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MANY ODD-POWER RECIPROCAL SUMS OF LUCAS SEQUENCES ARE IRRATIONAL

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ABSTRACT. Let $a \geq 3$, let $U_0 = 0$, $U_1 = 1$, $U_{n+1} = aU_n - U_{n-1}$, and let $\delta = \pm 1$. We prove that 1 and the numbers $\sum_{n\geq 1} \delta^n/U_n^l$, $l = 1, 3, \ldots, A-1$, span a $\mathbb{Q}$-vector space of dimension at least $(\pi/\sqrt{12+\pi^2} + o(1))\sqrt{A}$. So infinitely many of these sums with l odd are irrational, for instance infinitely many of $\sum_n 1/F_{2n}^l$. With the known linear independence of 1, $\sum_n \delta^n/U_n$ and $\sum_n \delta^n/U_n^3$, one of $\sum_n \delta^n/U_n^l$, $l = 5, 7, \ldots, 23$, lies outside the span of these three numbers. The proof adapts the author's construction for odd q-zeta values at real quadratic units, with a weighted bilateral sum. No single sum such as $\sum_n 1/U_n^5$ is decided, and sequences of norm -1, such as the Fibonacci numbers, are not covered.

1. INTRODUCTION

Fix an integer $a \geq 3$ and let $U_0 = 0$, $U_1 = 1$, $U_{n+1} = aU_n - U_{n-1}$. Then $U_n = (\eta^n - \eta^{-n})/(\eta - \eta^{-1})$ with $\eta = (a + \sqrt{a^2-4})/2 > 1$, a unit of norm $+1$ of $K = \mathbb{Q}(\sqrt{a^2-4})$, and every unit $\eta > 1$ of norm $+1$ of a real quadratic field arises in this way, with $a = \eta + \eta^{-1}$. Examples are $U_n = F_{2n}$ ($a = 3$), $U_n = F_{kn}/F_k$ for even k ($a = L_k$) and $U_n = \mathcal{P}_{2n}/2$, where $\mathcal{P}_n$ are the Pell numbers ($a = 6$). Let $\delta \in \{1, -1\}$ and, for even $A \geq 4$ and $1 \leq r < A/2$,

$$\tau(A, r) = \frac{4\pi^2 r(A-2r) - 12A}{12A + \pi^2(A+4r^2)}.$$

Theorem 1.1. *For every even $A \geq 4$ and every integer r with $1 \leq r < A/2$,*

$$\dim_{\mathbb{Q}}\Big(\mathbb{Q} + \sum_{\substack{1\leq l\leq A-1\\ l \text{ odd}}} \mathbb{Q}\sum_{n\geq 1}\frac{\delta^n}{U_n^l}\Big) \geq 1 + \tau(A, r).$$

As $A \to \infty$ through even integers, this dimension is at least $(\pi/\sqrt{12+\pi^2} + o(1))\sqrt{A} = (0.67178\ldots + o(1))\sqrt{A}$. In particular, at least $\lceil 1 + \tau(A, r)\rceil - 1$ of the numbers $\sum_n \delta^n/U_n^l$ with odd $l \leq A-1$ are irrational, and infinitely many of the numbers $\sum_n \delta^n/U_n^l$, l odd, are irrational.

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The function τ is that of the author's dimension bound for odd q-zeta values at real quadratic units [12, Theorem 1.1]. The bound depends neither on a nor on δ.

Corollary 1.2. *Let A_k be the smallest even A with $\max_r(1+\tau(A,r)) > k-1$, so that Theorem 1.1 gives $\dim \geq k$ for the sums with odd $l \leq A_k - 1$. For $2 \leq k \leq 11$ the values are those of Table 1, attained at $r = k-1$; at every even $A < A_k$ and every r, $1 + \tau(A,r) \leq k-1$.*

k	A_k	r	$1+\tau(A_k,r)$	k	A_k	r	$1+\tau(A_k,r)$
2	4	1	1.2438	7	88	6	6.0650
3	12	2	2.1601	8	118	7	7.0518
4	24	3	3.0948	9	152	8	8.0296
5	40	4	4.0358	10	190	9	9.0012
6	62	5	5.0635	11	234	10	10.0097

TABLE 1. Thresholds of Corollary 1.2.

For the first new cases we use the following result, proved with a Lean formalisation in [13]: *the numbers* 1, $\sum_n \delta^n/U_n$ *and* $\sum_n \delta^n/U_n^3$ *are linearly independent over* $\mathbb{Q}$. Let V_3 be their $\mathbb{Q}$-span, so $\dim_{\mathbb{Q}} V_3 = 3$.

Corollary 1.3. (1) *At least one of the numbers $\sum_n \delta^n/U_n^l$ with $l \in \{5,7,\dots,23\}$ lies outside V_3; in particular it is irrational.* (2) *At least two of the numbers $\sum_n \delta^n/U_n^l$, $l \in \{5,7,\dots,39\}$, are linearly independent modulo V_3.* (3) *In general, the $\sum_n \delta^n/U_n^l$ with l odd, $5 \leq l \leq A_k - 1$, span a space of dimension at least $k-3$ modulo V_3.*

At $A = 12$ Theorem 1.1 gives $\dim \geq 3$, which is contained in [13]; part (1), from $A = 24$, is the first finite statement that goes beyond it.

