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PRODUCTS OF TWO LOGARITHMS AND ARITHMETIC HOLONOMY BOUNDS

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Abstract. We prove, assuming the arithmetic holonomy bound of Calegari, Dimitrov and Tang (Theorem 7.0.1 of their preprint arXiv:2408.15403), that 1, $\log(1+1/q)$, $\log(1-1/q)$ and $\log(1+1/q)\log(1-1/q)$ are linearly independent over $\mathbb{Q}$ for every integer $26 \le q \le 53$. Together with Hata's theorem for $q \ge 54$ this gives the four-term independence for every $q \ge 26$. In particular the seven numbers $\log(1+1/N)\log(1+1/(N+1))$, $25 \le N \le 31$, are irrational; irrationality was previously known for $N \ge 32$ (Lysov). The proof feeds Lysov's explicit type II Hermite–Padé forms into the holonomy bound, and replaces the arithmetic of the forms by an analytic input: the monodromy of a minimal differential operator of order 13, which we show to be irreducible by interval-arithmetic certificates. Conditionally on the published form of the Calegari–Dimitrov–Tang argument for products of two logarithms, we also widen their Theorem C from $|m/n-1| < 10^{-6}$ to $|m/n-1| < 3\cdot 10^{-6}$, which gives infinitely many new irrational products $\log(1+1/m)\log(1+1/n)$ with $|m-n| \ge 2$, and we show by pooling several pairs that at most one exceptional pair class exists with $|m/n-1| < 5\cdot 10^{-5}$, and at most two with $|m/n-1| < 10^{-4}$. All certificates are exact or rigorous ball arithmetic and can be reproduced from the accompanying code at `https://github.com/telaviv/log-products-irrationality`.

1. Introduction

For an integer $q \ge 2$ put

$$\theta_q = \log\Big(1+\frac{1}{q}\Big)\log\Big(1-\frac{1}{q}\Big).$$

Hata [3, Cor. 6.3] showed that 1, $\log(1+1/q)$, $\log(1-1/q)$, θ_q are linearly independent over $\mathbb{Q}$ for every integer $q \ge 54$, with an effective linear independence measure; in particular $\theta_q \notin \mathbb{Q}$ for every integer $q \ge 54$ [3, Thm. 0.1]. Lysov [6] found an explicit formula for the common denominator of the type II Hermite–Padé approximants to $(\log(1+1/z), \log(1-1/z), \log(1+1/z)\log(1-1/z))$ and deduced that θ_q is irrational for every

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$q \geq 33$ [6, 7] ([6] is a preprint; we found no journal version as of 19 November 2026). Calegari, Dimitrov and Tang [1, Thm. C, Thm. 14.0.1] proved, by a new arithmetic holonomy bound, that 1, $\log(1+1/m)$, $\log(1+1/n)$ and $\log(1+1/m)\log(1+1/n)$ are linearly independent over $\mathbb{Q}$ whenever $m \neq n$ are integers in $\mathbb{Z}\backslash\{-1,0\}$ with $|m/n-1| < 10^{-6}$. In [1, §15.1.10] they suggest forming "the generating function of the special linear forms obtained from evaluating a regular sequence of functional Hermite–Padé approximants" and reading it through their holonomy bounds. This paper carries out that scheme for Lysov's forms, and pushes the constant 10^{-6} of [1, Thm. C].

Dependence on one preprint. Every result in this paper depends on the preprint [1] (arXiv:2408.15403v2, 16 September 2024; still the current version on 19 November 2026), which to our knowledge has not been refereed. Theorem 1 and Corollary 2 use only its general holonomy bound [1, Thm. 7.0.1], which we apply as stated. Theorems 3 and 4 also use the specific construction of [1, §14] for products of two logarithms, together with [1, Thm. 6.0.2, Cor. 9.0.19, Lemma 12.0.1]. That construction contains a step that is asserted rather than proved in [1], the irreducibility of a certain differential operator $\mathcal{L}_{a,b}$, and we inherit it; see Section 5. Each theorem below names exactly what it assumes.

1.1. **Results.**

Theorem 1. *Assume* [1, Thm. 7.0.1]. *For every integer q with $26 \leq q \leq 53$, the numbers*

$$1, \qquad \log\Big(1+\frac{1}{q}\Big), \qquad \log\Big(1-\frac{1}{q}\Big), \qquad \log\Big(1+\frac{1}{q}\Big)\log\Big(1-\frac{1}{q}\Big)$$

are linearly independent over $\mathbb{Q}$.

With Hata's theorem [3] for $q \geq 54$, the same holds for every integer $q \geq 26$. Precisely, [3, Cor. 6.3] states: let $q \geq 54$ be an integer; for every $\varepsilon > 0$ there is an effectively computable $H_1 = H_1(\varepsilon, q)$ such that

$$\Big|n_0 + n_1\log\Big(1+\frac{1}{q}\Big) + n_2\log\Big(1-\frac{1}{q}\Big) + n_3\theta_q + n_4\Lambda\Big(\frac{1}{q}\Big)\Big| \geq H^{-\sigma(q^2)/\tau(q^2)-\varepsilon}$$

for all $(n_0,\dots,n_4) \in \mathbb{Z}^5$ with $H = \max_{1\leq m\leq 4}|n_m| \geq H_1$. Here $\Lambda(x) = \mathrm{Li}_2((1+x)/2) - \mathrm{Li}_2((1-x)/2) - \log 2\cdot\log((1+x)/(1-x))$ [3, (3.5)], and σ, τ are explicit [3, (6.3), (6.4)]; the bound $q \geq 54$ comes from Hata's $\tau(k) > 0$ for every integer $k \geq 2909$ (and $53^2 < 2909 \leq 54^2$), and $\sigma(k)/\tau(k) \to 4$ as $k \to \infty$. Taking $n_4 = 0$ gives the four-term independence; in fact 1, $\log(1+1/q)$, $\log(1-1/q)$, θ_q, $\Lambda(1/q)$ are linearly independent over $\mathbb{Q}$. Hata states this for integers q only: Corollary 6.3 is the case $k = q^2$ of his Theorem 6.2, the analogous statement at $1/\sqrt{k}$ for every integer $k \geq 2909$.

Since $\log(1-1/q) = -\log(1+1/(q-1))$, Theorem 1 at $q = N+1$ contains the irrationality of the product of two consecutive logarithms $\log(1+1/N)\log(1+1/(N+1)) = -\theta_{N+1}$:

Corollary 2. *Assume* [1, Thm. 7.0.1]. *For every integer N with $25 \le N \le 31$,*

$$\log\Big(1+\frac{1}{N}\Big)\log\Big(1+\frac{1}{N+1}\Big) \notin \mathbb{Q}.$$

These are seven new irrational numbers. By Theorem 1 for $32 \le N \le 52$ and Hata's theorem for $N \ge 53$ (and, independently, Lysov's theorem [6] for $N \ge 32$), the product is irrational for every $N \ge 25$.

The second group of results concerns CDT's two-parameter family. For $m \in \mathbb{Z} \setminus \{-1, 0\}$ write $\ell_m = \log(1 + 1/m)$.

Theorem 3. *Assume the following parts of* [1] *as published: Theorem 6.0.2, Corollary 9.0.19, Lemma 12.0.1 and §14, including the irreducibility of the operator $\mathcal{L}_{a,b}$ and the reduction in the proof of Lemma 14.3.1. Let $m \ne n$ be integers in $\mathbb{Z} \setminus \{-1, 0\}$ with $|m/n - 1| < 3 \cdot 10^{-6}$. Then 1, ℓ_m, ℓ_n and $\ell_m\ell_n$ are linearly independent over $\mathbb{Q}$; in particular $\ell_m\ell_n$ is irrational.*

Theorem 4. *Make the assumptions of Theorem 3. Call a pair (m, n) of distinct integers in $\mathbb{Z} \setminus \{-1, 0\}$* exceptional *if $1, \ell_m, \ell_n, \ell_m\ell_n$ are linearly dependent over $\mathbb{Q}$, and take classes of pairs modulo $(m, n) \sim (n, m) \sim (-m - 1, -n - 1)$. Then:*

- (i) *at most one exceptional class has a representative with $|m/n - 1| < 5 \cdot 10^{-5}$;*
- (ii) *at most two exceptional classes have a representative with $|m/n - 1| < 10^{-4}$;*
- (iii) *every exceptional class has $|m/n - 1| \ge 3 \cdot 10^{-6}$ in all its representatives (Theorem 3);*
- (iv) *in the ranges of (i) and (ii), every exceptional class has $|m - n| \ge 2$, by Hata's theorem* [3, Cor. 6.3].

The relation $(m, n) \sim (-m - 1, -n - 1)$ comes from $\ell_{-k} = -\ell_{k-1}$: the two pairs give the same four numbers up to sign. For $1 \le N < M$ with $d = M - N$, the most permissive of the four representatives of $\{N, M\}$ is $(m, n) = (-N - 1, -M - 1)$, with $|m/n - 1| = d/(M + 1)$, so the pair is covered by Theorem 3 if and only if $d < 3 \cdot 10^{-6}(M + 1)$, and by [1, Thm. C] if and only if $d < 10^{-6}(M + 1)$. Consecutive pairs ($d = 1$) in either range have $q = N + 1 > 3 \cdot 10^5$ and are covered by Hata's theorem. So the new irrational numbers of Theorem 3 are

$$\ell_N\ell_M, \qquad d = M - N \ge 2, \qquad 10^{-6}(M + 1) \le d < 3 \cdot 10^{-6}(M + 1), \tag{1.1}$$

that is, $d \cdot 10^6/3 < M + 1 \le d \cdot 10^6$: about $\frac{2}{3}d \cdot 10^6$ numbers for each $d \ge 2$, infinitely many in all. For example: for $d = 2$, 666,664 $\le N \le$ 1,999,997; for $d = 3$, 999,997 $\le N \le$ 2,999,996; for $d = 10$, 3,333,323 $\le N \le$ 9,999,989 (exact integer computation over all four representatives, `code/thm3/f7_newpairs.py`). None of them is a known number in disguise: $(N + 1)/N$ is not a perfect power of a rational number, so ℓ_N and ℓ_M are multiplicatively independent and $\ell_N\ell_M$ is not of the form $c \log^2 r$. Theorem 4

names no new number; it is a statement about the family. For example, all but at most one of the numbers $\log(1+1/N)\log(1+1/(N+2))$ with $39{,}998 \le N \le 666{,}663$ are irrational, and all but at most two with $19{,}998 \le N \le 666{,}663$.

1.2. **Index conventions.** Four parametrisations are in use in the literature and in our sources. We use the following dictionary throughout.

convention	consecutive pair	number
positive form	$(N, N+1)$, $N \ge 1$	$\ell_N \ell_{N+1}$
Hata, Lysov, Thm. 1	$q = N+1$	$\theta_q = -\ell_N \ell_{N+1}$
CDT pair (m,n)	$(N, N+1) \sim (-N-2, -N-1)$	$\ell_m \ell_n$
CDT, odd form (a,b)	$(2N+1, 2N+3)$	$a = 2m+1$, $b = 2n+1$

Here $\ell_m = \log \frac{a+1}{a-1}$ when $a = 2m+1$. In Sections 3–4 the letter n is the index of Lysov's forms $(P_{j,n}, R_{j,n})$; in Sections 5–6 (m,n) is a CDT pair, as in the statements above. Lysov's paper writes m for our q, and Zeilberger and Zudilin [10] write $a = q-1$.

1.3. **What is new.**

- *New irrational numbers.* The seven products $\ell_N \ell_{N+1}$, $25 \le N \le 31$ (Corollary 2); no method gave their irrationality before. The products (1.1) (Theorem 3, conditional on [1, §14]), together with the four-term independence for each of them.
- *New linear independence at known irrational numbers.* The four-term independence of Theorem 1 for $26 \le q \le 32$ is new. For $33 \le q \le 53$ we know of no explicit statement; ours is the first, and its proof does not use Lysov's arithmetic. It may follow from Lysov's construction by the classical type II criterion, provided his refined denominator estimate covers the numerators $P_{1,n}$, $P_{2,n}$ as well (we have not checked this). Zeilberger and Zudilin [10, p. 8] state that the linear independence of $1, \log(a/(a+1)), \log((a+1)/(a+2))$, known for $a \ge 53$ by Hata, was "recently improved, though somewhat implicitly, by Volodya Lysov to $a \ge 32$"; that statement concerns the linear forms, not the product.
- *Nothing new for consecutive pairs in CDT's range or ours*: those are covered by Hata's theorem.
- *A family statement* (Theorem 4), which names no number.
- *By-products*: for every $q \ge 2$ the characteristic polynomial of Lysov's recurrence is an explicit quartic (Proposition 3.4); all local monodromies of the order-34 operator at its finite nonzero singular points are reflections (Proposition 4.3); an explicit form of the factor Φ_6 of [1, Lemma 14.3.1] and its 2-adic nonvanishing at odd pairs (Section 5.5); a misprinted factor in [1, (14.2.2), (14.2.3)].

