A SHAPE OF MAXIMAL DEGREE 150 IN THE PLANE JACOBIAN PROBLEM: THE POINT $(18/5, 8)$ IS NOT A REGULAR CORNER

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Abstract. In the list of Guccione, Guccione, Horruitiner and Valqui of the possible shapes of a counterexample to the plane Jacobian conjecture with degrees at most 150, one shape of maximal degree 150 consists of the standard $(3,2)$-pairs whose chain of corners is $(10,40)$, $(18/5,8)$, $(8/5,3)$. We show that $(18/5,8)$ is not a regular corner of any $(3,2)$-pair in $K[x^{\pm 1/5},y]$; hence, by a theorem of the four authors, no such pair exists. As in the note [4] on the degree pair $(126,84)$, the proof of the theorem uses only results of Guccione–Guccione–Valqui, applied to every root of the edge polynomial, together with elementary arithmetic, which is reproduced by an ancillary script. The argument has not yet been examined by specialists.

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1. Statement

Throughout, K is an algebraically closed field of characteristic zero, $L^{(l)} = K[x^{\pm 1/l}, y]$, $[P,Q] = P_xQ_y - P_yQ_x$ and $(m,n) = (3,2)$. We use the notation of [1]: $v_{\rho,\sigma}$, $\ell_{\rho,\sigma}$, $\mathrm{st}_{\rho,\sigma}$, $\mathrm{en}_{\rho,\sigma}$; the interval of directions $I =](1,-1),(1,0)]$; *(m,n)-pairs* in $L^{(l)}$ and their *regular corners* [1, Definitions 4.3 and 5.5]. For a regular corner $((a/l,b),(\rho,\sigma))$ of (P,Q) we write $\ell_{\rho,\sigma}(P) = x^{k/l}p(z)$ with $z := x^{-\sigma/\rho}y$, and we distinguish, as in [1, Section 5, before Remark 5.9]: Case I, $[\ell_{\rho,\sigma}(P), \ell_{\rho,\sigma}(Q)] \neq 0$; Case II, the bracket vanishes and p has at least two distinct linear factors, the factor z included (this is how we read the condition $\#\,\mathrm{factors}(p) > 1$ of [1]); Case III, the bracket vanishes and $p = \alpha(z-\beta)^r$ with $\beta \neq 0$. One always has $v_{1,-1}(\mathrm{st}_{\rho,\sigma}(P)) \neq 0$ (loc. cit., by [1, Theorem 2.6(4)]); Case II is called II.a) if this number is negative and II.b) if it is positive.

Theorem 1.1. *No $(3,2)$-pair in $L^{(5)}$ has a regular corner at the point $(18/5,8)$.*

Corollary 1.2. *There is no standard $(3,2)$-pair (P,Q) whose chain of corners, in the sense of* [2, Theorem 2.20]*, contains $A_i = (18/5,8)$ with $l_i = 5$. In particular the shape with chain $(10,40)$, $(18/5,8)$, $(8/5,3)$ and maximal degree 150 of* [2, Section 6] *does not occur.*

Proof. By items (3) and (6) (or (9)) of [2, Theorem 2.20], $(A_i,(\rho_i,\sigma_i))$ is a regular corner of the $(3,2)$-pair (P_i,Q_i) in $L^{(l_i)}$, which contradicts Theorem 1.1 when $l_i = 5$. For the chain $(10,40)$, $(18/5,8)$, $(8/5,3)$, which is (A_0,A_1,A_2) in the notation of that theorem: $l_0 = 1$, and the corner $(A_0,(\rho_0,\sigma_0))$ is of Case II (item (6)). In Case II.a) item (7) gives $l_1 = 1$, hence $A_1 \in \mathbb{Z}\times\mathbb{N}_0$, which is false; so we are in Case II.b) and item (8) applies: A_1 lies on the line of direction $(-\sigma_0/\rho_0, 1)$ through $(k_0/m, 0)$, and so does $A_0 = \frac{1}{m}\,\mathrm{en}_{\rho_0,\sigma_0}(P)$. Since $A_0 - A_1 = (32/5, 32)$, we get $(\rho_0,\sigma_0) = (5,-1)$ and $l_1 = \mathrm{lcm}(5,1) = 5$. □

Remark 1.3. If (P,Q) is a $(2,3)$-pair with a regular corner, then (Q,P) is a $(3,2)$-pair with the same regular corner [1, Definition 4.3, Proposition 4.6(2),(3), Corollary 5.7(2),(3)]; so Theorem 1.1 also holds for $(2,3)$-pairs. The list of [2, Section 6] gives only one of the two orders of (m,n); for the same chain with $(m,n) = (2,3)$, of degrees $(100,150)$, [2, Theorem 2.20] gives $(2,3)$-pairs (P_i,Q_i) in $L^{(5)}$ with a regular corner at $(18/5,8)$, which is excluded in the same way.