Corollary 1.4. *Theorem 1.1 and Corollaries 1.2 and 1.3 hold for $\sum_n 1/F_{2n}^l$ and $\sum_n (-1)^n/F_{2n}^l$, for $\sum_n \delta^n/F_{kn}^l$ with $k \geq 2$ even, and for $\sum_n \delta^n/\mathcal{P}_{2n}^l$. For instance, one of $\sum_n 1/F_{2n}^l$, $l \in \{5,7,\dots,23\}$, lies outside $\mathbb{Q} + \mathbb{Q}\sum_n 1/F_{2n} + \mathbb{Q}\sum_n 1/F_{2n}^3$.*

The Lambert form. For $0 < |\nu| < 1$ and odd $j \geq 1$ put

$$W_j(\nu) = \sum_{\substack{m \geq 1 \\ m \text{ odd}}} \frac{m^{j-1}\nu^m}{1-\nu^m} = \zeta_\nu(j) - 2^{j-1}\zeta_{\nu^2}(j), \qquad \zeta_q(j) = \sum_{m \geq 1} \frac{m^{j-1}q^m}{1-q^m}.$$

Theorem 1.5. *Let $d \geq 2$ be squarefree, $K = \mathbb{Q}(\sqrt{d})$, and let $\nu \in \mathcal{O}_K^\times$ have norm $+1$ and $0 < |\nu| < 1$. For every even $A \geq 4$ and every integer r with $1 \leq r < A/2$,*

$$\dim_{\mathbb{Q}}\Big(\mathbb{Q} + \sum_{\substack{1 \leq j \leq A-1 \\ j \text{ odd}}} \mathbb{Q}\sqrt{d}\, W_j(\nu)\Big) \geq 1 + \tau(A,r).$$

Theorems 1.1 and 1.5 are equivalent by an elementary change of basis (Lemma 2.1, Remark 2.2). Theorem 1.5 does not follow from [12, Theorem 1.1], which concerns $1, \sqrt{d}\,\zeta_q(3), \dots, \sqrt{d}\,\zeta_q(A-1)$ at one unit q: the $W_j(\nu)$ mix the units ν and ν^2, and $j = 1$ occurs.

The method. In [12] the rational function $R_n(T;q)$ of Krattenthaler, Rivoal and Zudilin [11, §3] is summed over q^k and q^{-n-k}, $k \geq 1$, with opposite signs. This gives linear forms in the odd $\zeta_q(l)$ whose coefficients are invariant or anti-invariant under $q \mapsto 1/q$, up to a power of q. At a unit of norm $+1$, where Galois conjugation is $q \mapsto 1/q$, integer forms in the numbers $\sqrt{d}\,\zeta_q(l)$ are then small at both real places, and Nesterenko's criterion applies. We keep the rational function and its partial-fraction coefficients, at $q = \nu^2$, and change only the sum:

$$S_n(\nu) = \sum_{k\geq 1} \nu^k R(\nu^{2k}) - \sum_{k\geq 1} \nu^{-n-k} R(\nu^{-2n-2k}).$$

The weights turn the q-zeta values into the $W_j(\nu)$, the minus sign removes the even j, and $j = 1$ survives (Lemma 4.1); the reciprocity under $\nu \mapsto 1/\nu$ holds for all odd j, $j = 1$ included (Lemma 4.2). Coefficients, decay and denominators change only by factors $e^{O(n)}$, so τ is unchanged.

We cite [12] for the properties of R_n and prove what is new. Nothing is formalised. Besides [12] the proofs use Nesterenko's criterion [16, 8, 11] and the asymptotic $\sum_{\ell\leq n} \phi_{\mathrm{E}}(\ell) = 3n^2/\pi^2 + O(n \log n)$ [9, Theorem 330]; Corollary 1.3 also uses [13].

2. The change of basis

Lemma 2.1. *Let $\eta > 1$ be a unit of norm $+1$ of a real quadratic field, with trace a and Lucas sequence (U_n), let $\delta \in \{1, -1\}$, $\nu = \delta/\eta$, and let d be the squarefree part of $a^2 - 4$. For odd $l \geq 1$,*

(2.1)

$$\sum_{n\geq 1} \frac{\delta^n}{U_n^l} = (\eta - \eta^{-1})^l\, V_l, \qquad V_l := \sum_{n\geq 1} \frac{\nu^{nl}}{(1-\nu^{2n})^l} = \sum_{\substack{m\geq 1\\ m \text{ odd}}} \mathcal{P}_l(m) \frac{\nu^m}{1-\nu^m},$$

where $\mathcal{P}_l(m) = \prod_i (m^2 - i^2)/(2^{l-1}(l-1)!)$, the product over odd i with $1 \leq i \leq l-2$. Hence $V_l = \sum_j a_{l,j} W_j(\nu)$, over odd $j \leq l$, with $a_{l,j} = [m^{j-1}]\mathcal{P}_l \in \mathbb{Q}$ and $a_{l,l} \neq 0$; and for every odd $L \geq 1$,

$$\operatorname{span}_{\mathbb{Q}}\Big(1, \sum_n \frac{\delta^n}{U_n}, \dots, \sum_n \frac{\delta^n}{U_n^L}\Big) = \operatorname{span}_{\mathbb{Q}}\big(1, \sqrt{d}\,W_1(\nu), \dots, \sqrt{d}\,W_L(\nu)\big).$$

Proof. Since $1/U_n^l = (\eta - \eta^{-1})^l \eta^{-nl}(1-\eta^{-2n})^{-l}$ and l is odd, $\delta^n \eta^{-nl} = \nu^{nl}$ and $\eta^{-2n} = \nu^{2n}$. Expanding $(1-\nu^{2n})^{-l}$ and summing over n (absolutely convergent), $V_l = \sum_{i\geq 0} \binom{i+l-1}{l-1} \nu^{l+2i}/(1-\nu^{l+2i})$. With $m = l + 2i$, $\binom{(m-l)/2+l-1}{l-1} = \prod_{t=1}^{l-1}(m-l+2t)/(2^{l-1}(l-1)!) = \mathcal{P}_l(m)$, since the factors $m - l + 2t$ are the numbers $m \pm i$, i odd, $i \leq l-2$. As $\mathcal{P}_l$ vanishes at

$m = 1, 3, \dots, l-2$, the sum may run over all odd m. $\mathcal{P}_l$ is even of degree $l-1$ with leading coefficient $1/(2^{l-1}(l-1)!)$, which gives the triangular expansion. Finally $(\eta - \eta^{-1})^2 = a^2 - 4$, so $(\eta - \eta^{-1})^l = (a^2-4)^{(l-1)/2} f\sqrt{d}$ with $f \in \mathbb{Q}^\times$. □

For example $V_3 = (W_3 - W_1)/8$ and $V_5 = (9W_1 - 10W_3 + W_5)/384$.