A literature search on 19 November 2026 (arXiv, Hexagon, GitHub, Zenodo, general web search) found no other work on products of two real logarithms

of this kind, on the constant 10^{-6} of [1, Thm. C], or on the Hata–Lysov family since [6]. The search is not exhaustive; in particular we could not find an abstract of a 2025 talk by Lysov on this subject.

1.4. **Status of the results.** The proofs are computer-assisted. Every computation that a proof uses is either an exact identity over $\mathbb{Q}$ (checked with SymPy or python-flint) or a rigorous ball-arithmetic enclosure (Arb, through python-flint). The chain has been checked by AI referees at each decision gate and once as a whole (Section 9); it has not yet been read by a human expert, and it is not formalised. In the notation of our working notes ("gate-refereed": checked by one adversarial AI referee when the result was accepted; "chain-refereed": covered by the final whole-chain AI referee; "owed": not yet done), the status labels are:

- Theorem 1 and Corollary 2: *proved, computer-assisted (exact symbolic identities and Arb certificates), given* [1, Thm. 7.0.1] *(preprint); gate-refereed once; chain-refereed; human read and Lean owed.*
- Theorem 3: *proof sketch plus certified computation; conditional on CDT as published (Thm. 6.0.2, Cor. 9.0.19, Lemma 12.0.1 and §14, in particular the irreducibility of $\mathcal{L}_{a,b}$ and the reduction in the proof of Lemma 14.3.1); Φ_6 factor of Lemma 14.3.1 via Section 5.5 (proved, exact computer algebra; not gate-refereed); gate-refereed once; chain-refereed; not in Lean, no human read.*
- Theorem 4: *proof sketch plus certified computation; conditional on CDT as published (Thm. 6.0.2, Cor. 9.0.19, Lemma 12.0.1 and §14, in particular the irreducibility of $\mathcal{L}_{a,b}$ and the reduction in the proof of Lemma 14.3.1, reused pair by pair); Φ_6 factor of Lemma 14.3.1 via Section 5.5 (proved, exact computer algebra; not gate-refereed); part (iv) adds Hata's theorem* [3, Cor. 6.3] *(checked against the paper); Lemma 6.1 proved, given* [1, Lemma 14.2.4]*; Lemma 5.1 ($\varepsilon \le 10^{-4}$) and the ray lemma proved; Lemma 6.2 proof sketch (classical); certificates exact (Arb and exact rationals); gate-refereed once; chain-refereed; not in Lean; no human read.*

1.5. **Method.** For Theorem 1 we start from Lysov's forms $R_{j,n}(q) = Q_n(q)f_j(q) - P_{j,n}(q)$, $j = 1, 2, 3$, with $(f_1, f_2, f_3) = (\log(1 + 1/q), \log(1 - 1/q), \theta_q)$, and their generating functions $H_j(y) = \sum_n R_{j,n}(q)y^n$. A rational relation $r_0 + a_1f_1 + a_2f_2 + a_3f_3 = 0$ makes $-\sum_j a_jH_j$ a power series with rational coefficients of controlled denominators, which continues analytically to $\mathbb{C} \setminus [y_2, \infty)$ for an explicit y_2. The holonomy bound [1, Thm. 7.0.1], applied with the Koebe map onto $\mathbb{C} \setminus [y_2, \infty)$ and the 14 functions $1, H, \vartheta H, \dots, \vartheta^{12}H$ ($\vartheta = y\,d/dy$), gives a contradiction once these 14 functions are linearly independent over $\mathbb{C}(y)$ and $q \ge 26$. Their independence follows from the irreducibility of the monodromy of the minimal operator L_{13} of the H_j (order 13), which we certify per q from rigorous

numerical analytic continuation, using that the local monodromies at the singular points y_2, y_3 are reflections. No arithmetic of the forms beyond the elementary denominator bound $\operatorname{lcm}(1,\dots,4n)\operatorname{lcm}(1,\dots,2n)$ is needed.

For Theorem 3 we keep CDT's construction and change only the map: we construct, by optimisation, a polynomial-exponential map ψ of the disc for which $h\circ\psi$ (h the hauptmodul of $\Gamma_0(2)$) is admissible for every pair with $|m/n-1|<3\cdot10^{-6}$ at once, and certify the holonomy inequality in its partition form [1, Thm. 6.0.2] by interval arithmetic. For Theorem 4 we put the functions of $K=2$ or 3 hypothetical exceptional pairs into one holonomy bound, after proving that they are jointly independent.

Outline. Section 2 recalls the holonomy bounds. Section 3 treats Lysov's forms: contact orders, denominators, recurrences and the differential equations. Section 4 proves Theorem 1 and Corollary 2. Sections 5 and 6 prove Theorems 3 and 4. Section 7 lists the computer-assisted steps and how to reproduce them, Section 8 the limits of the method, and Section 9 provenance and the use of AI tools.

2. The holonomy bounds

We use the notation of [1, §6, §7]. For $N\ge1$ write $[1,\dots,N]=\operatorname{lcm}(1,2,\dots,N)$. Let $\mathfrak{m}\ge1$, let $b=(b_{i,j})_{1\le i\le\mathfrak{m},\,1\le j\le r}$ be an array of nonnegative reals whose columns have the form $0=b_{1,j}=\cdots=b_{u_j,j}<b_{u_j+1,j}=\cdots=b_{\mathfrak{m},j}$, and let $e\in\mathbb{N}^{\mathfrak{m}}$. A formal power series $f_i\in\mathbb{Q}[[y]]$ has *denominator type* $(b_{i,\cdot},e_i)$ if

$$f_i(y)=a_{i,0}+\sum_{n\ge1}\frac{a_{i,n}\,y^n}{n^{e_i}\,[1,\dots,b_{i,1}n]\cdots[1,\dots,b_{i,r}n]},\qquad a_{i,n}\in\mathbb{Z}$$

Let $\sigma_i=\sum_j b_{i,j}$ be the row sums (nondecreasing in i, by the shape of the columns), and put

$$\tau(b)=\frac{1}{\mathfrak{m}^2}\sum_{i=1}^{\mathfrak{m}}(2i-1)\sigma_i,\qquad \tau(b;e)=\tau(b)+\tau^\sharp(e),$$

with $\tau^\sharp$ as in [1, Thm. 6.0.2, Def. 6.0.1]; $\tau^\sharp(0)=0$, and $\tau^\sharp(e)$ depends only on $\mathfrak{m}$, $\sum_i e_i$ and $\max_i e_i$.

For a holomorphic map φ from a neighbourhood of the closed unit disc $\overline{\mathbb{D}}$ to $\mathbb{C}$ with $\varphi(0)=0$ put

$$T(\varphi)=\iint_{\mathbb{T}^2}\log|\varphi(z)-\varphi(w)|\,d\mu_{\mathrm{Haar}}(z)\,d\mu_{\mathrm{Haar}}(w).$$

Theorem 2.1 ([1, Thm. 7.0.1])**.** *Let $f_1,\dots,f_{\mathfrak{m}}\in\mathbb{Q}[[y]]$ be linearly independent over $\mathbb{Q}(y)$, of denominator types given by (b,e), and let $\varphi:(\mathbb{D},0)\to(\mathbb{C},0)$ be holomorphic with $\log|\varphi'(0)|>\max\{\sigma_{\mathfrak{m}},\tau(b;e)\}$, such that every*

$f_i(\varphi(z))$ is the germ of a meromorphic function on $|z|<1$. Then

$$\mathfrak{m} \le \frac{T(\varphi)}{\log|\varphi'(0)| - \tau(b;e)}.$$

If all f_i are known to be holonomic, the hypothesis $\log|\varphi'(0)| > \max\{\sigma_{\mathfrak{m}}, \tau\}$ may be relaxed to $\log|\varphi'(0)| > \tau$.

(In [1] the map is defined on $\mathbb{D}$ and T is defined by a limit; we only use maps holomorphic on a neighbourhood of $\overline{\mathbb{D}}$.)

Theorem 2.2 ([1, Thm. 6.0.2], partition form)**.** *Let $f_1, \dots, f_{\mathfrak{m}}$ and (b,e) be as in Theorem 2.1, all f_i holonomic. Let $\varphi_0, \dots, \varphi_l$ be holomorphic maps $(\mathbb{D},0) \to (\mathbb{C},0)$ with $|\varphi_0'(0)| < \dots < |\varphi_l'(0)|$ and $\log|\varphi_l'(0)| > \tau(b;e)$, such that every $f_i(\varphi_k(z))$ is meromorphic on $|z|<1$. Let $0 = \vartheta_0 < \vartheta_1 < \dots < \vartheta_l < \vartheta_{l+1} = \mathfrak{m}$, and let $g : [0,1) \to \mathbb{R} \cup \{-\infty\}$ be $g(t) = \log|\varphi_k(e^{2\pi i(\mathfrak{m}t - \vartheta_k)/(\vartheta_{k+1}-\vartheta_k)})|$ for $t \in [\vartheta_k/\mathfrak{m}, \vartheta_{k+1}/\mathfrak{m})$. Then*

$$\mathfrak{m} \le \frac{\displaystyle\int_0^1\int_0^1 \max(g(s),g(t))\,ds\,dt + \frac{1}{\mathfrak{m}}\sum_{k=1}^{l} \vartheta_k^2 \log\frac{|\varphi_k'(0)|}{|\varphi_{k-1}'(0)|}}{\log|\varphi_l'(0)| - \tau(b;e)}.$$

With $l=0$ (the *trivial partition*) the numerator is the rearrangement integral $R(\varphi) = \iint \max(g(s),g(t))\,ds\,dt$ with $g(s) = \log|\varphi(e^{2\pi i s})|$. The double integral depends only on the distribution of g, which for a partition is the mixture $\sum_k \frac{\vartheta_{k+1}-\vartheta_k}{\mathfrak{m}} \cdot (\text{law of } \log|\varphi_k| \text{ on } \mathbb{T})$. We use Theorem 2.1 in Section 4 and Theorem 2.2 in Sections 5 and 6.

Lemma 2.3. *If φ is univalent on a neighbourhood of $\overline{\mathbb{D}}$ and $\varphi(0)=0$, then $T(\varphi) = \log|\varphi'(0)|$.*

Proof. The function $(z,w) \mapsto \log|(\varphi(z)-\varphi(w))/(z-w)|$ is pluriharmonic on a neighbourhood of $\overline{\mathbb{D}}^2$ (the quotient is holomorphic and zero-free there, by univalence), so its mean over $\mathbb{T}^2$ is its value $\log|\varphi'(0)|$ at $(0,0)$; and $\iint_{\mathbb{T}^2} \log|z-w| = 0$. □

3. Lysov's forms

3.1. **Definitions.** For integers $n \ge 0$ and l put

$$c_{n,l} = \sum_k (-1)^{n+l} \binom{n+l}{l}\binom{n}{l-k}\binom{2n+k}{2k}\binom{n+k}{k}\binom{4n+2k}{2n+k},$$

the sum over $0 \le k \le 2n$ with $0 \le l-k \le n$ (so $c_{n,l}=0$ unless $0 \le l \le 3n$), and

$$Q_n(z) = \sum_{l=0}^{3n} c_{n,l}\, z^{2l} \in \mathbb{Z}[z],$$

an even polynomial of degree $6n$ (Lysov's Q_{6n}, [6, formula (21) and Lemma 8]; we use the formula as a definition). Let $f_1(z) = \log(1+1/z)$,

$f_2(z) = \log(1 - 1/z)$ and $f_3 = f_1 f_2$, as Laurent series at $z = \infty$. Let $P_{j,n}$ be the polynomial part of $Q_n f_j$, and $R_{j,n} = Q_n f_j - P_{j,n} = O(z^{-1})$. Since Q_n is even and $f_2(z) = f_1(-z)$, we have $P_{2,n}(z) = P_{1,n}(-z)$. Explicitly, with $A_N = \sum_{i=1}^{N} (-1)^{i+1}/i$,

$$\begin{aligned} P_{1,n}(z) &= \sum_l c_{n,l} \sum_{m=1}^{2l} \frac{(-1)^{m+1}}{m} z^{2l-m}, \\ P_{3,n}(z) &= \sum_l c_{n,l} \sum_{m=1}^{l} \Big(-\frac{A_{2m-1}}{m}\Big) z^{2(l-m)}, \end{aligned} \tag{3.1}$$

because $[z^{-2m}] f_3 = -A_{2m-1}/m$ and the odd negative powers of z do not occur in f_3.

