THE DEGREE PAIR $(126, 84)$ IN THE PLANE JACOBIAN PROBLEM: THE POINT $(16/3, 10)$ IS NOT A REGULAR CORNER

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Abstract. In the list of Guccione, Guccione, Horruitiner and Valqui of the possible shapes of a counterexample to the plane Jacobian conjecture with degrees at most 150, two shapes have maximal degree 126. One of them consists of the standard $(3, 2)$-pairs of degrees $(126, 84)$ whose chain of corners is $(12, 30)$, $(16/3, 10)$, $(11/6, 3)$. We show that $(16/3, 10)$ is not a regular corner of any $(3, 2)$-pair in $K[x^{\pm 1/3}, y]$; hence no such pair exists, and the shape of maximal degree 144 of the same list whose chain passes through $(16/3, 10)$ does not occur either. The proof uses only the propositions of Guccione–Guccione–Valqui on regular corners, applied to every root of the edge polynomial at that corner and not only to the root which generates the next corner of the chain, together with elementary arithmetic, which is reproduced by an ancillary script. The other shape of maximal degree 126 is the subject of a companion note. The argument has not yet been examined by specialists.

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1. Statement

Throughout, K is an algebraically closed field of characteristic zero, $L^{(l)} = K[x^{\pm 1/l}, y]$, $[P, Q] = P_x Q_y - P_y Q_x$ and $(m, n) = (3, 2)$. We use the notation of [1]: $v_{\rho,\sigma}$, $\ell_{\rho,\sigma}$, $\mathrm{st}_{\rho,\sigma}$, $\mathrm{en}_{\rho,\sigma}$; the interval of directions $I =](1, -1), (1, 0)]$; *(m, n)-pairs* in $L^{(l)}$ and their *regular corners* [1, Definitions 4.3 and 5.5]. For a regular corner $((a/l, b), (\rho, \sigma))$ of (P, Q) we write $\ell_{\rho,\sigma}(P) = x^{k/l} p(z)$ with $z := x^{-\sigma/\rho} y$, and we distinguish, as in [1, Section 5, before Remark 5.9]: Case I, $[\ell_{\rho,\sigma}(P), \ell_{\rho,\sigma}(Q)] \neq 0$; Case II, the bracket vanishes and p has at least two distinct linear factors; Case III, the bracket vanishes and $p = \mu(z - \lambda)^r$ with $\lambda \neq 0$. One always has $v_{1,-1}(\mathrm{st}_{\rho,\sigma}(P)) \neq 0$ (loc. cit.); Case II is called II.a) if this number is negative and II.b) if it is positive.

Theorem 1.1. *No $(3, 2)$-pair in $L^{(3)}$ has a regular corner at the point $(16/3, 10)$.*

Corollary 1.2. *There is no standard $(3, 2)$-pair (P, Q) whose chain of corners, in the sense of* [2, Theorem 2.20]*, contains $A_i = (16/3, 10)$ with $l_i = 3$. In particular the following two shapes of* [2, Section 6] *do not occur:*

(i) the chain $(12, 30)$, $(16/3, 10)$, $(11/6, 3)$, of degrees $(126, 84)$;
(ii) the chain $(12, 36)$, $(12, 30)$, $(16/3, 10)$, $(11/6, 3)$, of maximal degree 144.

Proof. By items (3) and (6) (or (9)) of [2, Theorem 2.20], $(A_i, (\rho_i, \sigma_i))$ is a regular corner of the $(3, 2)$-pair (P_i, Q_i) in $L^{(l_i)}$, which contradicts Theorem 1.1 when $l_i = 3$. It remains to see that $l_i = 3$ in (i) and (ii). Let $A_{i-1} = (12, 30)$; in (i), $i = 1$ and $l_0 = 1$; in (ii), $i = 2$, and $l_1 = 1$ because l_1 is $l_0 = 1$ in Case II.a) and $\mathrm{lcm}(\rho_0, 1)$ in Case II.b), where $(\rho_0, \sigma_0) = (1, 0)$ since A_0 and A_1 have the same first coordinate and lie on the same edge. As $A_i \notin \mathbb{Z}^2$, the corner $(A_{i-1}, (\rho_{i-1}, \sigma_{i-1}))$ is not of Case II.a) (item (7)), so item (8) applies: A_i lies on the line of direction $(-\sigma_{i-1}/\rho_{i-1}, 1)$ through $(k_{i-1}/(m l_{i-1}), 0)$, and so does $A_{i-1} = \frac{1}{m} \mathrm{en}_{\rho_{i-1},\sigma_{i-1}}(P_{i-1})$. Since $A_{i-1} - A_i = (20/3, 20)$, we get $(\rho_{i-1}, \sigma_{i-1}) = (3, -1)$ and $l_i = \mathrm{lcm}(3, 1) = 3$. □

Remark 1.3. If (P, Q) is a $(2, 3)$-pair with a regular corner, then (Q, P) is a $(3, 2)$-pair with the same regular corner [1, Corollary 5.7]; so Theorem 1.1 also holds for $(2, 3)$-pairs.

