THE STARTING CORNER $(7, 35)$ IN THE PLANE JACOBIAN PROBLEM: THE POINT $(18/7, 4)$ IS NOT A REGULAR CORNER

DAVID NIEDBALA GIRAUDIN

Abstract. The note [3] excludes the standard $(2,3)$-pairs of degrees $(84, 126)$ with starting triple $((7,35),(2,0),(7,-1))$ under the additional condition that the polynomial h of degree 5 attached to the first edge has a single root; for the only other pattern left open by elementary arithmetic, a root of multiplicity 4 and a simple root, that note relied on the list of Guccione, Guccione, Horruitiner and Valqui [2]. We show that $(18/7, 4)$ is not a regular corner of any $(2,3)$-pair in $K[x^{\pm 1/7}, y]$. Hence the condition on h is automatic, and the theorem of [3] holds for every pair satisfying the other standing data of that note. The proof uses only the propositions of Guccione–Guccione–Valqui on regular corners, applied to every root of the edge polynomial, together with elementary arithmetic, which is reproduced by an ancillary script. The argument has not yet been examined by specialists.

hexagon:2611.00714v1

1. Statement

Throughout, K is an algebraically closed field of characteristic zero, $L^{(l)} = K[x^{\pm 1/l}, y]$, $[P, Q] = P_x Q_y - P_y Q_x$ and $(m, n) = (2, 3)$. We use the notation of [1]: $v_{\rho,\sigma}$, $\ell_{\rho,\sigma}$, $\mathrm{st}_{\rho,\sigma}$, $\mathrm{en}_{\rho,\sigma}$; the interval of directions $I =](1,-1),(1,0)]$; *(m,n)-pairs* in $L^{(l)}$ and their *regular corners* [1, Definitions 4.3 and 5.5]. For a regular corner $((a/l, b), (\rho, \sigma))$ of (P, Q) we write $\ell_{\rho,\sigma}(P) = x^{k/l} p(z)$ with $z := x^{-\sigma/\rho} y$, and we distinguish, as in [1, Section 5, before Remark 5.9]: Case I, $[\ell_{\rho,\sigma}(P), \ell_{\rho,\sigma}(Q)] \neq 0$; Case II, the bracket vanishes and p has at least two distinct linear factors, the factor z included (this is how we read the condition $\#\,\mathrm{factors}(p) > 1$ of [1]); Case III, the bracket vanishes and $p = \alpha(z - \beta)^r$ with $\beta \neq 0$. One always has $v_{1,-1}(\mathrm{st}_{\rho,\sigma}(P)) \neq 0$ (loc. cit., by [1, Theorem 2.6(4)]); Case II is called II.a) if this number is negative and II.b) if it is positive.

Theorem 1.1. *No $(2,3)$-pair in $L^{(7)}$ has a regular corner at the point $(18/7, 4)$.*

Corollary 1.2. *Let (P, Q) satisfy all the conditions of* [3, Standing data 1.1] *except the last one: (P, Q) is a standard $(2,3)$-pair in $L = K[x, y]$ with $[P, Q] \in K^\times$, $\deg P = 84$, $\deg Q = 126$, starting triple $((7,35),(2,0),(7,-1))$ and $(-1,1) < \mathrm{Succ}_P(1,0) < (-1,0)$, so that $\ell_{7,-1}(P) = \lambda_P(x^2 h(t))^2$ and $\ell_{7,-1}(Q) = \lambda_Q(x^2 h(t))^3$ with $t = xy^7$ and $h \in K[t]$ monic of degree 5, $h(0) \neq 0$. Then $h = (t - c)^5$ with $c \in K^\times$. Consequently, by* [3, Theorem 1.2]*, there is no such pair.*

Remark 1.3. If (P, Q) is a $(3,2)$-pair with a regular corner, then (Q, P) is a $(2,3)$-pair with the same regular corner [1, Proposition 4.6 and Corollary 5.7]; so Theorem 1.1 also holds for $(3,2)$-pairs.

2. The tools

We use the following statements, for an (m,n)-pair (P, Q) in $L^{(l)}$ and a regular corner $((a/l, b), (\rho, \sigma))$.

