hexagon:2611.00699v1

POWERFUL VALUES OF QUADRATIC POLYNOMIALS

DANNY PLYMESSER

Abstract. Let f be a quadratic polynomial with integer coefficients, positive leading coefficient and nonsquare discriminant. We show that the number of $n \le N$ with $f(n)$ powerful is $O(N^{\vartheta})$ for every $\vartheta > 293/1220$. A second bound has the exponent $389/1600$ and a constant that grows only polynomially with the coefficients of f; it is applied to pairs consisting of a 6-full or 8-full integer and a neighbouring powerful integer. A fixed-quadratic substitution also gives a bound for $1 + a^3t^3$. The proofs factor $f(n)$ in the quadratic field of its discriminant, apply a determinant method with two contact polynomials to the resulting equation of bidegree three and two, and count the points on the auxiliary curves with the lifted-curve estimate of [8]. The main quadratic and mixed-full bounds are also formalised in Lean, with the three external estimates recorded explicitly. AI systems produced and reviewed the results, proofs, and exposition. No human mathematical review has been performed. Section 13 describes the verification and its scope.

1. Introduction

A positive integer m is *powerful* if $p^2 \mid m$ for every prime $p \mid m$, and *k-full* if $p^k \mid m$ for every prime $p \mid m$. There are about $x^{1/2}$ powerful integers up to x. For integers a, b, c put

$$S(a,b,c;N) = \#\{1 \le n \le N : \ an^2+bn+c \ge 1, \ an^2+bn+c \text{ powerful}\}.$$

The values $an^2 + bn + c$ with $n \le N$ have size N^2, so counting powerful integers gives only $S(a,b,c;N) \ll N$. The elementary count of Section 4 gives $N^{2/5+\varepsilon}$, and the true order is expected to be far smaller. This paper proves upper bounds for $S(a,b,c;N)$ when the discriminant b^2-4ac is not a square. It develops the counting method of [8] for polynomial values, including families in which the coefficients vary.

1.1. Quadratic estimates.

Theorem 1.1 (fixed quadratic)**.** *Let $A, B, C \in \mathbb{Z}$ with $A > 0$ and $B^2 - 4AC$ not a square. For every $\varepsilon > 0$ there is K such that*

$$S(A,B,C;N) \le KN^{293/1220+\varepsilon} \qquad (N \ge 1).$$

Here K depends on A, B, C, ε, and N is real. Negative discriminants are allowed.

Theorem 1.2 (polynomial dependence on the height)**.** *There are absolute constants $K, \nu > 0$ such that for all $a, b, c \in \mathbb{Z}$ with $a \ge 1$ and $b^2 - 4ac$ not a square, every real $L \ge 2$ with $|a|, |b|, |c| \le L$, and every $N \ge 2$,*

$$S(a,b,c;N) \le KL^{\nu}N^{389/1600}.$$

Moreover there are absolute $\eta, C > 0$ such that, with $M_0 = \lfloor N^{\eta} \rfloor$,

$$\sum_{a=1}^{M_0} \sum_{|b| \le M_0} \sum_{\substack{|c| \le M_0 \\ b^2-4ac \text{ not a square}}} S(a,b,c;N) \le CN^{390/1600} \qquad (N \ge 2).$$

Neither bound has an ε. The exponent ν is not computed. The second bound follows by summing the first; one may take $\eta = 1/(1600(\nu+3))$. See Sections 8.3 and 8.4.

Date: Version 1.0.1, 18 November 2026.

Theorem 1.3 (the family n^2+a^2)**.** *Put*

$$\Theta(\kappa)=\begin{cases}(293+153\kappa)/1220, & \kappa\le 1/28,\\ (81+61\kappa)/340, & 1/28<\kappa\le 19/73,\\ (61+105\kappa)/310, & \kappa>19/73.\end{cases}$$

For every $0\le\kappa\le 289/1050$ and $\varepsilon>0$ there is K such that, for all $N\ge 2$ and all $a\in\mathbb{Z}$ with $1\le|a|\le N^{\kappa}$,

$$\#\{1\le n\le N : n^2+a^2 \text{ is powerful}\}\le KN^{\Theta(\kappa)+\varepsilon}.$$

The function Θ is continuous, with $\Theta(0)=293/1220$ and $\Theta(289/1050)=29/100$.

1.2. **Cubic values and full pairs.**

Corollary 1.4 (cubic values)**.** *There is an absolute $\nu_1\ge 0$ such that for every $\varepsilon>0$ there is C with*

$$\#\{1\le t\le T : 1+At^3 \text{ is powerful}\}\le CA^{\nu_1}T^{3/5+\varepsilon}\qquad (A\ge 1,\ T\ge 2).$$

For every $\varepsilon>0$ there is also C_ε such that

$$\#\{1\le t\le T : 1+a^3t^3 \text{ is powerful}\}\le C_\varepsilon(aT)^{293/1220+\varepsilon}\qquad (a\ge 1,\ T\ge 2).$$

Here A,a are positive integers and T is real.

In the following two theorems $s\in\{6,8\}$, and for real $z>0$

$$\delta_s(z)=\min\Big\{\frac{1}{6400(z+1)s(s+1)},\ \frac{1}{2}\Big(\frac{1}{s}-\frac{49}{400}\Big)\Big\},$$

which is positive. The trivial bound for the number of s-full $n\le Y$ is $\ll Y^{1/s}$.

Theorem 1.5 (a positive shift)**.** *For every integer $h\ge 1$ there are $K_h,C_h\ge 1$ such that*

$$\#\{1\le t\le T : At^2+h \text{ is powerful}\}\le K_hA^{C_h}T^{389/1600}\qquad (A\ge 1,\ T\ge 2),$$

and for all $s\in\{6,8\}$, $2\le r<s$ and $\varepsilon>0$ there is C with

$$\#\{1\le n\le Y : n \text{ is } s\text{-full},\ n+h \text{ is } r\text{-full}\}\le CY^{1/s-\delta_s(C_h)+\varepsilon}\qquad (Y\ge 2).$$

Theorem 1.6 (the reverse neighbour)**.** *There are functions $K_*,\nu_*:\mathbb{N}\to\mathbb{R}$, with $K_*(H')>0$ and $\nu_*(H')>0$ for every $H'\ge 1$, such that for all $A,H'\ge 1$ with AH' not a square and all $T\ge 2$,*

$$\#\{1\le t\le T : At^2-H'\ge 1,\ At^2-H' \text{ is powerful}\}\le K_*(H')A^{\nu_*(H')}T^{389/1600},$$

and for all $s\in\{6,8\}$, $2\le r<s$ and $\varepsilon>0$ there is C with

$$\#\{2\le n\le Y : n \text{ is } s\text{-full},\ n-1 \text{ is } r\text{-full}\}\le CY^{1/s-\delta_s(\nu_*(1))+\varepsilon}\qquad (Y\ge 2).$$

The proofs use only that the neighbour is powerful (Remark 11.4).

The coefficient exponents in these applications can be chosen from a single constant: Section 11 takes $C_h=\max(1,\nu)$ and $\nu_*(H')=\nu$, with ν from Theorem 1.2. Thus the shifts affect K_h and $K_*(H')$, but need not affect these exponents. The separate cubic argument in Section 10 permits the absolute choice $\nu_1=180$.

1.3. **Comparison with earlier work.** For a fixed quadratic with nonzero discriminant, Wongcharoenbhorn–Meemark [15, Theorem B] prove the exponent ≈ 0.4769, and the unrefereed manuscript [11, Theorem 1.3] states $2/5$. Both allow square discriminants, which Theorem 1.1 excludes. For n^2+a^2, [15, Theorem A] gives $29/100$ uniformly for $1 \le |a| \le N^{29/100}$; for n^2+1, Puchta [10] proved a bound of order $N^{2/5}$ up to logarithmic factors. Theorem 1.1 gives $293/1220$ for fixed $a \neq 0$. Theorem 1.3 improves $29/100$ in the shorter range $1 \le |a| \le N^{\kappa}$, $\kappa < 289/1050$, with equality at the endpoint.

Polynomial coefficient dependence alone is not new. Evertse–Silverman [4, Theorem 1] and Vaughan–Wooley [14, Lemma 3.5], with elementary class-number bounds, give $S(a,b,c;N) \ll_{\varepsilon} L^{39}N^{2/5+\varepsilon}$ in the range of Theorem 1.2. Appendix B derives this coarse coefficient power. The distinction in Theorem 1.2 is the exponent $389/1600$ with an absolute polynomial height loss; we found no earlier bound combining these features for all the stated quadratics. The coefficient average in [15, Remark 4.5], over heights at least $N^{2+\varepsilon}$, concerns a different range.

The cubic bounds are applications. The general estimate follows from Evertse–Silverman; the cube-coefficient estimate follows from Theorem 1.1 by factoring u^3+1. The same reduction and [11] give $(aT)^{2/5+\varepsilon}$. For the mixed-full applications, the cases $(r,s) = (2,6),(3,6),(2,8),(3,8),(4,8)$ improve $Y^{1/s}$; we found no stronger literature bound for these cases with the stated shifts and orientation. The other pairs illustrate the method without a best-known-bound claim.

Related results address different questions: infinitely many powerful n^2+k for almost all k [3], long blocks under Vojta's conjecture [7], and counts of norm-form or S-squareful values in several variables [13, 9]. The latter do not yield our one-variable estimates by fixing a coordinate. Under *abc* the quadratic count is $\ll N^{\varepsilon}$ [15, Proposition 3.7]; no such conjecture is assumed here. These comparisons are not an exhaustive literature search.

1.4. **Method and scope.** Put $\Delta = B^2-4AC$. The element $(2An+B)+\sqrt{\Delta}$ has norm $4Af(n)$. Factoring its ideal, choosing class representatives and balancing conjugates reduces a powerful value to $\gamma \mathsf{Y}^3\mathsf{X}^2 = (2An+B)+\sqrt{\Delta}$. In primitive coordinates $y, x \in \mathbb{Z}^2$ this gives

$$F(y;x) = e \neq 0, \qquad \deg F = (3,2). \tag{1.1}$$

This replaces the equation $e^3v^2 - d^3u^2 = h$ in [8]. Thue and Pell estimates handle boxes with a short side. In the other boxes the slopes lie within $O(1/n)$ of a real curve. Heath-Brown's determinant method [6, 5] places the points in each slope interval of length $X^{-\mu}$ on an auxiliary curve when $\mu > 57/305$. The threshold comes from two contact polynomials of bidegrees $(2,1)$ and $(1,2)$ and fifth-order vanishing. Low-degree components have $X^{o(1)}$ points; the remaining components are counted by the lifted-curve lemma of [8], using the affine-curve bound of [2]. Lattice moments then give $X^{1/10+3\mu/4+o(1)}$, with limiting exponent $293/1220$.

Sections 3–7 carry this out for a fixed quadratic. Section 8 makes the coefficient losses polynomial, using bounded-norm balancing elements and quantitative graph charts. Section 9 exploits the Gaussian structure of n^2+a^2. The applications follow in Sections 10 and 11. Split quadratics such as $n(n+1)$ are excluded here and treated in [8]. The fixed-quadratic constant is not effective; the uniform coefficient exponent is absolute but large and uncomputed. The coefficient average and mixed-full savings are therefore asymptotic, not practical numerical estimates. The exponent $389/1600$ is not optimized.

The main quadratic and mixed-full bounds also have proofs in Lean 4 with Mathlib. Three published estimates, stated in Section 2, enter that formalisation as explicit hypotheses. Their proofs lie outside the Lean development, so its conclusions are conditional on these inputs. The mathematical arguments invoke the published theorems. Section 13 and Appendix A describe the verification and authorship and map the paper's statements to the formal interfaces.

2. Notation and results from the literature

Counted variables are positive integers unless stated otherwise. An unspecified constant may depend on the parameters that precede it in the statement and on nothing else. We write $\omega(m)$ for the number of distinct prime divisors of m, $\tau(m)$ for the number of divisors, and $\|G\|$ for the largest absolute value of a coefficient of a polynomial G. A vector $y = (y_0, y_1) \in \mathbb{Z}^2$ is *primitive* if $\gcd(y_0, y_1) = 1$, and $\mathrm{ht}(y) = \max(|y_0|, |y_1|)$. A polynomial in two pairs of variables is *bihomogeneous of bidegree* (α, β) if it is homogeneous of degree α in the first pair and of degree β in the second.

We use the following three published estimates throughout. They are stated in the forms needed here. In the Lean development these statements are explicit hypotheses; the external theorems themselves are not formalised in this repository.

ST **(Stewart** [12, Theorem 1 and p. 796]**).** Let $F \in \mathbb{Z}[u, v]$ be a binary form of degree $\deg F \geq 3$ with coprime coefficients that is squarefree in $\mathbb{Q}[u, v]$. Then for every nonzero $m \in \mathbb{Z}$,

$$\#\{(u, v) \in \mathbb{Z}^2 : \gcd(u, v) = 1,\ F(u, v) = m\} \leq 2800\,(\deg F)^{1+\omega(|m|)}.$$

CC **(Castryck–Cluckers–Dittmann–Nguyen** [2, Theorem 3]**).** For every integer $w \geq 2$ there is c_w with the following property. Let $I \subset \mathbb{Q}[z_1, \ldots, z_w]$ be a prime ideal defining an affine curve of degree $d \geq 1$; that is, the quotient $\mathbb{Q}[z_1, \ldots, z_w]/I$ has Krull dimension one and, for all large k, the polynomials of degree at most k span a subspace of dimension $dk + k_0$ in it, for some integer k_0. Then for every $\Omega \geq 1$ the number of integer zeros P of I with $\max_i |P_i| \leq \Omega$ is at most

$$c_w\, d^3 \Omega^{1/d} (\log \Omega + d).$$

GE **(Gelfond;** [1, Lemma 1.6.11]**).** For every w there is $c \geq 0$ such that if $F, G \in \mathbb{Z}[z_1, \ldots, z_w]$, $F \neq 0$ and $G \mid F$, then $\|G\| \leq \exp(c \deg F)\, \|F\|$, where $\deg F$ is the total degree.

In ST reducible forms are allowed, and (u, v) and $(-u, -v)$ are counted separately. In CC the curve need not be geometrically irreducible. We use CC only for $w = 12$ and GE only for $w = 4$. Section 13 describes the comparison with the sources.

3. Arithmetic charts

Throughout Sections 3–7, fix integers $A > 0, B, C$ with nonsquare $\Delta = B^2 - 4AC$, and let $f(n) = An^2 + Bn + C$; constants may depend on them. Write $\theta = \sqrt{\Delta}$ and $\mathrm{N}(a + b\theta) = a^2 - \Delta b^2$, which vanishes only at zero. We call a, b the rational and $\sqrt{\Delta}$-coordinates. Using the finite class group and, for $\Delta > 0$, a norm-one unit, we construct finitely many integral forms F_χ of bidegree $(3, 2)$ whose primitive solutions $F_\chi(y; x) = e_\chi$ cover the powerful values. No hypothesis from the literature is used here.

3.1. Factorisation in the maximal order.

Let $\mathcal{O}$ be the maximal order of $\mathbb{Q}(\sqrt{\Delta})$. For $n \geq 1$ put $\alpha_n = (2An + B) + \theta$, which lies in $\mathcal{O}$ because $\Delta \equiv B^2 \pmod 4$, and $\bar{\alpha}_n = (2An + B) - \theta$. Then

$$\alpha_n \bar{\alpha}_n = (2An + B)^2 - \Delta = 4A\, f(n). \tag{3.1}$$

Every powerful integer is $d_0^3 u_0^2$ with $d_0, u_0 \geq 1$: take d_0 to be the product of primes occurring to odd exponent.

Lemma 3.1. *Let $n \geq 1$ with $f(n) > 0$ and $f(n)$ powerful. Then $(\alpha_n) = R\,\mathcal{A}^3\mathcal{B}^2$ with nonzero ideals $R, \mathcal{A}, \mathcal{B}$ of $\mathcal{O}$ such that R divides $(4A\Delta)$.*

Proof. Write $f(n) = d_0^3 u_0^2$. If $v_P(\alpha_n) = 1$, then either $\bar{\alpha}_n \in P$, giving $2\theta \in P$ and $P \mid (4\Delta)$, or $v_P(\bar{\alpha}_n) = 0$. In the latter case $P \nmid (4A)$ would give $1 = 3v_P(d_0) + 2v_P(u_0)$ by (3.1), a contradiction. Thus all exponent-one prime ideals divide $(4A\Delta)$. Put these into R, allocate an odd exponent $e \geq 3$ as $3 + 2((e-3)/2)$ to $\mathcal{A}^3\mathcal{B}^2$, and allocate even exponents to $\mathcal{B}^2$. □

Fix integral ideal representatives $\mathcal{R}$ of the finite class group and nonzero $\rho_I \in I$ for each $I \in \mathcal{R}$. Let $\mathcal{J}$ consist of triples $\pi = (R, I, J)$ with $R \mid (4A\Delta)$ and $I, J \in \mathcal{R}$. Choose a generator g_π of RI^3J^2 when principal, and put $g_\pi = 1$ otherwise; set $c_\pi = g_\pi/(\rho_I^3\rho_J^2) \neq 0$.

Lemma 3.2. *For every $n \geq 1$ with $f(n) > 0$ powerful there are $\pi \in \mathcal{J}$ and nonzero $\mathsf{Y}, \mathsf{X} \in \mathcal{O}$ with $\alpha_n = c_\pi \mathsf{Y}^3\mathsf{X}^2$.*

Proof. Write $\mathcal{A} = (\mathsf{Y}_0)I$, $\mathcal{B} = (\mathsf{X}_0)J$ using Lemma 3.1 and the class representatives. Then RI^3J^2 is principal and $\alpha_n = \epsilon g_\pi \mathsf{Y}_0^3\mathsf{X}_0^2$ for a unit ϵ. Set $\mathsf{Y} = \epsilon\mathsf{Y}_0\rho_I$ and $\mathsf{X} = \epsilon^{-1}\mathsf{X}_0\rho_J$. These are integral since $\mathsf{Y}_0\rho_I \in \mathcal{A}$, $\mathsf{X}_0\rho_J \in \mathcal{B}$; the powers of ϵ give the required identity. □

Fix $m_\mathcal{O} \geq 1$ with $\mathcal{O} \subset m_\mathcal{O}^{-1}\mathbb{Z}[\theta]$ and put $\iota(y) = (y_0 + y_1\theta)/m_\mathcal{O}$ for $y \in \mathbb{Z}^2$. Then $m_\mathcal{O}^2 \,\mathrm{N}\,\iota(y) = y_0^2 - \Delta y_1^2$ is integral, and nonzero for $y \neq 0$.

3.2. **Balancing.** Put $\|a + b\theta\|^2 = a^2 + |\Delta|b^2$. For $\Delta < 0$ this is $\mathrm{N}\,z$; for $\Delta > 0$ we normalize by units. For $\Delta > 0$, direct expansion gives

$$2\|z\|^2\|w\|^2 = \|zw\|^2 + \|\bar{z}w\|^2. \tag{3.2}$$

Consequently $\|zw\|^2 \leq 2\|z\|^2\|w\|^2$, $\|z^3\|^2 \leq 4\|z\|^6$, and, using $\bar{z}(zw) = (\mathrm{N}\,z)w$, $(\mathrm{N}\,z)^2\|w\|^2 \leq 2\|z\|^2\|zw\|^2$. Also $\|z^2\|^2 - \|z\|^4 = 4\Delta a^2b^2 \geq 0$.

Lemma 3.3. *Let $\Delta > 0$. There are a unit ϵ_0 of $\mathcal{O}$ of norm 1 and $U_0 \geq 1$ such that for every nonzero $\mathsf{Y} \in \mathcal{O}$ there is $k \in \mathbb{Z}$ with $\|\epsilon_0^{2k}\mathsf{Y}\|^2 \leq U_0\,|\mathrm{N}\,\mathsf{Y}|$.*

Proof. Choose a Pell unit $\epsilon_0 = x_1 + y_1\theta$ with integers $x_1 > 1$, $y_1 > 0$ and norm 1. Write $\epsilon_0^2 = U_0 + V_0\theta$, where $U_0 > 1$, $V_0 > 0$ and $U_0^2 - \Delta V_0^2 = 1$. Minimize the nonnegative integer $m_\mathcal{O}^2\|\epsilon_0^{2k}\mathsf{Y}\|^2$ over $k \in \mathbb{Z}$, and write the minimizing element as $a + b\theta$, with $s_+ = a^2 + \Delta b^2$ and $s_- = a^2 - \Delta b^2 = \mathrm{N}\,\mathsf{Y}$. The neighboring values are $(U_0^2 + \Delta V_0^2)s_+ \pm 4U_0V_0\Delta ab$; minimality gives $V_0s_+ \pm 2U_0ab \geq 0$. Therefore

$$U_0^2s_-^2 - s_+^2 = \Delta(V_0s_+ - 2U_0ab)(V_0s_+ + 2U_0ab) \geq 0,$$

which proves the bound. □

Lemma 3.4. *Let $n_0 = |B| + 2|\Delta| + 1$ and $G_0 = (2A + |B|)^2 + |\Delta| + 1$. For $n \geq n_0$,*

$$n^2 \leq 2\,\mathrm{N}\,\alpha_n, \qquad \mathrm{N}\,\alpha_n \leq G_0n^2, \qquad \|\alpha_n\|^2 \leq 3\,\mathrm{N}\,\alpha_n,$$

and hence $f(n) > 0$.

