hexagon:2611.00702v1

LINEAR INDEPENDENCE OF RECIPROCAL AND RECIPROCAL-CUBE SUMS OF LUCAS SEQUENCES

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Abstract. Let $\eta > 1$ be a unit of a real quadratic order, of norm $Q = \pm 1$, and let $U_n = (\eta^n - \bar{\eta}^n)/(\eta - \bar{\eta})$ be its Lucas sequence. We prove that for each sign $\delta = \pm 1$ the three numbers 1, $\sum_{n\geq 1} \delta^n/U_n$ and $\sum_{n\geq 1}(\delta Q)^n/U_n^3$ are linearly independent over $\mathbb{Q}$. For the Fibonacci numbers this says that 1, $\sum_n (-1)^n/F_n$, $\sum_n F_n^{-3}$ are linearly independent, and so are 1, $\sum_n F_n^{-1}$, $\sum_n (-1)^n/F_n^3$; for F_{2n} it gives 1, $\sum_n F_{2n}^{-1}$, $\sum_n F_{2n}^{-3}$. In particular $\sum_n (-1)^n/U_n^3$ is irrational for every such sequence. The proof is the Hankel-determinant construction of our earlier papers on $\sum_n U_n^{-3}$, run with the operator $c_3\theta(\theta+1)+c_1$, where the integers (c_3, c_1) come from a hypothetical linear relation, and with a sign attached to the weights. The whole argument is formalised in Lean 4 with Mathlib, using only the standard axioms.

1. Introduction

For integers P, Q let $U_n = U_n(P,Q)$ be the Lucas sequence $U_0 = 0$, $U_1 = 1$, $U_{n+1} = PU_n - QU_{n-1}$. When $Q = \pm 1$ and $D = P^2 - 4Q > 0$ is not a square, $U_n = (\eta^n - \bar{\eta}^n)/\sqrt{D}$ with $\eta = (P + \sqrt{D})/2$ a unit of the order $\mathbb{Z}[\eta]$ of norm $\eta\bar{\eta} = Q$. Every unit $\eta > 1$ of a real quadratic order arises in this way, with $P = \eta + \bar{\eta} \geq 1$ if $Q = -1$ and $P \geq 3$ if $Q = 1$; we call these pairs (P, Q) *admissible*. The Fibonacci numbers are $U_n(1, -1)$, the Pell numbers $\mathcal{P}_n$ are $U_n(2, -1)$, and $F_{kn}/F_k = U_n(L_k, (-1)^k)$, where L_k is the k-th Lucas number.

Theorem 1.1. *Let (P, Q) be admissible, $U_n = U_n(P, Q)$, and $\delta \in \{1, -1\}$. Then the numbers*

$$1, \qquad A_\delta = \sum_{n\geq 1} \frac{\delta^n}{U_n}, \qquad B_\delta = \sum_{n\geq 1} \frac{(\delta Q)^n}{U_n^3}$$

are linearly independent over $\mathbb{Q}$.

The four cases are:

2020 *Mathematics Subject Classification.* Primary 11J72; Secondary 11B39, 33D15, 41A21, 68V20.

Key words and phrases. Lucas sequences, Fibonacci numbers, Pell numbers, linear independence, irrationality, q-series, Hankel determinants, Lean.

$Q = \mathrm{N}(\eta)$	δ	A_δ	B_δ
$+1$ $(P \geq 3)$	$+1$	$\sum 1/U_n$	$\sum 1/U_n^3$
$+1$	-1	$\sum(-1)^n/U_n$	$\sum(-1)^n/U_n^3$
-1 $(P \geq 1)$	-1	$\sum(-1)^n/U_n$	$\sum 1/U_n^3$
-1	$+1$	$\sum 1/U_n$	$\sum(-1)^n/U_n^3$

So the signs of the two sums agree when $\mathrm{N}(\eta) = 1$ and differ when $\mathrm{N}(\eta) = -1$. The method forces this pairing; the other pairings, for example 1, $\sum_n F_n^{-1}$, $\sum_n F_n^{-3}$, are not covered.

Corollary 1.2. (i) *For every admissible* (P, Q), $\sum_{n\geq 1}(-1)^n/U_n(P,Q)^3$ *is irrational.*

(ii) *The numbers* 1, $\sum_n(-1)^n/F_n$, $\sum_n F_n^{-3}$ *are linearly independent over* $\mathbb{Q}$*, and so are* 1, $\sum_n F_n^{-1}$, $\sum_n(-1)^n/F_n^3$.

(iii) *For* $k \geq 1$ *and* $\delta = \pm 1$*, the numbers* 1, $\sum_n \delta^n/F_{kn}$, $\sum_n(\delta(-1)^k)^n/F_{kn}^3$ *are linearly independent over* $\mathbb{Q}$*. In particular* 1, $\sum_n F_{2n}^{-1}$, $\sum_n F_{2n}^{-3}$ *are linearly independent.*

(iv) *The numbers* 1, $\sum_n(-1)^n/\mathcal{P}_n$, $\sum_n \mathcal{P}_n^{-3}$ *are linearly independent over* $\mathbb{Q}$*, and so are* 1, $\sum_n \mathcal{P}_n^{-1}$, $\sum_n(-1)^n/\mathcal{P}_n^3$.

Proof. (i) Take $\delta = -Q$, so that $B_\delta = \sum_n(-1)^n/U_n^3$; a relation $B_\delta = r$ with $r \in \mathbb{Q}$ would be a nontrivial relation among $1, A_\delta, B_\delta$. (ii) and (iv) are the cases $(P, Q) = (1, -1)$ and $(2, -1)$ with $\delta = \mp 1$. (iii) The pair $(L_k, (-1)^k)$ is admissible ($L_k \geq 1$, and $L_k \geq 3$ for $k \geq 2$), and $U_n = F_{kn}/F_k$ by Binet's formula. So $A_\delta = F_k \sum_n \delta^n/F_{kn}$ and $B_\delta = F_k^3 \sum_n(\delta(-1)^k)^n/F_{kn}^3$, and multiplying the members of a family by nonzero rationals does not affect linear independence over $\mathbb{Q}$. □

Context. The weight-one sums are classical. André-Jeannin proved that $\sum_n x^n/w_n$ is irrational for a class of binary recurrences w_n and integers x that includes every $U_n(P, \pm 1)$ with $x = \pm 1$, in particular $\sum_n \delta^n/U_n \notin \mathbb{Q}$ [1]. Linear independence results for weight-one Lambert series at Pisot-type algebraic integers, with applications to reciprocal sums of Fibonacci and Lucas numbers, were proved by Bundschuh and Väänänen [2], and for reciprocal sums of Fibonacci and Lucas numbers along certain sequences of pairwise coprime indices, such as the squares of primes, by Duverney, Suzuki and Tachiya [5]. For the even powers, Duverney, Ke. Nishioka, Ku. Nishioka and Shiokawa proved that $\sum_n U_n^{-2s}$ $(s \geq 1)$ is transcendental for both signs of Q [4], and Elsner, Shimomura and Shiokawa determined the algebraic relations among such numbers [6, 7, 8]. These sums are values of quasi-modular forms at algebraic points, to which Nesterenko's theorem [12] applies (see [19, §3.4] for the Fibonacci zeta function). Wang [20, Cor. 5.2] expresses the generalised sums $\sum_n n^p U_n^{-s}$ through derivatives of Eisenstein series when $p \equiv s \pmod 2$ and $Q = 1$, and when p and s are even and $Q = -1$ (in his notation the sign is $\alpha\beta = \pm 1$), and derives algebraic independence results for them; his Cor. 5.2 concerns the non-alternating sums. The sums

of Theorem 1.1 have $p = 0$ and $s \in \{1, 3\}$, so the non-alternating ones are outside these parity classes. Wang also remarks that some alternating sums $\sum_n (-1)^{n-1} n^p U_n^{-s}$ admit such expressions under parity conditions that he does not state; we have not found such an expression for $p = 0$, $s \in \{1, 3\}$, and we know of no modular interpretation of the sums considered here.

