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A REMARK ON THE INDEPENDENCE STEP IN SHKREDOV'S APPROACH TO ZAREMBA'S CONJECTURE

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Abstract. The proof of Proposition 35 in Shkredov's preprint [1] applies a repulsion inequality (Corollary 34) to three denominators x, y, z, which must be $\vec{C}$-independent for the vector $\vec{C}$ of formula (116) of [1]. We show that, in the situation of that proof, this independence cannot hold: the identity (75) of [1], $d(y,z)x + d(z,x)y + d(x,y)z = 0$, is a nonzero integer relation whose coefficients lie in the box defined by $\vec{C}$ as soon as two of the three inequalities needed for the contradiction in that proof are satisfied. The same holds for the triple (x, y_0, z) to which Lemma 32 of [1] is applied. We also record that the deduction drawn from Proposition 35 in §5.2 of [1] does not address the second alternative of that proposition. These remarks concern the proofs only: they do not contradict the statements of the theorems of [1], nor of [2], whose §4 follows the same framework. Nothing in this note has been checked by a professional mathematician or by any third party.

1. Setting

We use the notation of [1] (version 2). Let q be large, $t = q^{1/2-\varepsilon}$, $N = q/t^2$ and $H = N^{9/20}$. If an integer $w = K(r_1, \dots, r_s)$ is the denominator of a convergent of a fraction, write $\widehat{w} = K(r_2, \dots, r_s)$ for the corresponding numerator, and put

$$d(x,y) = x\,\widehat{y} - y\,\widehat{x} \qquad [1,\ (70)].$$

For all such x, y, z one has the identity

$$d(y,z)\,x + d(z,x)\,y + d(x,y)\,z = 0 \qquad [1,\ (75)]. \tag{1}$$

By [1, Definition 24], elements x_1, x_2, x_3 of $\mathbb{Z}/q\mathbb{Z}$ are $\vec{C}$-*independent* if $\sum \alpha_j x_j \equiv 0 \pmod q$ with integers $|\alpha_j| \le C_j$ forces $\alpha_1 = \alpha_2 = \alpha_3 = 0$. A relation $\sum \alpha_j x_j = 0$ in $\mathbb{Z}$ is in particular such a congruence. The conclusion of [1, Lemma 32] is stated for equations $\alpha x + \beta y + \gamma z = 0$ in integers.

The step in question. In the proof of [1, Proposition 35] one has points $a < b < c$ of an interval I, with $I_* = c - a \le |I|$, and the following integers:

- $x = q_s(a)$, a convergent denominator of a/q;
- $y_0 = y_0(b)$ and $z = z(c)$, distinct $\widetilde{M}$-critical denominators of b/q and c/q supplied by [1, Lemma 32], with $\max\{y_0, z\} \in [tH, \sqrt{q}]$; the proof treats the case $\max\{y_0, z\} = y_0$;
- $y \ge y_0$, a convergent denominator $y = y_n(b) \le \sqrt{q}$ of b/q with $c_{n+1}(b)\,y > \sqrt{q}$, and the number $\widetilde{\mathcal{M}} \le c_{n+1}(b)$ defined by $\widetilde{\mathcal{M}} y = \sqrt{q}$.

Thus $z < y_0 \le y$. The bounds (121)–(122) of [1] are obtained through the numerators $\widehat{x}, \widehat{y}, \widehat{z}$ of these convergents and read

$$|bx - q\widehat{x}| \le X_1 := 2I_* x, \qquad |by - q\widehat{y}| \le X_2 := \frac{q}{\widetilde{\mathcal{M}} y} = \sqrt{q}, \qquad |bz - q\widehat{z}| \le X_3 := 2I_* z. \tag{2}$$

The vector $\vec{C}$ is the one given by [1, (116)] with $k = 3$:

$$C_1 = 2\Big(\frac{12X_2X_3}{tX_1}\Big)^{1/2}, \qquad C_2 = 2\Big(\frac{12X_1X_3}{tX_2}\Big)^{1/2}, \qquad C_3 = 2\Big(\frac{12X_1X_2}{tX_3}\Big)^{1/2}.$$

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The second part of Corollary 34 of [1] is applied in (123) to (x, y, z) and the point b, with this $\vec{C}$; its hypothesis includes the $\vec{C}$-independence of (x, y, z). It yields $C_1x + C_2y + C_3z \geq q/(4Mt)$, and the proof obtains a contradiction by establishing

$$C_1x < \frac{q}{12Mt}, \qquad C_2y < \frac{q}{12Mt}, \qquad C_3z < \frac{q}{12Mt} \tag{3}$$

through its inequalities (124)–(127). With (2) one computes

$$C_1^2 = \frac{48\sqrt{q}\,z}{t\,x}, \qquad C_2^2 = \frac{192\,I_*^2\,x\,z}{t\sqrt{q}}, \qquad C_3^2 = \frac{48\sqrt{q}\,x}{t\,z}, \tag{4}$$

so that the first two inequalities of (3) are equivalent to

$$t\,x\,z < \frac{q^{3/2}}{6912\,M^2} \qquad \text{and} \qquad I_*^2\,x\,y^2\,z\,t < \frac{q^{5/2}}{27648\,M^2}. \tag{5}$$