Proof. Let $m = 2An + B$. Since $A \geq 1$ and $n \geq |B|$ we have $n \leq m \leq (2A + |B|)n$, and $m^2 \geq n^2 \geq 2|\Delta|$. The three bounds follow from $\mathrm{N}\,\alpha_n = m^2 - \Delta$ and $\|\alpha_n\|^2 = m^2 + |\Delta|$. □

Lemma 3.5. *There is $U_1 \geq 1$ such that for every $n \geq n_0$ with $f(n)$ powerful there are $\pi \in \mathcal{J}$ and nonzero $\mathsf{Y}, \mathsf{X} \in \mathcal{O}$ with $\alpha_n = c_\pi\mathsf{Y}^3\mathsf{X}^2$, $\|\mathsf{Y}\|^2 \leq U_1|\mathrm{N}\,\mathsf{Y}|$ and $\|\mathsf{X}\|^2 \leq U_1|\mathrm{N}\,\mathsf{X}|$.*

Proof. For $\Delta < 0$ use Lemma 3.2 and $U_1 = 1$. For $\Delta > 0$, replace its pair (Y, X) by $(\epsilon_0^{2k}\mathsf{Y}, \epsilon_0^{-3k}\mathsf{X})$ using Lemma 3.3. The product is unchanged and $\|\mathsf{Y}\|^2 \leq U_0|\mathrm{N}\,\mathsf{Y}|$. With $c = c_\pi$, (3.2) and Lemma 3.4 give

$$(\mathrm{N}(c\mathsf{Y}^3))^2\|\mathsf{X}\|^4 \leq 2\|c\mathsf{Y}^3\|^2\|\alpha_n\|^2 \leq 48\|c\|^2U_0^3|\mathrm{N}\,\mathsf{Y}|^3|\mathrm{N}\,\alpha_n|.$$

Substitute $|\mathrm{N}\,\alpha_n| = |\mathrm{N}\,c|\,|\mathrm{N}\,\mathsf{Y}|^3|\mathrm{N}\,\mathsf{X}|^2$ and cancel to obtain $|\mathrm{N}\,c|\,\|\mathsf{X}\|^4 \leq 48\|c\|^2U_0^3|\mathrm{N}\,\mathsf{X}|^2$. One may take $U_1 = U_0 + 1 + \sum_\pi 48\|c_\pi\|^2U_0^3/|\mathrm{N}\,c_\pi|$. □

3.3. **The charts.** Clear denominators in the $\sqrt{\Delta}$-coordinate of $c_\pi \iota(y)^3\iota(x)^2$ to obtain an integral form F_π of bidegree $(3,2)$ and a denominator $e_\pi \geq 1$. A *chart* is $\chi = (\pi, g, g')$ with $\pi \in \mathcal{J}$, $1 \leq g, g' \leq G_1 = 1 + \sum_\pi e_\pi$. Set $\gamma_\chi = g^3 g'^2 c_\pi$, $e_\chi = e_\pi$ and $F_\chi = g^3 g'^2 F_\pi$, so

$$F_\chi(y;x) = e_\chi \cdot (\sqrt{\Delta}\text{-coordinate of } \gamma_\chi\, \iota(y)^3\iota(x)^2). \tag{3.3}$$

The set $\mathcal{X}$ of charts is finite. The form F_χ is not zero: otherwise (3.3) at $\iota(y) = \iota(x) = 1$ and at $\iota(y) = \theta$, $\iota(x) = 1$ (where $\theta^3 = \Delta\theta$) would show that both coordinates of γ_χ vanish.

Definition 3.6. Let $H \geq 1$, $\chi \in \mathcal{X}$ and $n \geq 1$. A pair (y, x) of primitive vectors in $\mathbb{Z}^2$ is an *arithmetic point of χ over n* if

$$\gamma_\chi\, \iota(y)^3\iota(x)^2 = (2An + B) + \sqrt{\Delta} \tag{3.4}$$

and, with $D_y = |\,\mathrm{N}\,\iota(y)|$ and $D_x = |\,\mathrm{N}\,\iota(x)|$,

$$H^{-1} \leq D_y, \quad H^{-1}D_y \leq \mathrm{ht}(y)^2 \leq HD_y, \quad H^{-1} \leq D_x, \quad H^{-1}D_x \leq \mathrm{ht}(x)^2 \leq HD_x,$$
$$H^{-1}n^2 \leq D_y^3 D_x^2 \leq Hn^2.$$

By (3.3) and (3.4) an arithmetic point satisfies $F_\chi(y;x) = e_\chi$.

Proposition 3.7. *There is $H \geq 1$ such that for every $n \geq n_0$ with $f(n)$ powerful there are a chart $\chi \in \mathcal{X}$ and an arithmetic point of χ over n.*

Proof. Represent the balanced pair of Lemma 3.5 by $y', x' \in \mathbb{Z}^2$. Since $F_\pi(y';x') = e_\pi$, their contents g, g' satisfy $g^3 g'^2 \mid e_\pi$. Thus $\chi = (\pi, g, g') \in \mathcal{X}$, and the primitive vectors $y = y'/g$, $x = x'/g'$ satisfy (3.4). Balancing is unchanged by scaling. Moreover

$$1 \leq m_{\mathcal{O}}^2 D_y = |y_0^2 - \Delta y_1^2| \leq (1 + |\Delta|)\,\mathrm{ht}(y)^2, \qquad \mathrm{ht}(y)^2 \leq m_{\mathcal{O}}^2 U_1 D_y,$$

and likewise for x. Taking norms gives $|\,\mathrm{N}\,\gamma_\chi| D_y^3 D_x^2 \in [n^2/2, G_0 n^2]$ by Lemma 3.4. A sufficiently large H, uniform over the finite chart list, gives all the required inequalities. □

Lemma 3.8. *If (y, x) is an arithmetic point of χ over n and over n', then $n = n'$; and $\mathrm{ht}(y), \mathrm{ht}(x) \leq H^3 n$. Consequently, for $X \geq 1$ let $\mathcal{W}(X)$ be the set of $n \geq n_0$ with $X < n \leq 2X$ and $f(n)$ powerful, and $\mathcal{S}(X)$ the set of all triples (χ, y, x) such that (y, x) is an arithmetic point of χ over some $n \in \mathcal{W}(X)$. Then $\mathcal{S}(X)$ is finite and $\#\mathcal{W}(X) \leq \#\mathcal{S}(X)$.*

Proof. The rational coordinate in (3.4) determines n since $A \neq 0$. As $HD_y, HD_x \geq 1$, $\mathrm{ht}(y)^2 \leq HD_y \leq H^5 D_y^3 D_x^2 \leq H^6 n^2$, and likewise for x. Thus $\mathcal{S}(X)$ is finite; choosing a triple for each n by Proposition 3.7 gives an injection. □

A given n may be represented by several triples; all bounds below are upper bounds for the number of triples.

3.4. **Slopes and the real curve.** Write $\sigma_\pm(a + b\theta) = a \pm b\sqrt{\Delta}$. A *dominant coordinate* d_y of primitive y has $|d_y| = \mathrm{ht}(y)$; the other coordinate divided by d_y is its *slope* $\xi \in [-1, 1]$. A *frame* ϕ chooses a chart χ and dominant indices for y, x, giving four frames per chart. Then $\sigma_\pm(\iota(y)) = d_y \ell_\pm(\xi)$ and $\sigma_\pm(\iota(x)) = d_x m_\pm(\zeta)$ for fixed linear forms (e.g. $\ell_\pm(\xi) = (\xi \pm \sqrt{\Delta})/m_{\mathcal{O}}$ when $d_y = y_1$). Put

$$R_\phi(\xi, \zeta) = \frac{\sigma_+(\gamma_\chi)\ell_+(\xi)^3 m_+(\zeta)^2 - \sigma_-(\gamma_\chi)\ell_-(\xi)^3 m_-(\zeta)^2}{2\sqrt{\Delta}}.$$

This real polynomial has degrees at most $(3, 2)$ and $F_\chi(y;x) = e_\chi d_y^3 d_x^2 R_\phi(\xi, \zeta)$. Choose $M_1 \geq 1$ bounding all $|\ell_\pm|, |m_\pm|$ on $[-1, 1]$ and set $\eta_0 = (HM_1)^{-1}$.

Lemma 3.9. *Let (y,x) be an arithmetic point of χ over n, in a frame ϕ of χ given by dominant coordinates d_y, d_x, with slopes (ξ,ζ). Then $|\ell_\pm(\xi)| \ge \eta_0$, $|m_\pm(\zeta)| \ge \eta_0$,*

$$R_\phi(\xi,\zeta) = \frac{1}{d_y^3 d_x^2} \qquad \text{and} \qquad |R_\phi(\xi,\zeta)| \le \frac{H^3}{n}.$$

Proof. The product $|\ell_+(\xi)\ell_-(\xi)| = D_y/\operatorname{ht}(y)^2 \ge H^{-1}$ and the upper bound M_1 give the lower bound for each factor; the same applies to $m_\pm$. Subtract the two embeddings of (3.4) and divide by $2\sqrt{\Delta}\, d_y^3 d_x^2$ for the identity. Finally $n^2 \le H D_y^3 D_x^2 \le H^6 d_y^6 d_x^4$. □

The set of $(\xi,\zeta) \in [-1,1]^2$ with $|\ell_\pm(\xi)|, |m_\pm(\zeta)| \ge \eta_0$ is the *balanced patch* of ϕ.

3.5. **Boxes and windows.** For a triple (χ, y, x) put $U = H|\operatorname{N}\iota(y)|$ and $V = H|\operatorname{N}\iota(x)|$.

Lemma 3.10. *If (y,x) is an arithmetic point of χ over $n \le 2X$, then $U \ge 1$, $V \ge 1$ and $U^3V^2 \le 4H^6X^2$.*

Proof. $U^3V^2 = H^5 D_y^3 D_x^2 \le H^6 n^2 \le 4H^6X^2$. □

Lemma 3.11. *Let $A_0 \ge 1$. There is $K_{\mathrm{p}} > 0$ such that for every $X \ge 2$ and every finite set $\mathcal{S}$ with functions $U, V : \mathcal{S} \to [1,\infty)$ satisfying $U^3V^2 \le A_0X^2$, the sets*

$$\mathcal{S}(D,E) = \{z \in \mathcal{S} :\ D/2 < U(z) \le D,\ E/2 < V(z) \le E\},$$

with D, E running over the powers 2^j, $j \ge 0$, partition $\mathcal{S}$; at most $K_{\mathrm{p}}(1+\log X)^2$ of them are nonempty; and each nonempty one has $D^3E^2 \le 32A_0X^2$.

Proof. Each z lies in exactly one set. If $z \in \mathcal{S}(D,E)$ then $D < 2U \le 2U^3V^2 \le 2A_0X^2$ and likewise for E, so there are at most $(2+\log_2(2A_0X^2))^2$ nonempty sets, and $D^3E^2 \le 2^3U^3 \cdot 2^2V^2 \le 32A_0X^2$. □

We use this with $A_0 = 4H^6$, so that $D^3E^2 \le 128H^6X^2$; the pair (D,E) is called a *box*.

Lemma 3.12. *Let $\vartheta > 0$, $C_{\mathrm{w}} \ge 0$, $X_0 \ge 1$, $n_1 \in \mathbb{N}$, and let $\mathcal{P}$ be a set of positive integers such that $\#\{n \in \mathcal{P} : n \ge n_1,\ X < n \le 2X\} \le C_{\mathrm{w}}X^\vartheta$ for all $X \ge X_0$. Then $\#\{n \in \mathcal{P} : n \le N\} \le KN^\vartheta$ for all $N \ge 1$, with $K = \max(2, 2X_0, 2n_1) + C_{\mathrm{w}}/(2^\vartheta - 1) + 1$.*

Proof. By strong induction on the integer $m = \lfloor N \rfloor$, the assertion is trivial for $m \le \max(2, 2X_0, 2n_1)$. Otherwise the prefix up to $\lfloor m/2 \rfloor$ and the window $(m/2, m]$ contribute at most $(K + C_{\mathrm{w}})(m/2)^\vartheta \le Km^\vartheta$, since $C_{\mathrm{w}} \le (2^\vartheta - 1)K$. □

4. The easy bound

This section bounds the number of points of a box by $\min(D,E)$, up to X^δ. From the literature it uses only ST, for binary cubic forms.

4.1. **The two fibrations.** Fix a chart and put $\gamma_\pm = \sigma_\pm(\gamma_\chi)$, $\mathsf{Y}_\pm(y) = (y_0 \pm \sqrt{\Delta}y_1)/m_{\mathcal{O}}$ and $\mathsf{X}_\pm(x) = (x_0 \pm \sqrt{\Delta}x_1)/m_{\mathcal{O}}$. Then (3.3) gives

$$\frac{2\sqrt{\Delta}}{e_\chi} F_\chi(y;x) = \gamma_+ \mathsf{Y}_+(y)^3 \mathsf{X}_+(x)^2 - \gamma_- \mathsf{Y}_-(y)^3 \mathsf{X}_-(x)^2. \tag{4.1}$$

Here $\gamma_+\gamma_- \ne 0$; each pair of linear forms is independent, and its values at nonzero integer vectors are nonzero, since their product is the corresponding norm.

Lemma 4.1. *Let $k_1, k_2 \in \mathbb{C}$ be nonzero and let $P_1, P_2 \in \mathbb{C}[t_0,t_1]$ be linearly independent linear forms. Then $k_1P_1^3 - k_2P_2^3$ is nonzero and squarefree.*

Proof. After rescaling P_1, P_2 by cube roots of k_1, k_2, the form becomes $P_1^3 - P_2^3$. Its three linear factors are distinct and nonzero because P_1, P_2 are independent. □

Lemma 4.2 (Cubic fibres)**.** *For every chart χ and every $x \neq 0$ in $\mathbb{Z}^2$ the number of primitive $y \in \mathbb{Z}^2$ with $F_\chi(y;x) = e_\chi$ is at most $K_{\mathrm{c}}(\chi) = 5600\,\tau(e_\chi)\,3^{1+\omega(e_\chi)}$.*

Proof. By (4.1) and Lemma 4.1, $F_\chi(\cdot\,;x)$ is a nonzero squarefree cubic. If solutions exist, its content g divides e_χ and $F_\chi(\cdot\,;x) = gQ_x$, with Q_x primitive and squarefree. For each of the $2\tau(e_\chi)$ signed divisors m of e_χ, hypothesis ST gives at most $2800 \cdot 3^{1+\omega(e_\chi)}$ primitive solutions of $Q_x = m$. This bounds in particular the level $m = e_\chi/g$. □

Lemma 4.3 (Quadratic fibres)**.** *Let $h \neq 0$ be an integer and $\varepsilon > 0$. There is $K_{\mathrm{q}} = K_{\mathrm{q}}(h,\varepsilon)$ such that for all real $H_1 \geq 1$, $Z_1 \geq 2$ and all integers a_1, a_2, a_3 of absolute value at most H_1 with $a_2^2 - 4a_1a_3 \neq 0$, the number of primitive $(u_1,u_2) \in \mathbb{Z}^2$ with $|u_1|, |u_2| \leq Z_1$ and $a_1u_1^2 + a_2u_1u_2 + a_3u_2^2 = h$ is at most $K_{\mathrm{q}}(H_1Z_1)^\varepsilon$.*

Proof. Write ψ for the form and $D_1 = a_2^2 - 4a_1a_3$. In each of the $4h^2$ residue classes modulo $2h$ containing solutions, fix one w. For another solution u in that class put

$$P = 2a_1u_1w_1 + a_2(u_1w_2 + u_2w_1) + 2a_3u_2w_2, \qquad R = u_1w_2 - u_2w_1.$$

Expansion gives $P^2 - D_1R^2 = 4h^2$. Both $P - 2h$ and R are divisible by $2h$, so $(p,r) = (P/(2h), R/(2h))$ is integral, satisfies $p^2 - D_1r^2 = 1$, and has $|p|, |r| \leq K_1 = \lceil 6H_1Z_1^2 \rceil$. The map is injective because $w_1P + (a_2w_1 + 2a_3w_2)R = 2hu_1$ and $w_2P - (2a_1w_1 + a_2w_2)R = 2hu_2$. For $D_1 < 0$ there are at most four such Pell solutions, and for square D_1 at most two. Otherwise they are $\pm\epsilon_1^j$ for a fundamental Pell unit with rational coordinate at least 2; the rational coordinate of ϵ_1^j is at least $2^{|j|}$. Thus there are at most $4\log K_1/\log 2 + 4$ per class, and $4h^2(4\log K_1/\log 2 + 4) \ll_{h,\varepsilon} (H_1Z_1)^\varepsilon$. □

Lemma 4.4. *For every chart χ and $y \neq 0$ in $\mathbb{Z}^2$, the binary quadratic form $x \mapsto F_\chi(y;x)$ has nonzero discriminant.*

Proof. Write $\mathsf{X}_\pm(x) = r_\pm x_0 + s_\pm x_1$ and $k_1 = \gamma_+\mathsf{Y}_+(y)^3$, $k_2 = \gamma_-\mathsf{Y}_-(y)^3$, which are nonzero. By (4.1) the form is a nonzero multiple of $k_1\mathsf{X}_+^2 - k_2\mathsf{X}_-^2$, whose discriminant is $4k_1k_2(r_+s_- - r_-s_+)^2 \neq 0$. □

4.2. **The bound.**

Proposition 4.5 (Easy bound)**.** *Let H be as in Proposition 3.7. For every $\delta > 0$ there is K_{e} such that for all $X \geq 2$ and $D, E \geq 1$ with $D^3E^2 \leq 128H^6X^2$, the number of triples $(\chi, y, x) \in \mathcal{S}(X)$ with $U \leq D$ and $V \leq E$ is at most $K_{\mathrm{e}}X^\delta \min(D,E)$.*

Proof. For a fixed chart, $\mathrm{ht}(y)^2 \leq D$, $\mathrm{ht}(x)^2 \leq E$, and both heights are at most $12H^3X$. There are at most $9E$ possible x; Lemma 4.2 bounds each fibre by $K_{\mathrm{c}}(\chi)$. There are at most $9D$ possible y. In each such primitive fibre, Lemma 4.4 gives nonzero discriminant, the quadratic coefficients are at most $4\|F_\chi\|(12H^3X)^3$, and the variables are bounded by $12H^3X$. Lemma 4.3 with $h = e_\chi$, $\varepsilon = \delta/4$ bounds this fibre by $K_\chi X^\delta$. Taking the smaller bound and summing over the finite chart list proves the proposition. □

5. The determinant method on a cell

We construct an auxiliary form of bidegree $(3q, 2q)$ vanishing on the points near a short branch segment of $R_\phi = 0$. This uses the chart bounds, with no hypothesis from the literature.

5.1. **Branches and cells.**

Proposition 5.1 (Branches)**.** *For every frame ϕ there are an integer $J \geq 0$, intervals $[a_j, b_j] \subset [-2,2]$ and functions g_j $(1 \leq j \leq J)$, and constants $r_0, c_{\mathrm{g}}, C_{\mathrm{g}}, \delta_0 > 0$, such that:*

(i) *g_j is smooth on $(a_j - 2r_0, b_j + 2r_0)$, and for $\xi \in [a_j - r_0, b_j + r_0]$ we have $R_\phi(\xi, g_j(\xi)) = 0$ and $|g_j'(\xi)| \geq c_{\mathrm{g}}$;*

(ii) *for every* $\varkappa \geq 0$ *there is* $M_\varkappa$ *with* $|g_j^{(i)}(\xi)| \leq M_\varkappa$ *for all* j, *all* $i \leq \varkappa + 1$ *and all* $\xi \in [a_j - r_0, b_j + r_0]$;
(iii) *every* (ξ, ζ) *in the balanced patch of* ϕ *with* $|R_\phi(\xi, \zeta)| \leq \delta_0$ *has an index* j *with* $\xi \in [a_j, b_j]$ *and* $|\zeta - g_j(\xi)| \leq C_{\mathrm{g}} |R_\phi(\xi, \zeta)|$.

Proof. Write $W_\pm = \sigma_\pm(\gamma_\chi)\ell_\pm^3 m_\pm^2$. At a zero of R_ϕ in the balanced patch, $W_+ = W_- \neq 0$, and

$$\ell_+\ell_-\,\partial_\xi(W_+ - W_-) = 3W_+(\ell'_+\ell_- - \ell'_-\ell_+), \qquad m_+m_-\,\partial_\zeta(W_+ - W_-) = 2W_+(m'_+m_- - m'_-m_+).$$

The brackets are nonzero constants by independence of the linear forms; thus both partial derivatives of R_ϕ are nonzero. At such a point (ξ_0, ζ_0) write $R_\phi(\xi, \zeta_0 + t) = \varphi_2(\xi)t^2 + \varphi_1(\xi)t + \varphi_0(\xi)$. Here $\varphi_0(\xi_0) = 0$ and $\varphi_1(\xi_0)\varphi_0'(\xi_0) \neq 0$. Locally the smooth function

$$g(\xi) = \zeta_0 - \frac{2\varphi_0(\xi)}{\varphi_1(\xi) + \operatorname{sgn}(\varphi_1(\xi_0))\sqrt{\varphi_1(\xi)^2 - 4\varphi_2(\xi)\varphi_0(\xi)}}$$

satisfies $R_\phi(\xi, g(\xi)) = 0$ and $g'(\xi_0) = -\varphi_0'(\xi_0)/\varphi_1(\xi_0) \neq 0$. The factorization

$$R_\phi(\xi, \zeta) = (\zeta - g(\xi))[\varphi_1(\xi) + \varphi_2(\xi)(\zeta + g(\xi) - 2\zeta_0)]$$

gives $|\zeta - g(\xi)| \leq 2|\varphi_1(\xi_0)|^{-1}|R_\phi(\xi, \zeta)|$ in a sufficiently small box.