For weight three, $\sum_n F_n^{-3} \notin \mathbb{Q}$ is the main result of [11], and $\sum_n U_n^{-3} \notin \mathbb{Q}$ for every admissible (P, Q) is proved in [10]; both papers have Lean formalisations. The alternating sums $\sum_n (-1)^{n-1} U_n^{-3}$, for the Lucas sequence of every quadratic unit $\eta > 1$, (and the analogous sums over U_{kn}) were shown to be irrational by twinheterodox [15], in a preprint that builds on [10], is described by its author as primarily AI-generated, and has no formal proof. Corollary 1.2(i) is that statement, and the case $\delta = Q$ of Theorem 1.1 with no weight-one term contains the main result of [10].

Mixed-weight independence results are known for Lambert-type series at integer bases: Bundschuh and Väänänen [3] prove that 1, $\sum_n 1/(q^n - 1)$, $\sum_n (-1)^{n-1}/(q^n - 1)^2$ are linearly independent over $\mathbb{Q}$ for suitable rational q, and a preprint posted while this paper was being completed [16] (described by its author as primarily AI-generated; no formal proof) claims the linear independence of 1 and $\sum_n (b^n - 1)^{-j}$, $j = 1, 2, 3$, for integers $b \geq 2$. Since $b^n - 1 = (b - 1) U_n(b + 1, b)$, these are reciprocal sums of Lucas sequences with $Q = b$, not ± 1, and they do not cover the sums considered here. For the Lucas sequences of quadratic units we are not aware of an earlier linear independence statement involving a weight-one and a weight-three reciprocal sum, such as 1, $\sum_n (-1)^n / F_n$, $\sum_n F_n^{-3}$.

The method. Suppose $q_0 + q_1 A_\delta + q_3 B_\delta = 0$ with integers q_i, not all zero. Then $(q_1, q_3) \neq (0, 0)$, and we put $(c_3, c_1) = (D q_3, 2 q_1)$. With $t = \eta^{-1}$, $\rho = \delta t$ and $w = Q t^2$, the number

$$\Xi = \sum_{N \geq 1} \rho^N \Big(\frac{2 c_3 w^N}{(1 - w^N)^3} + \frac{c_1}{1 - w^N} \Big) = \frac{2 c_3 B_\delta + c_1 D A_\delta}{D \sqrt{D}} \tag{1.1}$$

is then $-2 q_0 / \sqrt{D}$, a rational multiple of $\sqrt{D}$ (Lemma 2.2). We show that this is impossible for every $(c_3, c_1) \in \mathbb{Z}^2 \setminus \{0\}$, by the construction of [11, 10]: a moving-denominator Hankel determinant in the line of Uvarov [18], Krattenthaler [9], Prévost [13] and Zudilin [21, 22], which [11] adapted from an unrefereed, AI-assisted research draft on a q-zeta value [17, 14]. The linear functional is now

$$L(g) = \sum_{n \geq 1} \rho^n \, [\Theta g](w^n), \qquad \Theta = c_3 \theta(\theta + 1) + c_1, \quad \theta = s \tfrac{d}{ds},$$

and its moments against $1/(1 - w^a s)$ are $\rho^{-a}(\Xi - S_a)$, with partial sums S_a of (1.1). The Hankel determinants $\Delta_K(X)$ of L against $D_K(s) = \prod_{a \leq K} (1 - w^a s)$ satisfy four estimates:

	size (per $K^3 \log \eta$)	
real decay $\lvert\Delta_K(\Xi)\rvert$	$-2/9$	Proposition 3.1
conjugate $\lvert\sigma\Delta_K(-\Xi)\rvert$	-1	Proposition 5.1
denominators $\log \lvert \mathrm{N}\, C_K(t)\rvert$	$\le c_* = 1.1954\ldots < 11/9$	Propositions 4.5, 4.6
nonvanishing	infinitely many K	Proposition 6.1

Then $C_K(t)D^K\Delta_K(\Xi)$ is a nonzero element of $\mathbb{Z}[\eta]$ of norm < 1 for large K, a contradiction.

Compared with [10], which is the case $\delta = Q$, $(c_3, c_1) = (1, 0)$, the changes are these. The weight-one part c_1 of the operator has no factor θ; this costs a factor $|\rho|^m = t^m$ in the decay estimate instead of the factor $|w|^{m-R}$ that θ supplies (Section 3). The moment $\nu_0 = L(1)$ no longer vanishes, so the roots of $1 - \alpha_0 = 1 - \rho$ join the denominators (Section 4); the root types depend on δ, but the elementary root count used in the proof does not. In the conjugate estimate the generating function Γ of the tails acquires a weight-one part with $\Gamma(0) \neq 0$ (Section 5). The Cauchy transform of L has simple poles when $c_3 = 0$, and is still not rational (Section 6). Everything else is unchanged, and we cite [10] for the steps that are literally the same. The constants in the estimates do not depend on (P, Q, δ); they depend on (c_3, c_1) only through lower-order terms.

Applied to the unit η^k, whose Lucas sequence is U_{kn}/U_k, Theorem 1.1 gives the analogous statement for the sums over U_{kn}, with the sign δQ^k in place of δQ.

Formal verification. The theorem, its corollaries and every step of the proof are formalised in Lean 4 with Mathlib (`https://github.com/telaviv/lucas-reciprocal-independence`; Section 8), using only the standard axioms. The analytic ingredients are formalised for every real t with $0 < t \le \varphi^{-1}$, both signs Q, both signs δ and every $(c_3, c_1) \in \mathbb{Z}^2$ (every nonzero pair for the nonvanishing), and the integrality statement for every real $\tau \neq 0$ that is not a root of unity; only the final assembly uses that t is the inverse of a quadratic unit and chooses (c_3, c_1) from the relation. The weight-one case $q_3 = 0$ is proved by the same construction (with $c_3 = 0$), so no external irrationality result is used.

2. The functional and the determinants

Notation. Throughout, (P, Q) is admissible, $\delta \in \{1, -1\}$, and $(c_3, c_1) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$. Put $D = P^2 - 4Q$, $\eta = (P + \sqrt{D})/2$, $\bar\eta = Q/\eta$, $t = \eta^{-1}$, $\mathcal{O} = \mathbb{Z}[\eta]$, and let σ be the nontrivial automorphism of $\mathbb{Q}(\sqrt{D})$, so $\sigma t = Q/t$. We write $\varphi = (1 + \sqrt{5})/2$ and $\phi_{\mathrm{E}}(d)$ for Euler's function. Put $\delta' = \delta Q$ and

$$\rho = \delta t, \qquad w = Qt^2, \qquad \alpha_k = \rho w^k = \delta Q^k t^{2k+1} \ (k \ge 0),$$

$$\psi_N = \rho^N \Big(\frac{2c_3 w^N}{(1 - w^N)^3} + \frac{c_1}{1 - w^N} \Big), \qquad S_a = \sum_{N=1}^{a} \psi_N, \qquad \Xi = \sum_{N \ge 1} \psi_N,$$

$\theta = s\frac{d}{ds}$ and $\Theta = c_3\theta(\theta+1)+c_1$. Then $|\rho| = t$, $|w| = t^2$, $\rho w = \delta' t^3$ and $w/\rho = \delta' t$. We write h_m for the complete homogeneous symmetric polynomial of degree m ($h_m = 0$ for $m < 0$), and $K \ge 2$.

Lemma 2.1. (i) *D is not a square, so $\sqrt{D} \notin \mathbb{Q}$.* (ii) *$\eta \ge \varphi$, so $0 < t \le \varphi^{-1}$.* (iii) *t and σt are the roots of $\tau^2 - PQ\tau + Q$, and $t\,\sigma t = Q$.* (iv) *$\sqrt{D} = P - 2Qt$ and $P - 2Q\,\sigma t = -\sqrt{D}$.* (v) *$U_n = t^{-n}(1-w^n)/\sqrt{D}$.* (vi) *$\psi_N = \big(2c_3\delta'^N U_N^{-3} + c_1 D\delta^N U_N^{-1}\big)/(D\sqrt{D})$, so (1.1) holds.*

Proof. (i)–(v) are [10, Lemma 2.1]; they do not involve δ. (vi) By (v), $\rho^N/(1-w^N) = \delta^N t^N/(1-w^N) = \delta^N/(\sqrt{D}U_N)$, and $\rho^N w^N/(1-w^N)^3 = \delta^N Q^N t^{3N}/(1-w^N)^3 = \delta'^N/(D\sqrt{D}U_N^3)$. The series converge absolutely, since $|\psi_N| \le (2|c_3| + |c_1|)t^N/(1-t^2)^3$. □

Lemma 2.2. *Suppose $q_0 + q_1 A_\delta + q_3 B_\delta = 0$ with $q_0, q_1, q_3 \in \mathbb{Z}$ not all zero. Then $(c_3, c_1) := (Dq_3, 2q_1) \ne (0,0)$, and for this pair*

$$\Xi = -\frac{2q_0}{\sqrt{D}} = -\frac{2q_0(P-2Qt)}{D}, \qquad D\,\Xi \in \mathcal{O}, \qquad \sigma\Xi = -\Xi.$$

Proof. If $q_1 = q_3 = 0$ then $q_0 = 0$; and $D \ne 0$. By Lemma 2.1(vi), $D\sqrt{D}\,\Xi = 2Dq_3B_\delta + 2q_1 DA_\delta = -2Dq_0$. Then use Lemma 2.1(iv) and $\sigma\sqrt{D} = -\sqrt{D}$. □

So Theorem 1.1 follows once we show that, for every $(c_3, c_1) \in \mathbb{Z}^2 \setminus \{0\}$, Ξ is not of the form $-2q_0/\sqrt{D}$ with $q_0 \in \mathbb{Z}$. This is done in Section 7.