Remark 2.2. By Lemma 2.1, Theorem 1.1 for (a, δ) is Theorem 1.5 at $\nu = \delta/\eta$. Conversely, for ν as in Theorem 1.5, $\eta = 1/|\nu| > 1$ is a unit of norm $+1$, its trace $a = \eta + \eta^{-1}$ is an integer > 2, and $\nu = \delta/\eta$ with δ the sign of ν. Since $\eta \in K \setminus \mathbb{Q}$, $\mathbb{Q}(\sqrt{a^2-4}) = K$, so the squarefree d of Lemma 2.1 is that of Theorem 1.5.

3. What is used from [12]

From now on $A \geq 4$ is even, $1 \leq r < A/2$, $n \geq 1$ and $e = (A-2r)n/2 \in \mathbb{Z}$. Write $(x;q)_m = \prod_{i=0}^{m-1}(1 - xq^i)$, Φ_ℓ for the ℓ-th cyclotomic polynomial, $d_n(q) = \prod_{\ell \leq n} \Phi_\ell(q)$, ϕ_{E} for Euler's function and $\mathcal{T}(n) = \sum_{\ell \leq n} \phi_{\mathrm{E}}(\ell)$. Let

$$R(T) = R_n(T;q) = \frac{(q;q)_n^{A-2r}\,(q^{-rn}T;q)_{rn}\,(q^{n+1}T;q)_{rn}\,T^e}{(T;q)_{n+1}^A}$$

be the rational function [12, (2.1)], with partial fractions [12, (3.1)]

$$R(T) = \sum_{j=0}^{n} \sum_{s=1}^{A} \frac{c_{s,j}(q)}{(1-q^jT)^s}, \qquad c_{s,j} \in \mathbb{Q}(q).$$

We use the following statements of [12] verbatim. Items (a)–(c) are identities for an indeterminate q; (d) and (e) hold for every complex q with $0 < |q| < 1$, where $\kappa = |q|$; (f) holds for every unit q of norm $+1$ of a real quadratic field with $0 < |q| < 1$.

(a) [12, Lemma 2.1]: $R(q^{-n}/T) = q^{An/2}T^A R(T)$.
(b) [12, Corollary 3.1]: $c_{s,j}(1/q) = q^{-e} c_{s,n-j}(q)$.
(c) [12, Lemma 3.2]: $d_n(q)^{A-s} c_{s,j}(q) \in \mathbb{Z}[q^{\pm 1}]$; and $1 - q^{\pm m}$ divides $d_n(q)$ in $\mathbb{Z}[q^{\pm 1}]$ for $1 \leq m \leq n$ [12, §3].
(d) The proof of [12, Lemma 6.1]: with $c_\pm = \prod_{i \geq 1}(1 \pm \kappa^i)$ and $C = (c_+/c_-)^{2A}$, $R(q^k) = 0$ for $1 \leq k \leq rn$, and $C^{-1}\kappa^{ke} \leq |R(q^k)| \leq C\kappa^{ke}$ for $k \geq rn+1$.
(e) [12, Lemma 9.2]: with $\rho = (1-\kappa)/2$ and constants $c_+, \gamma_\pm > 0$ depending only on κ,

$$|c_{s,j}(q)| \leq \rho^{s - A(j+1)} c_+^{A-2r+1} \gamma_+ \gamma_-^{-A} (1+\rho)^{rn+e} \kappa^{-\mathcal{E}(j)},$$

where $\mathcal{E}(j) \leq (A + 4r^2)n^2/8 + rn/2$.
(f) [12, Lemma 7.1, Remark 7.2, and §9 before (9.2)]: $\sigma(d_n(q)) = d_n(1/q) = -q^{-\mathcal{T}(n)} d_n(q)$, $\log|d_n(q)| = O(n)$, and $\mathcal{T}(n) = 3n^2/\pi^2 + O(n \log n)$ [9, Theorem 330].

We also use the trace argument of [12, Lemma 9.3(a)]: $\mathrm{Tr}_{K/\mathbb{Q}}(x/\sqrt{d}) \in \mathbb{Z}$ for $x \in \mathcal{O}_K$, d squarefree; and Nesterenko's criterion as stated in [12, §9, proof of Theorem 1.1], from [16], [8, Theorem A], [11, Proposition 2]: *if $\vartheta_1, \dots, \vartheta_N$ are real and there are integers $p_{j,n}$ with $\frac{1}{n^2}\log|\sum_j p_{j,n}\vartheta_j| \to \alpha < 0$ and $\limsup_n \frac{1}{n^2}\log|p_{j,n}| \le \beta$ for every j, then $\dim_{\mathbb{Q}} \sum_j \mathbb{Q}\vartheta_j \ge 1 - \alpha/\beta$.*

We apply all of this at $q = \nu^2$ with ν a unit of norm $+1$, $0 < |\nu| < 1$; then q is a unit of norm $+1$ with $0 < q < 1$ and $\mathbb{Q}(q) = K$. The sums S_n and coefficients P_l, P_0 of [12] are not used; ours are defined below. (In [12, §9] the letter ν denotes $\log(1/|q|)$; here we write $\lambda = \log(1/\kappa)$.)