For each zero $z = (s_z, w_z)$ choose such an open box B_z with horizontal interval $(s_z - h_z, s_z + h_z)$, where $0 < h_z \leq 1/2$, so that g_z is smooth on $(s_z - 4h_z, s_z + 4h_z)$ and its equation and $|g_z'| \geq m_z > 0$ hold on $[s_z - 2h_z, s_z + 2h_z]$. Let C_z be the residual constant on B_z. The zero set in the compact balanced patch S has a finite subcover $B_1, \ldots, B_J$. Put $a_j = s_j - h_j$, $b_j = s_j + h_j$, $r_0 = \frac{1}{2}\min_j h_j$, $c_{\mathrm{g}} = \min_j m_j$ and $C_{\mathrm{g}} = 1 + \sum_j C_j$. Continuity on the compact intervals $[s_j - 2h_j, s_j + 2h_j]$ bounds all derivatives of each prescribed finite order, proving (i)–(ii). The compact set $S \setminus \bigcup_j B_j$ contains no zero; take δ_0 to be half its positive minimum of $|R_\phi|$, or 1 if this set is empty. This proves (iii). If the zero set is empty, take $J = 0$ and arbitrary positive graph constants, and choose δ_0 by the same minimum on S (or 1 if S is empty). □

The number of branches and the constants come from a compactness argument and are not effective.

For an integer $M \geq 1$ and a branch j put $\xi_{j,k} = a_j + k/M$ for $0 \leq k \leq (b_j - a_j)M$. The intervals $[\xi_{j,k}, \min(b_j, \xi_{j,k} + 1/M)]$ cover $[a_j, b_j]$, there are at most $5M$ of them, the points $\xi_{j,k}$ are $1/M$-separated, and by the mean value theorem and (i) the values $g_j(\xi_{j,k})$ are c_{g}/M-separated.

Definition 5.2. Let $57/305 < \mu < 1/5$, $X \geq 2$ and $M = \lceil X^\mu \rceil$. A *cell* $\mathcal{C}$ consists of a frame ϕ, a branch j and an index k as above; $\xi_{\mathcal{C}} = \xi_{j,k}$. Its set of points $\mathcal{S}_{\mathcal{C}}(X)$ consists of the triples $(\chi, y, x) \in \mathcal{S}(X)$ for which the frame given by some choice of dominant coordinates is ϕ and the slopes satisfy $\xi_{\mathcal{C}} \leq \xi \leq \min(b_j, \xi_{\mathcal{C}} + 1/M)$ and $|\zeta - g_j(\xi)| \leq C_{\mathrm{g}}|R_\phi(\xi, \zeta)|$. For such a point we write $p = (p_0, p_1, p_2, p_3)$ for the coordinates of y and x ordered so that $p_1 = d_y$ and $p_3 = d_x$ are the dominant ones; thus $\xi = p_0/p_1$ and $\zeta = p_2/p_3$.

If $X \geq H^3/\delta_0$, Lemma 3.9 and (iii) show that every triple of $\mathcal{S}(X)$ lies in $\mathcal{S}_{\mathcal{C}}(X)$ for at least one cell.

5.2. **Weighted jets.** Let $0 < \mu < 1/5$. In $\mathbb{R}[u, v]$ give the monomial $u^i v^j$ the *weight* $\mu i + j$, and say that a polynomial has weighted order at least t if all its monomials have weight at least t; weighted orders add under multiplication. Let $J_\mu(t) = \#\{(i, j) \in \mathbb{N}^2 : \mu i + j < t\}$. Fix a polynomial $g \in \mathbb{R}[u]$, and for $P \in \mathbb{R}[u, v]$ put $P^g(u, v) = P(u, g(u) + v)$. For integers $a, b \geq 0$ and real t let $W_t^{a,b}(g)$ be the space of P with $\deg_u P \leq a$, $\deg_v P \leq b$ such that P^g has weighted order at least t, and $W_{q,t}(g) = W_t^{3q,2q}(g)$. Since $W_t^{a,b}(g)$ is the kernel of the linear map sending P to the coefficients of P^g at the monomials of weight less than t,

$$\dim W_t^{a,b}(g) \geq (a + 1)(b + 1) - J_\mu(t). \tag{5.1}$$

Lemma 5.3 (Contact polynomials)**.** *For $(a,b)=(2,1)$ and for $(a,b)=(1,2)$ there is a nonzero $P\in\mathbb{R}[u,v]$ with $\deg_u P\le a$, $\deg_v P\le b$ such that P^g has weighted order at least 5μ.*

Proof. The six-dimensional space of such P contains a nonzero solution to the five conditions that the coefficients of $u^0,\dots,u^4$ in $P(u,g(u))$ vanish. The pure u terms of P^g then have weight at least 5μ, and every term involving v has weight at least $1>5\mu$. □

We fix such polynomials P_{21} and P_{12}.

Lemma 5.4 (Mixed-contact bound)**.** *Let $q,k,l\ge 0$ be integers with $2k+l\le 3q$ and $k+2l\le 2q$. For every real t,*

$$\dim W_{q,t}(g)\ge (3q-2k-l+1)(2q-k-2l+1)-J_\mu(t-5\mu(k+l)).$$

Proof. Multiplication by the nonzero $P_{21}^kP_{12}^l$, whose shift has weighted order at least $5\mu(k+l)$, injects $W^{3q-2k-l,\,2q-k-2l}_{t-5\mu(k+l)}(g)$ into $W_{q,t}(g)$. Apply (5.1) to the source. □

Lemma 5.5. *For $h\ge 0$, $J_\mu(h)\le h^2/(2\mu)+(1+1/\mu)h+1$.*

Proof. Let $n=\lfloor h\rfloor$. For each $0\le j\le n$ there are at most $(h-j)/\mu+1$ values of i, so $J_\mu(h)\le((n+1)h-n(n+1)/2)/\mu+n+1$, and the claimed bound exceeds this by $((h-n)^2+n)/(2\mu)+(h-n)\ge 0$. □

Lemma 5.6 (Layer-cake inequality)**.** *Let (C_{ij}) be a real matrix with rows $i\in I$, I finite, and columns $1\le j\le n$, let $w_i\ge 0$ be weights, and let $\mathrm{nul}(t)$ be the dimension of the space of $x\in\mathbb{R}^n$ with $\sum_j C_{ij}x_j=0$ for all i with $w_i<t$. If the rows $i_1,\dots,i_n$ form a nonsingular $n\times n$ matrix, then $\int_0^\infty \mathrm{nul}(t)\,dt\le w_{i_1}+\cdots+w_{i_n}$.*

Proof. On that space the map $x\mapsto(\sum_j C_{i_lj}x_j)_{l:\,w_{i_l}\ge t}$ is injective, so $\mathrm{nul}(t)\le\#\{l:w_{i_l}\ge t\}$; integrate over t. □

Proposition 5.7 (Weight surplus)**.** *Let $n_q=(3q+1)(2q+1)$, the number of monomials u^av^b with $a\le 3q$, $b\le 2q$. Let $57/305<\mu<1/5$, $\rho(\mu)=(305\mu-57)/18$ and $b_\mu=25\mu/3$, and let $q\ge 1$ be an integer with*

$$q>\frac{2}{\rho(\mu)}\Big(b_\mu^2(11+\tfrac{1}{\mu})+b_\mu(60\mu+11)+6\Big). \tag{5.2}$$

Let $g\in\mathbb{R}[u]$, let (b_j) be a basis of the space of polynomials of degree at most $3q$ in u and $2q$ in v, and let $m_1,\dots,m_{n_q}$ be monomials in u,v such that the matrix of the coefficients of m_i in b_j^g is nonsingular. Then the sum of the weights of the m_i is greater than $n_qq+\rho(\mu)q^3/2$.

Proof. Apply Lemma 5.6 to this matrix, with w_i the weight of m_i. If $P=\sum_j x_j\mathsf{b}_j$ lies in $W_{q,t}(g)$ then x lies in the space of the lemma, so $\dim W_{q,t}(g)\le\mathrm{nul}(t)$ and it suffices to bound $\int_0^{qb_\mu}\dim W_{q,t}(g)\,dt$ from below.

To select the contacts, write $r=s_1+s_2$ under the constraints $s_1,s_2\ge 0$, $2s_1+s_2\le 3$, $s_1+2s_2\le 2$. Then $0\le r\le 5/3$ and the residual area is $(3-r-s_1)(2-2r+s_1)$, with derivative $1+r-2s_1$. For fixed r it is maximized at $s_1=r$ if $r\le 1$, and at $s_1=(1+r)/2$ otherwise. Its maximum is

$$A(r)=\begin{cases}6-7r+2r^2, & 0\le r\le 1,\\ (5-3r)^2/4, & 1\le r\le 5/3.\end{cases}$$

At level z, maximize $A(r)-(z-5\mu r)_+^2/(2\mu)$. As A decreases, one may restrict to $r\le z/(5\mu)$. On the two ranges its derivatives are respectively

$$5z-7-(25\mu-4)r,\qquad \tfrac{1}{2}(10z-15-(50\mu-9)r).$$

They decrease and agree at $r=1$. Their zero, clipped at 0, gives the following path, with transitions at $z=7/5$, $5\mu+3/5$ and endpoint $r(b_\mu)=5/3$.

For $0 \le z \le b_\mu$ define

$$r(z) = \begin{cases} 0, & z \le \frac{7}{5}, \\ \dfrac{5z-7}{25\mu-4}, & \frac{7}{5} < z \le 5\mu + \frac{3}{5}, \\ \dfrac{10z-15}{50\mu-9}, & 5\mu+\frac{3}{5} < z \le b_\mu, \end{cases} \qquad s_1(z) = \begin{cases} r(z), & r(z) \le 1, \\ \frac{1+r(z)}{2}, & r(z) > 1, \end{cases}$$

and $s_2(z) = r(z) - s_1(z)$. Then $s_1, s_2 \ge 0$, $2s_1 + s_2 \le 3$, $s_1 + 2s_2 \le 2$, $0 \le z - 5\mu r \le b_\mu$, and the function $\Pi(z) = (3 - 2s_1 - s_2)(2 - s_1 - 2s_2) - (z - 5\mu r)^2/(2\mu)$ equals

$$6 - \frac{z^2}{2\mu}, \qquad 6 - \frac{z^2}{2\mu} + \frac{(5z-7)^2}{2(25\mu-4)}, \qquad \frac{9(b_\mu - z)^2}{2\mu(50\mu-9)}$$

on the three ranges (here $25\mu - 4 > 0$ and $50\mu - 9 > 0$). Take $k = \lfloor qs_1 \rfloor$ and $l = \lfloor qs_2 \rfloor$ in Lemma 5.4 with $t = qz$. The two sides of the rectangle are at least $q(3 - 2s_1 - s_2)$ and $q(2 - s_1 - 2s_2)$, and the level $h = qz - 5\mu(k + l)$ satisfies $0 \le h \le q(z - 5\mu r) + 10\mu$. By Lemma 5.5,

$$J_\mu(h) \le \frac{q^2(z - 5\mu r)^2}{2\mu} + q\, b_\mu\Big(11 + \frac{1}{\mu}\Big) + 60\mu + 11,$$

so $\dim W_{q,qz}(g) \ge q^2\Pi(z) - qb_\mu(11 + 1/\mu) - (60\mu + 11)$. Integrating the three displayed quadratic pieces gives $\int_0^{b_\mu} \Pi(z)\, dz = 17/6 + 305\mu/18 = 6 + \rho(\mu)$. Substituting $t = qz$,

$$\sum_i w_i \ge \int_0^{qb_\mu} \dim W_{q,t}(g)\, dt \ge q^3(6 + \rho(\mu)) - q^2 b_\mu^2\Big(11 + \frac{1}{\mu}\Big) - q\, b_\mu(60\mu + 11).$$

Since $n_q q = 6q^3 + 5q^2 + q$, the right side exceeds $n_q q + \rho(\mu)q^3/2$ by (5.2). □

Here 6 is the leading denominator-clearing cost in $n_q q$ and $57/305$ is the zero of ρ. The path optimizes the continuous bound for these two contact polynomials; this asserts no optimality among other determinant constructions. The bound holds for every g.

5.3. **Vanishing of the determinant.** To a point p of a cell attach the row

$$(p_0^a p_1^{3q-a} p_2^b p_3^{2q-b})_{0 \le a \le 3q,\ 0 \le b \le 2q} \in \mathbb{Z}^{n_q}.$$

Lemma 5.8 (Row cap)**.** *For a point of a cell, every entry of its row has absolute value at most* $(2H^3X)^q$.

Proof. The square of an entry is at most $\mathrm{ht}(y)^{6q}\,\mathrm{ht}(x)^{4q} \le (HD_y)^{3q}(HD_x)^{2q} = (H^5 D_y^3 D_x^2)^q \le (H^6 n^2)^q \le (2H^3X)^{2q}$, by Definition 3.6. □

Lemma 5.9 (Uniform Taylor estimate)**.** *Let* $\varkappa, q \ge 0$ *be integers and* $r_0, M_\varkappa > 0$. *There is* K_{T} *such that for every function* g *of class* $C^{\varkappa+1}$ *on* $(-r_0, r_0)$ *whose derivatives of order at most* $\varkappa + 1$ *are bounded by* $M_\varkappa$, *every real* c_0 *with* $|c_0| \le 2$ *and all* $a \le 3q$, $b \le 2q$, *the polynomial* $P_{ab}(u, v) = (c_0 + u)^a(T_\varkappa g(u) + v)^b$, *where* $T_\varkappa g$ *is the Taylor polynomial of* g *at* 0 *of degree* $\varkappa$, *has all coefficients at most* K_{T} *in absolute value and satisfies* $|(c_0 + u)^a(g(u) + v)^b - P_{ab}(u, v)| \le K_{\mathrm{T}}|u|^{\varkappa+1}$ *for* $|u| \le \min(r_0/2, 1)$ *and* $|v| \le 1$.

Proof. Put $M = M_\varkappa$, $T = T_\varkappa g$ and $S = 1 + (\varkappa + 1)M$. Each coefficient of T is bounded by M; submultiplicativity of the coefficient ℓ^1 norm therefore gives $\|P_{ab}\|_1 \le 3^{3q}S^{2q}$. Taylor's theorem on the segment from 0 to u gives $|g(u) - T(u)| \le M|u|^{\varkappa+1}/(\varkappa+1)!$. Since $|g(u) + v|, |T(u) + v| \le S$ and $|c_0 + u| \le 3$, the identity $s^b - t^b = (s - t)\sum_{j=0}^{b-1} s^{b-1-j}t^j$ bounds the difference by $3^{3q}(2q)S^{2q}M|u|^{\varkappa+1}$ for $b \ge 1$; for $b = 0$ it is zero. Thus $K_{\mathrm{T}} = 1 + 3^{3q}S^{2q}(1 + 2qM)$ works uniformly. □

Proposition 5.10 (Vanishing)**.** *Let $57/305 < \mu < 1/5$ and let $q \geq 1$ be an integer satisfying (5.2), chosen in terms of μ only. There is $X_1 \geq 2$ such that for all $X \geq X_1$, every cell $\mathcal{C}$ and all points $p^{(1)}, \dots, p^{(n_q)}$ of $\mathcal{S}_{\mathcal{C}}(X)$, the $n_q \times n_q$ integer matrix formed by their rows has determinant zero.*

Proof. Put $c_0 = \xi_{\mathcal{C}}$, $g(u) = g_j(c_0 + u)$, $u_i = \xi_i - c_0$ and $v_i = \zeta_i - g(u_i)$. Then $0 \leq u_i \leq X^{-\mu}$ and $|v_i| \leq C_{\mathrm{g}} H^3/X$. Set $\delta_1 = \rho(\mu) q^3/4$ and choose $\varkappa$ so that $\mu(\varkappa+1) > n_q q + \delta_1 + 1$. Lemma 5.9, with the uniform branch bounds, approximates the normalized matrix $\mathcal{E}_{i,(a,b)} = \xi_i^a \zeta_i^b$ by

$$\mathcal{E} = \mathcal{V}\mathcal{K} + \mathcal{E}', \qquad \mathcal{V}_{i,\epsilon} = u_i^{\epsilon_1} v_i^{\epsilon_2}, \quad \epsilon_1 + \epsilon_2 \leq \varkappa, \qquad |\mathcal{E}'_{i,(a,b)}| \ll X^{-\mu(\varkappa+1)}.$$

Here $\mathcal{K}$ is the bounded coefficient matrix of $P_{ab} = (c_0 + u)^a (T_\varkappa g(u) + v)^b$ restricted to those monomials. The omitted monomials have weight at least $\mu(\varkappa+1)$, which justifies their inclusion in the error. Expand $\det(\mathcal{V}\mathcal{K})$ by multilinearity in its rows. Each term is a coefficient determinant times $\prod_i u_i^{\epsilon_{i1}} v_i^{\epsilon_{i2}}$. If the coefficient determinant is nonzero, Proposition 5.7, applied to $T_\varkappa g$ and the basis $(c_0 + u)^a v^b$, makes the total weight greater than $n_q q + 2\delta_1$. The finitely many terms thus give $|\det(\mathcal{V}\mathcal{K})| \ll X^{-n_q q - 2\delta_1}$. All matrix entries are bounded; terms containing an error row give $O(X^{-\mu(\varkappa+1)})$. Consequently $|\det \mathcal{E}| \ll X^{-n_q q - \delta_1}$. Clearing row i by $p_1^{3q} p_3^{2q}$ multiplies the determinant by at most $(2H^3 X)^{q n_q}$ (Lemma 5.8). The integer determinant is therefore $O(X^{-\delta_1})$, uniformly over the finite branch list, and is zero for sufficiently large X. □

Proposition 5.11 (Auxiliary form)**.** *Let $57/305 < \mu < 1/5$ and q be as in Proposition 5.10. There is $X_{\mathrm{a}} \geq 2$ such that for every $X \geq X_{\mathrm{a}}$ and every cell $\mathcal{C}$ there is a nonzero form $G_{\mathcal{C}} \in \mathbb{Z}[p_0, p_1; p_2, p_3]$, bihomogeneous of bidegree $(3q, 2q)$, with $\|G_{\mathcal{C}}\| \leq X^{q n_q + 1}$ and $G_{\mathcal{C}}(p) = 0$ for every point p of $\mathcal{S}_{\mathcal{C}}(X)$.*

Proof. Take $X_{\mathrm{a}} = \max(X_1, n_q! (2H^3)^{q n_q})$. The distinct integer rows form a finite matrix $\mathcal{M}$ by Lemma 5.8; its rank $r < n_q$ by Proposition 5.10. If $r = 0$, take any coordinate vector c; otherwise choose a nonzero r-minor and one further column. The signed r-minors of this $r \times (r+1)$ block, extended by zero, give $0 \neq c \in \mathbb{Z}^{n_q}$ with $\mathcal{M}c = 0$: every row pairing is a vanishing $(r+1)$-minor. Moreover $|c_{(a,b)}| \leq r! (2H^3 X)^{qr} \leq X^{q n_q + 1}$. The form $\sum_{a,b} c_{(a,b)} p_0^a p_1^{3q-a} p_2^b p_3^{2q-b}$ has the required properties. □

After the permutation of coordinates given by the frame it is a form in $(y; x)$ vanishing at the triples of the cell.

6. Points on the auxiliary curve

We count $F_\chi = e_\chi$ on an auxiliary curve using the fibre bounds, the three literature hypotheses, and the lifted-curve count of [8]. For a chart χ and $U_y, U_x \in \mathrm{GL}_2(\mathbb{Z})$ put

$$F(\lambda; \tau) = F_\chi(U_y \lambda; U_x \tau), \qquad \lambda, \tau \in \mathbb{Z}^2.$$

This is again an integral form of bidegree $(3, 2)$, and by (4.1)

$$\frac{2\sqrt{\Delta}}{e_\chi} F(\lambda; \tau) = \gamma_+ \mathcal{L}_+(\lambda)^3 \mathcal{M}_+(\tau)^2 - \gamma_- \mathcal{L}_-(\lambda)^3 \mathcal{M}_-(\tau)^2 \tag{6.1}$$

with $\gamma_\pm \neq 0$ and linear forms $\mathcal{L}_\pm = \mathsf{Y}_\pm \circ U_y$, $\mathcal{M}_\pm = \mathsf{X}_\pm \circ U_x$ over $\mathbb{C}$; the forms $\mathcal{L}_+, \mathcal{L}_-$ are linearly independent, and so are $\mathcal{M}_+, \mathcal{M}_-$. The maps $\lambda \mapsto U_y \lambda$ and $\tau \mapsto U_x \tau$ are injective and preserve primitivity.

Proposition 6.1 (Component count)**.** *Let $D_0, \varpi \geq 1$ be integers and $\delta > 0$. There is K such that the following holds for every real $X \geq 2$, every chart $\chi \in \mathcal{X}$, all $U_y, U_x \in \mathrm{GL}_2(\mathbb{Z})$ with entries of absolute value at most X^{ϖ}, every nonzero $Q \in \mathbb{Z}[\lambda_1, \lambda_2; \tau_1, \tau_2]$, bihomogeneous of a bidegree (a_0, b_0) with $a_0 + b_0 \leq D_0$ and with $\|Q\| \leq X^{\varpi}$, and all real $\Lambda_1 \geq \Lambda_2 > 0$, $\Upsilon_1 \geq \Upsilon_2 > 0$ with $\Lambda_1, \Upsilon_1 \leq X^{\varpi}$: the set of pairs of primitive vectors (λ, τ) with*

$$|\lambda_i| \leq \Lambda_i, \quad |\tau_i| \leq \Upsilon_i \quad (i = 1, 2), \qquad F(\lambda; \tau) = e_\chi, \qquad Q(\lambda; \tau) = 0$$

is finite and has at most $KX^{\delta}(1+(\Lambda_1^3\Upsilon_1^2)^{1/10})$ *elements.*

Call the pairs in the displayed set *solutions*; K may depend on $(A,B,C,D_0,\varpi,\delta)$.