Definition 2.3. For g analytic on a neighbourhood of $|s| \le t^2$ put $L(g) = \sum_{n\ge1} \rho^n\,[\Theta g](w^n)$.

The series converges absolutely since $|\rho^n| = t^n$ and Θg is bounded on $|s| \le t^2$. (In [10] $\Theta g = O(s)$; here $\Theta g(0) = c_1 g(0)$ need not vanish, and the factor ρ^n alone gives convergence.)

Lemma 2.4. (i) *$L(s^k) = \nu_k := (c_3 k(k+1) + c_1)\,\alpha_k/(1-\alpha_k)$ for $k \ge 0$; in particular $\nu_0 = c_1\rho/(1-\rho)$.* (ii) *For $a \ge 1$, $L\big(1/(1-w^a s)\big) = \lambda_a(\Xi - S_a)$ with $\lambda_a = \rho^{-a}$.*

Proof. (i) $\Theta s^k = (c_3k(k+1)+c_1)s^k$, and $\sum_n \rho^n w^{nk} = \alpha_k/(1-\alpha_k)$ as $|\alpha_k| < 1$. (ii) $\theta(\theta+1)(1-cs)^{-1} = 2cs/(1-cs)^3$, so $\Theta(1-cs)^{-1} = 2c_3cs/(1-cs)^3 + c_1/(1-cs)$. With $c = w^a$, $s = w^n$ and $N = a+n$ the left side is $\rho^{-a}\sum_{N>a}\rho^N\big(2c_3w^N/(1-w^N)^3 + c_1/(1-w^N)\big) = \rho^{-a}\sum_{N>a}\psi_N$. □

Let $D_K(s) = \prod_{a=1}^K (1-w^a s)$, with poles $x_a = w^{-a}$, $|x_a| = \eta^{2a}$. For a polynomial f_0 of degree $< 2K-1$,

$$\frac{f_0}{D_K} = g_0 + \sum_{a=1}^K \frac{\mathrm{res}_a(f_0)}{1-w^a s}, \qquad \mathrm{res}_a(f_0) = \frac{f_0(x_a)}{\prod_{b\ne a}(1-w^{b-a})},$$

with g_0 a polynomial of degree $\le \deg f_0 - K$. Let L_X be the linear map with $L_X(s^k) = \nu_k$ and $L_X(1/(1-w^a s)) = \lambda_a(X - S_a)$, X an indeterminate, and

$$\Delta_K(X) = \det\big[L_X(s^{i+j}/D_K)\big]_{0\le i,j<K} \in \mathbb{Q}(\sqrt{D})[X], \qquad \deg \Delta_K \le K.$$

By Lemma 2.4, $\Delta_K(\Xi) = \det[L(s^{i+j}/D_K)]$.

The formal parameter. Every quantity above is the value at $\tau = t$ of a rational function of an indeterminate τ with rational coefficients, which depends on (P, Q, δ, c_3, c_1) only through (Q, δ, c_3, c_1): $\rho = \delta\tau$, $w = Q\tau^2$, $\alpha_k = \delta Q^k \tau^{2k+1}$, $\lambda_a = \delta^a \tau^{-a}$, $\psi_N = (\delta\tau)^N\big(2c_3(Q\tau^2)^N/(1-(Q\tau^2)^N)^3 + c_1/(1-(Q\tau^2)^N)\big)$. Write $\Delta_{K,\tau}(X) = \Delta_{K,\tau}^{(Q,\delta)}(X) \in \mathbb{Q}(\tau)[X]$ for the resulting polynomial (c_3, c_1 being fixed), and $w(\zeta)$, $\alpha_k(\zeta)$ for the values at $\tau = \zeta$. Since $\sigma t = Q/t$, the coefficientwise conjugate $\sigma\Delta_K$ is the value of $\Delta_{K,\tau}$ at $\tau = Q/t$. Both values are defined because no denominator vanishes at a real number that is neither 0 nor a root of unity.

For $(\delta, c_3, c_1) = (Q, 1, 0)$ the moments λ_a are half, and Ξ, S_a twice, those of [10], so $\Delta_K(X) = \Delta_K^{\mathrm{U}}(X/2)$, where Δ_K^{U} is the determinant of [10]; the cyclotomic multiplicities of the coefficients are the same.

Lemma 2.5. $\Delta_{K,\tau}^{(Q,-\delta)}(X) = \Delta_{K,-\tau}^{(Q,\delta)}(X)$.

Proof. The construction uses τ only through w and ρ: the moments are ν_k (a function of $\alpha_k = \rho w^k$), $\lambda_a = \rho^{-a}$ and ψ_N (a function of ρ^N and w^N), and D_K and the residues res_a depend on w only. Replacing τ by $-\tau$ leaves $w = Q\tau^2$ unchanged and maps $\rho = \delta\tau$ to $(-\delta)\tau$. □

3. Real decay

Proposition 3.1. $\log|\Delta_K(\Xi)| \le -\frac{2}{9}(K^3 + 3K^2 - 4)\log\eta + K\log(|c_3| + |c_1|) + O(K^2)$, *with an absolute implied constant.*

Proof. We follow [10, Proposition 3.1]; only the bound for the weight-one part is new. Let $V_j(s) = \prod_{u=1}^{j}(s - w^{\lceil u/3\rceil})$, $R_j = \lceil j/3\rceil$ and $a_j = \sum_{u\le j}\lceil u/3\rceil$. The change of basis from $1, s, \dots, s^{K-1}$ to $V_0, \dots, V_{K-1}$ is unitriangular, so $\Delta_K(\Xi) = \det[L(V_iV_j/D_K)]$.

Put $W = 1/D_K$ and $R = \max(R_i, R_j)$. Since $\Theta g(s_0)$ depends only on the 2-jet of g at s_0, the m-th term of $L(V_iV_jW)$ vanishes for $m < R$: then V_iV_j has a triple zero at w^m. This applies to both parts of Θ. For $m \ge R$ and $s = w^m$ every root β of V_j satisfies $|s| \le |\beta|$, so $|s - \beta| \le 2|\beta|$ and $\theta(s-\beta) = s$. Hence $|V_j(s)| \le 2^j|w|^{a_j}$ and $|\theta^u V_j(s)| \le j^u 2^j |w|^{a_j}|s|/|w|^{R_j}$ for $u = 1, 2$. On $|s| \le t^2$ we have $|w^a s| \le \varphi^{-2a-2}$, so $|W|$, $|\theta W|/|s|$ and $|\theta^2 W|/|s|$ are bounded by an absolute constant C_W.

The weight-three part. Since $\theta(\theta+1)$ kills constants, by Leibniz every term of $\theta(\theta+1)(V_iV_jW)$ contains at least one θ on some factor, hence a factor $\le |s|/|w|^R = |w|^{m-R}$. With $|\rho|^m \le 1$ and $\sum_{m\ge R}|w|^{m-R} \le (1-\varphi^{-2})^{-1}$ this part of $L(V_iV_jW)$ is at most $|c_3|C'(i+j+1)^2 4^{i+j}|w|^{a_i+a_j}$, as in [10].

