QUANTUM AFFINE REPRESENTATIONS AND CLUSTER CHARACTERS

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Abstract. We prove that cluster monomials in the shifted cluster algebra of finite simply laced type represent real simple classes. We establish the correspondence between rigid objects and real simple modules in every finite Hernandez–Leclerc window after removing frozen tensor factors, including the correspondence between indecomposable objects and prime modules, and prove the generic-kernel formula for their truncated q-characters. We disprove Hernandez's conjectural description of simple modules over truncated shifted quantum affine algebras by a type G_2 counterexample.

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1. Introduction

Quantum affine representations carry two related kinds of character data. Ordinary weights record the action of the finite Cartan algebra; rational ℓ-weights record the joint eigenvalues of all Drinfeld Cartan currents. Cluster structures on their Grothendieck rings relate tensor or fusion products to mutation, while quiver Grassmannians express character coefficients through Euler characteristics. We prove the shifted cluster-monomial assertion of Hernandez and the two real–rigid and generic-kernel conjectures of Duan–Schiffler. We also give a counterexample to Hernandez's finite-chain description of simple modules over truncated shifted quantum affine algebras.

1.1. **Characters and cluster categorification.** The quantum groups of Drinfeld and Jimbo arose from deformations of enveloping algebras and the Yang–Baxter equation [Dri87, Jim85]. For quantum affine algebras, the current presentation makes the Cartan eigenvalues into rational functions. Chari and Pressley classified finite-dimensional simple representations in rank one and then developed the general classification by Drinfeld polynomials [CP91, CP94]. This parametrization specifies the highest ℓ-weight, but the remaining ℓ-weights and the decomposition of tensor products require finer information.

Frenkel and Reshetikhin introduced the q-character homomorphism [FR99]. It takes a finite-dimensional representation to the sum of its ℓ-weight monomials, with their multiplicities, and is injective on the Grothendieck ring. Multiplication of q-characters records tensor product. The highest monomial of a simple finite-dimensional module is dominant, and its other terms are obtained by multiplying inverse simple-root monomials. In simply laced type, Nakajima's quiver-variety construction realizes representations and their character data geometrically, with intersection cohomology distinguishing simple objects from standard ones [Nak01]. This supplies both a deformation of the character ring and positive multiplicity polynomials.

Fomin and Zelevinsky introduced cluster algebras in their study of total positivity and canonical bases [FZ02]. Their exchange relations generate distinguished Laurent polynomials from an initial seed. Additive categorification gives a different interpretation of these relations: indecomposable summands of reachable cluster-tilting objects represent cluster variables, and sums of summands from one such object represent cluster monomials. Caldero and Chapoton expressed cluster variables through Euler characteristics of quiver Grassmannians in the Dynkin case [CC06]. Palu extended cluster characters and their multiplication formula to triangulated 2-Calabi–Yau categories [Pal08]. Amiot constructed the generalized cluster categories associated with Jacobi-finite quivers with potential [Ami09]. These constructions provide the additive categories and the F-polynomials used below.

Hernandez and Leclerc introduced monoidal categorification by quantum affine representations [HL10]. Here a simple module is *real* if its tensor square is simple. A nontrivial simple module is *prime* if it has no factorization into two nontrivial tensor factors. They introduced finite level subcategories $\mathcal{C}_\ell$ and related their Grothendieck rings to cluster algebras. Nakajima proved the corresponding results in the level-one geometric setting [Nak11]. Hernandez's tensor-product theorem shows that a product of finite-dimensional simple modules is simple exactly when every pairwise product is simple [Her10]. In particular, all positive tensor powers of a real simple module are simple.

Hernandez and Leclerc subsequently constructed a cluster algorithm for Kirillov–Reshetikhin q-characters and introduced the generic-kernel description in terms of a quiver with potential [HL16a]. Their formula identifies normalized characters with F-polynomials obtained from general maps between injective modules. The relation between real simple representations and arbitrary rigid objects goes beyond the objects produced by a specified sequence of mutations. The monoidal categorification results developed by Kashiwara, Kim, Oh, and Park [KKOP21], and the cluster structures considered by Duan and Schiffler [DS26], provide the cluster-monomial direction.

Duan and Schiffler allow an arbitrary height function ξ on the Dynkin diagram and define the finite categories $\mathcal{C}_\ell^{\leq\xi}$ [DS26]. They construct the compatible quantum seeds and their finite Jacobian algebras, prove the real–rigid correspondence at level one, and establish further cases of finite cluster type. Their journal Conjectures 6.1 and 6.2 ask for the correspondence and the generic-kernel formula at every finite level. These are Conjectures 5.21 and 5.22 in the second arXiv version. The correspondence includes all rigid objects and all real simples after removal of the frozen tensor factors.

The classification theorem of Duan and Schiffler in [DS25, Theorem 3.5] assumes multiplicative reachability; its Corollary 3.7 concerns reachable objects. Baur, Fu, and Li formulate a general correspondence between additive and monoidal categorifications and prove further relations and cases [BFL24]; the general real–rigid comparison remains their Conjecture 3.14. Neguţ's quiver-moduli construction gives character formulas involving critical K-theory [Neg25]. The formula in Duan–Schiffler Conjecture 6.2 uses the ordinary Euler characteristic of the prescribed generic kernel. Theorems 1.2 and 1.3 give these two original conclusions for every finite height window.

1.2. **Shifted algebras and truncations.** Hernandez and Jimbo constructed asymptotic representations and the prefundamental modules of quantum affine Borel category $\mathcal{O}$ [HJ12]. Their highest ℓ-weights are rational functions, and their transfer matrices enter Baxter relations. Hernandez and Leclerc identified cluster structures in suitable Grothendieck rings of this category [HL16b].

Finkelberg and Tsymbaliuk introduced shifted quantum affine algebras and their relations to quantized K-theoretic Coulomb branches [FT19]. Hernandez developed their category

$\mathcal{O}$, highest-weight classification, fusion products, finite-dimensional modules, and polynomial truncations [Her23]. A positive prefundamental shifted module is one-dimensional. The finite-dimensional shifted categories therefore provide initial seeds different from the finite-dimensional ordinary categories used in the Duan–Schiffler problem. Fusion gives multiplication on the sum of shifted Grothendieck groups; Hernandez and Zhang proved closure of finite-length modules under this multiplication [HZ25].

Geiss, Hernandez, and Leclerc constructed, in simply laced type, a family of shifted cluster structures indexed by the Weyl group [GHL24]. The longest Weyl group element gives the cluster algebra whose completion is the shifted Grothendieck ring. Its initial variables are the Weyl-indexed Q-variables related by extended QQ systems [FH24, GHL24]. Their general cluster-monomial simplicity assertion appears as Conjecture 9.16 in [GHL24] and as Conjecture 9.10 in Hernandez's ICM lecture [Her26].

The subsequent work of Kamnitzer, Labelle, Leroux-Lapierre, Pinet, and Weekes relates truncated shifted Yangians and the bi-infinite Bott–Samelson variety and establishes the relevant simply laced Q-variable results [KLLL$^+$26]. Geiss, Hernandez, and Leclerc extend the cluster constructions to nonsimply laced types and develop cluster expansions in [GHL26]. That paper distinguishes the simple Q-variables from the general cluster-monomial assertion, which appears as its Conjectures 11.11 and 11.18. Our first theorem concerns the original finite simply laced assertion, with the prescribed embedding and its weight normalization.

Truncations introduce a different question. For a tuple of polynomials P with constant term 1, the auxiliary Cartan series become polynomials of prescribed degrees. Hernandez related the resulting simple modules to mixed characters constructed using Langlands dual data [Her23]. The formulation in [Her26, Conjecture 11.4] permits a finite sequence of polynomial truncation parameters, each obtained from the mixed character of its predecessor. Its Example 11.7 already shows why one step does not suffice in general. Varagnolo and Vasserot construct geometric categories, completed-localization equivalences, and character comparisons for shifted affine quantum groups and Coulomb branches [VV25]. Their monomial-crystal identification remains Conjecture 4.4.7. Our type G_2 example constructs a simple in the original polynomial truncation whose highest label makes the two assertions of the ICM conjecture incompatible.

1.3. **Statements of the results.** Fix $q \in \mathbb{C}^*$ which is not a root of unity. In the first three theorems, $\mathfrak{g}$ is a finite-dimensional simple Lie algebra of type A, D, or E, with Dynkin vertex set I. For a highest ℓ-weight h, write $L(h)$ for the corresponding simple module. Multiplication in shifted category $\mathcal{O}$ means fusion; multiplication in the ordinary finite-dimensional category means tensor product.

For the first theorem, let P be the weight lattice, $\overline{P} = \frac{1}{2}P$, and let ω_i be the fundamental weights. Write $P_{i,r}$ for the rational ℓ-weight whose ith component is $1 - q^r x$ and whose other components are 1. For a pure divisor $p = \prod P_{i,r}^{e_{i,r}}$, put

$$\Omega(p) = \sum_{i,r} \frac{r}{2} e_{i,r} \omega_i.$$

The symbol $[\lambda]$ denotes the constant weight class. Let $\mathcal{A}_{w_0} = \mathcal{A}_{\Gamma'_\infty}$ be the shifted cluster algebra of [GHL24, Her26], and let $\mathcal{I}$ be its prescribed embedding in the shifted Grothendieck ring with coefficients extended from $\mathbb{Z}[P]$ to $\mathbb{Z}[\overline{P}]$. Its explicit seeds, formal series, and normalization are recalled in Section 2.

Theorem 1.1. *For every cluster monomial M of $\mathcal{A}_{\Gamma'_\infty}$ there is a pure highest divisor p_M such that*

$$\mathcal{I}(M) = [-\Omega(p_M)]\,[L(p_M)].$$

The fusion square of $L(p_M)$ is simple. Thus every cluster monomial represents a real simple class under the prescribed weight normalization.

For the next two theorems, a height function is a map $\xi : I \to \mathbb{Z}$ with $|\xi_i - \xi_j| = 1$ whenever i, j are adjacent. For $\ell \geq 1$, put

$$\widetilde{W} = \{(i,r) : r = \xi_i - 2\ell, \xi_i - 2\ell + 2, \ldots, \xi_i\}, \qquad W = \{(i,r) \in \widetilde{W} : r > \xi_i - 2\ell\}.$$

Let $\mathcal{C}_\ell^{\leq \xi}$ be the category of ordinary finite-dimensional type-one modules whose simple constituents have dominant highest monomials $m = \prod_{\widetilde{W}} Y_{i,r}^{u_{i,r}}$, $u_{i,r} \in \mathbb{N}$. Here $Y_{i,r} = Y_{i,q^r}$ is the usual fundamental ℓ-weight variable. The frozen highest monomials are

$$f_i = \prod_{r = \xi_i - 2\ell, \,\text{step}\, 2}^{\xi_i} Y_{i,r}.$$

Let $A = A_\ell^\xi$ be the finite mutable Jacobian algebra of [DS26, Section 3.2], let $\mathcal{C}_A$ be its Amiot category, and let T be its canonical cluster-tilting object. An object M is rigid if $\mathrm{Ext}^1_{\mathcal{C}_A}(M, M) = 0$. Its Duan–Schiffler degree is $g(M) = \mathrm{ind}_T(M[-1]) \in \mathbb{Z}^W$. The index is the difference of the multiplicities of the two terms in a minimal $\mathrm{add}(T)$ presentation. The quiver, potential, and these conventions are specified in Section 3.

For a highest monomial m, extend u by zero outside $\widetilde{W}$ and define

$$g_{i,r}(m) = u_{i,r} - u_{i,r-2}, \qquad (i,r) \in W. \tag{1.1}$$

For any $g \in \mathbb{Z}^W$, its unique nonnegative row multiplicities u satisfying (1.1) and $\min_r u_{i,r} = 0$ are obtained by integrating these differences and subtracting the row minimum.

Theorem 1.2. *The equality $g(M) = g(m)$ defines a bijection between isomorphism classes of rigid objects of $\mathcal{C}_A$ and real simple modules $L(m)$ in $\mathcal{C}_\ell^{\leq \xi}$ having no tensor factor $L(f_i)$. Under this bijection, indecomposable rigid objects correspond exactly to real prime simple modules other than the $L(f_i)$.*

Let $I_{i,r}$ be the indecomposable injective A-module with simple socle at (i,r). Define $K(m)$ as the kernel of a general map

$$\bigoplus_{(i,r)\in W} I_{i,r}^{[-g_{i,r}(m)]_+} \longrightarrow \bigoplus_{(i,r)\in W} I_{i,r}^{[g_{i,r}(m)]_+}, \qquad [a]_+ = \max(a, 0). \tag{1.2}$$

For a finite-dimensional A-module K, put

$$F_K(y) = \sum_{e \in \mathbb{N}^W} \chi(\mathrm{Gr}_e(K)) \prod_{(i,r)\in W} y_{i,r}^{e_{i,r}},$$

where $\mathrm{Gr}_e(K)$ is the projective variety of submodules of dimension vector e, and χ is the topological Euler characteristic. Define the ordinary root monomial

$$A_{i,r-1} = Y_{i,r-2} Y_{i,r} \prod_{j \sim i} Y_{j,r-1}^{-1}.$$

The truncation $\chi_q(L(m))^{\leq \xi}$ retains the monomials in the original Hernandez–Leclerc spectral window; its exact relation to the root variables is proved in Proposition 6.6.

Theorem 1.3. *For every real simple $L(m) \in \mathcal{C}_\ell^{\leq \xi}$ having no tensor factor $L(f_i)$, the general kernel in (1.2) satisfies*

$$m^{-1} \chi_q(L(m))^{\leq \xi} = F_{K(m)}\big((A_{i,r-1}^{-1})_{(i,r)\in W}\big).$$

For the last theorem, take $\mathfrak{g}$ of type G_2, with vertex 1 long, Cartan matrix $\left(\begin{smallmatrix} 2 & -1 \\ -3 & 2 \end{smallmatrix}\right)$, and $(q_1, q_2) = (q^3, q)$. The notation $z_{i,k}$ denotes the tuple with component $1-q^k x$ at i and component 1 at the other vertex.

Theorem 1.4. *Hernandez's Conjecture 11.4 is false. Its two assertions are incompatible for the polynomial parameters $P_L = z_{2,-3}$, $P_R = z_{2,1}$, and $P = P_L P_R$. The original polynomial truncation at P has the simple highest-weight module with rational weight*

$$\Psi_{*,1}(x) = q^4(1-q^{-6}x)(1-q^{-4}x)(1-x)(1-q^2x),$$
$$\Psi_{*,2}(x) = q^{-4}\frac{1-q^{-3}x}{(1-q^{-7}x)(1-q^{-1}x)}.$$

1.4. **Proofs.** The proof of Theorem 1.1 compares a finite set of mutations in an inserted shifted seed with mutations in the finite-dimensional shifted category. The comparison keeps the full exchange divisor at each relevant vertex. Its error is confined to spectral parameters below a movable lower boundary. The local shuffle description of Hernandez and Neguţ [HN26] implies that multiplication of a highest weight by a unit at a fixed finite spectral set leaves the corresponding normalized character coefficients unchanged. This identifies the prescribed QQ solution with actual simple characters and then treats a whole cluster monomial and its square.

For Theorem 1.2, tensor powers and local residue bounds first force strict inequalities on the nonhighest support of a real simple character. Quantum triangular and theta bases convert these inequalities into the absence of semistable walls next to its degree vector. A source-framing calculation detects the actual walls, including their critical multiplicities. The two-term silting chambers then give a rigid object with the required degree.

For the reverse direction, an exact category of projective modules over the Dynkin preprojective algebra, equipped with a degree-two operator, lifts every rigid object. A framing potential realizes the submodule varieties of its generic kernel as proper components of critical loci. The decomposition theorem gives nonnegative integer coefficients in the simple quantum-character basis. Hyperbolic restriction gives an exact tensor-power identity. A degree comparison in the quantum torus eliminates every lower simple constituent. The weight-one action on the potential then identifies the resulting character with the ordinary Euler F-polynomial. Restriction of injective copresentations and a maximal spectral-excess argument return this formula to the original finite window, proving Theorem 1.3. An evaluation-map calculation identifies direct-sum and prime decompositions.

The proof of Theorem 1.4 constructs a simple from the small resolution of the hypersurface $BA_L = 0$. Explicit current and Cartan formulas give its action through the original polynomial truncation. A rank-one descendant excludes its highest label from the product of the two one-factor highest-label sets. Finally, a nonnegative period-24 weight on spectral factors shows that P has no proper polynomial successor. The two assertions of the conjecture then force contradictory character supports.

2. Cluster monomials for shifted quantum affine algebras

Throughout this section, $\mathfrak{g}$ has finite type A, D, or E, and $q \in \mathbb{C}^*$ is not a root of unity. Let I be the set of vertices of its Dynkin diagram, $C = (c_{ij})$ its Cartan matrix, P its weight lattice, ω_i its fundamental weights, and $\alpha_i = \sum_j c_{ji}\omega_j$ its simple roots. Write $j \sim i$ when $c_{ij} = -1$, and let W and w_0 be its Weyl group and longest element. Multiplication of shifted representation classes means fusion multiplication.

2.1. **Highest weights, normalization, and formal series.** For $i \in I$ and $a \in \mathbb{C}^*$, let $\Psi_{i,a}$ be the rational ℓ-weight whose ith component is $1 - au$ and whose other components are 1. A *pure divisor* is a finite Laurent monomial in these $\Psi_{i,a}$. For $\lambda \in P$, the symbol $[\lambda]$ denotes the constant ℓ-weight with components $q^{\langle \lambda, \alpha_i^\vee \rangle}$, and also the corresponding invertible one-dimensional class. Set $\overline{P} = \frac{1}{2}P$ and extend scalars to $\mathbb{Z}[\overline{P}]$ when normalizing cluster variables. This is the coefficient extension in [GHL24, §7.1 and Theorem 9.15].

Every rational highest ℓ-weight h, regular and nonzero at $u = 0$, defines a simple shifted module $L(h)$. Its shift is determined by the degrees of the components of h. The highest ℓ-weight space has dimension one, and all its other ℓ-weights are obtained by multiplication by inverse root monomials

$$Y_{i,a} = [\omega_i] \frac{\Psi_{i,aq^{-1}}}{\Psi_{i,aq}}, \qquad A_{i,a} = Y_{i,aq^{-1}} Y_{i,aq} \prod_{j \sim i} Y_{j,a}^{-1}. \tag{2.1}$$

These facts, the fusion multiplication, and the injectivity of the q-character map are established in [Her23]; see also [GHL24, §9.1]. For any rational ℓ-weight h, the symbol $[h]$ denotes its basis monomial in the ℓ-weight group algebra. We use

$$F(h) = \frac{\chi_q L(h)}{[h]}. \tag{2.2}$$

Constant weight twists leave $F(h)$ unchanged.

Choose a bipartite coloring $\epsilon_i \in \{0, 1\}$ and put

$$V = \{(i, r) \in I \times \mathbb{Z} : r \equiv \epsilon_i \pmod 2\}, \qquad P_{i,r} = \Psi_{i,q^r}.$$

Let $\mathcal{P}_V$ be the free abelian group on the $P_{i,r}$, $(i, r) \in V$. For $p = \prod_{i,r} P_{i,r}^{e_{i,r}} \in \mathcal{P}_V$, define

$$\Omega(p) = \sum_{i,r} \frac{r}{2} e_{i,r} \omega_i, \qquad \nu(p) = [-\Omega(p)]p, \qquad \operatorname{supp}_\Psi(p) = \{r : e_{i,r} \neq 0 \text{ for some } i\}. \tag{2.3}$$

We set $\max \operatorname{supp}_\Psi(1) = -\infty$. The normalization ν is multiplicative. In particular, if

$$d_{i,r} = \frac{P_{i,r+2}}{P_{i,r-2}} \prod_{j \sim i} \frac{P_{j,r-1}}{P_{j,r+1}}, \tag{2.4}$$

then $\Omega(d_{i,r}) = \alpha_i$ and $\nu(d_{i,r}) = A_{i,q^r}^{-1}$.

Lemma 2.1. *The root monomials $A_{i,a}$ are multiplicatively independent for finite-support integer exponents. Consequently their nonnegative monoid is free and pointed.*

Proof. Partition $\mathbb{C}^*$ into its $q^{\mathbb{Z}}$-orbits. Each root monomial involves only one orbit, so it suffices to work on an orbit $a_0 q^{\mathbb{Z}}$. Encode a finite exponent sequence by a Laurent polynomial in z, with multiplication by z representing an increase of the spectral exponent by one. The map from the exponents of the inverse root monomials to their pure Ψ-divisors is the matrix

$$B(z)_{ii} = z^2 - z^{-2}, \qquad B(z)_{ij} = z^{-1} - z \quad (j \sim i), \qquad B(z)_{ij} = 0 \quad (i \neq j,\ j \not\sim i).$$

At $z = 1$ its expansion is $B(z) = 2(z-1)C + O((z-1)^2)$. Since $\det C \neq 0$, $\det B(z)$ is a nonzero Laurent polynomial. The matrix is therefore injective on $\mathbb{Z}[z, z^{-1}]^I$: multiply a vector in its kernel by its adjugate and use that $\mathbb{Z}[z, z^{-1}]$ is an integral domain. A multiplicative relation between full root monomials would induce a relation between their pure divisors, and thus has all exponents zero. A product of nonnegative powers can equal 1 only when every power is zero, which proves pointedness. □

Put $t_{i,a} = A_{i,a}^{-1}$. Let $\mathscr{M}$ be the free commutative monoid of finite-support nonnegative exponent vectors in the variables $t_{i,a}$, and use the ring

$$\mathscr{R} = \prod_{m \in \mathscr{M}} \mathbb{Z} m, \qquad (fg)_m = \sum_{m_1 m_2 = m} f_{m_1} g_{m_2}. \tag{2.5}$$

Each sum in this definition is finite. This ring allows infinitely many different spectral variables in one total degree. A series with constant coefficient 1 is invertible: in $(1+h)^{-1} = \sum_{k \geq 0} (-h)^k$, the coefficient of a monomial of total degree d receives contributions only from $k \leq d$. For a finite colored spectral set $S \subset I \times \mathbb{C}^*$, let

$$\pi_S : \mathscr{R} \longrightarrow \mathbb{Z}[[t_{i,a} : (i,a) \in S]]$$

set all variables outside S equal to zero. This is a ring homomorphism and commutes with inversion of constant-one series. The maps π_S jointly separate elements of $\mathscr{R}$, since every monomial has finite support. Formula (2.2) belongs to $\mathscr{R}$, has constant coefficient 1, and has nonnegative integer coefficients.

2.2. **Invariance under local units.** We will use the local description of shifted simple modules in [HN26, Corollary 3.9 and equations (132)–(136)]. We specify the coordinate convention. For an ℓ-weight $h(u)$, the corresponding function in the coordinate of that paper is $\psi(z) = h(z^{-1})$. Thus $1 - au$ becomes $1 - a/z$, and its zero is the local point $z = a$. The root parameter a in $A_{i,a}$ is unchanged.

Lemma 2.2 (Local units)**.** *Let h and g be rational ℓ-weights. Suppose that, for every $(i,a) \in S$, the function $g_i(z^{-1})$ is regular and nonzero at $z = a$. Then*

$$\pi_S F(hg) = \pi_S F(h). \tag{2.6}$$

In particular, if every $h_i(z^{-1})$ is regular and nonzero at the points of S of color i, then $\pi_S F(h) = 1$.

Proof. Fix a colored multiset $\mathbf{x} = ((i_1, x_1), \dots, (i_n, x_n))$ supported in S, with n_i entries of color i. Repetitions of a spectral point are allowed. Let $\mathcal{S}_{-\mathbf{n}}$ denote the corresponding negative shuffle space. It is the fixed space, independent of h, in the shuffle realization used in [HN26, §2]. Its elements are rational functions, symmetric in the variables of each color, with products of linear differences as denominators. The local quotient theorem identifies the multiplicity of $\prod_{b=1}^n t_{i_b,x_b}$ in $F(h)$ with

$$\dim_{\mathbb{C}} (\mathcal{S}_{-\mathbf{n}} / J(h)_{\mathbf{x}}).$$

Here $J(h)_{\mathbf{x}}$ consists of the shuffle functions f for which

$$\mathrm{Res}_{z_n = x_n} \cdots \mathrm{Res}_{z_1 = x_1} \frac{f(z_1, \dots, z_n)\, p(z_1, \dots, z_n) \prod_{b=1}^n h_{i_b}(z_b^{-1})}{\prod_{b<c} \zeta_{i_c i_b}(z_c / z_b)} = 0 \tag{2.7}$$

for every polynomial p and every ordering compatible with the colored multiset, where $\zeta_{ij}(z) = (z - q^{-c_{ij}})/(z-1)$. Residues are taken in the iterated local expansions of the cited formula. Using dz_b/z_b in place of dz_b multiplies the displayed integrand by regular units and has no effect on this argument.

We first verify the finite-jet assertion needed for (2.7). Set $y_b = z_b - x_b$, fix f and an ordering, and remove the polynomial p from the integrand. Near $\mathbf{x}$, its denominator is a product of powers of linear forms in the y_b, multiplied by an invertible regular function. Indeed, the shuffle denominators and the factors of ζ^{-1} are linear differences; poles of a component of h give powers of an individual y_b. A factor which does not vanish at $\mathbf{x}$ is a unit. Let D be the total degree of the vanishing denominator factors, counted with multiplicity. Expanding reciprocal linear forms in any of the prescribed orders preserves their homogeneous degree. The resulting

rational germ consequently has no homogeneous term of total degree less than $-D$. If a regular multiplier vanishes to order greater than $D+n$, every term of the product has degree greater than $-n$. Its coefficient of $y_1^{-1}\cdots y_n^{-1}$ is therefore zero. The iterated residue depends on only a finite Taylor jet of the multiplier. This reasoning also applies when several x_b coincide: factors $y_b - y_c$ still have homogeneous degree one.

Now put $G(z_1,\dots,z_n) = \prod_b g_{i_b}(z_b^{-1})$. Both G and G^{-1} are regular at $\mathbf{x}$. For fixed f and p, replace Gp by a polynomial having the required finite Taylor jet. If $f \in J(h)_{\mathbf{x}}$, equation (2.7) for that polynomial shows that the residue for hg and p vanishes. There are finitely many orderings; one can choose a jet order large enough for all of them. Thus $J(h)_{\mathbf{x}} \subseteq J(hg)_{\mathbf{x}}$. Apply the same argument to G^{-1} and hg to obtain the reverse inclusion. The annihilator subspaces, and hence their quotient dimensions, are equal. Lemma 2.1 identifies these dimensions with the coefficients of distinct root monomials. Varying $\mathbf{x}$ over all multisets in S proves (2.6) in every degree. Taking the reference highest weight to be 1, whose simple module is one-dimensional, gives the last assertion. □

If a pure divisor is supported on the $P_{i,r}$ with $r \le b$ and $b \to -\infty$, it is eventually a unit at every fixed finite set S. This follows from the infinite order of q: a fixed nonzero complex number equals at most one q^r. Lemma 2.2 applies to any finite product of such divisors, with positive or negative powers. Its conclusion does not depend on the number of factors.

2.3. **The prescribed shifted seeds.** The basic quiver Γ_e has vertex set V and arrows

$$(i,r) \longrightarrow (i,r+2), \qquad (i,r) \longrightarrow (j,r-1) \quad (j \sim i). \tag{2.8}$$

All vertices are mutable. Its initial variable at (i,r) has q-character $\nu(P_{i,r})$, the normalized one-dimensional positive prefundamental shifted class. Denote its cluster algebra by $\mathcal{A}_e$. The input from finite-dimensional representation theory is

$$\text{every cluster monomial of } \mathcal{A}_e \text{ is a real simple shifted class.} \tag{2.9}$$

This is [Her23, Theorem 8.10(ii)]; part (i) identifies the ring and its finite-dimensional shifted category. The same starting algebra is described in [GHL24, Theorem 9.18]. Consequently every finite mutation sequence from (2.8) consists of actual simple classes, and every monomial in any resulting cluster is an actual simple class.