6.1. **Components.**

Lemma 6.2 (Classification)**.** *Let c_4 be the constant in* GE *for four variables. Let Q be as in Proposition 6.1. There are at most a_0+b_0 polynomials $G\in\mathbb{Z}[\lambda;\tau]$, each irreducible, dividing Q and bihomogeneous of some bidegree (α,β) with $\alpha\le a_0$, $\beta\le b_0$, such that every integral zero of Q is a zero of one of them and $\|G\|\le\exp(c_4(a_0+b_0))\,\|Q\|$. Each G is irreducible over $\mathbb{Q}$ and is of exactly one of the following types.*

(a) Pure*: $\beta=0$ and $\alpha\ge 2$, or $\alpha=0$ and $\beta\ge 2$. Then G has no integral zero with $\lambda\ne 0$, respectively $\tau\ne 0$.*
(b) Lines*: $G=p\lambda_1+p'\lambda_2$, or $G=p\tau_1+p'\tau_2$.*
(c) Low mixed*: $\alpha,\beta\ge 1$ and $2\alpha+3\beta<10$, that is, $(\alpha,\beta)\in\{(1,1),(1,2),(2,1),(3,1)\}$. Then G is irreducible over $\mathbb{C}$.*
(d) High mixed*: $\alpha,\beta\ge 1$ and $2\alpha+3\beta\ge 10$.*

Proof. Take the nonconstant primitive irreducible factors of Q. Their total degrees sum to at most a_0+b_0; they are bihomogeneous, and GE bounds their heights. Gauss's lemma gives irreducibility over $\mathbb{Q}$. A binary form irreducible over $\mathbb{Q}$ of degree at least two has no nonzero rational zero, giving (a). If $\min(\alpha,\beta)=1$, the form is linear in one pair with coprime binary coefficients; these remain coprime over $\mathbb{C}$, so the form is absolutely irreducible. The positive bidegrees with $2\alpha+3\beta<10$ are exactly the four in (c); all remaining mixed factors belong to (d). □

Hypothesis GE enters the proof of Theorem 1.1 only here, and its height bound is used only for the components of type (c). Solutions are primitive, so a component of type (a) carries none.

Lemma 6.3 (Lines)**.** *In the setting of Proposition 6.1, a component of type* (b) *carries at most K_1X^{δ} solutions, with K_1 depending on the chart list, ϖ and δ.*

Proof. A nonzero linear form has at most two primitive zeros, differing by sign. If it fixes λ, then $y=U_y\lambda$ is primitive with $\mathrm{ht}(y)\le 2X^{2\varpi}$, and $x=U_x\tau$ is primitive with the same height bound. The quadratic fibre has nonzero discriminant and coefficients at most $32\|F_\chi\|X^{6\varpi}$; Lemma 4.3 with $\varepsilon=\delta/(8\varpi+1)$ gives $\ll X^{\delta}$ choices. If the line fixes τ instead, Lemma 4.2 gives at most $K_{\mathrm{c}}(\chi)$ choices for each sign. The unimodular maps are injective. □

6.2. **Low mixed components.** The next proposition is the analogue of [8, Proposition 5.3] for forms of the shape (6.1).

Proposition 6.4 (Graph components)**.** *Let $h\ne 0$ be an integer, $\alpha,\beta\ge 1$ integers with $\min(\alpha,\beta)=1$, and $\varepsilon>0$. There is $K_1=K_1(h,\alpha,\beta,\varepsilon)$ with the following property. Let F be an integral form of bidegree $(3,2)$ such that $\kappa_0F=\gamma_1\mathcal{L}_1(\lambda)^3\mathcal{M}_1(\tau)^2-\gamma_2\mathcal{L}_2(\lambda)^3\mathcal{M}_2(\tau)^2$ with nonzero $\kappa_0,\gamma_1,\gamma_2\in\mathbb{C}$, independent linear forms $\mathcal{L}_1,\mathcal{L}_2$ and independent linear forms $\mathcal{M}_1,\mathcal{M}_2$ over $\mathbb{C}$. Let $H_G\ge 1$ and let $G\in\mathbb{Z}[\lambda;\tau]$ be bihomogeneous of bidegree (α,β), irreducible over $\mathbb{C}$, with $\|G\|\le H_G$. Then the number of pairs of primitive vectors (λ,τ) with $G(\lambda;\tau)=0$ and $F(\lambda;\tau)=h$ is at most $K_1H_G^{\varepsilon}$.*

Proof. If $\beta=1$, write $G=\tau_1P(\lambda)+\tau_2Q(\lambda)$, put $r=\lambda$ and set $(a_G,b_G)=(r_1,r_2)$, $(c_G,e_G)=(-Q,P)$. If $\alpha=1$, use the analogous expression linear in λ, with $r=\tau$, $(a_G,b_G)=(-Q,P)$ and $(c_G,e_G)=(r_1,r_2)$. In either case these are coprime integral pairs of degrees β,α, with coefficients bounded by H_G, and r is primitive. Coprimality follows from absolute irreducibility. Every counted pair satisfies

$$\lambda_1b_G(r)=\lambda_2a_G(r),\qquad \tau_1e_G(r)=\tau_2c_G(r). \tag{6.2}$$

For coprime integral binary forms of degree n and height at most H_G, the resultant argument of [8, proof of Proposition 5.3] gives a nonzero integer of size at most $((2n)!)^2H_G^{4n}$ divisible by the gcd

of their values at every primitive r. Apply this to the two pairs to obtain $\mathrm{Res}_1, \mathrm{Res}_2$. Primitivity and (6.2) give $(a_G, b_G)(r) = g\lambda$, $(c_G, e_G)(r) = k\tau$, where $0 \neq g \mid \mathrm{Res}_1$ and $0 \neq k \mid \mathrm{Res}_2$. Thus the binary form $\Psi = F(a_G, b_G; c_G, e_G)$, of degree $d_R = 3\beta + 2\alpha$, takes the nonzero value g^3k^2h at r, a divisor of $\Xi = h\mathrm{Res}_1^3\mathrm{Res}_2^2$, with

$$|\Xi| \leq |h|((2\beta)!)^6((2\alpha)!)^4 H_G^{12\beta+8\alpha}.$$

Put $U_i = \mathcal{L}_i(a_G, b_G)$ and $V_i = \mathcal{M}_i(c_G, e_G)$. Independence makes (U_1, U_2) and (V_1, V_2) coprime pairs of degrees β, α. After absorbing cube roots of γ_i, $\kappa_0\Psi$ has the form $U_1^3V_1^2 - U_2^3V_2^2$. If there is a counted pair then $\Psi \neq 0$; as $\min(\alpha, \beta) = 1$, [8, Lemma 5.2] gives at least three distinct projective roots.

Let K_Ψ be the product of the distinct primitive integral irreducible factors of Ψ. By Gauss's lemma it has content 1, is squarefree of degree $3 \leq d_K \leq d_R$, and divides Ψ in $\mathbb{Z}[r_1, r_2]$; hence $K_\Psi(r) \mid \Xi$ at every counted point. For each of the $2\tau(|\Xi|)$ signed divisors m, hypothesis ST gives at most $2800d_K^{1+\omega(m)} \leq 2800d_R\tau(|\Xi|)^{d_R}$ primitive solutions of $K_\Psi(r) = m$. Each r determines one vector of the counted pair and the other up to sign by (6.2). Thus there are at most $11200d_R\tau(|\Xi|)^{d_R+1}$ pairs; the divisor bound gives $K_1H_G^{\varepsilon}$. □

The proof also gives the uniform version

$$\#\{(\lambda, \tau) : G = 0,\ F = h,\ \lambda, \tau \text{ primitive}\} \ll_{\alpha,\beta,\eta} (|h|H_G)^\eta \qquad (\eta > 0). \tag{6.3}$$

Apply the divisor bound above with exponent $\eta/((d_R + 1)(12\beta + 8\alpha))$: the resulting factor $|h|^{\eta/(12\beta+8\alpha)}H_G^\eta$ is at most $(|h|H_G)^\eta$, with a constant independent of F, h.

Corollary 6.5. *In the setting of Proposition 6.1, a component G of type* (c) *carries at most K_2X^δ solutions.*

Proof. By (6.1), F has the shape required in Proposition 6.4, G is irreducible over $\mathbb{C}$, and $\|G\| \leq \exp(c_4D_0)X^\varpi$ by Lemma 6.2. Apply Proposition 6.4 with $h = e_\chi$, $H_G = \exp(c_4D_0)X^\varpi$ and $\varepsilon = \delta/\varpi$, for each of the four bidegrees and each chart. □

6.3. **High mixed components and the lifted curve.**

Lemma 6.6 (Lemma P, [8, Lemma 5.5])**.** *Let c_{12} be the constant in CC for twelve variables. Let $F \in \mathbb{Z}[\lambda_1, \lambda_2; \tau_1, \tau_2]$ be bihomogeneous of bidegree $(3, 2)$, let e be a nonzero integer, and let $G \in \mathbb{Q}[\lambda_1, \lambda_2; \tau_1, \tau_2]$ be bihomogeneous of bidegree $(\alpha, \beta) \neq (0, 0)$ and irreducible over $\mathbb{Q}$. Let $\Lambda_1 \geq \Lambda_2 > 0$ and $\Upsilon_1 \geq \Upsilon_2 > 0$. Then the number of pairs of primitive vectors $(\lambda, \tau) \in \mathbb{Z}^2 \times \mathbb{Z}^2$ with $|\lambda_i| \leq \Lambda_i$, $|\tau_i| \leq \Upsilon_i$, $F(\lambda; \tau) = e$ and $G(\lambda; \tau) = 0$ is at most*

$$2\,c_{12}\,d^3\Omega^{1/d}(\log\Omega + d), \qquad d = 2\alpha + 3\beta, \quad \Omega = \max(1, \Lambda_1^3\Upsilon_1^2).$$

This is proved in [8]: the twelve monomials of bidegree $(3, 2)$ map the counted points, at most two-to-one, to integral points of height at most Ω on an affine curve of degree $2\alpha + 3\beta$ in twelve variables, to which CC applies. The form F is arbitrary, and G need not be irreducible over $\mathbb{C}$. We may and do assume $c_{12} \geq 1$.

Corollary 6.7. *In the setting of Proposition 6.1, a component G of type* (d) *carries at most $K_3X^\delta(1 + (\Lambda_1^3\Upsilon_1^2)^{1/10})$ solutions.*

Proof. Apply Lemma 6.6 with $e = e_\chi$. Here $10 \leq d \leq 3D_0$, so $\Omega^{1/d} \leq 1 + (\Lambda_1^3\Upsilon_1^2)^{1/10}$, while $\Omega \leq X^{5\varpi}$ gives $\log\Omega \leq (5\varpi/\delta)X^\delta$. Thus $K_3 = 54c_{12}D_0^3(5\varpi/\delta + 3D_0)$ suffices. □

This is the sole use of the lifted-curve count in Theorem 1.1. The exponent $1/10$ comes from the least remaining lifted degree, attained by bidegree $(2, 2)$.

Proof of Proposition 6.1. Every solution is a zero of one of the at most D_0 components of Lemma 6.2. Components of type (a) carry no solution, and the other types are bounded by Lemma 6.3 and Corollaries 6.5 and 6.7. Finiteness holds because the solutions lie in a box. So the proposition holds with $K = D_0(K_1 + K_2 + K_3)$. □

7. Reduced coordinates, moments and the proof of Theorem 1.1

Throughout, $57/305 < \mu < 1/5$, $X \geq 4$ and $M = \lceil X^\mu \rceil$, so that $1 \leq M \leq X$.

7.1. **Strips and reduced bases.** For $K_0 \geq 1$, $D > 0$ and a real slope ς, the *strip* $\Sigma(K_0, D, \varsigma)$ is the set of $v = (v_0, v_1) \in \mathbb{Z}^2$ with

$$|v_0 - \varsigma v_1| \leq K_0\sqrt{D}/M, \qquad |v_1| \leq K_0\sqrt{D},$$

and the *strip norm* $|v|_\varsigma$ of $v \in \mathbb{Z}^2$ is the Euclidean norm of the vector

$$(M(v_0 - \varsigma v_1)/(K_0\sqrt{D}),\ v_1/(K_0\sqrt{D})) \in \mathbb{R}^2.$$

The strip is the set of v whose image lies in the square $[-1, 1]^2$.

Lemma 7.1 (Strips)**.** *There is $K_\mathrm{s} \geq 1$ such that for every cell $\mathcal{C} = (\phi, j, k)$ and every point p of $\mathcal{S}_\mathcal{C}(X)$ with $U \leq D$ and $V \leq E$,*

$$(p_0, p_1) \in \Sigma(K_\mathrm{s}, D, \xi_\mathcal{C}), \qquad (p_2, p_3) \in \Sigma(K_\mathrm{s}, E, g_j(\xi_\mathcal{C})).$$

Proof. We have $|p_1| \leq \sqrt{D}$, $|p_3| \leq \sqrt{E}$ and $|p_0/p_1 - \xi_\mathcal{C}| \leq 1/M$. If M_d bounds first branch derivatives, the other slope satisfies $|p_2/p_3 - g_j(\xi_\mathcal{C})| \leq C_\mathrm{g}H^3/X + M_\mathrm{d}/M \leq (C_\mathrm{g}H^3 + M_\mathrm{d})/M$. Multiply by the dominant coordinates and take $K_\mathrm{s} = 1 + C_\mathrm{g}H^3 + M_\mathrm{d}$, maximized over frames. □

Proposition 7.2 (Reduced bases)**.** *Let $K_0 \geq 1$, let $D > M$ with M as above, and let $|\varsigma| \leq K_0$. There are $\mathsf{B} \in \mathrm{GL}_2(\mathbb{Z})$ and real numbers $\Lambda_1 \geq \Lambda_2 > 0$ with $\Lambda_1 \leq 2K_0\sqrt{D}$ such that the entries of B and of B^{-1} are at most $4(K_0 + 1)D$ in absolute value, the first column b_1 of B is a nonzero vector of least strip norm with $\Lambda_1 = 2/|b_1|_\varsigma$, and every primitive $v \in \Sigma(K_0, D, \varsigma)$ equals $\mathsf{B}\lambda$ for a primitive $\lambda \in \mathbb{Z}^2$ with $|\lambda_1| \leq \Lambda_1$ and $|\lambda_2| \leq \Lambda_2$.*

Proof. Let b_1 be a nonzero integer vector of least strip norm; it exists because the vectors of bounded strip norm lie in a finite box, and it is primitive. Choose $w \in \mathbb{Z}^2$ with $\det(b_1, w) = 1$ and put $b_2 = w - kb_1$ with k the integer nearest to $\langle b_1, w\rangle_\varsigma/|b_1|_\varsigma^2$, where $\langle\,,\rangle_\varsigma$ is the inner product of the strip norm. With $N_i = |b_i|_\varsigma^2$ and $P_0 = \langle b_1, b_2\rangle_\varsigma$ we get $N_1 \leq N_2$ and $|P_0| \leq N_1/2$, and therefore, for all real l_1, l_2,

$$|l_1b_1 + l_2b_2|_\varsigma^2 \geq \tfrac{1}{2}(l_1^2N_1 + l_2^2N_2). \tag{7.1}$$

Let B have columns b_1, b_2, and $\Lambda_i = 2/\sqrt{N_i}$. If $v = \mathsf{B}\lambda$ lies in the strip, its image has squared length at most 2, so (7.1) gives $\lambda_i^2N_i \leq 4$; and λ is primitive when v is. Write $b_1 = (a_1, c_1)$, $b_2 = (a_2, c_2)$ and $R_D = K_0\sqrt{D}$. Then $R_D^2N_1 = M^2(a_1 - \varsigma c_1)^2 + c_1^2 \geq 1$, which gives $\Lambda_1 \leq 2R_D$. The images of b_1, b_2 span a parallelogram of area M/R_D^2, so $N_1N_2 - P_0^2 = M^2/R_D^4$ and $3N_1N_2 \leq 4M^2/R_D^4$; hence $R_D^2N_2 \leq 2M^2$. So $|c_i| \leq 2M$ and $|a_i - \varsigma c_i| \leq 2$, which gives $|a_i| \leq 2 + 2K_0M \leq 4(K_0 + 1)D$. The entries of B^{-1} are those of B up to sign and order. □

The hypothesis $D > M$ is used for the bound on the entries. For each cell $\mathcal{C} = (\phi, j, k)$ we fix $\mathsf{B}_y(\mathcal{C})$ with $\Lambda_i(\mathcal{C})$ for the strip $\Sigma(K_0, D, \xi_\mathcal{C})$, and $\mathsf{B}_x(\mathcal{C})$ with $\Upsilon_i(\mathcal{C})$ for $\Sigma(K_0, E, g_j(\xi_\mathcal{C}))$, where $K_0 \geq K_\mathrm{s}$ also bounds 5, the slopes $\xi_\mathcal{C}$ and the values $g_j(\xi_\mathcal{C})$.

Lemma 7.3 (Per-cell count)**.** *Let q be as in Proposition 5.11 and $\delta > 0$. There are P_c and X_c such that for all $X \geq X_\mathrm{c}$, all $D, E > M$ with $D^3E^2 \leq 128H^6X^2$ and every cell $\mathcal{C}$, the number of points of $\mathcal{S}_\mathcal{C}(X)$ with $U \leq D$ and $V \leq E$ is at most $P_\mathrm{c}X^\delta(1 + (\Lambda_1(\mathcal{C})^3\Upsilon_1(\mathcal{C})^2)^{1/10})$.*

Proof. Let S_y, S_x be the coordinate permutations of the frame. Lemmas 7.1 and Proposition 7.2 give an injective parametrization $S_y y = \mathsf{B}_y\lambda$, $S_x x = \mathsf{B}_x\tau$ by primitive vectors in the reduced box. Set $U_y = S_y\mathsf{B}_y$, $U_x = S_x\mathsf{B}_x$ and $Q(\lambda;\tau) = G_{\mathcal{C}}(\mathsf{B}_y\lambda;\mathsf{B}_x\tau)$. Then $F_\chi(U_y\lambda;U_x\tau) = e_\chi$ and $Q = 0$; the latter form is nonzero of bidegree $(3q, 2q)$. As $D, E \le 128H^6X^2$, the matrix entries are at most X^3 for large X. Expanding the substitution gives

$$\|Q\| \le (5q+1)^4 2^{5q} X^{qn_q+1+15q} \le X^{\varpi}, \qquad \varpi = qn_q + 15q + 2.$$

Also $\Lambda_1, \Upsilon_1 \le X^2$. Proposition 6.1 with $D_0 = 5q$ and this ϖ now applies. □

7.2. **Moments.**

Lemma 7.4 (Tail count)**.** *Let $K_0 \ge 1$ and $c > 0$. There is C_{t} such that the following holds for all $D > 0$. Let $(\varsigma_i)_{i\in I}$ be real numbers with $|\varsigma_i| \le K_0$ and $|\varsigma_i - \varsigma_j| \ge c/M$ for $i \ne j$, with $\#I \le K_0M$; for each i let $b^{(i)}$ be a nonzero integer vector and $\ell_i = 2/|b^{(i)}|_{\varsigma_i}$. Then for every $\Lambda > 0$,*

$$\#\{i \in I : \ell_i \ge \Lambda\} \le C_{\mathrm{t}} D/\Lambda^2.$$

Proof. For $\Lambda < 8K_0\sqrt{D}/M$, the trivial bound K_0M suffices, since $M \ge 1$ and $M^2 < 64K_0^2D/\Lambda^2$. Otherwise put $Z = 2K_0\sqrt{D}/\Lambda \le M/4$. If $\ell_i \ge \Lambda$, its nonzero vector (p, p') satisfies $|p - \varsigma_i p'| \le Z/M$ and $|p'| \le Z$. Here $p' \ne 0$, since otherwise $|p| \le 1/4$; changing signs gives $1 \le p' \le Z$. Thus there are no such indices if $Z < 1$. For each integer $q = p' \le Z$, the bound $|p| \le K_0q + 1/4$ allows at most $(2K_0+2)q$ numerators. For each (p, q), admissible slopes lie in an interval of length $2Z/(Mq)$, which contains at most $1 + 2Z/(cq)$ of the separated ς_i. Therefore the number of indices is at most

$$(2K_0+2)\sum_{1\le q\le Z} q\left(1 + \frac{2Z}{cq}\right) \ll_{K_0,c} Z^2 = 4K_0^2D/\Lambda^2.$$

□

This is a double count of pairs consisting of an integer vector in a sector and a slope close to its direction; it uses neither primitivity nor reducedness of the $b^{(i)}$.