The weight-one part. The m-th term is $c_1\rho^m V_iV_jW(w^m)$, of absolute value at most $|c_1|C_W4^{i+j}|w|^{a_i+a_j}t^m$. There is no factor $|w|^{m-R}$ now; instead $\sum_{m\ge R}t^m\le(1-t)^{-1}\le\varphi^2$.

Hence $|L(V_iV_jW)|\le C_1(|c_3|+|c_1|)(i+j+1)^2 4^{i+j}|w|^{a_i+a_j}$ for an absolute constant C_1. Pull $|w|^{a_i}$ out of row i and column i and apply Hadamard's inequality: $|\Delta_K(\Xi)|\le|w|^{2\sum_{j<K}a_j}\big(C_1(|c_3|+|c_1|)(2K)^2 4^{2K}\sqrt{K}\big)^K$. Finally $2\sum_{j<K}a_j=(K^3+3K^2-\iota_K)/9$ with $\iota_K\in\{0,4,2\}$ according to K mod 3, and $|w|=\eta^{-2}$. □

4. Denominators

4.1. The integrality ring and the root types.

Lemma 4.1. *The coefficients of $\Delta_{K,\tau}(X)$ lie in $\mathbb{Z}[\tau,\tau^{-1},\Phi_d(\tau)^{-1}:d\in\mathcal{D}_K]$, where $\mathcal{D}_K$ is the set of orders of roots of unity ζ such that* (W) *$w(\zeta)=Q\zeta^2$ has order $e\le K$, or* (α) *$\alpha_k(\zeta)=1$ for some $0\le k\le K-2$.*

Proof. The inputs are the Laurent monomials w^a and $\lambda_a=\delta^a\tau^{-a}$; the factors $1/(1-w^{b-a})$, and $(1-w^N)^{-3}$, $(1-w^N)^{-1}$ ($N\le K$), where $1-w^n=1-Q^n\tau^{2n}$ is $\pm$ a product of cyclotomic polynomials $\Phi_d(\tau)$ with $w(\zeta)^n=1$; and $\nu_k=(c_3k(k+1)+c_1)\alpha_k/(1-\alpha_k)$, whose numerator has the integer coefficient $c_3k(k+1)+c_1$, and of which only $0\le k\le K-2$ occur (the quotient by D_K has degree $\le 2K-2-K$). Here $1-\alpha_k=1-\delta Q^k\tau^{2k+1}=\pm\prod\Phi_d(\tau)$, over the d with $\alpha_k(\zeta)=1$. The term $k=0$ occurs, since $\nu_0\ne0$ when $c_1\ne0$. Division by D_K is integral because its leading coefficient $\prod_a(-w^a)=\pm\tau^{K(K+1)}$ is a unit. No rational constant is inverted. □

For $d\in\mathcal{D}_K$ the type depends only on d: every primitive d-th root is a power ζ^j with j coprime to $2d$, hence odd, so $Q^j=Q$ and $\delta^j=\delta$, and therefore $w(\zeta^j)=w(\zeta)^j$ and $\alpha_k(\zeta^j)=\alpha_k(\zeta)^j$; powers coprime to the order preserve the order. Let $e(d)$ be the order of $w(\zeta)$; it depends only on Q, and is given in [10, §4.1]. In both cases $\sum_{e(d)=e}\phi_{\mathrm{E}}(d)=2\phi_{\mathrm{E}}(e)$: for each primitive e-th root ω the equation $Q\tau^2=\omega$ has two roots, and these $2\phi_{\mathrm{E}}(e)$ roots of unity are exactly those with $e(\tau)=e$. The α-condition is

$$\alpha_k(\zeta)=1\iff\zeta^{2k+1}=\delta Q^k,$$

which has exactly $2k+1$ solutions ζ, for all signs. The case $k=0$ is $\zeta=\delta$, i.e. $d=1$ if $\delta=1$ and $d=2$ if $\delta=-1$. Let $k_{\min}(d)$ be the least $k\ge0$ with $\alpha_k(\zeta)=1$, if there is one. All $d\in\mathcal{D}_K$ satisfy $d\le4K$: $w(\zeta)^e=1$ gives $\zeta^{4e}=1$, and $\alpha_k(\zeta)=1$ gives $\zeta^{4k+2}=1$.

Remark 4.2. The proof uses only the count $2k+1$, not the values of $k_{\min}$; for completeness we record them. If $\delta=Q$ they are those of [10, §4.1], except that $k_{\min}(1)=0$ (was 1) when $Q=1$ and $k_{\min}(2)=0$ (was 2) when $Q=-1$. For $Q=1$, $\delta=-1$ the condition is $\zeta^{2k+1}=-1$, so the α-roots have $d\equiv2\pmod4$, with $k_{\min}(d)=(d-2)/4$. In general, by Lemma 2.5, ζ is an α-root with index k for $(Q,-\delta)$ if and only if $-\zeta$ is one for (Q,δ),

and $w(-\zeta) = w(\zeta)$. If ζ has order d, then $-\zeta$ has order $d' = 2d$ (d odd), $d/2$ ($d \equiv 2 \bmod 4$) or d ($4 \mid d$); the map $d \mapsto d'$ is an involution with $\phi_{\mathrm{E}}(d') = \phi_{\mathrm{E}}(d)$.

Local setting. Fix ζ of order $d \in \mathcal{D}_K$, let $\varpi = \tau - \zeta$ and let $v = \mathrm{ord}_\varpi$ on $\mathbb{Q}(\zeta)((\varpi))$, extended to polynomials in X as the Gauss valuation (the minimum over the coefficients; it is multiplicative). Then:

(a) if $e(d) = e \le K$, then $x_a(\zeta) = x_b(\zeta)$ iff $e \mid a-b$, and then $v(x_a - x_b) = 1$: indeed $x_a - x_b = x_a(1 - w^{b-a})$, and $1 - w^n$ has only simple zeros, since $\frac{d}{d\tau} w^n \neq 0$ at $\tau \neq 0$;

(b′) if $e \mid N$, ψ_N has a pole of order at most 3 (exactly 3 if $c_3 \neq 0$, at most 1 if $c_3 = 0$), and ψ_N is regular if $e \nmid N$, by (a) and since ρ^N, w^N are units; so S_a has a pole of order ≤ 3, and is regular if $a < e$;

(c) $1 - \alpha_k = 1 - \delta Q^k \tau^{2k+1}$ is separable, so it has at most a simple zero at ζ;

(d) λ_a and x_a are Laurent monomials, hence units.

We call ζ a *W-root* if $e(d) \le K$ and an *α-root* if $\alpha_k(\zeta) = 1$ for some $0 \le k \le K - 2$; a *pure W-root* is a W-root that is not an α-root.

4.2. **Local bounds.** Put $c(m) = 2\sum_{j=0}^{m-2} \max(0, \frac{m+2}{2} - j)$ and $c'(m) = 2\sum_{j=0}^{m-1} \max(0, \frac{m+2}{2} - j)$; thus $c(1) = 0$, $c(m) = 4(m-1)$ for $m \le 6$, $c(m) \le (m+3)^2/4$, $c'(1) = 3$ and $0 \le c' - c \le 3$.

Proposition 4.3. *Let $e = e(d) \le K$ and partition $\{1, \dots, K\}$ into the clusters $I_r = \{a \equiv r \bmod e\}$ of sizes m_r, $1 \le r \le e$. Then*

$$-v(\Delta_{K,\tau}) \le \sum_{r=1}^{e-1} c(m_r) + c'(m_e) + \chi_\alpha \max(0, K - 1 - k_{\min}(d)),$$

with $\chi_\alpha = 1$ if ζ is also an α-root and $\chi_\alpha = 0$ otherwise.

Proof. The argument of [10, Proposition 4.2] applies. Its first three steps use only the facts (a), (d), the integrality of the quotient by D_K, and the following consequence of (b′): the Gauss valuation of $X - S_a$ is ≥ -3 if $a \ge e$ and ≥ 0 if $a < e$. These hold here; that ψ_N may now have a pole of order less than 3 only improves the valuation. We recall these steps, and then give the polynomial part, which changes.

Newton cluster basis. Let $F(s) = \prod_a (s - x_a)$, $F_r = \prod_{a \in I_r} (s - x_a)$, and for a cluster $I_r = \{a_1 < \cdots < a_m\}$ and $0 \le j < m$ put $N_{r,j} = (F/F_r) \prod_{l=m-j+1}^{m} (s - x_{a_l})$. At $\varpi = 0$ these reduce to $(F/F_r)(s - y_r)^j$ times units, $y_r = x_r(\zeta)$, which form a basis of the polynomials of degree $< K$; so the change of basis from $1, \dots, s^{K-1}$ has unit determinant, and $v(\Delta_{K,\tau})$ is the valuation of $G = [L_X(NN'/D_K)]$ in this basis.

