}$$

In the first case $\tau(|\widetilde{Q}(\mathbf{m})|) \ll (V+V_0)^\varepsilon$. Together with Lemmas 3.10 and 3.11 (the latter with $\mathbf{m}_1=\mathbf{0}$), and with $\kappa(\varrho_1)\leqslant\kappa(q_1)$ and $\varrho_1^\flat = q_1^\flat$, this gives

$$\sum_e\sum_{|\mathbf{m}-\mathbf{m}_0|\leqslant V}|S_{q_1,q_2}(\mathbf{m})| \ll (Q_1Q_2VV_0)^\varepsilon\,(g_1g_2)^{n/2+1}E^{(n+3)/2}(\varrho_1\varrho_2)^{n/2+1}\varrho_2^{1/2}$$
$$\times\Big\{\Big(q_1^\flat\Big)^{1/2}V^n+\kappa(q_1)^{n/3}+E^{1/2}\Big(\Big(q_1^\flat\Big)^{1/2}V^{n-1}\min\{V,\kappa(q_1)\}+\kappa(q_1)^{n/3}\Big)\Big\}.$$

Here $g_1\varrho_1 = q_1$ and $(g_2\varrho_2)^{n/2+1}\varrho_2^{1/2}E^{(n+3)/2} \ll Q_2^{(n+3)/2}g_2^{-1/2}$, and $E \leqslant 2Q_2$. The number of $\varrho_2 \mid \Delta^\infty$ with $\varrho_2 \leqslant 2Q_2$ is $O(Q_2^\varepsilon)$, and the sum of $g_2^{-1/2}$ over the integers $g_2$

all of whose prime factors divide $q_1$ is at most $\prod_{p|q_1}(1-p^{-1/2})^{-1} \ll q_1^{\varepsilon}$. Hence, for each $q_1$,

$$\sum_{Q_2<q_2\leqslant 2Q_2}\sum_{|\mathbf{m}-\mathbf{m}_0|\leqslant V}|S_{q_1,q_2}(\mathbf{m})| \ll (Q_1Q_2VV_0)^{\varepsilon}\, q_1^{n/2+1}Q_2^{(n+3)/2} \times\Big\{\big(q_1^{\flat}\big)^{1/2}V^n + Q_2^{1/2}\big(q_1^{\flat}\big)^{1/2}V^{n-1}\min\{V,\kappa(q_1)\} + Q_2^{1/2}\kappa(q_1)^{n/3}\Big\}. \tag{3.4}$$

It remains to sum over $q_1$. The number of $q_1 \leqslant 2Q_1$ with $q_1^{\flat}=\beta$ and $\mathrm{cf}(q_1^{\sharp})=\gamma$ is at most $2Q_1/(\beta\gamma)$, and for such $q_1$ we have $\kappa(q_1)=\beta\gamma\leqslant 2Q_1$. Here $\beta \mid \Delta^{\infty}$ and $\gamma$ is cube-full. First,

$$\sum_{q_1}\big(q_1^{\flat}\big)^{1/2} \ll Q_1\sum_{\beta}\beta^{-1/2}\sum_{\gamma}\gamma^{-1} \ll Q_1.$$

Secondly, $\sum_{\gamma}\gamma^{-1}\min\{1,\beta\gamma/V\} \ll \min\{1,(\beta/V)^{2/3}\}$, since there are $O(x^{1/3})$ cube-full integers up to $x$. Hence

$$\sum_{q_1}\big(q_1^{\flat}\big)^{1/2}\min\{V,\kappa(q_1)\} \ll Q_1V\sum_{\beta}\beta^{-1/2}\min\{1,(\beta/V)^{2/3}\} \ll Q_1V^{1/2+\varepsilon}.$$

Thirdly,

$$\sum_{q_1}\kappa(q_1)^{n/3} \ll Q_1\sum_{\beta\leqslant 2Q_1}\beta^{n/3-1}\sum_{\gamma\leqslant 2Q_1/\beta}\gamma^{n/3-1} \ll Q_1^{n/3+1/3}\sum_{\beta}\beta^{-1/3} \ll Q_1^{n/3+1/3}.$$

Inserting these three bounds into (3.4), with $q_1^{n/2+1}\ll Q_1^{n/2+1}$, we obtain the proposition. □

## 4. The dissection

We now begin the proof of Theorem 1.2. In this section we apply the $\delta$-symbol to the cubic equation, divide the range of the modulus $q_1$ and of the frequency $t$ into four parts, and dispose of the part in which the Weyl bound suffices.

### 4.1. **The basic identity.**

For $q_1\in\mathbb{N}$ we write $r_1 = q_1P^{-3/2}$.

**Lemma 4.1.** *With $r_0$ and $p_r$ as in §2,*

$$N_w(P) = \frac{c_*}{P^3}\sum_{q_1\leqslant r_0P^{3/2}}\sideset{}{^*}\sum_{a_1 \bmod q_1}\int_{\mathbb{R}} p_{r_1}(t)\,\mathcal{H}\Big(\frac{a_1}{q_1}+\frac{t}{P^3}\Big)\,dt,$$

*where $c_*$ is the constant $c_R$ of Lemma 2.2 for the level $R=P^{3/2}$; thus $c_*=1+O_N(P^{-N})$.*

*Proof.* We have $N_w(P)=\sum_{Q(\mathbf{x})=0}w(\mathbf{x}/P)\delta(C(\mathbf{x}))$. Apply Lemma 2.2 with $R=P^{3/2}$ and $k=C(\mathbf{x})$. If $w(\mathbf{x}/P)\neq 0$, then $|C(\mathbf{x})|P^{-3}\leqslant c_C$, so $h(q_1P^{-3/2},C(\mathbf{x})P^{-3})$ vanishes unless $q_1\leqslant r_0P^{3/2}$, and by Lemma 2.4 it equals $\int p_{r_1}(t)e(tC(\mathbf{x})P^{-3})\,dt$. The integral converges absolutely, and the sums are finite. □

Throughout the rest of the paper we write

$$\Theta=\Theta(t)=1+|t|.$$

We fix the parameters

$$\eta_0=\frac{1}{43},\qquad \delta_1=\frac{1}{40},\qquad \delta_2=\frac{1}{10},$$

and a small $\varepsilon > 0$, and divide the set of pairs $(q_1, t)$ with $q_1 \leqslant r_0 P^{3/2}$ and $t \in \mathbb{R}$ into four parts:

$$\begin{aligned}
\mathcal{R}_\infty &= \{(q_1,t) :\ |t| > P^{\varepsilon}/r_1\},\\
\mathfrak{M}_1 &= \{(q_1,t) :\ q_1 \leqslant P^{\delta_1},\ |t| \leqslant P^{\delta_1}\},\\
\mathcal{R}_W &= \{(q_1,t) \notin \mathcal{R}_\infty \cup \mathfrak{M}_1 :\ q_1^{n/8-2}\,\Theta^{n/8-1} \geqslant P^{2+\eta_0}\},\\
\mathcal{R}_\delta &= \{(q_1,t) \notin \mathcal{R}_\infty \cup \mathfrak{M}_1 :\ q_1^{n/8-2}\,\Theta^{n/8-1} < P^{2+\eta_0}\}.
\end{aligned}$$

If $(q_1,t) \in \mathfrak{M}_1$ then $|t| \leqslant P^{\delta_1} < P^{3/2-\delta_1} \leqslant P^{\varepsilon}/r_1$, so the four sets are disjoint. Accordingly

$$N_w(P) = N_\infty + N_{\mathfrak{M}} + N_W + N_\delta, \tag{4.1}$$

where each term is the part of the right-hand side of Lemma 4.1 with $(q_1,t)$ in the corresponding set. Since $|\mathcal{H}(\alpha)| \leqslant \sum_{\mathbf{x}} w(\mathbf{x}/P) \ll P^n$ and $p_{r_1}(t) \ll_N (r_1|t|)^{-N}$,

$$N_\infty \ll P^{n-3} \sum_{q_1 \ll P^{3/2}} q_1 \int_{|t|>P^{\varepsilon}/r_1} (r_1|t|)^{-N}\,dt \ll_N P^{-N}. \tag{4.2}$$

4.2. **The Weyl region.** In this subsection we prove the Weyl bound, Lemma 4.4, and deduce Proposition 4.5. The bound is a version of [3, Lemma 4.1], which rests on [8, Lemma 1]. We give a complete proof, for three reasons: in [3, §4] it is assumed that $n \geqslant 29$; we need the bound uniformly in $\beta$, and in §9 also in the presence of a linear phase; and we work with the smooth weight $w$. The argument is Davenport's [7, §§12–13], in the form given by Heath-Brown [13, §2]. For $\boldsymbol{\theta} \in \mathbb{R}^n$ we put

$$S(\alpha,\beta;\boldsymbol{\theta}) = \sum_{\mathbf{x}\in\mathbb{Z}^n} w(\mathbf{x}/P)\, e\Big(\alpha C(\mathbf{x}) + \beta Q(\mathbf{x}) + \boldsymbol{\theta}\cdot\mathbf{x}\Big),$$

so that $S(\alpha,\beta) = S(\alpha,\beta;\mathbf{0})$. In this subsection the implied constants depend only on $n$, $C$, $w$ and $\varepsilon$. In particular they do not depend on $\alpha$, $\beta$ or $\boldsymbol{\theta}$, and the form $Q$ plays no role.

**Lemma 4.2.** *Let $P \geqslant 1$, and let $V \in C^\infty(\mathbb{R}^n)$ be supported in $[-P,P]^n$, with $|\partial_1\cdots\partial_n V| \leqslant AP^{-n}$ on $\mathbb{R}^n$. Then, for all $\boldsymbol{\gamma} \in \mathbb{R}^n$,*

$$\Big|\sum_{\mathbf{x}\in\mathbb{Z}^n} V(\mathbf{x}) e(\boldsymbol{\gamma}\cdot\mathbf{x})\Big| \ll_n A \prod_{i=1}^n \min\Big\{P, \|\gamma_i\|^{-1}\Big\}.$$

*Proof.* For $1 \leqslant i \leqslant n$ and $y \in \mathbb{Z}$ let $E_i(y) = \sum_{u\in\mathbb{Z},\ -P-1<u\leqslant y} e(\gamma_i u)$, so that $|E_i(y)| \leqslant \min\{2P+2, (2\|\gamma_i\|)^{-1}\}$. Let $\Delta_i$ be the forward difference in the $i$-th coordinate, $\Delta_i g(\mathbf{x}) = g(\mathbf{x}+\mathbf{e}_i) - g(\mathbf{x})$. If $g\colon \mathbb{Z} \to \mathbb{C}$ vanishes outside $[-P,P]$, then $\sum_y g(y)e(\gamma_i y) = \sum_y g(y)(E_i(y) - E_i(y-1)) = -\sum_y (g(y+1)-g(y))E_i(y)$, all sums being finite. We apply this in each coordinate in turn and obtain

$$\sum_{\mathbf{x}} V(\mathbf{x}) e(\boldsymbol{\gamma}\cdot\mathbf{x}) = (-1)^n \sum_{\mathbf{x}} (\Delta_1\cdots\Delta_n V)(\mathbf{x}) \prod_{i=1}^n E_i(x_i).$$

Here $(\Delta_1\cdots\Delta_n V)(\mathbf{x}) = \int_{[0,1]^n} (\partial_1\cdots\partial_n V)(\mathbf{x}+\mathbf{s})\,d\mathbf{s}$, which has modulus at most $AP^{-n}$ and vanishes unless $\mathbf{x} \in [-P-1, P]^n$. The number of such $\mathbf{x}$ is $O(P^n)$, and the lemma follows. □

We write $C(\mathbf{x}) = \sum_{i,j,k} c_{ijk}x_ix_jx_k$ with coefficients $c_{ijk}$ that are symmetric in $i,j,k$. Since $C$ has integer coefficients, the numbers $6c_{ijk}$ are integers. We define the bilinear forms

$$B_i(\mathbf{y};\mathbf{z}) = 6\sum_{j,k=1}^{n} c_{ijk}y_jz_k \qquad (1 \leqslant i \leqslant n),$$

which have integer coefficients and satisfy $B_i(\mathbf{y};\mathbf{z}) = B_i(\mathbf{z};\mathbf{y})$. The vector $(B_i(\mathbf{y};\mathbf{z}))_i$ is $\mathbf{H}(\mathbf{y})\mathbf{z}$, where $\mathbf{H}(\mathbf{y}) = (6\sum_j c_{ijk}y_j)_{i,k}$ is the Hessian matrix of $C$ at $\mathbf{y}$, which is symmetric. We use two results from the literature. The first is Davenport's lemma [7, Lemma 12.6], in the form stated in [13, Lemma 2.2]: *let* $\mathbf{L}$ *be a real symmetric* $n \times n$ *matrix and* $a > 0$*, and for* $0 < Z \leqslant 1$ *let* $N(Z)$ *be the number of* $\mathbf{u} \in \mathbb{Z}^n$ *with* $|\mathbf{u}| < aZ$ *and* $\|(\mathbf{L}\mathbf{u})_i\| < a^{-1}Z$ *for* $1 \leqslant i \leqslant n$*. Then* $N(Z_2) \ll_n (Z_2/Z_1)^n N(Z_1)$ *for* $0 < Z_1 \leqslant Z_2 \leqslant 1$. The second is a result of Heath-Brown [11, Lemma 3], which is used in the same way in [3, §4]: *since* $C$ *is non-singular, for* $Y \geqslant 1$ *and* $0 \leqslant r \leqslant n$ *the number of* $\mathbf{y} \in \mathbb{Z}^n$ *with* $|\mathbf{y}| < Y$ *and* $\operatorname{rank}\mathbf{H}(\mathbf{y}) = r$ *is* $O(Y^r)$. For such $\mathbf{y}$ the solutions $\mathbf{z} \in \mathbb{R}^n$ of $\mathbf{H}(\mathbf{y})\mathbf{z} = \mathbf{0}$ form a subspace of dimension $n-r$, which contains $O(Y^{n-r})$ integral points with $|\mathbf{z}| < Y$. Summing over $r$, we obtain

$$\#\Big\{(\mathbf{y},\mathbf{z}) \in \mathbb{Z}^{2n} :\ |\mathbf{y}|, |\mathbf{z}| < Y,\ B_i(\mathbf{y};\mathbf{z}) = 0\ (1 \leqslant i \leqslant n)\Big\} \ll Y^n \qquad (Y \geqslant 1). \tag{4.3}$$

**Lemma 4.3.** *Let* $\varepsilon > 0$*, let* $\alpha, \beta \in \mathbb{R}$ *and* $\boldsymbol{\theta} \in \mathbb{R}^n$*, and suppose that* $S(\alpha,\beta;\boldsymbol{\theta}) \neq 0$*. Define* $T > 0$ *by* $|S(\alpha,\beta;\boldsymbol{\theta})| = P^nT^{-n}$*, and suppose that* $T^8 \leqslant P^{3/2}$*. If* $P$ *is sufficiently large in terms of* $n$*,* $C$*,* $w$ *and* $\varepsilon$*, then there are coprime integers* $s \geqslant 1$ *and* $b$ *such that*

$$\alpha = \frac{b}{s} + \varphi, \qquad s\big(1 + P^3|\varphi|\big) \ll P^\varepsilon T^8.$$

*Proof.* Write $W(\mathbf{x}) = w(\mathbf{x}/P)$ and $f(\mathbf{x}) = \alpha C(\mathbf{x}) + \beta Q(\mathbf{x}) + \boldsymbol{\theta}\cdot\mathbf{x}$, and let $\Delta_{\mathbf{h}}g(\mathbf{x}) = g(\mathbf{x}+\mathbf{h}) - g(\mathbf{x})$. By (U1), $W(\mathbf{x}) = 0$ unless $|\mathbf{x}| < P/2$. The variables $\mathbf{h}_1, \mathbf{h}_2$ below run over the integral vectors with $|\mathbf{h}_1|, |\mathbf{h}_2| < P$.

*Weyl differencing.* We have $|S|^2 = \sum_{\mathbf{h}_1}\sum_{\mathbf{x}} W(\mathbf{x}+\mathbf{h}_1)W(\mathbf{x})e(\Delta_{\mathbf{h}_1}f(\mathbf{x}))$, where $S = S(\alpha,\beta;\boldsymbol{\theta})$. There are $O(P^n)$ vectors $\mathbf{h}_1$, so by Cauchy's inequality

$$\begin{aligned}|S|^4 &\ll P^n \sum_{\mathbf{h}_1}\Big|\sum_{\mathbf{x}} W(\mathbf{x}+\mathbf{h}_1)W(\mathbf{x})e(\Delta_{\mathbf{h}_1}f(\mathbf{x}))\Big|^2\\ &\leqslant P^n \sum_{\mathbf{h}_1,\mathbf{h}_2}\Big|\sum_{\mathbf{x}} W_{\mathbf{h}_1,\mathbf{h}_2}(\mathbf{x})e\Big(\Delta_{\mathbf{h}_2}\Delta_{\mathbf{h}_1}f(\mathbf{x})\Big)\Big|,\end{aligned}$$

where $W_{\mathbf{h}_1,\mathbf{h}_2}(\mathbf{x}) = W(\mathbf{x})W(\mathbf{x}+\mathbf{h}_1)W(\mathbf{x}+\mathbf{h}_2)W(\mathbf{x}+\mathbf{h}_1+\mathbf{h}_2)$. A direct computation gives $\Delta_{\mathbf{h}_2}\Delta_{\mathbf{h}_1}C(\mathbf{x}) = \sum_i x_iB_i(\mathbf{h}_1;\mathbf{h}_2) + C_0(\mathbf{h}_1,\mathbf{h}_2)$, with $C_0$ independent of $\mathbf{x}$. Moreover $\Delta_{\mathbf{h}_2}\Delta_{\mathbf{h}_1}Q(\mathbf{x})$ does not depend on $\mathbf{x}$, and $\Delta_{\mathbf{h}_2}\Delta_{\mathbf{h}_1}(\boldsymbol{\theta}\cdot\mathbf{x}) = 0$. Thus $\beta$ and $\boldsymbol{\theta}$ disappear, and

$$|S|^4 \ll P^n \sum_{\mathbf{h}_1,\mathbf{h}_2}\Big|\sum_{\mathbf{x}} W_{\mathbf{h}_1,\mathbf{h}_2}(\mathbf{x})e\Big(\alpha\sum_{i=1}^{n} x_iB_i(\mathbf{h}_1;\mathbf{h}_2)\Big)\Big|.$$

The function $W_{\mathbf{h}_1,\mathbf{h}_2}$ is supported in $[-P,P]^n$, and by the product rule $\partial_1\cdots\partial_n W_{\mathbf{h}_1,\mathbf{h}_2} \ll P^{-n}$, uniformly in $\mathbf{h}_1,\mathbf{h}_2$. By Lemma 4.2,

$$|S|^4 \ll P^n \sum_{\mathbf{h}_1,\mathbf{h}_2}\prod_{i=1}^{n}\min\Big\{P, \|\alpha B_i(\mathbf{h}_1;\mathbf{h}_2)\|^{-1}\Big\}. \tag{4.4}$$

*Counting.* We follow the proof of [7, Lemma 13.2]. Fix $\mathbf{h}_1$, put $L_i(\mathbf{u}) = \alpha B_i(\mathbf{h}_1;\mathbf{u})$, and let $K = \lceil 2P\rceil$. For $\mathbf{r} \in \{0,\dots,K-1\}^n$ let $\mathcal{A}(\mathbf{r})$ be the set of $\mathbf{h}_2$ such that the fractional part of $L_i(\mathbf{h}_2)$ lies in $[r_i/K, (r_i+1)/K)$ for every $i$. If $\mathbf{h}_2, \mathbf{h}_2' \in \mathcal{A}(\mathbf{r})$, then

$\mathbf{u} = \mathbf{h}_2 - \mathbf{h}_2'$ satisfies $|\mathbf{u}| < 2P$ and $\|L_i(\mathbf{u})\| < K^{-1} \leqslant (2P)^{-1}$. Hence $\#\mathcal{A}(\mathbf{r}) \leqslant N_{\mathbf{h}_1}$, where

$$N_{\mathbf{h}_1} = \#\Big\{\mathbf{u} \in \mathbb{Z}^n : \ |\mathbf{u}| < 2P, \ \|\alpha B_i(\mathbf{h}_1; \mathbf{u})\| < (2P)^{-1} \ (1 \leqslant i \leqslant n)\Big\}.$$

For $\mathbf{h}_2 \in \mathcal{A}(\mathbf{r})$ we have $\|L_i(\mathbf{h}_2)\| \geqslant \min\{r_i, K-1-r_i\}/K$, and

$$\sum_{r=0}^{K-1} \min\Big\{P, \frac{K}{\min\{r, K-1-r\}}\Big\} \ll P \log P,$$

where $K/0$ is read as $+\infty$. Therefore the sum over $\mathbf{h}_2$ in (4.4) is $O((P \log P)^n N_{\mathbf{h}_1})$, and

$$|S|^4 \ll P^{2n} (\log P)^n \sum_{\mathbf{h}_1} N_{\mathbf{h}_1}.$$

*Shrinking.* Let $0 < Z \leqslant 1$. For fixed $\mathbf{h}_1$ the vector $(\alpha B_i(\mathbf{h}_1; \mathbf{u}))_i$ is $\alpha \mathbf{H}(\mathbf{h}_1)\mathbf{u}$, and $\alpha \mathbf{H}(\mathbf{h}_1)$ is symmetric. Davenport's lemma with $a = 2P$, $Z_2 = 1$ and $Z_1 = Z$ gives

$$N_{\mathbf{h}_1} \ll Z^{-n} \#\Big\{\mathbf{u} : \ |\mathbf{u}| < 2ZP, \ \|\alpha B_i(\mathbf{h}_1; \mathbf{u})\| < Z(2P)^{-1} \ \forall i\Big\}.$$

We sum over $\mathbf{h}_1$, enlarge its range to $|\mathbf{h}_1| < 2P$, and use the symmetry $B_i(\mathbf{h}_1; \mathbf{u}) = B_i(\mathbf{u}; \mathbf{h}_1)$. For fixed $\mathbf{u}$ the vector $(\alpha B_i(\mathbf{u}; \mathbf{h}_1))_i$ is $\alpha \mathbf{H}(\mathbf{u})\mathbf{h}_1$, and Davenport's lemma with $a = 2PZ^{-1/2}$, $Z_2 = Z^{1/2}$ and $Z_1 = Z^{3/2}$ applies to the conditions $|\mathbf{h}_1| < aZ_2 = 2P$ and $\|\alpha B_i(\mathbf{u}; \mathbf{h}_1)\| < a^{-1} Z_2 = Z(2P)^{-1}$. It gives a further factor $Z^{-n}$, and we obtain

$$|S|^4 \ll P^{2n} (\log P)^n Z^{-2n} M(Z), \tag{4.5}$$

where $M(Z)$ is the number of $(\mathbf{y}, \mathbf{z}) \in \mathbb{Z}^{2n}$ with $|\mathbf{y}|, |\mathbf{z}| < 2ZP$ and $\|\alpha B_i(\mathbf{y}; \mathbf{z})\| < Z^2(2P)^{-1}$ for $1 \leqslant i \leqslant n$.

