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Sharp finite cases of the Erdős–Surányi interval-product problem: $g(4) = 5$ and $g(5) = 6$

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Abstract

Let $g(n)$ be the Erdős–Surányi function of Erdős Problem #708: the least k such that, for every set A of n distinct integers at least 2 and every interval I of $\max A$ consecutive integers, some set of at most k elements of I has product divisible by $\prod_{a\in A} a$. We prove

$$g(4) = 5 \qquad \text{and} \qquad g(5) = 6.$$

The lower bound $g(n) \geq n + 1$ for $n \geq 3$ is due to Erdős and Surányi (1959). Our contribution is the matching universal upper bounds $g(4) \leq 5$ and $g(5) \leq 6$; to our knowledge these are the first exact values beyond $g(3) = 4$. The four-modulus proof combines a Hall-matching reduction with gcd/lcm collision repair and an anchored proper-divisor lemma. The five-modulus proof classifies a maximal antichain-admissible assignment of the moduli to interval multiples by its occupancy shape; the hardest shapes are a squarefree four-prime exception and a $(4, 1)$ configuration controlled by a propagation invariant. Both values, in the exact-cardinality reading of the original problem and in the at-most reading used by later work, are machine-checked in Lean 4. The formalization is registered in the Palomar registry as PALOMAR-2026-11-19-000019. The general question whether $g(n) \leq 2n$ remains open.

Keywords: Erdős Problem 708; Erdős–Surányi function; interval product-cover; Hall's theorem; divisibility antichains; Lean 4; formal verification.
MSC 2020: 11A05 (primary); 11B75, 05D15 (secondary).

Status of the results. Theorem 1.6 is formally verified: four Lean declarations stating it were mechanically checked at a pinned public commit, and the formalization is registered as PALOMAR-2026-11-19-000019 (Section 7). Sections 3–6 give a human-readable account of the same mathematics. Some auxiliary lemmas are organized differently from their Lean counterparts (Table 3), and the written text has not been refereed. The provenance of the work, including the use of AI systems, is disclosed at the end of the paper.

1 The problem and two cardinality conventions

For an integer L and a positive integer M, write

$$I(L, M) = \{L, L + 1, \ldots, L + M - 1\}.$$

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For a finite nonempty set $A \subseteq \{2, 3, \ldots\}$, put $M = \max A$ and $P(A) = \prod_{a \in A} a$. An *interval product-cover* of A in $I(L, M)$ is a subset $B \subseteq I(L, M)$ with

$$P(A) \mid \prod_{b \in B} b.$$

All products are integer products. Intervals may start at any integer; if $0 \in I(L, M)$, then $\{0\}$ is a product-cover. Erdős's 1992 formulation [3] uses positive starts. Reflecting an interval through 0 changes products only by a sign, so the two conventions give the same values.

Definition 1.1 (At-most reading)**.** Let $C(n, k)$ be the statement: for every n-element $A \subseteq \{2, 3, \ldots\}$ and every integer L there is a product-cover $B \subseteq I(L, \max A)$ with $|B| \leq k$. Put

$$g(n) = \min\{k : C(n, k)\}.$$

Here $n \geq 1$, and the set of admissible k is nonempty; for example, Chen proves $g(n) \leq 12n$ [2].

Definition 1.2 (Exact reading)**.** Let $C^{=}(n, k)$ be the same statement with $|B| = k$ in place of $|B| \leq k$. When some k satisfies $C^{=}(n, k)$, let $g^{=}(n)$ be the least such k.

The problem as printed in the tracker uses the exact condition [1]. Later work uses the at-most condition [2, 6, 10]. The two differ in general. For $A = \{2, 3, \ldots, n + 1\}$ every relevant interval has only $n + 1$ elements, so $C^{=}(n, k)$ fails for every $k > n + 1$. Hence, whenever $g(n) > n + 1$, no k satisfies $C^{=}(n, k)$ and $g^{=}(n)$ is undefined. By the asymptotic lower bound $g(n) \geq (2 - o(1))n$ of Erdős and Surányi [4], this happens for all large n.