(T1) [1, Proposition 5.19]: the point $(a/l, b)$ is the first entry of a regular corner of Case I or of Case II of a (possibly different) (m,n)-pair in $L^{(l)}$; in Case I, $l - a/b > 1$; in Case II, $\gcd(a, b) > 1$. The directions of these corners are not larger than (ρ, σ) (proof of that proposition, which uses [1, Proposition 5.17(3)]).

Date: 20 November 2026.
ORCID: 0009-0009-1526-1178.

(T2) Case I. By [1, Proposition 1.13], $v_{\rho,\sigma}(P)+v_{\rho,\sigma}(Q) = \rho+\sigma$; moreover $\mathrm{en}_{\rho,\sigma}(P) = m(a/l, b)$ by definition and $\mathrm{en}_{\rho,\sigma}(P) = \frac{m}{n}\,\mathrm{en}_{\rho,\sigma}(Q)$ [1, Corollary 5.7(2)]. Hence

$$m\Big(\frac{a}{l}\rho + b\sigma\Big) = \frac{m}{m+n}(\rho+\sigma), \tag{1}$$

which determines (ρ,σ), and $v_{\rho,\sigma}(P) = \frac{m}{m+n}(\rho+\sigma)$, $v_{\rho,\sigma}(Q) = \frac{n}{m+n}(\rho+\sigma)$. By [1, Proposition 5.13], either $\frac{1}{m}\,\mathrm{st}_{\rho,\sigma}(P) = (a'/l, 0)$ with $a' \in \mathbb{N}$ (statement and proof), or $\{\mathrm{st}_{\rho,\sigma}(P), \mathrm{st}_{\rho,\sigma}(Q)\} = \{(k/l, 0), (1-k/l, 1)\}$ with $k \in \mathbb{N}$.

(T3) Case II. By [1, Proposition 5.14], with $d := \gcd(a,b)$, $\bar a := a/d$, $\bar b := b/d$: $d > 1$, and there is $\mu \in \mathbb{N}$ with $d \nmid \mu$, $\mu \le l(bl-a)+1/\bar b$, such that (ρ,σ) is the direction orthogonal to $(\mu\bar a - l,\ \mu\bar b l - l)$; moreover $\rho/\gcd(\rho,l) \le b$. The point $A' := \frac{1}{m}\,\mathrm{st}_{\rho,\sigma}(P)$ lies in $\frac{1}{l}\mathbb{Z}\times\mathbb{N}_0$ (proof of that proposition), below $(a/l, b)$ on the line of direction $(-\sigma/\rho, 1)$ through it, and $v_{1,-1}(A') \neq 0$ as recalled above. One has $\ell_{\rho,\sigma}(P) = \lambda R^m$ for a (ρ,σ)-homogeneous $R \in L^{(l)}$ [1, Remark 3.1 and Corollary 5.7].

(T4) Case II.a). $(A', \mathrm{Pred}_P(\rho,\sigma))$ is a regular corner of (P,Q) [1, Remark 5.11].

(T5) Case II, roots. Let $\lambda \in K^\times$ be a root of p of multiplicity m_λ with

$$m_\lambda \ \ge\ \frac{m}{l}\cdot\frac{a\rho+bl\sigma}{\rho+\sigma}. \tag{2}$$

Then, with $l' := \mathrm{lcm}(\rho, l)$ and $\varphi(y) := y + \lambda x^{\sigma/\rho}$, the pair $(\varphi P, \varphi Q)$ is an (m,n)-pair in $L^{(l')}$ with a regular corner at $(\frac{k}{ml}, 0) + \frac{m_\lambda}{m}(-\frac{\sigma}{\rho}, 1)$, of direction smaller than (ρ,σ) [1, Proposition 5.18]. Equality in (2) cannot occur, since this corner would lie on the diagonal, which [1, Theorem 2.6(4)] forbids (proof of that proposition).

(T6) Case II.b). Some root $\lambda \in K^\times$ of p satisfies (2) [1, Proposition 5.16]. (That the root can be taken different from 0 uses the hypothesis $v_{1,-1}(\mathrm{st}_{\rho,\sigma}(P)) > 0$: the multiplicity of z in p is then smaller than the right-hand side of (2), by [1, Remark 5.15].)

