hexagon:2611.00695v1

$t^{\log t}$ IS TRANSCENDENTAL FOR ALMOST ALL INTEGERS t

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ABSTRACT. Let $\widetilde{\mathcal{L}} = \overline{\mathbb{Q}} + \overline{\mathbb{Q}}\mathcal{L}$ be the $\overline{\mathbb{Q}}$-vector space spanned by 1 and the logarithms of nonzero algebraic numbers. We prove that

$$\#\{2 \le t \le X : \ (\log t)^2 \in \widetilde{\mathcal{L}}\} \le \pi(X) + O(X^{3/4}).$$

Hence $t^{\log t} = e^{(\log t)^2}$ is transcendental for all integers $t \ge 2$ outside a set of density zero, and no relation $(\log t - B)(\log t - A) = C' + \gamma\nu$ with A, B, C', γ algebraic and ν a logarithm of an algebraic number holds outside that set. The proof uses Roy's strong six exponentials theorem only through configurations of exponent vectors in dimensions 2 and 3, where it makes the primitive exceptional integers a multiplicative Sidon set, and then Erdős's bound for such sets. The prime numbers show that the bound $\pi(X)$ is the limit of the method. We also show that the configurations of exponent vectors allowed by the strong six exponentials theorem can be much larger than in dimensions 2 and 3: there are such sets of 2^r points in $\mathbb{P}^{n-1}$ with $n = 1 + r(r+1)/2$. No individual number such as $2^{\log 2}$ is decided.

1. Introduction

Let $\mathcal{L} = \{\lambda \in \mathbb{C} : e^\lambda \in \overline{\mathbb{Q}}^\times\}$ be the set of logarithms of nonzero algebraic numbers, and let

$$\widetilde{\mathcal{L}} = \overline{\mathbb{Q}} + \overline{\mathbb{Q}}\mathcal{L}$$

be the $\overline{\mathbb{Q}}$-vector space spanned by 1 and $\mathcal{L}$; its elements are the *linear forms in logarithms* of Roy [12]. For an integer $t \ge 2$, $\log t$ denotes the real logarithm.

Whether $2^{\log 2} = e^{(\log 2)^2}$, or more generally $\alpha^{\log \alpha}$, is transcendental is open; Waldschmidt lists it as a consequence of the strong five exponentials conjecture [17, §3, after Conj. 3.5]. Diaz conjectured that $\lambda^2 \notin \widetilde{\mathcal{L}}$ for every $\lambda \in \widetilde{\mathcal{L}} \setminus \overline{\mathbb{Q}}$ (Conjecture (P3) of [3, §1], a consequence of the strong four exponentials conjecture) and remarked that he knew no λ for which this is proved [3, §1]. We show that the conjecture holds for $\lambda = \log t$ for almost all integers t.

2020 *Mathematics Subject Classification.* Primary 11J81; Secondary 11J86, 11B75, 11N25.

Key words and phrases. six exponentials theorem, linear forms in logarithms, multiplicative Sidon sets, transcendence, $\alpha^{\log \alpha}$.

Theorem 1.1. *For $X \geq 2$,*

$$E(X) := \#\{2 \leq t \leq X : \ (\log t)^2 \in \widetilde{\mathcal{L}}\} \leq \pi(X) + O(X^{3/4}).$$

The theorem is unconditional. Its inputs are Roy's strong six exponentials theorem [12, Cor. 2], Baker's theorem [1] and Erdős's bound for multiplicative Sidon sets [4]. With Erdős's later sharpening [5], [6, (4)] the error term becomes $O(X^{3/4}(\log X)^{-3/2})$ (Remark 4.2).

Corollary 1.2. *Let $\mathcal{E} \subset \mathbb{Z}_{\geq 2}$ be the set of t with $(\log t)^2 \in \widetilde{\mathcal{L}}$, so that $\#(\mathcal{E} \cap [2, X]) \leq (1 + o(1))X/\log X$. For every integer $t \geq 2$ with $t \notin \mathcal{E}$:*

1. *$t^{\log t} = e^{(\log t)^2}$ is transcendental;*
2. *$e^{(\log t)^2/\gamma} \notin \overline{\mathbb{Q}}^\times e^{\beta + \delta \log t}$ for all $\gamma \in \overline{\mathbb{Q}}^\times$ and $\beta, \delta \in \overline{\mathbb{Q}}$;*
3. *there are no $A, B, C', \gamma \in \overline{\mathbb{Q}}$ and $\nu \in \mathcal{L}$ with $(\log t - B)(\log t - A) = C' + \gamma\nu$.*