Let $\mathcal{R}_n = \mathbb{Q}[\nu, \nu^{-1}, d_n(\nu^2)^{-1}]$ for an indeterminate ν. By (c), $c_{s,j}(\nu^2) \in \mathcal{R}_n$. Since the identity $d_n(1/q) = -q^{-\mathcal{T}(n)}d_n(q)$ of (f) holds in $\mathbb{Q}(q)$ (see also [12, §4]), $\mathcal{R}_n$ is stable under $\nu \mapsto 1/\nu$, and evaluation at $\nu \in \mathbb{C}$ with $0 < |\nu| \ne 1$ is a ring homomorphism $\mathcal{R}_n \to \mathbb{C}$, since the roots of $d_n(\nu^2)$ are roots of unity.

4. The weighted bilateral forms

Let $\nu \in \mathbb{C}$, $0 < |\nu| < 1$, $q = \nu^2$ and $\kappa = |q|$, and let $S_n = S_n(\nu)$ be as in the introduction. No argument is a pole, and by (a), $\nu^{-n-k}R(q^{-n-k}) = \nu^{-n-k}q^{A(n/2+k)}R(q^k) = \nu^k\nu^{(A-1)(n+2k)}R(q^k)$, so by (d) both series converge absolutely. For $s \ge 1$ let $p_s(\mu) = \prod_{t=1}^{s-1}(\mu + 2t - 1)/(2^{s-1}(s-1)!)$, so that $p_s(2i+1) = \binom{i+s-1}{s-1}$; p_s has degree $s-1$, nonnegative coefficients, and $p_s(1) = 1$. For odd l with $1 \le l \le A-1$ put, with $c_{s,j} = c_{s,j}(\nu^2)$ and sums over $1 \le s \le A$, $0 \le j \le n$,

$$P_l = 2\sum_{s,j} c_{s,j}\nu^{-j}[\mu^{l-1}]p_s(\mu), \qquad P_0 = \sum_{s,j} c_{s,j}\nu^{-j}B_{s,j},$$

$$B_{s,j} = -\sum_{t=1}^{j} \nu^t(1-q^t)^{-s} + \sum_{m=1}^{n-j} \nu^{-m}(1-q^{-m})^{-s}.$$

By (c), $P_l, P_0 \in \mathcal{R}_n$.

Lemma 4.1 (expansion)**.** *For $0 < |\nu| < 1$, $S_n = P_0 + \sum_l P_l\, W_l(\nu)$, the sum over odd l with $1 \le l \le A-1$.*

Proof. Positive side. By the partial fractions and with $m = j + k$,

$$\sum_{k\ge1} \frac{\nu^k}{(1-q^{j+k})^s} = \nu^{-j}\Big[\sum_{m\ge1} - \sum_{m=1}^{j}\Big]\frac{\nu^m}{(1-\nu^{2m})^s},$$

$$\sum_{m\ge1} \frac{\nu^m}{(1-\nu^{2m})^s} = \sum_{\mu \text{ odd}} \frac{p_s(\mu)\nu^\mu}{1-\nu^\mu},$$

by expanding $(1-\nu^{2m})^{-s} = \sum_i \binom{i+s-1}{s-1}\nu^{2mi}$ and putting $\mu = 2i+1$.

Negative side. With $m = n + k - j$, so that $\nu^{-n-k} = \nu^{-j}\nu^{-m}$,

$$\sum_{k\geq 1} \nu^{-n-k}(1-q^{j-n-k})^{-s} = \nu^{-j}\Big[\sum_{m\geq 1} - \sum_{m=1}^{n-j}\Big]\nu^{-m}(1-q^{-m})^{-s}.$$

Here

$$\frac{\nu^{-m}}{(1-\nu^{-2m})^s} = \frac{(-1)^s\nu^{m(2s-1)}}{(1-\nu^{2m})^s} = (-1)^s\sum_{i\geq 0}\binom{i+s-1}{s-1}\nu^{m(2i+2s-1)}.$$

For $\mu = 2i+2s-1$, $p_s(-\mu) = (-1)^{s-1}\binom{i+s-1}{s-1}$, and $p_s(-\mu) = 0$ for odd $\mu \leq 2s-3$. Hence $\sum_{m\geq 1}\nu^{-m}(1-q^{-m})^{-s} = -\sum_{\mu \text{ odd}} p_s(-\mu)\nu^\mu/(1-\nu^\mu)$.

Difference. Altogether

$$S_n = \sum_{s,j} c_{s,j}\nu^{-j}\Big[\sum_{\mu \text{ odd}}\big(p_s(\mu)+p_s(-\mu)\big)\frac{\nu^\mu}{1-\nu^\mu} + B_{s,j}\Big],$$

and $p_s(\mu)+p_s(-\mu) = 2\sum_{\lambda \text{ even}}[\mu^\lambda]p_s\cdot\mu^\lambda$ with $\lambda \leq s-1 \leq A-1$, so $\lambda \leq A-2$. Summing over μ gives $\sum_l P_l W_l$ with $l = \lambda+1$. All rearrangements are of absolutely convergent double series. □

No relation such as $\sum_{s,j} c_{s,j} = 0$ is used, since the weights make every series converge. The term $\lambda = 0$ gives W_1; a one-sided weighted sum would keep the even W's as well.