3. Arithmetic

In this section $(m,n) = (2,3)$. For a point $(a/l, b)$ we say that *Case I is arithmetically impossible* if, for the direction given by (1), none of the starting points allowed in (T2) exists, and that *Case II is arithmetically impossible below* (ρ_*,σ_*) if no μ as in (T3) gives a direction of I smaller than (ρ_*,σ_*). Put $A := (18/7, 4)$ and

$$C_\mu := \big(\tfrac{2}{7}, 0\big) + \mu\big(\tfrac{4}{7}, 1\big), \qquad D_\mu := \big(\tfrac{10}{7}, 0\big) + \mu\big(\tfrac{2}{7}, 1\big), \qquad 0 \le \mu \le 4,$$

so that $C_4 = D_4 = A$, $C_1 = (6/7, 1)$, $C_2 = (10/7, 2)$, $D_2 = (2,2)$, $D_3 = (16/7, 3)$.

Lemma 3.1. *(a) At $A = (18/7, 4)$, $l = 7$: Case I is arithmetically impossible (direction $(133, -83)$); the directions allowed in Case II are $(7,-2)$ and $(7,-4)$.*

(b) At $C_1 = (6/7, 1)$, $l = 7$: $l - a/b = 1$ and $\gcd(6,1) = 1$.

(c) At $C_2 = (10/7, 2)$, $l = 7$: Case I is arithmetically impossible (direction $(63,-43)$); the only direction allowed in Case II below $(7,-4)$ is $(14,-9)$.

(d) At $D_3 = (16/7, 3)$, $l = 7$: Case I is arithmetically impossible (direction $(98,-73)$), and $\gcd(16,3) = 1$.

(e) At $(11/14, 1)$, $l = 14$: Case I is arithmetically impossible (direction $(56,-41)$), and $\gcd(11,1) = 1$.

(f) At $(17/7, 3)$, $l = 7$: Case I is arithmetically impossible (direction $(49,-39)$), and $\gcd(17,3) = 1$.

Proof. Direct computation, checked by the ancillary script; for instance in (a), (1) reads $(\frac{18}{7} - \frac{1}{5})\rho + (4 - \frac{1}{5})\sigma = 0$, so $(\rho,\sigma) = (133,-83)$, $v(P) = 20$, $v(Q) = 30$, and $\frac{133}{7}\cdot 2a' = 20$, $\frac{133}{7}k = 30$, $\frac{133}{7}k = 20$ have no integer solution. For Case II in (a): $d = 2$, $\bar a = 9$, $\bar b = 2$, and the direction attached to an odd μ is proportional to $(7(2\mu-1), -(9\mu-7))$; here $\mu \le 7\cdot 10 + \frac{1}{2}$. The greatest common divisor of $7(2\mu-1)$ and $9\mu-7$ divides 35, so $\rho/\gcd(\rho,7) \ge (2\mu-1)/5$ and the condition $\rho/\gcd(\rho,7) \le 4$ gives $\mu \le 10$; the odd values $\mu = 5, 7, 9$ give $\rho/\gcd(\rho,7) = 9, 13, 17$, which leaves $\mu = 1$ and $\mu = 3$. □

By (T1), (T2), (T3) and Lemma 3.1:

(3) none of the points C_1, D_3, $(17/7, 3)$ ($l = 7$) and $(11/14, 1)$ ($l = 14$) is the first entry of a regular corner of a $(2, 3)$-pair in $L^{(l)}$.