We recall explicitly the inserted seeds of [GHL24, §3]. In a region identified with Γ_e, an insertion of color i at height a has the following rule. Insert a vertex $*$ between (i,a) and $(i,a-2)$; replace $(i,a-2) \to (i,a)$ by $(i,a) \to * \leftarrow (i,a-2)$; and replace each $(i,a) \to (j,a-1)$, $j \sim i$, by $* \to (j,a-1)$. Give $*$ the label $(i,a-2)$ and decrease by two the second coordinate of each old vertex of color i below it. The upper vertex (i,a) is called red and the inserted vertex is called green. In particular, the vertex set remains V.

The pure braid operators are

$$T_i' P_{j,r} = P_{j,r} \quad (j \ne i), \qquad T_i' P_{i,r} = P_{i,r-2}^{-1} \prod_{j \sim i} P_{j,r-1}. \tag{2.10}$$

They act multiplicatively and satisfy the braid relations. For a reduced expression of w, let T_w' be the resulting product. The chamber label $T_w' P_{i,r}$ depends only on $w\omega_i$; these are the highest monomials in [GHL24, §7.1] and [GHL26, §§7–9].

Fix a reduced word $\mathbf{i} = (i_1,\dots,i_\ell)$ and its prefixes $w^{(m)} = s_{i_1}\cdots s_{i_m}$, with $w^{(0)} = e$. Put

$$n_j^{(m)} = \#\{b \le m : i_b = j\}.$$

Choose integers $r_1, \dots, r_\ell$ with $r_m \equiv \epsilon_{i_m} \pmod 2$ and large decreasing gaps. After the first $m-1$ insertions, use the tail coordinates

$$h(j,r) = r + 2n_j^{(m-1)}. \tag{2.11}$$

Perform the next insertion at tail height r_m, that is, at physical vertex $(i_m, r_m - 2n_{i_m}^{(m-1)})$. Choose the gaps so that its entire local pattern is below the preceding patterns. These are the separated adapted insertion seeds used below. Their pure labels are fixed by the following rule. Above a new insertion retain the old labels. In the old tail the labels are $T'_{w^{(m-1)}}P_{j,h}$; after the insertion, the new tail labels are $T'_{w^{(m)}}P_{j,h'}$, where $h' = r + 2n_j^{(m)}$ is the new tail coordinate. In particular, the red and green labels are

$$T'_{w^{(m-1)}}P_{i_m,r_m}, \qquad T'_{w^{(m)}}P_{i_m,r_m}, \tag{2.12}$$

respectively. Formula (2.10) shows that labels in the other columns are unchanged in the common tail.

For the original embedding, let $\mathcal{Q}_{w,i,r} \in \mathscr{R}$ be the constant-one inverse-root series of the GHL Q-variable with highest monomial $T'_wP_{i,r}$, and write

$$\mathbf{Q}_{w,i,r} = \nu(T'_wP_{i,r})\mathcal{Q}_{w,i,r}. \tag{2.13}$$

These are the identity-chamber components of the formal Q-variables constructed in [GHL24, Definitions 6.1 and 6.3, §7.1], with the weight normalization of its §7.2. In particular, $\mathcal{Q}_{e,i,r} = 1$, and the pair consisting of its highest monomial and its series depends only on the chamber $w\omega_i$. The family can equivalently be specified by these properties and the following QQ relations; their constant-one solution is unique by the coefficient argument in Proposition 2.7. The renormalized QQ relation is

$$\mathbf{Q}_{ws_i,i,r}\mathbf{Q}_{w,i,r-2} = \prod_{j\sim i}\mathbf{Q}_{w,j,r-1} + \mathbf{Q}_{w,i,r}\mathbf{Q}_{ws_i,i,r-2}, \qquad \ell(ws_i) = \ell(w)+1. \tag{2.14}$$

It is [GHL24, Proposition 7.2], also [GHL26, Proposition 9.3]. At each vertex of an inserted seed, assign the $\mathbf{Q}$ with the chamber and spectral label just defined. For $w = w_0$, these are the initial q-character values of the prescribed embedding

$$\mathcal{I} : \mathcal{A}_{w_0} \hookrightarrow \mathbb{Z}[\overline{P}] \otimes_{\mathbb{Z}[P]} K_0(\mathcal{O}_{\mathbb{Z}}) \tag{2.15}$$

of [GHL24, Theorems 7.4 and 9.15]. The category $\mathcal{O}_{\mathbb{Z}}$ consists of shifted category-$\mathcal{O}$ modules whose simple highest weights use the spectral factors $P_{i,r}$ with $(i,r) \in V$ and integral constant weights.

Mutating a red vertex at height a moves the insertion to $a+2$; mutating the green vertex moves it to $a-2$ [GHL24, Lemma 3.2]. Relation (2.14) gives these moves with exactly the normalized labels (2.13). Thus a finite change of insertion parameters preserves the embedded cluster algebra, as well as its quiver mutation class. Any fixed admissible parameters can be moved to separated parameters in finitely many such moves, keeping the insertions in their order [GHL24, Proposition 3.4]; see also [GHL26, Theorem 5.1 and Remark 9.5(ii)]. It is therefore enough to work with one separated seed for a fixed reduced expression of w_0.

2.4. **Exchange divisors and dominant highest monomials.** For a locally finite quiver with pure labels p_v, use signed arrows

$$a_{uv} = \#(u \longrightarrow v) - \#(v \longrightarrow u)$$

and define its *full exchange divisor* at k by

$$D_k = \prod_v p_v^{a_{kv}}. \tag{2.16}$$

The product includes every neighbor of k. Put $[b]_+ = \max(b, 0)$. The two numerator highest monomials in an exchange are

$$H_k^{\mathrm{out}} = \prod_v \nu(p_v)^{[a_{kv}]_+}, \qquad H_k^{\mathrm{in}} = \prod_v \nu(p_v)^{[-a_{kv}]_+}.$$

Their quotient is $\nu(D_k)$.

Lemma 2.3. *In an actual seed obtained by finitely many mutations from Γ_e, exactly one of H_k^{in} and H_k^{out} is the highest monomial of the left side of the exchange. The ratio of the other numerator highest monomial to it is a nontrivial monomial in the $t_{i,a}$.*

Proof. Write the exchange as $X_k X_k' = X^{\mathrm{in}} + X^{\mathrm{out}}$ and apply χ_q. The two factors on the left and the two cluster monomials on the right are actual simple classes by (2.9). The product on the left has highest monomial $H = \nu(p_k)\nu(p_k')$ with coefficient one. Every term on either side is its highest monomial times an inverse-root monomial, and all coefficients are nonnegative. Hence each right-hand highest monomial equals H times an inverse-root monomial. Conversely, the occurrence of H on the right forces one of those highest monomials to equal H: otherwise two nonnegative root exponent vectors would add to zero. This uses Lemma 2.1. Both right-hand highest monomials cannot equal H, since its coefficient on the left is one. The remaining ratio is therefore a nontrivial inverse-root monomial. □

If the incoming term is dominant, the highest-label update is

$$p_k' = p_k^{-1} \prod_v p_v^{[-a_{kv}]_+}. \tag{2.17}$$

For outgoing dominance replace $[-a_{kv}]_+$ by $[a_{kv}]_+$. In particular all actual normalized highest monomials remain of the form $\nu(p_v)$.

Lemma 2.4. *For the incoming update (2.17), matrix mutation and highest-label mutation give*

$$D_k' = D_k^{-1}, \qquad D_j' = D_j D_k^{[-a_{kj}]_+} \quad (j \neq k). \tag{2.18}$$

For the outgoing update they give

$$D_k' = D_k^{-1}, \qquad D_j' = D_j D_k^{[a_{kj}]_+} \quad (j \neq k). \tag{2.19}$$

These are identities for the full quiver.

Proof. Use additive notation x_v for the exponent vector of p_v and $d_j = \sum_v a_{jv} x_v$ for that of D_j. For $j, v \neq k$, matrix mutation is

$$a_{jv}' = a_{jv} + [a_{jk}]_+ [a_{kv}]_+ - [-a_{jk}]_+ [-a_{kv}]_+, \qquad a_{jk}' = -a_{jk}.$$

The incoming update is $x_k' = -x_k + \sum_v [-a_{kv}]_+ x_v$. Substitution yields

$$\begin{aligned} d_j' - d_j &= \sum_{v \neq k} \left([a_{jk}]_+ [a_{kv}]_+ - [-a_{jk}]_+ [-a_{kv}]_+ - a_{jk} [-a_{kv}]_+ \right) x_v \\ &= [a_{jk}]_+ \sum_v a_{kv} x_v = [-a_{kj}]_+ d_k. \end{aligned}$$

In the second line we used $[-b]_+ + b = [b]_+$. For the outgoing update the corresponding coefficient is

$$[a_{jk}]_+ [a_{kv}]_+ - [-a_{jk}]_+ [-a_{kv}]_+ - a_{jk} [a_{kv}]_+ = [-a_{jk}]_+ a_{kv},$$

which gives $d_j' - d_j = [a_{kj}]_+ d_k$. Finally, every arrow incident with k reverses, and $a_{kk} = 0$, so $d_k' = -d_k$. Local finiteness makes all sums finite. □

Consider a separated prefix seed, a tail chamber w, and a proposed next color i with $\ell(ws_i) = \ell(w)+1$. At a red front of tail height a, its outgoing labels are $T'_w P_{i,a+2}$ and $T'_w T'_i P_{i,a}$, and its incoming labels are $T'_w P_{j,a+1}$, $j \sim i$. Therefore

$$D^{\mathrm{red}}_{i,a} = T'_w \left(\frac{P_{i,a+2} T'_i P_{i,a}}{\prod_{j\sim i} P_{j,a+1}} \right) = T'_w d_{i,a}, \tag{2.20}$$

$$\frac{\prod_{j\sim i} T'_w P_{j,a+1}}{T'_w P_{i,a}} = T'_w T'_i P_{i,a+2}. \tag{2.21}$$

The basic tail at the same vertex has the same full divisor: its additional incoming vertical label $T'_w P_{i,a-2}$ replaces the green label in (2.20) by the identity (2.10).

Extend T'_i to constant weights by $T'_i[\lambda] = [s_i\lambda]$. On a pure divisor having total exponent zero in each color,

$$\Omega(T'_i p) = s_i \Omega(p). \tag{2.22}$$

Indeed, on a generator the difference is $\Omega(T'_i P_{j,r}) - s_i\Omega(P_{j,r}) = \delta_{ij}\alpha_i/2$; summing with the exponents of p gives zero. The zero-total-exponent property is preserved by every T'_i. Applying (2.22) successively to $d_{i,a}$ gives

$$\nu(T'_w d_{i,a}) = T'_w(A^{-1}_{i,q^a}), \qquad \mathrm{wt}(\nu(T'_w d_{i,a})) = -w\alpha_i. \tag{2.23}$$

Here wt takes a constant $[\lambda]$ to λ and each pure Ψ-divisor to zero. Since $w\alpha_i$ is a positive root, its negative belongs to $-Q_+ \setminus \{0\}$, where $Q_+ = \sum_i \mathbb{N}\alpha_i$. Whenever an actual front has divisor (2.20), Lemma 2.3 first shows that one of $\nu(D)$ and $\nu(D)^{-1}$ is a positive inverse-root monomial. Its ordinary weight then forces the incoming branch, and $\nu(D)$ has total inverse-root degree $\mathrm{ht}(w\alpha_i) > 0$.

2.5. **Finite sweeps and the entire upper region.** We prove a uniform comparison with an actual finite mutation seed. All vertices in the following statement retain their physical labels in V. An ideal inserted seed is the seed defined above; an actual seed is obtained from Γ_e by finitely many ordinary cluster mutations. Only the latter uses (2.9).

Proposition 2.5 (Finite patch)**.** *Fix a reduced word* $\mathbf{i} = (i_1, \dots, i_\ell)$ *and separated insertion heights* $r_1, \dots, r_\ell$*. Let* $(a^{(m)}, p^{(m)})$ *be the ideal seed after its first* m *insertions. There are constants* B_m, C_m*, independent of the cutoffs below, with the following property. Choose*

$$R_m \equiv r_m \pmod 2, \qquad R_1 \ll R_2 \ll \cdots \ll R_\ell,$$

all sufficiently negative. At stage m*, set* $i = i_m$ *and* $n = n_i^{(m-1)}$*, and, in the actual seed from the preceding stage, mutate successively at*

$$(i, R_m - 2n),\ (i, R_m - 2n + 2),\ \dots,\ (i, r_m - 2n - 2). \tag{2.24}$$

This is a finite sequence of $(r_m - R_m)/2$ *mutations. The resulting actual seed* $(a^{\mathrm{act}}, p^{\mathrm{act}})$ *satisfies, with* $U_m = \{(j,r) \in V : r \ge R_m + C_m\}$*:*

(i) $a^{\mathrm{act}}_{uv} = a^{(m)}_{uv}$ *for every pair* $u, v \in U_m$*;*
(ii) $\prod_u p_u^{\mathrm{act}\, a^{\mathrm{act}}_{vu}} = \prod_u (p_u^{(m)})^{a^{(m)}_{vu}}$ *for every* $v \in U_m$*;*
(iii) $\mathrm{supp}_\Psi(p_v^{\mathrm{act}}/p_v^{(m)}) \subseteq (-\infty, R_m + C_m]$ *for every* $v \in U_m$*;*
(iv) *for every* $(j,r) \in V$*,*

$$\max \mathrm{supp}_\Psi(p^{\mathrm{act}}_{j,r}) \le \max\{r, R_m\} + B_m. \tag{2.25}$$

The products in (ii) *include all neighbors, including those outside* U_m*.*

Proof. We first establish bounds for the ideal seeds, including the intermediate seeds in which the newest insertion moves upward. At most ℓ insertions occur. Each insertion changes a physical coordinate by either zero or two and introduces only the pattern in the insertion rule. Hence there is a constant L_q such that

$$a^{\mathrm{ideal}}_{(j,r),(k,s)} \neq 0 \quad \Longrightarrow \quad |r-s| \leq L_q \tag{2.26}$$

for every intermediate ideal seed. For example, increasing a bound by four at each insertion bounds the change of every old arrow length and the lengths of the newly introduced arrows. The bound does not depend on the location of the insertion. A label is obtained from a $P_{j,s}$ by at most ℓ operators (2.10), each of which changes a spectral index by at most two, and by at most ℓ coordinate changes of size two. Consequently there is a constant L_p such that

$$\max \operatorname{supp}_{\Psi}(p^{\mathrm{ideal}}_{j,r}) \leq r + L_p \tag{2.27}$$

for all these seeds. Choose the fixed gaps between the r_m large enough that the local pattern of each moving insertion, throughout its upward motion to r_m, stays below all preceding insertion patterns. The bounds just obtained allow one choice of gaps for the entire word.

Set $B_0 = C_0 = 0$. For $m = 0$, actual and ideal seeds are both Γ_e, with $p_{j,r} = P_{j,r}$. They have identical adjacency and full divisors, all label ratios are 1, and the global support bound is exactly r. Thus all four assertions begin with exact equalities. Suppose that they have been proved after stage $m-1$, and put $R = R_m$, $i = i_m$, and $w = w^{(m-1)}$. Choose R above the preceding boundary by a fixed amount so that

$$R - 2\ell \geq R_{m-1} + C_{m-1}. \tag{2.28}$$

For $m = 1$ this requirement is empty. Use the prefix heights $h(j,r) = r + 2n_j^{(m-1)}$. Let c_0 exceed the finitely many offsets in the insertion pattern and the bounds needed to exclude its other-color endpoints, and set

$$S_R = \{(i,r) : h(i,r) \geq R\} \ \cup \bigcup_{j \neq i} \{(j,r) : h(j,r) \geq R + c_0\}. \tag{2.29}$$

Every vertex mutated in (2.24) belongs to S_R. The prefix ideal seed and the ideal seed with a new insertion initially at height R have identical labels and principal matrices on S_R. The new green vertex and the shifted lower vertices are outside S_R, as are the changed other-color endpoints. By (2.28), the entire set S_R belongs to the preceding upper region. Actual and moving ideal principal matrices therefore agree on S_R before the sweep.

Principal matrices. For $u, v, k \in S_R$, the mutated entry a'_{uv} depends only on a_{uv}, a_{uk}, a_{kv}. Restricting to S_R consequently commutes with mutation at $k \in S_R$. Induction over the finite sequence (2.24) proves equality of the two principal matrices on *all* of S_R at every step. The moving ideal seed ends at the ideal insertion of height r_m, by the red-vertex mutation rule. This proves assertion (i) on any upper region contained in S_R.

Full front divisors and dominant branches. The current-color vertices at heights $R, R+2, \ldots, r_m$ form an upward path in the principal quiver on S_R. Only this color is mutated in the stage. At its first vertex all remaining path arrows point toward higher vertices. Upon mutating that vertex the one edge to the next vertex reverses; there is no directed two-path through it between two unmutated path vertices. Inductively, just before mutating the next vertex, its edges to the already mutated part point backward and its edge to the remaining part points forward. No arrow from a future front enters the current front. Equivalently, if k is the current front and j a future front, $a_{kj} \geq 0$. This path calculation remains valid in the full quiver, because principal restriction to the current-color set commutes with its mutations.

Initially, all actual front divisors equal the corresponding prefix ideal divisors by assertion (ii) of stage $m-1$ and (2.28). Each is $T'_w d_{i,a}$ at its height a. At the first ideal red front the same equality follows from (2.20). Apply Lemma 2.3 to the first actual exchange. Formula (2.23) selects the incoming branch. Formula (2.18), together with $a_{kj} \geq 0$, then leaves every future-front divisor unchanged. Thus at the next front the actual full divisor is still its original prefix divisor, and the same argument again selects the incoming branch. Induction gives, at every front of height a,

$$D_k^{\mathrm{act}} = D_k^{\mathrm{ideal}} = T'_w d_{i,a}, \qquad \nu(D_k^{\mathrm{act}}) \in \mathscr{M} \setminus \{1\}. \tag{2.30}$$

The ideal label update is (2.21), so its incoming updates are exactly the specified chamber labels. This proves (2.30) using the full divisors, including any actual neighbors below S_R.

Full divisors above a fixed strip. Before the sweep, actual and prefix ideal divisors agree throughout S_R. Comparing the prefix ideal seed with the newly inserted ideal seed changes full divisors only within a bounded strip of height R. To see this, above the strip the labels agree, and (2.26) places every ideal neighbor above the portion whose labels or arrows changed. Therefore $D_j^{\mathrm{act}}/D_j^{\mathrm{ideal}} = 1$ for every $j \in S_R$ above that strip. At a front the ratio is also one by (2.30). When a front k is mutated, the ratio at any other $j \in S_R$ is unchanged: the exponents in (2.18) agree by principal-matrix equality, and the multiplier D_k agrees by (2.30). At k both divisors invert. Thus all full divisors initially equal above the strip remain equal there throughout the sweep. One fixed upper threshold proves assertion (ii), uniformly over all vertices above it.

Global support. We next prove a bound independent of the number of fronts. For a current front $k = (i, r)$, every neighbor outside S_R has physical coordinate at most $R + c$ for a fixed constant c depending only on c_0 and ℓ. Such a neighbor is never mutated during this stage. By the preceding global bound, its entire label is supported below $R + c + B_{m-1}$; the inequality $R > R_{m-1}$ is used here. An incoming neighbor of another color inside S_R is also unchanged. Principal-matrix equality and (2.26) put its coordinate at most $r + L_q$, so its support is at most $\max\{r, R\} + L_q + B_{m-1}$. An already mutated same-color incoming neighbor, if present, has smaller physical coordinate than r. Its inductive bound is therefore at most $\max\{r, R\} + B_m$. The path argument excludes incoming same-color neighbors still to be mutated. Finally, the old denominator label p_k is unmutated and satisfies the previous bound.

Choose $B_m \geq B_{m-1} + \max\{c, L_q, 0\}$, increasing it by the fixed coordinate offsets if necessary. The incoming update (2.17) multiplies and divides Laurent monomials. The support of such a product is contained in the union of the supports of its factors; no new spectral position is introduced. The bound $\max\{r, R\} + B_m$ consequently holds at this front. Induction over the fronts uses the same B_m at each step. Unmutated vertices retain the previous bound, which is at most (2.25) because $R_m > R_{m-1}$ and $B_m \geq B_{m-1}$. This proves assertion (iv) on the whole quiver.

Ratios of labels. Initially, all actual-to-ideal ratios on S_R are supported below $R_{m-1} + C_{m-1} \leq R$. At a front the two incoming updates have identical exponents for neighbors in S_R. Their quotient is a product of previously existing internal ratios, divided by the old front ratio, and of factors from neighbors outside S_R. The latter actual factors are supported below $R + c + B_{m-1}$, and the ideal factors are supported below $R + c + L_p$, by (2.27). There is consequently a constant E_m, independent of the front and the sweep length, such that every one of these external factors is supported below $R + E_m$. Induction over the incoming updates preserves the same bound on all internal ratios, because multiplication and division only take unions of supports. Unmutated internal ratios already satisfy it. This proves assertion (iii) on S_R with threshold $R + E_m$.

Choose C_m large enough that $U_m \subseteq S_R$, that $R + C_m$ lies above the full-divisor exceptional strip, and that $C_m \geq E_m$. Assertions (i)–(iv) now hold together after stage m. Their constants depend on the fixed word and ideal patterns and not on any cutoff or sweep length. This completes the induction. □

Corollary 2.6. *In Proposition 2.5, take even integers $N \to \infty$ and*

$$R_m(N) = -(\ell - m + 1)N + \delta_m, \tag{2.31}$$

where the bounded integers δ_m impose the required parities. Every fixed finite set of vertices is eventually contained in U_ℓ. Any additional actual neighbor of such a set, absent from the ideal quiver, has its entire pure label supported below a bound tending to $-\infty$.

Proof. All fixed gap inequalities in the proposition hold for sufficiently large N, and $R_\ell(N) + C_\ell \to -\infty$. If both endpoints of an arrow lie in U_ℓ, its signed multiplicity agrees with the ideal one by assertion (i). Thus an additional neighbor has physical coordinate $r < R_\ell + C_\ell$. Assertion (iv) bounds its support by $R_\ell + \max\{C_\ell, 0\} + B_\ell$, which tends to $-\infty$. □

2.6. **Identification of the initial Q-variables.**

Proposition 2.7. *For every chamber label and admissible spectral exponent,*

$$\mathcal{Q}_{w,i,r} = F(T'_w P_{i,r}), \qquad \mathbf{Q}_{w,i,r} = [-\Omega(T'_w P_{i,r})]\chi_q L(T'_w P_{i,r}). \tag{2.32}$$

Proof. First derive the QQ identity for the simple characters themselves. Fix an ascent $ws_i > w$ and an exponent r. Choose a reduced word for w and separated insertions whose lower tail contains the local pattern of an additional color-i insertion with red height $r-2$. The preceding insertion heights can be chosen sufficiently large for every specified r. Apply Proposition 2.5 to this seed and retain the one red exchange at that vertex. Fix a finite colored spectral set S containing the root support of $T'_w(A^{-1}_{i,q^{r-2}})$. For sufficiently large N, the central labels have ratios to the ideal labels which are units at S, by assertion (iii). Every additional actual neighbor has a whole label which is a unit at S, by Corollary 2.6. After dividing the actual exchange by its dominant highest monomial, Lemma 2.2 identifies the normalized characters of the central factors with those of their ideal pure labels, and sends the normalized characters of additional factors to 1. Assertion (ii) supplies the exact coefficient

$$D_r = T'_w(A^{-1}_{i,q^{r-2}}).$$

The highest-label update at the red vertex is (2.21). Thus, writing $F_{w,i,r} = F(T'_w P_{i,r})$, the resulting equality is the restriction under π_S of

$$F_{ws_i,i,r} = \frac{\prod_{j\sim i} F_{w,j,r-1}}{F_{w,i,r-2}} + D_r \frac{F_{w,i,r}}{F_{w,i,r-2}} F_{ws_i,i,r-2}. \tag{2.33}$$

To justify the newly mutated label in this passage, its quotient with $T'_w T'_i P_{i,r}$ is the quotient of the two incoming highest updates. Its internal factors are the central label ratios, and its external factors are the additional neighbor labels. All are units at S. This proves the stated application of Lemma 2.2 to that label as well. Enlarging S by any finite set and using separation of coefficients proves (2.33) in $\mathscr{R}$.

The same normalized recurrence follows from (2.14) for $\mathcal{Q}_{w,i,r}$: the two dominant pure highest products agree by (2.21), and the ratio of the other product is D_r by (2.20). The actual front argument in the proof of Proposition 2.5 shows that D_r is a nontrivial inverse-root monomial of total degree $d = \mathrm{ht}(w\alpha_i) > 0$. We now prove uniqueness, which will also identify the prescribed series with the simple characters.

Proceed by reduced length. For $w = e$, both families are 1, since $L(P_{i,r})$ is one-dimensional. Suppose that all chamber series of length at most $\ell(w)$ have been identified. Appending s_i changes only the chamber $w\omega_i$, since $s_i\omega_j = \omega_j$ for $j \neq i$. Subtract the two instances of (2.33) and set

$$\Delta_r = F_{ws_i,i,r} - \mathcal{Q}_{ws_i,i,r}, \qquad V_r = F_{w,i,r}/F_{w,i,r-2}.$$

The already identified terms cancel, leaving $\Delta_r = D_r V_r \Delta_{r-2}$. For every $K \geq 1$,

$$\Delta_r = \left(\prod_{b=0}^{K-1} D_{r-2b} V_{r-2b} \right) \Delta_{r-2K}. \tag{2.34}$$

Each V_{r-2b} belongs to $\mathscr{R}$ and has constant term one. Every monomial on the right has total degree at least Kd. Given a fixed monomial of degree d_0, choose $K > d_0/d$; its coefficient in Δ_r is then zero. All coefficients vanish, proving (2.32). □

The ADE simplicity of these Q-variables is also a consequence of [KLLL$^+$26]; see [GHL26, Remark 11.15(iii)]. The preceding argument provides the initial-character identification needed for the finite-mutation proof below.

2.7. **Finite mutation words and full cluster monomials.**

Proposition 2.8. *Let Σ be a separated inserted seed of $\mathcal{A}_{w_0}$ and let $\boldsymbol{\mu} = (k_1, \ldots, k_t)$ be any finite mutation sequence. Every variable X of $\boldsymbol{\mu}(\Sigma)$ satisfies $\chi_q \mathcal{I}(X) = \nu(p)F(p)$ for a pure divisor p. Moreover, every cluster monomial M in that seed satisfies*

$$\chi_q \mathcal{I}(M) = \nu(p_M)F(p_M), \qquad p_M = \prod_v p_v^{e_v} \quad \text{if } M = \prod_v X_v^{e_v}. \tag{2.35}$$

Proof. Choose a finite set U containing all mutated vertices, all of their ideal neighbors at each step of $\boldsymbol{\mu}$, and all vertices occurring in M. Such a set is finite: the quiver is locally finite, each mutation changes finitely many entries, and M has finite support. Use the actual finite seeds of Proposition 2.5 with cutoffs (2.31), and then apply the same word $\boldsymbol{\mu}$ to them. For sufficiently large N, all vertices of U and their specified ideal neighbors lie in the entire upper region U_ℓ.

We give the simultaneous induction over this fixed word. Initially, the ideal variables have the stated simple-character values by Proposition 2.7. On the whole upper region the actual and ideal principal matrices agree, their full exchange divisors agree, and their pure highest-label ratios are supported below a bound tending to $-\infty$. No vertex outside U_ℓ is mutated. Its actual label is therefore unchanged during the word and is supported below the bound in Corollary 2.6. Principal restriction commutes with all mutations in the word, so the actual and ideal matrices continue to agree on the whole upper region. In particular, every additional actual neighbor of a vertex of U remains outside this region, even if that neighbor first becomes adjacent during the word.