The functions are Cauchy transforms: for $|z| > 1$,

$$f_1(z) = \int_{-1}^{0} \frac{dx}{z - x}, \qquad f_2(z) = -\int_{0}^{1} \frac{dx}{z - x}, \qquad f_3(z) = \int_{-1}^{1} \frac{\omega(x)\, dx}{z - x},$$

with ω odd and $\omega(x) = -\log(1 + 1/x)$ on $(0, 1)$. Write μ_j for the three (signed) measures; their total variations are $1, 1, 4\log 2$. Then $R_{j,n}(z) = \int Q_n(x)\, d\mu_j(x)/(z - x)$.

3.2. **Contact orders.**

Lemma 3.1 (Lysov)**.** *For all $n \geq 0$ and $j = 1, 2, 3$: $\int x^k Q_n(x)\, d\mu_j(x) = 0$ for $0 \leq k < 2n$. Equivalently $R_{j,n}(z) = O(z^{-2n-1})$ as $z \to \infty$.*

This is [6, Lemmas 7 and 8]. Since [6] is an unrefereed preprint we give the argument in full; it follows Lysov's.

Proof. Put $\mathfrak{d} = \frac{1}{x}\frac{d}{dx}$ and $\mathfrak{D} = \frac{d}{dx} \circ \frac{1}{x}$, so that $\mathfrak{d} x^m = m x^{m-2}$, $\mathfrak{D} x^m = (m-1) x^{m-2}$, both operators commute with $x \mapsto -x$, and for smooth f, g on $(0, 1]$

$$\int_\epsilon^1 f\, \mathfrak{d} g = \Big[\frac{fg}{x}\Big]_\epsilon^1 - \int_\epsilon^1 (\mathfrak{D} f) g, \qquad \int_\epsilon^1 f\, \mathfrak{D} g = \Big[\frac{fg}{x}\Big]_\epsilon^1 - \int_\epsilon^1 (\mathfrak{d} f) g.$$

Rodrigues form. We have

$$Q_n = \kappa_n\, \mathfrak{d}^n\big[(x^2 - 1)^n x^{2n}\, \tilde{U}\big], \qquad \kappa_n = \frac{1}{(2n)!\,(n!)^2},$$
$$\tilde{U} = \mathfrak{d}^n[x^{2n} U], \qquad U = \mathfrak{D}^{2n}[(x^2 - 1)^{2n} x^{4n}].$$

In the variable $w = x^2$ ($\mathfrak{d} = 2\, d/dw$, $\mathfrak{D}^{2n} = x\, \mathfrak{d}^{2n} x^{-1}$, $D = d/dw$) this is the pair of identities

$$\text{(i)} \quad Q_n(w) = \frac{1}{n!} D^n\big[w^n (w - 1)^n V_n(w)\big],$$
$$\text{(ii)} \quad V_n(w) = \frac{16^n}{(2n)!\, n!} D^n\big[w^{n+1/2} D^{2n}[w^{2n-1/2} (w - 1)^{2n}]\big],$$

with $V_n(w) = \sum_k (-1)^k B_k w^k$, $B_k = \binom{2n+k}{2k}\binom{n+k}{k}\binom{4n+2k}{2n+k}$. For (i), $\frac{1}{n!}D^n[w^{n+k}(w-1)^n] = \sum_j \binom{n}{j}(-1)^{n-j}\binom{n+k+j}{n} w^{k+j}$; multiplying by $(-1)^k B_k$ and collecting w^l gives $\sum_{k+j=l}(-1)^{n+l}\binom{n+l}{l}\binom{n}{j}B_k = c_{n,l}$. For (ii), the coefficient of w^k on the right is $\frac{16^n}{(2n)!\,n!}\binom{2n}{k}(-1)^k \frac{\Gamma(2n+k+\frac12)}{\Gamma(k+\frac12)}\frac{(n+k)!}{k!}$, which equals $(-1)^k B_k$ by the Legendre duplication formula for $0 \le k \le 2n$ (the ratio simplifies to 1; checked symbolically in `i1_contact.py`), and both sides vanish for $k > 2n$. The factors U, $x^{2n}U$, $\tilde U$ are even polynomials, and $x^{2n}U$ and $(x^2-1)^n x^{2n}\tilde U$ vanish to order $\ge 2n$ at 0.

Moments. By the Cauchy-transform representations, $[z^{-s-1}]R_{j,n} = \int x^s Q_n \, d\mu_j$. (For f_3 this is the identity $[z^{-2m}]f_3 = -A_{2m-1}/m = -2\int_0^1 x^{2m-1}\log(1+1/x)\,dx$, which follows from $\int_0^1 x^{N-1}\log(1+x)\,dx = A_N/N$ for even N and $\int_0^1 x^{N-1}(-\log x)\,dx = 1/N^2$; odd negative powers vanish on both sides.) Since Q_n is even, the lemma is equivalent to

$$\text{(O1)} \qquad \int_0^1 x^s Q_n \, dx = 0 \quad (0 \le s \le 2n-1),$$

$$\text{(O2)} \qquad \int_0^1 x^s Q_n \, \ell \, dx = 0 \quad (s \text{ odd},\ s \le 2n-1),$$

where $\ell(x) = \log(1+1/x)$. Let $h = x^s$ or $h = x^s\ell$ accordingly.

Step 1. Integrating by parts n times against $\mathfrak{d}$ gives $\int_0^1 Q_n h = \pm\kappa_n \int_0^1 (x^2-1)^n x^{2n}\tilde U \cdot r\,dx$ with $r = \mathfrak{D}^n h$. The boundary terms vanish: $\mathfrak{d}^{n-1-j}[(x^2-1)^n x^{2n}\tilde U]$ is divisible by $(x^2-1)^{j+1}$ and vanishes to order $2j+2$ at 0, while $\mathfrak{D}^j h = O(x^{-2j}|\log x|)$ at 0.

Step 2. If $h = x^s$ with s odd, then $r = 0$ ($\mathfrak{D}^n$ kills odd polynomials of degree $\le 2n-1$). If $h = x^s$ with s even, $r = c\,x^{s-2n}$. If $h = x^s\ell$, then by induction $\mathfrak{D}^j[x^s\ell] = \mathfrak{D}^j[x^s]\,\ell + (\text{rational})$, since $\ell' = -1/(x(x+1))$; as $\mathfrak{D}^n x^s = 0$, r is rational with poles only at 0 (order $\le 2n$) and -1 (order $\le n$), and $r = O(x^{-2})$ at ∞. Its odd part is $\mathfrak{D}^n[x^s\ell_{\text{even}}]$ with $\ell_{\text{even}} = \frac12\log(1-x^{-2}) = -\sum_{j\ge1} x^{-2j}/(2j)$; here $\mathfrak{D}^n x^{s-2j} = 0$ for $1 \le j \le (s-1)/2$, and the remaining powers are $\le x^{-2n-1}$, so the odd part of r is $O(x^{-2n-1})$. Put $B = x^{2n}(x^2-1)^n r$. In both nontrivial cases B is a polynomial divisible by $(x-1)^n$, of degree $\le 4n-2$, whose odd part $x^{2n}(x^2-1)^n r_{\text{odd}}$ has degree $\le 2n-1$; and $\int_0^1 Q_n h = \pm\kappa_n \int_0^1 \tilde U B\,dx$.

Step 3. Integrating by parts n times against $\mathfrak{d}$ ($\tilde U = \mathfrak{d}^n[x^{2n}U]$) gives $\int_0^1 \tilde U B = \pm\int_0^1 x^{2n}U\,\mathfrak{D}^n B$; the boundary terms vanish because $\mathfrak{D}^j B$ is divisible by $(x-1)^{n-j}$ and is $O(x^{-2j})$, while $\mathfrak{d}^{n-1-j}[x^{2n}U]$ vanishes to order $2j+2$ at 0.

Step 4. Integrating by parts $2n$ times against $\mathfrak{D}$ ($U = \mathfrak{D}^{2n}E$, $E = (x^2-1)^{2n}x^{4n}$) gives $\int_0^1 x^{2n}U\,\mathfrak{D}^n B = \pm\int_0^1 \mathfrak{d}^{2n}[x^{2n}\mathfrak{D}^n B]\cdot E$; the boundary terms vanish because $\mathfrak{D}^{2n-1-j}E$ is divisible by $(x^2-1)^{j+1}$ and vanishes to order $2j+2$ at 0, while $\mathfrak{d}^j$ of a polynomial is $O(x^{-2j})$.

Step 5. $x^{2n}\mathfrak{D}^n B$ is a polynomial of degree $\le 4n-2$ whose odd part $x^{2n}\mathfrak{D}^n[B_{\mathrm{odd}}]$ vanishes ($\deg B_{\mathrm{odd}} \le 2n-1$). An even polynomial of degree $\le 4n-2$ is killed by $\mathfrak{d}^{2n}$, so every integral is 0.

As sanity checks, `i1_contact.py` verifies (i) and (ii) exactly for small n, the moments of f_3 for $m \le 30$, and the contact orders exactly for $n \le 24$. □

Corollary 3.2. *$|Q_n(x)| \le (2n+1)2^{20n}$ on $[-1,1]$, and for every integer $q \ge 2$ and $j = 1,2,3$,*

$$|R_{j,n}(q)| \le (2n+1)\Big(\frac{2^{20}}{q^2}\Big)^n \frac{|\mu_j|}{q-1}.$$

Proof. We have

$$\begin{aligned}\sum_l |c_{n,l}| &\le \sum_k \max_l \binom{n+l}{l} 2^n \binom{2n+k}{2k}\binom{n+k}{k}\binom{4n+2k}{2n+k} \\ &\le (2n+1)2^{4n+n+4n+3n+8n}.\end{aligned}$$

By Lemma 3.1, $R_{j,n}(q) = q^{-2n}\int Q_n(x)x^{2n}\,d\mu_j(x)/(q-x)$. □

3.3. **Denominators.** The following lemma is [6, Lemma 9] at prime-power level; we give the proof in full, so that [6] is used only for the definition of $c_{n,l}$.

Lemma 3.3. *For every $n \ge 1$ and $j = 1,2,3$, the coefficients of $P_{j,n}$ have denominators dividing $[1,\dots,4n]\,[1,\dots,2n]$.*

Proof. Write $D_k = [1,\dots,k]$ and fix a prime p. We use Kummer's theorem in the form $v_p\binom{A}{B} = \sum_{i\ge1}(\lfloor A/p^i\rfloor - \lfloor B/p^i\rfloor - \lfloor (A-B)/p^i\rfloor)$, each term being 0 or 1. In every nonzero summand of $c_{n,l}$ we have $l-n \le k \le 2n$.

P_1 and P_2. By (3.1) the denominators of $P_{1,n}$ divide the numbers $m \le 2l \le 6n$, with coefficient $c_{n,l}$. If $p \le 2n$, then $v_p(m) \le \lfloor \log_p 6n\rfloor \le \lfloor \log_p 4n\rfloor + 1 \le v_p(D_{4n}D_{2n})$. If $2n < p \le 4n$, then $p^2 > 6n$, so $v_p(m) \le 1 = v_p(D_{4n})$. If $p > 4n$ and $p \mid m$, then $4n < p < 2l$ (p is odd). Then for every summand of $c_{n,l}$: $2n+k \le 4n < p$ and $p < 2l \le 2n+2k \le 4n+2k \le 8n < 2p$, so $\lfloor (4n+2k)/p\rfloor - 2\lfloor (2n+k)/p\rfloor = 1$ and $p \mid \binom{4n+2k}{2n+k}$. Hence $p \mid c_{n,l}$, and $c_{n,l}/m$ is p-integral. (The signs $(-1)^{m+1}$ play no role.) $P_{2,n}(z) = P_{1,n}(-z)$.

P_3. By (3.1) the term $c_{n,l}A_{2m-1}/m$ ($m \le l \le 3n$) has $v_p \ge v_p(c_{n,l}) - e_1 - e_2$ with $e_1 = v_p(m)$ and $e_2 = \lfloor \log_p(2m-1)\rfloor$. Now $e_1 \le \lfloor \log_p 2n\rfloor + \delta_1$, where $\delta_1 = 1$ if some $p^a \mid m$ with $p^a \in (2n, 3n]$ and $\delta_1 = 0$ otherwise (a higher excess is impossible since $2p^a > 3n$); and $e_2 \le \lfloor \log_p 4n\rfloor + \delta_2$, where $\delta_2 = 1$ if some $p^b \in (4n, 2m-1]$ and $\delta_2 = 0$ otherwise (again $2p^b > 6n$). If $\delta_1 + \delta_2 > 0$ then $m > 2n$, so $l > 2n$ and $k \ge l - n > n$ in every summand. If $\delta_1 = 1$: $n < p^a$, $k \le 2n < p^a$ and $n + k \ge l \ge m \ge p^a$, so $p \mid \binom{n+k}{k}$. If $\delta_2 = 1$: $2n + k \le 4n < p^b < 2m \le 2l \le 2n + 2k \le 4n + 2k$, so $p \mid \binom{4n+2k}{2n+k}$. These are different binomial factors, so $v_p(c_{n,l}) \ge \delta_1 + \delta_2$, and the term has $v_p \ge -v_p(D_{2n}D_{4n})$. □

(An exact check of the divisibility for $n \le 120$, and independently for selected $n \le 81$, agrees.)