*Conclusion.* The bound $|S| \leqslant \sum_{\mathbf{x}} W(\mathbf{x}) \ll P^n$ shows that $T \gg 1$. Let $Z = P^{\varepsilon-1} T^4$. Since $T^8 \leqslant P^{3/2}$, we have $Z \leqslant P^{\varepsilon - 1/4} \leqslant 1$ if $\varepsilon \leqslant 1/4$, which we may assume, and $Y = 2ZP = 2P^{\varepsilon} T^4 \geqslant 1$ if $P$ is large. Suppose first that $B_i(\mathbf{y}; \mathbf{z}) = 0$ for every $i$ and every $(\mathbf{y}, \mathbf{z})$ counted by $M(Z)$. Then $M(Z) \ll (ZP)^n$ by (4.3), and (4.5) gives

$$P^{4n} T^{-4n} = |S|^4 \ll P^{3n} (\log P)^n Z^{-n} = P^{4n - n\varepsilon} (\log P)^n T^{-4n},$$

which is false if $P$ is sufficiently large. Hence there are $(\mathbf{y}, \mathbf{z})$ counted by $M(Z)$ and an index $i$ with $m = B_i(\mathbf{y}; \mathbf{z}) \neq 0$. Let $c_B \geqslant 1$ be such that $|B_i(\mathbf{y}; \mathbf{z})| \leqslant c_B |\mathbf{y}||\mathbf{z}|$ for all $i$, $\mathbf{y}$, $\mathbf{z}$. Then $s_0 = |m|$ satisfies

$$1 \leqslant s_0 \leqslant 4c_B Z^2 P^2 = 4c_B P^{2\varepsilon} T^8, \qquad \|s_0 \alpha\| < \frac{Z^2}{2P} = \frac{P^{2\varepsilon} T^8}{2P^3}.$$

Let $b_0$ be an integer nearest to $s_0\alpha$, and $\varphi = \alpha - b_0/s_0$. Then $s_0 P^3 |\varphi| = P^3 \|s_0 \alpha\| < P^{2\varepsilon} T^8$, so $s_0(1 + P^3|\varphi|) \ll P^{2\varepsilon} T^8$. Writing $b_0/s_0 = b/s$ in lowest terms, we have $s \leqslant s_0$, and the lemma follows on replacing $\varepsilon$ by $\varepsilon/2$. □

**Lemma 4.4.** *Let $\alpha = a/q + z$ with $q \in \mathbb{N}$, $(a, q) = 1$, and put $\Xi = q(1 + P^3|z|)$. If $\Xi \leqslant P^{3/2+\varepsilon}$, then, uniformly in $\beta \in \mathbb{R}$ and $\boldsymbol{\theta} \in \mathbb{R}^n$,*

$$S(\alpha, \beta; \boldsymbol{\theta}) \ll P^{n+\varepsilon} \, \Xi^{-n/8}.$$

*Proof.* We may suppose that $S(\alpha, \beta; \boldsymbol{\theta}) \neq 0$, and define $T > 0$ by $|S(\alpha, \beta; \boldsymbol{\theta})| = P^n T^{-n}$. We show that $T^8 \gg P^{-3\varepsilon} \Xi$, which gives the lemma. If $T^8 > P^{3/2}$, this holds because $\Xi \leqslant P^{3/2+\varepsilon}$; so no restriction on the size of $T$ is needed. Let $T^8 \leqslant P^{3/2}$. By Lemma 4.3 there are coprime integers $s \geqslant 1$ and $b$ with

$$\alpha = \frac{b}{s} + \varphi, \qquad s(1 + P^3|\varphi|) \ll P^{\varepsilon} T^8.$$

If $b/s = a/q$, then $s = q$ and $\varphi = z$, so $\Xi \ll P^{\varepsilon}T^8$. Otherwise $1 \leqslant sq|a/q - b/s| \leqslant sq|\varphi| + sq|z|$. Here $sq|\varphi| \ll qP^{\varepsilon-3}T^8$ and $sq|z| \leqslant s\,\Xi P^{-3} \ll \Xi P^{\varepsilon-3}T^8$, and $q \leqslant \Xi$. Hence $T^8 \gg P^{3-\varepsilon}\Xi^{-1} \geqslant P^{-3\varepsilon}\Xi$, because $\Xi^2 \leqslant P^{3+2\varepsilon}$. In both cases $T^8 \gg P^{-3\varepsilon}\Xi$, and the lemma follows. □

**Proposition 4.5.** *We have* $N_W \ll P^{n-5-\eta_0+\varepsilon}$.

*Proof.* If $(q_1, t) \in \mathcal{R}_W$, then $\Xi = q_1\Theta \leqslant q_1 + P^{\varepsilon}q_1/r_1 \ll P^{3/2+\varepsilon}$. By Lemma 4.4 (with $2\varepsilon$ in place of $\varepsilon$), $|\mathcal{H}(a_1/q_1 + tP^{-3})| \leqslant \sup_\beta |S(a_1/q_1 + tP^{-3}, \beta)| \ll P^{n+\varepsilon}(q_1\Theta)^{-n/8}$. Since $p_{r_1}(t) \ll 1$,

$$N_W \ll P^{n-3+\varepsilon} \sum_{q_1} q_1 \int_{(q_1,t)\in\mathcal{R}_W} (q_1\Theta)^{-n/8}\, dt.$$

We divide the range into $O((\log P)^2)$ parts with $Q_1 < q_1 \leqslant 2Q_1$ and $T \leqslant \Theta < 2T$, where $Q_1$ and $T$ are powers of two. A part that meets $\mathcal{R}_W$ has $Q_1^{n/8-2}T^{n/8-1} \gg P^{2+\eta_0}$, and contributes $O(P^{n-3+\varepsilon}Q_1^{2-n/8}T^{1-n/8})$. This is $O(P^{n-5-\eta_0+\varepsilon})$, and the proposition follows on summing over the parts. □

## 5. The major arcs

In this section we evaluate $N_{\mathfrak{M}}$. For $q \in \mathbb{N}$ put

$$A(q) = q^{-n} \sum_{\substack{\mathbf{a} \bmod q \\ \gcd(q,a_3,a_2)=1}} S(\mathbf{a}, q; \mathbf{0}), \qquad \mathfrak{S} = \sum_{q=1}^{\infty} A(q), \qquad \mathfrak{J} = \int_{\mathbb{R}^2} J(\boldsymbol{\gamma}; \mathbf{0})\, d\boldsymbol{\gamma}.$$

**Lemma 5.1.** *Let* $n \geqslant 15$.

(i) *For* $Y \geqslant 1$,

$$\sum_{q>Y} q^{-n} \sum_{\substack{\mathbf{a} \bmod q \\ \gcd(q,a_3,a_2)=1}} |S(\mathbf{a}, q; \mathbf{0})| \ll Y^{7/6-n/12+\varepsilon}.$$

*In particular the series* $\mathfrak{S}$ *converges absolutely.*

(ii) *The integral* $\mathfrak{J}$ *converges absolutely, and* $\mathfrak{J} > 0$.

*Proof.* (i) By Corollary 3.4 with $\mathbf{m}_0 = \mathbf{0}$ and $V = 1$, $S(\mathbf{a}, q; \mathbf{0}) \ll q^{n/2+\varepsilon}(q^{(0)})^{1/2}(q^{(2)})^{n/3}$. Put $u = q^{(0)}q^{(1)}$ and $v = q^{(2)}$. Then $q = uv$, $(u, v) = 1$, $v$ is cube-full, and $q^{-n}S(\mathbf{a}, q; \mathbf{0}) \ll q^{\varepsilon}u^{(1-n)/2}v^{-n/6}$. There are at most $q^2$ pairs $\mathbf{a}$. Hence the sum in (i) is

$$\ll \sum_{\substack{u,v \\ uv>Y}} (uv)^{\varepsilon}u^{(5-n)/2}v^{2-n/6},$$

where $v$ runs over cube-full integers. The terms with $u > Y^{1/2}$ contribute $O(Y^{(7-n)/4+\varepsilon})$, and the terms with $v > Y^{1/2}$ contribute $O(Y^{(1/2)(1/3+2-n/6)+\varepsilon})$, since $n/6 - 2 > 1/3$.

(ii) The convergence follows from Lemma 2.8. By (U3) we may use $\mathbf{y} = (C(\mathbf{z}), Q(\mathbf{z}))$ and $\mathbf{z}'' = (z_i)_{i\neq i_1,i_2}$ as coordinates on a neighbourhood of $\mathcal{U}$. Then $J(\boldsymbol{\gamma}; \mathbf{0}) = \int_{\mathbb{R}^2} \Phi(\mathbf{y})e(\gamma_3y_1 + \gamma_2y_2)\, d\mathbf{y}$, where $\Phi(\mathbf{y})$ is the integral of $w$ over the fibre, against $d\mathbf{z}''$ divided by the absolute value of the minor in (U3). The function $\Phi$ is smooth, non-negative and compactly supported, so $\mathfrak{J} = \Phi(\mathbf{0})$ by Fourier inversion, and $\Phi(\mathbf{0}) > 0$ because $w \geqslant 0$ and $w(\mathbf{z}_0) > 0$. □

For $q_1, q_2 \in \mathbb{N}$ and $a_1, a_2 \in \mathbb{Z}$ let

$$S^*(a_1, q_1; a_2, q_2) = \sum_{\mathbf{k} \bmod q_1q_2} e\Big(\frac{a_1C(\mathbf{k})}{q_1} + \frac{a_2Q(\mathbf{k})}{q_2}\Big).$$

**Lemma 5.2.** *Let $q_1 \leqslant P^{\delta_1}$, $(a_1,q_1)=1$ and $|t| \leqslant P^{\delta_1}$, and put $\alpha = a_1/q_1 + tP^{-3}$. Let $q_2 \leqslant P$, $(a_2,q_2)=1$ and $\beta = a_2/q_2 + z_2$ with $|z_2| \leqslant (q_2P)^{-1}$. Then*

$$S(\alpha,\beta) = \frac{P^n}{(q_1q_2)^n}\, S^*(a_1,q_1;a_2,q_2)\, J\Big((t,z_2P^2);\mathbf{0}\Big) + O\Big(P^{n/2+8n\delta_1}\Big),$$

*and $S^*(a_1,q_1;a_2,q_2) \ll q_1^{2n}q_2^{n/2}$.*

*Proof.* Put $q=q_1q_2$, $L=P/q$, $t_2=z_2P^2$ and $\boldsymbol{\gamma}=(t,t_2)$. Splitting $\mathbf{x}$ into residue classes modulo $q$ and applying Poisson summation, as in [3, Lemma 6.1], we get

$$S(\alpha,\beta) = L^n \sum_{\mathbf{m}\in\mathbb{Z}^n} \mathfrak{S}(\mathbf{m})\, J(\boldsymbol{\gamma}; L\mathbf{m}), \qquad \mathfrak{S}(\mathbf{m}) = \sum_{\mathbf{k} \bmod q} e\Big(\frac{a_1C(\mathbf{k})}{q_1} + \frac{a_2Q(\mathbf{k})}{q_2} + \frac{\mathbf{k}\cdot\mathbf{m}}{q}\Big).$$

Write $\mathbf{k} = \mathbf{k}_1 + q_1\mathbf{l}$ with $\mathbf{k}_1$ modulo $q_1$ and $\mathbf{l}$ modulo $q_2$. For fixed $\mathbf{k}_1$ the sum over $\mathbf{l}$ is a product of $n$ sums $\sum_{l \bmod q_2} e_{q_2}(ul^2+vl)$ with $u = a_2d_iq_1^2$, each of modulus at most $(q_2(2u,q_2))^{1/2} \leqslant (2|d_i|q_1^2q_2)^{1/2}$. Hence $\mathfrak{S}(\mathbf{m}) \ll q_1^{2n}q_2^{n/2}$ for all $\mathbf{m}$; this includes the bound for $S^* = \mathfrak{S}(\mathbf{0})$. It remains to show that

$$L^n\, q_1^{2n}q_2^{n/2} \sum_{\mathbf{m}\neq\mathbf{0}} |J(\boldsymbol{\gamma}; L\mathbf{m})| \ll P^{n/2+8n\delta_1}. \tag{5.1}$$

Note that $|t_2| \leqslant P/q_2$ and $\|\boldsymbol{\gamma}\| \leqslant 2P^{\delta_1} + |t_2|$.

Suppose first that $q \leqslant P^{1-3\delta_1}$, so that $L \geqslant P^{3\delta_1}$. If $|t_2| \leqslant P^{2\delta_1}$, then $|L\mathbf{m}| \geqslant c_4\|\boldsymbol{\gamma}\|$ for all $\mathbf{m}\neq\mathbf{0}$, and by Lemma 2.7 the sum in (5.1) is $O(L^{-N})$. If $|t_2| > P^{2\delta_1}$, then $|t_2| \geqslant K_0(1+|t|)$, and we apply Lemma 2.10. The $\mathbf{m}\neq\mathbf{0}$ with $|L\mathbf{m}| < c_1|t_2|$ or $|L\mathbf{m}| > c_2|t_2|$ contribute $O(L^{-N})$. The number of the remaining $\mathbf{m}$ is $O((|t_2|/L)^n)$, and each contributes $O(P^{-N} + P^{\varepsilon}|t_2|^{-n/2})$. So the left-hand side of (5.1) is $O(1 + P^{\varepsilon}q_1^{2n}(q_2|t_2|)^{n/2}) = O(P^{n/2+2n\delta_1+\varepsilon})$.

Suppose now that $q > P^{1-3\delta_1}$. Then $q_2 > P^{1-4\delta_1}$, so $|t_2| < P^{4\delta_1}$ and $\|\boldsymbol{\gamma}\| \leqslant 3P^{4\delta_1}$. By (2.2) and Lemma 2.7, and since $L \geqslant P^{-\delta_1}$,

$$\sum_{\mathbf{m}\neq\mathbf{0}} |J(\boldsymbol{\gamma}; L\mathbf{m})| \ll \#\{\mathbf{m} : |L\mathbf{m}| < c_4\|\boldsymbol{\gamma}\|\} + \sum_{|L\mathbf{m}|\geqslant c_4\|\boldsymbol{\gamma}\|} |L\mathbf{m}|^{-N} \ll P^{5n\delta_1}.$$

The left-hand side of (5.1) is therefore $O(P^nq_1^nq_2^{-n/2}P^{5n\delta_1}) = O(P^{n/2+8n\delta_1})$. □

**Proposition 5.3.** *Let $n \geqslant 28$. Then $N_{\mathfrak{M}} = \mathfrak{S}\,\mathfrak{J}\,P^{n-5} + O(P^{n-5-1/40})$.*

*Proof.* Let $(q_1,t) \in \mathfrak{M}_1$ and $\alpha = a_1/q_1 + tP^{-3}$. Let $\mathfrak{F}(a_2/q_2)$, for $1 \leqslant a_2 \leqslant q_2 \leqslant P$ with $(a_2,q_2)=1$, be the arcs of the Farey dissection of order $P$. They are disjoint, they cover $\mathbb{R}/\mathbb{Z}$, and

$$\{\beta : |\beta - a_2/q_2| \leqslant (2q_2P)^{-1}\} \subset \mathfrak{F}(a_2/q_2) \subset \{\beta : |\beta - a_2/q_2| \leqslant (q_2P)^{-1}\}.$$

We integrate the formula of Lemma 5.2 over $\beta \in \mathfrak{F}(a_2/q_2)$ and sum over $a_2/q_2$. The error term contributes $O(P^{n/2+8n\delta_1})$. In the main term we use $S^* \ll q_1^{2n}q_2^{n/2}$ and $J((t,t_2);\mathbf{0}) \ll_N (1+|t_2|)^{-N}$. The arcs with $q_2 > P^{\delta_2}$ contribute

$$\ll P^{n-2}q_1^n \sum_{q_2>P^{\delta_2}} q_2^{1-n/2} \ll P^{n-2+n\delta_1-(n/2-2)\delta_2}.$$

For $q_2 \leqslant P^{\delta_2}$ we extend the integral over $z_2$ to $\mathbb{R}$; the part $|z_2| > (2q_2P)^{-1}$ has $|t_2| \geqslant P^{1-\delta_2}/2$ and is negligible. Hence

$$\mathcal{H}(\alpha) = P^{n-2} \sum_{q_2 \leqslant P^{\delta_2}} \sideset{}{^*}\sum_{a_2 \bmod q_2} \frac{S^*(a_1,q_1;a_2,q_2)}{(q_1q_2)^n} \int_{\mathbb{R}} J\Big((t,t_2);\mathbf{0}\Big)\, dt_2 + O\Big(P^{n-2+n\delta_1-(n/2-2)\delta_2} + P^{n/2+8n\delta_1}\Big).$$

We insert this into the definition of $N_{\mathfrak{M}}$. For $(q_1,t) \in \mathfrak{M}_1$ we have $r_1 \leqslant P^{\delta_1-3/2}$, so $p_{r_1}(t) = 1 + O(P^{-N})$ by Lemma 2.4, and $c_* = 1 + O(P^{-N})$. The error terms contribute

$$\ll P^{3\delta_1-3}\Big(P^{n-2+n\delta_1-(n/2-2)\delta_2} + P^{n/2+8n\delta_1}\Big) \ll P^{n-5+(n+3)\delta_1-(n/2-2)\delta_2},$$

which is $O(P^{n-5-17/40})$ for $n \geqslant 28$, since $(n+3)\delta_1 - (n/2-2)\delta_2 = (11-n)/40$. By Lemma 2.8, the integral of $J(\boldsymbol{\gamma};\mathbf{0})$ over $|t| \leqslant P^{\delta_1}$ and $t_2 \in \mathbb{R}$ is $\mathfrak{J} + O(P^{-N})$. Therefore

$$N_{\mathfrak{M}} = P^{n-5}\Big(\mathfrak{J} + O(P^{-N})\Big) \sum_{q_1 \leqslant P^{\delta_1}} \sum_{q_2 \leqslant P^{\delta_2}} \sideset{}{^*}\sum_{a_1 \bmod q_1} \sideset{}{^*}\sum_{a_2 \bmod q_2} \frac{S^*(a_1,q_1;a_2,q_2)}{(q_1q_2)^n} + O\Big(P^{n-5-17/40}\Big).$$

The quadruples $(q_1,a_1,q_2,a_2)$ with $(a_1,q_1) = (a_2,q_2) = 1$ correspond bijectively to the pairs $(q,\mathbf{a})$ with $\mathbf{a}$ modulo $q$ and $\gcd(q,a_3,a_2) = 1$, by $q = \operatorname{lcm}(q_1,q_2)$ and $\mathbf{a} = (a_1q/q_1, a_2q/q_2)$; and $(q_1q_2)^{-n}S^*(a_1,q_1;a_2,q_2) = q^{-n}S(\mathbf{a},q;\mathbf{0})$. The quadruples that are missing from the sum have $q > P^{\delta_1}$. By Lemma 5.1(i) and $n \geqslant 28$, the sum is $\mathfrak{S} + O(P^{-7\delta_1/6+\varepsilon})$, and $7\delta_1/6 > 1/40$. □

## 6. The nested $\delta$-symbol

In this section we estimate $N_\delta$. Throughout, $n \geqslant 28$. For $(q_1,t) \in \mathcal{R}_\delta$ we put

$$\Theta = 1 + |t|, \qquad R = R(q_1,t) = \frac{P}{(q_1\Theta)^{1/2}}.$$

Since $(q_1\Theta)^{n/8-2} \leqslant q_1^{n/8-2}\Theta^{n/8-1} < P^{2+\eta_0}$ and $n/8 - 2 \geqslant 3/2$, we have $q_1\Theta < P^{3/2}$ and $R > P^{1/4}$.