Lemma 7.5 (Moments)**.** *In the setting of Lemma 7.4, for $0 < \vartheta < 2$ there is K_{m} with $\sum_{i\in I}\ell_i^{\vartheta} \le K_{\mathrm{m}}M(D/M)^{\vartheta/2}$. If moreover $(\varsigma_i')_{i\in I}$, $E > 0$ and vectors with lengths ℓ_i' satisfy the same hypotheses with E in place of D, then*

$$\sum_{i\in I}\ell_i^{3/10}\ell_i'^{\,1/5} \le K_{\mathrm{m}}'M\Big(\frac{D}{M}\Big)^{3/20}\Big(\frac{E}{M}\Big)^{1/10} = K_{\mathrm{m}}'(D^3E^2)^{1/20}M^{3/4}.$$

Proof. Put $\Lambda_* = 4K_0\sqrt{D/M}$. Lengths at most Λ_* contribute at most $K_0M\Lambda_*^{\vartheta}$. Lemma 7.4 bounds the contribution from $[2^k\Lambda_*, 2^{k+1}\Lambda_*)$ by $C_{\mathrm{t}}M\Lambda_*^{\vartheta}2^{\vartheta}(2^{\vartheta}/4)^k$. Summing this convergent series proves the first assertion. Cauchy–Schwarz with moments $3/5$ and $2/5$ proves the second. □

Proposition 7.6 (Hard bound)**.** *Let $57/305 < \mu < 1/5$ and $\delta > 0$. There are K_{h} and X_{h} such that for all $X \ge X_{\mathrm{h}}$ and all $D, E > M$ with $D^3E^2 \le 128H^6X^2$, the number of triples $(\chi, y, x) \in \mathcal{S}(X)$ with $U \le D$ and $V \le E$ is at most $K_{\mathrm{h}}X^{\delta}(M + X^{1/10}M^{3/4})$.*

Proof. Take $X_{\mathrm{h}} \ge \max(4, n_0, X_{\mathrm{a}}, X_{\mathrm{c}}, H^3/\delta_0)$ for every frame, so all triples are covered by cells. On each fixed branch there are at most $5M$ cells; their two slope families are separated by $1/M$ and c_{g}/M. Apply Lemma 7.5 to the first basis columns, whose lengths are Λ_1, Υ_1, and sum Lemma 7.3: the branch contributes at most

$$P_{\mathrm{c}}X^{\delta}(5M + K_{\mathrm{m}}'(D^3E^2)^{1/20}M^{3/4}) \ll X^{\delta}(M + X^{1/10}M^{3/4}).$$

There are finitely many frames and branches. □

7.3. **The schedule.**

Lemma 7.7 (Schedule)**.** *Let $\varepsilon > 0$ and put $\eta = \min(1/1000, \varepsilon/4)$, $\mu = 57/305 + \eta$ and $\delta = \varepsilon/4$. For all $A_1, P > 0$ there is K_{sch} such that for every $X \geq 2$ and every family of at most $A_1(1+\log X)^2$ boxes (D_i, E_i) with counts $z_i \geq 0$ satisfying*

(i) *$z_i \leq PX^{\delta} \min(D_i, E_i)$ for all i, and*
(ii) *$z_i \leq PX^{\delta}(M + X^{1/10}M^{3/4})$ whenever $M < D_i$ and $M < E_i$,*

we have $\sum_i z_i \leq K_{\mathrm{sch}} X^{293/1220+\varepsilon}$.

Proof. Put $\vartheta = 1/10 + 3\mu/4 = 293/1220 + 3\eta/4 \geq \mu$. Since $M \leq 2X^{\mu}$, both cases (i)–(ii) give $z_i \leq 4PX^{\delta+\vartheta}$. The number of boxes is $\ll_{\varepsilon} A_1 X^{\varepsilon/4}$, and $3\eta/4 + \delta + \varepsilon/4 \leq \varepsilon$. □

The boxes with $\min(D, E) \leq M$ are those in which the reduced bases of Proposition 7.2 are not available; there the easy bound suffices.

7.4. **Proof of Theorem 1.1.** Choose μ, δ as in Lemma 7.7, and take the chart constants and the easy and hard bounds above. For $X \geq X_{\mathrm{h}}$, Lemma 3.8 gives $\#\mathcal{W}(X) \leq \#\mathcal{S}(X)$. Lemmas 3.10 and 3.11 split $\mathcal{S}(X)$ into $O((1+\log X)^2)$ boxes with $D^3E^2 \leq 128H^6X^2$. Propositions 4.5 and 7.6 supply the two bounds of Lemma 7.7, with $P = K_{\mathrm{e}} + K_{\mathrm{h}} + 1$. Hence

$$\#\mathcal{W}(X) \leq K_{\mathrm{w}} X^{293/1220+\varepsilon} \qquad (X \geq X_{\mathrm{h}}).$$

By Lemma 3.4, $f(n) > 0$ for $n \geq n_0$; thus these are exactly the large n in the window counted by $S(A, B, C; \cdot)$. Lemma 3.12 with $\vartheta = 293/1220 + \varepsilon$ now yields $S(A, B, C; N) \leq KN^{293/1220+\varepsilon}$ for every $N \geq 1$. □

The constant depends on (A, B, C, ε) and is not effective: class representatives, generators, branch neighborhoods and the thresholds $X_{\mathrm{a}}, X_{\mathrm{c}}$ were chosen existentially. The assumption $A > 0$ enters growth, and $A \neq 0$ also enters Lemmas 3.1 and 3.8. Nonsquareness gives the quadratic field, finite class group, nonvanishing norms, and the Pell unit when $\Delta > 0$.

8. The uniform bound: proof of Theorem 1.2

Let $f(n) = an^2 + bn + c$, where $a, b, c \in \mathbb{Z}$, $a \geq 1$, $\Delta = b^2 - 4ac$ is nonsquare, and $|a|, |b|, |c| \leq L$, $L \geq 2$. Constants are absolute unless a dependence is stated. The fixed-polynomial proof leaves its ideal representatives, unit and compactness bounds uncontrolled in L, and its determinant order grows as $\mu \to 57/305$. Here we give arithmetic and graph constructions with polynomial losses and fix $\mu_0 = 19/100$, $q_0 = 4096$. We reuse Proposition 5.7 and Lemmas 6.6, 3.11, 7.5 and 3.12.

8.1. **The trace equation.** Write $\Delta = \Delta_0 r_0^2$, with $r_0 \geq 1$ and Δ_0 squarefree, nonsquare and $|\Delta_0| \leq 5L^2$. Put $\omega_0 = (1 + \sqrt{\Delta_0})/2$ if $\Delta_0 \equiv 1 \pmod 4$ and $\omega_0 = \sqrt{\Delta_0}$ otherwise. Then $\mathcal{O} = \mathbb{Z} + \mathbb{Z}\omega_0$. For $y \in \mathbb{Z}^2$ set $\iota_0(y) = y_0 + y_1\omega_0$, and let $\sigma_{\pm}$ send $\sqrt{\Delta_0}$ to $\pm\sqrt{\Delta_0} \in \mathbb{C}$. The integral trace form

$$T_0(y; x) = 16 \operatorname{Tr}(\iota_0(y)^3 \iota_0(x)^2)$$

has bidegree $(3, 2)$. Also put $z_n = (2an + b) + r_0\sqrt{\Delta_0}$, so $z_n \bar{z}_n = 4af(n)$.

Lemma 8.1 (Balancing by a bounded-norm integer)**.** *Let $d \geq 2$ be a squarefree integer. For every $R > 0$ there is $\gamma = p + q\sqrt{d} \in \mathbb{Z}[\sqrt{d}]$ such that*

$$1 \leq |\operatorname{N}\gamma| \leq 3\sqrt{d}, \qquad \frac{R}{6d} \leq \left|\frac{p + q\sqrt{d}}{p - q\sqrt{d}}\right| \leq 6dR.$$

The integer γ is not required to be a unit.

Proof. Let p_j/q_j be the simple continued-fraction convergents of $u=\sqrt{d}$, starting with $p_0=\lfloor u\rfloor$, $q_0=1$ and $p_{-1}=1$, $q_{-1}=0$. The recurrences give $1\le q_j\le q_{j+1}$ and $q_{j+2}\ge q_{j+1}+q_j$, hence $q_j\to\infty$. If $u_{j+1}>a_{j+1}$ is the next complete quotient, then

$$u=\frac{p_ju_{j+1}+p_{j-1}}{q_ju_{j+1}+q_{j-1}},\qquad |p_jq_{j-1}-p_{j-1}q_j|=1,$$

so $q_{j+1}=a_{j+1}q_j+q_{j-1}$ implies

$$t_j:=|p_j-uq_j|\le q_{j+1}^{-1}. \tag{8.1}$$

Put $A_j=p_j+uq_j$, $B_j=|p_j^2-dq_j^2|$ and $\Lambda_j=A_j^2/B_j$. Since $p_j\ge uq_j-1>0$ and u is irrational,

$$1\le B_j=A_jt_j\le(2uq_j+t_j)t_j\le 2u+1\le 3u. \tag{8.2}$$

Thus $q_{j+1}\le A_j$ and $A_{j+1}\le 2uq_{j+1}+1\le 3uA_j$, giving

$$\Lambda_{j+1}\le 9u^2A_j^2=9u^2B_j\Lambda_j\le 27u^3\Lambda_j\le(6d)^2\Lambda_j. \tag{8.3}$$

Moreover $\Lambda_j\to\infty$ because $A_j\ge uq_j$, and $\Lambda_0\le 4d$. For $R\ge 1$ choose the first j with $\Lambda_j\ge R/(6d)$. If $j=0$, then $\Lambda_j\le 4d\le 6dR$; otherwise (8.3) gives $\Lambda_j<6dR$. Now $\gamma=p_j+q_j\sqrt{d}$ has embedding ratio Λ_j and absolute norm B_j. For $R<1$, apply this construction to R^{-1} and conjugate γ. □

Proposition 8.2 (Arithmetic reduction)**.** *There is an absolute constant $C_0\ge 1$ such that for every $n\ge 4L$ with $f(n)$ powerful there are an integer e and primitive vectors $y,x\in\mathbb{Z}^2$ with*

(i) $1\le e\le C_0L^{28}$ *and* $T_0(y;x)=e$;
(ii) $\iota_0(y)^3\iota_0(x)^2=\dfrac{e}{32\,\Delta_0^2r_0}\,(\sqrt{\Delta_0})^3z_n$;
(iii) $|\sigma_+(\iota_0y)|\le C_0L^4|\sigma_-(\iota_0y)|$ *and* $|\sigma_+(\iota_0x)|\le C_0L^6|\sigma_-(\iota_0x)|$, *and the same with σ_+ and σ_- exchanged;*
(iv) $n/(C_0L^9)\le \mathrm{ht}(y)^3\,\mathrm{ht}(x)^2\le C_0L^{40}n$.

The triple (e,y,x) determines n.

Proof. Write $d=\Delta_0$, $s=r_0$, $\theta=\sqrt{d}$ and $K=\mathbb{Q}(\theta)$. Parentheses denote principal fractional ideals; N denotes the field norm or absolute ideal norm. Set

$$z=(2an+b)+s\theta,\qquad w=\theta^3z=d^2s+d(2an+b)\theta,\qquad h_0=d^2s.$$

Since $\Delta=ds^2$ and $s\ge 1$,

$$1\le|d|\le|\Delta|\le 5L^2,\qquad 0<h_0\le\Delta^2\le 25L^4. \tag{8.4}$$

For $n\ge 4L$, the inequalities $L(n+1)\le n^2/2$ and $n^2+n+1\le 2n^2$ give

$$n^2/4\le f(n)\le 2Ln^2, \tag{8.5}$$

whence $z\neq 0$ and

$$\mathrm{N}\,w=-4ad^3f(n),\qquad n^2\le|\,\mathrm{N}\,w|\le 1000L^8n^2. \tag{8.6}$$

Write $f(n)=u^3v^2$ with $u,v\ge 1$, by expressing every prime exponent at least two as $3i+2j$. A prime ideal $\mathfrak{p}$ occurring exactly once in (z) divides $(4a\Delta)$: if $\bar z\in\mathfrak{p}$, then $2s\theta=z-\bar z\in\mathfrak{p}$ and $4\Delta\in\mathfrak{p}$; if neither $\bar z$ nor $4a$ belongs to $\mathfrak{p}$, valuations in $(z)(\bar z)=(4a)(u)^3(v)^2$ give the impossible equality $1=3i+2j$. Let $\mathfrak{R}$ be the product of these prime ideals. Splitting all remaining exponents into $3i+2j$ yields integral ideals

$$(z)=\mathfrak{R}\mathfrak{I}^3\mathfrak{J}^2,\qquad \mathfrak{R}\text{ squarefree.}$$

For $B_0=2a|\Delta|$, the identity $B_0^2=(4a\Delta)(a\Delta)$ and squarefreeness imply $\mathfrak{R}\mid(B_0)$, hence

$$r:=\mathrm{N}\,\mathfrak{R}\le B_0^2\le 100L^6. \tag{8.7}$$

The order $\mathbb{Z}[\theta]$ has discriminant $4d$, so $|D_K| \le 20L^2$. Minkowski supplies an integral ideal $\mathfrak{j}$ in the class of $\mathfrak{I}\mathfrak{J}$ with

$$N := \mathrm{N}\,\mathfrak{j} \le (4/\pi)^{r_2}\frac{2!}{2^2}\sqrt{|D_K|} \le 2\sqrt{|D_K|} \le 10L. \tag{8.8}$$

Choose $(\xi)\mathfrak{j} = \mathfrak{I}\mathfrak{J}$ and set $\alpha = z/\xi^2$, $\beta = N^3 r\xi^3/z$. These are integral:

$$(\alpha) = \mathfrak{R}\mathfrak{I}\mathfrak{j}^2, \qquad (\beta) = \frac{(N)^3(r)\mathfrak{J}}{\mathfrak{R}\mathfrak{j}^3} \subseteq \mathfrak{J}.$$

Here $(N) \subseteq \mathfrak{j}$, $(r) \subseteq \mathfrak{R}$, because an integral ideal contains its norm (the order of its additive quotient). The choices give the exact identity

$$\alpha^3\beta^2 = N^6 r^2 z. \tag{8.9}$$

For $d < 0$, set $m = 1$, $\alpha_1 = \alpha$, $\beta_1 = \beta$; conjugate embeddings have equal moduli. For $d > 0$ we have $d \ge 2$ and $2an + b \ge 7L \ge 3s\sqrt{d}$, since $s\sqrt{d} \le \sqrt{5}L$. Thus $Z := |\sigma_+(z)/\sigma_-(z)| \in [1/2, 2]$. Apply Lemma 8.1 to $R = A^{-1/2}$, $A = |\sigma_+(\alpha)/\sigma_-(\alpha)|$, obtaining

$$1 \le m := |\,\mathrm{N}\,\gamma| \le 3\sqrt{d} \le 7L,$$

$$F^{-1}A^{-1/2} \le G := |\sigma_+(\gamma)/\sigma_-(\gamma)| \le FA^{-1/2}, \quad F = 6d \le 30L^2.$$

Put $\alpha_1 = \gamma^2\alpha$, $\beta_1 = \bar\gamma^3\beta$. Then $\alpha_1^3\beta_1^2 = N^6r^2m^6z$ and its embedding ratios satisfy $F^{-2} \le A_1 = G^2A \le F^2$, $A_1^3B_1^2 = Z$. Since $\sqrt{2}F^3 \le 54000L^6$, in both signs of d we obtain

$$(900L^4)^{-1} \le A_1 \le 900L^4, \qquad (54000L^6)^{-1} \le B_1 \le 54000L^6. \tag{8.10}$$

Represent $\theta\alpha_1 = \iota_0(Y)$ and $\beta_1 = \iota_0(X)$ in the integral basis. Multiplication by θ preserves embedding ratios in absolute value. Thus, with $\kappa = N^6r^2m^6$,

$$1 \le \kappa \le ML^{24}, \qquad \iota_0(Y)^3\iota_0(X)^2 = \kappa w, \tag{8.11}$$

where an absolute $M \ge 54000$ bounds the ratios by ML^4, ML^6; for example $M = 20^6 \cdot 100^2 \cdot 7^6 + 100000$ suffices.

Divide Y, X by their positive coordinate contents g_Y, g_X. The resulting y, x are primitive, preserve the embedding ratios and satisfy

$$\iota_0(y)^3\iota_0(x)^2 = \lambda w, \qquad \lambda = \kappa/q, \quad q = g_Y^3 g_X^2 \ge 1. \tag{8.12}$$

The trace form is integral: in the half-integral basis the five factors have denominator dividing 2^5, and $16\,\mathrm{Tr}$ multiplies the rational coordinate by 32. Taking traces gives the positive integer $e = T_0(y; x)$ and $qe = 32\kappa h_0$, hence

$$1 \le e \le 800ML^{28}, \qquad \lambda = \frac{e}{32h_0}, \qquad \frac{1}{800L^4} \le \lambda \le ML^{24}. \tag{8.13}$$

This proves (i)–(iii) after enlarging C_0.

For $H(v) = \mathrm{ht}(v)$ and $A(v) = |\,\mathrm{N}\,\iota_0(v)|$, solving the two embedding equations and using $|\theta| \ge 1$, $|\omega_0|, |\bar\omega_0| \le 3L$ gives

$$H(v) \le 2\max_{\pm}|\sigma_\pm(\iota_0 v)|, \qquad A(v) \le 16L^2H(v)^2. \tag{8.14}$$

If the embedding ratio is bounded in both directions by Q, then also

$$H(v)^2 \le 4QA(v). \tag{8.15}$$

Put $V = H(y)^3H(x)^2$. Since $A(y)^3A(x)^2 = \lambda^2|\,\mathrm{N}\,w|$, (8.6), (8.13)–(8.15) give

$$\frac{n^2}{800^2\,16^5L^{18}} \le V^2 \le 1000\,4^5M^7L^{80}n^2. \tag{8.16}$$

Indeed the lower bound divides by 16^5L^{10} and uses $\lambda^2 \ge 800^{-2}L^{-8}$; the upper bound multiplies $4^5M^5L^{24}$, M^2L^{48} and $1000L^8n^2$. Square roots prove (iv) with an absolute C_0 also dominating

$800M$. Finally (ii) recovers w from (e, y, x), and its θ-coordinate $d(2an + b)$ determines n because $ad \neq 0$. □

8.2. **The count on a shell.** Let $\mathcal{P}(L, Z, \Delta_0, e)$ consist of primitive pairs (y, x) with $T_0(y; x) = e$, $Z \leq \mathrm{ht}(y)^3 \,\mathrm{ht}(x)^2 \leq 2Z$ and the balancing inequalities (iii) of Proposition 8.2. Choosing dominant coordinates gives four *orientations*, each with slopes $(\xi, \zeta) \in [-1, 1]^2$. Throughout the next three propositions, Δ_0 is squarefree and nonsquare with $|\Delta_0| \leq 5L^2$.

Proposition 8.3 (Fallback count)**.** *For $\delta > 0$ there are $K_{\mathrm{f}}, B_{\mathrm{f}} > 0$ such that for all $L, Z \geq 2$, all squarefree nonsquare Δ_0 with $|\Delta_0| \leq 5L^2$, all $1 \leq e \leq C_0 L^{28}$ and all $D, E \geq 1$ with $\sqrt{D}, \sqrt{E} \leq 4Z$, the number of pairs of primitive vectors with $|y_i| \leq \sqrt{D}$, $|x_i| \leq \sqrt{E}$ and $T_0(y; x) = e$ is at most $K_{\mathrm{f}} L^{B_{\mathrm{f}}} Z^{\delta} \min(D, E)$.*

Proof. Write $d = \Delta_0$, $h = e$, $P = \max(C_0, 4)LZ$. For a quadratic form Q of nonzero discriminant and coefficient height at most $A \geq 1$, Lemma 4.3 gives, uniformly in $h \neq 0$ and $X \geq 2$,

$$\#\{u \text{ primitive} : Q(u) = h,\ \mathrm{ht}(u) \leq X\} \ll_{\eta} |h|^2 (AX)^{\eta} \qquad (\eta > 0). \tag{8.17}$$

To track h, split into the $4h^2$ residue classes modulo $2h$. The injective map in that proof sends each nonempty class into $p^2 - \mathrm{disc}(Q)r^2 = 1$, $|p|, |r| \leq \lceil 6AX^2 \rceil$. This has $O(1 + \log(AX))$ solutions with absolute constant: negative or square discriminants give bounded counts, and positive nonsquare Pell solutions grow geometrically with ratio at least 2 in their positive rational coordinates. Absorb the logarithm to obtain (8.17).

For nonzero fixed x, the cubic in y is

$$16\{\sigma_+(\iota_0 x)^2 \sigma_+(\iota_0 y)^3 + \sigma_-(\iota_0 x)^2 \sigma_-(\iota_0 y)^3\}.$$

Its two coefficients are nonzero and its two linear forms independent, so it has three distinct projective roots. Divide by its positive content c; a nonempty fibre requires $c \mid h$. Hypothesis ST then bounds primitive solutions by $2800\, 3^{1+\omega(|h/c|)} \ll_{\eta} |h|^{\eta}$, uniformly in the cubic, using $3^{\omega(m)} \leq \tau(m)^2$ and the divisor bound (absorbing a sign into the form when necessary).

For fixed primitive y, expansion gives quadratic coefficient height $\ll (1 + |d|)^2 \,\mathrm{ht}(y)^3$ and discriminant

$$\begin{cases} -16d((2y_0 + y_1)^2 - dy_1^2)^3, & d \equiv 1 \pmod 4, \\ -4096d(y_0^2 - dy_1^2)^3, & d \not\equiv 1 \pmod 4. \end{cases}$$

It is nonzero because a nonzero integral vector has nonzero norm. Since $|d| \leq 5L^2$ and all coordinates are at most $4Z$, the heights and ranges are bounded by fixed powers of P. Choosing η sufficiently small in (8.17) bounds this fibre by $O_{\delta}(h^2 P^{\delta})$; the cubic fibre is $O_{\delta}(h^{\delta})$. Both are $O_{\delta}(h^{2+\delta} P^{\delta})$. Summing in either order, with at most $9D$ choices of y or $9E$ choices of x, gives

$$\#\{(y, x)\} \ll_{\delta} h^{2+\delta} P^{\delta} \min(D, E) \ll_{\delta, C_0} L^{56+29\delta} Z^{\delta} \min(D, E).$$

Thus $B_{\mathrm{f}} = 56 + 29\delta$ suffices. □

Proposition 8.4 (Graph cover)**.** *For every integer $\varkappa_0 \geq 0$ there are $C_{\mathrm{g}} \geq 2$ and an integer $k_{\mathrm{g}} \geq 1$, depending on $\varkappa_0$ and C_0, with the following property. Let $L, Z \geq 2$ with $H_{\mathrm{g}} := C_{\mathrm{g}} L^{k_{\mathrm{g}}} \leq Z$, let Δ_0 be squarefree and nonsquare with $|\Delta_0| \leq 5L^2$, and let $1 \leq e \leq C_0 L^{28}$. For each orientation there are at most H_{g} intervals $[a_j, b_j] \subset [-1, 1]$ and functions g_j of class $C^{\varkappa_0 + 1}$ on $(a_j - 2/H_{\mathrm{g}}, b_j + 2/H_{\mathrm{g}})$, with all derivatives of order at most $\varkappa_0 + 1$ bounded by H_{g} and $|g_j'| \geq 1/H_{\mathrm{g}}$ on $[a_j, b_j]$, such that every pair of $\mathcal{P}(L, Z, \Delta_0, e)$ in that orientation has slopes with $\xi \in [a_j, b_j]$ and $|\zeta - g_j(\xi)| \leq H_{\mathrm{g}}/Z$ for some j.*

Proof. Put $d = \Delta_0$. In a fixed orientation denote the dominant coordinates by y_1, x_1, so $\xi = y_0/y_1$, $\zeta = x_0/x_1$. Each embedding gives an affine factor $t + \omega$ or $1 + \omega t$. Write these as $A_{\pm}(s)$ for y and $B_{\pm}(w)$ for x. Then

$$\Phi(s, w) = 16\{A_+(s)^3 B_+(w)^2 + A_-(s)^3 B_-(w)^2\}$$

is real, and at a counted point

$$|\Phi(\xi,\zeta)| = \frac{e}{|y_1|^3|x_1|^2} \le C_0 L^{28}/Z. \tag{8.18}$$

Constants below may depend on $C_0, \varkappa_0$, but not on d, L, Z, e.