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2. The tools

We use the following statements, for an (m,n)-pair (P,Q) in $L^{(l)}$ and a regular corner $((a/l,b),(\rho,\sigma))$.

(T1) [1, Proposition 5.19]: the point $(a/l,b)$ is the first entry of a regular corner of Case I or of Case II of a (possibly different) (m,n)-pair in $L^{(l)}$; in Case I, $l-a/b>1$; in Case II, $\gcd(a,b)>1$. The directions of these corners are not larger than (ρ,σ) (proof of that proposition, which uses [1, Proposition 5.17(3)]).

(T2) Case I. By [1, Proposition 1.13], $v_{\rho,\sigma}(P)+v_{\rho,\sigma}(Q)=\rho+\sigma$; moreover $\mathrm{en}_{\rho,\sigma}(P)=m(a/l,b)$ by definition and $\mathrm{en}_{\rho,\sigma}(P)=\frac{m}{n}\,\mathrm{en}_{\rho,\sigma}(Q)$ [1, Corollary 5.7(2)]. Hence

$$m\Big(\frac{a}{l}\rho+b\sigma\Big)=\frac{m}{m+n}(\rho+\sigma), \tag{1}$$

which determines (ρ,σ), and $v_{\rho,\sigma}(P)=\frac{m}{m+n}(\rho+\sigma)$, $v_{\rho,\sigma}(Q)=\frac{n}{m+n}(\rho+\sigma)$. By [1, Proposition 5.13], either $\frac{1}{m}\,\mathrm{st}_{\rho,\sigma}(P)=(a'/l,0)$ with $a'\in\mathbb{N}$ (statement and proof), or $\{\mathrm{st}_{\rho,\sigma}(P),\mathrm{st}_{\rho,\sigma}(Q)\}=\{(k/l,0),(1-k/l,1)\}$ with $k\in\mathbb{N}$.

(T3) Case II. By [1, Proposition 5.14], with $d:=\gcd(a,b)$, $\bar a:=a/d$, $\bar b:=b/d$: $d>1$, and there is $\mu\in\mathbb{N}$ with $d\nmid\mu$, $\mu\le l(bl-a)+1/\bar b$, such that (ρ,σ) is the direction orthogonal to $(\mu\bar a-l,\ \mu\bar b l-l)$; moreover $\rho/\gcd(\rho,l)\le b$. The point $A':=\frac{1}{m}\,\mathrm{st}_{\rho,\sigma}(P)$ lies in $\frac{1}{l}\mathbb{Z}\times\mathbb{N}_0$ (proof of that proposition), below $(a/l,b)$ on the line of direction $(-\sigma/\rho,1)$ through it, and $v_{1,-1}(A')\neq 0$ as recalled above. One has $\ell_{\rho,\sigma}(P)=\lambda R^m$ for a (ρ,σ)-homogeneous $R\in L^{(l)}$ [1, Remark 3.1 and Corollary 5.7].

(T4) Case II.a). $(A',\mathrm{Pred}_P(\rho,\sigma))$ is a regular corner of (P,Q) [1, Remark 5.11].

(T5) Case II, roots. Let $\lambda\in K^\times$ be a root of p of multiplicity m_λ with

$$m_\lambda\ \ge\ \frac{m}{l}\cdot\frac{a\rho+bl\sigma}{\rho+\sigma}. \tag{2}$$

Then, with $l':=\mathrm{lcm}(\rho,l)$ and $\varphi(y):=y+\lambda x^{\sigma/\rho}$, the pair $(\varphi P,\varphi Q)$ is an (m,n)-pair in $L^{(l')}$ with a regular corner at $(\frac{k}{ml},0)+\frac{m_\lambda}{m}(-\frac{\sigma}{\rho},1)$, of direction smaller than (ρ,σ) [1, Proposition 5.18].

The use of (T5) for a root other than the one continuing a chain, followed by (T1) and (T2), is modelled on the proof of [2, Proposition 6.1].

3. Arithmetic

In this section $(m,n)=(3,2)$. For a point $(a/l,b)$ we say that *Case I is arithmetically impossible* if, for the direction given by (1), none of the starting points allowed in (T2) exists, and that *Case II is arithmetically impossible below* (ρ_*,σ_*) if no μ as in (T3) gives a direction of I smaller than (ρ_*,σ_*). Put

$$C_\mu:=(\tfrac25,0)+\mu(\tfrac25,1)=\Big(\frac{2+2\mu}{5},\,\mu\Big),\qquad 1\le\mu\le 8,$$

so that $C_8=(18/5,8)$.

Lemma 3.1. *(a) At* $(18/5,8)$, $l=5$*: Case I is arithmetically impossible (direction* $(39,-17)$*); the directions allowed in Case II are* $(5,-2)$ *and* $(15,-4)$*, and none of them is smaller than* $(5,-2)$.