Lemma 1.3 (Padding)**.** *Let $n \geq 1$ and $k \leq n + 1$. Then $C(n, k)$ holds if and only if $C^{=}(n, k)$ holds.*

Proof. An exact cover of size k is a cover of size at most k. Conversely, n distinct integers that are at least 2 have maximum $M \geq n + 1 \geq k$, so $I(L, M)$ has at least k elements. A cover B with $|B| < k$ can be enlarged by distinct elements of $I(L, M) \setminus B$ until $|B| = k$. Multiplying by further integer factors, including 0, preserves divisibility. □

Corollary 1.4. *If $g(n) \leq n + 1$, then $g^{=}(n)$ is defined and $g^{=}(n) = g(n)$.*

Proof. By Lemma 1.3, $C^{=}(n, g(n))$ holds, and for $k < g(n)$ the statement $C^{=}(n, k)$ fails because $C(n, k)$ fails. □

Remark 1.5. The exact reading already breaks down at $n = 6$. Chen's $K_{2,t}$ construction gives $g(6) \geq 8$ [2, 10]. By the observation above, $C^{=}(6, k)$ therefore fails for every k. This is why the formal statement of the general problem [6], and any future treatment of $g(6)$, must use the at-most reading.

Theorem 1.6 (Exact finite values)**.** *Let $M \geq 2$, and let I be any interval of M consecutive integers. For any four distinct integers in $[2, M]$, some set of at most five elements of I has a product divisible by the product of the four integers. For any five distinct integers in $[2, M]$, at most six elements suffice. Both bounds are optimal. Consequently*

$$g(4) = 5, \qquad g(5) = 6,$$

and, by Corollary 1.4, also $g^{=}(4) = 5$ and $g^{=}(5) = 6$.

The theorem allows the moduli to lie in $[2, M]$ without requiring that one of them equals M; this is slightly stronger than $C(4, 5)$ and $C(5, 6)$, and it is the form proved in Lean. Optimality follows from Theorem IV of Erdős and Surányi, which gives $g(n) \geq n + 1$ for every $n \geq 3$ [4]. We also give explicit extremal instances in Propositions 4.1 and 6.1.

Example 1.7 (Why six can be necessary)**.** Take $A = \{5, 10, 12, 15, 20\}$, so $M = 20$, and the window $I = \{51, \ldots, 70\}$. The target is $P(A) = 2^5 \cdot 3^2 \cdot 5^4$. Only four numbers in the window are divisible by 5, namely $55, 60, 65, 70$, and none is divisible by 25, so all four must be chosen to supply 5^4. Together they supply only $2^3 \cdot 3$ of the remaining factor $2^5 \cdot 3^2$. A single further number would have to be divisible by $2^2 \cdot 3 = 12$, and the only multiple of 12 in the window is 60, which is already used. So five numbers never suffice, while six do: add $63 = 3^2 \cdot 7$ and $64 = 2^6$. This illustrates the general difficulty: the moduli compete for few available multiples, and the forced choices do not supply enough of the other primes. Proposition 6.1 records this instance.

2 Context, attribution and organization

Gallai observed $g(2) = 2$. Erdős and Surányi proved $g(3) = 4$, the general lower bound $g(n) \geq n+1$ for $n \geq 3$, and the asymptotic lower bound $g(n) \geq (2-o(1))n$ [4]. Erdős asked whether $g(n) \leq (2+o(1))n$, or even $g(n) \leq 2n$ [1, 3]. General upper bounds have since improved substantially. Chen proved a Lean-verified bound $g(n) \leq 12n$ and showed that $2n$ suffices when the largest modulus is sufficiently large compared with n [2]. Further unformalized improvements are reported in public research notes [10]. On the lower side, the same notes report that Chen's $K_{2,t}$ family extends to every size, giving $g(n) \geq \lfloor 3n/2 \rfloor - 1$ for $n \geq 4$ [10]. For $n = 4, 5$ this coincides with $n + 1$, and from $n = 6$ on it exceeds $n + 1$. None of these bounds determines $g(4)$ or $g(5)$.

What is new. Our novelty claim is limited to the universal upper bounds $g(4) \leq 5$ and $g(5) \leq 6$, their proofs, and their formalization. The lower bounds are due to Erdős and Surányi. The five-modulus extremal instance is taken from Chen's project. The four-modulus instance adds the modulus 139 to a three-modulus instance found there [2]. A public survey dated 12 November 2026 describes $g(4) = 5$ as a conjecture, with the exact value of $g(4)$ open [10]. As read on 19 November 2026, the tracker lists no exact value beyond $g(3)$ [1]. These sources show the state of public knowledge before this work; they do not establish priority, and our literature search was not exhaustive.

Organization. Section 3 proves $g(4) \leq 5$, and Section 4 gives a four-modulus extremal instance. Section 5 proves $g(5) \leq 6$, and Section 6 gives a five-modulus extremal instance. Section 7 describes the Lean formalization, its statements, and its correspondence with the written lemmas. Section 8 discusses limitations and open problems.