Lemma 4.2 (reciprocity)**.** *In $\mathcal{R}_n$: $P_l(1/\nu) = \nu^{n-2e}P_l(\nu)$ for every odd l, $l = 1$ included, and $P_0(1/\nu) = -\nu^{n-2e}P_0(\nu)$.*

Proof. The substitution $\nu \mapsto 1/\nu$ is $q \mapsto 1/q$, so $c_{s,j}(\nu^{-2}) = \nu^{-2e}c_{s,n-j}(\nu^2)$ by (b). With $j' = n-j$,

$$P_l(1/\nu) = 2\sum_{s,j}\nu^{-2e}c_{s,n-j}\nu^j[\mu^{l-1}]p_s = \nu^{n-2e}\cdot 2\sum_{s,j'}c_{s,j'}\nu^{-j'}[\mu^{l-1}]p_s.$$

For P_0, term by term,

$$B_{s,j}(1/\nu) = -\sum_{t=1}^{j}\nu^{-t}(1-q^{-t})^{-s} + \sum_{m=1}^{n-j}\nu^m(1-q^m)^{-s} = -B_{s,n-j}(\nu),$$

so

$$P_0(1/\nu) = \sum_{s,j}\nu^{-2e}c_{s,n-j}\,\nu^j\big(-B_{s,n-j}(\nu)\big) = -\nu^{n-2e}P_0(\nu). \qquad \square$$

If ν is a unit of norm $+1$ in K, then $\sigma\nu = 1/\nu$, so $\sigma(f(\nu)) = f(1/\nu)$ for $f \in \mathcal{R}_n$, and

$$\sigma\big(P_l(\nu)\big) = \nu^{n-2e}P_l(\nu), \qquad \sigma\big(P_0(\nu)\big) = -\nu^{n-2e}P_0(\nu). \tag{4.1}$$

Heuristically, $W_j(1/\nu) = -W_j(\nu) - (1-2^{j-1})\zeta(1-j)$ formally, and the constant vanishes for every odd j, including $j = 1$ (where $1-2^0 = 0$): all W_j behave as anti-invariant, which is why $\sqrt{d}\,W_j$ is the right twist.

Lemma 4.3 (denominators)**.** *Put* $F_A = 2^{A-1}(A-1)!$. *Then, in* $\mathcal{R}_n$, $F_A\, d_n(\nu^2)^{A-l}P_l \in \mathbb{Z}[\nu^{\pm1}]$ *for odd* $1 \le l \le A-1$, *and* $d_n(\nu^2)^A P_0 \in \mathbb{Z}[\nu^{\pm1}]$. *Hence* $D_n := F_A\, d_n(\nu^2)^A$ *satisfies* $D_nP_l, D_nP_0 \in \mathbb{Z}[\nu^{\pm1}]$.

Proof. Write $d_n = d_n(\nu^2)$. Since $2^{s-1}(s-1)!\,p_s \in \mathbb{Z}[\mu]$ and $2^{s-1}(s-1)!$ divides F_A for $s \le A$, we have $F_A[\mu^\lambda]p_s \in \mathbb{Z}$, and $[\mu^\lambda]p_s = 0$ unless $\lambda \le s-1$. So only $s \ge l$ contributes to P_l, and then $F_A d_n^{A-l}c_{s,j}\nu^{-j}[\mu^{l-1}]p_s = d_n^{s-l}(d_n^{A-s}c_{s,j})\nu^{-j}F_A[\mu^{l-1}]p_s$ lies in $\mathbb{Z}[\nu^{\pm1}]$ by (c). For P_0: $1-q^t$ divides d_n for $1 \le t \le n$ by (c), and $(1-q^{-t})^{-s} = (-q^t)^s(1-q^t)^{-s}$, so $d_n^s B_{s,j} \in \mathbb{Z}[\nu^{\pm1}]$, and $d_n^A c_{s,j}\nu^{-j}B_{s,j} = (d_n^{A-s}c_{s,j})(d_n^s B_{s,j})\nu^{-j}$. □

5. Coefficients and decay

Lemma 5.1 (coefficients)**.** *Let* $\nu \in \mathbb{C}$, $0 < |\nu| < 1$, $\kappa = |\nu|^2$, $\rho = (1-\kappa)/2$ *and* $\Gamma_n = \sum_{j=0}^{n}\sum_{s=1}^{A}|\nu|^{-j}|c_{s,j}(\nu^2)|$. *For every odd* l *with* $1 \le l \le A-1$, $|P_l(\nu)| \le 2\Gamma_n$ *and* $|P_0(\nu)| \le n|\nu|(1-\kappa)^{-A}\Gamma_n$, *and*

$$\Gamma_n \le (n+1)A\,|\nu|^{-n}\rho^{-A(n+1)}c_+^{A-2r+1}\gamma_+\gamma_-^{-A}(1+\rho)^{rn+e}\kappa^{-(A+4r^2)n^2/8-rn/2}.$$

Consequently, with an implied constant depending only on A, r *and* $|\nu|$,

$$\log\max\bigl(|P_0(\nu)|, \max_l |P_l(\nu)|\bigr) \le \frac{A+4r^2}{8}\,n^2\lambda + O(n). \tag{5.1}$$