Suppose the assertions have been proved before the next mutation at k. Let D_k be the common full pure exchange divisor. The actual positive exchange and Lemma 2.3 imply that exactly one of $\nu(D_k)$ and $\nu(D_k)^{-1}$ is a nontrivial inverse-root monomial. Since D_k is the fixed ideal divisor, this choice is independent of N. Use the corresponding incoming or outgoing update to define the new ideal pure label p'_k. The quotient of the actual and ideal updated labels is a product of old internal label ratios and unchanged external actual labels, divided by the old label ratio at k. All have support below one bound tending to $-\infty$. Only finitely many such updates are made, so this assertion persists throughout the word. Formulas (2.18) and (2.19) propagate full-divisor equality at every vertex of U: their exponents agree by principal equality and their multiplier D_k agrees by the induction hypothesis. The divisor at k inverts on both sides.

It remains to identify the character of the new ideal label with the prescribed mutated variable. For incoming dominance the normalized actual exchange is

$$F(p_k^{\mathrm{act}})F((p'_k)^{\mathrm{act}}) = \prod_v F(p_v^{\mathrm{act}})^{[-a_{kv}^{\mathrm{act}}]_+} + \nu(D_k)\prod_v F(p_v^{\mathrm{act}})^{[a_{kv}^{\mathrm{act}}]_+}. \tag{2.36}$$

For outgoing dominance interchange the two products and replace $\nu(D_k)$ by $\nu(D_k)^{-1}$. Fix a finite colored spectral set S, enlarged to contain the support of this root coefficient. All central actual-to-ideal ratios and all external highest labels are units at S for sufficiently large N. Lemma 2.2 turns (2.36) into the same formula for the simple normalized characters $F(p_v)$ and $F(p'_k)$, with external factors equal to 1. By the induction hypothesis, the old ideal variables have their prescribed q-character values. Cluster mutation gives exactly this normalized formula for the new prescribed image. Division by $F(p_k)$ is legitimate in (2.5), and commutes with π_S. Thus the new prescribed q-character is $\nu(p'_k)F(p'_k)$ after restriction to every S, and hence without restriction. This closes the induction for all variables and all full divisors required by the word.

For the monomial assertion, retain the actual monomial with the same exponents in the final actual cluster. Put

$$p_{M,N} = \prod_v (p_{v,N}^{\mathrm{act}})^{e_v}, \qquad p_M = \prod_v p_v^{e_v}.$$

By (2.9), the actual monomial is a simple class. Its highest monomial is $\nu(p_{M,N})$, so multiplicativity of q-characters gives the equality of normalized series

$$F(p_{M,N}) = \prod_v F(p_{v,N}^{\mathrm{act}})^{e_v}. \tag{2.37}$$

The whole ratio $p_{M,N}/p_M$ is a finite product of the remote ratios from the induction, and is therefore a unit at each fixed S for sufficiently large N. Apply Lemma 2.2 to this whole ratio on the left of (2.37), and to each factor ratio on the right. It gives

$$\pi_S F(p_M) = \pi_S \prod_v F(p_v)^{e_v}.$$

Every root monomial is detected by some finite S, so $F(p_M) = \prod_v F(p_v)^{e_v}$ in $\mathscr{R}$. Multiplying by $\nu(p_M) = \prod_v \nu(p_v)^{e_v}$ proves (2.35). □

Proof of Theorem 1.1. Choose the reduced expression used by the original GHL initial seed, and change its insertion parameters to obtain a separated seed. Every cluster monomial of $\mathcal{A}_{w_0}$ is reached from this seed by a finite mutation sequence: changing the insertion parameters uses finitely many of the normalized QQ exchanges, and cluster mutation itself uses finite words. Proposition 2.8 and injectivity of the q-character map therefore give, for every cluster monomial M,

$$\mathcal{I}(M) = [-\Omega(p_M)]\,[L(p_M)].$$

Apply the same proposition to M^2, which is a monomial in the same cluster. Its pure highest weight is p_M^2, and hence

$$[L(p_M)]^2 = [L(p_M^2)].$$

The fusion square has finite length by [HZ25]. Its class is the class of a single simple module, so its length is one and $L(p_M)$ is real. The quiver and embedding in [Her26, Conjecture 9.10] are the primed quiver Γ'_∞ and the original embedding (2.15) for $\mathcal{A}_{w_0}$. □

3. Real modules and rigid objects in a finite window

3.1. **The finite quiver and its two degree vectors.** Let I be the vertex set of a simply laced finite Dynkin diagram, and write $i \sim j$ for adjacency. Fix a height function $\xi : I \longrightarrow \mathbb{Z}$, so that $|\xi_i - \xi_j| = 1$ whenever $i \sim j$, and an integer $\ell \geq 1$. Put

$$\begin{aligned} \widehat{I}_\xi &= \{(i,r) : r - \xi_i \in 2\mathbb{Z}\}, \\ \widetilde{W} &= \{(i,r) \in \widehat{I}_\xi : \xi_i - 2\ell \leq r \leq \xi_i\}, \\ W &= \{(i,r) \in \widehat{I}_\xi : \xi_i - 2\ell + 2 \leq r \leq \xi_i\}, \qquad W_{\mathrm{fr}} = \widetilde{W} \setminus W. \end{aligned} \tag{3.1}$$

The vertices in W_{fr} are frozen. The ice quiver on $\widetilde{W}$ has the arrows whose endpoints lie in this set:

$$\epsilon_{i,r} : (i, r-2) \longrightarrow (i,r), \qquad a_{ij,r} : (i,r) \longrightarrow (j, r-1) \quad (i \sim j).$$

Let Q be its full subquiver on W. Its potential is

$$\mathcal{W} = \sum_{i,r} \sum_{j \sim i} \big((i,r) \longrightarrow (j, r-1) \longrightarrow (i, r-2) \longrightarrow (i,r)\big), \tag{3.2}$$

where only triangles contained in W occur, and each cyclic path is included once. Path multiplication is composition of arrow maps. We use covariant representations of this displayed quiver; these are the right modules in the convention of [DS26, §3.2]. Set

$$A = A_\ell^\xi = \mathbb{C}Q/(\partial_a \mathcal{W} : a \in Q_1).$$

For $s \in W$, denote its indecomposable projective and injective modules by P_s and I_s.

The derivatives with respect to oblique arrows give

$$a_{ij,r}\epsilon_{i,r} + \epsilon_{j,r-1}a_{ij,r-2} = 0.$$

Each summand is omitted if one of its vertices lies outside W. Thus a vertical arrow can be moved past an adjacent oblique arrow; if the second path leaves W, the first is zero. Repeatedly applying these relations writes every nonzero path as a path of vertical arrows followed by a path of oblique arrows. Let R be the difference between the largest and smallest spectral coordinates in W. The first path has length at most $\lfloor R/2 \rfloor$, and the second at most R. Thus every path of length greater than $\lfloor R/2 \rfloor + R$ vanishes. In particular, the arrow ideal of A is nilpotent. This is the path calculation in [HL16a, proof of Proposition 4.17], with the boundary relations retained. The calculation uses only the endpoint inequalities, so it applies to every height function. That proposition also gives the rigidity, hence nondegeneracy, of the quiver with potential.

Let $\mathcal{C}_A$ be the Hom-finite Amiot cluster category of $(Q, \mathcal{W})$, and let $T = \bigoplus_{s \in W} T_s$ be its canonical cluster-tilting object, with $\mathrm{End}_{\mathcal{C}_A}(T) = A$ in the convention just fixed. For an object M, use the Duan–Schiffler degree

$$g_{\mathrm{DS}}(M) = \mathrm{ind}_T(M[-1]). \tag{3.3}$$

More explicitly, if

$$T^- \longrightarrow T^+ \longrightarrow M[-1] \longrightarrow T^-[1], \qquad T^\pm \in \mathrm{add}\, T,$$

is a minimal presentation triangle, its degree is $[T^+] - [T^-] \in K_0(\mathrm{add}\, T) = \mathbb{Z}^W$.

Write $Y_{i,r} = Y_{i,q^r}$, and put

$$A_{i,r-1} = Y_{i,r-2}Y_{i,r} \prod_{j \sim i} Y_{j,r-1}^{-1}, \qquad z_{i,r} = \prod_{s = r, r+2, \ldots, \xi_i} Y_{i,s}. \tag{3.4}$$

The category $\mathcal{C}_\ell^{\le\xi}$ consists of finite-dimensional type-one quantum affine modules whose simple constituents have dominant highest monomials in the $Y_{i,r}$, $(i,r)\in\widetilde{W}$. Its frozen simples are $L(z_{i,\xi_i-2\ell})$. For

$$m=\prod_{(i,r)\in\widetilde{W}} Y_{i,r}^{u_{i,r}},\qquad u_{i,r}\in\mathbb{N},$$

extend u by zero outside $\widetilde{W}$. Its exponent in the z-coordinates is

$$\widetilde{g}_{i,r}=u_{i,r}-u_{i,r-2}\quad((i,r)\in\widetilde{W}),\qquad g=\widetilde{g}|_W. \tag{3.5}$$

We reserve $g^{\mathrm{full}}_{i,r}=u_{i,r}-u_{i,r-2}$ for the same difference on the whole lattice. Thus $g^{\mathrm{full}}_{i,\xi_i+2}=-u_{i,\xi_i}$, whereas $\widetilde{g}$ has no coordinate there.

The truncated character $\chi_q^{\le\xi}L(m)$ is obtained by deleting monomials involving a Y-variable outside $\widetilde{W}$, as in [DS26, §3.1 and §4.3]. In the initial torus its leading monomial is $z^{\widetilde{g}}=m$. For $e\in\mathbb{N}^W$, put

$$\widehat{y}^e=\prod_{(i,r)\in W} A_{i,r-1}^{-e_{i,r}}.$$

Let $P=\widetilde{B}$ be the extended exchange matrix, with $b_{st}=\#(t\to s)-\#(s\to t)$. Direct substitution in (3.4) gives $\widehat{y}^e=z^{Pe}$. The mutable part of P is denoted by B. The generic kernel prescribed in the conjecture is

$$K(m)=\ker\left(\bigoplus_{s\in W} I_s^{[-g_s]_+}\longrightarrow\bigoplus_{s\in W} I_s^{[g_s]_+}\right), \tag{3.6}$$

where the map is general in this affine Hom space and $[a]_+=\max(a,0)$. For a finite-dimensional A-module K, its F-polynomial is

$$F_K(y)=\sum_{e\in\mathbb{N}^W}\chi(\mathrm{Gr}_e\,K)y^e.$$

Here $\mathrm{Gr}_e\,K$ is the projective variety of submodules of dimension vector e, and χ is the topological Euler characteristic.

3.2. **A residue bound for tensor powers.** We first prove a restriction on the support of an actual simple character. The parameter $q\in\mathbb{C}^\times$ is fixed throughout this argument and is not a root of unity.

Lemma 3.1. *Let $L(\psi^r)$ be the simple module of highest rational ℓ-weight ψ^r. Fix a nonzero colored spectral multiset*

$$e=\{(i_1,x_1),\dots,(i_N,x_N)\},\qquad x_a\neq 0.$$

Set $h_a=-\operatorname{ord}_{z=x_a}\psi_{i_a}(z)$ and $H=\sum_{a=1}^N h_a$. If $\rho_r(e)$ is the coefficient belonging to this root correction, then, with $B_N=2\binom{N}{2}$ and $T_r=rH+B_N-N$,

$$\rho_r(e)\le N!\binom{N+T_r}{N}\quad(T_r\ge 0),\qquad \rho_r(e)=0\quad(T_r<0). \tag{3.7}$$

Proof. We use the local residue realization of the actual simple module in [HN26, Corollary 3.9 and equations (127)–(136)]. Its standing parameter assumption is exactly the one fixed above; Example 2.8 identifies the finite-type shuffle quotient with the Drinfeld–Serre algebra.

In fixed colored degree N, the negative shuffle algebra is spanned by symmetrizations of

$$z_1^{d_1}\cdots z_N^{d_N}\prod_{a<b}\zeta_{i_bi_a}(z_b/z_a),\qquad \zeta_{ij}(t)=\frac{t-q^{-c_{ij}}}{t-1}.$$

Here $C = (c_{ij})$ is the symmetric Cartan matrix and $d_a \in \mathbb{Z}$. Consequently

$$D(z) = \prod_{a<b} (z_b - z_a)$$

is a common denominator. For every shuffle element F, the function $G = DF$ is Laurent polynomial and hence regular near the selected point. This remains true if spectral coordinates coincide.

Put $u_a = z_a/x_a - 1$. Locally,

$$\psi_{i_a}(z_a)^r = u_a^{-rh_a} v_a(u_a)^r, \qquad v_a \in \mathbb{C}[[u_a]]^\times.$$

The residue test following equation (132) of [HN26] pairs F against arbitrary Laurent polynomials, for each ordering. Apart from these regular test functions, the fixed rational factor is

$$D(z)^{-1} \prod_{a<b} \zeta_{i_b i_a}(z_b/z_a)^{-1}.$$

Every denominator is a unit at the selected point or a homogeneous linear form in the u_a. There are at most B_N possibly vanishing linear factors; this bound allows both D and the shuffle kernels and is independent of r. Numerator factors, units, test functions, and $dz_a/z_a = du_a/(1+u_a)$ contribute nonnegative Taylor degrees.

In every ordered Laurent expansion, an inverse homogeneous linear form has total degree -1. Thus, if $\operatorname{ord}_0 G = d$, every term of the integrand has total degree at least $d - B_N - rH$. The iterated residue extracts total degree $-N$. It therefore vanishes for every test function if $d > T_r$.

For $T_r \geq 0$, send F to the jets of DF of total degree at most T_r, at the at most $N!$ orderings and assignments of the colored multiset. The kernel of this linear map is contained in the local residue annihilator. The quotient by that annihilator is therefore a quotient of the jet-map image. Each jet space has dimension $\binom{N+T_r}{N}$. Equations (135)–(136) of [HN26] identify its local quotient dimension with the indicated simple-character contribution and give (3.7). If $T_r < 0$, every regular numerator has order greater than T_r, so the quotient is zero. □

The root corrections in this lemma are unambiguous. On one spectral $q^{\mathbb{Z}}$-orbit, a finite relation among the ordinary root monomials is a vector Laurent polynomial in the kernel of the quantum Cartan matrix

$$C(t) = (t + t^{-1})\,\mathrm{id} - (\delta_{i\sim j})_{i,j\in I}.$$

Here $\delta_{i\sim j}$ is one for adjacent vertices and zero otherwise. Its determinant is nonzero, since $C(1)$ is the finite Cartan matrix. Multiplication by its adjugate proves that the vector is zero. Different spectral orbits are independent, and the assumption on q identifies distinct integral spectral positions. Thus the coefficients in Lemma 3.1 are the coefficients of the specified inverse root monomials.

Proposition 3.2. *If $L(m)$ is real, every nonhighest monomial $m \prod A_{i,r-1}^{-e_{i,r}}$ of its full character satisfies*

$$\sum_{i,r} g_{i,r}^{\mathrm{full}} e_{i,r} < 0.$$

In particular, every nonzero correction of its truncated character satisfies $g \cdot e < 0$.

Proof. Hernandez's tensor-product theorem [Her10] gives $L(m)^{\otimes r} = L(m^r)$ for every $r \geq 1$. Let a nonhighest correction e have coefficient $a_e > 0$. The coefficient at the same correction in the r-th power of the normalized character is at least ra_e: choose this correction in one tensor factor and the highest monomial in all other factors.

The highest rational weight in the residue coordinate is

$$\psi_i(z) = q^{\sum_s u_{i,s}} \prod_s \left(\frac{1 - q^{s-1}/z}{1 - q^{s+1}/z} \right)^{u_{i,s}}.$$

Its signed pole order at $z = q^{r-1}$ is $u_{i,r-2} - u_{i,r} = -g^{\mathrm{full}}_{i,r}$. If $-g^{\mathrm{full}} \cdot e = 0$, Lemma 3.1 bounds the coefficient independently of r. If it is negative, that coefficient eventually vanishes. Both contradict ra_e. Thus $-g^{\mathrm{full}} \cdot e > 0$. Restriction to $e \in \mathbb{N}^W$ gives the last assertion. □

3.3. **The quantum torus and its geometric normalization.** The independent formal parameter in the quantum Grothendieck ring is v. Let Λ be the integral skew-symmetric form of the Duan–Schiffler seed. We use normalized monomials with multiplication

$$X^a X^b = v^{\Lambda(a,b)} X^{a+b}, \qquad P^T \Lambda = (2\,\mathrm{id} \mid 0), \qquad \Lambda(a, Pe) = -2a|_W \cdot e. \tag{3.8}$$

The second identity is [DS26, Proposition 3.6]; their parameter t_{DS} equals v^2. It implies that P has full column rank. Write $Y^e = X^{Pe}$. These Y^e are correction monomials in the quantum torus; the spectral variables $Y_{i,r}$ always carry both indices.

We record the precise conversion from the geometric multiplication used later. Write $e_{i,r}$ for a spectral coordinate basis vector. Let τ act on a finitely supported spectral vector by $(\tau u)_{i,r} = u_{i,r+1}$, and set $\mathsf{C} = \tau + \tau^{-1} - (\delta_{i\sim j})$. Let $C(z) = (z + z^{-1})\,\mathrm{id} - (\delta_{i\sim j})$, and let $\widetilde{C}_{ij}(k)$ be the coefficient of z^k in $C(z)^{-1}$, extended by zero for $k \le 0$. The inverse C^{-1} below is computed from the lower spectral boundary: its application to $e_{i,r}$ has coefficient $\widetilde{C}_{ji}(s-r)$ at (j,s). Set $\widehat{I}^{\circ}_{\xi} = \{(i, r-1) : (i,r) \in \widehat{I}_{\xi}\}$. For an index set S, $\mathbb{Z}^{(S)}$ denotes its finitely supported integral vectors.

Lemma 3.3. *Define alternating forms on spectral exponent vectors by*

$$\begin{aligned} \varepsilon(u,w) &= \langle \mathsf{C}^{-1}u, \tau^{-1}w\rangle - \langle \mathsf{C}^{-1}w, \tau^{-1}u\rangle, \\ \mathcal{N}(e_{i,r}, e_{j,s}) &= \widetilde{C}_{ij}(s-r+1) + \widetilde{C}_{ij}(r-s-1) - \widetilde{C}_{ij}(s-r-1) - \widetilde{C}_{ij}(r-s+1). \end{aligned} \tag{3.9}$$

Then $\beta = 2\varepsilon - \mathcal{N}$ vanishes when either argument lies in the finite root lattice $\mathsf{C}\mathbb{Z}^{(\widehat{I}^{\circ}_{\xi})}$. After setting the geometric shift parameter $t = v^2$, the multiplication $t^{\varepsilon(u,w)} e(u+w)$ becomes the Duan–Schiffler multiplication upon twisting homogeneous products by $v^{-\beta(u,w)}$. For $u = aw - \mathsf{C}e_1$ and $u' = bw - \mathsf{C}e_2$, this twist is one.

Proof. All pairings are finite because one argument in each pairing is finitely supported. The recurrence

$$\widetilde{C}_{ij}(k+1) + \widetilde{C}_{ij}(k-1) - \sum_{h\sim j} \widetilde{C}_{ih}(k) = \delta_{ij}\delta_{k0}$$

holds for every integer k. On basis vectors, $\varepsilon(e_{i,r}, e_{j,s}) = \widetilde{C}_{ij}(s-r+1) - \widetilde{C}_{ij}(r-s+1)$. Applying the recurrence to the second argument gives

$$\varepsilon(u, \mathsf{C}e) = \sum_{i,s} e_{i,s}(u_{i,s+1} - u_{i,s-1}),$$

$$\mathcal{N}(u, \mathsf{C}e) = 2\sum_{i,s} e_{i,s}(u_{i,s+1} - u_{i,s-1}).$$

Thus $\beta(u, \mathsf{C}e) = 0$; alternation gives the assertion in the first argument. The form β consequently descends to the quotient of spectral exponents by root exponents. Bilinearity makes $v^{-\beta}$ a multiplicative two-cocycle, so the indicated twist is associative. Its new exponent is $2\varepsilon - \beta = \mathcal{N}$. Under $X_{i,r} \mapsto z_{i,r}$, the latter form is exactly Λ, by [DS26, equations (3.1)–(3.3)]. Finally,

$$\beta(aw - \mathsf{C}e_1, bw - \mathsf{C}e_2) = ab\beta(w,w) = 0.$$

□

Nakajima's geometric simple characters have coefficients in $\mathbb{N}[t^{\pm 1}]$ by the IC-stalk construction of [Nak11, equation (3.18)]. Their multiplication constants have the same positivity property [Nak11, Theorem 3.22]. The twist in Lemma 3.3 is constant on each pair of pointed root cosets. It changes only the leading product phase. Thus it preserves positivity, bar invariance, and every normalized injective-standard expansion. More explicitly, in a normalized product of standard factors the cocycle multiplier is the same on every product of constituent monomials as on the product of highest monomials; normalization cancels it. The standard characters, and their unique bar-invariant triangular corrections, are unchanged as coefficient vectors. In particular, the actual simple characters in (3.8) have nonnegative Laurent coefficients, and their multiplication constants specialize at $v = 1$ to the usual composition multiplicities.

We invert the frozen classes. Normalized multiplication by them preserves actual simples, as proved in Lemma 3.14. Write L_a for the resulting normalized simple basis element of extended degree $a \in \mathbb{Z}^{\widetilde{W}}$. A localized simple class is called real if $L_a^2 = L_{2a}$; this agrees with reality of any dominant frozen translate.

Orient each Dynkin edge from higher to lower ξ, choose a Coxeter order $c = (i_1, \dots, i_{|I|})$ adapted to that orientation, and take the word $c^{\ell+1}$. Number the occurrences of each color by $p = 0, \dots, \ell$, and identify the occurrence (i, p) with $(i, \xi_i - 2p)$. The last occurrence is frozen. The initial interval module at this occurrence has highest monomial z_{i,ξ_i-2p}. A fundamental highest monomial $Y_{i,r}$ has degree

$$e_{(i,r)} - e_{(i,r+2)},$$

where the second term is zero above the window. Summing these degrees gives exactly $\widetilde{g}$ in (3.5). The positive-word seed $\dot{\mathbf{s}}(c^{\ell+1})$ has exchange matrix P: its vertical arrows are $(i, p+1) \to (i, p)$, and, when $\xi_i = \xi_j + 1$, its oblique arrows are $(i, p) \to (j, p)$ and $(j, p) \to (i, p+1)$, whenever the endpoints lie in the word. These are the arrows of (3.1). Theorem A.12 of [Qin25] uses the opposite seed, whose matrix is $-P$. Its canonical anti-isomorphism, followed by the inverse character anti-isomorphism of [Qin25, Remark A.9], cancels the two product reversals. The resulting algebra isomorphism identifies the positive-word seed with the present seed and sends its initial interval variables to the classes with highest monomials $z_{i,r}$. The compatible multiplication is (3.8), with the conversion of Lemma 3.3. Thus [Qin25, Theorems 6.7 and A.12] identify the common triangular basis with the L_a, including their degrees. This finite-word seed is injective-reachable. The equality of the ordinary and upper algebras for this finite word is [Qin25, Theorem 6.17(2)]. Hence [DM19, Proposition 4.14] puts each L_a, and each injective standard element, in the finite span of the quantum theta functions ϑ_p^v.

Each L_a is bar invariant, is pointed at X^a, and has an injective-standard expansion

$$L_a = I_a + \sum_{0 \neq f \in \mathbb{N}^W} b_{a,f}(v) I_{a+Pf}, \qquad b_{a,f}(v) \in v^{-1}\mathbb{Z}[v^{-1}]. \tag{3.10}$$

This equality is interpreted coefficientwise in positive correction degree. The theta expansion is finite. In every generic scattering chart a theta function is pointed at its label, with leading coefficient one, integral Laurent coefficients, and positive classical specialization. Transport preserves the bar involution. These assertions are respectively [Qin25, Lemma 6.2 and Definitions 6.3–6.4] and [DM19, Theorems 3.2 and 3.9, Proposition 3.12].

3.4. **Finite-degree transport and positive theta coefficients.** We give the coefficient argument with all frozen coordinates retained. For $c \in \mathbb{Z}^W$, define the automorphism

$$\sigma_c(Y^e) = v^{-4c \cdot e} Y^e.$$

All infinite products are taken in the positive correction completion over $\mathbb{Q}(v)$. A calculation modulo $\{e : e \not\leq E\}$ means equality of coefficients for $0 \leq e \leq E$, where $E \in \mathbb{N}^W$ and inequalities

are componentwise. This is a finite positive-degree quotient: the omitted set is upward closed, so products and inverses of units with constant term one respect it.

Let H be the scattering group element from the positive chamber to the negative chamber supplied by injective reachability. Normalize the injective variables by frozen monomials, so their mutable leading degrees are the negatives of the coordinate vectors. The injective standard of degree a is the normalized product of the positive part of a and the injective monomial for its negative part. Commuting its leading monomial past H^{-1} gives

$$I_a = X^a \sigma_{a_-}(H^{-1})H, \tag{3.11}$$

where $(a_-)_s = [-a_s]_+$ for $s \in W$. The frozen leading phase is removed by normalization, and compatibility in (3.8) makes it independent of correction degree.

Fix a box $0 \le e \le E$, and a vector $\nu \in \mathbb{Z}^W$ with $\nu \cdot e \neq 0$ for every nonzero e in that box. Extend it by zero in the frozen coordinates. Canonical factorization at a nearby generic endpoint gives

$$H = TF, \qquad F = 1 + \sum_{\nu\cdot d<0} f_d Y^d, \qquad T = 1 + \sum_{\nu\cdot d>0} t_d Y^d. \tag{3.12}$$

The equalities here and in the next calculation are in the fixed box. The chart transport is $\iota_\nu = \mathrm{Ad}_F$. Since $FX^a = X^a \sigma_{-a|_W}(F)$, equations (3.11)–(3.12) give the exact identity

$$\iota_\nu(I_a) = X^a \sigma_{-a|_W}(F)\sigma_{a_-}(H^{-1})T. \tag{3.13}$$

Lemma 3.4. *Fix E, a finite set of extended integral vectors h, and ν as above. For all sufficiently large N, uniformly for these h,*

$$\iota_\nu(L_{N\nu+h}) = X^{N\nu+h} \pmod{\{e : e \not\leq E\}}.$$

Proof. For a rational function of v, let $\deg_v$ mean its degree at infinity. Put $a = N\nu + h$, and take a term of correction degree $e = d_1 + d_2 + d_3$ in (3.13). The coefficients of F, H^{-1}, T and their internal multiplication phases are fixed functions of v. The remaining phase, relative to the normalized monomial X^{a+Pe}, is

$$4a|_W \cdot d_1 - 4a_- \cdot d_2 - 2a|_W \cdot e.$$

For sufficiently large N, the coefficient of N in this expression is

$$2(\nu \cdot d_1 - |\nu| \cdot d_2 - \nu \cdot d_3). \tag{3.14}$$

Here $|\nu|$ is taken coordinatewise. If a contribution is nonzero, $\nu \cdot d_1 \le 0$ and $\nu \cdot d_3 \ge 0$. Consequently

$$\nu \cdot d_1 - |\nu| \cdot d_2 - \nu \cdot d_3 \le -|\nu \cdot d_1| - |\nu \cdot d_2| - |\nu \cdot d_3| \le -|\nu \cdot e| < 0$$

for $e \neq 0$. There are finitely many splittings in the fixed box. All remaining degrees are bounded uniformly for the finite offset set. Every nonleading coefficient of the transported I_a therefore has negative degree for large N.