Proper parts. $G_{(r,j),(r',k)} = f^{\mathrm{pol}} + \sum_a \mathrm{res}_a(NN')\lambda_a(X - S_a)$, where f^{pol} is L_X of the polynomial part. If $r \neq r'$ all res_a vanish. If $r = r'$, $\mathrm{res}_{a_i} = 0$ unless $i \le m - \max(j,k)$, and then $v(\mathrm{res}_{a_i}) = j + k - (m-1)$, by (a). For $r < e$ the node $a_1 = r$ is regular and a_i is singular for $i \ge 2$; for $r = e$ every node

is singular. So the proper part of the (j,k) entry has $v \geq j+k-(m-1)-3$, and $v \geq j+k-(m-1)$ when $r<e$ and $\max(j,k)=m-1$. With the weights $\kappa_j = \max(0, \frac{m+2}{2} - j)$ (and $\kappa_{m-1}=0$ if $r<e$), scale row (r,j) by $\varpi^{\lceil \kappa_j \rceil}$ and column (r,j) by $\varpi^{\lfloor \kappa_j \rfloor}$. Every scaled proper part is integral, and the total scaling is $c(m_r)$ for $r<e$ and $c'(m_e)$ for $r=e$ [10, Proposition 4.2].

Polynomial part. Write $(\cdot)_+$ for the quotient by D_K. The quotient of NN' has degree $\leq K-2$ and coefficients in $\mathbb{Z}[\tau^{\pm 1}]$, so $f^{\mathrm{pol}} = \sum_{k=0}^{K-2} [s^k](NN'/D_K)_+ \, \nu_k$; the term $k=0$ is present now. At a pure W-root every ν_k, $0 \leq k \leq K-2$, is regular, since $c_3 k(k+1)+c_1 \in \mathbb{Z}$ and $1-\alpha_k(\zeta) \neq 0$; so the scaled matrix is integral, which gives the bound with $\chi_\alpha = 0$. At an α-root, put $\mathcal{K}_\alpha = \{0 \leq k \leq K-2 : \alpha_k(\zeta) = 1\}$ and write $\nu_k = \varpi^{-1} u_k$ with u_k regular for $k \in \mathcal{K}_\alpha$, by (c) (if $c_3 k(k+1)+c_1 = 0$ then $u_k = 0$). Split $f^{\mathrm{pol}} = g^{\mathrm{pol}} + \varpi^{-1} B$, where $B(f,g) = \sum_{k \in \mathcal{K}_\alpha} u_k(0)\, [s^k](fg/D_K)_+$ and g^{pol} collects $\varpi^{-1}(u_k - u_k(0))[s^k](\cdot)_+$ for $k \in \mathcal{K}_\alpha$ and $\nu_k[s^k](\cdot)_+$ for $k \notin \mathcal{K}_\alpha$; both have regular entries. If $\deg f \leq k_{\min}$, then $\deg(fg) \leq k_{\min} + K - 1$, so the quotient of fg by D_K has degree $< k_{\min}$ (it is 0 if $k_{\min} = 0$), every $[s^k]$ with $k \in \mathcal{K}_\alpha$ of it vanishes, and $B(f,\cdot) = 0$. So $\operatorname{rank} B \leq \max(0, K-1-k_{\min})$. After scaling the matrix is $A + \varpi^{-1} B'$, where A (the scaled proper parts and the scaled g^{pol}) is integral and B' is B in the Newton basis, scaled; B' has regular entries and $\operatorname{rank} B' = \operatorname{rank} B$, since the basis change and the scalings are invertible over the fraction field. Expanding the determinant multilinearly in the columns, a term taking more than $\operatorname{rank} B$ columns from $\varpi^{-1} B'$ vanishes by linear dependence, and every other term has valuation $\geq -\max(0, K-1-k_{\min})$. □

Lemma 4.4. *If $e(d) > K$ and ζ is an α-root, then $-v(\Delta_{K,\tau}) \leq \max(0, K-1-k_{\min}(d))$.*

Proof. Work in the monomial basis. Since $e > K$, no difference $|a-b| < K$ and no $N \leq K$ is divisible by e, so no factor $1-w^{b-a}$ and no $1-w^N$ vanishes at ζ; hence all res_a, λ_a and S_a ($a \leq K$) are regular at ζ. Only the ν_k, $k \in \mathcal{K}_\alpha$, carry poles, and the rank argument of Proposition 4.3 applies with no scaling. □

Proposition 4.5. *Let $b_{K,d}$ be the bound of Proposition 4.3 or Lemma 4.4 (0 if $d \notin \mathcal{D}_K$) and $C_K(\tau) = \prod_d \Phi_d(\tau)^{b_{K,d}}$. Then $C_K(\tau)\Delta_{K,\tau}(X) \in \mathbb{Z}[\tau^{\pm 1}][X]$; hence $C_K(t)\Delta_K(X) \in \mathcal{O}[X]$, and $C_K(\sigma t)\sigma\Delta_K(X) \in \mathcal{O}[X]$ is its conjugate.*

Proof. By Lemma 4.1 a coefficient is $f_1/(\tau^n \prod \Phi_d^{n_d})$ with $f_1 \in \mathbb{Z}[\tau]$. The bound $v_\zeta \geq -b_{K,d}$ at every primitive d-th root forces $\Phi_d^{n_d - b_{K,d}} \mid f_1$ in $\mathbb{Q}[\tau]$, hence in $\mathbb{Z}[\tau]$ since Φ_d is monic. Specialise $\tau \mapsto t$, a unit of $\mathcal{O}$, and $\tau \mapsto \sigma t$. □

4.3. **The size of $C_K(t)$.** Let $\mathcal{E}_1 = \sum_{a \geq 1} |\log(1-\varphi^{-a})| = 2.1136\ldots$. By [10, Lemma 4.5], which does not involve δ, for every $d \geq 1$

$$\left|\log|\Phi_d(t)\Phi_d(\sigma t)| - \phi_{\mathrm{E}}(d)\log\eta\right| \leq 2\mathcal{E}_1. \tag{4.1}$$

Let $\beta_7 = \prod_{p\le 7}(1-p^{-2}) = \frac{768}{1225}$ and $c_* = \frac{11}{8}\beta_7 + \frac{1}{3} = \frac{4393}{3675} = 1.19537\ldots$.

Proposition 4.6. $\log|\mathrm{N}\, C_K(t)| \le \big(c_* K^3 + O(K^2\log K)\big)\log\eta$, *with an absolute implied constant.*

Proof. By (4.1), $\log|\mathrm{N}\, C_K(t)| \le \log\eta \sum_d b_{K,d}\phi_{\mathrm{E}}(d) + 2\mathcal{E}_1 \sum_d b_{K,d}$, and $2\mathcal{E}_1 \le (2\mathcal{E}_1/\log\varphi)\log\eta$. Split $b_{K,d}$ into its W part and its α part. The W part is the same function of (K,d) as in [10]: it depends only on the cluster sizes and on $e(d)$, which depends only on Q. So [10, Proposition 4.6] applies verbatim: the sum of the W parts is $O(K^2\log K)$, and their contribution to $\sum_d b_{K,d}\phi_{\mathrm{E}}(d)$ is at most $\frac{11}{8}\beta_7 K^3 + O(K^2)$. The α part of $b_{K,d}$ is at most K, and nonzero only for the at most $4K$ orders $d \le 4K$, so $\sum_d b_{K,d} = O(K^2\log K)$.