(b) At $C_1=(4/5,1)$, $l=5$*:* $l-a/b=1$ *and* $\gcd(4,1)=1$.

(c) At $C_2=(6/5,2)$, $l=5$*: Case I is arithmetically impossible (direction* $(9,-5)$*); the only direction allowed in Case II below* $(5,-2)$ *is* $(2,-1)$.

(d) At $C_4=(10/5,4)$, $l=5$*: Case I is arithmetically impossible (direction* $(19,-9)$*), and so is Case II below* $(5,-2)$.

(e) At $C_5=(12/5,5)$ *and* $C_7=(16/5,7)$, $l=5$*: Case I is arithmetically impossible (directions* $(24,-11)$ *and* $(34,-15)$*), and* $\gcd(12,5)=\gcd(16,7)=1$.

(f) At $(7/10,1)$, $l=10$*: Case I is arithmetically impossible (direction* $(8,-5)$*), and* $\gcd(7,1)=1$.

(g) At $(14/5, 5)$, $l = 5$*: Case I is arithmetically impossible (direction* $(24, -13)$*), and* $\gcd(14, 5) = 1$.

Proof. Direct computation, checked by the ancillary script; for instance in (c), (1) reads $(\frac{6}{5} - \frac{1}{5})\rho + (2 - \frac{1}{5})\sigma = 0$, so $(\rho, \sigma) = (9, -5)$, $v(P) = \frac{12}{5}$, $v(Q) = \frac{8}{5}$, and $\frac{9}{5} \cdot 3a' = \frac{12}{5}$, $\frac{9}{5}k = \frac{8}{5}$, $\frac{9}{5}k = \frac{12}{5}$ have no integer solution. □

By (T1), (T2), (T3) and Lemma 3.1:

(3) none of the points C_1, C_5, C_7, $(14/5, 5)$ ($l = 5$) is the first entry of a regular corner of a $(3, 2)$-pair in $L^{(5)}$, and C_4 ($l = 5$) is not the first entry of a regular corner of direction smaller than $(5, -2)$.

Lemma 3.2. *No* $(3, 2)$*-pair in* $L^{(5)}$ *has a regular corner at* $C_2 = (6/5, 2)$ *of direction smaller than* $(5, -2)$.

Proof. By (T1) and Lemma 3.1(c) there would be a $(3, 2)$-pair (P', Q') in $L^{(5)}$ with a regular corner $(C_2, (2, -1))$ of Case II. Put $z := x^{1/2}y$. The point A' of (T3) is $C_2 - h(1/2, 1)$ with $h \in \{1, 2\}$ and lies in $\frac{1}{5}\mathbb{Z} \times \mathbb{N}_0$, so $h = 2$ and $A' = (1/5, 0)$. Thus $\ell_{2,-1}(P') = x^{3/5}p(z)$ with $\deg p = 6$, $p(0) \neq 0$. By (T3), $\ell_{2,-1}(P') = \lambda' R^3$ with $R = x^{1/5}r(z) \in L^{(5)}$, $\deg r = 2$, $r(0) \neq 0$; since $x^{1/5}z^i = x^{(2+5i)/10}y^i$ lies in $L^{(5)}$ only for i even, r is a multiple of $z^2 - c$ with $c \neq 0$, and $p = \lambda'(z^2 - c)^3$. A root λ of $z^2 = c$ has $m_\lambda = 3$, and the right-hand side of (2) is $\frac{3}{5} \cdot \frac{6 \cdot 2 - 2 \cdot 5}{1} = \frac{6}{5}$. By (T5) some $(3, 2)$-pair in $L^{(10)}$ has a regular corner at $(\frac{1}{5}, 0) + (\frac{1}{2}, 1) = (7/10, 1)$, which contradicts (T1), (T2) and Lemma 3.1(f). □

4. Proof of Theorem 1.1

Assume that some $(3, 2)$-pair in $L^{(5)}$ has a regular corner at $A_1 := (18/5, 8)$. By (T1) and Lemma 3.1(a) there is a $(3, 2)$-pair (P, Q) in $L^{(5)}$ with a regular corner $(A_1, (\rho, \sigma))$ of Case II, with $(\rho, \sigma) \in \{(5, -2), (15, -4)\}$. Let A' be the point of (T3).

The direction $(15, -4)$. Here $A' = A_1 - h(4/15, 1)$ with $0 < h \leq 8$, and $A' \in \frac{1}{5}\mathbb{Z} \times \mathbb{N}_0$ forces $h \in \{3, 6\}$: $A' = (14/5, 5)$ or $A' = (2, 2)$. The second is excluded by $v_{1,-1}(A') \neq 0$. In the first, $v_{1,-1}(A') < 0$, so we are in Case II.a) and $(14/5, 5)$ is the first entry of a regular corner by (T4), contrary to (3).