Proof. If $t^{\log t} = \alpha \in \overline{\mathbb{Q}}$, then $\alpha \neq 0$ and $(\log t)^2 \in \mathcal{L}$. If $e^{\lambda^2/\gamma} = \alpha e^{\beta + \delta\lambda}$ with $\lambda = \log t$, then $\lambda^2/\gamma - \beta - \delta\lambda \in \mathcal{L}$, so $\lambda^2 \in \widetilde{\mathcal{L}}$. In (3), $\lambda^2 = C' - AB + (A + B)\lambda + \gamma\nu \in \widetilde{\mathcal{L}}$. In each case $t \in \mathcal{E}$. The bound on $\mathcal{E}$ is Theorem 1.1 with the prime number theorem. □

In (3) the exceptional set $\mathcal{E}$ does not depend on A, B, C', γ, ν: the relation fails off $\mathcal{E}$ for all admissible constants at once, and ν is not restricted in height. The hypothesis that the constants are algebraic cannot be dropped: with complex constants, (3) holds at every t (take $\nu = \log 2$ and $\gamma = ((\log t - B)(\log t - A) - C')/\log 2$).

The method. Write $\log t = \sum_i a_i \log p_i$ with the exponent vector $a(t) \in \mathbb{Z}^n_{\geq 0}$. The strong six exponentials theorem, together with Baker's theorem, puts linear-algebra constraints on the set of exponent rays $[a]$ with $\lambda_a^2 \in \widetilde{\mathcal{L}}$ (Section 3): no three of them lie in a plane, and no four in a three-dimensional subspace. These two facts say exactly that the integers with primitive exponent vector and $(\log t)^2 \in \widetilde{\mathcal{L}}$ form a multiplicative Sidon set: all products $t_i t_j$ are distinct (Section 4). Erdős [4] proved that such a set in $[1, X]$ has at most $\pi(X) + O(X^{3/4})$ elements.

The primes themselves are compatible with all the constraints the method can see (Section 5), so $\pi(X)$ is the limit of the method: an improvement has to use more than these constraints, for instance information about individual numbers such as $p^{\log p}$. In dimension ≥ 4 the constraints allow much larger configurations: Proposition 6.1 gives 2^r admissible rays in dimension $1 + r(r + 1)/2$. This is why the proof counts through the multiplicative structure of the integers rather than by bounding the number of exceptional rays among y-smooth integers.

Related work. The six exponentials theorem was stated explicitly and proved independently by Lang [7, Ch. II] (see also [8]) and Ramachandra [10]; for its earlier history see [16, §1.3]. Roy [12] proved the strong form used

here, after Waldschmidt's version with coefficients in $\overline{\mathbb{Q}} + \mathcal{L}$ [15, Cor. 2.1]; see [16, §§1.3, 11.6].

The six exponentials theorem already says something about infinitely many integers at once: if β is irrational, then the integers $t \geq 2$ with t^β algebraic generate a multiplicative group of rank at most 2, and of the numbers $2^\pi, 3^\pi, 5^\pi, \ldots$ at most two are algebraic [8, Cor. 1, Cor. 3]. There the exponent β is fixed; in $t^{\log t}$ it moves with t.

The consequences of the strong six exponentials theorem that we know of are statements about a fixed small number of numbers. The closest to ours is Waldschmidt's [18, Cor. 2.4], restated by Diaz [3, Cor. 2 (P), point 1]: if $\lambda_1, \lambda_2, \lambda_3 \in \widetilde{\mathcal{L}} \setminus \overline{\mathbb{Q}}$ and $1, \lambda_2, \lambda_3$ are linearly independent over $\overline{\mathbb{Q}}$, then $\lambda_1\lambda_2$ and $\lambda_1\lambda_3$ do not both lie in $\widetilde{\mathcal{L}}$. Lemma 3.2 below is this corollary written in our notation, with Waldschmidt's proof. With $\lambda_1 = \lambda_2 = \log s$, $\lambda_3 = \log t$ and $(\log st)^2 = (\log s)^2 + 2\log s \log t + (\log t)^2$, it shows that $(\log s)^2$, $(\log t)^2$ and $(\log st)^2$ do not all lie in $\widetilde{\mathcal{L}}$ when $s, t \geq 2$ are multiplicatively independent integers; this is the dimension-2 case of Lemma 3.4 for three integers. The dimension-3 case and the passage to a count over all integers do not appear in [18] or [3].