3.4. **The recurrence.**

Proposition 3.4. *There are polynomials $p_0, \dots, p_4 \in \mathbb{Z}[n, w]$, of degree 34 in n and of degrees $3, 4, 5, 6, 3$ in w, such that for every integer $n \ge 0$ and every $z \in \mathbb{C}$, with $w = z^2$,*

$$\sum_{i=0}^{4} p_i(n, z^2)\, X_{n+i}(z) = 0 \qquad \text{for } X \in \{Q, P_1, P_2, P_3\},$$

hence also for $X = R_j$. Moreover

$$\sum_{j=0}^{4} \mathrm{lc}_n(p_j)(w)\, \lambda^j = 2^{59}(8w+1)(9w-1)(25w-16) \cdot \chi_w(-\lambda),$$

where $\chi_w(\lambda) = \lambda^4 + (16384w^3 - 20480w^2 + 5248w - 64)\lambda^3 + (135168w^2 + 159744w + 1536)\lambda^2 + (294912w - 16384)\lambda + 65536$.

Proof (computer-assisted, exact). Two relations for $c_{n,l}$. There are explicit polynomials a_0, a_1, a_2 and $\alpha, \beta_{a,i}$ with

$$\sum_{m=0}^{2} a_m(n,l) c_{n,l+m} = 0, \qquad \alpha(n) c_{n+1,l} = \sum_{a=0}^{3} \sum_{i=0}^{3} \beta_{a,i}(n)(l-i)^a c_{n,l-i},$$

$a_0 = -(l-3n)(l+n+1)(l+n+2)(2l+2n+1)$, $a_2 = -(l+2)^3(2l+3)$, $\alpha = -(n+1)^3(2n+1)^2(4n+1)(4n+3)$ (the others are in `runs/i1_relations.json`). They are proved by creative telescoping in k: with explicit certificates G_1, G_2 (hypergeometric terms given by factorial formulas), the summand identities $\sum_m a_m F(n, l+m, k) = G_1(n, l, k+1) - G_1(n, l, k)$ and its analogue hold at every integer point of the relevant regions. Over a common factorial base each reduces to one polynomial identity in $\mathbb{Q}[n, l, k]$, which is identically zero; at $k = -1$ the certificates vanish (factor k^2), they vanish for $k > 2n + 2$, so the sums over k telescope; the finitely many boundary cases are checked directly (`i1_certs.py`).

Order-4 operator. In generating-function form the relations say $D_n Q_n = 0$, with D_n an operator of order 4 in $w\, d/dw$ whose leading coefficient $-2(w-1)^2$ does not depend on n, and $\alpha(n) Q_{n+1} = E_n Q_n$ with E_n of order 3. Reducing $E_{n+i} \cdots E_n$ modulo D_n on the right expresses Q_{n+i}, $i = 0, \dots, 4$, through $Q_n, \dots, (w\, d/dw)^3 Q_n$; the cofactor vector of the resulting 4×5 matrix, made primitive, is $(p_0, \dots, p_4)$, and $\sum_i p_i Q_{n+i} = 0$ is re-verified exactly (`i1_opalg.py`).

The numerators. $\sum_i p_i(n, z^2) P_{j,n+i} = -\sum_i p_i(n, z^2) R_{j,n+i}$. The left side is a polynomial, and by Lemma 3.1 the right side is $O(z^{5-2n})$; so both vanish for $n \ge 3$, and $n = 0, 1, 2$ are checked directly.

The characteristic polynomial identity is exact (`i1_extras.py`). □

For $q \geq 2$ the prefactor $2^{59}(8w+1)(9w-1)(25w-16)$ does not vanish at $w = q^2$, so the characteristic roots of the recurrence are the roots λ of $\chi_{q^2}(-\lambda)$. For $2 \leq q \leq 60$ they are real, positive and distinct, and $1/\lambda_1 < 2^{-20}q^2$ (Arb root isolation, `fourterm/check_y1.py`); we order them $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4 > 0$ and put $y_k = 1/\lambda_k$, so that $0 < y_1 < y_2 < y_3 < y_4$. For example, for $q = 26$: $y_1 \approx 1.98 \cdot 10^{-13}$, $y_2 \approx 156.37$, $y_3 \approx 182.38$, $y_4 \approx 2703$. (For every $q \geq 2$ the roots are distinct, since the discriminant of c_{34} below does not vanish at $w = q^2$; we use their reality and order only for $26 \leq q \leq 53$.)

3.5. **The differential equations.** Fix an integer q with $2 \leq q \leq 60$ and put

$$H_j(y) = \sum_{n \geq 0} R_{j,n}(q)\, y^n \quad (j = 1, 2, 3), \qquad H_a = \sum_{j=1}^{3} a_j H_j \quad (a \in \mathbb{C}^3).$$

By Corollary 3.2, each H_j is holomorphic on $|y| < 2^{-20}q^2$, a disc containing 0 and y_1. Let $\vartheta = y\, d/dy$ and

$$L = \sum_{j=0}^{4} y^{4-j} p_j(\vartheta - j, q^2),$$

a differential operator of order 34. The coefficient of y^{N+4} in $L \sum_n X_n y^n$ is $\sum_j p_j(N) X_{N+j}$, so by Proposition 3.4, $LH_j = E_j$ with E_j a polynomial of degree ≤ 3. The ϑ^{34}-coefficient of L is $c_{34}(y) = \sum_j y^{4-j} \mathrm{lc}_n(p_j)$, which is a constant times $(8w+1)(9w-1)(25w-16)\, y^4 \chi_w(-1/y)$; its roots are $y_1, \ldots, y_4$, and $c_{34}(0) \neq 0$. Hence the singular points of L are $0, y_1, \ldots, y_4, \infty$, and L has no apparent singularities.

Proposition 3.5. *For every $a \in \mathbb{C}^3$, H_a continues analytically to a single-valued holomorphic function on $\mathbb{C} \setminus [y_2, \infty)$.*

Proof. H_a is holomorphic on the disc $|y| < 2^{-20}q^2$, which contains 0 and y_1, and satisfies $LH_a = E_a$; solutions of a linear differential equation continue along every path avoiding its singular points. The fundamental group of $\mathbb{C} \setminus ([y_2, \infty) \cup \{0, y_1\})$ is generated by loops around 0 and y_1 that lie in that disc, where H_a is single-valued; so the continuation is single-valued on $\mathbb{C} \setminus ([y_2, \infty) \cup \{0, y_1\})$ and, being holomorphic near 0 and y_1, extends holomorphically to $\mathbb{C} \setminus [y_2, \infty)$. □

3.6. **The minimal operator.** For each q with $26 \leq q \leq 53$ we computed an operator of recurrence order 25 and n-degree 13, common to the four sequences Q, P_1, P_2, P_3 at $z = q$ (modular nullspaces, Chinese remaindering and rational reconstruction; checked exactly for $n \leq 60$). Its associated differential operator $L_{13} = \sum_{j=0}^{25} y^{25-j} \tilde{p}_j(\vartheta - j)$ has order 13. Its leading coefficient has degree 25: its roots are $y_1, \ldots, y_4$ and 21 apparent singular points.

Proposition 3.6. *For each q with $26 \le q \le 53$ the order-25 recurrence holds for all $n \ge 0$ on Q, P_1, P_2, P_3 at $z = q$. Hence $L_{13} H_a = E'_a$ with E'_a a polynomial, for every $a \in \mathbb{C}^3$.*

Proof (computer-assisted, exact). Fraction-free right division in $\mathbb{Z}[n][S]$ (S the shift) gives $\mu(n) \cdot \mathrm{Rec}_{25} = \tilde{K} \cdot L_4$ with zero remainder, where $L_4 = \sum_i p_i(n, q^2) S^i$ and $\mu(n)$ is a nonzero integer times $\prod_{t=0}^{21} p_4(n+t, q^2)$. The polynomial $p_4(\cdot, q^2)$ has no root at a nonnegative integer, so $\mu(n) \neq 0$ for $n \ge 0$, and Proposition 3.4 gives the claim (`i1_rec25.py`). □

4. Proof of Theorem 1

Fix q with $26 \le q \le 53$. Write $f = (f_1(q), f_2(q), f_3(q))$, $P_n = (P_{1,n}(q), P_{2,n}(q), P_{3,n}(q))$ and $v_n = (Q_n(q), P_n) \in \mathbb{Q}^4$.

4.1. **Reduction.** Suppose $r_0 + a_1 f_1 + a_2 f_2 + a_3 f_3 = 0$ with $(r_0, a) \in \mathbb{Z}^4 \setminus \{0\}$. Then $a \neq 0$, and

$$G(y) := \sum_{n \ge 0} \big(r_0 Q_n(q) + a \cdot P_n\big) y^n = \sum_n \big(Q_n(q)(r_0 + a \cdot f) - a \cdot R_n\big) y^n = -H_a(y),$$

where $R_n = (R_{1,n}(q), R_{2,n}(q), R_{3,n}(q))$. By Lemma 3.3 (and $Q_n(q) \in \mathbb{Z}$), $G \in \mathbb{Q}[[y]]$ has constant term $r_0 \in \mathbb{Z}$ and denominator type $[1, \dots, 4n][1, \dots, 2n]$; so do $\vartheta^j G$ for all j (the coefficients are multiplied by n^j). By Proposition 3.5, G continues to $\mathbb{C} \setminus [y_2, \infty)$.

Let $\varphi(w) = 4y_2 w/(1+w)^2$, the Koebe map of $\mathbb{D}$ onto $\mathbb{C} \setminus [y_2, \infty)$, and $\varphi_r(w) = \varphi(rw)$ for $0 < r < 1$. Consider the $\mathtt{m} = 14$ functions $1, G, \vartheta G, \dots, \vartheta^{12} G$, with type rows $(0,0)$ for the constant 1 and $(4,2)$ for the others, $e = 0$. Then $\sigma_i = 6$ for $i \ge 2$ and $\tau = \frac{1}{196} \sum_{i=2}^{14} (2i-1) \cdot 6 = 6 \cdot \frac{195}{196}$. All pullbacks by φ_r are holomorphic on a neighbourhood of $\overline{\mathbb{D}}$, and φ_r is univalent, so $T(\varphi_r) = \log |\varphi_r'(0)| = L_q + \log r$ by Lemma 2.3, where

$$L_q = \log(4y_2).$$

If the 14 functions are linearly independent over $\mathbb{Q}(y)$, Theorem 2.1 gives $14 \le (L_q + \log r)/(L_q + \log r - \tau)$ for every $r < 1$ close to 1, hence $14 \le L_q/(L_q - \tau)$, that is

$$M_{14}(q) := \tfrac{13}{14} L_q - 6 \cdot \tfrac{195}{196} \le 0.$$

(The condition $L_q > \max\{6, \tau\} = 6$ of Theorem 2.1 holds.) But $M_{14}(q) > 0$ for every q with $26 \le q \le 53$; the Arb values are in Table 1, for example $M_{14}(26) = 0.0092251638 \pm 5 \cdot 10^{-12}$ and $M_{14}(53) = 1.3688032 \pm 3 \cdot 10^{-8}$ (`budget_4t.py`; y_2 is a root of the exact characteristic polynomial of Proposition 3.4). This contradiction proves Theorem 1, given:

Proposition 4.1. *Let $a \in \mathbb{C}^3 \setminus \{0\}$. The 14 functions $1, H_a, \vartheta H_a, \dots, \vartheta^{12} H_a$ are linearly independent over $\mathbb{C}(y)$, hence over $\mathbb{Q}(y)$.*

The rest of this section proves Proposition 4.1. For $a = (0,0,1)$ it is the case used for Corollary 2, which we also certified separately (Remark 4.6).