### 6.1. The identity.

**Lemma 6.1.** *Let $(q_1,t) \in \mathcal{R}_\delta$. Then*

$$\sideset{}{^*}\sum_{a_1 \bmod q_1} \mathcal{H}\Big(\frac{a_1}{q_1} + \frac{t}{P^3}\Big) = \frac{c_R\, P^n}{R^2} \sum_{q_2 \leqslant 4R} \frac{1}{(q_1q_2)^n} \sum_{\mathbf{m} \in \mathbb{Z}^n} S_{q_1,q_2}(\mathbf{m})\, I_{q_1,q_2}(t;\mathbf{m}),$$

*where*

$$I_{q_1,q_2}(t;\mathbf{m}) = \int_{\mathbb{R}^n} w(\mathbf{z})\, e\Big(tC(\mathbf{z})\Big)\, g_{q_2/R}\Big(\Theta\, Q(\mathbf{z})\Big)\, e\Big(-\frac{P\,\mathbf{m}\cdot\mathbf{z}}{q_1q_2}\Big) d\mathbf{z}.$$

*Proof.* Let $\mathbf{x} \in \mathbb{Z}^n$ with $q_1 \mid Q(\mathbf{x})$. Then $\delta(Q(\mathbf{x})) = \delta(Q(\mathbf{x})/q_1)\, B(Q(\mathbf{x})/(q_1R^2))$, because both sides vanish unless $Q(\mathbf{x}) = 0$, and $B(0) = 1$. We apply Lemma 2.2 with level $R$ to the integer $k = Q(\mathbf{x})/q_1$. If $B(k/R^2) \neq 0$, then $|k|/R^2 \leqslant 2$, so that $h(q_2/R, k/R^2)$ vanishes unless $q_2 \leqslant 4R$. Since $q_1R^2\Theta = P^2$, we have $k/R^2 = \Theta\, Q(\mathbf{x}/P)$. Therefore

$$\mathcal{H}(\alpha) = \frac{c_R}{R^2} \sum_{q_2 \leqslant 4R} \sideset{}{^*}\sum_{a_2 \bmod q_2} \sum_{\substack{\mathbf{x} \in \mathbb{Z}^n \\ q_1 \mid Q(\mathbf{x})}} w(\mathbf{x}/P)\, e\Big(\alpha C(\mathbf{x}) + \frac{a_2Q(\mathbf{x})}{q_1q_2}\Big) g_{q_2/R}\Big(\Theta Q(\mathbf{x}/P)\Big).$$

Take $\alpha = a_1/q_1 + tP^{-3}$ and sum over $a_1$. The function $\mathbf{x} \mapsto e_{q_1q_2}(a_1q_2C(\mathbf{x}) + a_2Q(\mathbf{x}))\mathbf{1}[q_1 \mid Q(\mathbf{x})]$ is periodic modulo $q_1q_2$. We split $\mathbf{x}$ into residue classes $\mathbf{k}$ modulo $q_1q_2$ and apply Poisson summation to the smooth function $\mathbf{x} \mapsto w(\mathbf{x}/P)e(tC(\mathbf{x}/P))g_{q_2/R}(\Theta Q(\mathbf{x}/P))$, whose Fourier transform at $\mathbf{m}/(q_1q_2)$ is $P^n I_{q_1,q_2}(t;\mathbf{m})$. □

By Lemma 4.1, Lemma 6.1 and the bounds $c_R \ll 1$, $p_{r_1}(t) \ll 1$,

$$N_\delta \ll P^{n-3} \sum_{q_1} \int_{(q_1,t)\in\mathcal{R}_\delta} \frac{1}{R^2} \sum_{q_2\leqslant 4R} \frac{1}{(q_1q_2)^n} \sum_{\mathbf{m}\in\mathbb{Z}^n} |S_{q_1,q_2}(\mathbf{m})|\,|I_{q_1,q_2}(t;\mathbf{m})|\,dt. \tag{6.1}$$

6.2. **The integrals.** We write $r_2 = q_2/R \leqslant 4$ and $\mathbf{u} = P\mathbf{m}/(q_1q_2)$. By Lemma 2.5 and Fubini's theorem,

$$I_{q_1,q_2}(t;\mathbf{m}) = \int_{\mathbb{R}} \widehat{g}_{r_2}(\tau)\, J\Big((t,\tau\Theta);\mathbf{u}\Big)\,d\tau, \qquad \|(t,\tau\Theta)\| = \Theta(1+|\tau|). \tag{6.2}$$

In the next three lemmas $(q_1,t) \in \mathcal{R}_\delta$, $q_2 \leqslant 4R$ and $\mathbf{m} \in \mathbb{Z}^n$, and $K_0$, $c_1$, $c_2$ are the constants of Lemma 2.10.

**Lemma 6.2.** *We have* $I_{q_1,q_2}(t;\mathbf{0}) \ll_N \Theta^{-N}$.

*Proof.* By Lemma 2.8, $J((t,\tau\Theta);\mathbf{0}) \ll (\Theta(1+|\tau|))^{-N-2}$, and $\widehat{g}_{r_2} \ll 1$. The lemma follows from (6.2). □

**Lemma 6.3.** *If* $\mathbf{m} \neq \mathbf{0}$ *and* $|\mathbf{u}| \geqslant P^\varepsilon\Theta/r_2$, *then* $I_{q_1,q_2}(t;\mathbf{m}) \ll_N P^{-N}|\mathbf{m}|^{-N}$.

*Proof.* Put $Y = r_2|\mathbf{u}|/\Theta$, so that $Y \geqslant P^\varepsilon$ by hypothesis. Since $\mathbf{u} = P\mathbf{m}/(q_1q_2)$ and $q_1R^2\Theta = P^2$, we have $Y = |\mathbf{m}|(q_1\Theta)^{-1/2} \geqslant |\mathbf{m}|/P$; moreover $|\mathbf{u}| \geqslant Y/4$ and $r_2^{-1} \leqslant R \leqslant P$. For the $\tau$ with $c_4\Theta(1+|\tau|) \leqslant |\mathbf{u}|$ we have $J((t,\tau\Theta);\mathbf{u}) \ll_M |\mathbf{u}|^{-M}$ by Lemma 2.7, and $\int_{\mathbb{R}} |\widehat{g}_{r_2}(\tau)|\,d\tau \ll r_2^{-1}$. The remaining $\tau$ satisfy $|\tau| \geqslant |\mathbf{u}|/(2c_4\Theta)$, and for them we use $\widehat{g}_{r_2}(\tau) \ll_M (r_2|\tau|)^{-M}$ and (2.2). Thus, for every $M \geqslant 2$,

$$I_{q_1,q_2}(t;\mathbf{m}) \ll_M r_2^{-1}\Big(|\mathbf{u}|^{-M} + Y^{1-M}\Big) \ll_M P\,Y^{1-M}.$$

Since $Y \geqslant \max\{P^\varepsilon, |\mathbf{m}|/P\}$, we have $Y^{1-M} \leqslant P^{-\varepsilon(M-1-N)}(|\mathbf{m}|/P)^{-N}$, and the claim follows on taking $M$ large enough in terms of $N$ and $\varepsilon$. □

**Lemma 6.4.** *If* $\mathbf{m} \neq \mathbf{0}$ *and* $|\mathbf{u}| < P^\varepsilon\Theta/r_2$, *then*

$$I_{q_1,q_2}(t;\mathbf{m}) \ll_N P^{-N} + \mathbf{1}\Big[|\mathbf{u}| \geqslant c_1K_0\Theta\Big]\frac{P^\varepsilon}{\Theta}\,|\mathbf{u}|^{1-n/2} + \int_{-K_0}^{K_0} \Big|J\Big((t,\tau\Theta);\mathbf{u}\Big)\Big|\,d\tau + \Theta^{-N}\mathbf{1}\Big[|\mathbf{u}| \leqslant P^\varepsilon\Theta\Big].$$

*Proof.* In (6.2) the range $|\tau| \leqslant K_0$ gives the third term. For $|\tau| > K_0$ the pair $\boldsymbol{\gamma} = (t,\tau\Theta)$ satisfies $|\gamma_2| \geqslant K_0(1+|\gamma_3|)$, and we apply Lemma 2.10. The $\tau$ with $|\mathbf{u}| < c_1|\tau|\Theta$ or $|\mathbf{u}| > c_2|\tau|\Theta$ contribute

$$\ll \int_{|\tau|>K_0} (|\tau|\Theta + |\mathbf{u}|)^{-N-1}d\tau \ll \Theta^{-1}(\Theta+|\mathbf{u}|)^{-N} \ll \Theta^{-N}\mathbf{1}\Big[|\mathbf{u}| \leqslant P^\varepsilon\Theta\Big] + P^{-\varepsilon N}.$$

The remaining $\tau$, with $|\tau| > K_0$ and $c_1|\tau|\Theta \leqslant |\mathbf{u}| \leqslant c_2|\tau|\Theta$, exist only if $|\mathbf{u}| \geqslant c_1K_0\Theta$. They form a set of measure $O(|\mathbf{u}|/\Theta)$, on which $J((t,\tau\Theta);\mathbf{u}) \ll P^{-N} + P^\varepsilon|\mathbf{u}|^{-n/2}$. Since $|\mathbf{u}| < P^\varepsilon\Theta/r_2 \leqslant P^3$, this gives the second term. □

6.3. **Dyadic blocks.** The terms of (6.1) with $\mathbf{m} \neq \mathbf{0}$ and $|\mathbf{u}| \geqslant P^{\varepsilon}\Theta/r_2$ contribute $O(P^{-N})$, by Lemma 6.3 and the trivial bound $|S_{q_1,q_2}(\mathbf{m})| \leqslant (q_1q_2)^{n+1}$. For the other terms the number of triples $(q_1, q_2, \mathbf{m})$ is $O(P^{4n})$, so the term $P^{-N}$ in Lemma 6.4 is also negligible. We divide $\mathcal{R}_\delta$ into $O((\log P)^2)$ blocks

$$\mathcal{B}(Q_1, T) = \{(q_1, t) \in \mathcal{R}_\delta : \ Q_1 < q_1 \leqslant 2Q_1, \ T \leqslant \Theta < 2T\},$$

where $Q_1 \geqslant 1/2$ and $T \geqslant 1$ are powers of two. If $\mathcal{B}(Q_1, T)$ is not empty, then

$$Q_1^{n/8-2}\, T^{n/8-1} < P^{2+\eta_0}, \qquad \text{and} \qquad Q_1 > \tfrac{1}{2}P^{\delta_1} \ \text{ or } \ T > \tfrac{1}{2}P^{\delta_1}, \tag{6.3}$$

the second statement because $\mathcal{R}_\delta$ is disjoint from $\mathfrak{M}_1$. Put $R_0 = P(Q_1T)^{-1/2}$, so that $R_0/2 < R \leqslant R_0$ on $\mathcal{B}(Q_1, T)$. By Lemmas 6.2 and 6.4, the contribution of $\mathcal{B}(Q_1, T)$ to the right-hand side of (6.1) is

$$\ll \mathcal{E}_{\mathrm{res}}(Q_1, T) + \mathcal{E}_{\mathrm{non}}(Q_1, T) + \mathcal{E}_{\mathrm{tail}}(Q_1, T) + P^{-N},$$

where, with the sums over $q_1$ and the integrals over $t$ restricted to $\mathcal{B}(Q_1, T)$, and with $q_2 \leqslant 4R$,

$$\mathcal{E}_{\mathrm{res}} = P^{n-3+\varepsilon} \sum_{q_1} \int \frac{1}{R^2\Theta} \sum_{q_2} \frac{1}{(q_1q_2)^n} \sum_{\substack{\mathbf{m}\neq\mathbf{0} \\ c_1K_0\Theta\leqslant|\mathbf{u}|<P^{\varepsilon}\Theta/r_2}} |S_{q_1,q_2}(\mathbf{m})|\, |\mathbf{u}|^{1-n/2}\, dt,$$

$$\mathcal{E}_{\mathrm{non}} = P^{n-3} \sum_{q_1} \int \frac{1}{R^2} \sum_{q_2} \frac{1}{(q_1q_2)^n} \sum_{\substack{\mathbf{m}\neq\mathbf{0} \\ |\mathbf{u}|<P^{\varepsilon}\Theta/r_2}} |S_{q_1,q_2}(\mathbf{m})| \int_{-K_0}^{K_0} \Big|J\Big((t, \tau\Theta); \mathbf{u}\Big)\Big|\, d\tau\, dt,$$

$$\mathcal{E}_{\mathrm{tail}} = P^{n-3} \sum_{q_1} \int \frac{1}{R^2\Theta^N} \sum_{q_2} \frac{1}{(q_1q_2)^n} \sum_{|\mathbf{u}|\leqslant P^{\varepsilon}\Theta} |S_{q_1,q_2}(\mathbf{m})|\, dt.$$

The sum in $\mathcal{E}_{\mathrm{tail}}$ includes $\mathbf{m} = \mathbf{0}$.

We estimate the three quantities in turn, in Propositions 6.5, 6.6 and 6.7. In these propositions and their proofs, $n \geqslant 28$, the numbers $Q_1, T$ satisfy (6.3), and the following conventions are in force. On $\mathcal{B}(Q_1, T)$ we have $R^{-2} = q_1\Theta P^{-2} \leqslant 4Q_1TP^{-2}$, and the $t$-integral has length $O(T)$. We divide the sum over $q_2$ into ranges $Q_2 < q_2 \leqslant 2Q_2$, where $Q_2 \geqslant 1/2$ is a power of two with $Q_2 \leqslant 4R_0$. Proposition 3.12 will be applied with these $Q_1, Q_2$; in each application $V$ and $V_0$ are bounded by a fixed power of $P$. Since all terms are non-negative, we may enlarge the ranges of summation. Finally, we put $M_0 = 2P^{\varepsilon}(Q_1T)^{1/2}$. Since $\mathbf{u} = P\mathbf{m}/(q_1q_2)$ and $r_2 = q_2/R$, the condition $|\mathbf{u}| < P^{\varepsilon}\Theta/r_2$, which occurs in $\mathcal{E}_{\mathrm{res}}$ and $\mathcal{E}_{\mathrm{non}}$, implies

$$|\mathbf{m}| < \frac{P^{\varepsilon}\Theta R q_1}{P} = P^{\varepsilon}(q_1\Theta)^{1/2} \leqslant M_0. \tag{6.4}$$

**Proposition 6.5.** *Let $Q_1, T$ satisfy (6.3). Then*

$$\mathcal{E}_{\mathrm{res}} \ll P^{\varepsilon}\Big\{P^{\frac{n}{2}-\frac{7}{2}} Q_1^{\frac{n+9}{4}} T^{\frac{n+5}{4}} + P^{\frac{n}{2}-3} Q_1^{\frac{n+7}{4}} T^{\frac{n+3}{4}} + P^{\frac{n}{2}-3} Q_1^{\frac{n}{3}+\frac{1}{3}} \min\Big\{1, \Big(\frac{P}{Q_1T}\Big)^{\frac{n}{2}-2}\Big\}\Big\}.$$

*Proof.* By (6.4), the conditions on $\mathbf{m}$ in the definition of $\mathcal{E}_{\mathrm{res}}$ imply

$$\frac{c_1K_0TQ_1Q_2}{P} \leqslant |\mathbf{m}| < M_0.$$

We divide the range of $\mathbf{m}$ into shells $M' < |\mathbf{m}| \leqslant 2M'$, where $M' \geqslant 1/2$ is a power of two with

$$\max\{1/2, \ c_1K_0TQ_1Q_2/(2P)\} \leqslant M' \leqslant M_0. \tag{6.5}$$

On such a shell $|\mathbf{u}|^{1-n/2} \ll (PM'/(Q_1Q_2))^{1-n/2}$. By Proposition 3.12 with $\mathbf{m}_0 = \mathbf{0}$ and $V = 2M'$, and since the length $O(T)$ of the $t$-integral cancels against the factor $1/\Theta \leqslant 1/T$,

$$\begin{aligned}\mathcal{E}_{\mathrm{res}} &\ll P^{n-3+\varepsilon} \cdot T \cdot \frac{Q_1 T}{P^2} \cdot \frac{1}{T} \sum_{Q_2, M'} \frac{(Q_1Q_2)^{\frac{n}{2}} Q_1^2 Q_2^{\frac{3}{2}}}{(Q_1Q_2)^n} \Big(\frac{PM'}{Q_1Q_2}\Big)^{1-\frac{n}{2}} \\ &\qquad \times \Big\{ M'^n \Big(1 + Q_2^{\frac{1}{2}} M'^{-\frac{1}{2}}\Big) + Q_1^{\frac{n}{3}-\frac{2}{3}} Q_2^{\frac{1}{2}} \Big\} \\ &= P^{\frac{n}{2}-4+\varepsilon} Q_1 T \sum_{Q_2, M'} \Big\{ Q_1 Q_2^{\frac{1}{2}} M'^{\frac{n}{2}+1} + Q_1 Q_2 M'^{\frac{n}{2}+\frac{1}{2}} + Q_1^{\frac{n}{3}+\frac{1}{3}} Q_2 M'^{1-\frac{n}{2}} \Big\}.\end{aligned}$$

The first two terms are largest for $Q_2 \asymp R_0$ and $M' \asymp M_0$, and give the first two terms of the proposition. In the third term let $\xi = TQ_1/P$. By (6.5), $Q_2 M'^{1-n/2} \ll Q_2 \min\{1, (\xi Q_2)^{1-n/2}\}$. If $\xi \leqslant 1$, the sum of this over $Q_2$ is $O(\xi^{-1})$, and if $\xi > 1$ it is $O(\xi^{1-n/2})$, since $Q_2 \geqslant 1/2$. So the third term is

$$\ll P^{\frac{n}{2}-4+\varepsilon} Q_1^{\frac{n}{3}+\frac{4}{3}} T \cdot \frac{P}{TQ_1} \min\Big\{1, \Big(\frac{P}{TQ_1}\Big)^{\frac{n}{2}-2}\Big\},$$

which is the third term of the proposition. □

**Proposition 6.6.** *Let $Q_1, T$ satisfy (6.3). Then $\mathcal{E}_{\mathrm{non}} \ll P^{\varepsilon}(\mathcal{E}_\alpha + \mathcal{E}_\beta)$, where*

$$\begin{aligned}\mathcal{E}_\alpha &= P^{\frac{n}{2}-\frac{7}{2}} Q_1^{\frac{n+9}{4}} T^{\frac{n+7}{4}} + P^{\frac{n}{2}-3} Q_1^{\frac{n+7}{4}} T^{\frac{n+6}{4}} + P^{n-5} Q_1^{\frac{7}{3}-\frac{n}{6}} T^{\frac{5}{2}} \max\Big\{1, \frac{P}{T^{1/2}Q_1}\Big\}^{2-\frac{n}{2}}, \\ \mathcal{E}_\beta &= P^{n-6} T^3 \sup_{Q_2} \Big\{ Q_1^{4-\frac{n}{2}} Q_2^{3-\frac{n}{2}} + Q_1^{\frac{10}{3}-\frac{n}{6}} Q_2^{3-\frac{n}{2}} \Big\},\end{aligned}$$

*the supremum being over $Q_2 \geqslant 1/2$ with $Q_1Q_2T \geqslant P^{1-\varepsilon}$.*

*Proof.* Fix $t$ and $\tau$ with $|\tau| \leqslant K_0$, and put $\boldsymbol{\gamma} = (t, \tau\Theta)$ and $\rho = P^{\varepsilon}\|\boldsymbol{\gamma}\|^{1/2}$; then $T \leqslant \|\boldsymbol{\gamma}\| \leqslant 2(1+K_0)T$. By Lemma 2.9,

$$\begin{aligned}(6.6)\qquad &\sum_{q_1,q_2} \sum_{0<|\mathbf{m}|\leqslant M_0} |S_{q_1,q_2}(\mathbf{m})| \, \big|J(\boldsymbol{\gamma}; \mathbf{u})\big| \\ &\qquad \ll P^{-N} + \int_{\mathcal{U}} \sum_{q_1,q_2} \sum_{\mathbf{m}\neq\mathbf{0}} |S_{q_1,q_2}(\mathbf{m})| \, \mathbf{1}\Big[\Big|\nabla f_{\boldsymbol{\gamma}}(\mathbf{z}) - \frac{P\mathbf{m}}{q_1q_2}\Big| \leqslant \rho\Big] \, d\mathbf{z},\end{aligned}$$

where $q_1$ runs over $(Q_1, 2Q_1]$, $q_2$ runs over $(Q_2, 2Q_2]$, and where we have used (6.4). Fix $\mathbf{z} \in \mathcal{U}$ and $q_1$, and let $\mathbf{w} = \nabla f_{\boldsymbol{\gamma}}(\mathbf{z})$, so that $|\mathbf{w}| \leqslant c_4\|\boldsymbol{\gamma}\|$. The condition in (6.6) says that $\mathbf{m}$ lies within $\rho q_1 q_2/P \leqslant V_1 := 4\rho Q_1Q_2/P$ of the point $(q_1q_2/P)\mathbf{w}$. As $q_2$ varies in $(Q_2, 2Q_2]$, these points lie on a segment of length at most $\Lambda := 2c_4 Q_1Q_2\|\boldsymbol{\gamma}\|/P$. Let $V = \max\{1, V_1\}$. Choosing points on the segment at distance at most $V$ from each other, we see that all the $\mathbf{m}$ in question, for all $q_2 \in (Q_2, 2Q_2]$, lie in the union of at most $\Lambda/V + 2$ balls of radius $2V$, whose centres depend only on $\mathbf{z}$ and $q_1$ (and on $\boldsymbol{\gamma}$ and $Q_2$). Moreover, if there is such an $\mathbf{m} \neq \mathbf{0}$, then $1 \leqslant |\mathbf{m}| \leqslant (4Q_1Q_2/P)(c_4\|\boldsymbol{\gamma}\| + \rho)$, hence

$$(6.7)\qquad Q_1Q_2T \geqslant P^{1-2\varepsilon}.$$

The number of balls is

$$\ll 1 + \min\Big\{\Lambda, \frac{\Lambda}{V_1}\Big\} \ll 1 + \min\Big\{\frac{TQ_1Q_2}{P}, T^{1/2}\Big\}.$$

For each ball we apply Proposition 3.12. Integrating over $\mathbf{z}$, $\tau$ and $t$ we obtain

$$\mathcal{E}_{\text{non}} \ll P^{n-5+\varepsilon}Q_1T^2\sum_{Q_2}\Big(1+\min\Big\{\frac{TQ_1Q_2}{P},T^{\frac12}\Big\}\Big)Q_1^{2-\frac n2}Q_2^{\frac32-\frac n2} \times\Big\{V^n\Big(1+Q_2^{\frac12}V^{-\frac12}\Big)+Q_1^{\frac n3-\frac23}Q_2^{\frac12}\Big\},$$

where the sum is over the $Q_2\leqslant 4R_0$ satisfying (6.7), and $V\asymp\max\{1,P^\varepsilon T^{1/2}Q_1Q_2/P\}$.