Counts of exceptions along an integer parameter, obtained by combining the six exponentials theorem with a combinatorial lemma, are also known for the numbers 2^{π^k}. By the six exponentials theorem one of $2^\pi, 2^{\pi^2}, 2^{\pi^3}$ is transcendental. As reported in Waldschmidt's survey [19, §4(a)], Srinivasan [13, 14] showed that at most $O(\sqrt{N})$ of the numbers 2^{π^k}, $1 \leq k \leq N$, are algebraic, and the implied constant was bounded by 2 by Ramachandra and Srinivasan [11] and by $\sqrt{2}$ by Balasubramanian and Ramachandra [2] (announced in [11, §2]). In [11, §2] the six exponentials theorem shows that every nonzero difference occurs at most twice among the exceptional exponents k, and an elementary counting lemma gives the bound. Our argument has the same structure: a transcendence theorem forces a Sidon-type property of the exceptional set, and an extremal bound for such sets finishes the proof; here the property is multiplicative and the bound is Erdős's. We could not find Theorem 1.1, or a density statement for $t^{\log t}$, in the sources we checked.

2. The inputs

Theorem 2.1 (Roy [12, Cor. 2]; see [16, Cor. 11.16])**.** *Let M be a 2×3 matrix with entries in $\widetilde{\mathcal{L}}$. If the rows of M are linearly independent over $\overline{\mathbb{Q}}$ and the columns of M are linearly independent over $\overline{\mathbb{Q}}$, then M has rank 2.*

Equivalently [12, p. 23]: if $x_1, x_2 \in \mathbb{C}$ and $y_1, y_2, y_3 \in \mathbb{C}$ are each linearly independent over $\overline{\mathbb{Q}}$, then one of the six products x_iy_j is not in $\widetilde{\mathcal{L}}$. Waldschmidt's earlier form [15, Cor. 2.1], for matrices with entries in $\overline{\mathbb{Q}} + \mathcal{L}$, does not suffice below: the hypothesis $(\log t)^2 \in \widetilde{\mathcal{L}}$ of Theorem 1.1 puts the entries $\lambda_v\lambda_u$ of the matrix in the proof of Lemma 3.2 in $\widetilde{\mathcal{L}}$, not in $\overline{\mathbb{Q}} + \mathcal{L}$.

Theorem 2.2 (Baker [1])**.** *If $\lambda_1, \dots, \lambda_n \in \mathcal{L}$ are linearly independent over $\mathbb{Q}$, then $1, \lambda_1, \dots, \lambda_n$ are linearly independent over $\overline{\mathbb{Q}}$.*

Theorem 2.3 (Erdős [4, §2])**.** *If $a_1 < \cdots < a_k \le X$ are positive integers such that the products $a_i a_j$ are pairwise distinct, then $k \le \pi(X) + O(X^{3/4})$.*

Erdős calls such sequences B sequences: "the product of any two of their members is different" [4, Introduction]. He later improved the error term to $O(X^{3/4}(\log X)^{-3/2})$ [5], which matches the lower bound of [4, §2], so that the maximum of k is

$$\pi(X) + \Theta\big(X^{3/4}(\log X)^{-3/2}\big);$$

see [6, (4)], where this is stated with references to [4, 5], and [9, (1.1)]. We apply Theorem 2.3 to a set with the strongest property: $t_1 t_2 = t_3 t_4$ implies $\{t_1, t_2\} = \{t_3, t_4\}$ as multisets, so the result applies whether or not products a_i^2 are counted.

3. Consistent configurations

Fix $n \ge 1$ and $\ell_1, \dots, \ell_n \in \mathcal{L}$ linearly independent over $\mathbb{Q}$; in Section 4 and Corollary 3.5, $\ell_i = \log p_i$ with $p_1, \dots, p_n$ the first n primes, which are linearly independent over $\mathbb{Q}$ by unique factorisation. Put $W = \overline{\mathbb{Q}}^n$ and, for $v \in W$, $\lambda_v = \sum_i v_i \ell_i$. By Theorem 2.2

$$\alpha + \lambda_v = 0 \ \text{ with } \alpha \in \overline{\mathbb{Q}},\ v \in W \quad \Longrightarrow \quad \alpha = 0,\ v = 0. \tag{3.1}$$

Let Sym_n be the space of symmetric $n \times n$ matrices over $\overline{\mathbb{Q}}$. For $v, u \in W$ put $v \circ u = \frac{1}{2}(vu^{\mathsf{T}} + uv^{\mathsf{T}})$, and for a subspace $U \subset W$ put $v \circ U = \{v \circ u : u \in U\}$. The linear map $\Lambda : \mathrm{Sym}_n \to \mathbb{C}$, $\Lambda(B) = \ell^{\mathsf{T}} B \ell$, satisfies $\Lambda(v \circ u) = \lambda_v \lambda_u$. Let

$$R = \Lambda^{-1}(\widetilde{\mathcal{L}}) \subset \mathrm{Sym}_n,$$

a $\overline{\mathbb{Q}}$-subspace.