Proof. Apply (e) at $q = \nu^2$. Since $\rho, \kappa < 1$ and $s \ge 1$, $\rho^{s-A(j+1)} \le \rho^{-A(n+1)}$ and $\kappa^{-\mathcal{E}(j)} \le \kappa^{-(A+4r^2)n^2/8-rn/2}$; with $|\nu|^{-j} \le |\nu|^{-n}$ and $(n+1)A$ terms this gives the bound for Γ_n. The coefficients of p_s are nonnegative with sum $p_s(1) = 1$, so $|[\mu^{l-1}]p_s| \le 1$ and $|P_l| \le 2\Gamma_n$. For $1 \le t \le n$, $|\nu^t(1-q^t)^{-s}| \le |\nu|(1-\kappa)^{-A}$ and $|\nu^{-t}(1-q^{-t})^{-s}| = |\nu|^{t(2s-1)}|1-q^t|^{-s} \le |\nu|(1-\kappa)^{-A}$; as $B_{s,j}$ has n terms, this gives the bound for P_0. All factors except $\kappa^{-(A+4r^2)n^2/8}$ are $e^{O(n)}$, since $|\nu|^{-n} = \kappa^{-n/2}$. □

Lemma 5.2 (decay)**.** *Fix* $\nu \in \mathbb{C}$ *with* $0 < |\nu| < 1$. *For all large* n, $S_n(\nu) \ne 0$ *and*

$$\log|S_n(\nu)| = -(rn+1)\bigl(e+\tfrac12\bigr)\lambda + O(1) = -\frac{r(A-2r)}{2}\,n^2\lambda + O(n).$$

Proof. By (d) and the computation at the start of Section 4, $S_n = \sum_{k \ge rn+1}\nu^k(1-\nu^{(A-1)(n+2k)})R(q^k)$. For large n, $|\nu|^{(A-1)(n+2rn+2)} \le \frac12$, so each factor $1-\nu^{(A-1)(n+2k)}$ has modulus in $[\frac12, \frac32]$, and $|\nu^k R(q^k)| \in [C^{-1}, C]\,\kappa^{k(e+1/2)}$ by (d). So the term $k = rn+1$ has modulus at least $(2C)^{-1}\kappa^{(rn+1)(e+1/2)}$, and the others have total modulus at most $\frac32 C\kappa^{(rn+2)(e+1/2)}/(1-\kappa^{e+1/2})$. The ratio of these bounds tends to 0 as $e \to \infty$, so $|S_n| \in [(4C)^{-1}, 4C]\,\kappa^{(rn+1)(e+1/2)}$ for large n; and $(rn+1)(e+\frac12) = \frac12 r(A-2r)n^2 + O(n)$. □

6. Proof of the theorems

Let d, K, ν be as in Theorem 1.5, $q = \kappa = \nu^2$, $\lambda = \log(1/\kappa)$, and $D_n = F_A d_n(q)^A \in \mathbb{Z}[\nu^{\pm 1}] \subset \mathcal{O}_K$ as in Lemma 4.3. By (f), $|\sigma D_n| = \kappa^{-A\mathcal{T}(n)}|D_n|$ and $\log|D_n| = O(n)$, so

$$\log|\mathrm{N}\, D_n| = \log|D_n\, \sigma D_n| = \frac{3A}{\pi^2} n^2\lambda + O(n\log n). \tag{6.1}$$

Balancing (as in [12, (9.3)]). Put $\varrho_n = |\nu|^{n-2e}|\sigma D_n|/|D_n|$, and let $m = m_n \in \mathbb{Z}$ be the unique integer with

$$2 \le |\nu|^{2m}\varrho_n < 2|\nu|^{-2}. \tag{6.2}$$

Put $D_n' = \nu^{-m} D_n \in \mathcal{O}_K$ and $M_n = |\nu|^{n-2e}|\sigma D_n'|$. Since $\sigma\nu = 1/\nu$, $M_n/|D_n'| = |\nu|^{2m}\varrho_n \in [2, 2|\nu|^{-2})$; so $|D_n'| \le M_n/2$ and $M_n^2 = (M_n/|D_n'|)\,|\nu|^{n-2e}|\,\mathrm{N}\, D_n|$, whence by (6.1)

$$\log M_n = \frac{3A}{2\pi^2} n^2\lambda + O(n\log n). \tag{6.3}$$

For odd l with $1 \le l \le A-1$ let $\ell_{0,n} = \mathrm{Tr}_{K/\mathbb{Q}}(D_n' P_0(\nu))$, $\ell_{l,n} = \mathrm{Tr}_{K/\mathbb{Q}}(D_n' P_l(\nu)/\sqrt{d})$ and $L_n = \ell_{0,n} + \sum_l \ell_{l,n}\sqrt{d}\, W_l(\nu)$.

Lemma 6.1. (a) $\ell_{0,n}, \ell_{l,n} \in \mathbb{Z}$. (b) $L_n = \big(D_n' - \nu^{n-2e}\sigma D_n'\big) S_n(\nu)$. (c) *For large* n, $L_n \ne 0$ *and* $\log|L_n| = \big(\frac{3A}{2\pi^2} - \frac{r(A-2r)}{2}\big) n^2\lambda + O(n\log n)$. (d) $\log\max\big(|\ell_{0,n}|, \max_l|\ell_{l,n}|\big) \le \big(\frac{3A}{2\pi^2} + \frac{A+4r^2}{8}\big) n^2\lambda + O(n\log n)$.