Lemma 3.2. *No $(2, 3)$-pair in $L^{(7)}$ has a regular corner at $C_2 = (10/7, 2)$ of direction smaller than $(7, -4)$.*

Proof. By (T1) and Lemma 3.1(c) there would be a $(2, 3)$-pair (P', Q') in $L^{(7)}$ with a regular corner $(C_2, (14, -9))$ of Case II. Put $z := x^{9/14}y$. The point A' of (T3) is $C_2 - e(9/14, 1)$ with $e \in \{1, 2\}$ and lies in $\frac{1}{7}\mathbb{Z} \times \mathbb{N}_0$, so $e = 2$ and $A' = (1/7, 0)$. Thus $\ell_{14,-9}(P') = x^{2/7}p(z)$ with $\deg p = 4$, $p(0) \neq 0$. By (T3), $\ell_{14,-9}(P') = \lambda' R^2$ with $R = x^{1/7}r(z) \in L^{(7)}$, $\deg r = 2$, $r(0) \neq 0$; since $x^{1/7}z^i = x^{(2+9i)/14}y^i$ lies in $L^{(7)}$ only for i even, r is a multiple of $z^2 - c$ with $c \neq 0$, and $p = \lambda'(z^2 - c)^2$. A root λ of $z^2 = c$ has $m_\lambda = 2$, and the right-hand side of (2) is $\frac{2}{7} \cdot \frac{10 \cdot 14 - 2 \cdot 7 \cdot 9}{5} = \frac{4}{5}$. By (T5) some $(2, 3)$-pair in $L^{(14)}$ has a regular corner at $(\frac{1}{7}, 0) + (\frac{9}{14}, 1) = (11/14, 1)$, which contradicts (3). □

4. Proof of Theorem 1.1

Assume that some $(2, 3)$-pair in $L^{(7)}$ has a regular corner at $A = (18/7, 4)$. By (T1) and Lemma 3.1(a) there is a $(2, 3)$-pair (P, Q) in $L^{(7)}$ with a regular corner $(A, (\rho, \sigma))$ of Case II, with $(\rho, \sigma) \in \{(7, -4), (7, -2)\}$. Let A' be the point of (T3). Since $\ell_{\rho,\sigma}(P)$ is a square by (T3), every root of p has even multiplicity.

The direction $(7, -4)$. Put $z := x^{4/7}y$. Here $A' = A - e(4/7, 1) = C_{b'}$ with $b' = 4 - e \in \{0, 1, 2, 3\}$, and $\ell_{7,-4}(P) = x^{4/7}p(z)$ with $p = z^{2b'}\tilde{p}(z)$, $\tilde{p}(0) \neq 0$,

$$\tilde{p}(z) = \lambda' \prod_i (z - c_i)^{2\mu_i}, \qquad c_i \in K^\times \text{ distinct}, \quad \mu_i \geq 1, \quad \sum_i \mu_i = 4 - b'.$$

The right-hand side of (2) is $\frac{2}{7} \cdot \frac{18 \cdot 7 - 4 \cdot 7 \cdot 4}{3} = \frac{4}{3}$, so every root of $\tilde{p}$, of multiplicity $2\mu_i \geq 2$, satisfies (2). A root with $\mu_i = 4$ would force $b' = 0$ and $p = \lambda'(z - c_i)^8$, which has a single linear factor and is not Case II; so $\mu_i \leq 3$. By (T5), for every i some $(2, 3)$-pair in $L^{(7)}$ has a regular corner at C_{μ_i}, of direction smaller than $(7, -4)$. By (3) and Lemma 3.2, $\mu_i \notin \{1, 2\}$, so $\mu_i = 3$ for every i. As $\sum_i \mu_i \leq 4$, there is exactly one root, and $b' = 1$: thus $A' = C_1 = (6/7, 1)$ and $v_{1,-1}(A') < 0$. We are in Case II.a) and, by (T4), C_1 is the first entry of a regular corner, contrary to (3).