2. The obstruction

Lemma 2.1. *Let b, q be integers and let x, y be convergent denominators with numerators $\widehat{x}, \widehat{y}$. Then*

$$q\,d(x, y) = y\,(bx - q\widehat{x}) - x\,(by - q\widehat{y}), \qquad \textit{hence} \qquad |d(x, y)| \leq \frac{x\,|by - q\widehat{y}| + y\,|bx - q\widehat{x}|}{q}.$$

Proof. The terms in b cancel, and what remains is $q(x\widehat{y} - y\widehat{x})$. □

Proposition 2.2. *Let x, y, z be convergent denominators with $y \neq z$, let $M \geq 1$, and assume (2) and the first two inequalities of (3). Put*

$$(\alpha, \beta, \gamma) = \big(d(y, z),\, d(z, x),\, d(x, y)\big).$$

Then (α, β, γ) is a nonzero vector of integers with $\alpha x + \beta y + \gamma z = 0$, and

$$|\alpha| \leq C_1, \qquad |\beta| \leq C_2, \qquad |\gamma| \leq C_3.$$

In particular (x, y, z) is not $\vec{C}$-independent, and Corollary 34 of [1] does not apply to it.

Proof. The relation is (1). If $\alpha = d(y, z) = 0$, then $\widehat{y}/y = \widehat{z}/z$ with both fractions in lowest terms, so $y = z$, which is excluded; hence $\alpha \neq 0$. By Lemma 2.1 and (2),

$$|\alpha| \leq \frac{z(\sqrt{q} + 2I_*y)}{q}, \qquad |\beta| \leq \frac{4I_*xz}{q}, \qquad |\gamma| \leq \frac{x(\sqrt{q} + 2I_*y)}{q}.$$

Comparing squares with (4):

- the bound for $|\beta|$ is at most C_2 if and only if $txz \leq 12\,q^{3/2}$;
- the bounds for $|\alpha|$ and $|\gamma|$ are at most C_1 and C_3 respectively if and only if $txz(\sqrt{q}+2I_*y)^2 \leq 48\,q^{5/2}$.

The first condition follows from the first inequality of (5). For the second, $(\sqrt{q}+2I_*y)^2 \leq 2q+8I_*^2y^2$, so (5) gives

$$txz(\sqrt{q} + 2I_*y)^2 < \frac{2\,q^{5/2}}{6912\,M^2} + \frac{8\,q^{5/2}}{27648\,M^2} < q^{5/2}.$$ □

Proposition 2.2 uses nothing beyond the bounds (2) and two of the inequalities (3) that the proof of [1, Proposition 35] needs for its contradiction. Hence, whenever that contradiction is available, the hypothesis of Corollary 34 fails for (x, y, z).

Proposition 2.3. *In the setting of Section 1, assume moreover $z < y_0 \le y$, $y_0 \ge tH$, $x \le H\sqrt{q}$, $\widetilde{M} \ge 1$ and $|by_0 - q\widehat{y}_0| \le q/(\widetilde{M}y_0)$. Then*

$$(\alpha_0, \beta, \gamma_0) = \big(d(y_0, z),\, d(z, x),\, d(x, y_0)\big)$$

is a nonzero vector of integers with $\alpha_0 x + \beta y_0 + \gamma_0 z = 0$ and $|\alpha_0| \le C_1$, $|\beta| \le C_2$, $|\gamma_0| \le C_3$, for the same vector $\vec{C}$.

Proof. As before $\alpha_0 \neq 0$ since $y_0 \neq z$, and the bound for β is unchanged. By Lemma 2.1,

$$|\alpha_0| \le \frac{2I_* y_0 z}{q} + \frac{z}{\widetilde{M}y_0}, \qquad |\gamma_0| \le \frac{2I_* y_0 x}{q} + \frac{x}{\widetilde{M}y_0}.$$

In each bound the first term is at most $C_1/2$, respectively $C_3/2$, if and only if $I_*^2\, x\, y_0^2\, z\, t \le 3\, q^{5/2}$, which follows from (5) and $y_0 \le y$. The second term is at most $C_1/2$, respectively $C_3/2$, if and only if $x\, z\, t \le 12\, \widetilde{M}^2 y_0^2 \sqrt{q}$, and indeed $x\, z\, t \le H\sqrt{q} \cdot y_0 \cdot t \le \sqrt{q}\, y_0^2$. □