Proof. (a) By Lemma 4.3, $D_n' P_0(\nu), D_n' P_l(\nu) \in \mathcal{O}_K$; use $\mathrm{Tr}(\mathcal{O}_K) \subset \mathbb{Z}$ and the trace argument of Section 3. (b) Write $\mathrm{Tr}(z) = z + \sigma z$. The terms z add up to $D_n' S_n$ by Lemma 4.1. By (4.1) and $\sigma\sqrt{d} = -\sqrt{d}$, the conjugate terms are $\sigma(D_n' P_0) = -\nu^{n-2e}\sigma(D_n')P_0$ and $\sigma(D_n' P_l/\sqrt{d})\sqrt{d}\, W_l = -\nu^{n-2e}\sigma(D_n')P_l W_l$, which add up to $-\nu^{n-2e}\sigma(D_n')S_n$. (c) $|D_n' - \nu^{n-2e}\sigma D_n'| \in [M_n - |D_n'|, M_n + |D_n'|] \subset [\frac{1}{2}M_n, \frac{3}{2}M_n]$; use Lemma 5.2 and (6.3). (d) $|\ell_{0,n}| \le (|D_n'| + M_n)|P_0| \le \frac{3}{2}M_n|P_0|$, and likewise $|\ell_{l,n}| \le \frac{3}{2}M_n|P_l|/\sqrt{d}$. Use (5.1) and (6.3). □

Proof of Theorem 1.5. Apply Nesterenko's criterion to the $A/2+1$ numbers 1 and $\sqrt{d}\, W_j(\nu)$, $j = 1, 3, \dots, A-1$, and to the forms L_n, which are defined for every $n \ge 1$. By Lemma 6.1(c), (d), whose first estimate is two-sided because Lemma 5.2 and the asymptotic in (f) are,

$$\alpha = \Big(\frac{3A}{2\pi^2} - \frac{r(A-2r)}{2}\Big)\lambda, \qquad \beta = \Big(\frac{3A}{2\pi^2} + \frac{A+4r^2}{8}\Big)\lambda > 0.$$

If $\alpha \ge 0$ then $\tau(A,r) \le 0$ and there is nothing to prove. Otherwise the dimension is at least

$$1 - \frac{\alpha}{\beta} = 1 + \frac{r(A-2r)/2 - 3A/(2\pi^2)}{3A/(2\pi^2) + (A+4r^2)/8} = 1 + \tau(A,r). \qquad \square$$

Proof of Theorem 1.1. By Lemma 2.1 at $\nu = \delta/\eta$, the span in Theorem 1.1 is that of Theorem 1.5, with d the squarefree part of a^2-4. The asymptotic is [12, proof of Corollary 1.2]: with $t = \sqrt{12+\pi^2}/(2\pi)$ and $r = \lceil t\sqrt{A}\rceil$ (admissible for $A \geq 6$), $\tau(A,r)/\sqrt{A} \to 4\pi^2 t/(12+\pi^2+4\pi^2t^2) = \pi/\sqrt{12+\pi^2}$. If at most u of the sums with odd $l \leq A-1$ were irrational, the span would have dimension at most $u+1$; this gives the count, and $\max_r \tau(A,r) \to \infty$. □

Proof of Corollaries 1.3 and 1.4. Let V be the span of Theorem 1.1 at $A = A_k$, so $\dim V \geq k$ by Corollary 1.2. As V is V_3 plus the span of the sums with $5 \leq l \leq A_k - 1$, and $\dim V_3 = 3$ by [13, Theorem 1.1] (its case $(P,Q) = (a,1)$, for both δ; in its Lean statement $(a,E) = (a,-1)$), $\dim V/V_3 \geq k-3$; (1) and (2) are $k=4$ and $k=5$. For Corollary 1.4: F_{2n} is the sequence with $a=3$; for even k, $F_{kn} = F_k U_n$ with $\eta = \varphi^k$ and $a = L_k$, so $\sum_n \delta^n/F_{kn}^l = F_k^{-l}\sum_n \delta^n/U_n^l$; and $\mathcal{P}_{2n} = 2U_n$ with $a=6$. □

7. Thresholds and computations

Proof of Corollary 1.2. Since $12A + \pi^2(A+4r^2) > 0$, the inequality $1 + \tau(A,r) > k-1$ is equivalent to

$$\pi^2\big[4r(A-2r) - (k-2)(A+4r^2)\big] > 12A(k-1).$$

For $2 \leq k \leq 11$, every even $A \leq A_k$ and every $1 \leq r < A/2$, this inequality was decided in exact rational arithmetic from $3.14159265358979 < \pi < 3.14159265358980$ (no case was left undecided), with the results stated; for example $\max_r(1+\tau(22,r)) = 2.9499 < 3$. □

From $k=7$ ($A_7 = 88$) on, two values of r reach the threshold at A_k (at $A=234$, $r=11$ gives 10.0208), and $A_{12} = 282$. The ratio $\max_r(1+\tau(A,r))/\sqrt{A}$ is 0.6475, 0.6635, 0.6691, 0.6709 at $A = 10^2, \ldots, 10^5$.

Computations. The ancillary files contain scripts (python-flint, mpmath) and logs in two independent sets, the author's (`anc/`) and those of the AI referee pass described under Provenance, written without the author's code (`anc/referee/`). Only the threshold certification (`manyu_opt.py`, Arb balls; `referee/ref_opt.py`, exact rationals) enters a proof. The other checks confirm Lemma 2.1 ($a \leq 7$, $l \leq 23$), Lemmas 4.2 and 4.3 exactly in $\mathbb{Q}(\nu)$ (35 triples (A,r,n), $A \leq 24$), and, at five units such as $\pm\varphi^{-2}$ and $2-\sqrt{3}$ with (A,r) up to $(24,3)$ (120 rows, 8000 bits), the integrality of the ℓ's, Lemmas 4.1, 5.1, 5.2 and 6.1(b); the referee repeated the last group in 44 rows and computed the denominator profiles of Remark 8.1(c).

8. Remarks and limits

Remark 8.1. (a) *Norm* -1. For F_n, $\mathcal{P}_n$, and F_{kn} with k odd, the unit has norm -1, Galois conjugation acts as $\nu \mapsto -1/\nu$, and Lemma 4.2 gives no relation between $\sigma(P_l(\nu))$ and $P_l(\nu)$. These sequences are not covered; for them the results we know concern $l \leq 3$ [1, 15, 14, 19, 13].