Nearby zeros and nonvanishing partial derivatives. For $d<0$, write $\omega = u+iv$, with $u=0$ or $1/2$, $v^2 \ge 1/4$, $v^2 \ge u^2$. Both possible affine factors have modulus at least $1/2$ on $\mathbb{R}$: their squared minima are respectively v^2 and $v^2/(u^2+v^2) \ge 1/2$. With $A = A_+$ and $B = B_+ = rw+t$, put $a = \mathrm{Re}(A^3 r^2)$, $b = \mathrm{Re}(A^3 rt)$, $c = \mathrm{Re}(A^3 t^2)$. Expansion gives

$$\Phi = 32(aw^2+2bw+c), \qquad b^2-ac = v^2|A|^6 \ge 1/256. \tag{8.19}$$

A quadratic $aw^2+2bw+c$ with $b^2-ac = \mathcal{D} > 0$ has a real root at distance at most $|aw^2+2bw+c|/\sqrt{\mathcal{D}}$: for $a \ne 0$ factor it and use that the farther root is at distance at least half the root separation $2\sqrt{\mathcal{D}}/|a|$; for $a=0$ use the linear equation. Consequently a root w_0 satisfies

$$|w-w_0| \le |\Phi(s,w)|, \tag{8.20}$$

and $|w_0| \le 3/2$ once $Z \ge 2C_0L^{28}$. At a zero, A^3B^2 is purely imaginary and $|\mathrm{Im}(A'/A)| = v/|A|^2$ (likewise for B), so

$$\Phi_w^2 = 4096v^2|A|^6, \qquad \Phi_s^2 = 9216v^2|A|^2|B|^4. \tag{8.21}$$

Thus $|\Phi_w| \ge 4$ and $|\Phi_s| \ge 6$.

For $d>0$, write $A_\pm = p_\pm + q_\pm s$, $B_\pm = r_\pm + t_\pm w$. Their coefficients are at most $5L$, and

$$|p_+q_- - p_-q_+|,\ |r_+t_- - r_-t_+| \ge 1,$$

since these are $|\omega_+ - \omega_-| = \sqrt{d}$ or $2\sqrt{d}$. By (8.14), two embeddings with ratio bounded by Q each have modulus at least $\mathrm{ht}(v)/(2Q)$. Dividing by the dominant coordinate and using (iii), weakened from L^4 to L^6, gives

$$\eta := 1/(2C_0L^6) \le |A_\pm(\xi)|, |B_\pm(\zeta)| \le U := 5L. \tag{8.22}$$

If $|\Phi(s,w)| < 32\eta^5$, the A factors have opposite signs; otherwise $|\Phi| \ge 32\eta^5$. The usual discriminant of $A_+^3(r_+ + t_+w)^2 + A_-^3(r_- + t_-w)^2$ is therefore

$$-4A_+^3A_-^3(r_+t_- - r_-t_+)^2 \ge 4\eta^6.$$

The same root-distance estimate gives

$$|w-w_0| \le |\Phi(s,w)|/(16\eta^3). \tag{8.23}$$

A polynomial lower threshold on Z makes this displacement at most $\min(1/2, \eta/(2U))$. Hence $|w_0| \le 3/2$, and all factors there lie in modulus between $\eta/2$ and $2U$. At such a zero abbreviate the factors to A, C, B, D; using $A^3B^2 = -C^3D^2$ gives

$$\Phi_s = 48\frac{A^2B^2}{C}(q_+C - q_-A), \qquad \Phi_w = 32\frac{A^3B}{D}(t_+D - t_-B).$$

The brackets are the constant determinants above, so both partials have modulus at least η^4/U.

Uniform analytic charts. In both cases the polynomial extension of Φ to complex variables and its partial derivatives of order at most two are bounded by $K = CL^5 \ge 1$ on $|s|, |w| \le 3$, by expansion of the five affine factors. For $d<0$ this means extending its real coefficients, not taking a real part at complex arguments. Put

$$\lambda = \frac{1}{160C_0^4L^{25}}, \qquad R_w = \frac{\lambda}{64K}, \qquad R_s = \frac{\lambda^2}{4096K^2}.$$

Both partials at our zeros are at least λ. At one such zero write

$$\Phi(s_0+z, w_0+v) = a(z)v^2 + b(z)v + c(z), \qquad b_0 = b(0).$$

Here $c(0)=0$, $|b_0|\ge\lambda$ and, on $|z|<R_s$, $|a|\le K$, $|b-b_0|,|c|\le K|z|$ (enlarge K absolutely if needed). In particular

$$|b-b_0|\le|b_0|/16,\quad |4ac|\le|b_0|^2/16,\quad |c|\le|b_0|R_w/8,\quad 2R_w|a|\le|b_0|/8.$$

Thus $\mathcal{D}=(b^2-4ac)/b_0^2$ satisfies $|\mathcal{D}-1|\le 1/4$. Take the square root equal to 1 at zero and set

$$G(z)=-\frac{2c(z)}{b(z)+b_0\sqrt{\mathcal{D}(z)}}. \tag{8.24}$$

The denominator has modulus at least $|b_0|$, so G is holomorphic, $G(0)=0$, $|G|\le R_w/4$, and $aG^2+bG+c=0$, even when $a=0$. It is the unique root with $|v|<R_w$, since $(v-G)\{a(v+G)+b\}=0$ and the second factor differs from b_0 by less than $|b_0|/4$. Uniqueness makes G real on the real axis. Along $g(s)=w_0+G(s-s_0)$ the second-derivative bounds give $|\Phi_s|,|\Phi_w|\ge\lambda/2$, whence $|g'|\ge\lambda/(2K)$. Cauchy's estimates on $|s-s_0|\le R_s/2$ give

$$|g(s)|\le 2,\qquad |g^{(n)}(s)|\le 2n!(2/R_s)^n\quad (n\ge 1).$$

All these bounds and reciprocal radii are polynomial in L for fixed n.

Finite cover. Partition $[-1,1]\times[-3/2,3/2]$ into rectangles of widths at most $R_s/16$ and heights at most $R_w/16$. From each rectangle containing a relevant zero choose one: every zero if $d<0$, and those with factors in modulus $[\eta/2,2U]$ if $d>0$. There are $O((R_sR_w)^{-1})$ choices. At each construct the graph above and retain $[s_0-R_s/4,s_0+R_s/4]\cap[-1,1]$. Every relevant zero is within $R_s/16,R_w/16$ of the chosen zero in its rectangle, so uniqueness places it on that graph. Equations (8.20) and (8.23) then put every counted point within CL^{46}/Z vertically of a selected graph. Choose $H_{\mathrm{g}}=C_{\mathrm{g}}L^{k_{\mathrm{g}}}$ to dominate the nearby-zero thresholds, the number of rectangles, $8/R_s$, $2K/\lambda$, CL^{46} and the derivative bounds through order $\varkappa_0+1$. This is a finite list of polynomial bounds. The enlarged intervals then remain in $|s-s_0|<R_s/2$, proving every assertion. □

Proposition 8.5 (Good-box count)**.** *There is an absolute integer $\varkappa_0$ such that for every integer $k\ge1$, every $C_2\ge1$ and every $\delta>0$ there are $C_{\mathrm{b}},B_{\mathrm{b}}>0$ and an integer K_{b} with the following property. Let $L,Z\ge2$ with $C_{\mathrm{b}}L^{K_{\mathrm{b}}}\le Z$, put $H_2=C_2L^k$ and $M=\lceil Z^{19/100}\rceil$, let Δ_0 be squarefree and nonsquare with $|\Delta_0|\le5L^2$, and let $1\le e\le C_2L^{28}$. Let $[a',b']\subset[-1,1]$ and let g be of class $C^{\varkappa_0+1}$ on $(a'-2r,b'+2r)$, $r\ge1/H_2$, with derivatives of order at most $\varkappa_0+1$ bounded by H_2 and $|g'|\ge1/H_2$ on $[a',b']$. Let $D,E>M$ with $D^3E^2\le H_2Z^2$. Then, in each orientation, the number of pairs of primitive vectors (y,x) with $|y_i|\le\sqrt{D}$, $|x_i|\le\sqrt{E}$, $T_0(y;x)=e$ and slopes satisfying $\xi\in[a',b']$ and $|\zeta-g(\xi)|\le H_2/Z$ is at most $C_{\mathrm{b}}L^{B_{\mathrm{b}}}Z^{97/400+\delta}$.*

Proof. Write $H=H_2=C_2L^k$ and fix

$$\mu=19/100,\quad q=4096,\quad J=(3q+1)(2q+1),\quad \gamma=19q^3/1440,\quad R=qJ+1.$$

Since $\rho(\mu)=19/360$, this q satisfies (5.2). Choose an absolute integer $\varkappa_0$ with

$$\mu(\varkappa_0+1)>qJ+\gamma+1. \tag{8.25}$$

Every threshold below is $Z\ge CH^A$, with C,A fixed before Z; finitely many such thresholds give $Z\ge C_{\mathrm{b}}L^{K_{\mathrm{b}}}$ with integral K_{b}.

Auxiliary forms and their heights. Use centres $s_j=a'+j/M$, $0\le j\le\lfloor M(b'-a')\rfloor$, and cells $[s_j,\min(s_j+1/M,b')]$. Write $u=\xi-s_j$, $v=\zeta-g(s_j+u)$, so $|u|\le Z^{-\mu}$, $|v|\le H/Z$. A polynomial threshold ensures $1/M\le r/2$ and $H/Z\le1$. Let T be the Taylor polynomial of $g(s_j+u)$ of degree $\varkappa_0$, whose coefficients have modulus at most H. For $0\le a\le3q$, $0\le b\le2q$,

$$P_{ab}(u,v)=(s_j+u)^a(T(u)+v)^b$$

has coefficients at most CH^{5q+1}, and Taylor's theorem and $|x^b-y^b|\le b\max(|x|,|y|)^{b-1}|x-y|$ give error

$$|(s_j+u)^a(g(s_j+u)+v)^b-P_{ab}(u,v)|\le CH^{5q+1}|u|^{\varkappa_0+1}.$$

Discarding terms of total degree greater than $\varkappa_0$ adds at most $CH^{A_0}Z^{-\mu(\varkappa_0+1)}$: their weights $\mu i+j$ are at least $\mu(\varkappa_0+1)$, and their number and degrees are fixed.

As in Proposition 5.10, expand the normalized evaluation determinant at any J points in the cell. By Proposition 5.7, nonzero coefficient determinants use monomials of total weight greater than $qJ+\rho(\mu)q^3/2 = qJ+2\gamma$. The finitely many ordinary terms are bounded by $CH^{A_1}Z^{-qJ-2\gamma}$; terms with an error row by $CH^{A_1}Z^{-\mu(\varkappa_0+1)}$. Thus

$$|\det \mathcal{E}| \le CH^{A_1}Z^{-qJ-\gamma}.$$

Clearing each row by $y_1^{3q}x_1^{2q}$, of modulus at most $(D^3E^2)^{q/2} \le H^{q/2}Z^q$, gives an integer determinant at most $CH^{A_1+qJ/2}Z^{-\gamma}$. A polynomial threshold makes it less than one, hence zero. The finite integer row set therefore has rank less than J and entries at most $H^{q/2}Z^q$. The signed-minor construction of Proposition 5.11, including rank zero, gives a nonzero integer null vector of height at most $J!H^{qJ/2}Z^{qJ} \le Z^R$ after another polynomial threshold. Homogenise to a form G_j of bidegree $(3q,2q)$ vanishing on the cell, with height at most Z^R; empty cells need no form.

Uniform component estimates. Suppose $P \ge 2$, $|d| \le 5P^2$, and two unimodular substitutions, an auxiliary form of degree at most $5q$, and the two larger box lengths all have entries or heights at most P^v, for fixed v. We claim the uniform version of Proposition 6.1:

$$C_{v,\delta}e^{2+\delta}P^{\delta}\{1+(\Lambda_1^3\Upsilon_1^2)^{1/10}\}, \qquad e \ge 1. \tag{8.26}$$

Factor the auxiliary form over $\mathbb{Q}$. There are at most $5q$ factors, whose heights are $O(P^v)$ by GE. Pure factors of degree at least two have no primitive zeros; a pure linear factor fixes its primitive pair up to sign. The uniform quadratic and cubic fibres proved in Proposition 8.3 then give $O(e^{2+\delta}P^{\delta})$: all transformed heights and ranges are polynomial in P, and their small exponents are chosen after v,δ.

For a mixed factor of bidegree (α,β) with $\min(\alpha,\beta)=1$, the resultant and squarefree-kernel argument of Proposition 6.4 gives

$$\#\text{solutions} \le 11200d_R\tau(|\Xi|)^{d_R+1}, \quad |\Xi| \le C|e|\,\|G\|^{12\beta+8\alpha}, \quad d_R = 3\beta+2\alpha.$$

Since the possible bidegrees are fixed, the divisor bound gives $O(|e|^{\delta}P^{\delta})$ with uniform constant; this includes $(4,1),(1,3)$ and all other factors of minimum bidegree one. The remaining factors have $\alpha,\beta \ge 2$, hence $10 \le 2\alpha+3\beta \le 3\deg G$. Lemma 6.6 gives $O(P^{\delta}\{1+(\Lambda_1^3\Upsilon_1^2)^{1/10}\})$, using $\log\max(1,\Lambda_1^3\Upsilon_1^2) \le 5v\log P$. Its constant is uniform in the transformed trace form and nonzero right-hand side. Adding the factors proves (8.26).

Taking $P = C_*LZ$, $C_* \ge \max(1,C_2)$, and $e \le C_2L^{28}$ gives

$$CL^{56+29\delta}Z^{\delta}\{1+(\Lambda_1^3\Upsilon_1^2)^{1/10}\}. \tag{8.27}$$

Here v changes the constant, not the displayed powers, because all fibre, divisor and logarithm exponents were chosen after v,δ.

Reduced bases. Impose $M \ge 3H$, for instance by $Z \ge (3H)^6$ since $6\mu > 1$. On a nondegenerate interval g' has constant sign, so the centres s_j are $1/M$-separated and $t_j = g(s_j)$ are $1/(HM)$-separated. There are at most $5M$ occupied cells, with $|s_j| \le 1$, $|t_j| \le 2$, since $|\zeta - t_j| \le H/Z + H/M \le 1$. A degenerate interval has a single cell and the same estimates, with separation vacuous. Put

$$Q = \lceil H \rceil + 1 \le 3H, \quad W = 3H, \quad M' = QM, \quad D' = W^2Q^2D, \quad E' = W^2Q^2E.$$

Both sets of centres are $1/M'$-separated. The cell inequalities

$$|y_1| \le \sqrt{D}, \quad |y_0 - s_jy_1| \le \sqrt{D}/M, \qquad |x_1| \le \sqrt{E}, \quad |x_0 - t_jx_1| \le W\sqrt{E}/M$$

place the points in strips of parameters (D',M'), (E',M'), because $\sqrt{D'}/M' = W\sqrt{D}/M$, and similarly for E. Their reduced integral bases give coordinate bounds $\Lambda_{1j} \ge \Lambda_{2j} > 0$, $\Upsilon_{1j} \ge \Upsilon_{2j} > 0$.

The estimates of Section 7 give

$$\|B_{yj}\|, \|B_{yj}^{-1}\| \le CW^2Q^2D, \qquad \Lambda_{1j} \le 10WQ\sqrt{D},$$

and analogously for x, E, Υ. Since $D, E > M \ge 1$ and $D^3E^2 \le HZ^2$, we have $D, E \le HZ^2$. Thus the matrix entries are $O(H^5Z^2)$ and lengths $O(H^3Z^2)$. Substituting into G_j preserves its bidegree and bounds its height by $C_qZ^R(CH^5Z^2)^{5q}$. All data in (8.27) are therefore bounded by a fixed power of C_*LZ. Unimodularity preserves primitivity and injectivity, so that estimate applies in every cell.

Cell sum. The centres are in a fixed bounded interval, are $1/M'$-separated, number at most $5M'$, and $D', E' > M'$. Lemma 7.5 gives

$$\begin{aligned}\sum_j \Lambda_{1j}^{3/10}\Upsilon_{1j}^{1/5} &\le CM'(D'/M')^{3/20}(E'/M')^{1/10}\\ &= CW^{1/2}Q^{5/4}M(D/M)^{3/20}(E/M)^{1/10}\\ &\le CW^2Q^2(D^3E^2)^{1/20}M^{3/4} \le 81CH^{4+1/20}Z^{1/10}M^{3/4}.\end{aligned}$$

Summing (8.27) therefore gives

$$\#\{(y,x)\} \le CL^{56+29\delta}Z^\delta H^5(M + Z^{1/10}M^{3/4}) \le 4CL^{56+29\delta}H^5Z^{97/400+\delta},$$

since $M \le 2Z^{19/100}$ and $19/100 < 97/400$. Thus $B_{\mathrm{b}} = 56 + 29\delta + 5k$ works; enlarge C_{b} and choose integral K_{b} to enforce all polynomial thresholds. □

Proposition 8.6 (Shell count)**.** *For every $\varepsilon > 0$ there are $C_1, B_1 > 0$ such that for all $L, Z \ge 2$, all squarefree nonsquare Δ_0 with $|\Delta_0| \le 5L^2$ and all $1 \le e \le C_0L^{28}$,*

$$\#\mathcal{P}(L, Z, \Delta_0, e) \le C_1L^{B_1}Z^{97/400+\varepsilon}.$$

Proof. Set $\delta = \varepsilon/2$. Choose the fallback constants, then the absolute Taylor order of Proposition 8.5, then its graph-cover constants $C_{\mathrm{g}}, k_{\mathrm{g}}$, and finally the good-box constants with $k = k_{\mathrm{g}}$, $C_2 = \max(128, C_0, C_{\mathrm{g}})$. Put $C_{\mathrm{cut}} = C_{\mathrm{g}} + C_{\mathrm{b}} + 2^{40}$ and $K_{\mathrm{cut}} = k_{\mathrm{g}} + K_{\mathrm{b}}$. Pairs in $\mathcal{P}$ have $1 \le \mathrm{ht}(y), \mathrm{ht}(x) \le 2Z$ and $\mathrm{ht}(y)^6\,\mathrm{ht}(x)^4 \le 4Z^2$. Lemma 3.11 partitions this finite set into $O((1+\log Z)^2)$ bins $D/2 < \mathrm{ht}(y)^2 \le D$, $E/2 < \mathrm{ht}(x)^2 \le E$, with $D^3E^2 \le 128Z^2$ and $\sqrt{D}, \sqrt{E} \le 4Z$.

If $Z < C_{\mathrm{cut}}L^{K_{\mathrm{cut}}}$, apply the fallback bound with $D = E = 4Z^2$ to obtain

$$\#\mathcal{P} \le 4K_{\mathrm{f}}L^{B_{\mathrm{f}}}Z^{\delta+2} \le 4K_{\mathrm{f}}C_{\mathrm{cut}}^2L^{B_{\mathrm{f}}+2K_{\mathrm{cut}}}Z^\delta.$$

Otherwise a bin with $\min(D, E) \le Z^{97/400}$ contributes at most $K_{\mathrm{f}}L^{B_{\mathrm{f}}}Z^{97/400+\delta}$. In every remaining bin, $M \le 2Z^{19/100} \le Z^{97/400} < D, E$, because $Z \ge 2^{40}$ implies $Z^{21/400} \ge 2$. Each of four orientations is covered by at most $H_{\mathrm{g}} = C_{\mathrm{g}}L^{k_{\mathrm{g}}}$ graphs. The good-box bound applies with $H_2 = C_2L^{k_{\mathrm{g}}} \ge H_{\mathrm{g}}$ and $D^3E^2 \le 128Z^2 \le H_2Z^2$, giving at most $4C_{\mathrm{g}}C_{\mathrm{b}}L^{k_{\mathrm{g}}+B_{\mathrm{b}}}Z^{97/400+\delta}$ pairs per bin. Absorb $(1+\log Z)^2 \ll_\varepsilon Z^{\varepsilon/2}$ and take $B_1 = 2B_{\mathrm{f}} + 3k_{\mathrm{g}} + B_{\mathrm{b}} + 2K_{\mathrm{b}}$. □

8.3. **Proof of Theorem 1.2.** Use Proposition 8.6 with $\varepsilon = 1/1600$, so its exponent is $\alpha_0 = 97/400 + 1/1600 = 389/1600$, and set $\nu = B_1 + 117$. For $X \ge 4C_0L^{10}$, Proposition 8.2 assigns each counted $n \in (X, 2X]$ a triple (e, y, x) injectively. There are at most C_0L^{28} choices of e. With $Z_1 = X/(C_0L^9) \ge 2$, the height product lies in $[Z_1, 2C_0L^{40}X]$, hence in $[jZ_1, 2jZ_1]$ for some integer $1 \le j \le (4C_0^2+2)L^{49}$. Since $(jZ_1)^{\alpha_0} \le (4C_0^2+2)L^{40}X^{\alpha_0}$, the shell bound gives

$$\#\{X < n \le 2X : f(n) \text{ powerful}\} \le C_0C_1(4C_0^2+2)^2L^{B_1+28+49+40}X^{\alpha_0} = K_{\mathrm{w}}L^\nu X^{\alpha_0}.$$

Lemma 3.12, with $X_0 = 4C_0L^{10}$, $n_1 = 1$ and $C_{\mathrm{w}} = K_{\mathrm{w}}L^\nu$, yields for $N \ge 1$

$$S(a, b, c; N) \le \left(8C_0L^{10} + \frac{K_{\mathrm{w}}L^\nu}{2^{\alpha_0}-1} + 1\right)N^{\alpha_0} \le KL^\nu N^{389/1600}, \quad K = 8C_0 + \frac{K_{\mathrm{w}}}{2^{\alpha_0}-1} + 1,$$

because $\nu \ge 10$.