Apply (3.10). At a target $e \le E$, only $f \le e$ can contribute. Enlarge the finite offset set by all Pf, $f \le E$. If $f < e$, the estimate just proved applies to the remaining nonzero correction $e - f$. If $f = e$, the coefficient is $b_{a,e}(v)$, already of negative degree. Thus every nonleading coefficient of $\iota_\nu(L_a)$ has negative degree.

The coefficients are Laurent polynomials: the ordinary cluster algebra lies in the finite quantum theta span, and its theta functions have integral expansions in this chart. They are bar invariant because L_a, its transport, and the normalized monomials are bar invariant. A Laurent polynomial invariant under $v \mapsto v^{-1}$ and of negative highest degree is zero. □

Proposition 3.5. *The classical theta expansion of every localized actual simple is finite and has the form*

$$L_a\big|_{v=1} = \vartheta_a + \sum_{0\neq e\in\mathbb{N}^W} c_e\,\vartheta_{a+Pe}, \qquad c_e\in\mathbb{N}. \tag{3.15}$$

Here $\vartheta_p = \vartheta_p^v|_{v=1}$. In particular, the Laurent support of ϑ_a in the initial torus is contained coefficientwise in that of $L_a|_{v=1}$.

Proof. Begin with the finite quantum theta expansion, allowing signed coefficients. Its labels lie in $a + P\mathbb{N}^W$. Indeed, among any labels outside that cone choose one minimal in the positive-correction order. Its leading monomial cannot be cancelled by a label in the cone, nor by a larger label, and minimality excludes a smaller offending label. It would then occur outside the pointed support of L_a, a contradiction. The coefficient of ϑ_a^v is one.

Fix a target $p = a + PE$, $E \neq 0$, and put

$$\nu = (1+|E|)p|_W + \mathbf{1}, \qquad |E| = \sum_s E_s.$$

For $0 < d \leq E$, an integral nonzero $p|_W \cdot d$ has the same sign as $\nu\cdot d$, since $|\mathbf{1}\cdot d| \leq |E| < 1+|E|$. If $p|_W \cdot d = 0$, then $\nu\cdot d = |d| > 0$. Choose the generic endpoint of ι_ν sufficiently close to ν to preserve these signs.

A broken line contributing X^p has reverse final segment $Q + tp$, $t \geq 0$. A last bend would lie on $d^\perp$ for a nonzero $d \leq E$. If $p|_W \cdot d \neq 0$, the pairing of this ray with d stays strictly of one sign. If it is zero, that pairing is the nonzero constant $Q|_W \cdot d$. Thus no last bend exists. Only the straight line with initial label p contributes. Writing $c_p(v) = [\vartheta_p^v]L_a$, we obtain

$$[X^p]\iota_\nu(L_a) = c_p(v).$$

Expand the actual product $L_aL_{N\nu}$. Its labels lie in $N\nu + a + P\mathbb{N}^W$, by pointedness and the same minimal-label argument. To read its coefficient at $N\nu + p$, only labels $N\nu + a + Pf$, $f \leq E$, can occur. Apply Lemma 3.4 simultaneously to this finite set and to $L_{N\nu}$. The latter becomes $X^{N\nu}$, and every possible output becomes its leading monomial in the box. Therefore

$$[L_{N\nu+p}](L_aL_{N\nu}) = v^{\Lambda(p,N\nu)}c_p(v). \tag{3.16}$$

This is an equality of Laurent polynomials. At $v = 1$ the left side is a nonnegative composition multiplicity. Hence $c_p(1) \geq 0$. The theta functions themselves have nonnegative classical coefficients, so (3.15) also gives the support assertion. □

3.5. **Positive tensor powers and simplicity.**

Theorem 3.6. *Let $a \in \mathbb{Z}^{\widetilde{W}}$. Suppose elements C_r of the localized quantum Grothendieck ring, for all $r \geq 1$, satisfy*

$$\begin{aligned} C_r &= C_1^r, \\ C_r &= L_{ra} + \sum_{0\neq f\in\mathbb{N}^W} \mu_{r,f}L_{ra+Pf}, \qquad \mu_{r,f}\in\mathbb{N}, \\ C_1 &= X^a + \sum_{\substack{0\neq e\in\mathbb{N}^W \\ a|_W\cdot e<0}} c_e(v)X^{a+Pe}. \end{aligned} \tag{3.17}$$

Each sum in the second line is finite; its multiplicities are ordinary integers. Then $C_r = L_{ra}$ for every $r \geq 1$. Consequently L_a is real and $L_a|_{v=1} = \vartheta_a$.

Proof. Fix a correction box $0 \le e \le E$. Choose a generic ν arbitrarily close to $a|_W$, preserving every nonzero sign of $a|_W \cdot d$ for $0 < d \le E$. Form $H = TF$ as in (3.12). For $k = ra + h$, with h in a fixed finite set, the coefficient of r in the phase of (3.13) is

$$2(a|_W \cdot d_1 - |a|_W| \cdot d_2 - a|_W \cdot d_3) \le 0. \tag{3.18}$$

Indeed d_1 has nonpositive $a|_W$-pairing and d_3 nonnegative pairing; the middle term is nonpositive. For each fixed target the remaining rational-function degrees are bounded independently of r.

Expand all actual basis terms in (3.17) by (3.10). At $ra + Pe$, only $f \le e$ and subsequent triangular corrections $f' \le e - f$ can contribute. Thus all offsets $h = P(f + f')$ belong to one finite set independent of r. The triangular coefficients have degree at most zero. The scalars $\mu_{r,f}$ have degree zero even when their numerical values grow with r. Equation (3.18) therefore gives a constant D_E such that

$$\deg_v [X^{ra+Pe}] \operatorname{Ad}_F(C_r) \le D_E \quad (e \le E) \tag{3.19}$$

for all sufficiently large r.

Set $G = \operatorname{Ad}_F(C_1)$ in the chosen box. The corrections of F and F^{-1} have nonpositive $a|_W$-pairing. Hence the same is true of the corrections of G. On the algebra supported in nonpositive pairing, the projection onto pairing zero is multiplicative: a sum of two nonpositive pairings is zero precisely when both are zero. The pairing-zero part of C_1 is X^a. Every pairing-zero monomial in F commutes with X^a, by compatibility. Thus the pairing-zero part of G is also exactly X^a.

If G has a nonleading term in the box, choose one of smallest positive total correction degree, say $b_e(v)X^{a+Pe}$. Its pairing $a|_W \cdot e$ is strictly negative. No expression of e as a sum of two nonzero supported corrections is possible, since each would have smaller total degree. In G^r the target correction therefore comes from exactly one insertion of this term. Commuting the $r - 1$ leading monomials on its two sides gives the coefficient

$$b_e(v) \sum_{j=0}^{r-1} v^{(2j-r+1)\Lambda(a,Pe)}. \tag{3.20}$$

The nonzero integer $\Lambda(a, Pe) = -2a|_W \cdot e$ implies that the displayed powers are distinct, and that the largest power has coefficient one. Hence its degree is

$$\deg_v b_e + (r-1)|\Lambda(a, Pe)|.$$

This contradicts (3.19), since $G^r = \operatorname{Ad}_F(C_r)$. It follows that

$$\operatorname{Ad}_F(C_1) = X^a, \qquad \operatorname{Ad}_F(C_s) = X^{sa} \pmod{\{e : e \not\le E\}} \tag{3.21}$$

for every fixed s.

By Proposition 3.5, the classical theta expansion of C_s is finite and nonnegative, and its coefficient at ϑ_{sa} is one. If a further label $sa + PE$ had positive coefficient, its theta function would contribute its leading monomial with positive coefficient in the generic chart used for this box. All other theta contributions are nonnegative. This contradicts (3.21). Thus $C_s|_{v=1} = \vartheta_{sa}$. Each nonzero $\mu_{s,f}$ in (3.17) would contribute at least $\mu_{s,f}$ to the coefficient of ϑ_{sa+Pf}. Therefore every such $\mu_{s,f}$ is zero. This proves $C_s = L_{sa}$ already over $\mathbb{Z}[v^{\pm 1}]$. Taking $s = 1, 2$ gives the last assertions. □

Corollary 3.7. *For every actual real simple of extended degree a,*

$$L_a^r = L_{ra} \quad (r \ge 1), \qquad L_a|_{v=1} = \vartheta_a.$$

Proof. Every ordinary tensor power is simple by [Her10]. Positivity of the quantum structure constants makes a quantum product whose classical specialization is one simple with multiplicity one a Laurent monomial times that simple. Its leading phase here is $\Lambda(ra, a) = 0$. Induction gives $L_a^r = L_{ra}$. Nonnegative quantum monomial coefficients cannot vanish on specializing $v = 1$, so Proposition 3.2 gives the strict quantum support required in Theorem 3.6. Apply that theorem to $C_r = L_{ra}$. □

3.6. **Nonzero critical invariants on a stability wall.** For a finite-dimensional A-module M, let $\dim M \in \mathbb{N}^W$ be its dimension vector. A real vector $\theta \in \mathbb{R}^W$ defines semistability by

$$\theta \cdot \dim M = 0, \qquad \theta \cdot \dim N \leq 0 \quad \text{for every } N \subset M.$$

Stability replaces the latter inequality by a strict inequality for every proper nonzero submodule. The semistable wall of M is the set of weights making it semistable.

Giving the vertical arrows of Q weight one and the oblique arrows weight zero gives an algebraic $\mathbb{C}^\times$-action. Every summand of $\mathcal{W}$ has weight one, and the quiver remaining after deletion of vertical arrows is acyclic. These two facts will provide nonvanishing of the critical invariant of a nonempty wall.

Lemma 3.8. *Let X be a smooth quasi-projective complex variety with a linearized algebraic $\mathbb{C}^\times$-action. Suppose every orbit has a limit at zero and the fixed locus is proper. Let $f : X \longrightarrow \mathbb{C}$ satisfy $f(tx) = tf(x)$. If $\mathrm{Crit}(f) \neq \varnothing$, the compactly supported weight polynomial of the shifted vanishing-cycle complex $\phi_f[-1]\mathbb{Q}_X$ is nonzero.*

Proof. Write $X_F^+ \to F$ for the attracting stratum of a connected fixed component and r_F for its affine-fiber rank. The Białynicki–Birula decomposition and the absolute motivic vanishing-cycle formula apply under the stated hypotheses [BBS13, Proposition 2.12 and Appendix B]. Locally over F, the attracting fiber admits homogeneous positive-weight coordinates. To obtain these coordinates, lift a homogeneous basis of its augmentation ideal modulo its square; graded Nakayama and smoothness identify the resulting graded algebra with a polynomial algebra. Every positive weight is an integer at least one. The restriction of a weight-one regular function is therefore linear in the weight-one coordinates and independent of the others. Write it as $\langle s_F, y_1 \rangle$, and put $Z_F = Z(s_F) \subset F$.

At a fixed point, df can be nonzero only on the weight-one tangent subspace. Hence $Z_F = F \cap \mathrm{Crit}(f)$. Over $F \setminus Z_F$, the zero and one fibers of the linear form are affine bundles of the same rank $r_F - 1$. Over Z_F only the zero fiber occurs, with rank r_F. In the Grothendieck ring of varieties,

$$[f^{-1}(0)] - [f^{-1}(1)] = \sum_F \mathbb{L}^{r_F}[Z_F], \tag{3.22}$$

where $\mathbb{L} = [\mathbb{A}^1]$. The left side is the absolute class of the shifted vanishing cycles by the cited motivic formula; weight one makes its character primitive.

The limit of a critical point is critical, since the critical locus is closed and invariant. Thus some Z_F is nonempty. All Z_F are proper. In their compactly supported Hodge–Deligne polynomials, the coefficient of the highest diagonal degree is the positive number of irreducible components of maximal dimension. Choose the maximal value of $r_F + \dim Z_F$ in (3.22). At its highest weight, all contributions are positive and lower-dimensional terms do not contribute. Their sum is nonzero. This proves the assertion after weight realization. A virtual shift or Tate twist multiplies it by an invertible sign and monomial. □

Proposition 3.9. *Suppose $\theta \in \mathbb{Q}^W$ admits a nonzero semistable A-module. There is a nonzero dimension vector d for which the actual critical weight polynomial of the θ-stable moduli is nonzero.*

Proof. Choose a nonzero semistable dimension vector d of smallest total dimension. Every semistable module of dimension d is stable: a proper Jordan–Hölder factor in the abelian category of semistable modules would have smaller positive dimension.

Choose positive integers m_s, $s \in W$, add a source vertex ∞, and add m_s arrows $\infty \to s$. Give the source dimension one and these arrows weight zero. The potential remains $\mathcal{W}$. Set $R = \sum_s m_s d_s > 0$, and choose a sufficiently small positive rational number ϵ. Put

$$\zeta_s = \theta_s - \epsilon m_s, \qquad \zeta_\infty = \epsilon R.$$

For every proper nonzero subdimension vector (e, η) with $\eta = 0, 1$, choose ϵ small enough that it preserves the sign of every nonzero $\theta \cdot e$. If $\theta \cdot e = 0$, its ζ-value is $-\epsilon m \cdot e < 0$ when $\eta = 0$, and $\epsilon m \cdot (d - e) > 0$ when $\eta = 1$, apart from the excluded zero and total vectors. Hence every free-quiver ζ-semistable representation of dimension $(d, 1)$ is stable. Its quotient X_{fr} is smooth: the projective base-change group acts freely on the stable locus. The function $f_{\mathrm{fr}} = \operatorname{Tr} \mathcal{W}$ descends to this smooth quotient and has weight one.

On its critical locus the underlying representation is θ-semistable. Otherwise a submodule with positive θ-weight, taken with source component zero, would remain destabilizing for ζ. Minimality of d makes the underlying critical representation stable. Conversely, a stable representation of dimension d and any nonzero framing is ζ-stable. Subobjects with a proper nonzero underlying submodule have strictly negative θ-weight, whose sign is preserved. The subobject with the entire underlying module and no source has weight $-\epsilon R$. A source-only subobject exists exactly for the zero framing and then has positive weight. This proves the description of the whole framed critical locus and its nonemptiness.

The GIT map from X_{fr} to the affine quotient of the free-quiver representation space is projective [Kin94, §2]. Its invariant functions are generated by traces of oriented cycles [LBP90]. The source belongs to no cycle, and every old cycle contains a vertical arrow. Every nonconstant trace invariant thus has positive cut weight. The cut action contracts the affine quotient to zero. The valuative criterion for the projective GIT map gives a limit at zero for every orbit in X_{fr}, and its fixed locus is contained in the projective zero fiber. The polarization is linearized. Lemma 3.8 applies, so the framed critical weight polynomial is nonzero.

It remains to remove the frame. Let X_{st} be the smooth moduli of free-quiver θ-stable representations of dimension d, with function $f_{\mathrm{st}} = \operatorname{Tr} \mathcal{W}$. The open part of X_{fr} where the underlying free representation is stable and the framing is nonzero is a Brauer–Severi bundle

$$\pi : U_{\mathrm{fr}} \longrightarrow X_{\mathrm{st}}, \qquad \pi^{-1}(x) \simeq \mathbb{P}^{R-1}.$$

It contains the entire critical locus just described, and $f_{\mathrm{fr}} = f_{\mathrm{st}}\pi$ there. Smooth pullback commutes with shifted vanishing cycles. The rational class $-c_1(\omega_\pi)/R$ restricts to the hyperplane class on every fiber. Its powers give the rational projective-bundle decomposition

$$R\pi_*\mathbb{Q} = \bigoplus_{j=0}^{R-1} \mathbb{Q}[-2j](-j).$$

Proper base change and the support in U_{fr} therefore multiply the unframed critical weight polynomial by $1 + t^2 + \cdots + t^{2R-2}$ to give the framed one. The unframed polynomial is consequently nonzero. □

3.7. **Critical wall crossing and the cluster diagram.** We now compare wall factors as elements of the completed charge group. This will retain charges central in the smaller mutable torus. Let

$$\chi(d, e) = \sum_{s \in W} d_s e_s - \sum_{s \to t \in Q_1} d_s e_t, \qquad \omega(d, e) = \chi(d, e) - \chi(e, d) = d^T B e.$$

For compactly supported, virtually normalized vanishing cycles, use the charge multiplication

$$x^d * x^e = \mathbb{L}^{\omega(d,e)/2} x^{d+e}.$$

This is the opposite of the product in [DM16, equation (48)]. We therefore reverse the increasing slope order in its compactly supported Harder–Narasimhan formula (70), obtaining decreasing slopes [DM16, Theorem 4.12]. Weight realization sends $\mathbb{L}^{1/2}$ to $-t$. Put $q_0(d) = \chi(d,d) \pmod 2$. Since

$$q_0(d+e) = q_0(d) + q_0(e) + \omega(d,e) \pmod 2,$$

the substitution $x^d \mapsto (-1)^{q_0(d)} Z^d$ converts this realization into

$$Z^d Z^e = t^{\omega(d,e)} Z^{d+e}. \tag{3.23}$$

Its positive rank-one factor is

$$\mathbb{E}(t,Z) = \sum_{n\geq 0} \frac{t^{n^2} Z^n}{[\mathrm{GL}_n]_{t^2}} = \exp\left(\sum_{n\geq 1} \frac{(-1)^{n-1} Z^n}{n(t^n - t^{-n})} \right), \tag{3.24}$$

where $[\mathrm{GL}_n]_{t^2} = \prod_{j=0}^{n-1} (t^{2n} - t^{2j})$ and the $n = 0$ coefficient is one.

The embedding of (3.23) into (3.8) is exactly

$$t = v^2, \qquad Z^d \longmapsto Y^d = X^{Pd}, \tag{3.25}$$

because $\Lambda(Pd, Pe) = 2d^T Be$. For a positive-to-negative crossing of the initial s-wall, the action for $a_s \geq 0$ is

$$\mathrm{Ad}_{\mathbb{E}(t, Y^{e_s})}(X^a) = X^a \prod_{j=1}^{a_s} (1 + t^{2j-1} Y^{e_s}). \tag{3.26}$$

Indeed $Y^{e_s} X^a = t^{2a_s} X^a Y^{e_s}$, and division of the exponential products in (3.24) gives

$$\frac{\mathbb{E}(t, t^{2a_s} Y)}{\mathbb{E}(t,Y)} = \prod_{j=1}^{a_s} (1 + t^{2j-1} Y).$$

At $v = 1$ this is $X^a (1 + Y^{e_s})^{a_s}$. Thus these are the quantum and classical cluster wall actions for P with incoming direction Pe_s.

Proposition 3.10. *For the finite quiver with potential $(Q, \mathcal{W})$, the actual critical semistable wall factors, with the normalization above, coincide with the positive quantum cluster wall factors in the full completed charge group.*

Proof. The cohomological Harder–Narasimhan theorem [DM16, Theorem 4.12] applies to the trace of a polynomial potential on the smooth stack of free-quiver representations, at arbitrary stability. Its vanishing cycles are supported on Jacobian representations. It gives a total element G_{crit} in the group completed by positive dimension vectors and a canonical factorization

$$G_{\mathrm{crit}} = G_+(\theta) G_0(\theta) G_-(\theta).$$

The middle factor is exactly the critical semistable series. This canonical factorization defines the consistent critical scattering diagram. For any bounded dimension vector only finitely many degrees occur, so the statement is an identity in the full completed group.

The finite adapted-word seed is injective reachable. Its reddening gallery runs from the positive to the negative chamber through finitely many reachable cluster chambers. The categorical mutation equivalences at a generic facet, [Mou19, Propositions 4.15–4.16 and

Theorem 4.17], identify its semistable category with the category of finite vector spaces at a vertex of the mutated loopless quiver. Consequently it is add E for a rigid brick E, with

$$\mathrm{End}_A(E) = \mathbb{C}, \qquad \mathrm{Ext}^1_A(E,E) = 0.$$

Its dimension vector d is the primitive positive normal to that facet, because mutations of the charge lattice are unimodular. The adjacency convention in that categorical description is $-B$; its direction map is $(-B)^T = B$. It therefore gives these same facet hyperplanes and directions, with the same covariant representations and stability inequalities.

We compute the critical factor at this facet at every multiple of d. Near E, the stable free-quiver quotient is smooth of dimension

$$r = 1 - \chi(d,d).$$

The tangent space of its critical scheme at $[E]$ is $\mathrm{Ext}^1_A(E,E) = 0$, so the normal Hessian is nondegenerate. The cut action preserves the unique critical point $[E]$. Its integral tangent weights are paired by the weight-one Hessian as w and $1-w$. No integer equals $1/2$. Thus r is even and $\chi(d,d)$ is odd.

For $n \geq 1$, the critical locus in the free semistable representation space of dimension nd consists of the orbit

$$\mathcal{O}_n = G_{nd}/\operatorname{GL}_n, \qquad G_{nd} = \prod_s \operatorname{GL}_{nd_s}.$$

Its tangent space equals the orbit tangent space, since $\mathrm{Ext}^1_A(E^n,E^n) = 0$. The orbit is a smooth subvariety of the critical scheme. Its dimension is therefore both a lower bound for the local dimension and the dimension of the Zariski tangent space. Equality makes the critical local ring regular and reduced. The normal Hessian has rank

$$r_n = (1-\chi(d,d))n^2.$$

The holomorphic Morse–Bott calculation gives a Tate contribution $\mathbb{L}^{r_n/2}$, tensored with its orientation sign local system. This local system is trivial: the function, Hessian, and vanishing cycles are G_{nd}-equivariant, and the stabilizer $\mathrm{Aut}(E^n) = \operatorname{GL}_n$ is connected. Equivariant rank-one sign local systems on this orbit factor through that stabilizer's component group.

The virtual normalization of the smooth quotient stack is $\mathbb{L}^{\chi(d,d)n^2/2}$. The coefficient at nd is therefore exactly

$$\frac{\mathbb{L}^{\chi(d,d)n^2/2}\mathbb{L}^{(1-\chi(d,d))n^2/2}}{[\operatorname{GL}_n]} = \frac{\mathbb{L}^{n^2/2}}{[\operatorname{GL}_n]}.$$

Weight realization and the quadratic sign give

$$\frac{(-t)^{n^2}(-1)^{q_0(nd)}}{[\operatorname{GL}_n]_{t^2}} = \frac{t^{n^2}}{[\operatorname{GL}_n]_{t^2}},$$

because $q_0(d) = 1$ and $q_0(nd) \equiv n^2 \pmod 2$. The entire semistable factor is thus $\mathbb{E}(t, Z^d)$, including every multiple of the primitive charge.

At each crossing of the reddening gallery the actual factor equals the cluster factor, with the inverse taken when the crossing direction is reversed. Their ordered products along the gallery are equal. Canonical factorization is unique: in the first nonzero total degree, assign each charge to its positive, zero, or negative θ-factor; divide out these terms and repeat in the next degree. This induction works in every finite positive-degree quotient and then in the completion. A consistent charge-valued diagram is recovered by these factorizations of its total positive-to-negative group element. Thus the equal total elements give equal factors at every θ. This comparison took place before passing to adjoint actions, so it includes the central part of the charge group. □

Corollary 3.11. *An open subset of stability space containing no essential classical cluster wall contains no nonzero semistable A-module.*

Proof. For one module, semistability is defined by the dimension vector and its finitely many possible subdimension vectors. Its wall is rational polyhedral. If it meets the given open set, a rational point on that wall lies in the open set. Choose a minimal nonzero semistable dimension there. Proposition 3.9 gives a nonzero critical weight polynomial in that degree. The stable representation stack is a $\mathbb{C}^\times$-gerbe over its coarse moduli, so its rational weight realization differs by the nonzero factor $(t^2-1)^{-1}$ and the invertible virtual monomial. There are no smaller nonzero semistable degrees, so the semistable series is nontrivial. The same is true of the quantum cluster factor by Proposition 3.10.

A stable module remains stable in a relatively open subset of its dimension hyperplane: its finitely many proper subdimension inequalities are strict. We may therefore choose a nearby generic point of this hyperplane within the given open set and away from the finitely many other walls of any fixed degree truncation detecting our factor. Quantum cluster wall positivity [DM19, Theorems 2.15 and 2.19] writes the nonzero normalized wall data as positive Laurent polynomials of fixed parity. Such a polynomial has positive value at one. Thus this factor has nontrivial classical action. The extended realization is faithful on charges, since $\Lambda(Pd, e_s) = 2d_s$ and $d \neq 0$. We have obtained an essential classical cluster wall in the open set, a contradiction. □

3.8. **A strict support inequality gives a silting cone.** Let $J : \mathbb{R}^W \to \mathbb{R}^{\widetilde{W}}$ be the coordinate inclusion, and use for valuations the rational skew form

$$\Lambda_{\mathrm{val}} = -\tfrac{1}{2}\Lambda, \qquad \Lambda_{\mathrm{val}} P = J, \qquad P^T \Lambda_{\mathrm{val}} = -J^T. \tag{3.27}$$

This is a rational Λ-structure for the seed datum (P, J) of [CMMM25, §5.1]. The valuation of the monomial X^{a+Pe} at $\Lambda_{\mathrm{val}} p$ is

$$(a+Pe)^T \Lambda_{\mathrm{val}} p = a^T \Lambda_{\mathrm{val}} p - e \cdot p|_W. \tag{3.28}$$

Lemma 3.12. *Suppose a classical theta function ϑ_a is a Laurent polynomial and each nonleading exponent $a + Pe$ in it satisfies $a|_W \cdot e < 0$. Then there is an open convex neighborhood U of a such that*

$$U \cap \{a + b : b|_W \in \mathbb{R}^W_{>0}\}$$

is disjoint from the closure of the semistable walls of A.

Proof. There are finitely many nonleading exponents. Choose U small and convex so that $p|_W \cdot e < 0$ for all of them and every $p \in U$. Equation (3.28) makes the leading exponent the unique minimizer of the initial theta expansion for each such valuation. By [CMMM25, Lemma 5.17], its minimum equals the minimum over broken lines with label a and any generic endpoint $p \in U$. The straight broken line exists and has final exponent a; it attains that minimum. Theorem 3.5 and Lemma 5.14 of that paper make this straight line Λ^T_{val}-taut. Lemma 5.12 requires such a line to take its largest possible bend at every nontrivial negative-to-positive wall crossing.

Suppose an essential conical wall meets the displayed open wedge. Choose a relative interior point $r = a + b$ in that intersection, with $b|_W > 0$, and let $d \in \mathbb{N}^W \setminus \{0\}$ be its positive normal. Then

$$a|_W \cdot d = -b|_W \cdot d < 0.$$

Take $s > 1$ sufficiently close to one that $p = a + sb$ still lies in U. The reverse straight ray

$$p + ta, \qquad t \geq 0,$$

meets the same conical wall at $t = s - 1$, where $p + (s-1)a = sr$. Its forward traversal is from the negative side to the positive side. A small generic perturbation retains this transverse crossing and the endpoint in U. The line has no bend, whereas maximal bending at a nontrivial crossing with $a|_W \cdot d \neq 0$ is nonzero. This contradicts tautness.

The wedge therefore contains no essential cluster wall. Corollary 3.11 excludes all semistable wall points. Since the wedge is open, it is disjoint from the closure of their union as well. □

Theorem 3.13. *Let $L(m) \in \mathcal{C}_\ell^{\leq \xi}$ be real, and let g be its mutable degree in (3.5). There is a unique rigid object $M \in \mathcal{C}_A$ with*

$$g_{\mathrm{DS}}(M) = g.$$

Its associated module is the exact generic kernel:

$$\mathrm{Hom}_{\mathcal{C}_A}(T, M) \simeq K(m).$$

Proof. By Proposition 3.2 and Corollary 3.7, the theta function $\vartheta_{\widetilde{g}}$ has the strict support property of Lemma 3.12. Project the resulting open convex wedge to mutable stability space. The projection is open and convex, avoids the closure of all semistable walls, and has g in its closure.