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2. The tools

We use the following statements, for an (m,n)-pair (P,Q) in $L^{(l)}$ and a regular corner $((a/l,b),(\rho,\sigma))$.

(T1) [1, Proposition 5.19]: the point $(a/l,b)$ is the first entry of a regular corner of Case I or of Case II of a (possibly different) (m,n)-pair in $L^{(l)}$; in Case I, $l-a/b>1$; in Case II, $\gcd(a,b)>1$. The directions of these corners are not larger than (ρ,σ) (proof of that proposition).

(T2) Case I. By [1, Proposition 1.13], $v_{\rho,\sigma}(P)+v_{\rho,\sigma}(Q)=\rho+\sigma$; moreover $\mathrm{en}_{\rho,\sigma}(P)=m(a/l,b)$ by definition and $\mathrm{en}_{\rho,\sigma}(P)=\frac{m}{n}\,\mathrm{en}_{\rho,\sigma}(Q)$ [1, Corollary 5.7(2)]. Hence

$$m\Big(\frac{a}{l}\rho+b\sigma\Big)=\frac{m}{m+n}(\rho+\sigma), \tag{1}$$

which determines (ρ,σ), and $v_{\rho,\sigma}(P)=\frac{m}{m+n}(\rho+\sigma)$, $v_{\rho,\sigma}(Q)=\frac{n}{m+n}(\rho+\sigma)$. By [1, Proposition 5.13], either $\frac{1}{m}\,\mathrm{st}_{\rho,\sigma}(P)=(a'/l,0)$ with $a'\in\mathbb{N}$ (statement and proof), or $\{\mathrm{st}_{\rho,\sigma}(P),\mathrm{st}_{\rho,\sigma}(Q)\}=\{(k/l,0),(1-k/l,1)\}$ with $k\in\mathbb{N}$.

(T3) Case II. By [1, Proposition 5.14], with $d:=\gcd(a,b)$, $\bar a:=a/d$, $\bar b:=b/d$: $d>1$, and there is $\mu\in\mathbb{N}$ with $d\nmid\mu$, $\mu\le l(bl-a)+1/\bar b$, such that (ρ,σ) is the direction orthogonal to $(\mu\bar a-l,\ \mu\bar bl-l)$; moreover $\rho/\gcd(\rho,l)\le b$. The point $A':=\frac{1}{m}\,\mathrm{st}_{\rho,\sigma}(P)$ lies in $\frac{1}{l}\mathbb{Z}\times\mathbb{N}_0$ (proof of that proposition), below $(a/l,b)$ on the line of direction $(-\sigma/\rho,1)$ through it, and $v_{1,-1}(A')\neq 0$ as recalled above. One has $\ell_{\rho,\sigma}(P)=\lambda R^m$ for a (ρ,σ)-homogeneous $R\in L^{(l)}$ [1, Remark 3.1 and Corollary 5.7].

(T4) Case II.a). $(A',\mathrm{Pred}_P(\rho,\sigma))$ is a regular corner of (P,Q) [1, Remark 5.11].

(T5) Case II, roots. Let $\lambda\in K^\times$ be a root of p of multiplicity m_λ with

$$m_\lambda\ \ge\ \frac{m}{l}\cdot\frac{a\rho+bl\sigma}{\rho+\sigma}. \tag{2}$$

Then, with $l':=\mathrm{lcm}(\rho,l)$ and $\varphi(y):=y+\lambda x^{\sigma/\rho}$, the pair $(\varphi P,\varphi Q)$ is an (m,n)-pair in $L^{(l')}$ with a regular corner at $(\frac{k}{ml},0)+\frac{m_\lambda}{m}(-\frac{\sigma}{\rho},1)$, of direction smaller than (ρ,σ) [1, Proposition 5.18].

The use of (T5) for a root other than the one continuing a chain, followed by (T1) and (T2), is the argument of [2, Proposition 6.1].

3. Arithmetic

In this section $(m,n)=(3,2)$. For a point $(a/l,b)$ we say that *Case I is arithmetically impossible* if, for the direction given by (1), none of the starting points allowed in (T2) exists, and that *Case II is arithmetically impossible below* (ρ_*,σ_*) if no μ as in (T3) gives a direction of I smaller than (ρ_*,σ_*).