Conventions. Throughout, $I = I(L, M)$ is an interval of M consecutive integers. Since $\{0\}$ is a cover whenever $0 \in I$, we assume $0 \notin I$. The original integers to be covered are called *moduli* or *originals*. A *site* is an element of I, and a set of sites *pays* an integer t if t divides the product of its elements. We use repeatedly that every interval of M consecutive integers contains at least $\lfloor M/d \rfloor$ multiples of $d \geq 1$. In particular, if $F \subseteq I$ and $M \geq (|F| + 1)d$, then some multiple of d lies in $I \setminus F$.

3 Four moduli: matching, gcds and anchors

Strategy. Each modulus $a \le M$ has at least one multiple in I. If the four moduli can be placed on four distinct multiples, those four sites already pay the product. The difficulty is collisions between moduli whose available multiples coincide. Two devices repair collisions while keeping the total at five sites. First, if a and b share a site, that site can pay $\operatorname{lcm}(a,b)$, which leaves only $\gcd(a,b)$ unpaid. The gcd is smaller, so it has more multiples in I, and one of them can be found elsewhere; if that one is occupied too, the same step repeats (Lemma 3.1). Second, if several moduli are confined to a single site x, we write each modulus as a prime times a proper divisor. The site x pays the primes, and the proper divisors, which are at most $M/2$, are placed at other sites (Lemmas 3.2 and 3.4). When no three moduli can be placed on distinct multiples, Hall's theorem shows that either three moduli share a single available site, or all four have only two available sites between them. Lemmas 3.4, 3.5 and 3.6 treat these configurations, and the end of the section assembles the cases.

In this section $a_1, \dots, a_4$ are distinct integers in $[2, M]$. For a modulus a, the set $I \cap a\mathbb{Z}$ is its *candidate set*. A point $x \in I$ is an *anchor* for a if $a \mid x$, and a *unique anchor* if $I \cap a\mathbb{Z} = \{x\}$. A modulus with a unique anchor exceeds $M/2$, because an interval of M consecutive integers contains at least two multiples of any $d \le M/2$.

Lemma 3.1 (One-defect repair)**.** *Let $a_1, \dots, a_t$ be distinct positive integers at most M. If $t-1$ of them have pairwise distinct multiples in I, then at most t sites pay $a_1 \cdots a_t$.*

Proof. Choose a multiple of the remaining modulus. If that site is unused, we have t distinct sites and we are done. Otherwise two moduli a, b share a site. Let that site pay $\operatorname{lcm}(a,b)$, and carry $\gcd(a,b)$ as a residual obligation. The identity $\operatorname{lcm}(a,b)\gcd(a,b) = ab$ preserves the required product. At every later stage, let S be the set of originals absorbed into the collision chain. The carried residual is $d = \gcd(S)$, and $|S| - 1$ sites are involved. The originals in S are $|S|$ distinct positive multiples of d that are at most M, so $|S|d \le M$. Hence I contains at least $\lfloor M/d \rfloor \ge |S|$ multiples of d, and one of them lies outside the involved sites. If that site is unused, it pays d and we are done. Otherwise it carries an original not yet in S; absorb that original and continue. Since $|S|$ strictly increases, the procedure terminates. □

Lemma 3.2 (Proper divisors off one anchor)**.** *Let $1 \le t \le 4$, and let $a_1, \dots, a_t$ be distinct integers in $[2, M]$. Let r_i be a proper positive divisor of a_i, and suppose $x \in I$ is divisible by every r_i. Then there are pairwise distinct sites $b_1, \dots, b_t \in I \setminus \{x\}$ with $r_i \mid b_i$.*

Proof. We verify Hall's condition [5] for the candidate families $(I \setminus \{x\}) \cap r_i\mathbb{Z}$. Every subfamily satisfies the same hypotheses, so it is enough to show that the union of all t families has at least t elements. Translate by $-x$; this preserves divisibility by every r_i because $r_i \mid x$. Reflect if necessary so that the nonnegative side of the translated interval reaches at least $\lfloor M/2 \rfloor$. Since $r_i \le a_i/2 \le M/2$, each family contains the positive point r_i.

If a label r occurs s times, the corresponding originals are s distinct proper multiples of r, so $M \ge (s+1)r$ and the interval contains at least s nonzero multiples of r. This settles the patterns in which all labels are equal. If all labels are distinct, the positive points r_i are already t distinct candidates. The mixed patterns are:

- (r, r, s). A deficient union would be exactly $\{r, s\}$. The second nonzero multiple of r forces $s = 2r$. Then $M \ge 2s = 4r$ gives a third nonzero multiple of r, a contradiction.
- (r, r, s, t). A deficient union would be exactly $\{r, s, t\}$, so all candidates are positive. The next two multiples of r force, after interchanging s and t, $s = 2r$ and $t = 3r$. Then $M \ge 2t = 6r$ gives further multiples, a contradiction.