Consider first the $Q_2$ with $T^{1/2}Q_1Q_2>P$. Then the number of balls is $O(T^{1/2})$ and $1\leqslant V\ll P^\varepsilon T^{1/2}Q_1Q_2/P$. The three terms in the braces contribute

$$P^{-5+\varepsilon}Q_1^{3+\frac n2}T^{\frac52+\frac n2}Q_2^{\frac32+\frac n2},\qquad P^{-\frac92+\varepsilon}Q_1^{\frac52+\frac n2}T^{\frac94+\frac n2}Q_2^{\frac32+\frac n2},\qquad P^{n-5+\varepsilon}Q_1^{\frac73-\frac n6}T^{\frac52}Q_2^{2-\frac n2}.$$

The first two are largest for $Q_2\asymp R_0=P(Q_1T)^{-1/2}$, the third for the smallest admissible $Q_2$, which is $\asymp\max\{1,P/(T^{1/2}Q_1)\}$. This gives $\mathcal{E}_\alpha$.

Consider now the $Q_2$ with $T^{1/2}Q_1Q_2\leqslant P$. Then $V\ll P^\varepsilon$, and by (6.7) the number of balls is $O(P^{2\varepsilon}TQ_1Q_2/P)$. The contribution is

$$\ll P^{n-6+\varepsilon}Q_1^{4-\frac n2}T^3Q_2^{\frac52-\frac n2}\Big\{1+Q_2^{\frac12}+Q_1^{\frac n3-\frac23}Q_2^{\frac12}\Big\}\ll P^\varepsilon\mathcal{E}_\beta.$$

This completes the proof. □

**Proposition 6.7.** *Let $Q_1,T$ satisfy (6.3). If $T\geqslant P^\varepsilon$, then $\mathcal{E}_{\text{tail}}\ll P^{-N}$. If $T<P^\varepsilon$, then*

$$\mathcal{E}_{\text{tail}}\ll P^\varepsilon\Big\{P^{\frac n2-\frac72}Q_1^{\frac{n+9}4}+P^{\frac n2-3}Q_1^{\frac{n+7}4}+P^{n-5}Q_1^{\frac73-\frac n6}\Big\}.$$

*Proof.* The condition $|\mathbf{u}|\leqslant P^\varepsilon\Theta$ implies $|\mathbf{m}|\leqslant 8P^\varepsilon TQ_1Q_2/P$. By Proposition 3.12 with $\mathbf{m}_0=\mathbf{0}$ and $V=\max\{1,8P^\varepsilon TQ_1Q_2/P\}$,

$$\mathcal{E}_{\text{tail}}\ll P^{n-5+\varepsilon}Q_1T^{2-N}\sum_{Q_2\leqslant 4R_0}Q_1^{2-\frac n2}Q_2^{\frac32-\frac n2}\Big\{V^n\Big(1+Q_2^{\frac12}V^{-\frac12}\Big)+Q_1^{\frac n3-\frac23}Q_2^{\frac12}\Big\}.$$

If $T\geqslant P^\varepsilon$ this is $O(P^{-N})$, since $N$ is arbitrary. Let $T<P^\varepsilon$; then $R_0\leqslant PQ_1^{-1/2}$. The $Q_2$ with $Q_1Q_2>P$ have $V\ll P^{2\varepsilon}Q_1Q_2/P$, and the terms with $V^n$ contribute

$$\ll P^{-5+\varepsilon}Q_1^{3+\frac n2}Q_2^{\frac32+\frac n2}\Big(1+(P/Q_1)^{\frac12}\Big)\ll P^\varepsilon\Big(P^{\frac n2-\frac72}Q_1^{\frac{n+9}4}+P^{\frac n2-3}Q_1^{\frac{n+7}4}\Big).$$

The $Q_2$ with $Q_1Q_2\leqslant P$ have $V\ll P^{2\varepsilon}$, and the terms with $V^n$ contribute $O(P^{n-5+\varepsilon}Q_1^{3-n/2})$. The last term in the braces contributes $O(P^{n-5+\varepsilon}Q_1^{7/3-n/6})$. Since $3-n/2\leqslant 7/3-n/6$, this proves the proposition. □

6.4. **The bookkeeping.** The next five lemmas evaluate the bounds of Propositions 6.5, 6.6 and 6.7, one term at a time. In all of them we suppose that

(6.8)
$$n\geqslant 28,\qquad 0<\eta_0\leqslant\frac{1}{20},\qquad Q_1\geqslant\frac12,\qquad T\geqslant 1,\qquad Z:=Q_1^{n/8-2}T^{n/8-1}<P^{2+\eta_0}.$$

The last condition is the first part of (6.3). For $n=28$ the exponents on the right-hand sides of the six bounds are 137/6, 68/3, 62/3, 152/7, 112/5 and 137/6, plus the multiples of $\eta_0$. In the proofs, each expression in braces is a decreasing function of $n$ for $n\geqslant 28$.

**Lemma 6.8.** *Suppose that (6.8) holds. Then*

$$P^{\frac n2-\frac72}Q_1^{\frac{n+9}4}T^{\frac{n+7}4}\ll P^{n-5-1/6+(37/6)\eta_0}.$$

*Proof.* Let $\lambda = 2(n+9)/(n-16) = 2 + 50/(n-16)$. In $Z^\lambda$ the exponent of $Q_1$ is $(n+9)/4$ and the exponent of $T$ is $(n+9)(n-8)/(4(n-16)) \geqslant (n+7)/4$. Since $T \geqslant 1$, we get $Q_1^{(n+9)/4}T^{(n+7)/4} \leqslant Z^\lambda < P^{\lambda(2+\eta_0)}$. Now $\lambda \leqslant 37/6$, and $n/2 - 7/2 + 2\lambda = n - 5 + \{11/2 - n/2 + 100/(n-16)\}$, where the expression in braces equals $-1/6$ for $n = 28$. □

**Lemma 6.9.** *Suppose that (6.8) holds. Then*

$$P^{\frac{n}{2}-3}Q_1^{\frac{n+7}{4}}T^{\frac{n+6}{4}} \ll P^{n-5-1/3+(35/6)\eta_0}.$$

*Proof.* Let $\lambda' = 2(n+7)/(n-16) = 2+46/(n-16)$. In $Z^{\lambda'}$ the exponent of $Q_1$ is $(n+7)/4$ and the exponent of $T$ is $(n+7)(n-8)/(4(n-16)) \geqslant (n+6)/4$. Hence $Q_1^{(n+7)/4}T^{(n+6)/4} \leqslant Z^{\lambda'} < P^{\lambda'(2+\eta_0)}$. Now $\lambda' \leqslant 35/6$, and $n/2 - 3 + 2\lambda' = n - 5 + \{6 - n/2 + 92/(n-16)\}$, where the expression in braces equals $-1/3$ for $n = 28$. □

**Lemma 6.10.** *Let $n \geqslant 28$, $Q_1 \geqslant 1/2$ and $T \geqslant 1$. Then*

$$P^{\frac{n}{2}-3}Q_1^{\frac{n}{3}+\frac{1}{3}} \min\Big\{1, \Big(\frac{P}{Q_1T}\Big)^{\frac{n}{2}-2}\Big\} \ll P^{n-5-7/3}.$$

*Proof.* If $Q_1T \leqslant P$, then $Q_1 \leqslant P$ and the left-hand side is at most $P^{5n/6-8/3}$. If $Q_1T > P$, it equals $P^{n-5}Q_1^{7/3-n/6}T^{2-n/2} \leqslant P^{n-5}(Q_1T)^{7/3-n/6} < P^{5n/6-8/3}$. Finally $5n/6 - 8/3 \leqslant n - 5 - 7/3$. □

**Lemma 6.11.** *Suppose that (6.8) holds. Then*

$$P^{n-5}Q_1^{\frac{7}{3}-\frac{n}{6}}T^{\frac{5}{2}} \max\Big\{1, \frac{P}{T^{1/2}Q_1}\Big\}^{2-\frac{n}{2}} \ll P^{n-5-9/7+(44/21)\eta_0}.$$

*Proof.* Suppose that $T^{1/2}Q_1 \geqslant P$. Then $Z \geqslant P^{n/8-2}T^{n/16}$, so $T^{n/16} < P^{4+\eta_0-n/8}$, and therefore $T < P^\theta$ with $\theta = (64+16\eta_0)/n - 2 \leqslant (2/7)(1+2\eta_0)$. Since $Q_1 \geqslant PT^{-1/2} \geqslant 1$ and $n/6 - 7/3 \geqslant 7/3$, the left-hand side is

$$P^{n-5}Q_1^{7/3-n/6}T^{5/2} \leqslant P^{n-5}(PT^{-1/2})^{-7/3}T^{5/2} = P^{n-5-7/3}T^{11/3} < P^{n-5-7/3+(22/21)(1+2\eta_0)},$$

and $7/3 - 22/21 = 9/7$. Suppose that $T^{1/2}Q_1 < P$. Then the left-hand side equals

$$P^{\frac{n}{2}-3}Q_1^{\frac{n}{3}+\frac{1}{3}}T^{\frac{n}{4}+\frac{3}{2}} = P^{\frac{n}{2}-3}(Q_1T^{1/2})^xZ^y, \qquad x = \frac{n}{6} + \frac{1}{3} + \frac{128}{3n}, \quad y = \frac{4}{3} + \frac{64}{3n},$$

which is less than $P^{n/2-3+x+y(2+\eta_0)} = P^{n-5+\{5-n/3+256/(3n)\}+y\eta_0}$. The expression in braces equals $-9/7$ for $n = 28$, and $y \leqslant 44/21$. □

**Lemma 6.12.** *Suppose that (6.8) holds, and let $Q_2 \geqslant 1/2$ with $Q_1Q_2T \geqslant P^{1-\varepsilon}$. Then*

$$P^{n-6}T^3Q_1^{4-\frac{n}{2}}Q_2^{3-\frac{n}{2}} \ll P^{n-5-3/5+(26/5)\eta_0+n\varepsilon},$$

$$P^{n-6}T^3Q_1^{\frac{10}{3}-\frac{n}{6}}Q_2^{3-\frac{n}{2}} \ll P^{n-5-1/6+(13/3)\eta_0+n\varepsilon}.$$

*Proof.* Since $Q_2 \geqslant 1/2$ and $Q_1Q_2 \geqslant P^{1-\varepsilon}/T$, the first expression is

$$O\Big(P^{n-6}T^3(Q_1Q_2)^{4-n/2}\Big) = O\Big(P^{n/2-2+n\varepsilon}T^{n/2-1}\Big).$$

Moreover $T^{n/8-1} \leqslant 2^{n/8-2}Z$, so $T \ll P^{8(2+\eta_0)/(n-8)}$. The exponent obtained is

$$\frac{n}{2} - 2 + n\varepsilon + \Big(4 + \frac{24}{n-8}\Big)(2+\eta_0) = n - 5 + \Big\{11 - \frac{n}{2} + \frac{48}{n-8}\Big\} + \Big(4 + \frac{24}{n-8}\Big)\eta_0 + n\varepsilon.$$

The expression in braces equals $-3/5$ for $n = 28$, and $4 + 24/(n-8) \leqslant 26/5$.

Let $E$ be the second expression, and put $\kappa = (n-20)/6$ and $G(n) = 1+13n/24-n^2/48$, so that $G(28) = -1/6$, $G(29) = -13/16$ and $G$ is decreasing for $n \geqslant 28$. Suppose first that $Q_1 T > P$. Since $Q_2 \geqslant 1/2$ and $Q_1 > P/T$, we have $E \ll P^{n-6-\kappa}T^{3+\kappa}$. Moreover $Z > (P/T)^{n/8-2}T^{n/8-1} = P^{n/8-2}T$, so $T < P^{4+\eta_0-n/8}$; this is possible only if $n \leqslant 32$. Hence

$$E \ll P^{n-6-\kappa+(3+\kappa)(4+\eta_0-n/8)} = P^{n-5+G(n)+(n-2)\eta_0/6}.$$

Suppose now that $Q_1 T \leqslant P$. Since $Q_2 \geqslant P^{1-\varepsilon}/(Q_1 T)$, we have

$$E \leqslant P^{n/2-3+n\varepsilon}\, T^{n/2} Q_1^{n/3+1/3}.$$

If $n \leqslant 33$, let $\lambda_2 = (n-2)/6$ and $\lambda_1 = -n^2/48 + 17n/24 - 1/3$; then $\lambda_1 \geqslant 0$ and

$$T^{n/2}Q_1^{n/3+1/3} = (Q_1 T)^{\lambda_1} Z^{\lambda_2} \leqslant P^{\lambda_1+\lambda_2(2+\eta_0)}, \qquad \frac{n}{2} - 3 + \lambda_1 + 2\lambda_2 = n - 5 + G(n).$$

If $n \geqslant 34$, then $T^{n/2}Q_1^{n/3+1/3} \ll Z^{4n/(n-8)}$, because $n(n-16)/(2(n-8)) \geqslant (n+1)/3$ for these $n$, and the exponent obtained is $n/2-3+n\varepsilon+(4n/(n-8))(2+\eta_0) < n-5-4+n\varepsilon$. In all cases the exponent is at most $n-5+G(n)+(n-2)\eta_0/6+n\varepsilon$ or less than $n-5-4+n\varepsilon$. For $n = 28$ this is the claim. For $29 \leqslant n \leqslant 33$ we have $G(n) \leqslant -13/16$ and $(n-2)\eta_0/6 \leqslant 31/120$, which is more than is needed. □

**Proposition 6.13.** *Let $n \geqslant 28$. Then $N_\delta \ll P^{n-5-1/43+\varepsilon}$.*

*Proof.* By (6.1) and the discussion following Lemma 6.4, $N_\delta$ is bounded by the sum of $\mathcal{E}_{\mathrm{res}} + \mathcal{E}_{\mathrm{non}} + \mathcal{E}_{\mathrm{tail}}$ over $O((\log P)^2)$ blocks satisfying (6.3), up to $O(P^{-N})$. We apply Lemmas 6.8–6.12 with $\eta_0 = 1/43$ to the bounds of Propositions 6.5, 6.6 and 6.7; the condition (6.8) holds by (6.3).

Since $T \geqslant 1$, the three terms of $\mathcal{E}_{\mathrm{res}}$ are at most the left-hand sides of Lemmas 6.8, 6.9 and 6.10. The three terms of $\mathcal{E}_\alpha$ are the left-hand sides of Lemmas 6.8, 6.9 and 6.11. The two terms of $\mathcal{E}_\beta$ are bounded by Lemma 6.12. The exponents obtained are $n-5-\sigma$ with

$$\sigma = \frac{1}{6} - \frac{37}{258}, \qquad \frac{1}{3} - \frac{35}{258}, \qquad \frac{7}{3}, \qquad \frac{9}{7} - \frac{44}{903}, \qquad \frac{3}{5} - \frac{26}{215}, \qquad \frac{1}{6} - \frac{26}{258},$$

and the smallest of these numbers is the first, which is equal to $1/43$.

Finally, if $T \geqslant P^\varepsilon$ then $\mathcal{E}_{\mathrm{tail}}$ is negligible. If $T < P^\varepsilon$, then $Q_1 > \frac{1}{2}P^{\delta_1}$ by (6.3), and $Q_1^{n/8-2} < P^{2+\eta_0}$. The first two terms of Proposition 6.7 are bounded by Lemmas 6.8 and 6.9 with $T = 1$, and the third is $O(P^{n-5-(n/6-7/3)\delta_1}) = O(P^{n-5-7/120})$. All these bounds are $O(P^{n-5-1/43+\varepsilon})$, as required. □

## 7. Proof of the theorems

It remains to show that the singular series is positive, and to deduce Theorems 1.1 and 1.2 from the results of the previous sections. In this section $n \geqslant 28$. We recall that $d_n \neq 0$ throughout §§2–7; the proofs of the two theorems for $d_n = 0$ are completed in §8.

### 7.1. The singular series.

For a prime $p$ and $k \geqslant 0$ let

$$\lambda(p^k) = \#\big\{\mathbf{x} \bmod p^k : \; C(\mathbf{x}) \equiv Q(\mathbf{x}) \equiv 0 \pmod{p^k}\big\}.$$

**Lemma 7.1.** *For every prime $p$ there is $\mathbf{x} \in \mathbb{Q}_p^n \setminus \{\mathbf{0}\}$ with $C(\mathbf{x}) = Q(\mathbf{x}) = 0$.*

*Proof.* The form $Q$ is non-degenerate of rank $n \geqslant 28$ over $\mathbb{Q}_p$. Every quadratic form over $\mathbb{Q}_p$ in at least five variables is isotropic [20, Chapter IV, Theorem 6]. Splitting off hyperbolic planes, we see that $Q$ is equivalent over $\mathbb{Q}_p$ to the orthogonal sum of at least twelve hyperbolic planes and a form in at most four variables. Hence there is a linear subspace $W \subset \mathbb{Q}_p^n$ of dimension 12 on which $Q$ vanishes. The restriction of $C$ to $W$ is a cubic form in 12 variables over $\mathbb{Q}_p$, and by the theorem of Lewis [14] a cubic form over $\mathbb{Q}_p$ in at least ten variables has a non-trivial zero. This is the argument used in [3] for $n \geqslant 29$. □

**Lemma 7.2.** *The function $A(q)$ is multiplicative, and*

$$\mathfrak{S} = \prod_p \sigma_p, \qquad \sigma_p = \sum_{k=0}^{\infty} A(p^k) = \lim_{k\to\infty} p^{-k(n-2)}\lambda(p^k),$$

*where the product converges absolutely. Moreover $\mathfrak{S} > 0$.*

*Proof.* Let $q = q'q''$ with $(q', q'') = 1$. By the Chinese remainder theorem the map $\mathbf{a} \mapsto (\mathbf{a}', \mathbf{a}'')$ with $\mathbf{a}' = (q''^2 a_3, q''a_2)$ modulo $q'$ and $\mathbf{a}'' = (q'^2 a_3, q'a_2)$ modulo $q''$ is a bijection between the $\mathbf{a}$ modulo $q$ with $\gcd(q, a_3, a_2) = 1$ and the pairs $(\mathbf{a}', \mathbf{a}'')$ with $\gcd(q', a_3', a_2') = \gcd(q'', a_3'', a_2'') = 1$. By (3.1) with $\mathbf{m} = \mathbf{0}$, $A(q) = A(q')A(q'')$. By Lemma 5.1(i), $\sum_q |A(q)| < \infty$. Hence $\mathfrak{S} = \prod_p \sigma_p$ with $\sigma_p = \sum_k A(p^k)$, and $\sum_p |\sigma_p - 1| < \infty$.