α part. $\max(0, K-1-k_{\min}(d))$ is the number of k with $k_{\min}(d) \le k \le K-2$, and for such k every primitive d-th root ζ satisfies $\alpha_j(\zeta) = 1$ for some $j \le k$. So

$$\sum_d \max(0, K-1-k_{\min}(d))\,\phi_{\mathrm{E}}(d) \le \sum_{k=0}^{K-2}\sum_{j=0}^{k} n_j = \sum_{k=0}^{K-2}(k+1)^2 = \frac{(K-1)K(2K-1)}{6} \le \frac{K^3}{3},$$

where $n_j = \#\{\zeta : \alpha_j(\zeta) = 1\} = 2j+1$, for all signs Q, δ. Compared with [10] only the terms $j = 0$ are new; they change the sum by exactly $K-1$. □

Remark 4.7. The following is a sketch and is not used in the proof. For $\delta = Q$, $b_{K,d}$ differs from its value in [10] only at $d = 1$ ($Q = 1$) or $d = 2$ ($Q = -1$), where $k_{\min}$ drops to 0; this changes $\sum_d b_{K,d}\phi_{\mathrm{E}}(d)$ by 1 or 2 once $K \ge 3$. For $\delta = -Q$, Lemma 2.5 and Remark 4.2 give $b_{K,d}^{(Q,-\delta)} = b_{K,d'}^{(Q,\delta)}$, and $d \mapsto d'$ is a bijection preserving ϕ_{E}, so $\sum_d b_{K,d}\phi_{\mathrm{E}}(d)$ is the same for δ and $-\delta$. Hence the sharper constants of [10, Remark 4.7], 1.130790 for $Q = -1$ and 1.100769 for $Q = 1$, replace c_* for both signs δ.

5. Smallness at the conjugate

Proposition 5.1. $\log|\sigma\Delta_K(-\Xi)| \le -K^2(K+1)\log\eta + K\log(\sqrt{K}\,M_K)$, *where* $M_K = C_0(|c_3| + |c_1|)(\varphi^3K^2 + \varphi^2)$ *and* $C_0 = \prod_{i\ge 1}(1-\varphi^{-2i})^{-1} = 2.0770906\ldots$.

Proof. At $\tau = \sigma t = Q/t$ the data become as follows. First, $\sigma w = Q/t^2 = w^{-1}$ and $\sigma\rho = \delta Q/t = \delta'/t$, so $\lambda_a \mapsto (\sigma\rho)^{-a} = \delta'^a t^a$. Second, $\sigma\psi_N = -\psi_N$; this is an identity in $\mathbb{Q}(\tau)$ under $\tau \mapsto Q/\tau$: using $1-w^{-N} = -w^{-N}(1-w^N)$,

$$\sigma\psi_N = \Big(\frac{\delta'}{t}\Big)^N\Big(\frac{2c_3w^{-N}}{(1-w^{-N})^3} + \frac{c_1}{1-w^{-N}}\Big) = -\Big(\frac{\delta'}{t}\Big)^N w^N\Big(\frac{2c_3w^N}{(1-w^N)^3} + \frac{c_1}{1-w^N}\Big),$$

and $(\delta'/t)^N w^N = \delta^N Q^{2N} t^N = \rho^N$. (Equivalently: $U_N \in \mathbb{Q}$ and $\sigma\sqrt{D} = -\sqrt{D}$ in Lemma 2.1(vi).) So $S_a \mapsto -S_a$. Third, $\alpha_k\,\sigma\alpha_k = \delta t\, w^k \cdot \delta' t^{-1} w^{-k} = \delta\delta' = Q$, so $\sigma\nu_k = (c_3 k(k+1) + c_1)Q/(\alpha_k - Q)$, and since $|\alpha_k| = t^{2k+1} \le t$,

$$|\sigma\nu_k| \le \frac{|c_3|k(k+1) + |c_1|}{1-t} \qquad (k \ge 0).$$

Let L_X^σ be the corresponding formal functional. The conjugate poles are $r_a = w^a$, $|r_a| = t^{2a}$, and $\sigma D_K(s) = \prod_a (1 - w^{-a}s) = (-1)^K w^{-\binom{K+1}{2}} F^\sigma(s)$ with $F^\sigma(s) = \prod_a (s - r_a)$. Hence

$$\sigma\Delta_K(X) = (-1)^{K^2} w^{K\binom{K+1}{2}} \det[Y_{i+j}(X)], \qquad Y_k(X) = L_X^\sigma(s^k/F^\sigma).$$

The pole moments at $X = -\Xi$. Since $1/(1 - w^{-a}s) = -r_a/(s - r_a)$, $L_X^\sigma(1/(s - r_a)) = -r_a^{-1}\delta'^a t^a (X + S_a)$, and at $X = -\Xi$ this is $r_a^{-1}\delta'^a t^a \sum_{N>a} \psi_N$. Write $N = a + M$; then $w^N = r_a w^M$ and $r_a^{-1}\delta'^a t^a \rho^N = r_a^{-1}(\delta'\delta)^a t^{2a} \delta^M t^M = r_a^{-1} w^a \delta^M t^M = \delta^M t^M$. Hence the pole moment is $\Gamma(r_a)$, where

$$\Gamma(r) = 2c_3 r \sum_{M\ge1} \frac{\delta'^M t^{3M}}{(1 - rw^M)^3} + c_1 \sum_{M\ge1} \frac{\delta^M t^M}{1 - rw^M} = \sum_{n\ge0} \gamma_n r^n,$$

using $\delta^M t^M w^M = \delta'^M t^{3M}$. The first sum is the function Γ of [10, Proposition 5.1] with the sign δ'^M; the second is new. Γ is analytic on $|r| < \eta^2$, and $\gamma_n = \gamma_n^{(3)} + \gamma_n^{(1)}$ with

$$\gamma_n^{(3)} = c_3 n(n+1) \sum_{M\ge1} \delta'^M t^{3M} w^{M(n-1)} \ (n \ge 1), \qquad \gamma_0^{(3)} = 0,$$

$$\gamma_n^{(1)} = c_1 \sum_{M\ge1} \delta^M t^M w^{Mn} \ (n \ge 0).$$

So $|\gamma_n^{(3)}| \le |c_3| n(n+1) t^{2n+1}/(1 - t^3)$ and $|\gamma_n^{(1)}| \le |c_1| t^{2n+1}/(1 - t)$. Unlike in [10], $\Gamma(0) = \gamma_0 = c_1\delta t/(1 - \delta t)$ is nonzero when $c_1 \ne 0$. Partial fractions give

$$\frac{s^k}{F^\sigma(s)} = \sum_{j=0}^{k-K} h_{k-K-j}(r)\, s^j + \sum_{a=1}^{K} \frac{r_a^k}{(F^\sigma)'(r_a)(s - r_a)},$$

and with $\sum_a r_a^m/(F^\sigma)'(r_a) = h_{m-K+1}(r)$ and $|r_a| \le t^2 < \eta^2$ we get

$$Y_k(-\Xi) = \sum_{j=0}^{k-K} h_{k-K-j}(r)\, \sigma\nu_j + \sum_{n\ge0} \gamma_n h_{k+n-K+1}(r).$$

The term $n = 0$ is $\gamma_0 h_{k-K+1}(r)$, which vanishes for $k < K - 1$.

Bounds. As $h_m(r_1, \dots, r_K) = w^m h_m(1, w, \dots, w^{K-1})$, $|h_m(r)| \le C_0 t^{2m} \le C_0$ for $m \ge 0$, using $\prod_{i\ge1}(1 - t^{2i})^{-1} \le C_0$ for $t \le \varphi^{-1}$. In the first sum $j \le k - K \le K - 2$, so $|\sigma\nu_j| \le (|c_3|K^2 + |c_1|)/(1 - t)$, and the first sum is at most $C_0(|c_3|K^2 + |c_1|)/((1 - t)(1 - t^2))$. For the second, $\sum_n n(n+1)t^{2n+1} = 2t^3/(1 - t^2)^3 \le 2$ and $1/(1 - t^3) \le \varphi^2/2$, so $\sum_n |\gamma_n^{(3)}| \le |c_3|\varphi^2$;

and $\sum_{n\ge0}|\gamma_n^{(1)}|\le|c_1|t/((1-t)(1-t^2))\le|c_1|\varphi^2$. Since $(1-t)(1-t^2)\ge\varphi^{-3}$ and all these quantities increase with t, $|Y_k(-\Xi)|\le C_0\big((|c_3|K^2+|c_1|)\varphi^3+(|c_3|+|c_1|)\varphi^2\big)\le M_K$ for $0\le k\le 2K-2$. Hadamard's inequality and $|w|^{K\binom{K+1}{2}}=\eta^{-K^2(K+1)}$ finish the proof. □

6. Nonvanishing

Proposition 6.1. *For $(c_3,c_1)\ne(0,0)$, $\Delta_K(\Xi)\ne0$ for infinitely many K.*

Proof. Let

$$\mathcal{C}(z)=L\Big(\frac{1}{z-s}\Big)=\sum_{n\ge1}\rho^n\Big(\frac{2c_3w^nz}{(z-w^n)^3}+\frac{c_1}{z-w^n}\Big)\qquad(|z|>t^2),$$

using $\theta(\theta+1)(z-s)^{-1}=2sz/(z-s)^3$. Since $|\rho^n|=t^n$, the series converges locally uniformly on $\mathbb{C}\setminus(\{0\}\cup\{w^n\})$, so $\mathcal{C}$ is meromorphic on $\mathbb{C}\setminus\{0\}$; it is analytic at ∞ and $O(1/z)$.