For the coefficient sum take $\kappa_1 = 1/(1600(\nu+3))$, $M_0 = \lfloor N^{\kappa_1} \rfloor$, $N \geq 2$, and apply this bound with $L = 2N^{\kappa_1}$. Each term is at most $K2^{\nu} N^{\kappa_1\nu + 389/1600}$ and there are at most $M_0(2M_0+1)^2 \leq 9N^{3\kappa_1}$ triples. Since $\kappa_1(\nu+3) = 1/1600$, the sum is at most $9K2^{\nu}N^{390/1600}$. □

8.4. **The exponent and the constants.** The identity

$$\frac{389}{1600} = \frac{1}{10} + \frac{3}{4} \cdot \frac{19}{100} + \frac{1}{1600}$$

separates the cell-sum exponent from the fixed slack absorbing divisor bounds and bins. The first two terms exceed $293/1220$ by $57/24400$. Choosing $\mu_0 = 19/100 \in (57/305, 1/5)$ fixes absolute determinant and Taylor orders and permits polynomial dependence on L. The exponent is convenient, not optimal. No useful numerical value of ν or the resulting averaging range is claimed; the point is a fixed $\eta > 0$ permitting a family of coefficients growing with N.

9. Sums of two squares with a varying shift: proof of Theorem 1.3

We adapt Sections 3–7 to $\mathbb{Z}[i]$, choosing every constant before a. Throughout $a \geq 1$; the result depends only on $|a|$. Theorem 1.2 is not used.

9.1. **The equation.**

Lemma 9.1 (Gaussian labels)**.** *Let $n \geq 1$ with $n^2 + a^2$ powerful. There are an integer $\ell \geq 1$ dividing a, a Gaussian integer γ with $1 \leq \mathrm{N}\,\gamma \leq a^2$, and primitive $y, x \in \mathbb{Z}^2$ such that*

$$n + ia = \ell\,\gamma\,(y_0 + iy_1)^3 (x_0 + ix_1)^2. \tag{9.1}$$

For given a the pair (ℓ, γ) takes at most $4\tau(a)^3$ values, and (ℓ, γ, y, x) determines n.

Proof. Let P divide $n + ia$ exactly once in $\mathbb{Z}[i]$, above p. If p is inert, then $p \mid a$; if $p = 2$, the norm has valuation one, contradicting powerfulness. If p splits, powerfulness forces the conjugate prime also to divide $n + ia$, so again $p \mid a$. Take γ to be a unit times the product of these P. Its norm divides $\mathrm{rad}(a)^2$, and there are at most $4 \cdot 4^{\omega(a)} \leq 4\tau(a)^2$ choices. Every remaining exponent $e \geq 2$ is $2(e/2)$ if even, and $3 + 2((e-3)/2)$ if odd; hence $n + ia = \gamma \mathsf{Y}^3 \mathsf{X}^2$. Dividing the coordinates of Y, X by their gcds ℓ_1, ℓ_2 gives (9.1), with $\ell = \ell_1^3 \ell_2^2 \mid a$. This adds at most $\tau(a)$ choices. □

No class representatives or balancing are needed. Put $F_\gamma(y; x) = \mathrm{Im}(\gamma(y_0 + iy_1)^3 (x_0 + ix_1)^2)$. This integral $(3,2)$-form satisfies $F_\gamma(y; x) = a/\ell$, while n is ℓ times the real part; $\mathrm{N}\,\gamma \leq a^2$ controls its coefficients.

9.2. **Small values and one shell.**

Lemma 9.2 (A quadratic estimate uniform in the right-hand side)**.** *For every $\eta > 0$ there is C_η such that, for $H \geq 1$, $Z \geq 2$, integers A, B, h with $A^2 + B^2 \neq 0$, $h \neq 0$, $|A|, |B| \leq H$, the number of primitive integer pairs (u, v) satisfying*

$$|u|, |v| \leq Z, \qquad B(u^2 - v^2) + 2Auv = h$$

is at most $C_\eta (HZ|h|)^\eta$.

Proof. If $B = 0$, then $u, v \mid h$, giving at most $4\tau(|h|)^2 \ll_\eta |h|^\eta$ pairs. Suppose $B \neq 0$. The injective map $(u, v) \mapsto (x, y) = (Bu + Av, v)$ gives

$$x^2 - dy^2 = Bh, \qquad d = A^2 + B^2 \geq 1, \qquad |x|, |y| \leq Z_0 = \lceil 2HZ \rceil \leq 3HZ.$$

Vaughan–Wooley [14, Lemma 3.5], with coefficients $(1, -d, Bh)$, bounds the solutions with $1 \leq |x|, |y| \leq Z_0$ by $O_\xi((d|Bh|Z_0)^\xi)$, allowing four sign choices. The coefficients and right-hand side are nonzero; solutions on the coordinate axes add at most four. Since

$$d|Bh|Z_0 \leq 6H^4 Z|h| \leq 6(HZ|h|)^4,$$

taking $\xi = \eta/4$ proves the assertion, with constant depending only on η. □

Proposition 9.3 (Fallback)**.** *For every $\varepsilon > 0$ there is K such that for all $R_1 \geq 2$, all Gaussian integers $\gamma = \gamma' + i\gamma'' \neq 0$ and all integers $h \neq 0$ with $|\gamma'|, |\gamma''|, |h| \leq R_1$, the number of pairs of primitive vectors with $F_\gamma(y; x) = h$ and $|y|^6|x|^4 \leq R_1^2$, where $|y|^2 = y_0^2 + y_1^2$, is at most $KR_1^{2/5+\varepsilon}$.*

Proof. Put $U = |y|^2, V = |x|^2 \geq 1$. Since $U^3V^2 \leq R_1^2$, either U or V is at most $R_1^{2/5}$; each short disc contains at most $9R_1^{2/5}$ vectors. All coordinates are at most R_1. For fixed $x \neq 0$, Lemma 4.1 makes $F_\gamma(\cdot\,; x)$ nonzero and squarefree. The content-divisor argument of Lemma 4.2 gives at most $5600\tau(|h|)3^{1+\omega(|h|)} \leq 16800\tau(|h|)^3 \ll_\varepsilon R_1^\varepsilon$ primitive y, uniformly in x, γ, h. For fixed $y \neq 0$, write $P + iQ = \gamma(y_0 + iy_1)^3 \neq 0$. The equation in x is $Q(x_0^2 - x_1^2) + 2Px_0x_1 = h$, with $|P|, |Q| \leq 8R_1^4$. Lemma 9.2, with $H = 8R_1^4$, $Z = R_1$ and exponent $\varepsilon/6$, gives $O_\varepsilon(R_1^\varepsilon)$ solutions since $HZ|h| \leq 8R_1^6$. Sum cubic fibres over short x and quadratic fibres over short y. □

Let $X \geq 2$ with $a^2 \leq X$ and write $a = X^\upsilon$, so $0 \leq \upsilon \leq \frac{1}{2}$. Define

$$m_0(\upsilon) = \max\Big\{\frac{57 + 51\upsilon}{305},\ \frac{47 + 61\upsilon}{255},\ \frac{4 + 14\upsilon}{31}\Big\}. \tag{9.2}$$

Proposition 9.4 (Shell bound)**.** *For $0 < \delta \leq 1/100$ and $\varepsilon > 0$ there are K and X_2, depending only on δ and ε, such that for all $X \geq X_2$, all $1 \leq a \leq \sqrt{X}$ and all labels (ℓ, γ), the number of pairs of primitive (y, x) satisfying (9.1) for some $n \in (X, 2X]$ is at most*

$$KX^\varepsilon(M + X^{1/10}M^{3/4}), \qquad M = \lceil X^{m_0(\upsilon)+\delta}\rceil.$$

Proof. Uniform phase charts. For $c^2 + d^2 = 1$ put

$$A(s) + iB(s) = (c + id)(s + i)^3, \quad R(s) = (1 + s^2)^{3/2}, \quad P_{c,d}(s, w) = B(s)(w^2 - 1) + 2A(s)w.$$

Since $A^2 + B^2 = R^2$, wherever the denominators are nonzero we have

$$f_\pm(s) = \frac{\pm B(s)}{R(s) \pm A(s)}, \qquad P_{c,d}(s, w) = (w - f_\pm(s))(B(s)w + A(s) \pm R(s)).$$

If $|w| \leq 1$, $|P_{c,d}(s, w)| \leq 1/4$ and $D = Bw + A$, then $D^2 = R^2 + BP_{c,d}$ and $R \geq 1$, $|B| \leq R$ imply $|D| \geq R/2$. Choose the sign of D. As $|Bw| \leq R$, the corresponding denominator satisfies $R \pm A \geq R/2 \geq 1/2$ and $|D \pm R| \geq 1$; hence $|w - f_\pm(s)| \leq |P_{c,d}(s, w)|$.

On $[-2, 2]$, $|R'| < 14$, $|A'| \leq 21$. Take $h_0 = 1/1000$ and centers $t_j = -1 + j/1000$, $0 \leq j \leq 2000$. Retain *each* root formula whose denominator at t_j is at least $1/4$, on $[t_j - h_0, t_j + h_0]$. These at most 4002 charts are smooth on $(t_j - 3h_0, t_j + 3h_0)$: there the denominator exceeds $1/4 - 105/1000 > 1/8$, and $|f_\pm| < 96$ since $|B| \leq R < 12$. Every $s \in [-1, 1]$ is within $h_0/2$ of a center, where the residual-selected denominator is still at least $1/4$. Thus the charts cover the stated small-residual set uniformly.

At a root $(Bw + A)^2 = R^2 > 0$, so $\partial_w P \neq 0$ and implicit differentiation gives

$$f'(s) = -\frac{3}{2}\frac{1 + f(s)^2}{1 + s^2}. \tag{9.3}$$

In particular $|f'| \geq 3/10 > 1/4$ on the enlarged intervals. Define $H_0(s, w) = w$ and $H_{k+1} = \partial_s H_k - \frac{3}{2}(1 + w^2)(1 + s^2)^{-1}\partial_w H_k$. The chain rule gives $f^{(k)}(s) = H_k(s, f(s))$; maxima of these smooth functions on $[-2, 2] \times [-100, 100]$ give phase-independent bounds $M_\varkappa$ for $k \leq \varkappa + 1$.

The arithmetic tube. Rotate y, x by units so that their dominant coordinates $d_y = y_1, d_x = x_1$ are positive, absorbing inverse units into γ. There are at most 16 cases, preserving $\mathrm{N}\,\gamma$. With $G = |\gamma|$ and $c + id = \gamma/G$, the slopes $s, w \in [-1, 1]$ satisfy

$$P_{c,d}(s, w) = \frac{a}{\ell G d_y^3 d_x^2}. \tag{9.4}$$

Indeed, take imaginary parts in (9.1). Dominance gives $n \le \ell G|y|^3|x|^2 \le \sqrt{32}\,\ell G d_y^3 d_x^2$, so $|P_{c,d}(s,w)| \le 8a/X$. For uniformly large X, the charts therefore give $|w - f(s)| \le 8X^{-(1-\upsilon)}$, since $a \le \sqrt{X}$.

Uniform determinant orders. Put $\sigma = 1-\upsilon$, $\mu = m_0(\upsilon) + \delta$. Then $\mu \ge 57/305 + \delta$, $\sigma \ge 1/2$, $\mu < \sigma$ and $t_0 = \mu/\sigma \in (57/305, 1)$. Multiplying the three bounds of Proposition 9.5 by σ gives leading coefficients

$$(305\mu + 51\sigma)/18, \quad (255\mu + 61\sigma)/18, \quad (31\mu + 14\sigma)/3.$$

Each is at least $6+(31/3)\delta$ by (9.2). Choose $q \ge 1$, $q > (c_5+6)/\delta$, and then $\varkappa$ with $(57/305+\delta)(\varkappa+1) > n_q q + \delta q^3 + 1$. Thus every nonsingular minor has two-scale weight at least $(6+(31/3)\delta)q^3 - c_5 q^2 > n_q q + \delta q^3$. Both orders depend only on δ. On a cell, $|u| \le X^{-\mu}$, $|v| \le 8X^{-\sigma}$. Lemma 5.9 gives the expansion of Proposition 5.10 with weight $\mu i + \sigma j$; discarded terms have weight at least $\mu(\varkappa+1)$ because $\mu < \sigma$. The main minors have surplus δq^3 and the error a larger saving. Clearing denominators costs at most $(16X)^q$ per row, since $|y|^6|x|^4 \le 5X^2$. Every integer determinant therefore vanishes beyond a uniform threshold. Proposition 5.11's cofactor construction gives a nonzero $(3q, 2q)$-form of height at most X^{qn_q+1}.

All scales and component bounds. There are $O((1+\log X)^2)$ dyadic norm boxes, with $D^3E^2 \ll X^2$. The derivative bounds and $X^{-\sigma} \le X^{-\mu} \le 2/M$ give strips with one absolute constant. For any $D > 0$, apply Proposition 7.2 at $R_D = D + M + 1 > M$. The norms differ by a scalar, so the same basis is reduced and $\Lambda_i(D) = \sqrt{D/R_D}\,\Lambda_i(R_D)$. Its entries and inverse entries are $O(D+M+1)$ and $\Lambda_1(D) \le 2K_0\sqrt{D}$; likewise for E. Thus all entries, coordinate lengths, and the transported auxiliary form and its factors have height at most X^P, for fixed $P = P(\delta)$, using substitution and GE and increasing the uniform threshold.

Here is the required uniformity in the component count. Put $R = X^P$, also bounding $|h| = a/\ell$ and the coefficients of γ. Transported original coordinates are at most $Z = 2R^2$; the parts of $\gamma(y_0 + iy_1)^3$ are at most $H = 8RZ^3$, so $HZ|h| \le 128R^{10}$. Lemma 9.2 with exponent $\eta/(10P+1)$ gives $O_{\delta,\eta}(X^\eta)$ per quadratic fibre; the cubic fibres give the same bound by $16800\tau(|h|)^3$. Pure lines fix one primitive vector up to sign; higher-degree irreducible pure factors have no primitive rational zero. For the four low mixed bidegrees use (6.3): the complex presentation survives unimodular substitution, and $|h|$ and factor height are polynomially bounded in X. For high mixed factors, Lemma 6.6 is uniform in the transported form and $h \ne 0$. Summing at most $5q$ factors gives the per-cell bound $O_{\delta,\eta}(X^\eta(1 + \Lambda_1^{3/10}\Upsilon_1^{1/5}))$.

Summation. Use the regular $1/M$ grid on each branch. Its graph values are $1/(4M)$-separated by (9.3). The mixed moment extends to all $D, E > 0$: apply it to the same reduced bases with both scales $R = M+1$, then multiply by $(D/R)^{3/20}(E/R)^{1/10}$. The powers of R cancel, giving

$$\sum_{\mathcal{C}} \Lambda_1(\mathcal{C})^{3/10}\Upsilon_1(\mathcal{C})^{1/5} \ll M(D/M)^{3/20}(E/M)^{1/10} \ll X^{1/10}M^{3/4}.$$

The constant term contributes $O(M)$. Sum over the bounded number of branches and rotations, and over the dyadic boxes; choose $\eta = \varepsilon/2$ and absorb the logarithms in $X^{\varepsilon/2}$. All constants and thresholds depend only on δ, ε. □

9.3. **The weight of a minor with two scales.** Give $u^i v^j$ weight $t_0 i + j$, $0 < t_0 < 1$. For $g \in \mathbb{R}[u]$ let $w_q(t_0)$ be the least total weight of monomials whose coefficient matrix in the shifts b_j^g is nonsingular, where (b_j) is a basis of the $(3q, 2q)$ rectangle, as in Proposition 5.7.

Proposition 9.5. *There is an absolute c_5 such that for every $g \in \mathbb{R}[u]$ with $g'(0) \ne 0$ and every integer $q \ge 1$:*

(i) *if $\frac{57}{305} < t_0 \le \frac{1}{5}$, then $w_q(t_0) \ge \frac{305t_0+51}{18}q^3 - c_5q^2$;*
(ii) *if $\frac{1}{3} \le t_0 < 1$, then $w_q(t_0) \ge \frac{31t_0+14}{3}q^3$;*

(iii) *if* $\frac{1}{5} \le t_0 \le \frac{1}{3}$, *then* $w_q(t_0) \ge \frac{255t_0+61}{18}q^3 - c_5q^2$.

Proof. (i) The computation in Proposition 5.7 gives $(17/6 + 305t_0/18)q^3 - O(q^2)$ uniformly on $57/305 < t_0 < 1/5$. The weight of each fixed family is affine in t_0, so its bound passes to $t_0 = 1/5$.

(ii) Write $g = g_0 + g_1u + g_2u^2 + \cdots$, $g_1 \neq 0$, and put

$$W_1 = v - g_0, \quad G_1 = (1 - (g_2/g_1)u)(v - g_0) - g_1u, \quad P_{a,b} = u^a W_1^{b-j} G_1^j, \quad j = \min(3q - a, b),$$

for $0 \le a \le 3q$, $0 \le b \le 2q$. Shifting by g gives $W_1^g = v + g_1u + O(u^2)$ and $G_1^g = v(1 - (g_2/g_1)u) + O(u^3)$, of weighted orders t_0 and at least 1 because $3t_0 \ge 1$. Thus $P_{a,b}$ lies in the rectangle and its shift has order at least $t_0(a + b - j) + j$.

These polynomials form a basis. Use the invertible shift by the tangent $g_0 + g_1u$ and an irrational auxiliary weight $\lambda i + j$, $1/2 < \lambda < 1$. Then $W_1 \mapsto g_1u + v$ and $G_1 \mapsto v - g_2u^2 - (g_2/g_1)uv$, so $P_{a,b}$ has unique initial term $g_1^{b-j}u^{a+b-j}v^j$. Its exponent (i, j) recovers $a = \min(3q, i+j) - j$, $b = i + j - a$; the initial weights are therefore distinct. The least-weight term in any nonzero linear relation cannot cancel. Independence of these n_q polynomials proves the claim. Their coefficient matrix at any admissible monomial family is nonsingular. A nonzero determinant term matches each polynomial to a monomial of at least its order, giving

$$\sum_i w(m_i) \ge \sum_{a,b} \big(t_0(a + b - j) + j\big) = \frac{31t_0 + 14}{3}q^3 + \Big(\frac{19t_0}{2} + 3\Big)q^2 + \Big(\frac{13t_0}{6} + \frac{1}{3}\Big)q.$$

(iii) Write $t_0 = c'/5 + c''/3$, $c', c'' \ge 0$, $c' + c'' = 1$. For each admissible family, affinity of its weight and (i),(ii) give at least $c'(56q^3/9 - c_5q^2) + c''73q^3/9$. Since $(56c' + 73c'')/9 = (255t_0 + 61)/18$, the claimed bound follows. □

The chart slope bound supplies $g'(0) \neq 0$. The three leading coefficients after multiplication by $1 - v$ exceed 6 precisely at the three thresholds in (9.2); the middle range is the interpolation in (iii).

9.4. **The exponent and the assembly.** Put $\Theta_0(v) = 1/10 + 3m_0(v)/4$ for $0 \le v \le 1/2$. Since $m_0(v) + \delta \le 11/31 + 1/100 < 2/5$, the ceiling bound $M \le 2X^{m_0(v)+\delta}$ gives $M \ll X^{1/10}M^{3/4}$. Proposition 9.4 is therefore $\ll X^{\Theta_0(v)+3\delta/4+\varepsilon}$ per label and shell. The three terms defining Θ_0 are

$$\frac{293 + 153v}{1220}, \qquad \frac{81 + 61v}{340}, \qquad \frac{61 + 105v}{310}.$$

Their maximum switches at $1/28$ and $19/73$, where $m_0(v)/(1 - v) = 1/5$ and $1/3$, respectively. Hence $\Theta_0 = \Theta$ on $[0, 1/2]$.

Lemma 9.6 (Perspective inequality)**.** *If* $0 \le \beta_1 \le \kappa \le \frac{1}{2}$ *and* $0 < t_1 \le 1$ *with* $\beta_1/t_1 \le \frac{1}{2}$, *then* $t_1\,\Theta_0(\beta_1/t_1) \le \Theta_0(\kappa)$.