- (r, r, r, s). Deficiency forces exactly three nonzero multiples of r, so $M < 5r$. Since $2s \le M$ and $s \ne r$, we get $s = 2r$. The three distinct originals for r must be $2r, 3r, 4r$, while the original for s is $4r$, contradicting distinctness.
- (r, r, s, s) with $r < s$. If there are at least three nonzero multiples of r, then s must be one of them, and $M \ge 3s \ge 6r$ gives more multiples. Otherwise $M < 4r$, so $r < s < 4r/3$ because $M \ge 3s$. Write $r = du$ and $s = dv$ with $\gcd(u, v) = 1$ and $u < v$. If $\operatorname{lcm}(r, s) = rv < 4r$, then $v \le 3$ and so $v/u \ge 3/2$, contradicting $s/r < 4/3$. Thus $\operatorname{lcm}(r, s) \ge 4r > M$, the two candidate families are disjoint, and each has two elements.

Hence the union always has at least t elements, and Hall's theorem gives distinct representatives. □

Lemma 3.3 (Common gcds in the upper half)**.** *Two distinct integers in $(M/2, M]$ have gcd at most $M/3$. Three distinct integers in this range have gcd at most $M/5$.*

Proof. After division by the common gcd d, the quotients are distinct integers in $(M/(2d), M/d]$. If $M/d < 3$ this range contains at most one integer, and if $M/d < 5$ it contains at most two. □

Lemma 3.4 (Four moduli at one anchor)**.** *If four distinct moduli have the same unique anchor x, then five sites suffice.*

Proof. All four moduli exceed $M/2$, so they form a divisibility antichain. Let $Q = \operatorname{lcm}(a_1, \dots, a_4)$, so $Q \mid x$. Connect a_i to the prime units (p, j) of Q with $j \le v_p(a_i)$ (prime units are defined before Lemma 5.7). By Lemma 5.7, Hall's theorem assigns distinct units to the four moduli. Let p_i be the prime of the unit assigned to a_i; then $\prod_i p_i \mid Q$. Put $r_i = a_i/p_i$. By Lemma 3.2, there are four distinct multiples of the proper divisors r_i off x. Their product pays $\prod_i r_i$, and x pays $\prod_i p_i$. Together the five sites pay $a_1a_2a_3a_4$. □

Lemma 3.5 (Triple-anchor assembly)**.** *Suppose three of the moduli, a, b, c, have the same unique anchor x, and the fourth has an anchor $y \ne x$. Then five sites suffice.*

Proof. Write $h = \gcd(a, b, c)$ and write the three pairwise gcds as hu, hv, hw. Comparing prime valuations shows that u, v, w are pairwise coprime and

$$abc = \operatorname{lcm}(a, b, c)\, h^2uvw. \tag{1}$$

Each pairwise gcd is at most $M/3$ by Lemma 3.3, so it has a multiple outside $\{x, y\}$; choose one for each. Three distinct choices pay h^3uvw. If exactly two sites occur, the shared site pays the lcm of two of the labels and the other site pays the third, which gives h^2uvw. If all three choices coincide, that site pays $\operatorname{lcm}(hu, hv, hw) = huvw$, and since $h \le M/5$ there is another multiple of h outside the three occupied sites. In every case at most three sites outside $\{x, y\}$ pay h^2uvw. The anchor x pays $\operatorname{lcm}(a, b, c)$ and y pays the fourth modulus, so by (1) at most five sites suffice. □

Lemma 3.6 (Two-anchor completion)**.** *Suppose the union of the candidate sets of the four moduli is exactly $\{x, y\}$, and no three moduli have a common unique anchor. Then five sites suffice.*

Proof. Put $D = |y - x|$. A modulus dividing both x and y whose candidate set is $\{x, y\}$ must equal D; otherwise a multiple strictly between x and y would lie in I. Call such a modulus a *bridge*. Since the moduli are distinct, there is at most one bridge.

No bridge. Then every modulus has a unique anchor, and since no three share one, two moduli a, b are anchored at x and two moduli c, d at y. The gcds $r = \gcd(a, b)$ and $s = \gcd(c, d)$ are at most $M/3$, so each has a multiple off $\{x, y\}$. If the chosen sites differ, they pay rs, and with x and

y (paying $\operatorname{lcm}(a,b)$ and $\operatorname{lcm}(c,d)$) this covers $abcd$. If they coincide, that site pays $\operatorname{lcm}(r,s)$. The gcd $q=\gcd(r,s)$ divides four distinct originals, so $M\ge 4q$, and a multiple of q exists outside the three used sites.