4.2. **An independence lemma.** Fix a base point $y_* \in (y_1, y_2)$ which is a regular point of L_{13}, and consider germs at y_*; γ runs over closed paths at y_* avoiding all singular points of L_{13}. Let

$$U_a = \operatorname{span}_{\mathbb{C}}\{(\gamma - 1)H_a\}, \qquad S = \operatorname{Sol}(L_{13}) \ (\dim S = 13).$$

By Proposition 3.6, $U_a \subseteq S$. U_a is invariant under monodromy, since $\delta(\gamma - 1)H_a = (\delta\gamma - 1)H_a - (\delta - 1)H_a$.

Lemma 4.2. *If* $\dim U_a = u$, *then* $1, H_a, \vartheta H_a, \dots, \vartheta^{u-1} H_a$ *are linearly independent over* $\mathbb{C}(y)$.

Proof. Suppose $c_0 + K H_a = 0$ with $K = \sum_{j<u} c_{j+1}\vartheta^j$, $c_i \in \mathbb{C}(y)$. Continuation along γ (perturbed off the poles of the c_i) gives $c_0 + K(\gamma H_a) = 0$, so K kills $(\gamma - 1)H_a$ for every γ, hence the u-dimensional space U_a. A nonzero operator of order $\le u - 1$ cannot do that, so $K = 0$, and then $c_0 = 0$. □

So Proposition 4.1 follows from $U_a = S$, which in turn follows from $U_a \neq 0$ (Lemma 4.4) and the irreducibility of S under the monodromy (Proposition 4.5).

4.3. **Local exponents.**

Proposition 4.3. *For every integer* $q \ge 2$, *each of the four roots* y_k *of* c_{34} *is a regular singular point of* L *with exponents* $0, 1, \dots, 32$ *and* $\frac{1}{2}$. *Every local solution of* $Lf = E$ (E *holomorphic at* y_k) *has the form* $h_0(t) + c\,t^{1/2}h(t)$, $t = y - y_k$, *with* h_0, h *holomorphic and* $h(0) \neq 0$. *In particular the local monodromy of the affine solution space of* $Lf = E$ *at* y_k *is a reflection, and a solution is single-valued around* y_k *if and only if it is holomorphic there.*

Proof (computer-assisted, exact). Let c_{34}, c_{33} be the ϑ^{34}- and ϑ^{33}-coefficients of L. In d/dy form, $a_{34} = y^{34}c_{34}$ has a simple zero at y_k (the discriminant of c_{34} is a constant times $w^3(w-1)^2(8w+1)^8(9w-1)^8(25w-16)^8 \neq 0$) and $a_{33}(y_k) = y_k^{33}c_{33}(y_k)$. So y_k is regular singular with exponents $0, \dots, 32$ and $\beta_k = 33 - c_{33}(y_k)/(y_k c'_{34}(y_k))$. The pseudo-remainder of $2c_{33} - 65\,y\,c'_{34}$ by c_{34} in $\mathbb{Z}[w][y]$ is identically zero, so $\beta_k = \frac{1}{2}$ for all k and all q (cross-check: $\operatorname{Res}_y(c_{34}, (33-b)yc'_{34} - c_{33})$ is a constant times $w^3(w-1)^2(8w+1)^{10}(9w-1)^{10}(25w-16)^{10}(2b-1)^4$). In the power-series recursion at y_k the pivot vanishes only at exponent $\frac{1}{2}$, never at an integer, which gives 33 holomorphic homogeneous solutions, a holomorphic particular solution and one solution $t^{1/2}h(t)$ (`fourterm/exponents.py`). □

These statements concern L, not L_{13} (which has its own exponents and apparent points); they are transferred to S only after $S \subseteq \operatorname{Sol}(L)$ has been certified (step (a) below).

4.4. **No combination is single-valued.**

Lemma 4.4. *For every* $a \in \mathbb{C}^3 \setminus \{0\}$, $U_a \neq 0$.

Proof. Suppose $U_a = 0$, so that H_a extends to a single-valued holomorphic function on $\mathbb{C} \setminus \{0, y_1, \dots, y_4\}$.

(i) It is holomorphic at 0 and y_1 (where it equals its power series) and at y_2, y_3, y_4 by Proposition 4.3. So H_a is entire.

(ii) E_a is a polynomial of degree ≤ 3, so H_a is killed by $L' = \vartheta(\vartheta-1)(\vartheta-2)(\vartheta-3)L$. With $u = 1/y$, the ϑ_u-leading coefficient of L' at $u = 0$ is $\pm\mathrm{lc}(p_0) \neq 0$, so ∞ is a regular singular point. A single-valued solution of a Fuchsian equation near ∞ has moderate growth, hence is meromorphic at ∞: H_a is a polynomial.

(iii) $Ly^D = \sum_j p_j(D-j)y^{D+4-j}$, so if $H_a \not\equiv 0$ has degree $D \ge 0$, the coefficient of y^{D+4} in LH_a is $p_0(D, q^2)$ times the leading coefficient of H_a; $p_0(\cdot, q^2)$ has no nonnegative integer root for $2 \le q \le 60$. Since $\deg E_a \le 3 < D+4$, $H_a \equiv 0$, that is, $(a \cdot f)Q_n - a \cdot P_n = 0$ for all n.

(iv) The Casoratian $\det[v_0, v_1, v_2, v_3]$ equals, as a function of $z = q$,

$$-\frac{z}{291060}\big(7427354003333880z^6 - 4691165040913890z^4 - 141637476029121z^2 + 63093459176677\big),$$

which has no integer root other than 0. So $v_0, \dots, v_3$ are independent and $(a \cdot f, -a) = 0$, i.e. $a = 0$. □

4.5. **Irreducibility of the monodromy.** *The data.* For each q, the script `mono13.py` computes rigorous enclosures of the monodromy of $g = y^5 H_3$ (multiplication by y^5 is equivariant, and shifts the exponents of L_{13} at 0 to make the Taylor recursion well-conditioned) on the 14-dimensional affine module spanned by g and $y^5 S$. Coordinates: the 13 scaled Taylor coefficients at the base point $y_* = \mathrm{round}(0.42y_2)$ and an affine coordinate. The generators are: A, the circle $|y| = y_*$ (which encloses 0, y_1 and some apparent points); 2 and 3, loops around y_2 and y_3 conjugated by paths that avoid all 21 apparent points. The computation:

- initial data at $b = 2^{-21}$ from the power series of g, with exact $Q_n(q)$, $P_{3,n}(q)$ and the tail bound of Corollary 3.2;
- Taylor steps $c \mapsto c + h$ along the paths, $|h| \le 0.2\delta$ (δ the distance to the nearest singular point of L_{13}): coefficients of an approximate solution by the recurrence at a midpoint, the residual $L_{13}\hat{y} - E$ computed in balls, and the error $y - \hat{y}$ bounded by a Cauchy majorant whose constants $\max|A_i/A_{13}|$ on a circle are enclosed by arc subdivision using factored forms of the coefficients and certified root balls of the leading coefficient; the step matrix is enclosed and applied as a ball product;
- working precision 2200 bits, truncation target 1700 bits per step; all radii of the generator matrices stay below 10^{-308} (typically 10^{-400}).

The output is three 14×14 ball matrices g_A, g_2, g_3 whose last row is exactly $(0, \dots, 0, 1)$, and the coordinate vector v of g. As a check, $g_A v = v$ holds

within the balls (H_3 is holomorphic inside $|y| < y_2$). The 13×13 blocks are the actions on S.

The reflection criterion. Let $u = (g_2 - 1)v \in S$.

(a) If the 13 vectors $g_w u$ (w words in g_A, g_2, g_3) are linearly independent, then $S = \operatorname{span}\{g_w u\}$. Each $g_w u$ is a continuation of $(\gamma_2 - 1)H_3$, a solution of the homogeneous equation $Lf = 0$, so $S \subseteq \operatorname{Sol}(L)$, and loops act on S through their classes in $\pi_1(\mathbb{C} \setminus \{0, y_1, \dots, y_4\})$. In particular g_2 acts on the affine module $\mathbb{C}H_3 \oplus S$ as a conjugate of the local loop at y_2, which is a reflection by Proposition 4.3. Its image is $\mathbb{C}u$, so $g_2|_S = 1 + u \otimes \phi$ for some $\phi \in S^*$.

(b) The row $\psi =$ row i_0 of $(g_2 - 1)|_S$ equals $u_{i_0}\phi$; one entry ball excludes 0, so $\phi \neq 0$ and $u_{i_0} \neq 0$.

(c) If moreover the 13 covectors ψg_w are linearly independent, S is irreducible under $\langle g_A, g_2, g_3 \rangle$: for an invariant subspace $W \neq 0$, either $\phi(W) \neq 0$, and then $u = (g_2 x - x)/\phi(x) \in W$ for some $x \in W$, so $W \supseteq \operatorname{span}\{g_w u\} = S$; or $\phi(W) = 0$, and then $W \subseteq \bigcap_w \ker(\phi g_w) = 0$.

Proposition 4.5. *For every q with $26 \leq q \leq 53$, the 13×13 determinants of (a) and (c) exclude 0. Hence S is irreducible under the monodromy group of L_{13}.*

Proof (computer-assisted, Arb). The words were chosen greedily on midpoints (a heuristic that does not affect rigour); at every q both lists are $\varnothing, A, A^2, A^3, A^4, 3A^3, A^5, A3A^3, A^6, A^2 3A^3, A^7, A^3 3A^3, A^8$ (in the notation of the certificate files). The determinants are evaluated in Arb at 2200 bits from the ball entries (`fourterm/irred.py`); their imaginary parts are listed in Table 1, with relative radii at most $4 \cdot 10^{-30}$; the real parts are balls around 0 of radius at most $2 \cdot 10^{-219}$. Irreducibility under the subgroup $\langle g_A, g_2, g_3 \rangle$ implies irreducibility under the whole monodromy group. □

Proof of Proposition 4.1, Theorem 1 and Corollary 2. By Lemma 4.4, U_a is a nonzero monodromy-invariant subspace of S, so $U_a = S$ by Proposition 4.5; then Lemma 4.2 with $u = 13$ gives Proposition 4.1, and the reduction above gives Theorem 1. Corollary 2 is the case $26 \leq q \leq 32$ of Theorem 1: if $\theta_q = u/v$ with $u, v \in \mathbb{Z}$, $v \neq 0$, then $(r_0, a) = (-u, 0, 0, v)$ is a nonzero relation. □

Remark 4.6 (Independent certificate for Corollary 2). Corollary 2 was first proved by a different final step: for $26 \leq q \leq 32$ the 14 continuations of H_3 along the words $\varnothing, 2, 3, A2, A3, A^2 2, A^2 3, A^3 2, A^3 3, A^4 2, A^4 3, A^5 2, A^6 2, A^7 2$ have a nonzero 14×14 Arb determinant (`cert_det.py`), for example

$$[257906186217.156323475542228360 \pm 3.7 \cdot 10^{-19}]\, i \quad (q = 26),$$

$$[992625.090850805422458198252898 \pm 3.9 \cdot 10^{-25}]\, i \quad (q = 32).$$

TABLE 1. Certificates for Theorem 1: the least $\mathfrak{m}$ with $M_{\mathfrak{m}}(q) > 0$, the margin $M_{14}(q)$ (midpoint of an Arb ball of radius $\leq 5 \cdot 10^{-8}$, truncated to six decimals), y_2, the base point y_*, and the midpoints of the two 13×13 determinants of Proposition 4.5 (`code/thm1_cor2/fourterm/summary_4t.json`). The jump between $q = 37$ and $q = 38$ is a change of scaled coordinates (base point and scaling), not a failure.