For $k \geqslant 0$ we detect the two congruences by additive characters, and sort the pairs $\mathbf{a}$ modulo $p^k$ according to the value $p^{k-j}$ of $\gcd(p^k, a_3, a_2)$:

$$\begin{aligned}\lambda(p^k) &= p^{-2k} \sum_{\mathbf{a} \bmod p^k} \sum_{\mathbf{x} \bmod p^k} e_{p^k}\Big(a_3 C(\mathbf{x}) + a_2 Q(\mathbf{x})\Big) \\ &= p^{-2k} \sum_{j=0}^{k} p^{(k-j)n} \sum_{\substack{\mathbf{a} \bmod p^j \\ \gcd(p^j, a_3, a_2) = 1}} S(\mathbf{a}, p^j; \mathbf{0}) = p^{k(n-2)} \sum_{j=0}^{k} A(p^j).\end{aligned}$$

This proves the formula for $\sigma_p$.

Since $\sum_p |\sigma_p - 1| < \infty$, it remains to show that $\sigma_p > 0$ for every $p$. By Lemma 7.1 and homogeneity there is a primitive vector $\mathbf{x}_0 \in \mathbb{Z}_p^n$ with $C(\mathbf{x}_0) = Q(\mathbf{x}_0) = 0$. Since $X$ is non-singular, the vectors $\nabla C(\mathbf{x}_0)$ and $\nabla Q(\mathbf{x}_0)$ are linearly independent. So there are indices $i < j$ for which the minor $M(\mathbf{x}) = \partial_i C\, \partial_j Q - \partial_j C\, \partial_i Q$ does not vanish at $\mathbf{x}_0$; let $\mu$ be the $p$-adic valuation of $M(\mathbf{x}_0)$. Let $k > 2\mu$, and let $\mathbf{y} \in \mathbb{Z}_p^n$ with $\mathbf{y} \equiv \mathbf{x}_0 \pmod{p^{2\mu+1}}$. Then $C(\mathbf{y}) \equiv Q(\mathbf{y}) \equiv 0 \pmod{p^{2\mu+1}}$ and $M(\mathbf{y})$ has valuation $\mu$. By Hensel's lemma for two equations in the two unknowns $x_i$, $x_j$, there is $\mathbf{x} \in \mathbb{Z}_p^n$ with $C(\mathbf{x}) = Q(\mathbf{x}) = 0$ and $x_l = y_l$ for all $l \neq i, j$. The vectors $\mathbf{y}$ give $p^{(k-2\mu-1)(n-2)}$ different values of $(y_l)_{l\neq i,j}$ modulo $p^k$, and therefore at least as many solutions $\mathbf{x}$ modulo $p^k$. So $\lambda(p^k) \geqslant p^{(k-2\mu-1)(n-2)}$ for all $k > 2\mu$, and $\sigma_p \geqslant p^{-(2\mu+1)(n-2)} > 0$. □

7.2. **Proof of Theorem 1.2.** By (4.1), (4.2) and Propositions 4.5, 5.3 and 6.13, with $\eta_0 = 1/43$,

$$N_w(P) = \mathfrak{S}\,\mathfrak{J}\,P^{n-5} + O\Big(P^{n-5-1/40}\Big) + O\Big(P^{n-5-1/43+\varepsilon}\Big).$$

Here $\mathfrak{J} > 0$ by Lemma 5.1(ii) and $\mathfrak{S} > 0$ by Lemma 7.2. This proves Theorem 1.2 for $d_n \neq 0$.

7.3. **Proof of Theorem 1.1.** Let $X$ be as in Theorem 1.1. If $X(\mathbb{Q}) \neq \emptyset$ then $X(\mathbb{R}) \neq \emptyset$. Conversely, suppose that $X(\mathbb{R}) \neq \emptyset$. As explained in §2, after a linear change of variables over $\mathbb{Q}$ we may suppose that $X$ is given by forms $C$ and $Q$ as in (1.1). By Theorem 1.2, which has just been proved for $d_n \neq 0$ and is proved in §8 for $d_n = 0$, we have $N_w(P) > 0$ if $P$ is large. By (U1) the weight $w$ vanishes in a neighbourhood of the origin, so there is $\mathbf{x} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$ with $C(\mathbf{x}) = Q(\mathbf{x}) = 0$. Hence $X(\mathbb{Q}) \neq \emptyset$.

## 8. Quadratic forms of rank $n-1$

In this section we prove Theorem 1.2, and with it Theorem 1.1, in the case that $d_n = 0$. Thus

$$Q(\mathbf{x}) = \sum_{i=1}^{n-1} d_i x_i^2, \qquad d_1 \cdots d_{n-1} \neq 0,$$

and the variable $x_n$ does not occur in $Q$. We write $\mathbf{x} = (\mathbf{x}', x_n)$ with $\mathbf{x}' = (x_1, \dots, x_{n-1})$, and similarly for other vectors. In this section $D = d_1 \cdots d_{n-1}$ and $\Delta = 2D$, and $q^\flat$, $q^\sharp$, $\mathrm{cf}(q)$ and $\kappa(q)$ are defined as in §1 with this $\Delta$. The parameters $\eta_0$, $\delta_1$, $\delta_2$ and the four sets of pairs $(q_1, t)$ are those of §4, and $n \geqslant 28$.

Several results of the preceding sections do not involve the coefficient $d_n$, and remain valid with the same proofs. These are Lemmas 2.2–2.6, the bound (2.2) and Lemmas 2.7, 2.8 and 2.9; the relation (3.1), Lemmas 3.1–3.3, Corollary 3.4 and Lemmas 3.5 and 3.6, where we note that the results of [3] quoted in (3.1) and Lemmas 3.1–3.3 are stated for $d_1 \cdots d_{n-1} \neq 0$; all of §4; Lemma 5.1; and Lemma 6.1. We shall use them freely. What has to be replaced is Lemma 2.10, the estimates for the sums $S_{q_1,q_2}(\mathbf{m})$ in Lemmas 3.7–3.11 and Proposition 3.12, and those parts of §§5–7 that depend on them.

8.1. **The weight function and the integrals.** Let $\mathbf{e}_n = (0, \dots, 0, 1)$. Then $Q(\mathbf{e}_n) = 0$ and $\nabla Q(\mathbf{e}_n) = \mathbf{0}$. If $C(\mathbf{e}_n)$ were zero, then $\mathbf{e}_n$ would define a singular point of $X$. Hence $C(\mathbf{e}_n) \neq 0$. In particular a real zero $\mathbf{z}_0$ of $C$ and $Q$ is not a multiple of $\mathbf{e}_n$, that is, $\mathbf{z}_0' \neq \mathbf{0}$. In the construction of §2.2 we choose $\rho_0$ so small that, in addition to (U1)–(U3), we have $|\mathbf{z}'| \geqslant |\mathbf{z}_0'|/2$ for $\mathbf{z} \in \mathcal{U}$. Then there are constants $0 < \kappa_1 \leqslant \kappa_2$ with $\kappa_1 \leqslant |(d_i z_i)_{i<n}| \leqslant \kappa_2$ for $\mathbf{z} \in \mathcal{U}$.

**Lemma 8.1.** *There are constants $K_0 \geqslant 1$ and $0 < c_1 < c_2$, depending only on $C$, $Q$ and $\mathcal{U}$, with the following property. Suppose that $|\gamma_2| \geqslant K_0(1 + |\gamma_3|)$. If $|\mathbf{u}| < c_1|\gamma_2|$ or $|\mathbf{u}| > c_2|\gamma_2|$, then $J(\boldsymbol{\gamma}; \mathbf{u}) \ll_N (|\gamma_2| + |\mathbf{u}|)^{-N}$. For all $\mathbf{u}$ and every $P \geqslant 1$,*

$$J(\boldsymbol{\gamma}; \mathbf{u}) \ll_{N,\varepsilon} P^{-N} + P^{\varepsilon} |\gamma_2|^{-(n-1)/2}.$$

*Proof.* We have $\nabla f_\gamma(\mathbf{z}) = 2\gamma_2(d_1 z_1, \dots, d_{n-1} z_{n-1}, 0) + \gamma_3 \nabla C(\mathbf{z})$. If $K_0$ is large enough, then $\kappa_1|\gamma_2| \leqslant |\nabla f_\gamma(\mathbf{z})| \leqslant 3\kappa_2|\gamma_2|$ for $\mathbf{z} \in \mathcal{U}$, and the first assertion follows as in the proof of Lemma 2.10, with $c_1 = \kappa_1/2$ and $c_2 = 6\kappa_2$. Moreover, if $\mathbf{z}, \mathbf{w} \in \mathcal{U}$ have the same last coordinate, then the first $n-1$ coordinates of $\nabla f_\gamma(\mathbf{z}) - \nabla f_\gamma(\mathbf{w})$ form a vector of norm at least $|\gamma_2|\,|\mathbf{z} - \mathbf{w}|$, because $|d_i| \geqslant 1$ for $i < n$ and $\nabla C$ is Lipschitz on $\mathcal{U}$. Let $\rho = P^{\varepsilon}\|\boldsymbol{\gamma}\|^{1/2}$. For each value of $z_n$, the set of the $\mathbf{z}'$ with $\mathbf{z} = (\mathbf{z}', z_n) \in \mathcal{U}$ and $|\nabla f_\gamma(\mathbf{z}) - \mathbf{u}| \leqslant \rho$ therefore has diameter at most $2\rho/|\gamma_2| \ll P^{\varepsilon}|\gamma_2|^{-1/2}$. Since $z_n$ runs over an interval of length at most 1, the measure of the set in Lemma 2.9 is $O(P^{n\varepsilon}|\gamma_2|^{-(n-1)/2})$. □

With Lemma 8.1 in place of Lemma 2.10, the proofs of Lemmas 6.2, 6.3 and 6.4 give the following: Lemmas 6.2 and 6.3 hold as stated, and Lemma 6.4 holds with $|\mathbf{u}|^{3/2-n/2}$ in place of $|\mathbf{u}|^{1-n/2}$ in the second term on the right.

8.2. **Exponential sums.** The next lemma is the source of the saving in the present case.

**Lemma 8.2.** *Let $q_1, q_2 \in \mathbb{N}$ and $\mathbf{m} \in \mathbb{Z}^n$. Then $S_{q_1,q_2}(\mathbf{m}) = 0$ unless $q_2 \mid m_n$.*

*Proof.* Let $j \in \mathbb{Z}$. In the definition of $S_{q_1,q_2}(\mathbf{m})$ we replace $\mathbf{k}$ by $\mathbf{k}+q_1 j\mathbf{e}_n$; this permutes the residue classes modulo $q_1q_2$. Since $x_n$ does not occur in $Q$, the value of $Q(\mathbf{k})$ does not change. Since $C(\mathbf{k}+q_1 j\mathbf{e}_n) \equiv C(\mathbf{k}) \pmod{q_1}$, the factor $e_{q_1q_2}(a_1q_2C(\mathbf{k})) = e_{q_1}(a_1C(\mathbf{k}))$ does not change either. The factor $e_{q_1q_2}(\mathbf{k}\cdot\mathbf{m})$ is multiplied by $e_{q_2}(jm_n)$. Hence $S_{q_1,q_2}(\mathbf{m}) = e_{q_2}(jm_n)S_{q_1,q_2}(\mathbf{m})$ for all $j$. □

**Lemma 8.3.** *For all $e \in \mathbb{N}$ and $\mathbf{m} \in \mathbb{Z}^n$ we have $|S_{1,e}(\mathbf{m})| \leqslant (2^{n-1}|D|)^{1/2}\, e^{(n+3)/2}$.*

*Proof.* With $G(u,v) = \sum_{k \bmod e} e_e(uk^2+vk)$ we have

$$S_{1,e}(\mathbf{m}) = \sideset{}{^*}\sum_{a \bmod e} \Big(\prod_{i=1}^{n-1} G(ad_i, m_i)\Big) \sum_{k \bmod e} e_e(km_n).$$

The last sum has modulus at most $e$. Squaring out, as in the proof of Lemma 3.7, we get $|G(u,v)|^2 \leqslant e\,(2u,e)$, and $(2ad_i,e) \leqslant 2|d_i|$ for $(a,e)=1$. There are at most $e$ values of $a$. □

Thus the sum over $a_2$ is estimated trivially. For the part of $q_2$ that has a common factor with $q_1$ we use the estimates of [3], in the following form.

**Lemma 8.4.** *Let $q \in \mathbb{N}$ and $\mathbf{a} \in \mathbb{Z}^2$ with $\gcd(q, a_3, a_2) = 1$. Let $\mathbf{m}_0' \in \mathbb{R}^{n-1}$, $m_n \in \mathbb{Z}$ and $V \geqslant 1$. Then*

$$\sum_{\substack{\mathbf{m}' \in \mathbb{Z}^{n-1} \\ |\mathbf{m}'-\mathbf{m}_0'| \leqslant V}} |S(\mathbf{a}, q; (\mathbf{m}', m_n))| \ll q^{n/2+\varepsilon}\Big(q^{(0)}\Big)^{1/2}\Big(V^{n-1}\Big(q^{(2)}\Big)^{1/3} + \Big(q^{(2)}\Big)^{n/3}\Big).$$

*Proof.* As in the proof of Corollary 3.4, by (3.1) and Lemmas 3.1 and 3.2 it suffices to show that, for a cube-full modulus $q$ with $\gcd(q,a_3) \leqslant C_\Delta$, the sum on the left is $O(q^{n/2+\varepsilon}(V^{n-1}q^{1/3} + q^{n/3}))$. Let $U = \min\{V, q^{1/3}\}$, so that $1 \leqslant U \leqslant V$. The set of the $\mathbf{m}'$ with $|\mathbf{m}' - \mathbf{m}_0'| \leqslant V$ is covered by at most $(2V/U)^{n-1}$ sets of the form $\{\mathbf{m}' : |\mathbf{m}' - \mathbf{c}'| \leqslant U\}$. For each of them we apply Lemma 3.3 to the cube $\{\mathbf{m} : |\mathbf{m} - (\mathbf{c}', m_n)| \leqslant U\}$ in $\mathbb{R}^n$. This gives the bound

$$(2V/U)^{n-1}\, q^{n/2+\varepsilon}\Big(U^n + q^{n/3}\Big).$$

If $q^{1/3} \leqslant V$, then $U = q^{1/3}$ and this is $O(q^{n/2+\varepsilon}V^{n-1}q^{1/3})$. If $q^{1/3} > V$, then $U = V$ and it is $O(q^{n/2+\varepsilon}q^{n/3})$. □

In the next two lemmas $q_1, \varrho_2 \in \mathbb{N}$ are such that every prime factor of $\varrho_2$ divides $q_1$. Let $\mathcal{A}$ be the set of primes $p \mid q_1$ with $v_p(\varrho_2) \geqslant 1 + \nu_p$, and put

$$w_0 = \prod_{p \in \mathcal{A}} p^{v_p(q_1\varrho_2)}, \qquad w_2 = \prod_{\substack{p \notin \mathcal{A} \\ v_p(q_1\varrho_2) \geqslant 3}} p^{v_p(q_1\varrho_2)}. \tag{8.1}$$

**Lemma 8.5.** *For all $\mathbf{m}_0' \in \mathbb{R}^{n-1}$, $m_n \in \mathbb{Z}$ and $V \geqslant 1$,*

$$\sum_{|\mathbf{m}'-\mathbf{m}_0'| \leqslant V} |S_{q_1,\varrho_2}(\mathbf{m}', m_n)| \ll (q_1\varrho_2)^{n/2+1+\varepsilon}\, w_0^{1/2}\Big(V^{n-1}w_2^{1/3} + w_2^{n/3}\Big).$$

*Proof.* Put $q = q_1\varrho_2$. By Lemma 3.6, $|S_{q_1,\varrho_2}(\mathbf{m})| \leqslant q_1^{-1}\sum_{b,a_1,a_2}|S(\mathbf{a},q;\mathbf{m})|$ with $\mathbf{a} = (a_1\varrho_2, a_2+b\varrho_2)$ and $\gcd(q, a_1\varrho_2, a_2+b\varrho_2) = 1$, and there are at most $q_1^2\varrho_2$ triples $(b,a_1,a_2)$. Since $(a_1,q_1)=1$ and every prime factor of $q$ divides $q_1$, we have $v_p(a_1\varrho_2) = v_p(\varrho_2)$ for $p \mid q$. So in the factorisation $q = q^{(0)}q^{(1)}q^{(2)}$ attached to $\mathbf{a}$ we have $q^{(0)} = w_0$ and $q^{(2)} = w_2$, for all triples. The claim follows from Lemma 8.4. □

**Lemma 8.6.** *Let $q_1 \in \mathbb{N}$, and let $w_0$ and $w_2$ be defined by (8.1). Then, the sums being over all $\varrho_2$ whose prime factors divide $q_1$,*

$$\sum_{\varrho_2}\varrho_2^{-3/2}w_0^{1/2}w_2^{1/3} \ll q_1^{\varepsilon}\,\kappa(q_1)^{1/2}, \qquad \sum_{\varrho_2}\varrho_2^{-3/2}w_0^{1/2}w_2^{n/3} \ll q_1^{\varepsilon}\,\kappa(q_1)^{n/3}.$$

*Proof.* Each of the two sums is a product over the primes $p \mid q_1$ of sums of local factors. Let $r = v_p(q_1) \geqslant 1$ and $s = v_p(\varrho_2) \geqslant 0$, and let $j = 1/3$ or $j = n/3$. If $s \geqslant 1+\nu_p$, the local factor is $p^{-3s/2}p^{(r+s)/2} = p^{r/2-s}$. If $s \leqslant \nu_p$, it is $p^{-3s/2}p^{(r+s)j}$ if $r+s \geqslant 3$, and $p^{-3s/2}$ otherwise. Suppose first that $p \nmid \Delta$ and $r \leqslant 2$. Then $\nu_p = 0$, the factor for $s = 0$ is 1, and the sum of the local factors over $s \geqslant 0$ is at most $1 + \sum_{s\geqslant 1}p^{1-s} \leqslant 3$. In the remaining case $p^r$ is the exact power of $p$ dividing $\kappa(q_1)$, and the sum over $s$ is at most

$$(\nu_p+1)\,p^{\nu_p j}\,p^{rj} + \sum_{s>\nu_p}p^{r/2-s} \leqslant c_p\,p^{r\max\{j,1/2\}},$$

where $c_p = 3$ for $p \nmid \Delta$, and $c_p$ depends only on $p$ and $n$ for $p \mid \Delta$. Since $3^{\omega(q_1)} \ll q_1^{\varepsilon}$, this proves the lemma. □

**Proposition 8.7.** *Let $Q_1, Q_2 \geqslant 1/2$ and $V \geqslant 1$, and for all $q_1, q_2$ let $\mathbf{m}_0(q_1,q_2) \in \mathbb{R}^n$. Then*

$$\sum_{Q_1<q_1\leqslant 2Q_1}\ \sum_{Q_2<q_2\leqslant 2Q_2}\ \sum_{\substack{\mathbf{m}\in\mathbb{Z}^n\\ |\mathbf{m}-\mathbf{m}_0(q_1,q_2)|\leqslant V}}|S_{q_1,q_2}(\mathbf{m})|$$
$$\ll (Q_1Q_2)^{n/2+\varepsilon}Q_1^2Q_2^{3/2}\,(Q_2+V)\Big\{V^{n-1}+Q_1^{n/3-2/3}\Big\}.$$

In contrast to Proposition 3.12, the centres $\mathbf{m}_0(q_1,q_2)$ are arbitrary. For $V \geqslant Q_2$ the bound is $V^n + VQ_1^{n/3-2/3}$ in place of the expression in braces in Proposition 3.12; for $V < Q_2$ it is weaker than that bound, by a factor of at most $Q_2^{1/2}$.