Definition 3.1. A subspace $S \subset \mathrm{Sym}_n$ is *consistent* if it contains no $v \circ U$ with $v \ne 0$ and $\dim U = 2$. A finite set $T \subset \mathbb{P}(W)$ is consistent if $S_T = \mathrm{span}\{v \circ v : [v] \in T\}$ is consistent, and $c(n) = \sup\{|T| : T \subset \mathbb{P}^{n-1}(\overline{\mathbb{Q}}) \text{ consistent}\}$.

For $v \ne 0$ put $K_S(v) = \{u \in W : v \circ u \in S\}$. Then S is consistent if and only if $\dim K_S(v) \le 1$ for every $v \ne 0$. Subspaces of consistent subspaces, and subsets of consistent sets, are consistent.

Lemma 3.2. *R is consistent.*

This is Waldschmidt's [18, Cor. 2.4] applied to $\lambda_v, \lambda_{u_1}, \lambda_{u_2}$; we include the proof, which is his.

Proof. Suppose $v \neq 0$ and u_1, u_2 are linearly independent with $v \circ u_1, v \circ u_2 \in R$, that is, $\lambda_v \lambda_{u_1}, \lambda_v \lambda_{u_2} \in \widetilde{\mathcal{L}}$. Consider

$$M = \begin{pmatrix} 1 & \lambda_{u_1} & \lambda_{u_2} \\ \lambda_v & \lambda_v \lambda_{u_1} & \lambda_v \lambda_{u_2} \end{pmatrix}.$$

Its entries lie in $\widetilde{\mathcal{L}}$, since $\lambda_u \in \overline{\mathbb{Q}}\mathcal{L}$, and it has rank 1. If α times the first row plus β times the second vanishes, then $\alpha + \beta\lambda_v = 0$, so $\beta = 0$ by (3.1) (as $v \neq 0$), and then $\alpha = 0$. If $\alpha + \beta\lambda_{u_1} + \gamma\lambda_{u_2} = 0$, then $\alpha + \lambda_{\beta u_1 + \gamma u_2} = 0$, so $\beta u_1 + \gamma u_2 = 0$ and $\alpha = 0$ by (3.1), whence $\beta = \gamma = 0$. The columns of M are the products of $(1, \lambda_v)^{\mathsf{T}}$ with $1, \lambda_{u_1}, \lambda_{u_2}$, so they are independent too. This contradicts Theorem 2.1. □

Consequently every finite subset of

$$Z = \{[v] \in \mathbb{P}(W) : \lambda_v^2 \in \widetilde{\mathcal{L}}\}$$

is consistent, because $S_T \subset R$ for $T \subset Z$.

Lemma 3.3 (Consistency is intrinsic)**.** *Let $W' \subset W$ be a subspace and $S \subset \operatorname{Sym}^2 W' := \operatorname{span}\{v \circ u : v, u \in W'\}$. If $v \neq 0$ and $\dim K_S(v) \geq 1$, then $v \in W'$ and $K_S(v) \subset W'$.*

Proof. A change of basis P of W acts by $B \mapsto PBP^{\mathsf{T}}$ and respects $\circ$, so we may assume $W' = \operatorname{span}(e_1, \ldots, e_m)$; then S consists of matrices supported on the upper left $m \times m$ block. Let $u \in K_S(v)$, so $v_i u_j + u_i v_j = 0$ whenever $i > m$ or $j > m$. If $v_i \neq 0$ for some $i > m$, then taking $j = i$ gives $u_i = 0$, and then $v_i u_j = -u_i v_j = 0$ for all j, so $u = 0$. Hence if $K_S(v) \neq 0$, then $v \in W'$. In that case pick $j \leq m$ with $v_j \neq 0$; for $i > m$ we get $u_i v_j = -v_i u_j = 0$, so $u \in W'$. □

In particular $c(n)$ is nondecreasing in n, and the consistency of a set of points depends only on the points, not on the ambient space.

Lemma 3.4. *Let T be a consistent set.*

1. *No three points of T lie on a projective line.*
2. *No four points of T lie in a projective plane.*

Hence $c(2) = 2$ and $c(3) = 3$.