q	$\mathfrak{m}_{\min}$	$M_{14}(q)$	y_2	y_*	det (a)	det (c)
26	14	0.009225	156.368	66	$1.634 \cdot 10^{14}\, i$	$2.307 \cdot 10^{89}\, i$
27	12	0.082012	169.118	71	$1.605 \cdot 10^{13}\, i$	$1.657 \cdot 10^{91}\, i$
28	11	0.152055	182.368	77	$1.985 \cdot 10^{12}\, i$	$1.121 \cdot 10^{93}\, i$
29	10	0.219554	196.119	82	$2.151 \cdot 10^{11}\, i$	$5.668 \cdot 10^{94}\, i$
30	9	0.284686	210.369	88	$2.871 \cdot 10^{10}\, i$	$2.739 \cdot 10^{96}\, i$
31	8	0.347612	225.119	95	$4.584 \cdot 10^{9}\, i$	$1.253 \cdot 10^{98}\, i$
32	7	0.408477	240.369	101	$6.520 \cdot 10^{8}\, i$	$6.072 \cdot 10^{99}\, i$
33	7	0.467411	256.119	108	$1.094 \cdot 10^{8}\, i$	$5.079 \cdot 10^{101}\, i$
34	7	0.524534	272.370	114	$1.669 \cdot 10^{7}\, i$	$2.784 \cdot 10^{103}\, i$
35	6	0.579952	289.120	121	$2.964 \cdot 10^{6}\, i$	$1.637 \cdot 10^{105}\, i$
36	6	0.633764	306.370	129	$6.013 \cdot 10^{5}\, i$	$1.007 \cdot 10^{107}\, i$
37	6	0.686062	324.120	136	$1.120 \cdot 10^{5}\, i$	$4.311 \cdot 10^{108}\, i$
38	5	0.736928	342.370	144	$7.143 \cdot 10^{27}\, i$	$3.168 \cdot 10^{92}\, i$
39	5	0.786438	361.120	152	$1.535 \cdot 10^{27}\, i$	$6.101 \cdot 10^{93}\, i$
40	5	0.834663	380.370	160	$3.367 \cdot 10^{26}\, i$	$1.076 \cdot 10^{95}\, i$
41	5	0.881668	400.120	168	$7.537 \cdot 10^{25}\, i$	$1.747 \cdot 10^{96}\, i$
42	5	0.927512	420.371	177	$1.865 \cdot 10^{25}\, i$	$2.758 \cdot 10^{97}\, i$
43	5	0.972252	441.121	185	$4.342 \cdot 10^{24}\, i$	$3.843 \cdot 10^{98}\, i$
44	4	1.015940	462.371	194	$1.110 \cdot 10^{24}\, i$	$5.232 \cdot 10^{99}\, i$
45	4	1.058624	484.121	203	$2.884 \cdot 10^{23}\, i$	$9.447 \cdot 10^{100}\, i$
46	4	1.100350	506.371	213	$8.133 \cdot 10^{22}\, i$	$2.221 \cdot 10^{102}\, i$
47	4	1.141158	529.121	222	$2.178 \cdot 10^{22}\, i$	$4.203 \cdot 10^{103}\, i$
48	4	1.181090	552.371	232	$6.302 \cdot 10^{21}\, i$	$8.202 \cdot 10^{104}\, i$
49	4	1.220180	576.121	242	$1.847 \cdot 10^{21}\, i$	$1.468 \cdot 10^{106}\, i$
50	4	1.258466	600.371	252	$5.482 \cdot 10^{20}\, i$	$2.422 \cdot 10^{107}\, i$
51	4	1.295978	625.121	263	$1.740 \cdot 10^{20}\, i$	$4.090 \cdot 10^{108}\, i$
52	4	1.332747	650.371	273	$5.295 \cdot 10^{19}\, i$	$5.802 \cdot 10^{109}\, i$
53	4	1.368803	676.122	284	$1.717 \cdot 10^{19}\, i$	$8.436 \cdot 10^{110}\, i$

So $\dim \operatorname{span}\{\gamma H_3\} = 14$, and the elementary analogue of Lemma 4.2 gives the independence of $1, H_3, \ldots, \vartheta^{12} H_3$. (The passage from coordinate vectors to germs uses $E \neq 0$.) Independent reruns at $q = 26$ and $q = 32$ with initial data that do not use Lemma 3.1 (base $b = 2^{-22}$, the crude bound $(2n+1)2^{20n}|\mu_3|/(q-1)$) and a different discretisation reproduce these determinants to all 30 printed digits, and give the same irreducibility certificates.

Remark 4.7. Smaller holonomy budgets suffice for larger q ($\mathfrak{m}_{\min}$ in Table 1), but the proof uses $\mathfrak{m} = 14$ at every q. No case analysis on the relation vector and no input from linear forms in logarithms is needed: only $a \neq 0$ is used.

5. Widening CDT's range: proof of Theorem 3

5.1. **CDT's construction.** We recall [1, §14]. For an odd integer a with $|a| \geq 3$ let $A(a,x) = (1 - 2ax + x^2)^{-1/2} = \sum_k u_k(a)x^k$ and $H(a,x) = A(a,x)\int_0^x A(a,t)\,dt = \sum_k v_k(a)x^k$, so that $v_k(a)/u_k(a) \to \frac{1}{2}\log\frac{a+1}{a-1}$. For distinct odd $a = 2m+1$, $b = 2n+1$ put $\lambda_a = \frac{1}{2}\ell_m$, $\lambda_b = \frac{1}{2}\ell_n$, and suppose $r_0 + r_a\lambda_a + r_b\lambda_b + r_{ab}\lambda_a\lambda_b = 0$ with $r \in \mathbb{Z}^4$, $(r_a, r_b, r_{ab}) \neq 0$. CDT form

$$P(x) = \sum_k \big(r_0 u_k(a)u_k(b) + r_a v_k(a)u_k(b) + r_b u_k(a)v_k(b) + r_{ab}v_k(a)v_k(b)\big)x^k$$

(a combination of Hadamard products) and symmetrise: $G(y) = P(x) + P(x/(x-1))$ with $y = x^2/(x-1)$ [1, Def. 14.1.9]. By [1, Prop. 14.1.11], G has denominator type $[1,\dots,2k]^2$, satisfies a differential equation $\mathcal{L}_{a,b}$ whose singular points are $0, 4, \infty$ and the four points $Z_4(a,b)$, the y-images of the four products $x = (a \pm \sqrt{a^2-1})(b \pm \sqrt{b^2-1})$ (plus apparent singularities at the roots of a polynomial R_{10}), and G is overconvergent at the image α of the smallest product. With the seven "pure" functions $B_1,\dots,B_7$ of [1, §10] and ten functions built from G (G, G', G'', G''' and six primitives), [1, Lemma 14.3.1] gives $\mathfrak{m} = 17$ functions, linearly independent over $\mathbb{Q}(y)$, with

$$\sigma = (0,2,2,4,\dots,4), \qquad e = (0,0,1;0,0,0,0,0,0;1,1,1,1,1,1,1,1),$$
$$\tau = \frac{1136}{289} + \frac{78419}{242760} = \frac{1032659}{242760} = 4.2538\dots$$

[1, (14.5.2)–(14.5.4)]. The order of the functions is CDT's: B_1, B_2, B_3; $B_4, B_5, G, G', G'', G'''$; B_6, B_7 and the six primitives. The map is $\varphi = h \circ \psi$, with

$$h(\mathfrak{q}) = -256\,\mathfrak{q}\prod_{k\geq 1}(1+\mathfrak{q}^k)^{24} = -256\Big(\frac{\eta(2\tau)}{\eta(\tau)}\Big)^{24},$$

the hauptmodul of $\Gamma_0(2)$ in the cusp coordinate $\mathfrak{q} = e^{2\pi i\tau}$, and $\psi : \mathbb{D} \to \mathbb{D}$ holomorphic with $\psi^{-1}(0) = \{0\}$; admissibility of φ is checked through [1, Cor. 9.0.19]. The constant 10^{-6} of [1, Thm. C] enters only through [1, Lemma 14.2.4] (which needs $|a/b \mp 1| < \frac{1}{2}$), [1, Lemma 14.2.6] (which needs $0 < |a/b - 1| < 10^{-3}$), and the location of the singularities, [1, §14.4], which determines the admissible maps. We change only the map.

5.2. **Location of the singular values.**

Lemma 5.1. *Let $0 < \varepsilon \leq 10^{-4}$ and let $m \neq n$ be integers in $\mathbb{Z}\setminus\{-1,0\}$ with $|m/n - 1| < \varepsilon$; put $a = 2m+1$, $b = 2n+1$. Then the three genuine singular values (the points of $Z_4(a,b)$ other than α) are real with $|y| \geq Y_1 := 1/\varepsilon - 6$, and α is real with $|\alpha| \leq Y_0 := \varepsilon^4/16$.*

Proof. Let $A = |a|$, $B = |b|$, $u = a + \operatorname{sgn}(a)\sqrt{a^2-1}$, v likewise; the products are $x \in \{uv, u/v, v/u, 1/(uv)\}$, all real since $A, B \geq 3$, and $y = x^2/(x-1) = (x-1)+2+1/(x-1)$. By the mean value theorem for arccosh, $\delta := |\log|u| - \log|v|| \leq |A-B|/\sqrt{\min(A,B)^2-1}$. For positive pairs $|A-B|/\min(A,B) < \varepsilon/(1-\varepsilon)$, for negative pairs $\leq 2\varepsilon|n|/(2|n|(1-\varepsilon)-1) \leq \varepsilon/(1-1.5\varepsilon)$, using $|n| \geq 1/\varepsilon$; and $\min(A,B) \geq 2/\varepsilon - 3$, so the square-root factor is at most $1+\varepsilon^2/7$. Hence $\delta \leq \varepsilon(1+1.6\varepsilon)$, and $x \in \{u/v, v/u\}$ has $|x-1| \leq e^\delta - 1 \leq \varepsilon(1+3\varepsilon)$, so $|y| \geq (1-3\varepsilon)/\varepsilon - 2 - 2\varepsilon > 1/\varepsilon - 6$. For $x = uv$, $|x| \geq (2\min(A,B)-1)^2$ gives $y > 1/\varepsilon$; for $x = 1/(uv)$, $|\alpha| \leq 2/|uv|^2 < \varepsilon^4/16$. □

The constant 6 cannot be dropped entirely: CDT's "$|y| > 1/\varepsilon$" fails by $O(1)$. At $(m,n) = (-10^6, -10^6-1)$ the smallest genuine singular value has $|y| = 10^6 - 1.50$; at $\varepsilon = 3\cdot 10^{-6}$ the worst case is $1/\varepsilon - 1.83$, at $(m,n) = (-333333, -333334)$. This does not affect [1, Thm. C], whose map has room.

5.3. **Admissibility of one map for the whole range.** Let $\varphi = h \circ \psi$ with ψ entire, $\psi^{-1}(0) = \{0\}$, and suppose

(a) $|\psi| < 1$ on $\mathbb{T}$;
(b) $|\varphi| > 2Y_0$ on $\mathbb{T}$;
(c′) $|\varphi| < Y_1$ on $\mathbb{T}$.

By (a) and the maximum principle $\psi(\overline{\mathbb{D}}) \subset \mathbb{D}$, so φ is holomorphic on a neighbourhood of $\overline{\mathbb{D}}$. By (c′), $\varphi(\mathbb{D}) \subset D(0, Y_1)$, which by Lemma 5.1 misses every genuine singular value of every pair in the range. Since φ vanishes only at 0, simply, and $|\varphi| > 2Y_0$ on $\mathbb{T}$, the argument principle shows that every point of $U = D(0, 2Y_0)$ has exactly one preimage in $\mathbb{D}$; this is the univalent leaf over $U \ni \alpha$ required by [1, Cor. 9.0.19], with $Y^0 = \{\alpha\}$ and Y^1 the three genuine singular values. (Cor. 9.0.19 leaves U free, subject to being a contractible neighbourhood of 0 in $\mathbb{C} \setminus \{4\}$ containing α on which G is holomorphic; G is holomorphic on its principal disc, which contains $D(0, 2Y_0)$.) So the 17 pullbacks $f_i \circ \varphi$ are meromorphic on $\mathbb{D}$, for every pair with $|m/n - 1| < \varepsilon$ at once.

5.4. **The certificate.** We take $\psi(z) = z\exp(\sum_{k=0}^{96} c_k z^k)$ with dyadic coefficients $c_k \in 2^{-40}\mathbb{Z}$ (the certified object; stored in `code/thm3/runs/cert3_p4_C_eps3.0e-06_eps3.0e-06_l3.json`), found by an admissibility-aware L-BFGS optimisation of $L - T/17$ (started from lune-shaped maps and from CDT's map). Then $L = \log|\varphi'(0)| = \log 256 + c_0$ exactly. We apply Theorem 2.2 with the four maps $\varphi_k(z) = \varphi(r_k z)$, where

$$(r_0, r_1, r_2, r_3) = \Big(\frac{1}{2048}, \frac{99}{256}, \frac{215}{256}, 1\Big), \quad (\vartheta_1, \vartheta_2, \vartheta_3) = \Big(\frac{61}{200}, \frac{231}{200}, \frac{1109}{500}\Big);$$

the φ_k are restrictions of φ, so they are admissible.

Certification (`cert3.py`, `cert3d.py`). The circle is cut into dyadic intervals, adaptively. On each interval, an Arb value of $\log\varphi$ at the midpoint (via η-quotients) and a rigorous bound for $|d\log\varphi/d\vartheta|$, namely $(1 + 24\sum_k k\rho^k/(1-\rho^k)) \cdot \sup|1 + zg'(z)|$ with $\rho \geq \sup|\psi|$ on the interval and

$g = \sum c_k z^k$, give an upper step function for $\log|\varphi|$, hence an exact rational upper bound for the rearrangement integral of the mixture, and decide (a), (b), (c′). The results at $\varepsilon = 3 \cdot 10^{-6}$:

- (a), (b), (c′) hold; $\max_{\mathbb{T}} \log|\varphi| \le 13801311154345/2^{40} = 12.55222 < \log Y_1 = 12.71688$;
- $L = 4.7983738127618231\ldots$, mixture integral ≤ 9.08327, penalty $\frac{1}{17}\sum \vartheta_k^2 \log(r_k/r_{k-1}) = 2.51411\ldots/17$, so that the numerator is $[9.23115527305872 \pm 1.6 \cdot 10^{-15}]$;
- hence $17 \le \text{numerator}/(L-\tau) \le 16.95200$ would be needed: contradiction, with margin $L - \text{numerator}/17 - \tau \ge 0.0015378$.