Remark 2.4. In [1] the hypotheses of Proposition 2.3 come from the statement of Lemma 32 ($y_0 = \max\{y_0, z\} \ge tH$, and y_0 is $\widetilde{M}$-critical) and from the range of x: the critical denominators of a considered there are at most $T = \sqrt{q}\, N^{2/7}$, and $x = q_s(a)$ lies at most m^2 steps beyond them with partial quotients bounded by $M_* \le N^{10^{-10}}$, so that $x \le H\sqrt{q}$ for large q. Thus the conclusion of Lemma 32, in the form in which it is used in the proof of Proposition 35, cannot hold in this range of parameters.

The remark following (77) in [1] anticipates small relations coming from (1) when a, b, c are close, and the points are separated for this reason. Propositions 2.2 and 2.3 show that this separation does not suffice here: the coefficients of (1) are controlled by the residues modulo q at the single point b, and the inequalities that produce the contradiction are those that place the coefficients in the box.

3. The deduction drawn from Proposition 35

Proposition 35 of [1] concludes with an alternative: either I does not satisfy the δ-Assumption, or some $a \in I \cap Z_M^{-1}(t)$ has a large partial quotient (at least M_*) at a denominator in $[\sqrt{qN}/H, \sqrt{qN}]$. In §5.2, Proposition 35 is invoked to conclude that subintervals of the relevant length "cannot satisfy δ-Assumption". This requires the second alternative to fail for every $a \in I \cap Z_M^{-1}(t)$. The sentence preceding the deduction gives, for points of $Z_* \cap Z_*^{-1}$, an upper bound on the denominators $x(a)$; it does not treat the partial quotients of the points of $I \cap Z_M^{-1}(t)$ at denominators in $[\sqrt{qN}/H, \sqrt{qN}]$. The text before Proposition 35 remarks that one should work with the smaller set $(Z_M(t) \cap Z_{M_*}(tH)) \cap (Z_M(t) \cap Z_{M_*}(tH))^{-1}$; we did not find this restriction carried out in the statement or proof of Proposition 35, nor an argument excluding the second alternative in §5.2. We record this as a point that we could not complete from the text, independently of Section 2.

4. Scope

(1) Section 2 is a statement about the proof of [1, Proposition 35] as written in version 2: the triples to which Lemma 32 and Corollary 34 are applied there carry the relation (1) with coefficients in the box of $\vec{C}$. The results of [1] whose proofs pass through Proposition 35 (§5.2, and §5.4 which uses its conclusion) depend on a replacement for this step.
(2) The introduction of [2] states that its §4 outlines Shkredov's argument and that its expansion theorem is implemented in the framework of the proof of Theorem 8 of [1]. To that extent Theorem 1.2 of [2] depends on the same step. We have not examined [2] beyond this.
(3) These remarks do not contradict the statements of the theorems of [1] or [2]. Results that do not use this step, such as the bound $M = O(\log q/\log\log q)$ of [3], are not concerned.

(4) Status. Lemma 2.1 and Propositions 2.2 and 2.3 are proved above. Their application to [1] rests on our reading of formulas (70), (75), (116), (121)–(127), of Definition 24, Lemma 32 and Corollary 34 of that paper.
(5) **No external check.** Nothing in this note has been checked by a professional mathematician or by any third party: neither the proofs, nor the reading of [1], nor the ancillary script. The only checks are those carried out by the author with the AI model named below.

Ancillary file

`anc/cramer_identity.py` checks in exact integer arithmetic: Lemma 2.1 and identity (1) on convergents of random fractions (2993 configurations with $q = 10^{12} + 39$); and the inequalities $|\alpha| \leq C_1$, $|\beta| \leq C_2$, $|\gamma| \leq C_3$ of Proposition 2.2 on 5000 random parameter sets satisfying (5), together with those of Proposition 2.3 on the 4957 of these sets that satisfy $xzt \leq \sqrt{q}\, y_0^2$. Its output is in `anc/cramer_identity_output.log`. These checks support the algebra; they are not part of the proofs.

Use of AI

This note was prepared with the assistance of an AI model (Claude, Anthropic), which carried out the reading of [1], the computations and the drafting under the author's direction. The author takes responsibility for the content.

References

[1] I. D. Shkredov, *On some results of Korobov and Larcher and Zaremba's conjecture*, arXiv:2603.14116v2 (24 June 2026).
[2] X. Zhang, *Expansion in* $\mathrm{SL}_2(\mathbb{Z}/q\mathbb{Z})$ *and Zaremba's conjecture*, arXiv:2605.02518v2 (8 May 2026).
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