Proof. Each term of Θ_0 is $c + c'v$ with $c, c' \ge 0$, and $t_1(c + c'\beta_1/t_1) = ct_1 + c'\beta_1 \le c + c'\kappa$. Take their maximum. □

Proof of Theorem 1.3. Choose δ and the auxiliary ε small enough for the desired final exponent. Fix $1 \le a \le N^\kappa$ and a label. For $n \le \max(2a^2, 2X_2)$, Proposition 9.3 applies with $h = a/\ell$ and $R_1 \asymp a^2 + X_2$: this dominates $|\gamma|$, $|h|$ and $|y|^3|x|^2 \le \sqrt{n^2 + a^2}$. It gives $\ll N^{4\kappa/5+\varepsilon}$, and $4\kappa/5 \le \Theta(\kappa)$ in the stated range. The remaining $n \le N$ lie in $O(1 + \log N)$ shells with $a^2 \le X$. For each use Proposition 9.4 and Lemma 9.6 with $t_1 = \log X/\log N$, $\beta_1 = \log a/\log N$, obtaining $X^{\Theta_0(v)} \le N^{\Theta(\kappa)}$. Lower shells may have $v > \kappa$, which is why all three terms of (9.2) are needed. Sum the shells and $4\tau(a)^3 \ll N^\varepsilon$ labels, using Lemma 9.1 to recover n. □

The endpoint $289/1050$ is where $\Theta = 29/100$; the proof uses the restriction on κ only for $4\kappa/5 \le \Theta(\kappa)$ and $\kappa \le 1/2$.

10. Cubic applications

Proof of the cube-coefficient bound in Corollary 1.4. Put $u = at$, $Q(u) = u^2 - u + 1$. Since

$$u^3 + 1 = (u+1)Q(u), \qquad Q(u) = (u-2)(u+1) + 3,$$

we have $\gcd(u+1, Q(u)) \mid 3$. If u^3+1 is powerful, every prime $p \neq 3$ dividing $Q(u)$ occurs there to order at least two; multiplying by 9 also ensures this at 3. Thus $9Q(u)$ is powerful. This fixed positive quadratic has nonsquare discriminant -243. The injection $t \mapsto at$, followed by dropping $a \mid u$, gives

$$\#\{1 \le t \le T : 1 + a^3t^3 \text{ powerful}\} \le S(9, -9, 9; aT) \ll_\varepsilon (aT)^{293/1220+\varepsilon}$$

by Theorem 1.1, with constant independent of a. □

Lemma 10.1. *Put $P_A(t) = 1 + At^3$, where $A \ge 1$ is an integer. For every $\epsilon > 0$ there is C_ϵ, independent of A, q, k, such that for all positive integers q, k,*

$$\#\{t \in \mathbb{Z} : P_A(t) \in q\mathbb{Z}^2 \setminus \{0\}\} \le C_\epsilon A^{178} q^\epsilon,$$
$$\#\{t \in \mathbb{Z} : P_A(t) \in k\mathbb{Z}^3 \setminus \{0\}\} \le C_\epsilon A^{62} k^\epsilon.$$

Proof. Over $\mathbb{Q}$, Evertse–Silverman [4, Theorem 1] bounds integral arguments giving nonzero squares or cubes of a separable polynomial by, respectively,

$$7^{M(4+9s)} 4^{\rho_2(L)}, \qquad 17^{M(6+s)} 3^{2Ms+\rho_3(L)}.$$

Here the degree-M field L contains at least three roots in the square case and two in the cube case; s counts infinity and the primes of the polynomial discriminant. The class-group ranks satisfy $4^{\rho_2(L)} \le h_L^2$, $3^{\rho_3(L)} \le h_L$.

Take $L = \mathbb{Q}(\alpha, \zeta_3)$, $\alpha^3 = -A^2$, containing the roots $\alpha\zeta_3^j/A$ of P_A. For noncube A, the degree-three and degree-two fields are linearly disjoint, so the order $\mathbb{Z}[\alpha, \zeta_3]$ has absolute discriminant $(27A^4)^2 3^3$. For cube A, $L = \mathbb{Q}(\zeta_3)$. In either case $M \le 6$ and $|D_L| \le 3^9 A^8$. Minkowski gives an integral ideal of norm at most $c\sqrt{|D_L|}$ in every class, uniformly in degree at most six. Hence

$$h_L \le c^2 |D_L| \zeta_L(2) \le c^2 |D_L| \zeta(2)^6 \ll A^8.$$

The square fibre gives $qP_A(t) = (qv)^2$, with discriminant supported on $3Aq$, so $s \le 2 + \omega(A) + \omega(q)$. Similarly the cube fibre uses $k^2 P_A(t) = (kv)^3$ and $s \le 2 + \omega(A) + \omega(k)$. Scaling preserves the distinct roots. The two estimates become

$$CA^{16} 7^{54\omega(A)} 7^{54\omega(q)} \ll_\epsilon A^{178} q^\epsilon, \qquad CA^8 (17^6 3^{12})^{\omega(A)+\omega(k)} \ll_\epsilon A^{62} k^\epsilon.$$

Indeed $7^{54\omega(A)} \le A^{162}$ and $17^6 3^{12} < 2^{54}$; for the fibre parameters use $C_1^{\omega(m)} \ll_\epsilon m^\epsilon$ with fixed C_1. All constants are uniform in A, q, k. □

Proof of the general cubic bound in Corollary 1.4. For $1 \le t \le T$, $T \ge 2$, write a powerful value as $P_A(t) = e^2 d^3 \le 2AT^3$. With $U = (2AT^3)^{1/5}$, either $e \le U$ or $d \le U$. Apply Lemma 10.1 to $k = e^2$ in the first case and to $P_A(t) = d(ed)^2$, $q = d$, in the second. Summing gives

$$\#\{1 \le t \le T : P_A(t) \text{ powerful}\} \ll_\xi A^{62} \sum_{e \le U} e^{2\xi} + A^{178} \sum_{d \le U} d^\xi \ll_\xi A^{178+1/5+2\xi/5} T^{3/5+6\xi/5}.$$

Take $\xi = \min(5\varepsilon/6, 1)$: then $6\xi/5 \le \varepsilon$ and $178 + 1/5 + 2\xi/5 < 180$, proving the first assertion with $\nu_1 = 180$ independently of ε. □

11. Consequences of the uniform bound

In this section K and ν are the constants of Theorem 1.2.

Corollary 11.1. *For all integers $A, h, H' \geq 1$ and all $T \geq 2$,*

$$\#\{1 \leq t \leq T : At^2 + h \text{ is powerful}\} \leq K(2h)^\nu A^\nu T^{389/1600},$$

and, if AH' is not a square, the number of $t \leq T$ with $At^2 - H' \geq 1$ powerful is at most $K(H'+3)^\nu A^\nu T^{389/1600}$.

Proof. Apply Theorem 1.2 to $(A, 0, h)$ with $L = 2Ah$ (for $h = 1$, $L = 2A$) and to $(A, 0, -H')$ with $L = A + H' + 2 \leq (H'+3)A$. The discriminants are $-4Ah < 0$ and $4AH'$. □

These give the first assertions of Theorems 1.5 and 1.6, with $C_h = \max(1, \nu)$, $K_h = \max(1, K(2h)^\nu)$, $\nu_*(H') = \nu$ and $K_*(H') = K(H'+3)^\nu$.

Lemma 11.2 (Canonical row)**.** *Let $k \geq 2$. Every k-full $n \geq 1$ can be written as $n = \prod_{i=0}^{k-1} a_i^{k+i}$, where for $i \geq 1$ the integer a_i is the product of the primes p with $v_p(n) \equiv i \pmod k$, and the exponent of p in a_0 is $\lfloor v_p(n)/k \rfloor$, reduced by one if $k \nmid v_p(n)$. The row $(a_0, \dots, a_{k-1})$ determines n.*

Proof. Let $v_p(n) = ck + j$ with $0 \leq j < k$. If $j > 0$ then $c \geq 1$ because n is k-full, and $(c-1)k + (k+j) = ck + j$. If $j = 0$ the exponent is ck. □

Proposition 11.3. *Let $s \in \{6, 8\}$ and let h be a nonzero integer. Suppose there are $K', z > 0$ such that for all integers $A \geq 1$ that are not squares and all $T \geq 2$,*

$$\#\{1 \leq t \leq T : At^2 + h \geq 1,\ At^2 + h \text{ is powerful}\} \leq K' A^z T^{389/1600}. \tag{11.1}$$

Then for every $\varepsilon > 0$ there is C such that, for every real $Y \geq 2$, the number of $n \leq Y$ such that n is s-full, n is not a square, and $n + h$ is powerful is at most $CY^{1/s - \delta_s(z) + \varepsilon}$. If (11.1) holds for all $A \geq 1$, the same bound holds without the condition that n is not a square.

Proof. For $Y < 4$ the bound is trivial, so let $Y \geq 4$. Write $\delta = \delta_s(z)$, $\kappa_2 = 1/(1600(z+1))$ and $T = \sqrt{Y} \geq 2$; then $s(s+1)\delta \leq \kappa_2/2$ and $39/320 \leq 1/s - \delta$. Let $(a_0, \dots, a_{s-1})$ be the canonical row of n and put

$$P = \prod_i a_i, \qquad E_1 = \prod_i a_i^i, \qquad H_3 = \prod_{i \geq 1} a_i^{s+i},$$

so that $n = P^s E_1 = a_0^s H_3$ and $H_3 \leq E_1^{s+1}$.

If $P \leq Y^{1/s-\delta}$, we count rows: there are at most $\tau(P)^s \ll Y^\varepsilon$ rows with a given product P, hence $\ll Y^{1/s-\delta+\varepsilon}$ such n. The condition on $n + h$ is not used here.

If $P > Y^{1/s-\delta}$, then $E_1 = n/P^s \leq Y^{s\delta}$ and $H_3 \leq Y^{s(s+1)\delta} \leq T^{\kappa_2}$. Since s is even, $n = H_3 t^2$ with $t = a_0^{s/2} \leq T$, and H_3 is not a square if n is not. For each H_3, the number of $t \leq T$ with $H_3 t^2 + h$ powerful is at most $K' H_3^z T^{389/1600}$ by (11.1). Summing over the integers $H_3 \leq T^{\kappa_2}$ gives

$$\leq K' T^{\kappa_2(z+1) + 389/1600} = K' T^{390/1600} = K' Y^{39/320} \leq K' Y^{1/s-\delta}.$$ □

Proof of the second assertions of Theorem 1.5. For $h \geq 1$, Corollary 11.1 gives (11.1) for every $A \geq 1$, with $z = \nu$ (respectively $z = C_h$). An r-full integer with $r \geq 2$ is powerful, so Proposition 11.3 applies. □

Proof of the second assertion of Theorem 1.6. The split is according to whether n is a square. If n is not a square, take $h = -1$: Corollary 11.1 with $H' = 1$ (so $L = A + 3$) gives (11.1) for nonsquare A with $z = \nu$, and Proposition 11.3 applies. Let $n = m^2$. Since n is s-full with $s \geq 4$, every prime dividing m divides it at least twice, so m is powerful. Also $n - 1 = (m-1)(m+1)$ is powerful and coprime to m. Hence $(m-1)m(m+1)$ is powerful, and $m - 2 \leq \sqrt{Y}$ is counted by the width-three theorem of [8, Theorem 1.2], which gives $\ll Y^{77/640 + \varepsilon/2}$ such m. Finally $\frac{1}{2}(\frac{77}{320} + \varepsilon) \leq \frac{49}{400} + \varepsilon$ and $\frac{49}{400} \leq \frac{1}{s} - \delta_s(\nu)$, by the second term in the definition of δ_s. □

As used here, the width-three theorem requires CC and GE as well as ST.

Remark 11.4. The proofs bound the number of $n \le Y$ with n s-full and $n+h$ powerful for each fixed $h \ge 1$, and the number with $n-1$ powerful. The hypothesis $r < s$ is not used, r enters only through $r \ge 2$, and the constants do not depend on r. The restriction to $s \in \{6, 8\}$ enters as follows: s is even, so that $n = H_3 t^2$; $1/s > 39/320$; and, for Theorem 1.6, $1/s \ge 49/400$. Here $39/320$ is half of $390/1600$, $77/640$ is half of $77/320$, and $49/400$ lies between $77/640$ and $1/8$. In the definition of δ_s the first term is the smaller one for every $z > 0$, so $\delta_6(\nu) = 1/(268800(\nu+1))$ and $\delta_8(\nu) = 1/(460800(\nu+1))$. Since ν is not explicit, these savings cannot be evaluated numerically.

12. Remarks on the limits of the method

Remark 12.1 (The exponent $293/1220$). For a cell width $X^{-\mu}$ the method gives the exponent $\frac{1}{10} + \frac{3}{4}\mu$, and the sufficient determinant condition proved here is $\mu > 57/305$. So $293/1220$ is the infimum of the exponents obtained from Lemma 7.7 and Proposition 5.7. The term $\frac{1}{10}$ is the exponent of Lemma 6.6 at the least lifted degree 10, attained by components of bidegree $(2, 2)$. A smaller exponent for these components, or a better path (k, l) in Proposition 5.7, would lower $293/1220$. The condition $\mu < 1/5$ is used for the contact polynomials.

13. Formal verification and authorship

Theorems 1.1, 1.2, 1.3, 1.5 and 1.6 have formal proofs in Lean 4 with Mathlib. Their formal statements take the three estimates of Section 2 as hypotheses. These estimates are published mathematical results; their proofs are outside the formal development. The written cubic arguments of Section 10 are recorded separately in Appendix A, including the stronger cube-coefficient consequence of Theorem 1.1.

Independent statement comparisons and a second kernel replay check the formal declarations. The remaining links are mathematical ones: that the external estimates imply their formal versions, and that the formal statements express the claims in the paper. These links, and the written proofs, have been reviewed by AI systems. Formal verification does not certify novelty or replace review of the exposition. Appendix A gives the exact source mapping, trust base and reproduction information.

Use of AI systems. All results, proofs (written and formal), computation scripts and the text of this manuscript were produced by AI systems. The results and their proofs come from large language models running as agents in the Jocelyn research harness, together with Lean; the writing and revision of the manuscript, its reviews and the resulting corrections were done with AI coding assistants, operated by the author. The author chose the problem, operated the harness, set its objectives, decided the content of the manuscript and commissioned the reviews, and takes responsibility for the manuscript. All reviews were carried out by AI systems; no human has reviewed the results.

Appendix A. Formal statements and source mapping

The paper uses its own numbering. The identifiers in the shared formal library are stable names and do not prescribe the order of exposition.

In `A3`, the coefficient-average exponent is the particular choice $\eta = 1/(1600(\nu+3))$ used in the proof of Theorem 1.2. The cube-coefficient bound of Corollary 1.4 is the written consequence of `A1` proved in Section 10; that stronger conclusion has not been separately replayed. The library also retains the weaker cube-coefficient bound $C(a^3)^{\nu_2} T^{389/1600}$ in `A4`. The general cubic estimate has a formal proof there using Stewart's estimate for non-cube coefficients and `A3`, with all three external inputs, for cube coefficients. The classical proof given here uses Evertse–Silverman and supplies the coarse choice $\nu_1 = 180$. This numerical coefficient exponent is not an additional replayed claim.

Paper statement	Formal interface
Theorem 1.1	`A1`
Theorem 1.2	`A3`
Theorem 1.3	`A2`
Corollary 1.4, general bound	first part of `A4`
Theorem 1.5	`A12`; `A11` is its $h = 1$ case
Theorem 1.6	`A13`

TABLE 1. Formal interfaces for the statements of the paper.

The written proof of Lemma 9.2 uses Vaughan–Wooley [14, Lemma 3.5]; the formal development derives the corresponding quadratic-fibre estimate by a direct Pell argument.

Verification data. The pins are Lean 4.33.1 and Mathlib commit `0df444a360eaa60ab8c11dca51a86af692955474`. The formal interfaces lie in `JocelynWorkspace.Formalization.PowerfulCounting.ConsecutiveCounting`, with explicit hypotheses `StewartHyp`, `CCDNThm3` and `GelfondHyp`. These are theorem arguments, not new axioms. The standard axiom set is `propext`, `Classical.choice` and `Quot.sound`. The independent statement file is `replay/statements/consecutive-counting/Statements.lean`. The integrated receipt `replay/receipts/consecutive-counting.json` covers 38 statements in the shared repository, including the interfaces above. The same-source negative control `replay/receipts/consecutive-counting-negative.json` records rejection of a deliberately false strengthening of `A13`. The sources, pinned dependencies, build commands and replay instructions are available at https://github.com/dannyply/math-frontier.

The source comparisons use Stewart [12, Theorem 1, p. 795 and its consequence on p. 796], the affine-curve theorem of [2, Theorem 3], and the factor-height inequality of [1, Lemma 1.6.11, p. 27]. Stewart's consequence is used with $g = |m|$, squarefreeness supplying the nonzero discriminant. The curve theorem is used for a prime ideal with linear Hilbert function. The factor-height statement follows by applying the product-height inequality to the factor and its integral cofactor. These comparisons, like the manuscript reviews, were performed by AI systems.

The intermediate declaration map is supplied with the repository as `verification/consecutive-counting/manuscript-source-map.json`. It distinguishes exact statements, composed proof routes and written refinements, including the more general tail and moment bounds and the compressed Gaussian label count. Independent AI readers compared the mathematical statements and formal sources in both directions, fixing their initial interpretations before comparison. The formal declarations belong to the dependency closure of the replayed interfaces; this source review is not a separate kernel replay of each written lemma. All mathematical proofs needed here are in the manuscript or the cited literature.

APPENDIX B. A CLASSICAL COEFFICIENT-UNIFORM COMPARISON

We justify the comparison used in Section 1.3: for the quadratics in Theorem 1.2, the classical estimates give

$$S(a, b, c; N) \ll_\varepsilon L^{39} N^{2/5+\varepsilon}. \tag{B.1}$$

The coefficient power is deliberately coarse. Put $Q(n) = an^2 + bn + c$ and $\Delta = b^2 - 4ac$. Then $0 < |\Delta| \le 5L^2$, and the quadratic root field has absolute discriminant at most $20L^2$, hence class number $\ll L^2$ by the Minkowski argument in Lemma 10.1.

For $k \ge 1$, the equation $Q(n) = kv^3$ gives $k^2 Q(n) = (kv)^3$. The cube estimate of Evertse–Silverman used in that lemma applies with field degree two and $s \le 1 + \omega(\Delta) + \omega(k)$. Since

$17^2 3^4 < 2^{18}$, its bound is

$$\#\{n \in \mathbb{Z} : Q(n) \in k\mathbb{Z}^3 \setminus \{0\}\} \ll_\xi L^2 |\Delta|^{18} k^\xi \ll_\xi L^{38} k^\xi.$$

The roots are distinct because $\Delta \neq 0$.

For a square fibre $Q(n) = qv^2$ with $q \geq 1$ and $1 \leq n \leq N$, completion of the square gives

$$(2an + b)^2 - 4aqv^2 = \Delta, \qquad |2an + b|, |v| \leq 3LN.$$

Vaughan–Wooley [14, Lemma 3.5] bounds the solutions by $C_\xi (L^4 qN)^\xi$, uniformly in these parameters. Its nonzero right-hand-side condition follows from $\Delta \neq 0$; signs and a zero first coordinate change the count only by a constant.

Write each counted powerful value as $Q(n) = e^2 d^3 \leq 3LN^2$, and put $U = (3LN^2)^{1/5}$. Either $e \leq U$ or $d \leq U$. Summing the cube fibres with $k = e^2$ gives $C_\xi L^{38} U^{1+2\xi}$; summing the square fibres with $q = d$ gives $C_\xi L^{4\xi} N^\xi U^{1+\xi}$. Thus the two contributions are bounded by

$$C_\xi L^{38+1/5+2\xi/5} N^{2/5+4\xi/5}, \qquad C_\xi L^{1/5+21\xi/5} N^{2/5+7\xi/5}.$$

Taking $\xi = \min(5\varepsilon/7, 1/4)$ proves (B.1). This is a consequence of the cited estimates, not a claim that those sources state this coefficient power.

References

[1] E. Bombieri and W. Gubler, *Heights in Diophantine Geometry*, Cambridge University Press, 2006.

[2] W. Castryck, R. Cluckers, P. Dittmann and K. H. Nguyen, *The dimension growth conjecture, polynomial in the degree and without logarithmic factors*, Algebra Number Theory **14** (2020), 2261–2294; arXiv:1904.13109v1.

[3] J.-M. De Koninck, N. Doyon and F. Luca, *Powerful values of quadratic polynomials*, J. Integer Seq. **14** (2011), Article 11.3.3.

[4] J.-H. Evertse and J. H. Silverman, *Uniform bounds for the number of solutions to* $Y^n = f(X)$, Math. Proc. Cambridge Philos. Soc. **100** (1986), 237–248.

[5] D. R. Heath-Brown, *Square-free values of* $n^2 + 1$, Acta Arith. **155** (2012), 1–13.

[6] D. R. Heath-Brown, *The density of rational points on curves and surfaces*, Ann. of Math. (2) **155** (2002), 553–595.

[7] H. Pasten, *Powerful values of polynomials and a conjecture of Vojta*, J. Number Theory **133** (2013), 2964–2998.

[8] D. Plymesser, *Counting powerful products of consecutive integers*, https://doi.org/10.5281/zenodo.23614742.

[9] S. S. Pohl, *S-squareful values of binary quadratic forms*, arXiv:2610.20316v1, preprint.

[10] J.-C. Puchta, *On the powerfull part of* $n^2 + 1$, Arch. Math. (Brno) **39** (2003), 187–189.

[11] Y. Saito, *Shared prime support, small radicals, and square-full polynomial values*, unrefereed manuscript, https://doi.org/10.5281/zenodo.21942579.

[12] C. L. Stewart, *On the number of solutions of polynomial congruences and Thue equations*, J. Amer. Math. Soc. **4** (1991), 793–835.

[13] S. Streeter, *Campana points and powerful values of norm forms*, Math. Z. **301** (2022), 627–664.

[14] R. C. Vaughan and T. D. Wooley, *Further improvements in Waring's problem*, Acta Math. **174** (1995), 147–240.

[15] W. Wongcharoenbhorn and Y. Meemark, *Square-full values of quadratic polynomials*, Bull. Aust. Math. Soc. **113** (2026), 40–54.