The margin is thin (0.048 in the function count) but certified, and all slack of the step functions is in the safe direction. Certificates for $\varepsilon = 2 \cdot 10^{-6}$ and $2.5 \cdot 10^{-6}$ (trivial partition; margins ≥ 0.01138 and ≥ 0.00131) and a certificate for CDT's own map at 10^{-6} (margin ≥ 0.0308) serve as cross-checks; only the $3 \cdot 10^{-6}$ certificate is needed, since Lemma 5.1 at $\varepsilon = 3 \cdot 10^{-6}$ covers every pair below it. An independent non-rigorous sampling with a q-product instead of η gives numerator ≈ 9.2297.

5.5. **The ε-dependent steps of CDT's chain.** Lemma 14.2.4 of [1] needs $|a/b \mp 1| < \frac{1}{2}$, and Lemma 14.2.6 needs $0 < |a/b - 1| < 10^{-3}$; here $|a/b - 1| < \varepsilon(1 + 2\varepsilon) \le 10^{-5}$. The proof of Lemma 14.2.6 uses only $b - a = 2(n - m) \in \mathbb{Z} \setminus \{0\}$, so $|m - n| \ge 2$ changes nothing. [1, Lemma 14.3.1] needs a 10×10 determinant $\det(a, b)$ to be nonzero. We recomputed it exactly from the operator of [1, Prop. 14.1.11] and the reduction [1, (14.3.2)–(14.3.5)], with numerators S_k of the shape [1, (14.3.6)] (one for each basis vector of the unknown Q), and proved (Proposition 5.2 below)

$$\det(a,b)(a-b)^{86} = \kappa\, a^6 b^6 (a+b)^{24} Q_1^4 Q_2^4 Q_3^4 Q_4^2 Q_5^2\, \Phi_4 \Phi_6 \Phi_{14}^3 \Phi_{26} \Phi_{79}^3,$$

with $\kappa = 645700815/37778931862957161709568$, $Q_1 = 4a^2 - 4ab + 1$, $Q_2 = 4b^2 - 4ab + 1$, $Q_3 = 4a^2 - 4ab + 4b^2 - 3$, $Q_4 = 16a^2 - 40ab + 16b^2 + 9$, $Q_5 = 80a^2 - 128ab + 80b^2 - 45$, and Φ_d irreducible of a-degree d. The factors are handled as follows.

- a, b, $a \pm b$ are nonzero, and the Q_i are odd for odd a, b.
- $\Phi_4 = 80a^4 - 88a^3b + 24a^2b^2 - 399a^2 - 88ab^3 + 786ab + 80b^4 - 399b^2 + 9$ is odd for odd a, b, hence nonzero. (CDT's Lemma 14.2.6 asserts $\mathrm{Res}(R_{10}, y(4-y)) \ne 0$, but its proof treats only Φ_{14} and Φ_{79}.)
- For Φ_{14} and Φ_{79} we use the argument in the proof of CDT's Lemma 14.2.6, which uses only $|b - a| \ge 1$ and $|a/b - 1| \le 10^{-3}$; a larger $|b - a|$ only shrinks the error terms.
- Φ_{26}: Lemma 14.2.4 gives $\mathrm{Res}(R_4, R_{10}) \ne 0$ by irreducibility and division, for $|a/b - 1| < \frac{1}{2}$.

- Φ_6 divides none of $\Delta_y(R_{10})$, $\mathrm{Res}_y(R_{10},R_4)$, $\mathrm{Res}_y(R_{10},y(y-4))$, contrary to the text of [1], so neither lemma covers it. Explicitly,

$$\begin{aligned}\Phi_6(a,b) = {}& 59290a^6b^2 - 185493a^5b^3 + 249865a^4b^4 - 185493a^3b^5 + 59290a^2b^6 \\ & + 8491a^5b - 65541a^4b^2 + 119686a^3b^3 - 65541a^2b^4 + 8491ab^5 \\ & - 6090a^4 + 15225a^3b - 21000a^2b^2 + 15225ab^3 - 6090b^4 \\ & + 975a^2 - 4260ab + 975b^2 + 1755,\end{aligned}$$

  and $\Phi_6(a,b) \equiv 16 \pmod{32}$ for all odd a,b (check over the 16×16 odd residue classes mod 32). So $\Phi_6(a,b)\neq 0$ at every odd pair. (Also $|\Phi_6(a,b)|/|a|^8 \geq 1864$ whenever $a\neq b$ are integers with $|a/b-1|<1/20$, so Φ_6 has no integer zeros near the diagonal.)

Hence, given CDT's set-up, [1, Lemma 14.3.1] holds for all odd $a\neq b$ with $|a/b-1|<10^{-3}$, with no extra hypothesis. The set of irreducible factors agrees with CDT's printed list; the multiplicities differ, which does not matter. A misprint in [1]: the factor "$1+4b^2-2ab$", printed in both (14.2.2) and (14.2.3), should read $1+4b^2-4ab$.

Proposition 5.2. *Given the shape* [1, (14.2.1)] *of the operator of* [1, Prop. 14.1.11] *and the reduction* [1, (14.3.2)–(14.3.5)]*, the displayed identity for* $\det(a,b)(a-b)^{86}$ *holds in* $\mathbb{Q}(a,b)$*, hence at every pair of integers* $a\neq b$*.*

Proof (computer-assisted, exact). The operator. CDT print only R_4, R_{10} and c_3/c_4 of their operator $\mathcal{L}=\sum_{i\leq 4} c_i D^i$. We interpolate a candidate $\tilde{\mathcal{L}}\in\mathbb{Q}[a,b,y][D]$ with $c_4=(y-4)^2y^3R_4R_{10}$ and verify it: the coefficients of y^k, $k<63$, of $\tilde{\mathcal{L}}(G_A)$ vanish identically in $\mathbb{Q}[a,b]$, where $G_A=\sum_n P_n(a)P_n(b)p_n(y)$ (P_n Legendre polynomials, $p_n(x+w)=x^n+w^n$ for $w=x/(x-1)$), and these 63 conditions on the 63 unknowns of CDT's ansatz have rank 63 (already at $(a,b)=(7,5)$). So the ansatz has a unique solution over $\mathbb{Q}(a,b)$, and CDT's operator is $\tilde{\mathcal{L}}$; also c_3/c_4 agrees symbolically with CDT's printed formula (`f1_Lsym.py`).

An a priori degree bound. With $\tilde{\mathcal{L}}$, the 10 numerators $S_k=y^4R_{10}^4\sum_i a_i y^i$ obtained from $b_3=(y-4)^2R_4y^k/R_{10}$, $k=0,\dots,9$ (CDT's (14.3.6) is the case of a general $Q=\sum_k q_k y^k$, with one numerator S_{42}), are explicit in $\mathbb{Q}[a,b,y]$. In the variables $t=a-b$ and b, the leading coefficient of R_{10} is $-32t^4$, and the remainders of S_k modulo R_{10} give $\det(a,b)=\det M'(t,b)/t^{200}$ with M' a polynomial matrix. Summing the largest entry degrees column by column gives $\deg_b\det M'\leq 690$ (and $\deg_t\det M'\leq 685$), using the entries only (`f1_detsym.py`).

The identity. It is equivalent to $\det M'(t,b)-\kappa\, t^{114}\Pi(t+b,b)=0$, where $\kappa\Pi$ is the right side; this polynomial has b-degree at most $\max(690,\deg\Pi)=690$. For each of the 692 integers $b_0=-346,\dots,345$ we computed $\det M'(t,b_0)\in\mathbb{Q}[t]$ exactly (fraction-free elimination, every division checked exact) and found it equal to $\kappa t^{114}\Pi(t+b_0,b_0)$ (`f1_`

`identity.py`, about 35 minutes). A polynomial of b-degree ≤ 690 that vanishes on 692 lines $b = b_0$ is zero. □

An earlier version of this computation interpolated $\det(a, b_0)(a - b_0)^{86}$ on ten lines from 420 values each, which gives the right answer (it is a consequence of Proposition 5.2) but, by the bound above, did not by itself prove it (`code/thm3/f1/`).

Proof of Theorem 3. A relation for a pair with $|m/n - 1| < 3 \cdot 10^{-6}$ gives, by [1, Prop. 14.1.11, Lemma 14.3.1] and Section 5.5, 17 functions as in Section 5.1, linearly independent over $\mathbb{Q}(y)$ and holonomic. The map of Section 5.4 is admissible for them, and Theorem 2.2 gives $17 \le 16.952$, a contradiction. □

6. Pooling pairs: proof of Theorem 4

Suppose there are K pairwise inequivalent exceptional classes with representatives (m_k, n_k) in the range. Each gives a G-function G_k and its ten associated functions (the *G-family* of the pair); the pure functions $B_1, \dots, B_7$ do not depend on the pair. The idea is to put all $7 + 10K$ functions into one holonomy bound.

6.1. **Joint independence.**

Lemma 6.1. *Assume* [1, Lemma 14.2.4]. *Let (a, b), (c, d) be integer pairs with $|a|, |b|, |c|, |d| \ge 2$, $a \ne b$, $c \ne d$, $|a/b - 1| < \frac{1}{2}$, $|c/d - 1| < \frac{1}{2}$. If $Z_4(a, b) \cap Z_4(c, d) \ne \varnothing$, then $\{c, d\} = \{a, b\}$ or $\{c, d\} = \{-a, -b\}$. Moreover $0, 4 \notin Z_4(a, b)$.*

Proof. $Z_4(a, b)$ is the root set of the quartic $R_4(a, b; y)$ of [1, (14.2.1)], which is irreducible over $\mathbb{Q}$ by [1, Lemma 14.2.4], with constant term 1 and leading coefficient $4(a - b)^2$. Two irreducible quartics with a common root and constant term 1 are equal. With $\sigma = a^2 + b^2$ and $\pi = ab$ the coefficients of y^4 and y^3 are $c_4 = 4\sigma - 8\pi$ and $c_3 = 32\pi^2 - 16\pi\sigma + 16\pi - 8\sigma$, and $c_3 = -2c_4 - 4\pi c_4$ identically. So R_4 determines π and σ, hence $(a + b)^2$ and $(a - b)^2$, hence $\{a, b\}$ up to a common sign. Finally $R_4(0) = 1$ and $R_4(4) = (16a^2 - 40ab + 16b^2 + 9)^2$, which is odd (`code/thm4/lemmaD.py`). □

Lemma 6.2. *Every branch of every function in the G-family of (a, b) is holomorphic on $\mathbb{C} \setminus (\{0, 4\} \cup Z_4(a, b))$, and every branch of $B_1, \dots, B_7$ is holomorphic on $\mathbb{C} \setminus \{0, 4\}$.*

Proof sketch. $P(x)$ is a combination of Hadamard products $f \star g$ with $f \in \{A(a, \cdot), H(a, \cdot)\}$ and $g \in \{A(b, \cdot), H(b, \cdot)\}$, each factor continuable along every path avoiding its two singular points. By Hadamard's multiplication theorem in its global form (deform the contour in $f \star g(x) = \frac{1}{2\pi i} \oint f(t) g(x/t)\, dt/t$; [2] for the star-domain form, and [5, Thm. A] (preprint) for the path-wise form: if f continues along every path avoiding a closed discrete set S and g along every path avoiding T, then $f \star g$ continues along

every path avoiding $\{0\} \cup S \cdot T$), $f \star g$ continues along every path avoiding 0 and the four products. In y, the two roots of $x^2 - yx + y = 0$ are distinct off $y \in \{0, 4\}$ and avoid $\{0\} \cup \{\text{products}\} \cup \{p/(p-1)\}$ exactly off $\{0\} \cup Z_4$, so $G = P(x) + P(x/(x-1))$ continues along every path in $\mathbb{C} \setminus (\{0, 4\} \cup Z_4)$; derivatives and primitives inherit this. The B_i are functions on $\mathbb{P}^1 \setminus \{0, 4, \infty\}$ [1, §10]. □