Lemma 3.1. *(a) At* $(16/3,10)$, $l=3$*: Case I is arithmetically impossible (direction* $(21,-11)$*); the directions allowed in Case II are* $(12,-5)$ *and* $(2,-1)$.

(b) At $C_1:=(5/6,1)$, $l=6$*:* $l-a/b=1$ *and* $\gcd(5,1)=1$.

(c) At $C_2:=(8/6,2)$, $l=6$*: Case I is arithmetically impossible (direction* $(27,-17)$*); the only direction allowed in Case II below* $(2,-1)$ *is* $(12,-7)$.

(d) At $(9/12,1)$, $l=12$*: Case I is arithmetically impossible (direction* $(16,-11)$*), and* $\gcd(9,1)=1$.

(e) At $C_4:=(14/6,4)$, $l=6$*: Case I is arithmetically impossible (direction* $(57,-32)$*), and so is Case II below* $(2,-1)$.

(f) At $C_5:=(17/6,5)$, $l=6$*: Case I is arithmetically impossible (direction* $(144,-79)$*), and* $\gcd(17,5)=1$.

(g) At $(7/3,4)$, $l=3$*: Case I is arithmetically impossible (direction* $(57,-32)$*), and* $\gcd(7,4)=1$.

(h) At $(11/3,6)$, $l=3$*: Case I is arithmetically impossible (direction* $(87,-52)$*), and* $\gcd(11,6)=1$.

Proof. Direct computation; for instance in (c), (1) reads $(\frac{4}{3}-\frac{1}{5})\rho+(2-\frac{1}{5})\sigma=0$, so $(\rho,\sigma)=(27,-17)$, $v(P)=6$, $v(Q)=4$, and $\frac{27}{6}\cdot 3a'=6$, $\frac{27}{6}k=4$, $\frac{27}{6}k=6$ have no integer solution; in Case II, $d=2$, $(\bar a,\bar b)=(4,1)$, the direction is orthogonal to $(4\mu-6,6\mu-6)$ with μ odd, $\mu\le 25$: $\mu=1$ gives no direction of I, $\mu=3$ gives $(2,-1)$, $\mu=5$ gives $(12,-7)$ with $\rho/\gcd(\rho,6)=2\le 2$, and $7\le\mu\le 25$ violates $\rho/\gcd(\rho,6)\le 2$. All items are checked by the ancillary script. □

By (T1), (T2), (T3) and Lemma 3.1(b),(e),(f),(g),(h):

(3) none of the points C_1, C_5 $(l=6)$, $(7/3,4)$, $(11/3,6)$ $(l=3)$ is the first entry of a regular corner of a $(3,2)$-pair in $L^{(l)}$, and C_4 $(l=6)$ is not the first entry of a regular corner of direction smaller than $(2,-1)$.

Lemma 3.2. *No $(3,2)$-pair in $L^{(6)}$ has a regular corner at $C_2=(8/6,2)$ of direction smaller than $(2,-1)$.*

Proof. By (T1) and Lemma 3.1(c) there would be a $(3,2)$-pair (P',Q') in $L^{(6)}$ with a regular corner $(C_2,(12,-7))$ of Case II. Put $z:=x^{7/12}y$. The point A' of (T3) is $C_2-h(7/12,1)$ with $h\in\{1,2\}$ and lies in $\frac{1}{6}\mathbb{Z}\times\mathbb{N}_0$, so $h=2$ and $A'=(1/6,0)$. Thus $\ell_{12,-7}(P')=x^{1/2}p(z)$ with $\deg p=6$, $p(0)\neq 0$. Since $x^{1/2}z^i\in L^{(6)}$ only for i even, and since $\ell_{12,-7}(P')$ is a cube by (T3), $p=\lambda'(z^2-c)^3$ with $c\neq 0$. A root λ of $z^2=c$ has $m_\lambda=3$, and the right-hand side of (2) is $\frac{3}{6}\cdot\frac{8\cdot12-2\cdot6\cdot7}{5}=\frac{6}{5}$. By (T5) some $(3,2)$-pair in $L^{(12)}$ has a regular corner at $(\frac{1}{6},0)+(\frac{7}{12},1)=(9/12,1)$, which contradicts (T1), (T2) and Lemma 3.1(d). □

4. Proof of Theorem 1.1

Assume that some $(3,2)$-pair in $L^{(3)}$ has a regular corner at $A_1:=(16/3,10)$. By (T1) and Lemma 3.1(a) there is a $(3,2)$-pair (P,Q) in $L^{(3)}$ with a regular corner $(A_1,(\rho,\sigma))$ of Case II, with $(\rho,\sigma)\in\{(12,-5),(2,-1)\}$. Let $A'=(a'/3,b')$ be the point of (T3).