Asai's chamber theorem [Asa19, Theorem 3.17] identifies the connected components of this complement with the open cones

$$\left\{ \sum_j c_j [U_j] : c_j > 0 \right\},$$

where $U = \bigoplus_j U_j$ ranges over basic two-term silting complexes. The connected projected wedge lies in one such cone. Hence g is a nonnegative real combination of the $[U_j]$. These classes form a basis of the integral Grothendieck group [AIR12, Proposition 1.6 and Theorem 5.1]. Its coefficients are therefore nonnegative integers. The corresponding direct sum of the U_j is a two-term presilting complex of class g.

Write its minimal differential as $f : P^- \to P^+$, in degrees $-1, 0$. The two terms have no common indecomposable projective. Indeed, if they shared P_s, the map with an identity component on this summand would be nonzero modulo the radical in $\mathrm{Hom}_A(P^-, P^+)$. Since f is minimal, every map $af - fb$ is radical. That component would survive in

$$\mathrm{Hom}_{K^b(\mathrm{proj}\, A)}(f, f[1]) = \mathrm{Hom}_A(P^-, P^+)/\{af - fb : a \in \mathrm{End}(P^+),\, b \in \mathrm{End}(P^-)\},$$

contrary to presilting. Thus

$$P^- = \bigoplus_s P_s^{[-g_s]_+}, \qquad P^+ = \bigoplus_s P_s^{[g_s]_+}.$$

The same quotient is the tangent cokernel of the change-of-basis orbit of f. It vanishes, so the orbit is open in the affine space $\mathrm{Hom}_A(P^-, P^+)$. That space is irreducible and hence has a unique open orbit. Write $D = \mathrm{Hom}_{\mathbb{C}}(-, \mathbb{C})$ for linear duality. The Nakayama equivalence $\nu_A = D\,\mathrm{Hom}_A(-, A)$ identifies this Hom space and its automorphism action with those for the injectives. Therefore $\nu_A(f)$ is a general map in (3.6).

Lift f to $\mathrm{add}\, T$ and complete it to a triangle

$$T^- \xrightarrow{f} T^+ \longrightarrow X \longrightarrow T^-[1].$$

The two-term presilting–rigid correspondence [AIR12, Proposition 4.6 and Theorem 4.7] makes X rigid. Set $M = X[1]$. The rotated triangle is

$$M \longrightarrow T^-[2] \xrightarrow{f[2]} T^+[2] \longrightarrow M[1].$$

On $\operatorname{add} T$, the functor $\operatorname{Hom}(T,-[2])$ is the Nakayama functor, by the 2-Calabi–Yau property. Moreover $\operatorname{Hom}(T,T^+[1])=0$. The long exact sequence thus identifies

$$\operatorname{Hom}(T,M)=\ker\bigl(\nu_A\operatorname{Hom}(T,T^-)\longrightarrow\nu_A\operatorname{Hom}(T,T^+)\bigr)=K(m).$$

The presentation triangle gives $\operatorname{ind}_T(M[-1])=[T^+]-[T^-]=g$. Finally, the index determines a rigid object up to isomorphism [DK08, §2.3], applied to $M[-1]$. □

The generic kernel here is itself rigid as a module. To see this directly, put $u=\nu_A(f):J^0\to J^1$ and $K=\ker u$. The orbit tangent calculation gives $\operatorname{Hom}(J^0,J^1)=\{au-ub\}$. Every $h:K\to J^1$ extends to J^0, since J^1 is injective, and restricting $au-ub$ to K gives $h=-ub|_K$. Thus $\operatorname{Hom}(K,J^0)\to\operatorname{Hom}(K,J^1)$ is surjective. The kernel of the next differential in an injective resolution is a subspace of $\operatorname{Hom}(K,J^1)$, so the first self-extension group vanishes.

3.9. **The frozen normalization.**

Lemma 3.14. *For each mutable integral vector $g\in\mathbb{Z}^W$, there is exactly one dominant monomial m in the finite window such that $L(m)$ has no frozen simple tensor factor and has mutable degree g. Explicitly, number a column from bottom to top by $j=0,\dots,\ell$, put*

$$s_{i,0}=0,\qquad s_{i,j}=\sum_{k=1}^{j}g_{i,k},\qquad u_{i,j}=s_{i,j}-\min_{0\le k\le\ell}s_{i,k}.$$

Then $m=\prod_{i,j}Y_{i,\xi_i-2\ell+2j}^{u_{i,j}}$.

Proof. The equations $u_{i,j}-u_{i,j-1}=g_{i,j}$ have all solutions $u_{i,j}=c_i+s_{i,j}$. Dominance is the inequality $c_i\ge-\min_j s_{i,j}$. Increasing c_i by one multiplies m by the full-column monomial $f_i=z_{i,\xi_i-2\ell}$.

Normalized multiplication by a frozen variable preserves the common triangular basis. Indeed its pairing with every correction Pe is zero, so its multiplication has one constant phase on a pointed element. Removing that leading phase makes the product bar invariant. Its injective-standard expansion is the translation of (3.10), with the same negative-degree coefficients. Uniqueness of the bar-invariant triangular element identifies it with the basis element of the translated degree. Under the actual-simple identification, specialization at one gives

$$L(f_i)\otimes L(m')=L(f_im')$$

for every dominant m' in the same window. Conversely, such an actual tensor factor multiplies highest monomials by f_i. Thus it occurs precisely when every $u_{i,j}$ in that column is positive. The absence of this factor is $\min_j u_{i,j}=0$, which is exactly (3.14). □

Theorem 3.13 and Lemma 3.14 give an injection from the real simples without frozen factors to all rigid objects of $\mathcal{C}_A$. The shift in (3.3) retains the decoration: degree e_s gives $T_s[1]$, even though its module in (3.6) is zero. The converse construction and the character formula are proved in the next section.

4. Rigid operators and the original finite category

Throughout Sections 4–7, the Dynkin diagram is of type ADE. Choose an orientation Q and write $\overline{Q}$ for its doubled quiver, with reverse arrow h^* for $h\in Q_1$. Let e_i be the vertex idempotents and let $\Pi=\mathbb{C}\overline{Q}/(\sum_{h\in Q_1}(hh^*-h^*h))$ be the preprojective algebra. Give every doubled arrow degree -1, and put

$$B=\Pi[u],\qquad\deg u=2.$$

The variable u is central. All module maps and extension groups in these sections preserve the displayed grading unless a superscript specifies another degree. Write $Q_i = \Pi e_i$ and $J_i = D(e_i\Pi)$, where $D = \mathrm{Hom}_{\mathbb{C}}(-,\mathbb{C})$; the generator of Q_i and the socle of J_i have degree zero. The shift $\langle s\rangle$ raises degrees by s. A finite-dimensional graded B-module whose restriction to Π is projective is a pair (P,U), where P is graded projective over Π and $U \in \mathrm{Hom}^2_{\Pi}(P,P)$.

4.1. **An exact category of operators.** Fix a height function $b = (b_i)$, so $|b_i - b_j| = 1$ on each edge, and an integer $\ell \geq 0$. Set

$$a_i = b_i - 2\ell, \qquad \Omega = \{(i, b_i - 2k) : i \in I,\ 0 \leq k \leq \ell\}.$$

Let $\mathcal{E}_\Omega$ be the full subcategory of finite-dimensional graded B-modules whose restrictions to Π are projective and whose graded Π-tops are supported on Ω. Its conflations are short exact sequences of graded B-modules with terms in this subcategory. Every such sequence splits over Π. Conversely, an extension of two objects has a graded Π-splitting, so its top is the direct sum of the two endpoint tops. Consequently $\mathcal{E}_\Omega$ is extension closed and

$$\mathrm{Ext}^1_{\mathcal{E}_\Omega}(M,N) = \mathrm{Ext}^{1,0}_B(M,N). \tag{4.1}$$

A nonzero map $Q_i\langle s\rangle \to Q_j\langle t\rangle$ is represented by a path of length $t-s$, and therefore requires $t - s \geq \mathrm{dist}(i,j)$, where dist is graph distance. The height inequality $|b_i - b_j| \leq \mathrm{dist}(i,j)$ gives, on the chosen parity lattice,

$$\mathrm{Hom}^0_{\Pi}(Q_i\langle s\rangle, M) = 0 \quad (s > b_i), \qquad \mathrm{Hom}^0_{\Pi}(M, Q_i\langle t\rangle) = 0 \quad (t < a_i) \tag{4.2}$$

for $M \in \mathcal{E}_\Omega$. For the first equality, a component into a summand $Q_j\langle t\rangle$ of M would satisfy $t - s \leq b_j - b_i - 2 < \mathrm{dist}(i,j)$. For the second, a component from $Q_j\langle s\rangle$ would satisfy $t - s \leq a_i - a_j - 2 < \mathrm{dist}(i,j)$.

For $(i,s) \in \Omega$ define

$$P_i(s) = \bigoplus_{r=0}^{(b_i-s)/2} Q_i\langle s + 2r\rangle,$$

$$I_i(s) = \bigoplus_{r=0}^{(s-a_i)/2} Q_i\langle s - 2r\rangle.$$

In each expression u is the identity from a summand to the summand two degrees higher, and is zero on the highest summand. Evaluation on the first summand in $P_i(s)$ and projection onto the highest summand in $I_i(s)$ give

$$\mathrm{Hom}^0_B(P_i(s), M) = \mathrm{Hom}^0_{\Pi}(Q_i\langle s\rangle, M), \qquad \mathrm{Hom}^0_B(M, I_i(s)) = \mathrm{Hom}^0_{\Pi}(M, Q_i\langle s\rangle). \tag{4.3}$$

Indeed, a map in the first right-hand side extends uniquely by successive applications of U_M; its last required vanishing is (4.2). For a map $c : M \to Q_i\langle s\rangle$, the component of the second extension in $Q_i\langle s - 2r\rangle$ is cU_M^r, with its prescribed shift. Its last boundary condition is the second equality in (4.2). Both right-hand functors are exact on Π-split sequences. Thus $P_i(s)$ is projective and $I_i(s)$ is injective in $\mathcal{E}_\Omega$.

Choose a decomposition of $M|_\Pi$ into shifted Q_i's. Extending its inclusions by the first equality in (4.3) produces an epimorphism from a sum of $P_i(s)$'s to M. Its underlying Π-map has a section supplied by the initial summands, so its kernel belongs to $\mathcal{E}_\Omega$. Extending the projections of the decomposition by the second equality produces an admissible monomorphism from M to a sum of $I_i(s)$'s. Its underlying Π-map has a retraction, so its cokernel also belongs to $\mathcal{E}_\Omega$. There are enough projectives and injectives.

An endomorphism of either interval acts by a single scalar on its Π-top: the top is a one-dimensional chain at one color. The kernel of this scalar map consists of endomorphisms with image in the Π-radical; it is nilpotent. Hence each interval has local endomorphism ring and is indecomposable. Every projective or injective is a summand of the respective displayed sums. Comparison of their tops then identifies all projective-injectives:

$$F_i = P_i(a_i) = I_i(b_i). \tag{4.4}$$

Lemma 4.1. *For a graded B-module (P,U) with P projective over Π, the sequence*

$$0 \longrightarrow (B \otimes_\Pi P)\langle 2\rangle \xrightarrow{u\otimes 1-1\otimes U} B \otimes_\Pi P \longrightarrow (P,U) \longrightarrow 0$$

is a projective resolution in the graded module category. In particular $\mathrm{Ext}^{j,0}_B((P,U),N) = 0$ *for* $j \geq 2$*, and*

$$\mathrm{Ext}^{1,0}_B((P,U),(P,U)) = \mathrm{coker}\big(\mathrm{End}^0_\Pi(P) \xrightarrow{H\mapsto HU-UH} \mathrm{Hom}^2_\Pi(P,P)\big). \tag{4.5}$$

Proof. The last map sends $u^r \otimes x$ to $U^r x$. If a nonzero polynomial $\sum_{r=0}^{d} u^r \otimes x_r$ has $x_d \neq 0$, its image under the first map has highest coefficient x_d in degree $d+1$; the first map is injective. Modulo its image, $u^r \otimes x = 1 \otimes U^r x$, so the kernel of the last map equals that image. Induction from a projective Π-module gives a projective B-module. Applying graded Hom proves the assertions, including the stated differential. □

The category $\mathcal{E}_\Omega$ is hereditary as an exact category. To check this inside the category, represent a two-fold extension by

$$0 \to N \to E_1 \to H \to 0, \qquad 0 \to H \to E_2 \to M \to 0.$$

The ambient vanishing in Lemma 4.1 lifts the first class to an extension of E_2 by N. That extension is Π-split, and its top remains in Ω; it is therefore a conflation in $\mathcal{E}_\Omega$. The lift trivializes the two-fold Yoneda class there. Moreover, the terminal segment of a full interval gives a conflation

$$0 \longrightarrow P_i(s) \longrightarrow F_i \longrightarrow I_i(s-2) \longrightarrow 0, \tag{4.6}$$

where the last term is zero for $s = a_i$. Thus every projective embeds in a projective-injective. For $s < b_i$, the same sequence with index $s+2$ expresses $I_i(s)$ as a quotient of F_i with projective kernel; $I_i(b_i) = F_i$. Thus every injective is a quotient of a projective-injective. These properties make $\mathcal{E}_\Omega$ a Hom-finite, Krull–Schmidt, algebraic 0-Auslander exact category in the sense of [GNP23, Definition 3.7].

4.2. **The finite Jacobian and all rigid lifts.** Let C be the graded cover of B, with vertices (i,k) corresponding to degrees $b_i - 2k$. A doubled Π-arrow keeps k fixed or raises it by one; a u-arrow lowers k by one. Let C_+ be the support quotient on $k \geq 0$. A path starting at level k which visits a negative level contains at least $k+1$ copies of u. Since u is central, these can be commuted to the beginning. Conversely, a term divisible by u^{k+1} factors through a negative level when those copies are put first. Hence the projective at (i,k) in C_+ is

$$(Be_i)\langle b_i - 2k\rangle / u^{k+1}(Be_i)\langle b_i - 2k\rangle = P_i(b_i - 2k).$$

Write e_Ω for the sum of the idempotents at $0 \leq k \leq \ell$ and e_ℓ for the sum at $k = \ell$. With left-projective conventions,

$$\mathrm{End}_{\mathcal{E}_\Omega}\Big(\bigoplus_{(i,s)\in\Omega} P_i(s)\Big)^{\mathrm{op}} = e_\Omega C_+ e_\Omega.$$

A path between vertices at $0 \le k < \ell$ which reaches a level $k \ge \ell$ first meets level ℓ. Thus

$$(e_\Omega C_+ e_\Omega)/(e_\ell) = C\big|_{0\le k<\ell} =: \Lambda_\Omega. \tag{4.7}$$

Here the restriction denotes the quotient by the complementary idempotents. Fujita–Murakami identify C with the Hernandez–Leclerc Jacobian cover; their height automorphism adjusts the signs of the potential [FM23, Remark 4.3 and Propositions 4.4 and 4.6]. It fixes vertices and rescales arrows, so it preserves these support quotients. Therefore Λ_Ω is the finite mutable Jacobian algebra of the original Hernandez–Leclerc strip after the relabeling below.

Chen's algebraic 0-Auslander quotient theorem gives an equivalence of extriangulated categories

$$\Phi : \mathcal{E}_\Omega/[\mathrm{inj} \to \mathrm{proj}] \xrightarrow{\sim} K^{[-1,0]}(\mathrm{proj}\,\Lambda_\Omega) \tag{4.8}$$

[Che24, Theorem 5.7]. The ideal is generated by maps from injectives to projectives. The construction of this quotient retains the extension bifunctor [FGP+26, Section 2.2 and Theorem 2.8]. Together with (4.1), this yields

$$\mathrm{Ext}^{1,0}_B(M,N) \simeq \mathrm{Hom}_{K^b(\mathrm{proj}\,\Lambda_\Omega)}(\Phi M, \Phi N[1]). \tag{4.9}$$

All hypotheses of the theorem hold by the explicit projectives, injectives, heredity, and (4.6); algebraicity holds because $\mathcal{E}_\Omega$ is an exact category of modules.

If an identity belongs to the quotient ideal, it factors through a finite sum of injectives and also through a finite sum of projectives. Its object is a direct summand of both sums, hence is projective-injective. For any other indecomposable, the ideal in its local endomorphism ring is proper and lies in the radical. An isomorphism between two such objects in the quotient lifts to maps whose composites are units; these maps give an isomorphism already in $\mathcal{E}_\Omega$. Consequently (4.8) kills exactly the F_i's and induces a bijection on all remaining indecomposables. Equation (4.9) preserves and reflects rigidity.

The standard correspondence between rigid objects of the Amiot category and two-term presilting complexes [AIR12, Proposition 4.6 and Theorem 4.7] is taken with the convention $g(X) = \mathrm{ind}_T(X[-1])$ used in [DS26] and Section 3. It includes shifted projective summands. Thus the density of (4.8) lifts every rigid object, with its decoration, to a rigid object of $\mathcal{E}_\Omega$.

For the original dominant monomial write

$$m = \prod_{i\in I}\prod_{k=0}^{\ell} Y_{i,\xi_i-2k}^{w_{i,k}}, \qquad P_w = \bigoplus_{i,k} J_i\langle \xi_i - 2k + 2\rangle^{w_{i,k}}. \tag{4.10}$$

Let h be the Coxeter number and let $i \mapsto i^*$ be the Nakayama permutation. The graded Frobenius identity is

$$J_i \simeq Q_{i^*}\langle h-2\rangle \qquad \text{[FM23, Corollary 3.4(1)]}. \tag{4.11}$$

Hence the summand at (i,k) in (4.10) has top at $(i^*, \xi_i + h - 2k)$. Choose $b_{i^*} = \xi_i + h$. This is a height function, and the map $(i,r) \mapsto (i^*, r+h)$ identifies its strip with the original one. Its parity is correct: the nonzero socle path from i to i^* has length $h-2$. We use this relabeling when comparing (4.7) with the original seed.

The construction of (4.8) fixes the projective coordinates:

$$\Phi(P_{i^*}(b_{i^*} - 2p)) = \Lambda_\Omega e_{(i,p)}[0] \quad (0 \le p < \ell), \qquad \Phi(F_{i^*}) = 0.$$

If this two-term complex represents X in the fundamental domain of the Amiot category, then $M = X[1]$ has $g_{\mathrm{DS}}(M) = \mathrm{ind}_T X$. The interval $P_{i^*}(b_{i^*} - 2p)$ has one copy of the projective top at each level $0 \le k \le p$. Heredity and enough projectives give every operator a projective resolution of length at most one. Taking its Π-top is additive on conflations, since these split

over Π. The interval-top vectors form a triangular integral basis. Thus, if $c_{i,p}$ are its projective Grothendieck coordinates,

$$w_{i,k} = \sum_{p \geq k} c_{i,p}, \qquad c_{i,k} = w_{i,k} - w_{i,k+1}, \qquad w_{i,\ell+1} = 0. \tag{4.12}$$

Quotienting by the full interval removes the coordinate $k = \ell$. The remaining differences are precisely the original mutable Kirillov–Reshetikhin coordinates.

Proposition 4.2. *Every rigid object of the original finite category has a unique rigid operator lift without full-interval summands. Its dominant vector in* (4.10) *is characterized by the given mutable differences and*

$$w_{i,k} \geq 0, \qquad \min_{0 \leq k \leq \ell} w_{i,k} = 0 \quad (i \in I). \tag{4.13}$$

Indecomposability is preserved by this lift.

Proof. Fix the graded projective module P_w. Its graded automorphism group acts by conjugation on $\mathrm{Hom}^2_\Pi(P_w, P_w)$. Lemma 4.1 identifies the normal space to its orbit with $\mathrm{Ext}^{1,0}_B(K, K)$. A rigid operator therefore has an open orbit, and an irreducible affine space has at most one open orbit.

On the Π-top, U induces, for each color, a chain

$$W_{i,\ell} \longrightarrow W_{i,\ell-1} \longrightarrow \cdots \longrightarrow W_{i,0}, \qquad \dim W_{i,k} = w_{i,k}.$$

The matrices of these arrows are independent linear coordinates in the operator space: any prescribed matrix lifts by scalar identity maps between adjacent shifts of Q_{i^*}. Put $d_i = \min_k w_{i,k}$. The condition that the entire composition have rank d_i is a nonempty invariant open condition. Nonemptiness follows by choosing a common d_i-dimensional subspace at every level and making the successive maps identity on it. The open orbit of a rigid operator lies in this locus.

Choose d_i bottom vectors on which the entire composition is injective. Formula (4.3) gives a B-map $f : \bigoplus_i F_{i^*}^{d_i} \to K$ whose map on Π-tops is injective at every level. Choose a left inverse on these tops. Since $K|_\Pi$ is projective, it lifts to a Π-map $H : K \to \bigoplus_i F_{i^*}^{d_i}$. The map Hf is the identity modulo the radical, and is invertible. Replacing H by $(Hf)^{-1}H$ gives a Π-retraction. Thus f is an inflation in $\mathcal{E}_\Omega$. Injectivity of the F_{i^*}'s splits it over B:

$$K \simeq \bigoplus_i F_{i^*}^{d_i} \oplus K_0.$$

No larger full-interval multiplicity is possible, since each copy contributes one at every level. Its multiplicity is therefore exactly $\min_k w_{i,k}$.

Density and (4.9) first give a rigid lift of any rigid object. Remove the full intervals. The resulting vector satisfies (4.13). For each color, successive differences determine all $w_{i,k}$ up to addition of a constant; their minimum determines that constant. Thus the vector is unique. For that vector the rigid operator is unique by the open-orbit argument. The indecomposable statement follows from the preceding analysis of the quotient ideal. □

In the common triangular basis, multiplication by a frozen basis element preserves the actual simple basis, as established in Section 3. The full-column monomial $\prod_{k=0}^{\ell} Y_{i,\xi_i-2k}$ therefore divides an actual simple as a frozen tensor factor exactly when every $w_{i,k}$ is positive. Consequently (4.13) is also the frozen-free normalization on the representation side.

4.3. **The full generic injective kernel.** From now on write $w_{i,r}$ for the multiplicity of $Y_{i,r}$ in m, and extend w by zero at every other degree. Define

$$g^{\mathrm{full}}_{i,s} = w_{i,s} - w_{i,s-2} \qquad (i \in I,\ s \in \mathbb{Z}). \tag{4.14}$$

In particular $g^{\mathrm{full}}_{i,\xi_i+2} = -w_{i,\xi_i}$. Its restriction to the mutable vertices is the finite degree vector in (4.12). We retain the full vector in all ambient Hom and Ext pairings.

Let

$$\mathbb{I}(i,s) = D(e_iB)\langle s\rangle = \bigoplus_{n\geq 0} J_i\langle s-2n\rangle.$$

This is an injective object of the locally finite graded module category, since $\mathrm{Hom}^0_B(N, \mathbb{I}(i,s)) \simeq DN_{i,s}$. For $P = P_w$ put

$$C(P) = \bigoplus_{n\geq 0} P\langle -2n\rangle,$$

with u the left shift S. The homogeneous map U commutes with S. Finite dimensionality and its positive degree make U nilpotent. There is an exact sequence

$$0 \longrightarrow K = (P,U) \xrightarrow{\eta} C(P) \xrightarrow{S-U} C(P)\langle -2\rangle \longrightarrow 0, \qquad \eta(x) = (x, Ux, U^2x, \ldots). \tag{4.15}$$

For the kernel, the equations $x_{n+1} = Ux_n$ give precisely $\eta(P)$. For surjectivity, given a finitely supported sequence y, set $x_0 = 0$ and $x_{n+1} = y_n + Ux_n$. This sequence is eventually zero by nilpotence of U and maps to y. The injective terms in (4.15) are

$$I^0 = \bigoplus_{i,r} \mathbb{I}(i, r+2)^{w_{i,r}}, \qquad I^1 = \bigoplus_{i,r} \mathbb{I}(i,r)^{w_{i,r}}.$$

In particular K has injective dimension at most one. The relevant Hom spaces between these terms are finite-dimensional: each is dual to a fixed homogeneous component of B, and the degrees in Π are bounded. Write $[z]_+ = \max(z, 0)$.

Proposition 4.3. *If $K = (P_w, U)$ is rigid, it is the kernel of a surjective generic map*

$$f : I_- = \bigoplus_{i,s} \mathbb{I}(i,s)^{[-g^{\mathrm{full}}_{i,s}]_+} \longrightarrow I_+ = \bigoplus_{i,s} \mathbb{I}(i,s)^{[g^{\mathrm{full}}_{i,s}]_+}. \tag{4.16}$$

For every finite-dimensional graded B-module M,

$$g^{\mathrm{full}} \cdot \dim M = \dim \mathrm{Ext}^{1,0}_B(M,K) - \dim \mathrm{Hom}^0_B(M,K). \tag{4.17}$$

For every nonzero submodule $N \subset K$,

$$\mathrm{Ext}^{1,0}_B(N,K) = 0, \qquad g^{\mathrm{full}} \cdot \dim N = -\dim \mathrm{Hom}^0_B(N,K) < 0. \tag{4.18}$$

Proof. First take the map $f_0 : I^0 \to I^1$ of (4.15). For any $h : I^0 \to I^1$, the restriction $h\eta$ lifts through f_0 because $\mathrm{Ext}^{1,0}_B(K,K) = 0$: write $h\eta = f_0\beta$. Injectivity of I^0 extends $\beta : K \to I^0$ to $b : I^0 \to I^0$. Then $h - f_0 b$ vanishes on K and factors through the surjection f_0, say $h - f_0 b = a f_0$. Thus

$$\mathrm{Hom}^0_B(I^0, I^1) = \mathrm{End}^0_B(I^1) f_0 + f_0 \,\mathrm{End}^0_B(I^0).$$

This is the surjectivity of the differential of the change-of-basis action. The orbit of f_0 in its finite-dimensional affine Hom space is open.

Modulo the categorical radical of the finite additive category of the injectives occurring here, a map consists of an ordinary matrix for each common indecomposable type. The locus where all these matrices have maximal possible rank is a nonempty invariant open subset. The open orbit of f_0 meets it. An invertible matrix pivot in such a block is an isomorphism of injectives; elementary row and column operations split it off as an identity summand. Iterating removes

all common indecomposables. The remaining multiplicities are the negative and positive parts of the difference $[I^1]-[I^0]$, which is (4.14). The reduced map is surjective, has kernel K, and has an open orbit by the same lifting argument. This proves (4.16).

Applying $\mathrm{Hom}_B^0(M,-)$ to that injective resolution and using dual evaluation gives (4.17). For $N\subset K$, any map $\alpha: N\to I_+$ extends to $h: I_-\to I_+$ by injectivity of I_+. The orbit calculation writes $h=af+fb$. Since $f|_N=0$, $\alpha=f(b|_N)$. Hence $\mathrm{Hom}_B^0(N,I_-)\to\mathrm{Hom}_B^0(N,I_+)$ is surjective and $\mathrm{Ext}_B^{1,0}(N,K)=0$. The inclusion $N\hookrightarrow K$ is a nonzero element of $\mathrm{Hom}_B^0(N,K)$ when $N\neq 0$, proving the strict inequality. □

Lemma 4.4. *For a finite set S of spectral vertices, let B_S be the quotient of the graded cover by the idempotents outside S, and let $\Gamma_S N$ be the largest submodule of N supported on S. Then Γ_S is left exact and*

$$\Gamma_S\mathbb{I}(i,s)=\begin{cases}\mathbb{I}_S(i,s), & (i,s)\in S,\\ 0, & (i,s)\notin S,\end{cases}$$

where $\mathbb{I}_S(i,s)$ is the injective of the finite support quotient. Restriction induces a surjection

$$\mathrm{Hom}_B^0(I_-,I_+)\longrightarrow\mathrm{Hom}_{B_S}(\Gamma_S I_-,\Gamma_S I_+). \tag{4.19}$$

The kernel of a generic map on the right is $\Gamma_S K$.

Proof. The inclusion of modules supported on S into ambient modules is exact, and Γ_S is its right adjoint. Thus it is left exact and sends injectives to injectives. The dual evaluation formula identifies the stated injectives when $(i,s)\in S$ and gives zero otherwise. For a map on the right of (4.19), compose with $\Gamma_S I_+\hookrightarrow I_+$ and extend from $\Gamma_S I_-\subset I_-$ using injectivity of I_+. Restricting the extension gives the original map, proving surjectivity.