Proof. (1) Let $[w_1], [w_2], [w_3]$ be distinct points of $\mathbb{P}(W')$, $\dim W' = 2$. In a basis of W', write $w_i = (1, s_i)$ or $(0, 1)$; the matrices $w_i w_i^{\mathsf{T}}$ are linearly independent (Vandermonde), so they span $\operatorname{Sym}^2 W'$, which contains $w_1 \circ W'$.

(2) Let $[w_1], \ldots, [w_4]$ be distinct points of $\mathbb{P}(W')$, $\dim W' \leq 3$. By (1) we may assume that $\dim W' = 3$ and that no three are collinear. Then w_1, w_2, w_3 is a basis of W' and $w_4 = aw_1 + bw_2 + cw_3$ with $abc \neq 0$; replacing w_1, w_2, w_3 by aw_1, bw_2, cw_3 (the same points), in this basis the points are e_1, e_2, e_3 and $(1, 1, 1)$. The span of $e_1 e_1^{\mathsf{T}}, e_2 e_2^{\mathsf{T}}, e_3 e_3^{\mathsf{T}}$ and the all-ones matrix is

$$S_4 = \{B \in \operatorname{Sym}_3 : B_{12} = B_{13} = B_{23}\}.$$

Take $v = (0, 1, 1)$. For $u \in \overline{\mathbb{Q}}^3$ the off-diagonal entries of $v \circ u$ are $\frac{1}{2}u_1$, $\frac{1}{2}u_1$ and $\frac{1}{2}(u_2 + u_3)$, so $v \circ u \in S_4$ for all u in the plane $U = \{u_1 = u_2 + u_3\}$. Thus $S_4 \supset v \circ U$ with $\dim U = 2$.

For the values of c: frames are consistent (Lemma 5.1), so $c(n) \geq n$, and (1), (2) give the upper bounds for $n = 2, 3$. □

The proof of (2) is the general one in coordinates: four points in general position in $\mathbb{P}^2$ lie on a pencil of conics, the pencil contains a singular member, and its vertex v gives $v \circ U \subset S$ with $\dim U = 2$.

Corollary 3.5. *At least one of the numbers* $(\log 2)^2, (\log 3)^2, (\log 6)^2$*, and at least one of the numbers* $(\log 2)^2, (\log 3)^2, (\log 5)^2, (\log 30)^2$*, is not in* $\widetilde{\mathcal{L}}$*. In particular one of* $2^{\log 2}, 3^{\log 3}, 6^{\log 6}$ *and one of* $2^{\log 2}, 3^{\log 3}, 5^{\log 5}, 30^{\log 30}$ *is transcendental.*

Proof. Take $\ell = (\log 2, \log 3, \log 5)$. The exponent vectors are $e_1, e_2, e_1 + e_2$ (three points on a line) and $e_1, e_2, e_3, e_1 + e_2 + e_3$ (four points in a plane). If all numbers in one group were in $\widetilde{\mathcal{L}}$, these points would lie in Z, so they would form a consistent set by Lemma 3.2, against Lemma 3.4. □

4. Proof of Theorem 1.1

Let $X \geq 2$ and $n = \pi(X)$. For $1 \leq t \leq X$ let $a(t) \in \mathbb{Z}^n_{\geq 0}$ be its exponent vector, so that $\log t = \lambda_{a(t)}$. Call $t \geq 2$ *exceptional* if $(\log t)^2 \in \widetilde{\mathcal{L}}$, that is, $[a(t)] \in Z$. This is a property of the ray, since $\lambda^2_{ka} = k^2\lambda^2_a$. Call t *primitive* if the gcd of the entries of $a(t)$ is 1, and let F_0 be the set of primitive exceptional $t \leq X$. Two distinct primitive vectors in $\mathbb{Z}^n_{\geq 0} \setminus \{0\}$ are not proportional over $\overline{\mathbb{Q}}$ (a proportionality factor would be a positive rational, hence 1), so distinct elements of F_0 give distinct points of Z.