(b) *Single values.* The theorems decide no individual sum such as $\sum_n 1/U_n^5$, and do not say which l give irrational values. We see no way to eliminate W_1 and W_3 from the forms L_n without multiplying their coefficients by factors e^{cn^2} with $c > 0$.

(c) *Sharper thresholds.* At $q = 1/p$ the bounds of [11] were improved by saving one power of d_n [10] and by a refinement of Nesterenko's criterion [8]; we have not adapted either. A full power of $d_n(\nu^2)$ cannot be saved in Lemma 4.3: in every case we computed exactly (in $\mathbb{Q}(\nu)$, $A \le 24$) the denominator of P_0 contains $\Phi_{2n}(\nu)^A$, so $d_n(\nu^2)^{A-1}P_0 \notin \mathbb{Z}[\nu^{\pm 1}]$ there. In these cases the exponent A occurs only at $\Phi_{2\ell}(\nu)$ with ℓ in roughly $(2n/3, n]$ and at a few small indices, so a partial saving is not excluded.

(d) *Odd A.* We have not proved a version of the construction for odd A (then e is a half-integer for odd n). Such a version, with the same function τ, would give nothing new. For fixed $r \ge 1$,

$$\tau(A, r) = \frac{(4\pi^2 r - 12)A - 8\pi^2 r^2}{(12 + \pi^2)A + 4\pi^2 r^2}$$

is a Möbius function of A whose denominator is positive for $A > 0$ and whose determinant $(4\pi^2 r - 12) \cdot 4\pi^2 r^2 + 8\pi^2 r^2(12 + \pi^2)$ is positive, so it is strictly increasing in A. Moreover $r < A/2$ implies $r < (A + 1)/2$. Hence $\tau(A, r) < \tau(A + 1, r)$, and Theorem 1.5 at the even number $A + 1$ concerns the same numbers $\sqrt{d}\,W_1, \sqrt{d}\,W_3, \ldots, \sqrt{d}\,W_A$.

9. Related work

André-Jeannin proved that $\sum_n x^n/w_n$ is irrational for a class of binary recurrences that includes every $U_n(P, \pm 1)$ with $x = \pm 1$, in particular $\sum_n \delta^n/U_n \notin \mathbb{Q}$ [1]. Tachiya [18] proved irrationality results for weight-one Lambert series at algebraic integers of Pisot type; Bundschuh and Väänänen [2] proved linear independence results for such series, with applications to reciprocal sums of Fibonacci and Lucas numbers, and [3] treats mixed weights one and two at rational bases; Duverney, Suzuki and Tachiya [5] proved linear independence results for reciprocal sums of Fibonacci and Lucas numbers along certain sequences of indices. For both signs of the norm the even powers $\sum_n U_n^{-2s}$ are transcendental [4], and their algebraic relations are known [6, 7]; these are values of quasi-modular forms, to which Nesterenko's theorem [17] applies. Wang [24] expresses $\sum_n n^p U_n^{-s}$ through derivatives of Eisenstein series when $p \equiv s \pmod 2$ (norm $+1$) and when p, s are even (norm -1); the sums $\sum_n U_n^{-l}$, l odd, of this note are outside these parity classes, and we know of no such expression for them or for their alternating versions.

At $q = 1/p$ with $p \in \mathbb{Z} \setminus \{0, \pm 1\}$, Krattenthaler, Rivoal and Zudilin proved a lower bound of order $\sqrt{A}$ for the dimension of the span of $1, \zeta_q(3), \zeta_q(5), \ldots, \zeta_q(A - 1)$ [11], refined in [10, 8]. The analogue at real quadratic units for $\sqrt{d}\,\zeta_q(l)$ is [12], of which this note is a variant.

For weight three, the author proved $\sum_n 1/F_n^3 \notin \mathbb{Q}$ and $\sum_n 1/U_n^3 \notin \mathbb{Q}$ for all real quadratic units [15, 14], and the linear independence of $1, \sum_n \delta^n/U_n, \sum_n (\delta\varepsilon)^n/U_n^3$, with ε the norm of the unit [13], all with Lean formalisations. The Hexagon contributor twinheterodox posted neighbouring results of weight at most three: the alternating sums $\sum_n (-1)^{n-1}/U_n^3$ [19], the weight-three Lambert sum $\sum_n n^2/(b^n - 1)$ at Pisot, Salem and other algebraic bases [22, 20], a weighted reciprocal Lucas sum [23] and the reciprocal Mersenne cubes [21]; these preprints are described by their author as primarily AI-generated, have no formal proofs, and were not checked by us. None of these works treats an odd power $l \geq 5$, and in our searches of Hexagon, arXiv, GitHub and Zenodo (November 2026) we found no earlier result on $\sum_n \delta^n/U_n^l$ for an odd $l \geq 5$.

Provenance

The construction was found by varying the summation in [12, §9], choosing weights so that the resulting values are those of Lemma 2.1. The proof was checked line by line in a separate whole-chain referee pass, which rebuilt the exact computations of Section 7 with independent code and found no error affecting a statement. Nothing in this note is formalised. Its external inputs are those of [12, Theorem 1.1] and the lemmas listed in Section 3; the Lean-verified result of [13] enters only Corollary 1.3. The referee pass was itself carried out by an AI model (a separate Claude session, without access to the author's code); it is not a peer review.

Use of AI tools. This work was carried out with substantial assistance from AI systems (Anthropic's Claude models), which were used to search the literature, to develop and check the proofs, to run the computations and to draft this text. The author directed the project and is responsible for its content.

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