Restriction of automorphisms is an automorphism, so the image of the ambient open orbit lies in one finite-window orbit. A surjective linear map is open; this image contains an open subset of the finite Hom space. Its orbit is therefore open. Left exactness gives $\ker(\Gamma_S f)=\Gamma_S\ker f=\Gamma_S K$, as asserted. □

One may choose an enlarged Hernandez–Leclerc strip containing the finite support of K and the socle vertices of both terms in (4.16). In that strip $\Gamma_S K=K$, the finite mutable degree is g^{full}, and the generic kernel is exactly K. The same strip works for every direct sum $K^{\oplus r}$, since its support and its injective socle vertices do not change.

5. Framing potentials and positive critical characters

5.1. **The potential of a graded operator.** Fix a rigid $K=(P,U)$ as above. Since Π is self-injective, P is also injective. Let $R=\bigoplus_i \mathbb{C}e_i$, and put $W=\mathrm{soc}_\Pi P$. A choice of its graded injective decomposition identifies

$$P=\mathrm{Hom}_R(\Pi,W). \tag{5.1}$$

The left Π-action on the right is $(a\varphi)(p)=\varphi(pa)$. Write $\mathrm{ev}: P\to W$ for evaluation on the vertex idempotents. Choose homogeneous path representatives for a basis of Π. The coinduction adjunction expresses $\mathrm{ev}\,U$ uniquely in the form

$$\mathrm{ev}\,U=\sum_p C_p\,\mathrm{ev}\,p. \tag{5.2}$$

Here C_p is a map between the appropriate graded vertex spaces of W, of degree $2+\mathrm{length}(p)$. This degree makes both sides of (5.2) homogeneous of degree two.

We use the following framing convention for Nakajima varieties. The doubled-arrow maps α_h have degree -1, the map $a : V \to W$ has degree zero, and $b : W \to V$ has degree -2. The moment equation and stability condition are

$$\mu(\alpha, a, b) = \sum_{h \in Q_1} [\alpha_h, \alpha_{h^*}] + ba = 0, \qquad \text{no nonzero } \alpha\text{-invariant subspace lies in } \ker a. \tag{5.3}$$

The stable moment fiber is smooth and the change-of-basis group of V acts freely. Its quotient is the smooth Nakajima variety. Taking the specified graded fixed component gives $X_e = \mathfrak{M}^\bullet(e, W)$, where $e = (\dim V_{i,s})$, and a proper map

$$\pi_e : X_e \longrightarrow Y(W) = \mathfrak{M}_0^\bullet(W).$$

We use the standard finite stratification of this affine base and the proper-map decomposition from [Nak11, Sections 3.1–3.3].

Define a regular invariant function on the moment fiber by

$$A_U(\alpha, a) = \sum_p C_p a p(\alpha), \qquad f_U(\alpha, a, b) = \operatorname{Tr}_V\big(bA_U(\alpha, a)\big). \tag{5.4}$$

It descends to the affine base. The degree of C_p in (5.2) makes every trace in (5.4) have spectral weight zero, so the function restricts to the graded base. We fix these path representatives throughout; for $U^{\oplus r}$ we use their block-diagonal copies.

The invariant generators of the affine quotient are the matrix entries of the framing paths $ap(\alpha)b$ and traces of doubled-quiver cycles [CY23, Theorem 2.2 and Proposition 3.2]. Every cycle strictly lowers spectral degree, so its trace vanishes identically on the graded locus. Stability makes the combined map

$$V \longrightarrow \bigoplus_p W, \qquad x \longmapsto (ap(\alpha)x)_p \tag{5.5}$$

injective for a finite set of paths. Indeed, the common kernel over all paths is α-invariant and lies in $\ker a$; finite dimensionality then reduces to finitely many paths. On a chart where a maximal minor of (5.5) is invertible, this map has a left inverse. The equations $ap(\alpha)b = 0$ consequently imply $b = 0$ over arbitrary base schemes. The converse follows from the invariant generators and the vanishing cycle traces. Therefore the central fiber is scheme-theoretically the locus $b = 0$.

At $b = 0$, the moment equation is the preprojective relation, and (5.5) becomes the coinduced embedding

$$\iota : V \longrightarrow P, \qquad \iota(x)(p) = a(px). \tag{5.6}$$

It identifies the central fiber with the graded quiver Grassmannian $\mathrm{Gr}_\Pi(e, P)$, including its scheme structure.

Lemma 5.1. *The scheme-theoretic central critical locus of $f_U\pi_e$ is*

$$Z_e := \mathrm{Crit}(f_U\pi_e) \cap \pi_e^{-1}(0) = \mathrm{Gr}_B(e, K). \tag{5.7}$$

Proof. The differential of a spectral-invariant function vanishes on every nonzero-weight tangent space at a fixed point. Thus being critical on the fixed component is equivalent to being critical on the ambient smooth Nakajima variety. On the stable moment fiber the differential of the moment map is surjective, and its dual is injective. The Lagrange multiplier for criticality is therefore unique. At $b = 0$, differentiation of $f_U - \operatorname{Tr}_V(\varepsilon\mu)$ gives

$$[\varepsilon, \alpha_h] = 0 \quad (h \in \overline{Q}_1), \qquad a\varepsilon = A_U(\alpha, a). \tag{5.8}$$

The arrow derivatives of f_U contain b and vanish there. The b-derivative gives the second equation; the a-derivatives vanish. Homogeneity gives $\deg \varepsilon = 2$.

The first equations say that ε is Π-linear. Both $U\iota$ and $\iota\varepsilon$ are then Π-linear maps to the coinduced module. Applying evaluation and using (5.2) shows that their equality is precisely the second equation in (5.8). Thus criticality is equivalent to U-stability of the submodule in (5.6). All these equations and the left inverses of (5.5) hold in families. They identify the functors of flat graded subbundles with the prescribed dimensions, which proves the equality of schemes. □

5.2. **Isolation of the central critical component.** For $H \in \mathrm{End}^0_\Pi(P)$ use the same construction to define f_H. Its coefficient at a path of length d has degree d, so f_H has spectral weight -2. The ambient symplectic form also has weight -2. Its Hamiltonian vector field X_H therefore has weight zero and is tangent to every graded fixed component. Fix the global Poisson sign so that the following identity has the displayed sign.

Lemma 5.2. *Let $\mathfrak{m}$ be the maximal ideal of the origin in $Y(W)$. On X_e,*

$$X_H(f_U) = f_{HU-UH} \pmod{\pi_e^*\mathfrak{m}^2}. \tag{5.9}$$

The classes of f_T, for $T \in \mathrm{Hom}^2_\Pi(P,P)$, span $\mathfrak{m}/\mathfrak{m}^2$.

Proof. For a coefficient C and a path p, put $F(C,p) = \mathrm{Tr}_W(Cap(\alpha)b)$. Framing contractions in the canonical Poisson bracket give

$$\{F(C,p), F(D,q)\}_{a,b} = F(CD, qp) - F(DC, pq). \tag{5.10}$$

For example, the derivatives with respect to a and b are $p(\alpha)bC$ and $Cap(\alpha)$, respectively. Their two products, after cyclic permutation of the trace, are the two terms on the right. Under coinduction, composition of $C\,\mathrm{ev}\,p$ and $D\,\mathrm{ev}\,q$ has coefficient $CD\,\mathrm{ev}(qp)$. Summing (5.10) therefore gives the commutator term.

An arrow contraction can occur when $p = p_2hp_1$ and $q = q_2h^*q_1$. Its trace, up to its orientation sign, is

$$\mathrm{Tr}_W\big((Cap_2q_1b)(Daq_2p_1b)\big).$$

Writing the trace as a sum of products of matrix entries exhibits two framing-path factors. Thus it belongs to the square of the ideal of framing-path coordinates. Reducing a word modulo a preprojective relation has the same effect: on (5.3), the relation is $-ba$, and its insertion introduces a second framing pair. After restriction to the graded locus, a factor of nonzero spectral weight vanishes; a product of two weight-zero factors remains in $\mathfrak{m}^2$. This proves (5.9) for the fixed path representatives.

The invariant generator theorem quoted above gives framing-path entries and cycle traces. The latter vanish. Modulo $\mathfrak{m}^2$, the replacement of a preprojective relation by $-ba$ shows that every framing path may be reduced in Π. Its weight-zero matrix entries are exactly the linear functions obtained by varying the coefficients of degree $2 + \mathrm{length}(p)$ in (5.2). Coinduction identifies those coefficients with $\mathrm{Hom}^2_\Pi(P,P)$. Their classes span $\mathfrak{m}/\mathfrak{m}^2$. □

Proposition 5.3. *The projective scheme Z_e of (5.7) is open and closed in* $\mathrm{Crit}(f_U\pi_e)$.

Proof. Fix $x \in Z_e$ and let J be the ideal generated by $\pi_e^*\mathfrak{m}$ in the local ring $\mathcal{O}_{\mathrm{Crit}(f_U\pi_e),x}$. Choose finitely many f_T spanning $\mathfrak{m}/\mathfrak{m}^2$. Nakayama's lemma at the base origin says that these generate its local maximal ideal. Rigidity and (4.5) give, for each chosen T, an H with $HU - UH = T$. Every derivative $X_H(f_U)$ vanishes on the critical scheme. By Lemma 5.2, each generator of J lies in J^2, whence $J = J^2$. The ideal is finitely generated and contained in the local maximal ideal. Nakayama's lemma gives $J = 0$.

Thus the critical scheme is contained in the central fiber near each point of Z_e. Lemma 5.1 identifies that intersection with Z_e scheme-theoretically. It is therefore open in the critical

scheme. It is closed and projective as a quiver Grassmannian of the finite-dimensional module K. □

5.3. **An integer grading and a weight-one action.** Give the cover algebra the second grading

$$\deg_{\mathrm{cut}} u = 1, \qquad \deg_{\mathrm{cut}} \alpha_h = 0.$$

The relations are homogeneous. Choose a finite support quotient containing the support of K. Any extension of K by itself is zero at every vertex where both endpoint spaces are zero; hence its graded Ext^1 is the same in this finite quotient and in the full cover. Rigidity gives an open orbit in its representation variety.

Lemma 5.4. *The module K admits a $\mathbb{Z}$-grading for this cut, compatible with its spectral grading.*

Proof. Twisting the action of a homogeneous element a by $a \mapsto t^{\deg_{\mathrm{cut}} a} a$ gives an algebraic family of modules through K at $t = 1$. The inverse image of the open orbit contains a nonempty Zariski-open subset of $\mathbb{C}^*$. Choose in it a number α which is not a root of unity. An isomorphism to the twist, with its direction chosen appropriately, gives an invertible vertex-preserving linear map ψ such that

$$\psi(ax) = \alpha^{\deg_{\mathrm{cut}} a} a\psi(x).$$

It follows that a homogeneous a of cut degree d sends the generalized λ-eigenspace of ψ to the generalized $\alpha^d\lambda$-eigenspace. Partition the nonzero eigenvalues into their $\alpha^{\mathbb{Z}}$-orbits, choose one representative λ_0 in each orbit, and assign degree j to the generalized eigenspace with eigenvalue $\lambda_0\alpha^j$. Infinite order makes j unique. This is the required integer grading. Since the vertex idempotents have cut degree zero, each piece retains its spectral vertex decomposition. This is the open-orbit gradability argument of [AO10, Theorem 4.2 and Corollary 4.5] in the present setting. □

Every Π-arrow preserves the cut grading and U raises it by one. Each cut piece of P is a Π-direct summand, hence is injective. We may choose (5.1) compatibly with this grading on W. All C_p then have cut degree one. If $h(t)$ is the resulting cocharacter of the framing group, then $h(t)C_p h(t)^{-1} = tC_p$. The framing action sends (a, b) to $(h(t)a, bh(t)^{-1})$, so

$$f_U(h(t)\cdot x) = \sum_p \mathrm{Tr}_W\big(h(t)^{-1}C_p h(t) a p(\alpha) b\big) = t^{-1} f_U(x).$$

The inverse cocharacter therefore gives an algebraic action commuting with the spectral action and satisfying

$$f_U(t\cdot x) = t f_U(x). \tag{5.11}$$

5.4. **Positive integer multiplicities.** For $r \geq 1$, write $K_r = K^{\oplus r}$, $W_r = W^{\oplus r}$ and $U_r = U^{\oplus r}$. Direct sums preserve rigidity. Put

$$X_{r,e} = \mathfrak{M}^\bullet(e, W_r), \qquad D_{r,e} = \dim_{\mathbb{C}} X_{r,e}, \qquad \pi_{r,e} : X_{r,e} \to Y(W_r),$$
$$F_{r,e} = R\pi_{r,e*}\mathbb{Q}_{X_{r,e}}[D_{r,e}], \qquad E_{r,e} = \phi^p_{f_{U_r}} F_{r,e}, \qquad \phi^p_f = \phi_f[-1].$$

Empty varieties contribute zero. By proper base change and Proposition 5.3,

$$H_{r,e} := (E_{r,e})_0 = R\Gamma\big(\mathrm{Gr}_B(e, K_r), \phi^p_{f_{U_r}\pi_{r,e}}\mathbb{Q}_{X_{r,e}}[D_{r,e}]\big). \tag{5.12}$$

The decomposition theorem, in Nakajima's normalization, gives

$$F_{r,e} \simeq \bigoplus_S \mathrm{IC}_S \otimes M_{r,e,S}. \tag{5.13}$$

The IC_S are the simple perverse sheaves on the relevant affine strata, extended to their closures; $M_{r,e,S}$ are bounded complexes of finite-dimensional vector spaces. Record their cohomology by

$$a_{r,e,S}(t) = \sum_k t^{-k} \dim H^k(M_{r,e,S}).$$

These are Nakajima's simple-character coefficients [Nak11, equation (3.18) and Theorems 3.19 and 3.22]. In the quantum parameter convention of Section 3, their coefficients are $a_{r,e,S}(v^2)$; the exact multiplication adjustment is Lemma 3.3 and is applied explicitly below.

For a dominant highest monomial m, denote its normalized quantum simple class by $L_v(m)$ and write $\chi_{q,v}$ for the quantum character homomorphism in the convention of Section 3. At $v = 1$ these specialize to $[L(m)]$ and its ordinary q-character.

Proposition 5.5. *There is a finite class C_r in the actual quantum Grothendieck ring whose character is*

$$\chi_{q,v}(C_r) = \sum_e P_{r,e}(v^2)\mathbf{m}(W_r, e), \qquad P_{r,e}(t) = \sum_k t^{-k} \dim H^k(H_{r,e}). \tag{5.14}$$

Here $\mathbf{m}(W_r, e)$ is the normalized torus monomial with commutative image $m^r \prod_{i,s} A_{i,s-1}^{-e_{i,s}}$; the translation from the framing convention is verified in (6.13). The class has the expansion

$$C_r = L_v(m^r) + \sum_{m' < m^r} \mu_{r,m'} L_v(m'), \qquad \mu_{r,m'} \in \mathbb{N}. \tag{5.15}$$

The inequality $m' < m^r$ means that $m' = m^r \prod_{i,s} A_{i,s-1}^{-e_{i,s}}$ for a nonzero finitely supported vector $e \geq 0$. Its character support is contained in the dimension vectors of submodules of K_r.

Proof. For each e, the complement of the central projective critical component in the full critical locus is closed by Proposition 5.3. Its proper image under $\pi_{r,e}$ is closed and avoids zero. Thus $E_{r,e}$ is supported only at zero in a neighborhood of zero. Apply ϕ^p to the direct sum (5.13). The support of a direct summand is contained in the support of the direct sum. Therefore every $\phi^p_{f_{U_r}} \mathrm{IC}_S$ occurring there is supported at zero near zero. Perverse-normalized vanishing cycles are exact for the perverse t-structure. A perverse sheaf supported at one point is an ordinary vector space in degree zero. Consequently

$$V_{r,S} := (\phi^p_{f_{U_r}} \mathrm{IC}_S)_0 \quad \text{is in degree zero,} \qquad \mu_{r,S} := \dim V_{r,S} \in \mathbb{N}.$$

Taking stalks yields

$$H_{r,e} \simeq \bigoplus_S V_{r,S} \otimes M_{r,e,S}, \qquad P_{r,e}(t) = \sum_S \mu_{r,S} a_{r,e,S}(t). \tag{5.16}$$

This also applies to every relevant S: each appears in at least one standard complex. For the zero stratum, $\mathrm{IC}_{\{0\}} = \mathbb{Q}_{\{0\}}$ and $f_{U_r}|_{\{0\}} = 0$, so its normalized vanishing cycles are $\mathbb{Q}_{\{0\}}$; hence $\mu_{r,0} = 1$.

The finite IC/simple dictionary of [Nak11, Theorem 3.17, equation (3.18), and Theorem 3.19] now turns (5.16) into a positive linear combination of actual simple classes, with highest class $L_v(m^r)$ and coefficient one. In this identification we use the spectral translation described below in (6.13). Formula (5.12) makes $H_{r,e} = 0$ unless $\mathrm{Gr}_B(e, K_r)$ is nonempty; there are only finitely many such e. Thus the character and its positive simple expansion are finite, and have the asserted support. The coefficients $\mu_{r,m'}$ are the dimensions just computed, and contain no powers of v. □

6. Tensor powers and the exact character

6.1. **Derived hyperbolic restriction.** For positive integers a, b, split $W_{a+b} = W_a \oplus W_b$ and let $\lambda(s)$ act as $1 \oplus s$. The fixed path coefficients are block diagonal, so $f_{U_{a+b}}$ is λ-invariant and its restriction to the fixed base is $f_{U_a} \boxplus f_{U_b}$. Let

$$Y^+ \xrightarrow{\iota} Y(W_{a+b}), \qquad \kappa : Y^+ \longrightarrow Y(W_{a+b})^\lambda$$

be the inclusion of the attracting locus and its limit map. The direct-sum map

$$j : Y(W_a) \times Y(W_b) \longrightarrow Y(W_{a+b})^\lambda \tag{6.1}$$

is finite and bijective [VV01, Lemma 3.5.1]. Over $\mathbb{C}$ it is a proper homeomorphism, so its pushforward identifies the constructible derived categories. We retain j_* in the formulas.

Lemma 6.1. *Let Y have a $\mathbb{C}^*$-action fixing 0, and let E be an equivariant constructible complex whose support near 0 is contained in $\{0\}$. Then*

$$(R\kappa_! \iota^* E)_0 \simeq E_0.$$

Proof. If $y \neq 0$ in $\kappa^{-1}(0)$ had a nonzero stalk, equivariance would give a nonzero stalk at every $\lambda(s)y$. As s tends to zero, these points enter any neighborhood of 0, contradicting the support condition. Hence the restriction to $\kappa^{-1}(0)$ is supported at its single point 0. Compactly supported base change then identifies its cohomology with E_0. □

For $E_{a+b,e}$, equivariance follows from that of the proper map, the constant complex, and the invariant function. The complement of the central critical component has a closed invariant proper image avoiding zero, as in Proposition 5.5. Its complement is an invariant neighborhood with the required support property. Lemma 6.1 thus gives

$$(R\kappa_! \iota^* E_{a+b,e})_{(0,0)} \simeq H_{a+b,e}. \tag{6.2}$$

We state the derived restriction formula with its ranks. Use the symmetric framing $W^{\mathrm{sym}} = W\langle -1\rangle$, in which both framing maps have degree -1, and write $\omega_j = \dim W_j^{\mathrm{sym}}$. Use τ and C from Lemma 3.3, with the pairing $\langle x, y\rangle = \sum_{i,s} x_{i,s} y_{i,s}$. Let $\mathrm{Adj} = (\delta_{i\sim j})$ be the adjacency matrix. Then $\mathsf{C}^* = \mathsf{C}$ and $\tau^* = \tau^{-1}$. For $e = e_1 + e_2$ put

$$\begin{aligned} d_{12} &= \langle e_1, \tau^{-1}(\omega_2 - \mathsf{C}e_2)\rangle + \langle e_2, \tau\omega_1\rangle, \\ d_{21} &= \langle e_2, \tau^{-1}(\omega_1 - \mathsf{C}e_1)\rangle + \langle e_1, \tau\omega_2\rangle. \end{aligned} \tag{6.3}$$

For the action $1 \oplus s$ on the two framing blocks, the positive normal directions are maps from the first block to the second. The degree-zero deformation complex gives their rank as

$$d_{12} = \langle e_1, \tau^{-1}\omega_2\rangle + \langle e_2, \tau\omega_1\rangle + \langle e_1, \mathrm{Adj}\, \tau^{-1} e_2\rangle - \langle e_1, e_2\rangle - \langle e_1, \tau^{-2} e_2\rangle.$$

This expands the first formula in (6.3); interchanging the two blocks gives d_{21}. These are the attracting and repelling normal ranks. The exact standard-complex formula is

$$R\kappa_! \iota^* F_{a+b,e} \simeq \bigoplus_{e_1+e_2=e} j_*(F_{a,e_1} \boxtimes F_{b,e_2})[d_{21} - d_{12}]. \tag{6.4}$$

This is [VV01, Lemma 4.1(a)], with the rank identities of its Lemma 3.5.2; it is restated in [Nak11, Section 3.5]. In these conventions the total dimension difference is $d_{12} + d_{21}$. Integration along the attracting affine bundle of dimension d_{12} shifts the constant complex by $-2d_{12}$. After normalizing the two factors by their dimensions, the total shift is therefore

$$(d_{12} + d_{21}) - 2d_{12} = d_{21} - d_{12},$$

as in (6.4). The quoted theorem gives a direct-sum isomorphism of complexes.

Proposition 6.2. *There is an isomorphism of bounded complexes of vector spaces*

$$H_{a+b,e} \simeq \bigoplus_{e_1+e_2=e} (H_{a,e_1} \otimes H_{b,e_2})[d_{21} - d_{12}]. \tag{6.5}$$

Consequently

$$P_{a+b,e}(t) = \sum_{e_1+e_2=e} t^{d_{21}-d_{12}} P_{a,e_1}(t) P_{b,e_2}(t). \tag{6.6}$$

Proof. Hyperbolic restriction commutes with normalized vanishing cycles for an invariant function and a monodromic constructible complex [Des25, Theorem 3.7 and the specialization following its proof]. Here the function is invariant, the complexes are equivariant, and the affine-base action is linearizable by a finite set of homogeneous coordinate generators. Thus

$$R\kappa_! \iota^* \phi^p_{f_{U_{a+b}}} F_{a+b,e} \simeq \phi^p_{f_{U_a} \boxplus f_{U_b}} R\kappa_! \iota^* F_{a+b,e}.$$

Apply this to (6.4). Proper pushforward through j commutes with vanishing cycles. The derived Thom–Sebastiani theorem for arbitrary constructible complexes gives

$$\left(\phi^p_{f_{U_a} \boxplus f_{U_b}} (F_{a,e_1} \boxtimes F_{b,e_2})\right)_{(0,0)} \simeq H_{a,e_1} \otimes H_{b,e_2}$$

[Mas99, Section 0, theorem]. Its normalization is $\phi^p = \phi[-1]$, so this equation has no additional shift. It applies on the possibly singular affine bases. Taking the stalk and using (6.2) proves (6.5). Finally, a shift $[d]$ multiplies $\sum_k t^{-k} \dim H^k$ by t^d, which gives (6.6). □

6.2. **The quantum multiplication factor.** Take the formal inverse

$$\mathsf{C}^{-1} = \tau^{-1}(1 - \tau^{-1} \operatorname{Adj} + \tau^{-2})^{-1},$$

expanded in nonnegative powers of τ^{-1} inside the parentheses, from the lower spectral boundary toward higher degrees. For finitely supported x, y define

$$A(x,y) = \langle \mathsf{C}^{-1} x, \tau^{-1} y \rangle, \qquad \varepsilon(x,y) = A(x,y) - A(y,x). \tag{6.7}$$

The pairing is finite because its second argument has finite support. Put $\nu_j = \omega_j - \mathsf{C} e_j$. Expanding all terms and using the adjoint identities gives

$$\begin{aligned} A(\nu_1,\nu_2) &= A(\omega_1,\omega_2) - \langle \mathsf{C}^{-1}\omega_1, \tau^{-1}\mathsf{C}e_2\rangle - \langle e_1, \tau^{-1}\omega_2\rangle + \langle e_1, \tau^{-1}\mathsf{C}e_2\rangle \\ &= A(\omega_1,\omega_2) - \langle e_2, \tau\omega_1\rangle - \langle e_1, \tau^{-1}(\omega_2 - \mathsf{C}e_2)\rangle \\ &= A(\omega_1,\omega_2) - d_{12}. \end{aligned} \tag{6.8}$$

Subtracting the equation with 1 and 2 interchanged yields

$$\varepsilon(\nu_1,\nu_2) = \varepsilon(\omega_1,\omega_2) + d_{21} - d_{12}. \tag{6.9}$$

In Nakajima's cohomological convention, multiplication of normalized monomials of exponent ν_1, ν_2 is multiplied by $t^{\varepsilon(\nu_1,\nu_2)}$. In the convention of Section 3, let $\mathcal{N}$ be the skew form in the Y-monomial torus. Lemma 3.3 gives

$$t = t_{\mathrm{DS}} = v^2, \qquad \beta = 2\varepsilon - \mathcal{N}, \qquad \beta(\mathsf{C}e, x) = \beta(x, \mathsf{C}e) = 0. \tag{6.10}$$

Here t records geometric cohomological shift and t_{DS} is the deformation parameter of [DS26]; the normalized DS product has factor $t_{\mathrm{DS}}^{\mathcal{N}/2} = v^{\mathcal{N}}$. Thus its monomial multiplication differs by the explicit central factor $v^{-\beta(\nu_1,\nu_2)}$. For the present framing split, $\omega_1 = a\omega$ and $\omega_2 = b\omega$, so

$$\beta(\nu_1,\nu_2) = \beta(a\omega, b\omega) = ab\beta(\omega,\omega) = 0, \qquad \varepsilon(a\omega, b\omega) = 0.$$

Equations (6.9)–(6.10) therefore give the exact exponent

$$\mathcal{N}(\nu_1,\nu_2) = 2(d_{21} - d_{12}). \tag{6.11}$$

Substituting $t = v^2$ in (6.6) and comparing each normalized monomial now proves

$$C_a C_b = C_{a+b}, \qquad C_r = C_1^r. \tag{6.12}$$

The comparison uses the injective quantum character homomorphism of [Nak11, Theorems 3.19 and 3.22], with precisely the parameter adjustment in (6.10).

The spectral convention in these formulas corresponds to the original highest monomial as follows. A summand $J_i\langle r+2\rangle$ of P has asymmetric framing at degree $r+2$ and symmetric framing at degree $r+1$. Its native highest variable is $Y_{i,r+1}$. A vertex (i,s) of V contributes the native root $A_{i,s}^{-1}$. Translating every spectral index by -1 gives

$$\left(\prod_{i,r} Y_{i,r+1}^{w_{i,r}}\right)\prod_{i,s} A_{i,s}^{-e_{i,s}} \longmapsto m\prod_{i,s} A_{i,s-1}^{-e_{i,s}}. \tag{6.13}$$

The normalized torus representatives obey the same dictionary, and

$$A_{i,s-1} = Y_{i,s-2}Y_{i,s}\prod_{j\sim i} Y_{j,s-1}^{-1}.$$

This translation preserves all pairings above, tensor products, and simplicity. We use the right-hand monomials from now on.