$\mathcal{C}$ is not rational. The nodes w^n are pairwise distinct, as $0<|w|<1$, and near w^n all terms but the n-th are analytic. If $c_3\ne0$, $\mathcal{C}$ has a pole of order exactly 3 at w^n, with leading coefficient $2c_3\rho^nw^{2n}\ne0$ (the weight-one term has order ≤1). If $c_3=0$, then $c_1\ne0$ and $\mathcal{C}$ has a simple pole at w^n with residue $c_1\rho^n\ne0$. Either way $\mathcal{C}$ has infinitely many poles.

The rest is the argument of [10, Proposition 6.1], which uses only that $\mathcal{C}$ is not rational, that L is a linear combination of 2-jets at the nodes w^n with weights $O(t^n)$, that $0<|w|<1$, and that the exterior nodes $x_a=w^{-a}$ are distinct with $|x_a|\to\infty$; we repeat it. For $\deg R\le n-1$ put $H_{n,R}(z)=L\big(R(s)/((z-s)D_n(s))\big)$, analytic for $|z|>t^2$ and in particular at every x_a, and

$$T_{n,R}(z)=L\Big(\frac{R(z)D_n(s)-D_n(z)R(s)}{(z-s)D_n(s)}\Big),$$

a polynomial in z of degree $\le n-1$. Then $\mathcal{C}R-T_{n,R}=D_nH_{n,R}$. If $\Delta_n(\Xi)=0$, a kernel vector gives $R_n\ne0$ of degree $\le n-1$ with $L(s^jR_n/D_n)=0$ for $j<n$. Put $T_n=T_{n,R_n}$ and $E_n=\mathcal{C}R_n-T_n$. Expanding $(z-s)^{-1}=\sum_j s^jz^{-j-1}$, uniformly with two s-derivatives on $|s|\le t^2$ for $|z|>t^2+\epsilon_0$ ($\epsilon_0>0$), gives $H_{n,R_n}(z)=O(z^{-n-1})$, so $E_n=O(z^{-1})$ and $E_n(x_a)=0$ for $a\le n$. If $\Delta_n(\Xi)=\Delta_{n+1}(\Xi)=0$, then $V_0=T_{n+1}R_n-T_nR_{n+1}=R_{n+1}E_n-R_nE_{n+1}$ is a polynomial, $O(z^{n-1})$, vanishing at $x_1,\dots,x_n$; so $V_0=0$ and $T_n/R_n=T_{n+1}/R_{n+1}$. If $\Delta_n(\Xi)=0$ for all $n\ge N$, all T_n/R_n equal one reduced fraction T/R_0, so $R_n=U_n'R_0$, $T_n=U_n'T$, and $E:=\mathcal{C}R_0-T\ne0$ because $\mathcal{C}$ is not rational. Then $U_n'E=E_n=O(1/z)$ forces $E=c_0z^{-q}(1+o(1))$ with $c_0\ne0$, $q\ge1$, and $\deg U_n'\le q-1$; but $E(x_a)\ne0$ for all large a while $E_n(x_a)=0$, so $U_n'(x_a)=0$ for all large $a\le n$, too many zeros once n is large. □

7. Proof of Theorem 1.1

Suppose $q_0 + q_1 A_\delta + q_3 B_\delta = 0$ with integers q_i, not all zero (a rational relation becomes one after clearing denominators). Fix $(c_3, c_1) = (Dq_3, 2q_1) \neq (0,0)$ as in Lemma 2.2, so that $D\Xi \in \mathcal{O}$ and $\sigma\Xi = -\Xi$. For K in the infinite set of Proposition 6.1 put

$$y_K = C_K(t)\, D^K\, \Delta_K(\Xi)$$

(here D^K is the K-th power of the discriminant, not the polynomial D_K). By Proposition 4.5, $C_K(t)\Delta_K(X) = \sum_{j \le K} b_j X^j$ with $b_j \in \mathcal{O}$, and $D^K \Xi^j = D^{K-j}(D\Xi)^j \in \mathcal{O}$; so $y_K \in \mathcal{O}$. Moreover $y_K \neq 0$, because $\Delta_K(\Xi) \neq 0$ and $C_K(t) \neq 0$ (t is not a root of unity). Since $\Xi \in \mathbb{Q}(\sqrt{D})$, $\sigma(\Delta_K(\Xi)) = (\sigma\Delta_K)(\sigma\Xi) = \sigma\Delta_K(-\Xi)$, so $\sigma y_K = C_K(\sigma t) D^K \sigma\Delta_K(-\Xi)$. The norm $y_K\, \sigma y_K$ of the nonzero algebraic integer y_K is a nonzero rational integer, so $|y_K\, \sigma y_K| \ge 1$, and by Propositions 3.1, 5.1 and 4.6

$$\begin{aligned} 0 \le \log|\,\mathrm{N}\, C_K(t)| + 2K\log D + \log|\Delta_K(\Xi)| + \log|\sigma\Delta_K(-\Xi)| \\ \le \big(c_* - \tfrac{2}{9} - 1\big) K^3 \log\eta + O(K^2 \log K), \end{aligned}$$

where the implied constant depends on D and (c_3, c_1), which are fixed. Since $c_* - \frac{11}{9} = \frac{11}{8}\beta_7 - \frac{8}{9} = -0.02684\ldots < 0$, this fails for large K. □

Remark 7.1. Not used in the proof. With the constants of Remark 4.7 the margin is at least 0.0914 ($Q = -1$) and 0.1214 ($Q = 1$) per $K^3 \log\eta$, for both δ and all (c_3, c_1). We checked the construction numerically, as a cross-check only. (1) The exact multiplicities of the cyclotomic factors in the denominators of $\Delta_{K,\tau}(X)$, computed in $\mathbb{Q}(\tau)[X]$, never exceed the bounds $b_{K,d}$: for all four (Q, δ), several pairs (c_3, c_1) and $K \le 9$ (1645 pairs (K, d)), and in an independent computation for $K \le 10$ (2388 pairs); more than half of the bounds are attained. The terms $k = 0$ are needed: for $Q = -1$ and $c_1 \neq 0$ the multiplicity of Φ_2 ($\delta = -1$) or Φ_1 ($\delta = 1$) is 4 at $K = 2$ and 8 at $K = 3$, above the values 3 and 7 of the bound without these terms. (2) At finite K, with exact determinants over $\mathbb{Q}(\sqrt{D})$ and interval arithmetic, the left side above without the term $2K \log D$, and with the norm of the ideal generated by the coefficients of Δ_K in place of $C_K(t)$, divided by $K^3 \log\eta$, is negative for $6 \le K \le 11$ in all cases tried; at $K = 11$ it is about -0.22 for the Fibonacci numbers ($\delta = 1$, $(c_3, c_1) = (1, 2)$), -0.21 for the Pell numbers ($\delta = 1$, $(1, 1)$) and -0.25 for $(P, Q) = (3, 1)$ ($\delta = -1$, $(3, -2)$). For $c_3 = 0$ the measured decay of $\Delta_K(\Xi)$ is much faster than $-2/9$, about -0.70 per $K^3 \log\eta$. The scripts are available from the author.