Lemma 6.3 (cross-pair independence)**.** *Make the assumptions of Theorem 3. Let (a_k, b_k), $k = 1, \dots, K$, be odd pairs with $0 < |a_k/b_k - 1| < 10^{-3}$, pairwise inequivalent ($\{a_j, b_j\} \neq \pm\{a_k, b_k\}$ for $j \neq k$), each with a relation vector as above. Then $B_1, \dots, B_7$ and the K G-families are linearly independent over $\mathbb{C}(y)$.*

Proof sketch. Suppose $\sum_i S_i B_i + \sum_k \sum_l R_{kl} F_{kl} = 0$ with $S_i, R_{kl} \in \mathbb{C}(y)$, F_{kl} the G-family of pair k. Fix k. CDT's proof of Lemma 14.3.1 (following Lemma 12.1.1) uses a loop γ_k, a conjugate of a small loop around a point $s \in Z_4(a_k, b_k)$, with $G_k \neq \gamma_k G_k$. By Lemma 6.1, $s \notin \{0, 4\} \cup Z_4(a_j, b_j)$ for $j \neq k$, so by Lemma 6.2 every branch of every F_{jl} ($j \neq k$) and of every B_i is holomorphic at s and fixed by γ_k. Applying $1 - \gamma_k$ leaves $\sum_l R_{kl}(1 - \gamma_k) F_{kl} = 0$, which is exactly the relation reached in CDT's single-pair proof (the varied primitives $(1 - \gamma_k) \int y^e (G_k - T)$ are primitives of $y^e (1 - \gamma_k) G_k$, and CDT's argument does not depend on the choice of primitive). CDT's argument, with the determinant of Section 5.5, gives $R_{kl} = 0$. Doing this for every k leaves $\sum S_i B_i = 0$, so $S_i = 0$ [1, Lemma 12.0.1]. □

Pairs sharing an entry, (a, b) and (a, c), are inequivalent, so Lemma 6.3 covers them. As a falsification test (not used in the proof), the exact rank modulo two primes $p > 2008$ of the union of the series, truncated at y^{1000}, with polynomial coefficients of degree $\leq D$, is full for 30 instances ($K = 1, 2, 3$, including shared entries, pairs in the range and degenerate relation vectors), while duplicated pairs give exactly the expected nullity $10(D + 1)$ (`code/thm4/xrank.py`, `runs/xrank.out`).

6.2. **The pooled holonomy inequality.** By Lemma 6.3 we have $\mathfrak{m} = 7 + 10K$ functions, holonomic and linearly independent over $\mathbb{Q}(y)$, with $\sigma = (0, 2, 2, 4, 4, 4, 4 \mid 4^{10K})$ and $e = (0, 0, 1, 0, 0, 1, 1 \mid (0^4, 1^6)^K)$ (now in the order $B_1, \dots, B_7$, then for each pair G, G', G'', G''' and the six primitives; for $K = 1$ this is CDT's list reordered within the block $\sigma_i = 4$, which changes neither $\tau(b)$ nor $\tau^\sharp(e)$). Evaluating $\tau^\sharp$ of [1, (6.0.6)] at the explicit point $\xi = 169/100$ (an upper bound for the minimum, hence a valid τ) gives

$$\tau_2 = \frac{1106977681}{262702440} = 4.2138081\dots, \qquad \tau_3 = \frac{29601442477}{7082066992} = 4.1797744\dots$$

for $K = 2$ and $K = 3$ (`tau_pool.py`; CDT's τ for $K = 1$ is reproduced by the same code).

For the maps of this section the disc condition (c′) fails, and we use instead:

Lemma 6.4 (ray lemma)**.** *Let φ be holomorphic on a neighbourhood of $\overline{\mathbb{D}}$ and $\Gamma = (-\infty, -Y_1] \cup [Y_1, \infty)$. If $\varphi(\mathbb{T}) \cap \Gamma = \varnothing$, then $\varphi(\overline{\mathbb{D}}) \cap \Gamma = \varnothing$.*

Proof. For $y_0 \in \Gamma$ the number of zeros of $\varphi - y_0$ in $\mathbb{D}$ is the winding number of $\varphi(\mathbb{T})$ about y_0. It is constant on each half-line of Γ (connected subsets of $\mathbb{C} \setminus \varphi(\mathbb{T})$), and 0 for $|y_0| > \max_{\mathbb{T}} |\varphi|$. □

With Lemma 5.1 (real singular values), (a), (b) and "$\varphi(\mathbb{T}) \cap \Gamma = \varnothing$" make φ admissible for all pairs in the range simultaneously. The certificate decides $\varphi(s) \notin \Gamma$ on each interval either by $|\varphi| < Y_1$ or by a lower bound for $|\sin \arg \varphi|$. The results (`cert_pool.py`, trivial partition, $\delta_{\max} = 0.002$):

range ε	K	$\mathfrak{m}$	L	numerator $\le$	certified margin $L - R^+/\mathfrak{m} - \tau$
10^{-5}	2	27	4.7056374	8.795131	≥ 0.16608
$3 \cdot 10^{-5}$	2	27	4.5683356	8.064226	≥ 0.05585
$5 \cdot 10^{-5}$	2	27	4.5159123	7.819914	≥ 0.01247
10^{-4}	3	37	4.4006730	7.326966	≥ 0.02287

The $5 \cdot 10^{-5}$ map was re-optimised for $L - T/27$ (`reopt.py`); the 10^{-5} map is an earlier optimised map dilated by $r = 1 - 2^{-14}$ (the undilated map crosses the ray $[Y_1, \infty)$ near $1/\varepsilon$, which the certificate detects); the other two are stored optimised maps. Only the rows $5 \cdot 10^{-5}$ and 10^{-4} are needed. Lemma 5.1 is proved for $\varepsilon \le 10^{-4}$; at $\varepsilon = 5 \cdot 10^{-5}$ the map has little room ($\varphi(\mathbb{T})$ crosses the negative axis at ≈ -19988, inside $Y_1 = 19994$), and an exhaustive high-precision check of Lemma 5.1 on a boundary window found the closest genuine value at $Y_1 + 4.25$.

Proof of Theorem 4. Suppose there were $K = 2$ (resp. 3) inequivalent exceptional classes with representatives in $|m/n - 1| < 5 \cdot 10^{-5}$ (resp. 10^{-4}). CDT's construction gives a G-function for each, and Lemma 6.3 (applicable since $|a/b - 1| < 10^{-3}$) gives $7 + 10K$ independent functions. The certified map is admissible for all of them, and Theorem 2.2 gives $27 \le 25.885$ (resp. $37 \le 33.169$), a contradiction. This proves (i) and (ii); (iii) is Theorem 3. For (iv): a consecutive pair $(N, N+1)$ in these ranges has $1/(N+2) < 10^{-4}$, so $q = N + 1 \ge 10^4 > 54$, and Hata's theorem gives the four-term independence. □

7. What is computer-assisted, and how to reproduce it

The folder `code/` of the accompanying repository `https://github.com/telaviv/log-products-irrationality` contains every script and certificate output used above, in one directory per result; `code/README.md` maps each number of this paper to a script and an output file. Software: Python 3.12, python-flint 0.9.0 (FLINT/Arb ball arithmetic), SymPy 1.14, mpmath 1.3, NumPy and SciPy (optimisation only). The computer-assisted steps are:

step	kind	script (folder)	time
Relations for $c_{n,l}$; order-4 recurrence; char. polynomial (Prop. 3.4)	exact	`i1_*.py` (`thm1_cor2`)	minutes
Contact orders, sanity checks (Lemma 3.1)	exact	`i1_contact.py`	10 s
L_{13} and I1′ per q (Prop. 3.6)	exact	`findop.py`, `i1_rec25.py`, `fourterm/mkrec.py`	seconds per q
Budget $M_{14}(q)$	Arb	`fourterm/budget_4t.py`	seconds
Local exponents, Casoratian, p_0 roots (Prop. 4.3, Lemma 4.4)	exact	`exponents.py` (`fourterm`)	15 s
Certified continuation, g_A, g_2, g_3	Arb	`mono13.py`	$\approx$ 40 min per q
Irreducibility (Prop. 4.5)	Arb	`fourterm/irred.py`	2 s per q
Rank-14 cross-check (Remark 4.6)	Arb	`cert_det.py`	seconds
Map certificate, Theorem 3	Arb + exact	`cert3.py`, `cert3d.py` (`thm3`)	$\approx$ 8 min
$\det(a,b)$ identity and Φ_6 (Prop. 5.2)	exact	`f1_Lsym.py`, `f1_detsym.py`, `f1_identity.py`, `f1_phi6_nonvanish.py` (`thm3/f1`)	35 min
New pairs (1.1)	exact	`f7_newpairs.py`	seconds
Lemma 6.1; pooled τ; falsifier	exact	`lemmaD.py`, `tau_pool.py`, `xrank.py` (`thm4`)	minutes
Pooled map certificates (Theorem 4)	Arb + exact	`cert_pool.py`	$\approx$ 4 min each

The map optimisations (`lib_opt.py`, `lib_range.py`, `reopt.py`) and the falsifier are not part of any proof: they produce candidates that the certificates then check. The referees (Section 9) re-ran the main certificates bit for bit on separate copies (among them the budget and irreducibility certificates of Theorem 1, the rank certificate at $q = 29$, the $3\cdot10^{-6}$ map certificate and the $5 \cdot 10^{-5}$ pooled certificate), and re-ran the continuation step independently, with different initial data and discretisation, at $q = 26, 32$ (Remark 4.6; outputs in `code/thm1_cor2/runs/referee_rerun/`) and at $q = 40, 53$ (the gate referee's reruns of Theorem 1, which reproduce both determinants of Proposition 4.5 to all 30 printed digits; their outputs are not included in `code/`).

What the computer establishes is, in each case, a finite statement: an identity of explicit polynomials, a determinant ball that excludes 0, or an exact rational upper bound for an integral. What it does not establish: the

hand arguments (Lemmas 3.3, 4.2, 4.4, the reflection criterion, Lemmas 5.1, 6.1, 6.4, the reductions), the theorems of [1] we use, and the correctness of python-flint's Arb routines (in particular `acb.modular_eta` and matrix determinants), which we take on trust. Arb [4] is now part of FLINT [8]; python-flint [9] 0.9.0, as used, is linked against FLINT 3.6.0.

8. Limits

- $q \le 25$ *is out of reach of this route.* At $q = 25$, $M_{14}(25) \approx -0.0665 < 0$, and 14 is the largest budget available for this module (order 13 plus the constant). Neither the rank nor the irreducibility can help; a new family of forms, a better denominator type or a better map would be needed.
- 10^{-5} *is not reached.* For Theorem 3 the best range margin we found at $4 \cdot 10^{-6}$ is -0.013 (partition form), and even for a single pair the best margin at 10^{-5} is -0.0086 [numerical, local optima]. We have no certified ceiling: the only rigorous ceiling we know is far from tight. This agrees with [1, Remark 14.0.4], which says that the constant could be improved but probably not beyond about 10^{-4}.
- *Pooling stops near* $6 \cdot 10^{-5}$ *for* $K = 2$ on the maps we explored (margins -0.009 at $6 \cdot 10^{-5}$, -0.031 at $7 \cdot 10^{-5}$, -0.069 at 10^{-4}, numerical).
- *The main inherited assumption*, the irreducibility of $\mathcal{L}_{a,b}$ (asserted in CDT's template Lemma 12.1.1 and implicit in Lemma 14.3.1), is open. Its local system is the push-forward of a rank-2 system along the double cover $x \mapsto y$; by Mackey's criterion it would follow from the irreducibility of the rank-2 system and its non-isomorphism with its conjugate under $x \mapsto x/(x-1)$ [heuristic].

9. Provenance and use of AI tools

Provenance. The mathematical inputs are: Lysov's explicit formula for Q_n [6], used as a definition (his Lemmas 7–9 are re-proved above); the holonomy bounds and the construction for products of two logarithms of Calegari, Dimitrov and Tang [1], used as stated; Hata's theorem [3], used only for the extension to $q \ge 54$ and for Theorem 4(iv). The scheme of Section 4 is the one suggested in [1, §15.1.10]. The recurrence of Proposition 3.4 was found by modular linear algebra and then proved by creative telescoping; the minimal operators were found the same way. Apart from the preprints cited above ([1], [6] and, for Lemma 6.2, [5]), nothing in this paper depends on unpublished work by others.

Use of AI tools. This work was carried out with substantial assistance from AI systems (Anthropic's Claude models), which were used to search the literature, to develop the proofs, to write the scripts and produce the certificates, to referee the arguments, and to draft this text. The referee

reports were also produced by Claude models, working adversarially on separate copies of the files; they are not a substitute for a human expert's reading. The author directed the project and is responsible for its content. None of the results is verified in Lean or another proof assistant.

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