6.3. **The scalar-positive ray and real simplicity.**

Theorem 6.3. *Let $K = (P_w, U)$ be a finite-dimensional graded B-module, with P_w as in (4.10), and suppose $\operatorname{Ext}_B^{1,0}(K,K) = 0$. Then $L(m)$ is real and*

$$C_r = L_v(m^r), \qquad L_v(m)^r = L_v(m^r) \quad (r \geq 1). \tag{6.14}$$

Moreover, for every e,

$$P_{r,e}(t) = a_{r,e,0}(t). \tag{6.15}$$

Proof. Choose one finite Hernandez–Leclerc strip containing the support of K, the injective socle vertices in (4.16), and all Y-indices in (6.13) for $e \leq \dim K$. The same set of indices contains the monomial support for K_r for every r. Lemma 4.4 identifies K_r there with its exact generic kernel, and its degree is rg^{full}.

We verify the inputs of Theorem 3.6 in this fixed seed. Proposition 5.5 gives a finite expansion of each C_r in actual simple classes with nonnegative integer coefficients and leading coefficient one. Formula (6.13) makes it pointed at degree rg^{full}. Every leading monomial of a simple term with positive multiplicity occurs in this positive character. It therefore has degree $rg^{\mathrm{full}} + Pf$ for a nonnegative correction vector f supported in the chosen mutable strip, in the notation of (3.8). The newly added frozen row lies below the support of w, so its degree coordinates are zero. Thus all the terms belong to the required actual simple basis in this one finite seed. Equation (6.12) gives $C_r = C_1^r$ in the quantum ring of that seed, with the exact parameter convention verified in (6.11). Finally, if a nonzero e occurs in the character of C_1, then (5.12) implies $\mathrm{Gr}_B(e,K) \neq \varnothing$. Proposition 4.3 gives $g^{\mathrm{full}} \cdot e < 0$. These are all the hypotheses of the scalar-positive ray theorem. It follows that $C_r = L_v(m^r)$ and that every power of $L_v(m)$ is the displayed simple class.

Specialization at $v = 1$ is the ordinary Grothendieck ring, whose simple classes form a basis and whose tensor-product coefficients are composition multiplicities. The equality $[L(m)^{\otimes r}] = [L(m^r)]$ therefore gives an object of length one. In particular $L(m)$ is real. Comparing the monomial coefficients of $C_r = L_v(m^r)$ gives $P_{r,e}(v^2) = a_{r,e,0}(v^2)$. The substitution $\mathbb{Z}[t,t^{-1}] \to \mathbb{Z}[v,v^{-1}]$, $t \mapsto v^2$, is injective; this proves (6.15). □

6.4. **Euler characteristic and the full character.**

Lemma 6.4. *Let X be smooth of pure complex dimension D, let $f : X \to \mathbb{C}$ be regular, and suppose that a $\mathbb{C}^*$-action satisfies $f(t \cdot x) = tf(x)$. If $Z \subset f^{-1}(0)$ is a proper invariant union of critical components, then*

$$\chi(Z, \phi_f[-1]\mathbb{Q}_X[D]) = (-1)^D \chi(Z). \tag{6.16}$$

Proof. At a fixed critical point x, linearize the compact circle action holomorphically near x and choose an invariant sufficiently small Milnor ball. The weight-one action transports the Milnor fiber at δ to that at $e^{i\theta}\delta$. At $\theta = 2\pi$ it is the identity, so its monodromy is the identity. If the germ of f is nonzero, criticality gives $f \in \mathfrak{m}_x^2$. A'Campo's Lefschetz-number theorem gives zero for the Lefschetz number of its monodromy [A'C73, Théorème 1 bis]; see also the precise statement in [GLN23, Theorem 0.1]. Thus its Milnor fiber has Euler characteristic zero. If the germ is identically zero, its nearby fiber is empty and has the same Euler characteristic.

The vanishing-cycle triangle gives at x

$$\chi((\phi_f[-1]\mathbb{Q}_X[D])_x) = (-1)^{D-1}(\chi(\text{Milnor fiber at } x) - 1) = (-1)^D.$$

The stalk Euler function is constructible and constant along the $\mathbb{C}^*$-orbits. Euler integration localizes it to the fixed locus: each nonfixed orbit has Euler characteristic zero, and an algebraic stratification by such orbit fibrations gives the localization identity. The same identity applies to the constant function on Z. Therefore

$$\chi(Z, \phi_f[-1]\mathbb{Q}_X[D]) = (-1)^D \chi(Z^{\mathbb{C}^*}) = (-1)^D \chi(Z),$$

which proves (6.16). □

Define the ordinary quiver Grassmannian polynomial of a finite graded B-module by

$$F_K(y) = \sum_e \chi(\mathrm{Gr}_B(e, K)) \prod_{i,s} y_{i,s}^{e_{i,s}}, \tag{6.17}$$

where χ is the topological Euler characteristic.

Theorem 6.5. *For every rigid (P_w, U) as in Theorem 6.3,*

$$\chi_q L(m) = m \sum_e \chi(\mathrm{Gr}_B(e, K)) \prod_{i,s} A_{i,s-1}^{-e_{i,s}}. \tag{6.18}$$

Proof. Apply Lemma 6.4 to $X_{1,e}$, the action (5.11), and the proper component (5.7). Equation (5.12) gives

$$P_{1,e}(-1) = (-1)^{D_{1,e}} \chi(\mathrm{Gr}_B(e, K)). \tag{6.19}$$

Theorem 6.3 has already identified $P_{1,e}(t) = a_{1,e,0}(t)$. The parity theorem for the zero-stratum coefficient is

$$a_{1,e,0}(t) \in t^{D_{1,e}} \mathbb{Z}_{\geq 0}[t^{-2}] \qquad \text{[Nak11, equation (3.18) and Remark 3.20].} \tag{6.20}$$

It implies $a_{1,e,0}(-1) = (-1)^{D_{1,e}} a_{1,e,0}(1)$. Combining this with (6.19) gives

$$\chi(\mathrm{Gr}_B(e, K)) = a_{1,e,0}(1).$$

The right-hand side is the ordinary simple-character coefficient. The monomial identification is (6.13). There is no collision between distinct finite dimension vectors: the exponent map is $-\mathsf{C}$, whose formal inverse was displayed above. This proves (6.18) coefficient by coefficient. □

6.5. **The original truncation.** Return to the original highest-monomial window

$$\{(i,r) : \xi_i - 2\ell \le r \le \xi_i, \quad r \equiv \xi_i \pmod 2\}$$

and its mutable vertex set

$$W_{\ell,\xi} = \{(i,s) : \xi_i - 2\ell + 2 \le s \le \xi_i, \quad s \equiv \xi_i \pmod 2\}. \tag{6.21}$$

Let $K(m)$ denote the finite generic injective kernel prescribed by the mutable degree vector in [DS26]. It is distinct in notation from the full module K.

Proposition 6.6. *For a rigid full kernel K of highest monomial m,*

$$K(m) \simeq \Gamma_{W_{\ell,\xi}} K, \qquad \chi_q^{\le \xi} L(m) = m\, F_{K(m)}(y_{i,s} = A_{i,s-1}^{-1}). \tag{6.22}$$

The truncation here is the original deletion of monomials containing a Y-variable outside the highest-monomial window.

Proof. Lemma 4.4 restricts the reduced cofree presentation to the positive and negative parts of the original mutable degree vector. Its generic kernel is $\Gamma_{W_{\ell,\xi}} K$, proving the first assertion. A submodule of K supported on $W_{\ell,\xi}$ lies in this largest supported submodule. Consequently their quiver Grassmannians, for these dimension vectors, agree as varieties.

We check the monomial deletion explicitly. A degree-t component of J_j at color i is dual to paths of length $t \ge \operatorname{dist}(i,j)$. If $w_{j,r} > 0$, then $r \ge \xi_j - 2\ell$. Every degree s at color i in $J_j\langle r+2\rangle$ therefore satisfies

$$s \ge \xi_j - 2\ell + 2 + \operatorname{dist}(i,j) \ge \xi_i - 2\ell + 2.$$

Thus K has no support below the lower boundary of (6.21).

Let e be the dimension vector of a submodule. Suppose that some $e_{i,s} > 0$ has $s > \xi_i$. Among these coordinates choose one maximizing the excess $E = s - \xi_i$, and among the coordinates with that excess choose ξ_i maximal. For a neighbor j, a positive $e_{j,s+1}$ would have excess $E+2$ if $\xi_j = \xi_i - 1$. If $\xi_j = \xi_i + 1$, it would have the same excess E and larger height. Both contradict the choice. Hence all $e_{j,s+1}$ vanish. The exponent of the out-of-window variable $Y_{i,s}$ in $m\prod_{j,t} A_{j,t-1}^{-e_{j,t}}$ is then

$$-e_{i,s} - e_{i,s+2} + \sum_{j\sim i} e_{j,s+1} < 0.$$

This monomial is deleted by the original truncation.

Conversely, suppose that e is supported on $W_{\ell,\xi}$. For (i,s) in that set, the two same-color variables of $A_{i,s-1}$ have indices $s-2, s$ in the highest-monomial window. For a neighbor with $\xi_j = \xi_i - 1$, its index $s-1$ lies between $\xi_j - 2\ell$ and ξ_j; for a neighbor with $\xi_j = \xi_i + 1$, the same inequalities follow from $\xi_i - 2\ell + 2 \le s \le \xi_i$. Thus all neighboring variables also lie in that window. The monomial is retained. Applying this criterion to (6.18) proves (6.22). □

Corollary 6.7. *The frozen-free real simple objects are in bijection with all rigid objects of the original finite category, including their decoration. For every real simple $L(m)$ in the original category, let M_m be the unique rigid object with mutable degree $g(m)$. Then*

$$\chi_q^{\le \xi} L(m) = m\, F_{K(m)}(y_{i,s} = A_{i,s-1}^{-1}), \qquad K(m) \simeq \operatorname{Hom}_{\mathcal{C}_A}(T, M_m). \tag{6.23}$$

The corresponding full rigid operator has underlying module P_w and satisfies (6.18).

Proof. Given a rigid categorical object X, take its normalized lift by Proposition 4.2. Theorem 6.3 produces the real simple $L(m)$ at exactly that column-minimum-zero dominant vector. Conversely, Theorem 3.13 gives for a real simple the unique rigid object with its mutable degree vector and identifies its Hom module with the prescribed finite generic kernel. The two

constructions are inverse: on the categorical side uniqueness is the rigidity/index statement in that theorem; on the dominant side the mutable differences and column minima determine w uniquely by (4.12). This proves the frozen-free bijection.

For an arbitrary real $L(m)$, put $d_i = \min_k w_{i,k}$ and remove d_i full columns. By Lemma 3.14, the result is an actual simple $L(m_0)$ and $L(m)$ is its tensor product with the corresponding frozen simples. Multiplication by each frozen simple permutes the actual simple basis up to its specified quantum normalization. At $v = 1$, this implies that a composition series for $L(m_0)^{\otimes 2}$ remains a composition series after multiplying by the squared frozen factor, with distinct simple classes. Since $L(m)^{\otimes 2}$ is simple, $L(m_0)^{\otimes 2}$ has length one. Thus $L(m_0)$ is real.

Lift its rigid categorical object to K_0, and add $\bigoplus_i F_{i^*}^{d_i}$. These are projective-injective in the same exact category, so (4.1) shows that their direct sum K with K_0 remains rigid. Its underlying projective type is exactly P_w. Theorems 6.3 and 6.5, followed by Proposition 6.6, give the asserted formulas for $L(m)$. The finite generic kernel and the Hom module are the same ones identified in Theorem 3.13. □

7. Tensor factorization and indecomposability

Lemma 7.1. *Let K be a rigid full kernel and let M be a finite-dimensional graded B-module. Then*

$$\max_{N \subset M} g_K^{\mathrm{full}} \cdot \dim N = \dim \mathrm{Ext}_B^{1,0}(M, K). \tag{7.1}$$

Proof. Put $d = \dim \mathrm{Hom}_B^0(M, K)$ and form the evaluation map $M \to K^{\oplus d}$ using a basis of this Hom space. Let U be its kernel and V its image. From $0 \to V \to K^{\oplus d} \to Q \to 0$, rigidity and injective dimension at most one give $\mathrm{Ext}_B^{1,0}(V, K) = 0$. Applying $\mathrm{Hom}_B^0(-, K)$ to $0 \to U \to M \to V \to 0$ now shows that $\mathrm{Hom}_B^0(M, K) \to \mathrm{Hom}_B^0(U, K)$ is surjective. Every map from M to K vanishes on U by its definition, so $\mathrm{Hom}_B^0(U, K) = 0$. The same long exact sequence, using $\mathrm{Ext}_B^{2,0}(V, K) = 0$, gives $\mathrm{Ext}_B^{1,0}(M, K) \simeq \mathrm{Ext}_B^{1,0}(U, K)$. Equation (4.17) therefore makes U attain the right-hand side of (7.1).

For any $N \subset M$, the exact sequence with quotient M/N and injective dimension one make $\mathrm{Ext}_B^{1,0}(M, K) \to \mathrm{Ext}_B^{1,0}(N, K)$ surjective. Again by (4.17),

$$g_K^{\mathrm{full}} \cdot \dim N \le \dim \mathrm{Ext}_B^{1,0}(N, K) \le \dim \mathrm{Ext}_B^{1,0}(M, K).$$

This is the required upper bound. □

Lemma 7.2. *If e is a vertex of the convex hull of the dimension vectors of submodules of M, then M has exactly one submodule of dimension e, and $\chi(\mathrm{Gr}_B(e, M)) = 1$.*

Proof. For two such submodules N, N', the equality $2e = \dim(N \cap N') + \dim(N + N')$ expresses e as the midpoint of two points of that convex hull. Extremality makes both vectors equal to e. Thus $N = N \cap N' = N'$. The Grassmannian has one underlying point and Euler characteristic one. □

Proposition 7.3. *If $L(m_1)$ and $L(m_2)$ are real and their tensor product is simple, their full rigid kernels satisfy*

$$\mathrm{Ext}_B^{1,0}(K_1, K_2) = \mathrm{Ext}_B^{1,0}(K_2, K_1) = 0.$$

The full kernel of their tensor product is $K_1 \oplus K_2$.

Proof. The pairwise-simple tensor theorem [Her10, Theorem 1.1] implies that every mixed product $L(m_1)^{\otimes a} \otimes L(m_2)^{\otimes b}$ is simple and real: all pairwise factors are simple by realness and the given simple product, and the same test applies to its square. Its highest monomial is $m_1^a m_2^b$.

Suppose that the maximum of $g_2^{\mathrm{full}} \cdot \dim N$ for $N \subset K_1$ is positive. A linear function on the finite submodule polytope attains its maximum at a vertex e. Lemma 7.2 and (6.18) give coefficient one for the monomial of exponent e in $L(m_1)$. Taking the highest monomial from $L(m_2)^{\otimes b}$ gives a positive coefficient at the same root exponent e in the simple product of highest monomial $m_1 m_2^b$. Corollary 6.7 identifies this coefficient with a Grassmannian coefficient of its full rigid kernel. Since it is nonzero, that Grassmannian is nonempty. Proposition 4.3 consequently gives

$$(g_1^{\mathrm{full}} + b g_2^{\mathrm{full}}) \cdot e < 0 \qquad (b \geq 1).$$

The positive coefficient of b contradicts this inequality for large b. The maximum is therefore zero, since the zero submodule attains zero. Lemma 7.1 gives $\mathrm{Ext}_B^{1,0}(K_1, K_2) = 0$. Reversing the two modules proves the other vanishing.

Their direct sum is rigid and has underlying projective type $P_{w_1+w_2}$. The full kernel of the real tensor product has the same type and is rigid by Corollary 6.7. Both operators lie in the open orbit of the same irreducible affine operator space, so they are isomorphic. □

Lemma 7.4. *For finite-dimensional graded B-modules, $F_{K_1 \oplus K_2} = F_{K_1} F_{K_2}$.*

Proof. For a submodule $N \subset K_1 \oplus K_2$, record $N_1 = N \cap K_1$ and its image $N_2 \subset K_2$. Once these are fixed, N/N_1 is the graph of a unique B-map $N_2 \to K_1/N_1$. Conversely, each such map gives a submodule with that intersection and image. The fibers are therefore the affine spaces $\mathrm{Hom}_B^0(N_2, K_1/N_1)$.

On the product of the two quiver Grassmannians, this Hom space is the kernel of a morphism of vector bundles: write a homogeneous linear map between the universal graded bundles and impose the finitely many equations of commuting with the arrows. Stratify by the ranks of those equations. On each stratum its kernel is a vector bundle, so the fibers have Euler characteristic one and Euler characteristic is multiplicative. Summing over strata and over $\dim N_1 + \dim N_2 = e$ gives the coefficient of y^e in the claimed identity. □

Proof of Theorems 1.2 and 1.3. The bijection and exact character formula are Corollary 6.7. It remains to identify prime objects.

Suppose a frozen-free real simple factors as a tensor product of two nontrivial simples $L(m_1)$ and $L(m_2)$. Applying the only-if direction of [Her10, Theorem 1.1] to the four factors in the square of this product shows that each factor is real. Column minima show that both are frozen-free: a zero entry of $w_1 + w_2$ is a zero entry of each nonnegative vector. Proposition 7.3 identifies its full kernel with $K_1 \oplus K_2$. Both summands are nonzero, since their highest monomials are nontrivial, and neither contains a full interval. Proposition 4.2 then gives a nontrivial decomposition of the associated rigid categorical object.

Conversely, decompose a normalized rigid lift as $K = K_1 \oplus K_2$ with two nonzero summands. Each summand is rigid. Its underlying projective multiplicities are nonnegative, and the zero column minima of their sum force zero column minima for each. Theorem 6.3 supplies real simples $L(m_1)$ and $L(m_2)$ with $m = m_1 m_2$. Their full characters and Lemma 7.4 give

$$\chi_q L(m_1)\, \chi_q L(m_2) = \chi_q L(m).$$

Injectivity of the q-character map [FR99] identifies the corresponding Grothendieck classes. A tensor product with the class of one simple has length one, so $L(m_1) \otimes L(m_2) \simeq L(m)$. Both factors are nontrivial because K_1, K_2 are nonzero. Indecomposability of the categorical object is equivalent to that of its normalized lift by Proposition 4.2. Thus primality is equivalent to indecomposability, completing both theorems. □

8. A counterexample for truncated shifted algebras

In this section the Cartan matrix and symmetrizing integers are

$$C = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}, \qquad (d_1, d_2) = (3, 1), \qquad q_i = q^{d_i}. \tag{8.1}$$

Thus vertex 1 is long. A rational ℓ-weight is a pair of nonzero rational functions regular and nonzero at zero. Its normalization divides each component by its value at zero. For $b \in \mathbb{C}^*$, let $z_{i,b}$ have component $1 - bx$ at i and component 1 at the other vertex. We abbreviate z_{i,q^k} to $z_{i,k}$. All products of these symbols below have finite support. Define the normalized truncation roots by

$$a_{1,k} = \frac{z_{1,k} z_{1,k+6}}{z_{2,k+3}}, \qquad a_{2,k} = \frac{z_{2,k} z_{2,k+2}}{z_{1,k-1} z_{1,k+1} z_{1,k+3}}. \tag{8.2}$$

The same formulas with q^k replaced by b define $a_{i,b}$ on every $q^{\mathbb{Z}}$-orbit. We write $\Psi \preceq P$ when P/Ψ is a product of nonnegative integral powers of the $a_{i,b}$.

8.1. **Polynomial truncations and the finite-chain assertion.** Let $P = (P_i(x))$ have polynomial components with constant term 1, put $\lambda_i = \deg P_i$, and let

$$\mu_i = \lambda_i - \sum_j a_j c_{ji}, \qquad a_j \in \mathbb{N}. \tag{8.3}$$

In the adjoint shifted algebra, the invertible generators $\phi_i^{\pm}$ and the auxiliary series $A_i^{\pm}$ satisfy

$$A_{i,0}^{+} = (\phi_i^{+})^{-1}, \qquad A_{i,0}^{-} = z_i'(\phi_i^{-})^{-1}, \qquad \prod_j (z_j')^{c_{ji}} = \mathrm{lc}(P_i). \tag{8.4}$$

Here $\mathrm{lc}(P_i)$ is the leading coefficient, and the two Cartan leading terms are $\prod_j (\phi_j^{+})^{c_{ji}}$ and $\prod_j (\phi_j^{-})^{c_{ji}}$, respectively. The positive Cartan recursion is

$$\psi_i^{+}(x) = P_i(x) \frac{\displaystyle\prod_{j \neq i} \prod_{p=1}^{-c_{ji}} A_j^{+}(q_j^{c_{ji}+2p} x)}{A_i^{+}(x) A_i^{+}(q_i^2 x)}. \tag{8.5}$$

An empty product is 1. Hernandez's polynomial truncation imposes

$$\begin{gathered} A_i^{+}(x) \in U[x], \quad \deg A_i^{+} = a_i, \qquad A_i^{-}(x) \in U[x^{-1}], \quad \deg_{x^{-1}} A_i^{-} = a_i, \\ A_{i,0}^{+} A_{i,0}^{-} = (-q_i)^{a_i}, \qquad A_i^{+}(x) = (x q_i^{-1})^{a_i} A_i^{-}(x). \end{gathered} \tag{8.6}$$

We use the original algebra of [Her23, Definition 10.3]. In (8.6), U denotes that quotient, and the degree conditions include invertible extreme coefficients.

Let $Q(P)$ be the set of normalized highest ℓ-weights of its simple objects in shifted category $\mathcal{O}$, allowing every μ in (8.3). On a highest vector the commuting coefficients of A_i^{+} have a common polynomial eigenvalue with nonzero constant term. Factoring these polynomials in (8.5) gives

$$\Psi \in Q(P) \quad \Longrightarrow \quad P/\Psi = \prod_{i,b} a_{i,b}^{n_{i,b}}, \qquad n_{i,b} \in \mathbb{N}. \tag{8.7}$$

The root of an A_i^{+} eigenpolynomial is counted with its algebraic multiplicity. This is also the order statement of [Her23, Theorem 11.9].

Let $\chi(P)$ denote the mixed character of [Her26, Section 11]. It is multiplicative in the linear factors of P. Write $P \longrightarrow \Psi$ if its support contains the normalized weight Ψ. These labels suppress the constant parts, as in [Her26, Remark 10.3]; the full constants are fixed by (8.4) and (8.6). Conjecture 11.4 asserts

(i) $P \longrightarrow \Psi$ implies $\Psi \in Q(P)$;
(ii) if $\Psi \in Q(P)$, there are polynomial parameters $P = P_0, P_1, \dots, P_N$ with $P_j \longrightarrow P_{j+1}$ and $P_N \longrightarrow \Psi$.

The argument below applies to any character assignment multiplicative in the linear factors and satisfying these two assertions.

Set

$$P_L = z_{2,-3}, \qquad P_R = z_{2,1}, \qquad P = P_L P_R, \qquad t = \frac{z_{1,-6} z_{1,-4} z_{1,0} z_{1,2} z_{2,-3}}{z_{2,-7} z_{2,-1}}. \tag{8.8}$$

Multiplying the three expressions in (8.2) gives

$$\frac{P}{t} = a_{1,-8} a_{2,-7} a_{2,-1}. \tag{8.9}$$

We will prove

$$t \in Q(P), \qquad t \notin Q(P_L)Q(P_R), \qquad P' \preceq P \text{ and } P' \text{ polynomial } \implies P' = P. \tag{8.10}$$

8.2. **Root independence and polynomial isolation.**

Lemma 8.1. *The truncation roots $a_{i,b}$ are multiplicatively independent among finite products.*

Proof. Different $q^{\mathbb{Z}}$-orbits of b do not share a linear factor, since q is not a root of unity. Fix one orbit and encode a finite exponent sequence by a Laurent polynomial in s, with multiplication by s shifting an index by one. The map from root exponents to linear-factor exponents is

$$M(s) = \begin{pmatrix} 1 + s^6 & -(s^{-1} + s + s^3) \\ -s^3 & 1 + s^2 \end{pmatrix}.$$

Its determinant is

$$(1 + s^6)(1 + s^2) - s^3(s^{-1} + s + s^3) = 1 - s^4 + s^8.$$

This is nonzero in the integral domain $\mathbb{Z}[s, s^{-1}]$. Multiplication by the adjugate shows that $M(s)f = 0$ implies $f = 0$. Apply this to each of the finitely many orbits present in a relation. □

Lemma 8.2. *The only polynomial parameter P' with $P' \preceq P$ is P.*

Proof. On the component containing the even long indices and odd short indices, assign the following nonnegative weights to the linear factors, periodically with period 24:

k	0	2	4	6	8	10	12	14	16	18	20	22
$w(z_{1,k})$	0	2	1	5	4	6	6	4	5	1	2	0
$w(z_{2,k+1})$	0	4	5	6	10	7	10	6	5	4	0	3

For all even k the table gives

$$w(z_{1,k}) + w(z_{1,k+6}) - w(z_{2,k+3}) = 1,$$

and, for all odd k, it gives

$$w(z_{2,k}) + w(z_{2,k+2}) - w(z_{1,k-1}) - w(z_{1,k+1}) - w(z_{1,k+3}) = 1.$$

Each identity follows for the twelve residue classes by reading the table; periodicity covers every integer. On the opposite parity component, and on every other $q^{\mathbb{Z}}$-orbit, assign the constant weights 3 at the long vertex and 5 at the short vertex. The two root weights there are $2 \cdot 3 - 5 = 1$ and $2 \cdot 5 - 3 \cdot 3 = 1$. Thus $w(a_{i,b}) = 1$ for every truncation root, while $w(P) = w(z_{2,21}) + w(z_{2,1}) = 0$.

If $P/P' = \prod a_{i,b}^{n_{i,b}}$, put $N = \sum n_{i,b}$. Every linear factor of the polynomial tuple P' has nonnegative multiplicity and nonnegative weight. Consequently

$$0 \le w(P') = w(P) - N = -N.$$

It follows that every $n_{i,b}$ is zero and $P' = P$. □

8.3. **A geometric simple module.** We first construct a simple over the geometric algebra and compute its entire support. The fixed-point and completed-localization constructions used here are those of [VV25, Sections 2–4].

In the D_4 unfolding of (8.1), retain the short center c and the long leaf a of residue 4 modulo 6. Put

$$V_c = \mathbb{C}^2, \qquad V_a = \mathbb{C}, \qquad W_c = \mathbb{C}^2, \qquad W_a = 0.$$

The nonzero arrow is $c \to a$. Hence the representation space is

$$N = \operatorname{Hom}(W_c, V_c) \oplus \operatorname{Hom}(V_c, V_a) \simeq \mathbb{A}^6.$$

Write its coordinates as $A = (A_L, A_R)$ and B, where $A_L, A_R \in \mathbb{C}^2$ and $B \in (\mathbb{C}^2)^*$. The original flavor and gauge cocharacters are

$$\rho_2 = (-5/2, -1/2), \qquad \gamma_1 = -4/3, \qquad \gamma_2 = (-7/2, -1/2). \tag{8.11}$$

The flavor index is $2d_i + 2d_i\rho_{i,s}$, so the two framing factors are $z_{2,-3}$ and $z_{2,1}$. The gauge index is $2d_i\gamma_{i,s}$, giving -8 at the long vertex and $-7, -1$ at the short vertex.

The short-vertex regrading following [VV25, (3.1.72)] subtracts 2 from its geometric degrees. Thus the red degrees are $-7, -3$, and the target black degrees are

$$c(-9), \quad a(-8), \quad c(-3). \tag{8.12}$$

The original torus characters remain those of (8.11). The flag conditions in these degrees are

$$A(W^{\le k}) \subseteq F^{\le k}, \qquad B(F^{\le k}) \subseteq F^{<k}. \tag{8.13}$$

They can also be checked before regrading: the first framing vector has degree -5 and must lie in the center line of degree -7; the arrow condition is $B(V_c^{\le k}) \subset V_a^{<k-1}$. At $k = -7$ the long degree -8 is excluded by the strict inequality, so B kills that line. At $k = -1$ it is allowed.