Lemma 4.1. *If* $t_1, t_2, t_3, t_4 \in F_0$ *and* $t_1t_2 = t_3t_4$*, then* $\{t_1, t_2\} = \{t_3, t_4\}$ *as multisets.*

Proof. If the two sides share an element, cancelling it gives the claim. So assume $\{t_1, t_2\} \cap \{t_3, t_4\} = \emptyset$ and write $a_i = a(t_i)$. If $t_1 = t_2$ and $t_3 = t_4$, then $t_1 = t_3$, which is excluded. If $t_1 = t_2$ and $t_3 \neq t_4$, then $2a_1 = a_3 + a_4$, so the three distinct points $[a_1], [a_3], [a_4]$ of Z lie on a line, against Lemma 3.4(1). (The case $t_3 = t_4$, $t_1 \neq t_2$ is symmetric.) Otherwise the four points $[a_i]$ are distinct and $a_1 + a_2 = a_3 + a_4$, so they lie in a projective plane, against Lemma 3.4. □

Proof of Theorem 1.1. By Lemma 4.1 the products t_it_j of elements of $F_0 \subset [2, X]$ are pairwise distinct, so $|F_0| \leq \pi(X) + O(X^{3/4})$ by Theorem 2.3. An exceptional $t \leq X$ that is not primitive is a perfect power t_0^k with $k \geq 2$, and there are at most $\sum_{2 \leq k \leq \log_2 X} X^{1/k} = O(\sqrt{X})$ of these. Hence $E(X) \leq \pi(X) + O(X^{3/4})$. □

Remark 4.2. With the upper bound of [5] (see [6, (4)]) in place of Theorem 2.3, the same proof gives $E(X) \le \pi(X) + O(X^{3/4}(\log X)^{-3/2})$, since the perfect powers contribute only $O(\sqrt{X})$.

Remark 4.3. Only Lemma 3.4, that is, consistency in dimensions 2 and 3, enters the proof. Part (1) alone does not give a bound $\pi(X) + o(\pi(X))$. It says that no three elements of F_0 have linearly dependent exponent vectors, and the integers $pq \le X$ with primes $p < q$, of which there are $\asymp X \log\log X/\log X$, have this property. Indeed, in a relation $\alpha a_1 + \beta a_2 + \gamma a_3 = 0$ between the exponent vectors of three of them, a prime dividing only one of the three forces its coefficient to vanish, and then the other two vanish because two distinct such vectors are not proportional; if there is no such prime, the three integers are pq, qr, pr, whose exponent vectors are linearly independent (the determinant is 2).

Remark 4.4. The same bound holds for the other branches of the logarithm: for a fixed integer $k \neq 0$,

$$\#\{2 \le t \le X : \ (\log t + 2\pi i k)^2 \in \widetilde{\mathcal{L}}\} \le \pi(X) + O(X^{3/4}).$$

To see this, apply Section 3 to the $n+1$ logarithms $2\pi i, \log p_1, \dots, \log p_n$, which are linearly independent over $\mathbb{Q}$ (compare real and imaginary parts), and to the vectors $b(t) = (k, a(t)) \in \mathbb{Z}^{n+1}$, so that $\log t + 2\pi i k = \lambda_{b(t)}$. Since $k \neq 0$, distinct t give non-proportional vectors $b(t)$, so no reduction to primitive vectors is needed, and $t_1 t_2 = t_3 t_4$ gives $b(t_1) + b(t_2) = b(t_3) + b(t_4)$. The proof of Lemma 4.1, with b in place of a, shows that the set of all such $t \le X$ is a multiplicative Sidon set, and Theorem 2.3 applies.

5. What the method cannot do

Lemma 5.1. *If $w_1, \dots, w_m \in W$ are linearly independent, then the set $\{[w_1], \dots, [w_m]\}$ is consistent.*

Proof. By Lemma 3.3 and a change of basis we may assume $w_i = e_i$ and $m = n$, so S is the space of diagonal matrices. Let $v \neq 0$ and $u \in K_S(v)$, so $v_i u_j + u_i v_j = 0$ for $i \neq j$. Fix i with $v_i \neq 0$. Then $u_j = -u_i v_j / v_i$ for every $j \neq i$, so u is determined by u_i and $\dim K_S(v) \le 1$. □

Applied to the coordinate rays, Lemma 5.1 says that the configuration in which every prime is exceptional, $(\log p)^2 \in \widetilde{\mathcal{L}}$ for all $p \le X$, satisfies every constraint that Lemma 3.2 and linear algebra impose. So no argument that uses only the consistency of the exceptional rays can give a bound below $\pi(X)$, and Theorem 1.1 is sharp for this method up to the error term. A bound $o(\pi(X))$ needs input beyond the consistency of the exceptional rays, for instance information about the individual numbers $p^{\log p}$.