*Proof.* Let $q_1 \in (Q_1, 2Q_1]$ and $q_2 \in (Q_2, 2Q_2]$, and write $\mathbf{m}_0 = \mathbf{m}_0(q_1,q_2)$. Put

$$\varrho_2 = \prod_{p\mid q_1}p^{v_p(q_2)}, \qquad e = \frac{q_2}{\varrho_2}.$$

Then $(e, q_1\varrho_2) = 1$, and $S_{q_1,q_2}(\mathbf{m}) = S_{q_1,\varrho_2}(\mathbf{m})\,S_{1,e}(\mathbf{m})$ by Lemma 3.5. By Lemma 8.2 only the $\mathbf{m}$ with $q_2 \mid m_n$ contribute. The number of integers $m_n$ with $q_2 \mid m_n$ and $|m_n - m_{0,n}| \leqslant V$ is at most $1+2V/q_2 \leqslant 1+4V/Q_2$. For each of them we use Lemmas 8.3 and 8.5, with $w_0$ and $w_2$ as in (8.1). This gives

$$\sum_{|\mathbf{m}-\mathbf{m}_0|\leqslant V}|S_{q_1,q_2}(\mathbf{m})| \ll \Big(1+\frac{V}{Q_2}\Big)e^{(n+3)/2}(q_1\varrho_2)^{n/2+1+\varepsilon}\,w_0^{1/2}\Big(V^{n-1}w_2^{1/3}+w_2^{n/3}\Big).$$

We sum over $q_2$. For given $q_1$ and $\varrho_2$ the integer $e$ is at most $2Q_2/\varrho_2$, so that $\sum_e e^{(n+3)/2} \ll (Q_2/\varrho_2)^{(n+5)/2}$, and $\varrho_2^{n/2+1}\varrho_2^{-(n+5)/2} = \varrho_2^{-3/2}$. By Lemma 8.6,

$$\sum_{Q_2<q_2\leqslant 2Q_2}\ \sum_{|\mathbf{m}-\mathbf{m}_0(q_1,q_2)|\leqslant V}|S_{q_1,q_2}(\mathbf{m})|$$
$$\ll (Q_1Q_2)^{\varepsilon}\,q_1^{n/2+1}Q_2^{(n+3)/2}(Q_2+V)\Big\{\kappa(q_1)^{1/2}V^{n-1}+\kappa(q_1)^{n/3}\Big\}.$$

It remains to sum over $q_1$. As in the proof of Proposition 3.12, the number of $q_1 \leqslant 2Q_1$ with $q_1^\flat = \beta$ and $\mathrm{cf}(q_1^\sharp) = \gamma$ is at most $2Q_1/(\beta\gamma)$, and then $\kappa(q_1) = \beta\gamma$. Hence

$$\sum_{q_1} \kappa(q_1)^{1/2} \ll Q_1 \sum_{\beta} \beta^{-1/2} \sum_{\gamma} \gamma^{-1/2} \ll Q_1, \qquad \sum_{q_1} \kappa(q_1)^{n/3} \ll Q_1^{n/3+1/3},$$

where $\beta$ runs over the integers with $\beta \mid \Delta^\infty$ and $\gamma$ over the cube-full integers. This proves the proposition. □

8.3. **The major arcs.**

**Lemma 8.8.** *The statements of Lemma 5.2 hold with the error term $O(P^{(n+1)/2+8n\delta_1})$ and with the bound $S^*(a_1, q_1; a_2, q_2) \ll q_1^{2n} q_2^{(n+1)/2}$. Moreover, Proposition 5.3 holds.*

*Proof.* We follow the proof of Lemma 5.2. The sum over $\mathbf{l}$ is a product of $n$ sums. Those with $i < n$ are estimated as before, and the sum with $i = n$ has modulus at most $q_2$. Hence $\mathfrak{S}(\mathbf{m}) \ll q_1^{2n} q_2^{(n+1)/2}$ for all $\mathbf{m}$, and it remains to show that

$$L^n\, q_1^{2n} q_2^{(n+1)/2} \sum_{\mathbf{m}\neq\mathbf{0}} |J(\boldsymbol{\gamma}; L\mathbf{m})| \ll P^{(n+1)/2+8n\delta_1}.$$

If $q \leqslant P^{1-3\delta_1}$ and $|t_2| \leqslant P^{2\delta_1}$, the sum over $\mathbf{m}$ is $O(L^{-N})$, as before. If $q \leqslant P^{1-3\delta_1}$ and $|t_2| > P^{2\delta_1}$, we use Lemma 8.1. The number of the $\mathbf{m}$ that are not negligible is $O((|t_2|/L)^n)$, and each of them contributes $O(P^{-N} + P^\varepsilon |t_2|^{-(n-1)/2})$. So the left-hand side is $O(1 + P^\varepsilon q_1^{2n} (q_2|t_2|)^{(n+1)/2}) = O(P^{(n+1)/2+2n\delta_1+\varepsilon})$. If $q > P^{1-3\delta_1}$, the sum over $\mathbf{m}$ is $O(P^{5n\delta_1})$ as before, and the left-hand side is

$$\ll P^n q_1^n q_2^{(1-n)/2} P^{5n\delta_1} \ll P^{n+6n\delta_1-(n-1)(1-4\delta_1)/2} \ll P^{(n+1)/2+8n\delta_1}.$$

In the proof of Proposition 5.3 the arcs with $q_2 > P^{\delta_2}$ now contribute

$$\ll P^{n-2} q_1^n \sum_{q_2 > P^{\delta_2}} q_2^{(3-n)/2} \ll P^{n-2+n\delta_1-(n-5)\delta_2/2},$$

and the error terms contribute

$$\ll P^{3\delta_1-3}\Big(P^{n-2+n\delta_1-(n-5)\delta_2/2} + P^{(n+1)/2+8n\delta_1}\Big) = P^{n-5-(n-13)/40} + P^{n-5-(12n-103)/40},$$

which is $O(P^{n-5-3/8})$ for $n \geqslant 28$. The rest of the proof is unchanged; in particular the sum over the quadruples is $\mathfrak{S} + O(P^{-7\delta_1/6+\varepsilon})$ by Lemma 5.1(i). □

8.4. **The nested $\delta$-symbol.** Lemma 6.1 and the bound (6.1) hold as before. We define the blocks $\mathcal{B}(Q_1, T)$ and the quantities $\mathcal{E}_{\mathrm{res}}$, $\mathcal{E}_{\mathrm{non}}$ and $\mathcal{E}_{\mathrm{tail}}$ as in §6, except that $|\mathbf{u}|^{1-n/2}$ is replaced by $|\mathbf{u}|^{3/2-n/2}$ in the definition of $\mathcal{E}_{\mathrm{res}}$. By the remark following Lemma 8.1, the contribution of $\mathcal{B}(Q_1, T)$ to the right-hand side of (6.1) is again $O(\mathcal{E}_{\mathrm{res}} + \mathcal{E}_{\mathrm{non}} + \mathcal{E}_{\mathrm{tail}} + P^{-N})$. The next three propositions correspond to Propositions 6.5, 6.6 and 6.7. In their proofs Proposition 8.7 takes the place of Proposition 3.12, and we use the notation and the conventions introduced before Proposition 6.5.

**Proposition 8.9.** *Let $Q_1, T$ satisfy (6.3). Then*

$$\mathcal{E}_{\mathrm{res}} \ll P^\varepsilon \Big\{ P^{\frac{n}{2}-\frac{7}{2}} Q_1^{\frac{n+9}{4}} T^{\frac{n+7}{4}} + P^{\frac{n}{2}-\frac{5}{2}} (Q_1 T)^{\frac{n+5}{4}} + P^{\frac{5n}{6}-\frac{8}{3}} \Big\}.$$

*Proof.* We proceed as in the proof of Proposition 6.5. With the shells $M' < |\mathbf{m}| \leqslant 2M'$ as before, we have $|\mathbf{u}|^{3/2-n/2} \ll (PM'/(Q_1Q_2))^{3/2-n/2}$ on a shell, and Proposition 8.7 with $V = 2M'$ gives

$$\begin{aligned}\mathcal{E}_{\mathrm{res}} &\ll P^{n-3+\varepsilon}\cdot\frac{Q_1T}{P^2}\sum_{Q_2,M'}\frac{(Q_1Q_2)^{\frac{n}{2}}Q_1^2Q_2^{\frac{3}{2}}}{(Q_1Q_2)^n}\Big(\frac{PM'}{Q_1Q_2}\Big)^{\frac{3}{2}-\frac{n}{2}}\\ &\qquad\qquad\times(Q_2+M')\Big\{M'^{n-1}+Q_1^{\frac{n}{3}-\frac{2}{3}}\Big\}\\ &= P^{\frac{n}{2}-\frac{7}{2}+\varepsilon}Q_1T\sum_{Q_2,M'}\Big\{Q_1^{\frac{1}{2}}M'^{\frac{n}{2}+\frac{3}{2}}+Q_1^{\frac{1}{2}}Q_2M'^{\frac{n}{2}+\frac{1}{2}}\\ &\qquad\qquad+Q_1^{\frac{n}{3}-\frac{1}{6}}Q_2M'^{\frac{3}{2}-\frac{n}{2}}+Q_1^{\frac{n}{3}-\frac{1}{6}}M'^{\frac{5}{2}-\frac{n}{2}}\Big\}.\end{aligned}$$

The first two terms are largest for $M' \asymp M_0$ and $Q_2 \asymp R_0$, and give

$$P^{\frac{n}{2}-\frac{7}{2}+\varepsilon}Q_1^{\frac{n+9}{4}}T^{\frac{n+7}{4}} \qquad\text{and}\qquad P^{\frac{n}{2}-\frac{5}{2}+\varepsilon}Q_1^{\frac{n+5}{4}}T^{\frac{n+3}{4}}.$$

In the last two terms let $\xi = TQ_1/P$, and recall (6.5). If $\xi \leqslant 1$, the sum of $Q_2M'^{3/2-n/2}$ over $Q_2$ and $M'$ is $O(\xi^{-1})$, and the sum of $M'^{5/2-n/2}$ is $O(P^{\varepsilon})$. The two terms then contribute $O(P^{n/2-5/2+\varepsilon}Q_1^{n/3-1/6})$, which is $O(P^{5n/6-8/3+\varepsilon})$ since $Q_1 \leqslant P$. If $\xi > 1$, the two sums are $O(\xi^{3/2-n/2})$ and $O(\xi^{5/2-n/2})$, and the two terms contribute

$$\ll P^{\frac{n}{2}-\frac{7}{2}+\varepsilon}Q_1^{\frac{n}{3}+\frac{5}{6}}T\Big(\frac{P}{TQ_1}\Big)^{\frac{n}{2}-\frac{5}{2}} = P^{n-6+\varepsilon}Q_1^{\frac{10}{3}-\frac{n}{6}}T^{\frac{7}{2}-\frac{n}{2}} \leqslant P^{n-6+\varepsilon}(Q_1T)^{\frac{10}{3}-\frac{n}{6}} < P^{\frac{5n}{6}-\frac{8}{3}+\varepsilon}.$$

This completes the proof. □

**Proposition 8.10.** *Let $Q_1, T$ satisfy (6.3). Then $\mathcal{E}_{\mathrm{non}} \ll P^{\varepsilon}(\mathcal{E}'_{\alpha}+\mathcal{E}'_{\beta})$, where*

$$\begin{aligned}\mathcal{E}'_{\alpha} &= P^{\frac{n}{2}-\frac{7}{2}}Q_1^{\frac{n+9}{4}}T^{\frac{n+7}{4}}+P^{\frac{n}{2}-\frac{5}{2}}(Q_1T)^{\frac{n+5}{4}}+\mathcal{D},\\ \mathcal{D} &= P^{n-5}Q_1^{\frac{7}{3}-\frac{n}{6}}T^{\frac{5}{2}}\max\Big\{1,\frac{P}{T^{1/2}Q_1}\Big\}^{\frac{5}{2}-\frac{n}{2}}\max\Big\{1,\frac{T^{1/2}Q_1}{P}\Big\},\\ \mathcal{E}'_{\beta} &= P^{n-6}\,T^3\sup_{Q_2}\Big\{Q_1^{4-\frac{n}{2}}Q_2^{\frac{7}{2}-\frac{n}{2}}+Q_1^{\frac{10}{3}-\frac{n}{6}}Q_2^{\frac{7}{2}-\frac{n}{2}}\Big\},\end{aligned}$$

*the supremum being over $Q_2 \geqslant 1/2$ with $Q_1Q_2T \geqslant P^{1-\varepsilon}$.*

*Proof.* We proceed as in the proof of Proposition 6.6. The argument leading to (6.7) and to the bound for the number of balls is unchanged. Proposition 8.7 is applied to each ball, and we obtain

$$\mathcal{E}_{\mathrm{non}} \ll P^{n-5+\varepsilon}Q_1T^2\sum_{Q_2}\Big(1+\min\Big\{\frac{TQ_1Q_2}{P},T^{\frac{1}{2}}\Big\}\Big)Q_1^{2-\frac{n}{2}}Q_2^{\frac{3}{2}-\frac{n}{2}}(Q_2+V)\Big\{V^{n-1}+Q_1^{\frac{n}{3}-\frac{2}{3}}\Big\},$$

with the same range of $Q_2$ and the same $V$ as before. Consider first the $Q_2$ with $T^{1/2}Q_1Q_2 > P$, for which $V \ll P^{\varepsilon}T^{1/2}Q_1Q_2/P$. The terms $V^n$, $Q_2V^{n-1}$, $Q_2Q_1^{n/3-2/3}$ and $VQ_1^{n/3-2/3}$ contribute

$$\begin{aligned}&P^{-5+\varepsilon}Q_1^{3+\frac{n}{2}}T^{\frac{5}{2}+\frac{n}{2}}Q_2^{\frac{3}{2}+\frac{n}{2}}, \qquad P^{-4+\varepsilon}Q_1^{2+\frac{n}{2}}T^{2+\frac{n}{2}}Q_2^{\frac{3}{2}+\frac{n}{2}},\\ &P^{n-5+\varepsilon}Q_1^{\frac{7}{3}-\frac{n}{6}}T^{\frac{5}{2}}Q_2^{\frac{5}{2}-\frac{n}{2}}, \qquad P^{n-6+\varepsilon}Q_1^{\frac{10}{3}-\frac{n}{6}}T^3Q_2^{\frac{5}{2}-\frac{n}{2}}.\end{aligned}$$

The first two are largest for $Q_2 \asymp R_0$, where they are the first two terms of $\mathcal{E}'_{\alpha}$. The last two are largest for the smallest admissible $Q_2$, which is $\asymp \max\{1, P/(T^{1/2}Q_1)\}$; their

sum is then $O(P^{\varepsilon}\mathcal{D})$. For the $Q_2$ with $T^{1/2}Q_1Q_2 \leqslant P$ we have $V \ll P^{\varepsilon}$, the number of balls is $O(P^{2\varepsilon}TQ_1Q_2/P)$, and the contribution is

$$\ll P^{n-6+\varepsilon}Q_1^{4-\frac{n}{2}}T^3Q_2^{\frac{5}{2}-\frac{n}{2}}\Big\{Q_2+Q_2Q_1^{\frac{n}{3}-\frac{2}{3}}\Big\} \ll P^{\varepsilon}\mathcal{E}'_{\beta}.$$

This completes the proof. □

**Proposition 8.11.** *Let $Q_1, T$ satisfy (6.3). If $T \geqslant P^{\varepsilon}$, then $\mathcal{E}_{\mathrm{tail}} \ll P^{-N}$. If $T < P^{\varepsilon}$, then*

$$\mathcal{E}_{\mathrm{tail}} \ll P^{\varepsilon}\Big\{P^{\frac{n}{2}-\frac{7}{2}}Q_1^{\frac{n+9}{4}} + P^{\frac{n}{2}-\frac{5}{2}}Q_1^{\frac{n+5}{4}} + P^{n-5}Q_1^{\frac{7}{3}-\frac{n}{6}} + P^{\frac{5n}{6}-\frac{8}{3}}\Big\}.$$

*Proof.* As in the proof of Proposition 6.7, $\mathcal{E}_{\mathrm{tail}}$ is negligible if $T \geqslant P^{\varepsilon}$. Let $T < P^{\varepsilon}$. With $V = \max\{1, 8P^{\varepsilon}TQ_1Q_2/P\}$ we have

$$\mathcal{E}_{\mathrm{tail}} \ll P^{n-5+\varepsilon}Q_1 \sum_{Q_2 \leqslant 4R_0} Q_1^{2-\frac{n}{2}}Q_2^{\frac{3}{2}-\frac{n}{2}}(Q_2+V)\Big\{V^{n-1}+Q_1^{\frac{n}{3}-\frac{2}{3}}\Big\}.$$

For the $Q_2$ with $Q_1Q_2 > P$ the terms $V^n$ and $Q_2V^{n-1}$ contribute

$$O\Big(P^{-5+\varepsilon}Q_1^{3+n/2}Q_2^{3/2+n/2}\Big) \qquad \text{and} \qquad O\Big(P^{-4+\varepsilon}Q_1^{2+n/2}Q_2^{3/2+n/2}\Big);$$

with $Q_2 \ll PQ_1^{-1/2}$ these are the first two terms of the proposition. For the $Q_2$ with $Q_1Q_2 \leqslant P$ the same terms contribute $O(P^{n-5+\varepsilon}Q_1^{3-n/2})$, which is $O(P^{n-5+\varepsilon}Q_1^{7/3-n/6})$. The terms with $Q_1^{n/3-2/3}$ contribute

$$\ll P^{n-5+\varepsilon}Q_1^{\frac{7}{3}-\frac{n}{6}}\sum_{Q_2} Q_2^{\frac{3}{2}-\frac{n}{2}}(Q_2+V) \ll P^{n-5+\varepsilon}Q_1^{\frac{7}{3}-\frac{n}{6}}\Big(1+\frac{Q_1}{P}\Big),$$

and $P^{n-6}Q_1^{10/3-n/6} \leqslant P^{5n/6-8/3}$ if $Q_1 > P$. □

The next three lemmas take the place of Lemmas 6.9, 6.11 and 6.12. In their proofs we use the notation of the proofs of Lemmas 6.8–6.12.

**Lemma 8.12.** *Suppose that (6.8) holds. Then*

$$P^{\frac{n}{2}-\frac{5}{2}}(Q_1T)^{\frac{n+5}{4}} \ll P^{n-5-1/2+(11/2)\eta_0}.$$

*Proof.* Let $\lambda'' = 2(n+5)/(n-16) = 2 + 42/(n-16)$. In $Z^{\lambda''}$ the exponent of $Q_1$ is $(n+5)/4$ and the exponent of $T$ is $(n+5)(n-8)/(4(n-16)) \geqslant (n+5)/4$. Hence $(Q_1T)^{(n+5)/4} \leqslant Z^{\lambda''} < P^{\lambda''(2+\eta_0)}$. Now $\lambda'' \leqslant 11/2$, and $n/2 - 5/2 + 2\lambda'' = n - 5 + \{13/2 - n/2 + 84/(n-16)\}$, where the expression in braces is decreasing in $n$ and equals $-1/2$ for $n = 28$. □

**Lemma 8.13.** *Suppose that (6.8) holds, and let $\mathcal{D}$ be as in Proposition 8.10. Then*

$$\mathcal{D} \ll P^{n-5-9/7+(44/21)\eta_0}.$$

*Proof.* Suppose that $T^{1/2}Q_1 \geqslant P$. Then $\mathcal{D} = P^{n-6}Q_1^{10/3-n/6}T^3$ and, as in the proof of Lemma 6.11, $T < P^{\theta}$ with $\theta \leqslant (2/7)(1+2\eta_0)$. Since $Q_1 \geqslant PT^{-1/2} \geqslant 1$ and $n/6 - 10/3 \geqslant 4/3$,

$$\mathcal{D} \leqslant P^{n-6}(PT^{-1/2})^{-4/3}T^3 = P^{n-5-7/3}T^{11/3},$$

which is the expression estimated there. Suppose that $T^{1/2}Q_1 < P$. Then, with $x$ and $y$ as in the proof of Lemma 6.11,

$$\mathcal{D} = P^{\frac{n}{2}-\frac{5}{2}}Q_1^{\frac{n}{3}-\frac{1}{6}}T^{\frac{n}{4}+\frac{5}{4}} = P^{\frac{n}{2}-\frac{5}{2}}(Q_1T^{1/2})^{x-\frac{1}{2}}Z^y < P^{\frac{n}{2}-3+x+y(2+\eta_0)},$$

since $x \geqslant 1/2$. This is the bound obtained there. □

**Lemma 8.14.** *Suppose that (6.8) holds, and let $Q_2 \geqslant 1/2$ with $Q_1Q_2T \geqslant P^{1-\varepsilon}$. Then*

$$P^{n-6}T^3Q_1^{4-\frac{n}{2}}Q_2^{\frac{7}{2}-\frac{n}{2}} \ll P^{n-5-3/5+(26/5)\eta_0+n\varepsilon},$$
$$P^{n-6}T^3Q_1^{\frac{10}{3}-\frac{n}{6}}Q_2^{\frac{7}{2}-\frac{n}{2}} \ll P^{n-5-1/6+(13/3)\eta_0+n\varepsilon}.$$

*Proof.* Since $Q_2 \geqslant 1/2$, we have $Q_2^{7/2-n/2} \ll Q_2^{4-n/2}$, so that the first expression is $O(P^{n-6}T^3(Q_1Q_2)^{4-n/2})$, which was estimated in the proof of Lemma 6.12. Let $E'$ be the second expression. If $Q_1T > P$, then $E' \ll P^{n-6-\kappa}T^{3+\kappa}$, and the argument given there applies. Suppose that $Q_1T \leqslant P$. Since $Q_2 \geqslant P^{1-\varepsilon}/(Q_1T)$, we have $E' \leqslant P^{n/2-5/2+n\varepsilon}\,T^{(n-1)/2}Q_1^{n/3-1/6}$. If $n \leqslant 32$, then with $\lambda_1$ and $\lambda_2$ as there we have $\lambda_1 \geqslant 1$ and

$$T^{(n-1)/2}Q_1^{n/3-1/6} = (Q_1T)^{\lambda_1-1/2}Z^{\lambda_2} \leqslant P^{\lambda_1-1/2+\lambda_2(2+\eta_0)},$$

so that the exponent obtained is again $n-5+G(n)+(n-2)\eta_0/6+n\varepsilon$. If $n \geqslant 33$, then $T^{(n-1)/2}Q_1^{n/3-1/6} \ll Z^{4(n-1)/(n-8)}$, because $(n-1)(n-16)/(2(n-8)) \geqslant n/3-1/6$ for these $n$, and the exponent obtained is $n/2-5/2+n\varepsilon+(4(n-1)/(n-8))(2+\eta_0) < n-5-3+n\varepsilon$. We conclude as in the proof of Lemma 6.12. □