For (8.12), the variety of flags over N is therefore

$$X = \{(L, A_L, A_R, B) : L \in \mathbb{P}(V_c),\ A_L \in L \subseteq \ker B\}. \tag{8.14}$$

The choices for A_L, A_R, B over L have dimensions $1, 2, 1$. Thus X is a rank-four vector bundle on $\mathbb{P}^1$, smooth of dimension 5. Its projective projection has image

$$Z = \{BA_L = 0\} \subset N.$$

The equation $b_1a_1 + b_2a_2 = 0$ is irreducible. Its singular locus is $S = \{A_L = B = 0\} \simeq \mathbb{A}^2_{A_R}$, of codimension 3 in Z. Away from S, the line is $\mathbb{C}A_L$ if $A_L \neq 0$ and $\ker B$ if $B \neq 0$; these agree when both are defined. Over S the fiber is $\mathbb{P}^1$. The map is small since $3 > 2\dim \mathbb{P}^1$. Consequently

$$\pi_*\mathbb{C}_X[5] = \mathrm{IC}_Z\,. \tag{8.15}$$

The local system on the smooth locus of Z has rank one and is constant.

For each loading ν, let $K_\nu = (\pi_\nu)_*\mathbb{C}_{X_\nu}[\dim X_\nu]$. The convolution algebra is the locally unital sum of the equivariant groups $\operatorname{Ext}^\bullet_G(K_\nu, K_{\nu'})$, with $G = \mathrm{GL}(V_c) \times \mathrm{GL}(V_a)$. The decomposition theorem expresses each K_ν as a finite sum of shifted simple perverse sheaves. After these decompositions, a morphism has a well-defined perverse Ext degree; negative degrees vanish and degrees add under composition. Quotienting by positive perverse Ext degrees leaves matrix

algebras on the multiplicity spaces of the simple perverse summands. The block belonging to IC_Z therefore defines a simple module S_Z and

$$e_\nu S_Z \neq 0 \quad \Longleftrightarrow \quad \mathrm{IC}_Z \text{ occurs, up to shift, in } K_\nu. \tag{8.16}$$

Indeed, the degree-zero maps between its copies supply every matrix unit, so any nonzero vector generates the multiplicity module. This also proves the assertion for the locally unital direct sum over all loadings. Each individual multiplicity space is finite dimensional. The construction is the convolution–Ext description of [Gin98, Proposition 5.1 and Theorem 5.2]. Positive-degree equivariant parameters act by positive perverse Ext and annihilate S_Z. In particular this is a nilpotent module for the completed-localization correspondence.

The module descends through the full cyclotomic ideal. To see this, set

$$U = \{(A, B) : \det A \neq 0, B \neq 0\} \subset N.$$

The set $U \cap Z$ is dense and nonempty; it contains $A = 1$, $B = (0, 1)$. At every point of U, the framing generates $V_c \oplus V_a$. A loading with a black degree greater than the maximum red degree -3 would give a proper subspace $F^{\leq -3}$ containing $A(W_c)$ and stable under B. But $A(W_c) = V_c$ and $B(V_c) = V_a$, so this is impossible. Hence $X_\nu|_U$ is empty for every violating loading. Since $\mathrm{IC}_Z\,|_U \neq 0$, (8.16) shows that every such e_ν, and its generated two-sided ideal, annihilates S_Z. The target loading is integral and its weight space is nonzero by (8.15); restriction to the integral idempotent corner is thus a nonzero simple module. This last assertion follows directly: if $eM \neq 0$ and M is simple, then any nonzero eAe-submodule $L \subseteq eM$ satisfies $AL = M$ and $eAL = eAeL = L$, so $L = eM$.

Lemma 8.3. *The supported integral loadings have short degrees $x \leq y$ and long degree ℓ precisely as follows:*

$$x \leq -7 < y \leq -3, \qquad x \leq \ell < y, \qquad x, y \text{ odd}, \quad \ell \equiv 4 \pmod 6. \tag{8.17}$$

Each supported loading has multiplicity one. Equivalently,

$$\ell = -8 - 6n, \quad x = -9 - 6n - 2m, \quad y = -3 - 2\epsilon, \qquad n, m \in \mathbb{N}, \quad \epsilon \in \{0, 1\}. \tag{8.18}$$

Proof. If $x = y$, there is no line in the flag. Its image is a linear subspace obtained by setting whole framing columns or B to zero. If that subspace meets U, the image is N and its only simple summand is IC_N. Thus IC_Z cannot occur.

Suppose $x < y$ and denote the center line by L. Nonemptiness over U forces $x \leq -7$, $y \leq -3$, and $\ell < y$. The first inequality prevents $A_L = 0$, the second prevents the two independent framing columns from being confined to L, and the third prevents $B = 0$. The right column is then free. The four possible remaining conditions are

Framing	Arrow	Condition on L	Map over U
$y \leq -7$	$\ell < x$	none	$U \times \mathbb{P}^1 \to U$
$-7 < y$	$\ell < x$	$A_L \in L$	isomorphism onto U
$y \leq -7$	$x \leq \ell$	$BL = 0$	isomorphism onto U
$-7 < y$	$x \leq \ell$	$A_L \in L \subseteq \ker B$	isomorphism onto $U \cap Z$

In the first three rows the direct image over the smooth connected set U is a sum of shifts of its constant sheaf. It cannot contain a summand with proper support $U \cap Z$. In the fourth row the global variety is exactly (8.14), and (8.15) gives multiplicity one. This proves (8.17). The allowed odd values of y are $-5, -3$. The congruence for ℓ then gives $\ell = -8 - 6n$, and the odd values $x \leq \ell$ give $x = \ell - 1 - 2m$. These are (8.18). □

The completed-localization isomorphism [VV25, Theorem 4.3.1] now transfers S_Z to the completed geometric algebra. Restriction to the original algebra preserves simplicity. A vector and its image under a chosen completed action involve finitely many generalized Cartan blocks. Smoothness makes the action on each source block factor through a finite quotient by a Cartan-ideal power. Density gives an uncompleted representative modulo these powers in the chosen target blocks. Its action has finite support; Chinese remainders in the Cartan subalgebra retain the chosen output blocks and annihilate the other output blocks, to their respective nilpotent orders. Thus every completed image of a vector is already an uncompleted image. The generated submodule is the same, so simplicity follows. This is the comparison of [VV25, Proposition 4.2.9]; all its support bounds are explicit here. In the original cocharacter coordinates,

$$\sum_{i,s} d_i\gamma_{i,s} = -8 - 6n - m - \epsilon \leq -8.$$

8.4. **The current action and the polynomial relations.** Let $\mathfrak{A} = \mathcal{A}^{\rho}_{\mu}$ be the specialized K-theoretic Coulomb algebra of [VV25, Sections 3.2 and 4.2], with determinant covers, $\lambda = (0, 2)$, $\mu = (4, -1)$, and the flavor cocharacter (8.11). We construct the shifted action in this unlocalized algebra. The gauge dimensions are $a_1 = 1, a_2 = 2$, and the orientation is $2 \to 1$. Put $D_i = \det(V_i)^{1/2}$ and $\check{z}_{i,s} = q^{2d_i\rho_{i,s}}$. Let $S_{i,N}$ and $Q_{i,N}$ be the classes of the Nth tautological line powers on the negative and positive minuscule correspondences, respectively, for $N \in \mathbb{Z}$.

For the following calculation define

$$J_i^- = \prod_{j\to i} D_j^{-c_{ji}}, \qquad J_i^+ = \prod_{i\to j} D_j^{-c_{ji}}, \qquad b_i = -\sum_{i\to j} a_j c_{ji}, \qquad \eta_i = -\sum_{i\to j} a_j c_{ij}.$$

In the present orientation these are

$$(J_1^-, J_1^+, b_1, \eta_1) = (D_2^3, 1, 0, 0), \qquad (J_2^-, J_2^+, b_2, \eta_2) = (1, D_1, 1, 3).$$

Define currents, for all $r \in \mathbb{Z}$, by

$$\begin{aligned} \widehat{E}_{i,r} &= (-1)^{a_i+1}(J_i^-)^{-1} q_i^{2(r+a_i)} S_{i,r+a_i}, \\ \widehat{F}_{i,r} &= (-1)^{b_i+1} q_i^{\eta_i - 1} J_i^+ q_i^{2(r-b_i)} Q_{i,r-b_i}. \end{aligned} \tag{8.19}$$

Explicitly,

$$\begin{aligned} \widehat{E}_{1,r} &= q^{6r+6} D_2^{-3} S_{1,r+1}, \quad & \widehat{F}_{1,r} &= -q^{6r-3} Q_{1,r}, \\ \widehat{E}_{2,r} &= -q^{2r+4} S_{2,r+2}, & \widehat{F}_{2,r} &= q^{2r} D_1 Q_{2,r-1}. \end{aligned} \tag{8.20}$$

The Drinfeld generators will act by $x^+_{i,r} = (q_i - q_i^{-1})^{-1}\widehat{E}_{i,r}$ and $x^-_{i,r} = (q_i - q_i^{-1})^{-1}\widehat{F}_{i,r}$.

To check all relations, use the faithful geometric embedding of [VV25, Lemma 5.1.2]. Let $w_{i,s}$ be the gauge characters and let $T_{i,s}$ be the difference shifts, so

$$T_{i,s} w_{j,t}^{1/2} = q_i^{\delta_{ij}\delta_{st}} w_{j,t}^{1/2} T_{i,s}.$$

Define the following finite products:

$$B_i(x) = \prod_s (1 - w_{i,s}x), \qquad W_{i,s} = \prod_{t\neq s}(1 - w_{i,t}/w_{i,s}),$$

$$W^-_{i,s} = \prod_{j\to i}\prod_t \prod_{p=1}^{-c_{ji}} (1 - q_j^{c_{ji}+2p} w_{j,t}/w_{i,s}),$$

$$W^+_{i,s} = \prod_{i\to j}\prod_t \prod_{p=1}^{-c_{ji}} (1 - q_i^{-2} q_j^{c_{ji}+2p} w_{j,t}/w_{i,s}), \qquad Z_i(u) = P_i(u^{-1}).$$

The embedding sends the geometric classes to

$$\iota(S_{i,N}) = -(-q_i^2)^{-a_i} \sum_s (q_i^{-2} w_{i,s})^{N-a_i} D_i^2 \frac{Z_i(w_{i,s}) W_{i,s}^-}{W_{i,s}} T_{i,s}^{-1}, \tag{8.21}$$

$$\iota(Q_{i,N}) = (-q_i^2)^{b_i} q_i^{-\eta_i} \sum_s w_{i,s}^{N+b_i} (J_i^+)^{-2} \frac{W_{i,s}^+}{W_{i,s}} T_{i,s}. \tag{8.22}$$

All factors in $J_i^\pm$ have color different from i, and thus commute with the color-i shifts. Substituting $N = r + a_i$ and $N = r - b_i$ gives

$$\iota(\widehat{E}_{i,r}) = D_i^2 (J_i^-)^{-1} \sum_s w_{i,s}^r \frac{Z_i(w_{i,s}) W_{i,s}^-}{W_{i,s}} T_{i,s}^{-1}, \tag{8.23}$$

$$\iota(\widehat{F}_{i,r}) = -q_i^{-1} (J_i^+)^{-1} \sum_s (q_i^2 w_{i,s})^r \frac{W_{i,s}^+}{W_{i,s}} T_{i,s}. \tag{8.24}$$

The scalar simplifications, valid for every integer r, are

$$(-1)^{a_i+1} q_i^{2(r+a_i)} [-(-q_i^2)^{-a_i}] q_i^{-2r} = 1,$$
$$(-1)^{b_i+1} q_i^{\eta_i - 1} q_i^{2(r-b_i)} (-q_i^2)^{b_i} q_i^{-\eta_i} = -q_i^{2r-1}.$$

In [FT19, Theorem 7.1], set its quantum parameter equal to q and its flavor variables to $\mathsf{z}_{i,s} = q_i \check{z}_{i,s}$. Its framing polynomial is then $\prod_s (1 - q_i \mathsf{z}_{i,s}/u) = \prod_s (1 - q_i^2 \check{z}_{i,s}/u) = Z_i(u)$. The incoming and outgoing factors are the ones above, and its current scalars satisfy

$$(q_i - q_i^{-1}) \frac{-q_i}{1 - q_i^2} = 1, \qquad (q_i - q_i^{-1}) \frac{1}{1 - q_i^2} = -q_i^{-1}.$$

Thus the images of $x_{i,r}^\pm$ are exactly those in that theorem. Its scope is every coweight satisfying (8.3) [FT19, Section 7(i)]. The complete Drinfeld and Serre relations therefore hold after the faithful embedding and hence in $\mathfrak{A}$. The calculation takes place with universal gauge variables. The freeness in [VV25, Proposition 3.2.2(a)] permits the identities to be checked after scalar localization. Clearing scalar denominators then specializes them to every non-root-of-unity q.

The two Cartan expansions are those of the rational functions

$$\Psi_i(x) = K_i P_i(x) \frac{\displaystyle\prod_{j \neq i} \prod_{p=1}^{-c_{ji}} B_j(q_j^{c_{ji}+2p} x)}{B_i(x) B_i(q_i^2 x)}, \qquad K_i = D_i^2 \prod_{j \neq i} D_j^{c_{ji}}. \tag{8.25}$$

Their coefficients lie in the gauge representation ring because the extreme coefficients of the denominators are invertible determinant characters.

Lemma 8.4. *The current action just constructed maps onto* $\mathfrak{A}$.

Proof. The leading Cartan terms are $K_1 = D_1^2 D_2^{-3}$ and $K_2 = D_1^{-1} D_2^2$. Hence

$$D_1 = K_1^2 K_2^3, \qquad D_2 = K_1 K_2^2. \tag{8.26}$$

For $m \geq 1$, put $p_{i,m} = \sum_s w_{i,s}^m$. Multiplying the coefficient of x^m in $\log(\Psi_i/(K_i P_i))$ by m gives two linear equations in $p_{1,m}, p_{2,m}$, with matrix

$$\begin{pmatrix} 1 + q^{6m} & -(q^{-m} + q^m + q^{3m}) \\ -q^{3m} & 1 + q^{2m} \end{pmatrix}.$$

Its determinant is $1-q^{4m}+q^{8m}$. If this vanished, multiplication by $1+q^{4m}$ would give $1+q^{12m}=0$, contrary to the assumption on q. Both power sums therefore belong to the image. Newton identities and (8.26) recover the symmetric gauge polynomials and their inverse determinants. For these dimensions it suffices to use $m=1$ for color 1 and $m=1,2$ for color 2.

Equations (8.19) then recover every $S_{i,N}$ and $Q_{i,N}$, since their prefactors are invertible and r ranges over $\mathbb{Z}$. The geometric generation in the proof of [VV25, Proposition 4.2.2] uses the gauge representation ring and all representation-ring decorations of the positive and negative minuscule correspondences. Such decorations are generated over the symmetric gauge ring by powers of the singled-out tautological line. Explicitly, in the GL_2 factor, if its two characters are x, y and $D_2^2=xy$, then

$$x^u y^v D_2^w = D_2^{2v+w} x^{u-v} \qquad (u,v,w\in\mathbb{Z}).$$

The GL_1 assertion is immediate. Thus every required decoration is already in the image. The generation statement follows in the source from signed basic cocharacters in the abelianized algebra (Lemma 4.2.4), passage to the Iwahori algebra with the nil-Hecke algebra (Lemma 4.2.5), and the full spherical idempotent (Lemma 4.2.3). It proves the asserted surjectivity. □

Choose $z'=(q^{-6},q^{-4})$. Then $\prod_j (z'_j)^{c_{ji}}=\mathrm{lc}(P_i)$, namely $1, q^{-2}$. Install the adjoint generators by

$$\phi_i^+ = D_i, \qquad \phi_i^- = z'_i(-q_i)^{-a_i} D_i^{-1}. \tag{8.27}$$

Their quasi-commutations follow from the displayed shift relation. At zero the leading Cartan relation is $K_i=\prod_j(\phi_j^+)^{c_{ji}}$. At infinity (8.25) has leading coefficient

$$\mathrm{lc}(P_i)\prod_j(-q_j)^{-a_j c_{ji}} D_j^{-c_{ji}} = \prod_j(\phi_j^-)^{c_{ji}}.$$

For a neighbor with $-c_{ji}=r$, the exponent calculation uses $\sum_{p=1}^{r}(-r+2p)=r$; the denominator contributes $q_i^{2a_i}D_i^4$. These account for every sign and power. The auxiliary polynomials are

$$A_i^+(x)=D_i^{-1}\prod_s(1-w_{i,s}x), \qquad A_i^-(x)=(-q_i)^{a_i}D_i\prod_s(1-w_{i,s}^{-1}x^{-1}). \tag{8.28}$$

They give (8.5), and their constants are precisely (8.4). Since $\prod_s w_{i,s}=D_i^2$,

$$(xq_i^{-1})^{a_i}(-q_i)^{a_i}D_i\prod_s(1-w_{i,s}^{-1}x^{-1}) = D_i^{-1}\prod_s(1-w_{i,s}x).$$

Together with $A_{i,0}^+A_{i,0}^-=(-q_i)^{a_i}$, this proves all relations (8.6). The action thus factors through the specified Hernandez polynomial truncation.

8.5. **The highest weight and category $\mathcal{O}$.** On the target loading, $D_1=D_2=q^{-4}$. Substitution in (8.25) gives

$$\begin{aligned}\Psi_{*,1}(x)&=q^4(1-q^{-6}x)(1-q^{-4}x)(1-x)(1-q^2x),\\ \Psi_{*,2}(x)&=q^{-4}\frac{1-q^{-3}x}{(1-q^{-7}x)(1-q^{-1}x)}.\end{aligned} \tag{8.29}$$

Its shift is $(4,-1)$ by (8.3). At infinity the leading coefficients are $(q^{-4},-q)$; at zero they are (q^4,q^{-4}). Their products are $(1,-q^{-3})$. Formula (8.27) gives $\phi^+=(q^{-4},q^{-4})$ and $\phi^-=(-q^{-5},q^{-2})$, in agreement with both Cartan leading terms.

The normalized ordinary inverse q-character root satisfies

$$(A_{i,q^k}^{\mathrm{ord}})^{-1}=a_{i,k}/a_{i,k-2d_i}. \tag{8.30}$$

For full ℓ-weights the constants in this ratio are (q^{-6}, q^3) for $i = 1$ and (q^3, q^{-2}) for $i = 2$. These follow either from (8.25) or from the Cartan action of a negative Drinfeld generator. In the original gauge indices, Lemma 8.3 gives

$$(1, -8 - 6n), \quad (2, -7 - 6n - 2m), \quad (2, -1 - 2\epsilon).$$

Let $\Psi_{n,m,\epsilon}$ denote the full ℓ-weight of this loading. Telescoping (8.30) gives

$$\frac{\Psi_{n,m,\epsilon}}{\Psi_*} = \prod_{r=0}^{n-1} (A^{\mathrm{ord}}_{1,q^{-8-6r}})^{-1} \prod_{s=0}^{3n+m-1} (A^{\mathrm{ord}}_{2,q^{-7-2s}})^{-1} (A^{\mathrm{ord}}_{2,q^{-1}})^{-\epsilon}. \tag{8.31}$$

Thus its ordinary Cartan weight is

$$(q^{4+3n+3m+3\epsilon}, q^{-4-3n-2m-2\epsilon}). \tag{8.32}$$

A fixed such weight determines n and $m + \epsilon$, since the two linear forms have independent coefficient vectors and q is not a root of unity. Its multiplicity is at most 2. When both values of ϵ occur, their normalized weights differ: the exponent of $z_{2,1}$ is ϵ. Every individual ℓ-weight space is therefore one dimensional by Lemma 8.3.

All ordinary weights are below the target by (8.31), and each ordinary weight space is finite dimensional. The positive and negative Cartan leading operators differ, after inversion, by the fixed central scalars computed above. Hence the module belongs to Hernandez's category $\mathcal{O}_{(4,-1)}$. The target is the unique maximal ordinary weight, so every positive Drinfeld current kills its one-dimensional weight space. Lemma 8.4 makes the pullback of the geometric simple simple. A nonzero target vector thus generates $L(\Psi_*)$. Equations (8.28) show that it is an object of the polynomial truncation at P. Normalizing (8.29) gives t in (8.8), proving

$$t \in Q(P). \tag{8.33}$$

8.6. **Excluding a product of the two one-factor parameters.** We need a polynomiality statement at a vector highest for one color. Its proof also specifies the mode convention. Put $[A, B]_s = AB - sBA$.

Lemma 8.5. *Let M be a module over a polynomial truncation. Suppose that v is a common Cartan eigenvector killed by $x^+_{i,0}$, that its rational Cartan eigenvalue is $\Psi_i(x)$, and that the eigenvalue of $A^+_i(x)$ on v is $a_i(x)$. Then $a_i(x)\Psi_i(x)$ is a polynomial.*

Proof. Define

$$x^{-,+}_i(x) = -\sum_{r>0} x^-_{i,r} x^r, \qquad C_i(x) = (q_i - q_i^{-1}) x^{-,+}_i(x) A^+_i(x) = \sum_{r>0} C_{i,r} x^r.$$

The Cartan–current relation [Her23, (11.24)] is

$$(w - x)[A^+_i(x), C_i(w)]_{q_i} = (q_i - q_i^{-1})(xC_i(w)A^+_i(x) - wC_i(x)A^+_i(w)).$$

Its coefficient of w gives

$$-x[A^+_i(x), C_{i,1}]_{q_i^{-1}} = -(q_i - q_i^{-1}) C_i(x) A^+_{i,0}.$$

Since A^+_i is polynomial and $A^+_{i,0}$ is invertible, $C_i(x)$ is polynomial as an operator. Now $A^+_i(x)v = a_i(x)v$ and $x^+_{i,0}v = 0$. The positive–negative current commutator yields

$$x^+_{i,0} C_i(x) v = -a_i(x) \left(\Psi_i(x) - \Psi^+_{i,0} - \sum_{0<r\le\mu_i} \Psi^-_{i,r} x^r \right) v.$$

The sum is empty if $\mu_i \le 0$. The left side and the subtracted terms on the right are polynomial. Since $v \neq 0$, so is $a_i(x)\Psi_i(x)$. □

Proposition 8.6. *The element t does not belong to $Q(P_L)Q(P_R)$.*

Proof. Suppose that $t = uv$ with $u \in Q(P_L)$ and $v \in Q(P_R)$. By (8.7), (8.9), and Lemma 8.1, the roots assigned to the left factor have the form

$$P_L/u = a_{1,-8}^{a} a_{2,-7}^{b} a_{2,-1}^{c}, \qquad a, b, c \in \{0, 1\}.$$

The normalized A_i^+ eigenpolynomials on its highest vector are $(1 - q^{-8}x)^a$ and $(1 - q^{-7}x)^b(1 - q^{-1}x)^c$. By (8.5), the exponents of $A_i^+ u_i$ at $(2, 1)$, $(1, -2)$, and $(2, -5)$ are, respectively,

$$-c, \qquad c - a, \qquad a - b.$$

For example, at $(2, 1)$ the denominator comes only from $a_{2,-1}^{c}$; at $(1, -2)$ the denominator of $a_{1,-8}^{a}$ and the numerator contributed by $a_{2,-1}^{-c}$ give $c - a$; at $(2, -5)$ the factors $a_{1,-8}^{-a}$ and $a_{2,-7}^{-b}$ give $a - b$. Lemma 8.5 makes these exponents nonnegative. It follows successively that $c = 0$, $a = 0$, and $b = 0$. Hence $u = P_L$ and the right normalized highest weight is forced to be

$$r = \frac{t}{P_L} = \frac{z_{1,-6} z_{1,-4} z_{1,0} z_{1,2}}{z_{2,-7} z_{2,-1}}. \tag{8.34}$$

Its long A_1^+ eigenpolynomial is a nonzero scalar times $1 - q^{-8}x$.

Let v_0 be a highest vector of a putative simple with this label, and let $N_2 = U_2^- v_0$, where U_2^- is generated by all negative currents of color 2. Cartan–current commutators make N_2 stable under the full Cartan algebra. The cross-color relation $[x_{1,s}^+, x_{2,t}^-] = 0$ makes every vector in N_2 highest for color 1. The fundamental Cartan commutators in [Her23, proof of Theorem 9.12] show that A_1^+ commutes with all color-2 currents. Thus its eigenpolynomial throughout N_2 remains

$$\kappa(1 - q^{-8}x), \qquad \kappa \neq 0. \tag{8.35}$$

As a color-2 highest module, N_2 has a unique simple highest quotient. Its normalized highest rational function is $(1 - q^{-7}x)^{-1}(1 - q^{-1}x)^{-1}$. By [Her23, Theorems 5.4–5.5 and Example 5.2(iii)], that quotient is the fusion product of the two negative prefundamentals, with its nonzero highest constant. The rank-one character includes the three-step prefix of the factor at -1, and hence contains the weight lowered by

$$(A_{2,q^{-1}}^{\mathrm{ord}})^{-1}(A_{2,q^{-3}}^{\mathrm{ord}})^{-1}(A_{2,q^{-5}}^{\mathrm{ord}})^{-1} = a_{2,-1}/a_{2,-7} \tag{8.36}$$

after normalization. Indeed, the rank-one negative prefundamental at an index k has one term for each length $d \geq 0$, whose normalized factor is $\prod_{j=0}^{d-1}(A_{2,q^{k-2j}}^{\mathrm{ord}})^{-1}$; taking $d = 3$ at $k = -1$ and $d = 0$ at $k = -7$ gives the stated term.

The depth-three ordinary weight space of N_2 is finite dimensional, as it lies in the category-$\mathcal{O}$ module under consideration. Generalized eigenspaces for the color-2 Cartan operators pass surjectively to the corresponding quotient space. The nonzero lifted generalized eigenspace is stable under all commuting Cartan operators, and therefore contains a common eigenvector. The triangular q-character statement [Her23, Theorem 5.11] restricts its full weight ratio to three ordinary inverse roots, all of color 2. Its color-2 coordinate determines the full ratio uniquely. On each spectral $q^{\mathbb{Z}}$-orbit separately, for a finite color-2 truncation-root exponent sequence h_k, the color-2 linear-factor exponents are $h_k + h_{k-2}$. Their vanishing gives $h_k = -h_{k-2}$, whose only finite-support solution is zero. Combined with the finite difference in (8.30), this proves injectivity on finite ordinary color-2 root products as well.

Consequently the full lifted weight is obtained from (8.34) by (8.36). The contributions to its long coordinate are

$$\frac{z_{1,-4}}{z_{1,2}}, \qquad \frac{z_{1,-6}}{z_{1,0}}, \qquad \frac{z_{1,-8}}{z_{1,-2}},$$

so that coordinate is

$$r'_1 = \frac{z_{1,-8} z_{1,-6}^2 z_{1,-4}^2}{z_{1,-2}}.$$

Apply Lemma 8.5 to its common Cartan eigenvector at color 1. In view of (8.35), the asserted polynomial is a nonzero scalar multiple of

$$\frac{(1-q^{-8}x)^2(1-q^{-6}x)^2(1-q^{-4}x)^2}{1-q^{-2}x}.$$

The denominator has no common root with the numerator because q is not a root of unity. This contradiction proves the proposition. □

8.7. **The finite-chain contradiction.**

Proof of Theorem 1.4. Suppose that assertions (i) and (ii) of Conjecture 11.4 hold. By (i), every weight in the support of $\chi(P_L)$ belongs to $Q(P_L)$, and every weight in the support of $\chi(P_R)$ belongs to $Q(P_R)$. Multiplicativity and Proposition 8.6 imply that t is absent from $\chi(P)$.

On the other hand, (8.33) and (ii) give a finite chain of polynomial parameters $P = P_0, P_1, \dots, P_N$ with t in $\chi(P_N)$. Assertion (i) and (8.7) give $P_{j+1} \preceq P_j$ for each step. Transitivity gives $P_j \preceq P$, and Lemma 8.2 forces $P_j = P$ for every j. Hence t belongs to $\chi(P)$, a contradiction. □

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