The method decides no individual instance: $2^{\log 2}$, $\alpha^{\log \alpha}$ and e^{π^2} remain open. The same holds for products. In Sym_2, with $\ell = (\log 2, \log 3)$, consider

the involution $\sigma(i,j)=(j,i)$ and

$$S_\sigma=\operatorname{span}\{u\circ\sigma(u):u\in\overline{\mathbb{Q}}^2\}=\left\{\begin{pmatrix}a&b\\b&a\end{pmatrix}\right\}.$$

If $v\circ e_1, v\circ e_2\in S_\sigma$, then the diagonal entries give $v_1=0$ and $v_2=0$; so S_σ is consistent, and by Lemma 3.3 it is consistent in every Sym_n. It contains $e_1\circ e_2$, $(1,1)\circ(1,1)$ and $(1,-1)\circ(1,-1)$. Thus the relations

$$\log 2\cdot\log 3\in\widetilde{\mathcal{L}},\qquad(\log 6)^2\in\widetilde{\mathcal{L}},\qquad(\log(2/3))^2\in\widetilde{\mathcal{L}},$$

and more generally $\log(2^i3^j)\log(2^j3^i)\in\widetilde{\mathcal{L}}$ for all i,j, are jointly compatible with Lemma 3.2: strong six exponentials and Baker, used through $\overline{\mathbb{Q}}$-linear relations among the products $\lambda_u\lambda_w$, cannot exclude any of them. (By Corollary 3.5 one of $(\log 2)^2,(\log 3)^2,(\log 6)^2$ is not in $\widetilde{\mathcal{L}}$; consistently, $e_1\circ e_1\notin S_\sigma$.)

6. Large consistent configurations

Lemma 3.4 bounds consistent sets in dimensions 2 and 3. In higher dimension they can be large. We include this to explain why the proof of Theorem 1.1 does not proceed by bounding $c(n)$: among y-smooth integers $t\le X$, a bound of the form $c(\pi(y))\log_2 X$ for the number of exceptional t is useless once $y\ge(\log X)^{2+\varepsilon}$, by the following result.

Proposition 6.1. *For $r\ge 1$ let $n_r=1+r(r+1)/2$ and let $T_r\subset\mathbb{P}^{n_r-1}(\mathbb{Q})$ be the set of 2^r points*

$$\big(1,\ (x_i)_{1\le i\le r},\ (x_ix_j)_{1\le i<j\le r}\big),\qquad x\in\{\pm1\}^r.$$

Then T_r is consistent. Hence $c(n)\ge 2^{\lfloor(\sqrt{8n-7}-1)/2\rfloor}$ for all $n\ge1$, and $c(n)$ is not bounded by any polynomial in n.

The 2^r points are distinct, since their coordinates x_i differ. The bound follows from the first statement because $8n_r-7=(2r+1)^2$ and c is non-decreasing. The proof of consistency uses an exhaustive computer check for $2\le r\le 5$; the cases $r\ge6$ are reduced to $r=5$ by hand.

Hankel structure. Let $\mathcal{A}\subset\mathbb{F}_2^r$ contain 0, let $W=\overline{\mathbb{Q}}^{\mathcal{A}}$, and for $x\in\mathbb{F}_2^r$ let $t_x=(\chi_x(a))_{a\in\mathcal{A}}$ with $\chi_x(a)=(-1)^{x\cdot a}$, and put $T_{\mathcal{A}}=\{[t_x]:x\in\mathbb{F}_2^r\}$. For $\mathcal{A}_r=\{S\subset\{1,\dots,r\}:|S|\le2\}$ (subsets as vectors) the points of $T_{\mathcal{A}_r}$ are the 2^r points of T_r.

Lemma 6.2. *$S_{T_{\mathcal{A}}}=\operatorname{span}\{t_xt_x^{\mathsf T}\}$ is the space $H_{\mathcal{A}}$ of symmetric $\mathcal{A}\times\mathcal{A}$ matrices whose (a,b) entry depends only on $a+b$.*

Proof. $t_xt_x^{\mathsf T}=[\chi_x(a+b)]_{a,b}$. Every $B\in H_{\mathcal{A}}$ has the form $[f(a+b)]$ for a function f on $\mathbb{F}_2^r$, and the characters span all such functions. □