8. Formal verification

The Lean development is the repository

`https://github.com/telaviv/lucas-reciprocal-independence`

(Lean code as of commit `73a88f1`); all file names below refer to it. It (library `LucasInd`, Lean 4.35.0, Mathlib v4.35.0, about 10 100 lines) generalises the formalisation of [10] and proves

```
theorem LucasInd.linearIndependent_sums (hA : Admissible a E)
                    (hδ :  δ = 1 ∨ δ = -1) :
   LinearIndependent ℚ ![(1 :  ℝ), sumA a E δ, sumB a E δ],
```

where `Admissible a E` means $E = 1$, $a \geq 1$ or $E = -1$, $a \geq 3$; `Useq a E` is the sequence with $U_{n+2} = aU_{n+1} + EU_n$, so $(a, E) = (P, -Q)$; and `sumA a E δ` and `sumB a E δ` are the series $\sum_{n\geq 0} \delta^{n+1}/U_{n+1}$ and $\sum_{n\geq 0}(\delta \cdot (-E))^{n+1}/U_{n+1}^3$, written with a `tsum`. Thus $\varepsilon := -E = Q$ and `sumB` is B_δ. `LucasInd.rat_relation_trivial` restates the theorem as: a relation $q_0 + q_1 A_\delta + q_3 B_\delta = 0$ with $q_i \in \mathbb{Q}$ forces $q_0 = q_1 = q_3 = 0$. The proof uses only the axioms `propext`, `Classical.choice` and `Quot.sound`; there is no `sorry`, no `axiom` declaration, no `native_decide` and no `implemented_by`.

What has to be trusted. For the theorem itself, only the definitions of `Admissible`, `Useq`, `sumA` and `sumB` in `LucasInd/Defs.lean` (and Mathlib's `tsum` and `LinearIndependent`) have to be read. A divergent `tsum` has the value 0, which would make the family dependent, so the statement cannot hold for a degenerate reason; moreover `LucasInd.summable_sums` shows that both series converge. The corollaries, in `LucasInd/Final.lean`, are `LucasInd.irrational_alt` (Corollary 1.2(i)), `LucasInd.fib_twin_one` and `LucasInd.fib_twin_two` (part (ii), written with `Nat.fib`), `LucasInd.fib_mul` and `LucasInd.fib_two_mul` (part (iii)), and `LucasInd.pell_indep` (part (iv), with `pell = Useq 2 1`). In them the term of index $m = n + 1$ carries the sign $(-1)^m$, as in Corollary 1.2. The identity $F_{kn} = F_k U_n(L_k, (-1)^k)$ is `LucasInd.fib_mul_eq`, with the Lucas numbers `lucasNum`, in `LucasInd/Corollaries.lean`.

Definitions and the chain. `LucasInd/Defs.lean` defines the construction of Section 2 over an arbitrary field with signs $\varepsilon = Q$ and δ, integers c_3, c_1 and a parameter τ (`wpar` $= \varepsilon\tau^2$, `alpha`, `nu`, `psi`, `S`, `node`, `F`, `res`, `wrho` $= \varepsilon\delta\tau = w/\rho$, `mConst`, `mLin`, `nrm`, `Delta`); the operator `opP` $= \Theta$, the real functional `Lfun` and `XiR` as functions of $(\varepsilon, \delta, c_3, c_1, t)$; and the clearing polynomial `CKpoly` $= \prod_{d\leq 4K} \Phi_d^{b_{K,d}}$ with exponents `nExp` $=$ `nW` $+$ `nAlpha`, where `nAlpha` uses the α-roots with $0 \leq k \leq K-2$ (`IsAlphaRoot`). `LucasInd/Statements.lean` states the ingredients as propositions; the proofs assume `Good` ε δ `t`, i.e. $\varepsilon, \delta \in \{\pm 1\}$ and $0 < t \leq \varphi^{-1}$, and hold for every $(c_3, c_1) \in \mathbb{Z}^2$ (for `Nonvanishing`, every $(c_3, c_1) \neq (0, 0)$). The conjugate point is $\tau = \varepsilon/t$. `LucasInd/Unit.lean` contains Lemma 2.1 (part (vi) is `LucasInd.psi_eq_U`), and `LucasInd.indep_of` (in `LucasInd/Main.lean`) proves Lemma 2.2 and derives the theorem from the ingredients at $\varepsilon = -E$, $t = 1/\eta$, $(c_3, c_1) = (Dq_3, 2q_1)$; the contradiction of Section 7 is `LucasInd.false_of_xi_eq`.

Paper	Lean statement	Lean proof
Lemma 2.4, entries	`EntryIdentity`	`LucasInd.Analytic.entryIdentity`
Prop. 3.1	`DecayBound`	`LucasInd.Analytic.decayBound`
Lemma 4.1–Prop. 4.5	`Integrality`	`LucasInd.Denom.integrality`
(4.1), Prop. 4.6	`CostBound`	`LucasInd.Cost.costBound`
Prop. 5.1	`ConjBound`	`LucasInd.Conj.conjBound`
Prop. 6.1	`Nonvanishing`	`LucasInd.NonVan.nonvanishing`
Lemma 2.2, Section 7		`LucasInd.indep_of`

`EntryIdentity` is used only inside the proof of `DecayBound`, not by the assembly; the summability of ψ_N, used to identify Ξ with $(2c_3B_\delta + c_1DA_\delta)/(D\sqrt{D})$ (`LucasInd.XiR_eq`), is `LucasInd.Conj.summable_psi`. The formal statements are the propositions in the form the assembly needs, with constants that may depend on $(\varepsilon, \delta, c_3, c_1, t)$; the uniformity claimed in Propositions 3.1 and 4.6 and the explicit M_K of Proposition 5.1 are not expressed by them and are not needed, since (P, Q) and (c_3, c_1) are fixed in the proof. `DecayBound` is $|\Delta_K(\Xi)| \le C^{K^2} t^{2b_K}$ with $b_K = 2\sum_{j<K} a_j$ (proved with $C = 4096(|c_3| + |c_1| + 1)$); `ConjBound` is $|\sigma\Delta_K(-\Xi)| \le C^{K^2} t^{K^2(K+1)}$; and `CostBound` is $\log|C_K(t)C_K(\varepsilon/t)| \le \frac{121}{100}K^3 \log t^{-1} + CK^2(1 + \log K)^2$, whose proof follows Proposition 4.6 with $c_* = \frac{4393}{3675} < \frac{121}{100} < \frac{11}{9}$ for all four (ε, δ). `Nonvanishing` is proved under the hypothesis $(c_3, c_1) \neq (0, 0)$, working with the Cauchy transform $\mathcal{C}$ directly; the two pole types are `LucasInd.NonVan.cauchyF_cubic_pole` and `LucasInd.NonVan.cauchyF_simple_pole`. `Integrality` is stated for every $(c_3, c_1) \in \mathbb{Z}^2$ and K, with one integer polynomial for all real $\tau \neq 0$ that are not roots of unity. Its proof, in the folder `LucasInd/Denom/`, works in the local rings of $\mathbb{C}(\tau)$ at the roots of unity, as in [10]. It uses the root types only through `LucasInd/Roots.lean`, whose lemmas `LucasInd.orderOf_w_of_primitive` and `LucasInd.alpha_eq_one_iff_of_primitive` show that the type of a root of unity depends only on its order. The changes against [10] are local: `LucasInd/Denom/Localization.lean` inverts $1 - \alpha_k$ for $0 \le k \le K - 2$ (the new root $k = 0$), with `LucasInd.Denom.quotient_coeff_eq_zero` showing that no other ν_k enters; `LucasInd.Denom.alpha_eq_sgn` writes $\alpha_k = s\tau^{2k+1}$ with $s = \delta\varepsilon^k$, so that $1 - \alpha_k = -s(\tau^{2k+1} - s)$; the mixed-weight fact that $\varpi^3 S_a$ is regular at every ζ is `LucasInd.Denom.uniformizer_cube_mul_S_mem`; and the rank correction in `LucasInd/Denom/Integrality.lean` allows $k_{\min}(d) = 0$, matching the definition of the exponents exactly. The assembly is `LucasInd.Denom.localBounds` and `LucasInd.Denom.integrality_of_local_bounds`. The proof of `CostBound` uses no involution $d \mapsto d'$: as in Proposition 4.6, the root counts it needs do not depend on the signs.

Provenance. The formal development generalises that of [10], which generalises that of [11]; parts of the latter are adapted from the Lean formalisation QZeta3 [14] of [17] (Apache License 2.0). Every adapted file says so in its header, and the file `NOTICE` records the reuse.

Use of AI tools. This work was carried out with substantial assistance from AI systems (Anthropic's Claude models), which were used to search the literature, to develop and check the proof, to write the Lean formalisation and to draft this text. The author directed the project and is responsible for its content. The formal proof is checked by the Lean kernel.

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