**Proposition 8.15.** *We have $N_\delta \ll P^{n-5-1/43+\varepsilon}$.*

*Proof.* We argue as in the proof of Proposition 6.13, with $\eta_0 = 1/43$. The first term in Propositions 8.9 and 8.10 is the left-hand side of Lemma 6.8, and the term $P^{5n/6-8/3}$ is at most $P^{n-5-7/3}$. The remaining terms of $\mathcal{E}_{\mathrm{res}}$, $\mathcal{E}'_\alpha$ and $\mathcal{E}'_\beta$ are estimated by Lemmas 8.12, 8.13 and 8.14. The exponents obtained are $n-5-\sigma$ with

$$\sigma = \frac{1}{6}-\frac{37}{258},\qquad \frac{1}{2}-\frac{11}{86},\qquad \frac{7}{3},\qquad \frac{9}{7}-\frac{44}{903},\qquad \frac{3}{5}-\frac{26}{215},\qquad \frac{1}{6}-\frac{26}{258},$$

and the smallest of these numbers is the first, which is equal to $1/43$. If $T < P^\varepsilon$, then $Q_1 > \frac{1}{2}P^{\delta_1}$, the first two terms of Proposition 8.11 are bounded by Lemmas 6.8 and 8.12 with $T = 1$, and the third is $O(P^{n-5-7/120})$. □

## 8.5. Proof of Theorems 1.1 and 1.2 for $d_n = 0$.

**Lemma 8.16.** *For every prime $p$ there is $\mathbf{x} \in \mathbb{Q}_p^n \setminus \{\mathbf{0}\}$ with $C(\mathbf{x}) = Q(\mathbf{x}) = 0$.*

*Proof.* The form $Q$ is a non-degenerate quadratic form in the $n-1 \geqslant 27$ variables $x_1, \dots, x_{n-1}$. As in the proof of Lemma 7.1, it is equivalent over $\mathbb{Q}_p$ to the orthogonal sum of at least twelve hyperbolic planes and a form in at most four variables. So $Q$ vanishes on a linear subspace of $\mathbb{Q}_p^n$ of dimension 12, and the theorem of Lewis applies to the restriction of $C$ to this subspace. □

The proof of Lemma 7.2 uses the forms $C$ and $Q$ only through (3.1), Lemma 5.1(i), the existence of a non-trivial zero in $\mathbb{Q}_p^n$ and the non-singularity of $X$. With Lemma 8.16 in place of Lemma 7.1, it shows that $\mathfrak{S} = \prod_p \sigma_p$ and $\mathfrak{S} > 0$ in the present case as well. By (4.1), (4.2), Proposition 4.5, Lemma 8.8 and Proposition 8.15,

$$N_w(P) = \mathfrak{S}\,\mathfrak{J}\,P^{n-5} + O\Big(P^{n-5-1/43+\varepsilon}\Big),$$

and $\mathfrak{J} > 0$ by Lemma 5.1(ii). This proves Theorem 1.2 for $d_n = 0$. Theorem 1.1 follows as in §7.

*Remark* 8.17. We have checked the bounds of Propositions 8.9, 8.10 and 8.11, with $\eta_0 = 1/43$, by solving the corresponding linear programmes in exact arithmetic for $28 \leqslant n \leqslant 200$; again this is a check only, and it is not used in the proofs. The smallest

saving is again $1/43$, for $n = 28$, and it comes from the first term of $\mathcal{E}_{\mathrm{res}}$ and $\mathcal{E}'_\alpha$, which is the same as in the case of rank $n$.

## 9. Weak approximation

In this section we prove Theorem 1.3. We keep the assumptions of Theorem 1.2: $n \geqslant 28$, the forms $C$ and $Q$ are as in (1.1), $X$ is non-singular and $X(\mathbb{R}) \neq \emptyset$. Both cases $d_n \neq 0$ and $d_n = 0$ are allowed; we give the argument for $d_n \neq 0$, and indicate at the appropriate places what changes for $d_n = 0$. The weight $w$ is as in §2.2 (with the additional condition of §8 if $d_n = 0$). We recall that the point $\mathbf{z}_0$ in its construction may be any real point with $0 < |\mathbf{z}_0| \leqslant 1/4$ and $C(\mathbf{z}_0) = Q(\mathbf{z}_0) = 0$, since every point of $X(\mathbb{R})$ is non-singular, and that $\rho_0$ may be replaced by any smaller positive number, since (U1)–(U3) remain valid for a smaller ball. We fix $H \in \mathbb{N}$. In this section the implied constants are allowed to depend on $H$, and $P$ is assumed to be sufficiently large in terms of $H$.

9.1. **The asymptotic formula with a congruence condition.** For $\mathbf{b} \in \mathbb{Z}^n$ let

$$N_w(P; \mathbf{b}, H) = \sum_{\substack{\mathbf{x} \in \mathbb{Z}^n,\ \mathbf{x} \equiv \mathbf{b} \bmod H \\ C(\mathbf{x}) = Q(\mathbf{x}) = 0}} w(\mathbf{x}/P).$$

For a prime $p$ put $h_p = v_p(H)$, and for $k \geqslant h_p$ let

$$\lambda_{\mathbf{b}}(p^k) = \#\Big\{\mathbf{x} \bmod p^k :\ \mathbf{x} \equiv \mathbf{b} \pmod{p^{h_p}},\ C(\mathbf{x}) \equiv Q(\mathbf{x}) \equiv 0 \pmod{p^k}\Big\}.$$

If $p \nmid H$ this is the number $\lambda(p^k)$ of §7.

**Theorem 9.1.** *Let $\mathbf{b} \in \mathbb{Z}^n$. The limits $\sigma_p(\mathbf{b}) = \lim_{k\to\infty} p^{-k(n-2)}\lambda_{\mathbf{b}}(p^k)$ exist, the product $\mathfrak{S}(\mathbf{b}) = \prod_p \sigma_p(\mathbf{b})$ converges absolutely, and for every $\varepsilon > 0$*

$$N_w(P; \mathbf{b}, H) = \mathfrak{S}(\mathbf{b})\,\mathfrak{J}\,P^{n-5} + O_\varepsilon\Big(P^{n-5-1/43+\varepsilon}\Big).$$

*Suppose that for every prime $p \mid H$ there is $\mathbf{y} \in \mathbb{Z}_p^n \setminus \{\mathbf{0}\}$ with $\mathbf{y} \equiv \mathbf{b} \pmod{p^{h_p}}$ and $C(\mathbf{y}) = Q(\mathbf{y}) = 0$. Then $\mathfrak{S}(\mathbf{b}) > 0$.*

9.2. **Additive characters.** For $\boldsymbol{\theta} \in \mathbb{R}^n$ we recall the sum $S(\alpha, \beta; \boldsymbol{\theta})$ of §4, and we define

$$\mathcal{H}(\alpha; \boldsymbol{\theta}) = \sum_{\substack{\mathbf{x} \in \mathbb{Z}^n \\ Q(\mathbf{x}) = 0}} w(\mathbf{x}/P)\, e\Big(\alpha C(\mathbf{x}) + \boldsymbol{\theta} \cdot \mathbf{x}\Big) = \int_0^1 S(\alpha, \beta; \boldsymbol{\theta})\, d\beta,$$

$$N_w(P; \boldsymbol{\theta}) = \sum_{\substack{\mathbf{x} \in \mathbb{Z}^n \\ C(\mathbf{x}) = Q(\mathbf{x}) = 0}} w(\mathbf{x}/P)\, e(\boldsymbol{\theta} \cdot \mathbf{x}).$$

By the orthogonality of the additive characters modulo $H$,

$$N_w(P; \mathbf{b}, H) = \frac{1}{H^n} \sum_{\mathbf{c} \bmod H} e_H(-\mathbf{b} \cdot \mathbf{c})\, N_w(P; \mathbf{c}/H). \tag{9.1}$$

From now on we fix $\boldsymbol{\theta} = \mathbf{c}/H$ with $\mathbf{c} \in \mathbb{Z}^n$. We have $N_w(P; \boldsymbol{\theta}) = \sum_{Q(\mathbf{x})=0} w(\mathbf{x}/P) e(\boldsymbol{\theta} \cdot \mathbf{x}) \delta(C(\mathbf{x}))$, and the proof of Lemma 4.1 gives the formula of that lemma with $N_w(P; \boldsymbol{\theta})$ and $\mathcal{H}(\alpha; \boldsymbol{\theta})$ in place of $N_w(P)$ and $\mathcal{H}(\alpha)$. With the four sets of §4 we write, as in (4.1),

$$N_w(P; \boldsymbol{\theta}) = N_\infty(\boldsymbol{\theta}) + N_{\mathfrak{M}}(\boldsymbol{\theta}) + N_W(\boldsymbol{\theta}) + N_\delta(\boldsymbol{\theta}).$$

Since $|\mathcal{H}(\alpha; \boldsymbol{\theta})| \leqslant \sum_{\mathbf{x}} w(\mathbf{x}/P)$, the bound (4.2) holds for $N_\infty(\boldsymbol{\theta})$.

9.3. **The Weyl region.**

**Lemma 9.2.** *We have* $N_W(\boldsymbol{\theta}) \ll P^{n-5-\eta_0+\varepsilon}$.

*Proof.* Lemma 4.4 holds uniformly in $\boldsymbol{\theta} \in \mathbb{R}^n$, and $|\mathcal{H}(\alpha;\boldsymbol{\theta})| \leqslant \sup_\beta |S(\alpha,\beta;\boldsymbol{\theta})|$. The proof of Proposition 4.5 therefore applies without change. □

9.4. **The major arcs.** For $q \in \mathbb{N}$ put

$$A(q;\boldsymbol{\theta}) = \mathbf{1}[q\boldsymbol{\theta} \in \mathbb{Z}^n]\; q^{-n} \sum_{\substack{\mathbf{a} \bmod q \\ \gcd(q,a_3,a_2)=1}} S(\mathbf{a},q;q\boldsymbol{\theta}), \qquad \mathfrak{S}(\boldsymbol{\theta}) = \sum_{q=1}^{\infty} A(q;\boldsymbol{\theta}).$$

By Corollary 3.4 with $\mathbf{m}_0 = \mathbf{m}$ and $V = 1$, the bound for $S(\mathbf{a},q;\mathbf{0})$ that is used in the proof of Lemma 5.1(i) holds for $S(\mathbf{a},q;\mathbf{m})$, uniformly in $\mathbf{m} \in \mathbb{Z}^n$. That proof therefore gives, for $Y \geqslant 1$,

$$\sum_{q>Y} q^{-n} \sum_{\substack{\mathbf{a} \bmod q \\ \gcd(q,a_3,a_2)=1}} \max_{\mathbf{m}\in\mathbb{Z}^n} |S(\mathbf{a},q;\mathbf{m})| \ll Y^{7/6-n/12+\varepsilon}. \tag{9.2}$$

In particular the series $\mathfrak{S}(\boldsymbol{\theta})$ converges absolutely.

**Lemma 9.3.** *We have* $N_{\mathfrak{M}}(\boldsymbol{\theta}) = \mathfrak{S}(\boldsymbol{\theta})\,\mathfrak{J}\,P^{n-5} + O(P^{n-5-1/40})$.

*Proof.* We go through the proofs of Lemma 5.2 and Proposition 5.3. With the notation used there, Poisson summation is now applied to a function that carries the additional factor $e(\boldsymbol{\theta}\cdot\mathbf{x})$, and gives

$$S(\alpha,\beta;\boldsymbol{\theta}) = L^n \sum_{\mathbf{m}\in\mathbb{Z}^n} \mathfrak{S}(\mathbf{m})\, J\Big(\boldsymbol{\gamma}; L(\mathbf{m}-q\boldsymbol{\theta})\Big),$$

with the same sums $\mathfrak{S}(\mathbf{m})$, for which $\mathfrak{S}(\mathbf{m}) \ll q_1^{2n} q_2^{n/2}$. A term with $\mathbf{m} = q\boldsymbol{\theta}$ occurs only if $q\boldsymbol{\theta} \in \mathbb{Z}^n$. For all other $\mathbf{m}$ we have $|\mathbf{m} - q\boldsymbol{\theta}| \geqslant 1/H$, because $H(\mathbf{m}-q\boldsymbol{\theta})$ is a non-zero integer vector. The proof of (5.1) uses the vectors $L\mathbf{m}$ with $\mathbf{m} \neq \mathbf{0}$ only through the bound $|\mathbf{m}| \geqslant 1$ and through the number of $\mathbf{m}$ with $|L\mathbf{m}|$ in a given range. It therefore applies to the vectors $L(\mathbf{m}-q\boldsymbol{\theta})$ with $\mathbf{m} \neq q\boldsymbol{\theta}$, if $P$ is large enough in terms of $H$. Hence the formula of Lemma 5.2 holds for $S(\alpha,\beta;\boldsymbol{\theta})$, with $S^*(a_1,q_1;a_2,q_2)$ replaced by

$$S^*(a_1,q_1;a_2,q_2;\boldsymbol{\theta}) = \mathbf{1}[q_1q_2\boldsymbol{\theta} \in \mathbb{Z}^n] \sum_{\mathbf{k} \bmod q_1q_2} \psi(\mathbf{k}),$$

$$\psi(\mathbf{k}) = e\Big(\frac{a_1C(\mathbf{k})}{q_1} + \frac{a_2Q(\mathbf{k})}{q_2} + \boldsymbol{\theta}\cdot\mathbf{k}\Big),$$

which is again $O(q_1^{2n}q_2^{n/2})$. The proof of Proposition 5.3 now shows that

$$N_{\mathfrak{M}}(\boldsymbol{\theta}) = P^{n-5}\Big(\mathfrak{J} + O(P^{-N})\Big) \sum_{q_1\leqslant P^{\delta_1}} \sum_{q_2\leqslant P^{\delta_2}} \sideset{}{^*}\sum_{a_1 \bmod q_1} \sideset{}{^*}\sum_{a_2 \bmod q_2} \frac{S^*(a_1,q_1;a_2,q_2;\boldsymbol{\theta})}{(q_1q_2)^n} + O\Big(P^{n-5-17/40}\Big).$$

Let $(q,\mathbf{a})$ correspond to $(q_1,a_1,q_2,a_2)$ as in that proof, so that $q = \operatorname{lcm}(q_1,q_2)$. The function $\psi$ on $\mathbb{Z}^n$ satisfies $\psi(\mathbf{k}+q\mathbf{j}) = \psi(\mathbf{k})e(q\boldsymbol{\theta}\cdot\mathbf{j})$ for $\mathbf{j} \in \mathbb{Z}^n$. Suppose that $q_1q_2\boldsymbol{\theta} \in \mathbb{Z}^n$. Then $q_1q_2$ is a period of $\psi$, and $(q_1q_2)^{-n}\sum_{\mathbf{k} \bmod q_1q_2}\psi(\mathbf{k})$ is the mean value of $\psi$. If $q\boldsymbol{\theta} \in \mathbb{Z}^n$, this mean value is $q^{-n}S(\mathbf{a},q;q\boldsymbol{\theta})$. If $q\boldsymbol{\theta} \notin \mathbb{Z}^n$, it vanishes, because $\mathbf{j} \mapsto e(q\boldsymbol{\theta}\cdot\mathbf{j})$

is then a non-trivial character of $(\mathbb{Z}/(q_1q_2/q)\mathbb{Z})^n$. Since $q\boldsymbol{\theta} \in \mathbb{Z}^n$ implies $q_1q_2\boldsymbol{\theta} \in \mathbb{Z}^n$, we conclude that in all cases

$$\frac{S^*(a_1, q_1; a_2, q_2; \boldsymbol{\theta})}{(q_1q_2)^n} = \mathbf{1}[q\boldsymbol{\theta} \in \mathbb{Z}^n]\, q^{-n} S(\mathbf{a}, q; q\boldsymbol{\theta}).$$

The quadruples that are missing from the sum have $q > P^{\delta_1}$. By (9.2) the sum is $\mathfrak{S}(\boldsymbol{\theta}) + O(P^{-7\delta_1/6+\varepsilon})$, and the lemma follows.

If $d_n = 0$, the same argument applies to the proof of Lemma 8.8, with the bounds $O(q_1^{2n} q_2^{(n+1)/2})$ for $\mathfrak{S}(\mathbf{m})$ and for $S^*(a_1, q_1; a_2, q_2; \boldsymbol{\theta})$, and with the error term $O(P^{n-5-3/8})$ in the last display but one. □

9.5. **The nested $\delta$-symbol.** For $\mathbf{v} \in \mathbb{R}^n$ we define $I_{q_1,q_2}(t; \mathbf{v})$ by the formula of Lemma 6.1, with $\mathbf{v}$ in place of $\mathbf{m}$.

**Lemma 9.4.** *Let $(q_1, t) \in \mathcal{R}_\delta$. Then*

$$\sum_{a_1 \bmod q_1}^{*} \mathcal{H}\Big(\frac{a_1}{q_1} + \frac{t}{P^3}; \boldsymbol{\theta}\Big) = \frac{c_R\, P^n}{R^2} \sum_{q_2 \leqslant 4R} \frac{1}{(q_1q_2)^n} \sum_{\mathbf{m} \in \mathbb{Z}^n} S_{q_1,q_2}(\mathbf{m})\, I_{q_1,q_2}(t; \mathbf{m} - q_1q_2\boldsymbol{\theta}).$$

*Moreover, Lemmas 6.2, 6.3 and 6.4 hold for $I_{q_1,q_2}(t; \mathbf{v})$ with any $\mathbf{v} \in \mathbb{R}^n$ in place of $\mathbf{m}$, where now $\mathbf{u} = P\mathbf{v}/(q_1q_2)$.*

*Proof.* The proof of Lemma 6.1 applies, with the factor $e(\boldsymbol{\theta} \cdot \mathbf{x})$ attached to the smooth function. The Fourier transform of $\mathbf{x} \mapsto w(\mathbf{x}/P)e(tC(\mathbf{x}/P))g_{q_2/R}(\Theta Q(\mathbf{x}/P))e(\boldsymbol{\theta} \cdot \mathbf{x})$ at $\mathbf{m}/(q_1q_2)$ is $P^n I_{q_1,q_2}(t; \mathbf{m} - q_1q_2\boldsymbol{\theta})$, and the complete sums are not affected. In the proofs of Lemmas 6.2, 6.3 and 6.4 it is not used that $\mathbf{m}$ is an integer vector. □

The centre of the range of $\mathbf{m}$ is now $q_1q_2\boldsymbol{\theta}$, which depends on $q_2$. If $d_n = 0$ this causes no difficulty, since Proposition 8.7 allows arbitrary centres. If $d_n \neq 0$ we need the following variant of Proposition 3.12.