The direction $(12,-5)$. Here $A'=A_1-h(5/12,1)$ with $0<h\le 10$, and $A'\in\frac{1}{3}\mathbb{Z}\times\mathbb{N}_0$ forces $h\in\{4,8\}$: $A'=(11/3,6)$ or $A'=(2,2)$. The second is excluded by $v_{1,-1}(A')\neq 0$. In the first, $v_{1,-1}(A')<0$, so we are in Case II.a) and $(11/3,6)$ is the first entry of a regular corner by (T4), contrary to (3).

The direction $(2,-1)$. Put $z:=x^{1/2}y$. Now $A'=A_1-h(1/2,1)$ with h even, so $b'\in\{0,2,4,6,8\}$ and $A'=(\frac{1}{3}+\frac{b'}{2},b')$. The line of the edge meets the first axis at $(1/3,0)$, so $\ell_{2,-1}(P)=x\,p(z)$ with $p=z^{3b'}\tilde p(z)$, $\tilde p(0)\neq 0$, $\deg\tilde p=3(10-b')$. Since $x\,z^i\in L^{(3)}$ only for i even, and since $\ell_{2,-1}(P)$ is a cube by (T3),

$$\tilde p(z)=\lambda'\prod_i(z^2-c_i)^{3\mu_i},\qquad c_i\in K^\times\text{ distinct},\quad \mu_i\ge 1,\quad \sum_i\mu_i=\frac{10-b'}{2}\le 5.$$

The right-hand side of (2) is $\frac{3}{3}\cdot\frac{16\cdot2-10\cdot3}{1}=2$, so every root λ of $\tilde p$, of multiplicity $3\mu_i\ge 3$, satisfies (2). By (T5), for every i some $(3,2)$-pair in $L^{(6)}$ has a regular corner at

$$C_{\mu_i}=(\tfrac{1}{3},0)+\mu_i(\tfrac{1}{2},1)=\Big(\frac{2+3\mu_i}{6},\,\mu_i\Big),$$

of direction smaller than $(2,-1)$. By (3) and Lemma 3.2, $\mu_i\notin\{1,2,4,5\}$, so $\mu_i=3$ for every i. Then $\frac{10-b'}{2}$ is a positive multiple of 3, hence $b'=4$ and $A'=(7/3,4)$. Since $v_{1,-1}(A')<0$ we are in Case II.a), and $(7/3,4)$ is the first entry of a regular corner by (T4), contrary to (3). □

Remark 4.1. The corner $C_3=(11/6,3)$, which continues the chain of [2], passes the tests used here: Case I is arithmetically possible there, with direction $(12,-7)$ and $\mathrm{st}(Q)=(1/6,0)$, $\mathrm{st}(P)=(5/6,1)$. In the shapes of Corollary 1.2 one has $b'=0$ (Case II.b)), the multiplicities μ_i add up to 5, and the contradiction comes from the roots other than the one continuing the chain, whose multiplicities add up to 2.

5. Scope

- **Proved here, relative to the cited results:** Theorem 1.1 and Corollary 1.2. The inputs are (T1)–(T5), that is [1, Propositions 1.13, 5.13, 5.14, 5.18, 5.19, Corollary 5.7, Remarks 3.1, 5.11], and [2, Theorem 2.20] for the corollary.
- **Verified by exact computation:** Lemma 3.1 and the numerical values in the proofs (ancillary file `verify_126_84.py` in the folder `anc`; Python standard library only).
- **Not claimed:** a statement on the plane Jacobian conjecture itself. The other shape of maximal degree 126 in [2], of degrees $(84, 126)$, is treated in the companion note [3], by a different method which rests on more inputs. The completeness of the list of [2] is not re-examined here.
- The argument has been checked only by the author with AI assistance (Claude, Anthropic); it has not been validated by professionals.

References

[1] C. Valqui, J. A. Guccione, J. J. Guccione, *On the shape of possible counterexamples to the Jacobian Conjecture*, J. Algebra **471** (2017), 13–74, doi:10.1016/j.jalgebra.2016.08.039; arXiv:1401.1784v3. Statement numbers refer to the arXiv version v3.

[2] J. A. Guccione, J. J. Guccione, R. Horruitiner, C. Valqui, *Some algorithms related to the Jacobian Conjecture*, arXiv:1708.07936v1.

[3] D. Niedbala Giraudin, *The degree pair* $(84, 126)$ *in the plane Jacobian problem: exclusion of the shape with final corner* $(19/7, 5)$, companion note (2026).

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