The direction $(5, -2)$. Put $z := x^{2/5}y$. Now $A' = A_1 - h(2/5, 1) = C_{b'}$ with $b' = 8 - h \in \{0, \dots, 7\}$ (and $C_0 := (2/5, 0)$). The line of the edge meets the first axis at $(2/5, 0)$, so $\ell_{5,-2}(P) = x^{6/5}p(z)$ with $p = z^{3b'}\tilde{p}(z)$, $\tilde{p}(0) \neq 0$, $\deg \tilde{p} = 3(8 - b')$. Since $\ell_{5,-2}(P)$ is a cube by (T3),

$$\tilde{p}(z) = \lambda' \prod_i (z - c_i)^{3\mu_i}, \qquad c_i \in K^\times \text{ distinct}, \quad \mu_i \geq 1, \quad \sum_i \mu_i = 8 - b'.$$

The right-hand side of (2) is $\frac{3}{5} \cdot \frac{18 \cdot 5 - 8 \cdot 5 \cdot 2}{3} = 2$, so every root of $\tilde{p}$, of multiplicity $3\mu_i \geq 3$, satisfies (2). A root with $\mu_i = 8$ would force $b' = 0$ and $p = \lambda'(z - c_i)^{24}$, which has a single linear factor and is not Case II; so $\mu_i \leq 7$. By (T5), for every i some $(3, 2)$-pair in $L^{(5)}$ has a regular corner at C_{μ_i}, of direction smaller than $(5, -2)$. By (3) and Lemma 3.2, $\mu_i \notin \{1, 2, 4, 5, 7\}$, so $\mu_i \in \{3, 6\}$ for every i. Then $8 - b' \in \{3, 6\}$, that is, $A' = C_5 = (12/5, 5)$ or $A' = C_2 = (6/5, 2)$. In both cases $v_{1,-1}(A') < 0$, so we are in Case II.a) and, by (T4), A' is the first entry of a regular corner of (P, Q) of direction smaller than $(5, -2)$. This contradicts (3) for C_5 and Lemma 3.2 for C_2. □

Remark 4.1. The corner $C_3 = (8/5, 3)$, which ends the chain of [2], passes the tests used here: Case I is arithmetically possible there, with direction $(2, -1)$. The contradiction comes from the other roots of the edge polynomial and from the starting point of the edge.

5. Scope

- **Proved here, relative to the cited results:** Theorem 1.1 and Corollary 1.2. The inputs are (T1)–(T5), that is [1, Theorem 2.6(4), Remark 2.8, Definitions 4.3, 5.5, Propositions 1.13, 4.6, 5.13, 5.14, 5.17(3), 5.18, 5.19, Corollary 5.7, Remarks 3.1, 5.11, and the discussion of the cases before Remark 5.9], and [2, Theorem 2.20] for the corollary. The reference [2] is an arXiv preprint; of its Theorem 2.20 we use items (3), (6), (8), (9), and item (7) only at the first corner, where $l_0 = 1$. Three of these inputs are taken from proofs rather than statements, as indicated in Section 2: the monotonicity of the directions in (T1), the integrality of a' in (T2), and the lattice of A' in (T3).
- **Verified by exact computation:** Lemma 3.1 and the numerical values in the proofs (ancillary file `verify_150.py` in the folder `anc`; Python standard library only).
- **Not claimed:** a statement on the plane Jacobian conjecture itself, or on the other shapes of [2]; a new bound on the degree of a counterexample (the published one, [3], concerns degrees below 125 and does not involve this shape). The completeness of the list of [2] is not re-examined here.
- The argument has been checked only by the author with AI assistance (Claude, Anthropic); it has not been validated by professionals.

References

[1] C. Valqui, J. A. Guccione, J. J. Guccione, *On the shape of possible counterexamples to the Jacobian Conjecture*, J. Algebra **471** (2017), 13–74, doi:10.1016/j.jalgebra.2016.08.039; arXiv:1401.1784v3. Author order as in the journal; statement numbers refer to the arXiv version v3.

[2] J. A. Guccione, J. J. Guccione, R. Horruitiner, C. Valqui, *Some algorithms related to the Jacobian Conjecture*, arXiv:1708.07936v1.

[3] J. A. Guccione, J. J. Guccione, R. Horruitiner, C. Valqui, *Increasing the degree of a possible counterexample to the Jacobian Conjecture from 100 to 108*, arXiv:2204.14178.

[4] D. Niedbala Giraudin, *The degree pair* $(126, 84)$ *in the plane Jacobian problem: the point* $(16/3, 10)$ *is not a regular corner*, Hexagon `hexagon:2611.00709` (submitted 20 November 2026).

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