Reduction to a graph. Fix $v \neq 0$, let $P = \operatorname{supp} v$ and $Y = \mathcal{A} \setminus P$, and let $K(v) = \{u : v \circ u \in H_{\mathcal{A}}\}$. On the diagonal class $a + b = 0$ the condition says $v_a u_a = c$ is constant. If $Y = \emptyset$, then $u_a = c/v_a$ and $\dim K(v) \leq 1$. If $Y \neq \emptyset$, then $c = 0$ and u vanishes on P; the (a, b) entry of $v \circ u$ is $\frac{1}{2} v_a u_b$ for $a \in P$, $b \in Y$, and 0 when a, b are on the same side. Group the unordered pairs $\{a, b\}$, $a \neq b$, into classes by $a + b$, and call a pair *cross* if it meets both P and Y. If a class contains a pair with both elements on the same side, all its cross values $v_a u_b$ vanish, and we say that the Y-ends b of its cross pairs are *killed*. If a class consists of cross pairs $(a, b), (a', b'), \ldots$ only, then $v_a u_b = v_{a'} u_{b'} = \ldots$, and we join $b, b', \ldots$ by edges. On each component of the resulting graph on Y, u is determined by one value, since the coefficients v_a are nonzero, and it vanishes on a component containing a killed vertex. So $\dim K(v) \leq \beta(P)$, the number of components without a killed vertex (*alive* components; the others are *dead*). Hence:

Lemma 6.3. *If $\beta(P) \leq 1$ for every partition $\mathcal{A} = P \sqcup Y$ with $P, Y \neq \emptyset$, then $T_{\mathcal{A}}$ is consistent.*

The finite check. For $\mathcal{A}_r$ with $r = 2, 3, 4, 5$ ($|\mathcal{A}_r| = 4, 7, 11, 16$) an exhaustive computer enumeration of all $2^{|\mathcal{A}_r|} - 2$ partitions gives $\beta(P) \leq 1$. (The script `anc/toric_check.py` accompanying this note runs in a few seconds; the case $r = 1$ is immediate, since T_1 is a frame.)

From $r = 5$ to all r. Let $r \geq 6$ and fix a partition of $\mathcal{A}_r$. For $R \subset \{1, \ldots, r\}$ the set $\mathcal{A}_R$ of subsets of R of size ≤ 2 is contained in $\mathcal{A}_r$, and each class of $\mathcal{A}_R$ is contained in the class of $\mathcal{A}_r$ with the same sum. Hence a vertex killed for the restricted partition is killed for $\mathcal{A}_r$, and an edge of the graph of $\mathcal{A}_R$ is either an edge for $\mathcal{A}_r$ or has both ends killed for $\mathcal{A}_r$. Following a path, we get: (i) two vertices in one component for $\mathcal{A}_R$ lie in one component for $\mathcal{A}_r$, or both lie in dead components for $\mathcal{A}_r$; (ii) if b lies in an alive component for $\mathcal{A}_r$, then its component for $\mathcal{A}_R$ is alive.

Let $b, b' \in Y$ lie in alive components for $\mathcal{A}_r$. If some $a \in P$ has $|a \cup b \cup b'| \leq 5$, choose $R \supset a \cup b \cup b'$ with $|R| = 5$. The restricted partition is nontrivial ($a \in P$, $b \in Y$), and $\mathcal{A}_R \cong \mathcal{A}_5$, so by (ii) and the finite check b, b' lie in one component for $\mathcal{A}_R$; by (i) they lie in one component for $\mathcal{A}_r$, since their components for $\mathcal{A}_r$ are alive. Otherwise $|b \cup b'| \geq 4$, so b, b' are disjoint 2-sets and every $a \in P$ is a 2-set disjoint from both; $P \neq \emptyset$, say $b = \{1, 2\}$, $b' = \{3, 4\}$, $a = \{5, 6\} \in P$. The class of $\{1, 2, 5, 6\}$ contains the cross pair $(\{5, 6\}, \{1, 2\})$ and the pair $(\{1, 5\}, \{2, 6\})$, whose elements meet b and so lie in Y. So b is killed, a contradiction. Hence $\beta(P) \leq 1$, and Lemma 6.3 proves Proposition 6.1.

Remark 6.4. The enumeration was also run at $r = 6$ ($4.2 \cdot 10^6$ partitions), with the same result. Proposition 6.1 gives $c(n) \geq 2^{(1+o(1))\sqrt{2n}}$. We do not discuss upper bounds for $c(n)$, $n \geq 4$, here; whether $c(n)$ grows exponentially is open.

Provenance

The results were found in an exploratory search for density versions of the strong six exponentials theorem. Each proof was checked line by line in separate referee passes, and the finite enumeration of Section 6 was rerun independently, including the case $r = 6$. Theorem 1.1 uses the strong six exponentials theorem in Roy's form [12, Cor. 2]; the logarithms $\log t$ are real throughout, except in Remark 4.4, and the constants in Corollary 1.2 are algebraic.

Use of AI tools. This work was carried out with substantial assistance from AI systems (Anthropic's Claude models), which were used to search the literature, to develop and check the proofs, to run the computer enumeration and to draft this text. The author directed the project and is responsible for its content.

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