The direction $(7, -2)$. Put $z := x^{2/7}y$. Here $A' = A - e(2/7, 1) = D_{b'}$ with $b' = 4 - e \in \{0, 1, 2, 3\}$, and $\ell_{7,-2}(P) = x^{20/7}p(z)$ with $p = z^{2b'}\tilde{p}(z)$, $\tilde{p}(0) \neq 0$, $\deg \tilde{p} = 2(4 - b')$. The case $b' = 2$ is excluded by $v_{1,-1}(A') \neq 0$, since $D_2 = (2, 2)$. If $b' = 3$, then $A' = D_3$ and $v_{1,-1}(A') < 0$: we are in Case II.a) and D_3 is the first entry of a regular corner by (T4), contrary to (3). Let $b' \in \{0, 1\}$. Then $v_{1,-1}(A') > 0$, we are in Case II.b), and the right-hand side of (2) is $\frac{2}{7} \cdot \frac{18 \cdot 7 - 4 \cdot 7 \cdot 2}{5} = 4$. By (T6) some root $\lambda \neq 0$ of p has multiplicity $2\mu \geq 4$, with $\mu \leq 4 - b'$, and by (T5) some $(2, 3)$-pair in $L^{(7)}$ has a regular corner at D_μ. The value $\mu = 2$ is the case of equality in (2), which cannot occur. The value $\mu = 3$ contradicts (3). The value $\mu = 4$ requires $b' = 0$ and $p = \lambda'(z - \lambda)^8$, which has a single linear factor and is not Case II. □

5. Proof of Corollary 1.2

By the definition of the starting triple [1, Definition 6.2], $((7, 35), (7, -1))$ is a regular corner of the $(2, 3)$-pair (P, Q) in $L^{(1)}$, with $\frac{1}{2}\operatorname{st}_{7,-1}(P) = (2, 0)$; so $\ell_{7,-1}(P) = x^4 p(z)$ with $z = x^{1/7}y$ and $p(z) = \lambda_P h(z^7)^2$. The bracket of $\ell_{7,-1}(P)$ and $\ell_{7,-1}(Q)$ vanishes, both being powers of $x^2 h(t)$ up to constants [3, (D1)]; p has at least seven distinct roots, and $v_{1,-1}(2, 0) > 0$: we are in Case II.b). A root c of h of multiplicity k gives seven roots of p of multiplicity $2k$. The right-hand side of (2) is $2 \cdot \frac{7 \cdot 7 - 35}{6} = \frac{14}{3}$, so by (T6) some root of h has multiplicity $k \geq 3$, and by (T5) some $(2, 3)$-pair in $L^{(7)}$ has a regular corner at $(2, 0) + k(\frac{1}{7}, 1)$. For $k = 3$ this point is $(17/7, 3)$, contrary to (3); for $k = 4$ it is $(18/7, 4)$, contrary to Theorem 1.1. Hence $k = 5$

and $h = (t-c)^5$. The last assertion is then [3, Theorem 1.2], since (P, Q) now satisfies all its standing data. □

6. Scope

- **Proved here, relative to the cited results:** Theorem 1.1 and Corollary 1.2. The inputs are (T1)–(T6), that is [1, Theorem 2.6(4), Remark 2.8, Definitions 4.3, 5.5, Propositions 1.13, 4.6, 5.13, 5.14, 5.16, 5.17(3), 5.18, 5.19, Corollary 5.7, Remarks 3.1, 5.11, and the discussion of the cases before Remark 5.9], and, for the corollary, [1, Definition 6.2] and the theorem of [3], which itself rests on the results listed in the scope section of that note and on an exact computation.
- **Verified by exact computation:** Lemma 3.1 and the numerical values in the proofs (ancillary file `verify_7_35.py` in the folder `anc`; Python standard library only).
- **Not claimed:** a statement on the plane Jacobian conjecture itself; a new bound on the degree of a counterexample; a statement on the pairs with starting corner $(7, 35)$ and another starting triple.
- The argument has been checked only by the author with AI assistance (Claude, Anthropic); it has not been validated by professionals.

References

[1] C. Valqui, J. A. Guccione, J. J. Guccione, *On the shape of possible counterexamples to the Jacobian Conjecture*, J. Algebra **471** (2017), 13–74, doi:10.1016/j.jalgebra.2016.08.039; arXiv:1401.1784v3. Author order as in the journal; statement numbers refer to the arXiv version v3.

[2] J. A. Guccione, J. J. Guccione, R. Horruitiner, C. Valqui, *Some algorithms related to the Jacobian Conjecture*, arXiv:1708.07936v1.

[3] D. Niedbala Giraudin, *The degree pair* $(84, 126)$ *in the plane Jacobian problem: exclusion of the shape with final corner* $(19/7, 5)$, Hexagon `hexagon:2611.00710` (submitted 20 November 2026).

Independent researcher, France