**Proposition 9.5.** *Suppose that $d_n \neq 0$. Let $Q_1, Q_2 \geqslant 1/2$ and $V \geqslant 1$, and for each $q_1$ let $\mathbf{m}_0(q_1) \in \mathbb{R}^n$ with $|\mathbf{m}_0(q_1)| \leqslant V_0$. Then the bound of Proposition 3.12 holds for the sum*

$$\sum_{Q_1 < q_1 \leqslant 2Q_1} \sum_{Q_2 < q_2 \leqslant 2Q_2} \sum_{\substack{\mathbf{m} \in \mathbb{Z}^n \\ |\mathbf{m} - \mathbf{m}_0(q_1) - q_1q_2\boldsymbol{\theta}| \leqslant V}} |S_{q_1,q_2}(\mathbf{m})|.$$

*Proof.* We follow the proof of Proposition 3.12. Fix $q_1$, write $\mathbf{m}_0 = \mathbf{m}_0(q_1)$, and define $g_2$, $\varrho_2$, $e$, $g_1$ and $\varrho_1$ as there. We fix $g_2$ and $\varrho_2$ and a residue class $e_0$ modulo $H$, and consider the $q_2 = g_2\varrho_2 e$ with $e \equiv e_0 \pmod H$. Put

$$\mathbf{v}_1 = q_1g_2\varrho_2e_0\,\boldsymbol{\theta}, \qquad \mathbf{l} = q_1g_2\varrho_2\,\frac{e - e_0}{H}\,\mathbf{c} \in \mathbb{Z}^n,$$

so that $q_1q_2\boldsymbol{\theta} = \mathbf{v}_1 + \mathbf{l}$. Writing $\mathbf{m} = \mathbf{m}' + \mathbf{l}$, we see that $\mathbf{m}'$ runs over the integer vectors with $|\mathbf{m}' - \mathbf{m}_0 - \mathbf{v}_1| \leqslant V$, and this range does not depend on $e$. Since $\varrho_1\varrho_2$ divides $q_1\varrho_2$, we have $\mathbf{l} \equiv \mathbf{0} \pmod{\varrho_1\varrho_2}$, and therefore $S_{\varrho_1,\varrho_2}(\mathbf{m}) = S_{\varrho_1,\varrho_2}(\mathbf{m}')$. Next, $H\mathbf{m} = H\mathbf{m}' + q_1g_2\varrho_2(e - e_0)\mathbf{c} \equiv H(\mathbf{m}' - \mathbf{v}_1) \pmod e$, where $H(\mathbf{m}' - \mathbf{v}_1) = H\mathbf{m}' - q_1g_2\varrho_2e_0\mathbf{c}$ is an integer vector. Hence

$$H^2\,\widetilde{Q}(\mathbf{m}) \equiv \widetilde{Q}_H(\mathbf{m}') \pmod e, \qquad \widetilde{Q}_H(\mathbf{m}') = \widetilde{Q}\Big(H(\mathbf{m}' - \mathbf{v}_1)\Big) = H^2\,\widetilde{Q}(\mathbf{m}' - \mathbf{v}_1) \in \mathbb{Z}.$$

Let $e'$ be the largest divisor of $e$ all of whose prime factors divide $\Delta H$, and let $e'' = e/e'$. Then $e^\flat$ divides $e'$ and $(e'', H) = 1$, so that

$$\Big(e^\flat\Big)^{1/2}\Big(e^\sharp, \widetilde{Q}(\mathbf{m})\Big)^{1/2} \leqslant e'^{1/2}\Big(e'', \widetilde{Q}(\mathbf{m})\Big)^{1/2} = e'^{1/2}\Big(e'', \widetilde{Q}_H(\mathbf{m}')\Big)^{1/2}.$$

As in the proof of Proposition 3.12 we obtain

$$|S_{q_1,q_2}(\mathbf{m})| \leqslant |S_{\varrho_1,\varrho_2}(\mathbf{m}')| \cdot A^{\omega(e)} e^{(n+1)/2} e'^{1/2} \Big(e'', \widetilde{Q}_H(\mathbf{m}')\Big)^{1/2} \cdot 2^{\omega(g_1)} (g_1 g_2)^{n/2+1}.$$

The sum of the middle factor over $e \leqslant E$ is estimated as before, with $\Delta H$ in place of $\Delta$: it is $O(E^{(n+3)/2+\varepsilon}\tau(|\widetilde{Q}_H(\mathbf{m}')|))$ if $\widetilde{Q}_H(\mathbf{m}') \neq 0$, and $O(E^{(n+4)/2+\varepsilon})$ if $\widetilde{Q}_H(\mathbf{m}') = 0$. Here $|\widetilde{Q}_H(\mathbf{m}')| \ll (V+V_0)^2$, because $|\mathbf{m}' - \mathbf{v}_1| \leqslant V + V_0$, and $\widetilde{Q}_H(\mathbf{m}') = 0$ if and only if $\widetilde{Q}(\mathbf{m}' - \mathbf{v}_1) = 0$. We now sum over $\mathbf{m}'$ by means of Lemmas 3.10 and 3.11, with $\mathbf{m}_0 + \mathbf{v}_1$ in place of $\mathbf{m}_0$ and with $\mathbf{m}_1 = \mathbf{v}_1$. This gives the bound obtained in the proof of Proposition 3.12 for the sum over $e$ and $\mathbf{m}$, now for the $e$ in the class $e_0$ modulo $H$. The sum over the classes $e_0$ contributes a factor $H$, and the rest of the proof is unchanged. □

**Proposition 9.6.** *We have* $N_\delta(\boldsymbol{\theta}) \ll P^{n-5-1/43+\varepsilon}$.

*Proof.* By Lemma 9.4, the bound (6.1) holds for $N_\delta(\boldsymbol{\theta})$ with $I_{q_1,q_2}(t; \mathbf{m} - q_1q_2\boldsymbol{\theta})$ in place of $I_{q_1,q_2}(t;\mathbf{m})$. We go through the argument of §6, writing now

$$\mathbf{v} = \mathbf{m} - q_1q_2\boldsymbol{\theta}, \qquad \mathbf{u} = \frac{P\,\mathbf{v}}{q_1q_2}.$$

Since $H\mathbf{v} \in \mathbb{Z}^n$, we have $|\mathbf{v}| \geqslant 1/H$ whenever $\mathbf{v} \neq \mathbf{0}$. By the second part of Lemma 9.4 and this lower bound, the terms with $\mathbf{v} \neq \mathbf{0}$ and $|\mathbf{u}| \geqslant P^\varepsilon\Theta/r_2$ contribute $O(P^{-N})$, and the contribution of a block $\mathcal{B}(Q_1, T)$ is bounded by $\mathcal{E}_{\text{res}} + \mathcal{E}_{\text{non}} + \mathcal{E}_{\text{tail}} + P^{-N}$. Here the three quantities are defined as in §6 with the present $\mathbf{u}$, with the condition $\mathbf{m} \neq \mathbf{0}$ replaced by $\mathbf{v} \neq \mathbf{0}$, and with the term $\mathbf{v} = \mathbf{0}$, if there is one, included in $\mathcal{E}_{\text{tail}}$. We claim that the bounds of Propositions 6.5, 6.6 and 6.7 remain valid. The proposition then follows from Lemmas 6.8–6.12, as in the proof of Proposition 6.13.

In the proof of Proposition 6.5 the conditions on $\mathbf{m}$ give $c_1K_0TQ_1Q_2/P \leqslant |\mathbf{v}| < M_0$. The $\mathbf{v}$ with $|\mathbf{v}| > 1/2$ are divided into shells $M' < |\mathbf{v}| \leqslant 2M'$ as before. If there are $\mathbf{v}$ with $0 < |\mathbf{v}| \leqslant 1/2$, then $M' = 1/2$ satisfies (6.5); these $\mathbf{v}$ have $|\mathbf{v}| \geqslant 1/H$, hence $|\mathbf{u}|^{1-n/2} \ll (P/(Q_1Q_2))^{1-n/2}$, and we include them in the shell with $M' = 1/2$. On each shell $|\mathbf{m} - q_1q_2\boldsymbol{\theta}| \leqslant 2M'$, and we apply Proposition 9.5 with $\mathbf{m}_0 = \mathbf{0}$ and $V = 2M'$ in place of Proposition 3.12.

In the proof of Proposition 6.6 the condition in (6.6) becomes $|\nabla f_\gamma(\mathbf{z}) - \mathbf{u}| \leqslant \rho$. It says that $\mathbf{m}$ lies within $\rho q_1q_2/P$ of the point $(q_1q_2/P)\mathbf{w} + q_1q_2\boldsymbol{\theta}$. The points $(q_1q_2/P)\mathbf{w}$ are covered as before, and we find that all the $\mathbf{m}$ in question, for all $q_2 \in (Q_2, 2Q_2]$, lie in the union of at most $\Lambda/V + 2$ sets of the form $\{\mathbf{m} : |\mathbf{m} - \mathbf{m}_0 - q_1q_2\boldsymbol{\theta}| \leqslant 2V\}$, where the points $\mathbf{m}_0$ depend only on $\mathbf{z}$ and $q_1$ (and on $\boldsymbol{\gamma}$ and $Q_2$). If there is such an $\mathbf{m}$ with $\mathbf{v} \neq \mathbf{0}$, then $1/H \leqslant |\mathbf{v}| \leqslant (4Q_1Q_2/P)(c_4\|\boldsymbol{\gamma}\| + \rho)$, so that (6.7) holds with an additional factor $1/H$ on the right; this is immaterial, since $\varepsilon$ is arbitrary. To each of the sets we apply Proposition 9.5, and the rest of the proof is unchanged.

In the proof of Proposition 6.7 the condition $|\mathbf{u}| \leqslant P^\varepsilon\Theta$ implies $|\mathbf{m} - q_1q_2\boldsymbol{\theta}| \leqslant 8P^\varepsilon TQ_1Q_2/P$, and Proposition 9.5 is applied with $\mathbf{m}_0 = \mathbf{0}$.

If $d_n = 0$, we argue in the same way with the quantities $\mathcal{E}_{\text{res}}$, $\mathcal{E}_{\text{non}}$ and $\mathcal{E}_{\text{tail}}$ of §8, and claim that the bounds of Propositions 8.9, 8.10 and 8.11 remain valid; the proposition then follows as in the proof of Proposition 8.15. In the proofs of these three propositions the ranges of $\mathbf{m}$ are now centred at the points $\mathbf{m}_0 + q_1q_2\boldsymbol{\theta}$ in place of $\mathbf{m}_0$, which is permitted in Proposition 8.7, and the remarks made above on the $\mathbf{v}$ with $0 < |\mathbf{v}| \leqslant 1/2$ and on (6.7) apply without change. □

9.6. **The singular series.**

**Lemma 9.7.** *Let $\mathbf{b} \in \mathbb{Z}^n$. The limits $\sigma_p(\mathbf{b})$ exist, the product $\mathfrak{S}(\mathbf{b}) = \prod_p \sigma_p(\mathbf{b})$ converges absolutely, and*

$$\frac{1}{H^n} \sum_{\mathbf{c} \bmod H} e_H(-\mathbf{b}\cdot\mathbf{c})\, \mathfrak{S}(\mathbf{c}/H) = \mathfrak{S}(\mathbf{b}).$$

*Moreover, the last assertion of Theorem 9.1 holds.*

*Proof.* For $q \in \mathbb{N}$ put $g = (q, H)$ and

$$B(q) = \sum_{\substack{\mathbf{a} \bmod q \\ \gcd(q,a_3,a_2)=1}} \Big(\frac{g}{q}\Big)^n \sum_{\substack{\mathbf{y} \bmod q \\ \mathbf{y}\equiv\mathbf{b} \bmod g}} e_q\Big(a_3C(\mathbf{y}) + a_2Q(\mathbf{y})\Big).$$

Let $\mathbf{c}$ be an integer vector. If $A(q;\mathbf{c}/H) \neq 0$, then $H$ divides $q\mathbf{c}$, that is, $\mathbf{c} = (H/g)\mathbf{c}'$ with an integer vector $\mathbf{c}'$, which is determined modulo $g$ by $\mathbf{c}$ modulo $H$. In this case $e_H(-\mathbf{b}\cdot\mathbf{c}) = e_g(-\mathbf{b}\cdot\mathbf{c}')$ and $S(\mathbf{a},q;q\mathbf{c}/H) = \sum_{\mathbf{y} \bmod q} e_q(a_3C(\mathbf{y}) + a_2Q(\mathbf{y}))e_g(\mathbf{c}'\cdot\mathbf{y})$. Since $\sum_{\mathbf{c}' \bmod g} e_g(\mathbf{c}'\cdot(\mathbf{y}-\mathbf{b}))$ is $g^n$ if $\mathbf{y} \equiv \mathbf{b} \pmod g$ and vanishes otherwise, we obtain

$$\frac{1}{H^n} \sum_{\mathbf{c} \bmod H} e_H(-\mathbf{b}\cdot\mathbf{c})\, A(q;\mathbf{c}/H) = \frac{B(q)}{H^n}. \tag{9.3}$$

The same computation shows that $|B(q)| \leqslant g^n q^{-n} \sum_{\mathbf{a}} \max_{\mathbf{m}} |S(\mathbf{a},q;\mathbf{m})|$, so that $\sum_q |B(q)| < \infty$ by (9.2).

The function $B$ is multiplicative. Indeed, let $q = q'q''$ with $(q', q'') = 1$. The residues $\mathbf{y}$ modulo $q$ correspond to the pairs of residues $(\mathbf{y}', \mathbf{y}'')$ modulo $q'$ and $q''$, and $\mathbf{y} \equiv \mathbf{b} \pmod{(q,H)}$ if and only if $\mathbf{y}' \equiv \mathbf{b} \pmod{(q',H)}$ and $\mathbf{y}'' \equiv \mathbf{b} \pmod{(q'',H)}$. Moreover $e_q(a_3C(\mathbf{y}) + a_2Q(\mathbf{y})) = e_{q'}(a_3'C(\mathbf{y}') + a_2'Q(\mathbf{y}'))\, e_{q''}(a_3''C(\mathbf{y}'') + a_2''Q(\mathbf{y}''))$, where $\mathbf{a}' \equiv \overline{q''}\mathbf{a} \pmod{q'}$ and $\mathbf{a}'' \equiv \overline{q'}\mathbf{a} \pmod{q''}$, and $\mathbf{a} \mapsto (\mathbf{a}', \mathbf{a}'')$ is a bijection between the pairs that occur in $B(q)$ and those that occur in $B(q')$ and $B(q'')$. Hence $\sum_q B(q) = \prod_p \sum_{j\geqslant 0} B(p^j)$, and the product converges absolutely.

Let $p$ be a prime, $h = h_p$ and $k \geqslant h$. We detect the congruences $C(\mathbf{x}) \equiv Q(\mathbf{x}) \equiv 0 \pmod{p^k}$ by additive characters and sort the pairs $\mathbf{a}$ modulo $p^k$ as in the proof of Lemma 7.2:

$$\lambda_{\mathbf{b}}(p^k) = p^{-2k} \sum_{j=0}^{k} \sum_{\substack{\mathbf{a} \bmod p^j \\ \gcd(p^j,a_3,a_2)=1}} \sum_{\substack{\mathbf{x} \bmod p^k \\ \mathbf{x}\equiv\mathbf{b} \bmod p^h}} e_{p^j}\Big(a_3C(\mathbf{x}) + a_2Q(\mathbf{x})\Big).$$

The summand depends only on $\mathbf{x}$ modulo $p^j$. There are $p^{(k-h)n}$ vectors $\mathbf{x}$ in the inner sum, and they are equally distributed among the classes $\mathbf{y}$ modulo $p^j$ with $\mathbf{y} \equiv \mathbf{b} \pmod{p^{\min\{h,j\}}}$. Since $(p^j, H) = p^{\min\{h,j\}}$, the sum over $\mathbf{a}$ and $\mathbf{x}$ is $p^{(k-h)n}B(p^j)$. Therefore

$$\sum_{j=0}^{k} B(p^j) = p^{hn}\, p^{-k(n-2)} \lambda_{\mathbf{b}}(p^k) \qquad (k \geqslant h).$$

So the limit $\sigma_p(\mathbf{b})$ exists, $\sum_{j\geqslant 0} B(p^j) = p^{hn}\sigma_p(\mathbf{b})$, and $\sum_q B(q) = H^n \prod_p \sigma_p(\mathbf{b})$. Summing (9.3) over $q$ we obtain the formula of the lemma. For $p \nmid H$ we have $\sigma_p(\mathbf{b}) = \sigma_p$, so the product over $p$ converges absolutely, by Lemma 7.2.

It remains to prove that $\mathfrak{S}(\mathbf{b}) > 0$ under the hypothesis of Theorem 9.1. For $p \nmid H$ we have $\sigma_p(\mathbf{b}) = \sigma_p > 0$ by Lemma 7.2. Let $p \mid H$, $h = h_p$, and let $\mathbf{y} \in \mathbb{Z}_p^n \setminus \{\mathbf{0}\}$ with $\mathbf{y} \equiv \mathbf{b} \pmod{p^h}$ and $C(\mathbf{y}) = Q(\mathbf{y}) = 0$. Since $X$ is non-singular, the vectors $\nabla C(\mathbf{y})$ and $\nabla Q(\mathbf{y})$ are linearly independent. As in the proof of Lemma 7.2 there

are indices $i < j$ for which the corresponding minor does not vanish at $\mathbf{y}$; let $\mu$ be its $p$-adic valuation there, and put $k_0 = \max\{2\mu+1, h+\mu\}$. Let $k \geqslant k_0$ and let $\mathbf{y}' \in \mathbb{Z}_p^n$ with $\mathbf{y}' \equiv \mathbf{y} \pmod{p^{k_0}}$. Then $C(\mathbf{y}') \equiv Q(\mathbf{y}') \equiv 0 \pmod{p^{k_0}}$, and the minor has valuation $\mu$ at $\mathbf{y}'$. By Hensel's lemma for two equations in the two unknowns $x_i$, $x_j$, there is $\mathbf{x} \in \mathbb{Z}_p^n$ with $C(\mathbf{x}) = Q(\mathbf{x}) = 0$, with $x_l = y_l'$ for all $l \neq i, j$, and with $\mathbf{x} \equiv \mathbf{y}' \pmod{p^{k_0-\mu}}$. Since $k_0 - \mu \geqslant h$, we have $\mathbf{x} \equiv \mathbf{b} \pmod{p^h}$. The vectors $\mathbf{y}'$ give $p^{(k-k_0)(n-2)}$ different values of $(y_l')_{l\neq i,j}$ modulo $p^k$. Hence $\lambda_{\mathbf{b}}(p^k) \geqslant p^{(k-k_0)(n-2)}$ for $k \geqslant k_0$, and $\sigma_p(\mathbf{b}) \geqslant p^{-k_0(n-2)} > 0$. □

9.7. **Proof of Theorem 9.1.** Let $\boldsymbol{\theta} = \mathbf{c}/H$ with $\mathbf{c} \in \mathbb{Z}^n$. By the splitting of $N_w(P;\boldsymbol{\theta})$, the bound (4.2) for $N_\infty(\boldsymbol{\theta})$, Lemmas 9.2 and 9.3, and Proposition 9.6,

$$N_w(P;\boldsymbol{\theta}) = \mathfrak{S}(\boldsymbol{\theta})\,\mathfrak{J}\,P^{n-5} + O\Big(P^{n-5-1/43+\varepsilon}\Big).$$

We insert this into (9.1) and apply Lemma 9.7. This proves Theorem 9.1.

9.8. **Proof of Theorem 1.3.** Let $X$ be as in Theorem 1.1. If $X(\mathbb{R}) = \emptyset$ there is nothing to prove. A linear change of variables over $\mathbb{Q}$ induces a bijection on the rational points and homeomorphisms on the real and the $p$-adic points, so that we may suppose that $X$ is given by forms $C$ and $Q$ as in (1.1). Let $\mathcal{S}$ be a finite set of primes, let $\xi_\infty \in X(\mathbb{R})$ and $\xi_p \in X(\mathbb{Q}_p)$ for $p \in \mathcal{S}$, and let neighbourhoods $\mathcal{W}_\infty$ of $\xi_\infty$ in $X(\mathbb{R})$ and $\mathcal{W}_p$ of $\xi_p$ in $X(\mathbb{Q}_p)$ be given. By the definition of the product topology, it suffices to find a point of $X(\mathbb{Q})$ that lies in all of these neighbourhoods.

We write $\xi_\infty = [\mathbf{z}_0]$ with $\mathbf{z}_0 \in \mathbb{R}^n$ and $0 < |\mathbf{z}_0| \leqslant 1/4$, and $\xi_p = [\mathbf{b}_p]$ with a primitive vector $\mathbf{b}_p \in \mathbb{Z}_p^n$. There are $\rho_0 > 0$ and $k \in \mathbb{N}$ with the following properties: if $\mathbf{z} \in \mathbb{R}^n$ with $|\mathbf{z} - \mathbf{z}_0| \leqslant \rho_0$ and $C(\mathbf{z}) = Q(\mathbf{z}) = 0$, then $[\mathbf{z}] \in \mathcal{W}_\infty$; and if $p \in \mathcal{S}$ and $\mathbf{y} \in \mathbb{Z}_p^n$ with $\mathbf{y} \equiv \mathbf{b}_p \pmod{p^k}$ and $C(\mathbf{y}) = Q(\mathbf{y}) = 0$, then $[\mathbf{y}] \in \mathcal{W}_p$. We may take $\rho_0$ so small that the construction of §2.2 applies with this $\mathbf{z}_0$ and $\rho_0$ (together with the additional condition of §8 if $d_n = 0$), and we let $w$ be the corresponding weight. Put $H = \prod_{p\in\mathcal{S}} p^k$, and choose $\mathbf{b} \in \mathbb{Z}^n$ with $\mathbf{b} \equiv \mathbf{b}_p \pmod{p^k}$ for all $p \in \mathcal{S}$. The hypothesis of Theorem 9.1 holds, with $\mathbf{y} = \mathbf{b}_p$ for $p \in \mathcal{S}$. Hence $\mathfrak{S}(\mathbf{b}) > 0$, and $\mathfrak{J} > 0$ by Lemma 5.1(ii). By Theorem 9.1, $N_w(P;\mathbf{b},H) > 0$ if $P$ is large. So there is $\mathbf{x} \in \mathbb{Z}^n$ with $C(\mathbf{x}) = Q(\mathbf{x}) = 0$, $\mathbf{x} \equiv \mathbf{b} \pmod{H}$ and $|\mathbf{x}/P - \mathbf{z}_0| \leqslant \rho_0$. Then $\mathbf{x} \neq \mathbf{0}$, the point $[\mathbf{x}] = [\mathbf{x}/P]$ of $X(\mathbb{Q})$ lies in $\mathcal{W}_\infty$, and it lies in $\mathcal{W}_p$ for every $p \in \mathcal{S}$. This proves Theorem 1.3.

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