One bridge. Write the bridge as D, the moduli anchored only at x as a,b, and the modulus anchored only at y as c. Put $r=\gcd(a,b)$ and $s=\gcd(c,D)$. If both have off-anchor multiples, the argument of the previous paragraph applies: x pays $\operatorname{lcm}(a,b)$, y pays $\operatorname{lcm}(c,D)$, and a collision is repaired by a multiple of $\gcd(r,s)$, which divides four distinct originals. The remaining obstruction is that s has no multiple outside $\{x,y\}$. Since $s\mid D$, this forces $s=D$, so $D\mid c$. The interval contains exactly two multiples of D, so $M<3D$, and $c\neq D$ gives $c=2D$. Neither a nor b is divisible by D, since the only candidates would be D and $2D$. Put $u=\gcd(a,D)$ and $v=\gcd(b,D)$. Then $u,v\le D/2\le M/4$ (as $2D=c\le M$), $r\le M/3$, and comparing valuations gives

$$\frac{abD}{\operatorname{lcm}(a,b,D)}=r\,\operatorname{lcm}(u,v).$$

The anchors pay $\operatorname{lcm}(a,b,D)\cdot c$, so distinct multiples of r, u and v off $\{x,y\}$ suffice. Their off-anchor candidate families have at least 1, 2 and 2 elements. A failure of Hall's condition would force the families of u and v to coincide in exactly two points. Since u and v divide x and y, their complete sets of interval multiples would coincide, forcing $u=v$. Then u divides the three upper-half originals a,b,c, so $u\le M/5$ by Lemma 3.3, which gives at least three off-anchor multiples of u, a contradiction. Hall's theorem gives the representatives. □

Proof of $g(4)\le 5$. Form the bipartite graph joining each modulus to its candidate set. Every modulus has a neighbor. A matching of size four gives four sites paying the product, and a matching of size three is completed by Lemma 3.1. If the maximum matching has size at most two, the deficiency version of Hall's theorem gives a set S of moduli with $|S|-|N(S)|\ge 2$. Since $N(S)\neq\emptyset$, either three moduli have a common unique anchor, or all four have candidates in a set of at most two sites. In the first case, use Lemma 3.4 if the fourth modulus has no other anchor and Lemma 3.5 otherwise. In the second case, use Lemma 3.4 if the candidate union is a single site, Lemma 3.5 if three moduli share a unique anchor, and Lemma 3.6 otherwise. These cases are exhaustive. □

4 A sharp four-modulus instance

Proposition 4.1. *Let*

$$A_4=\{77,91,139,143\},\qquad I_4=\{5935,5936,\dots,6077\}.$$

No subset of I_4 with at most four elements has product divisible by $P(A_4)$, whereas a five-element subset does.

Proof. Here $|I_4|=143=\max A_4$ and $P(A_4)=7^2\cdot 11^2\cdot 13^2\cdot 139$. The only element of I_4 divisible by two of $7,11,13$ is $6006=2\cdot 3\cdot 7\cdot 11\cdot 13$, since $77\cdot 77=5929$, $77\cdot 79=6083$, $91\cdot 65=5915$, $91\cdot 67=6097$, $143\cdot 41=5863$ and $143\cdot 43=6149$ all lie outside I_4. No element of I_4 is divisible by 13^2, because $13^2\cdot 35=5915$ and $13^2\cdot 36=6084$. A cover must raise the exponents of 7, 11 and 13 to two. If 6006 is chosen, three further distinct sites are needed for one more factor each of 7, 11 and 13. If 6006 is not chosen, two sites are needed for 13^2, and the sites supplying 7 and 11 are different from these and from each other. Either way at least four sites divisible by 7, 11 or 13 are used. The only multiple of 139 in I_4 is $5977=43\cdot 139$, which is coprime to $7\cdot 11\cdot 13$. So it is a fifth site, and at least five sites are needed. Conversely

$$B_4=\{5936,5940,5941,5977,6006\}=\{7\cdot 848,\ 11\cdot 540,\ 13\cdot 457,\ 139\cdot 43,\ 6006\}$$

is a five-element cover. □

5 Five moduli: the universal six-site theorem

Throughout this section $m_1, \dots, m_5$ are distinct integers in $[2, M]$.

Strategy. With five moduli a matching argument alone is not enough, so we start from an *anchor assignment*: every modulus is placed on one of its multiples, and the moduli sharing a site must form a divisibility antichain. This restriction is what makes the gcd bounds of Lemma 5.2 available. We take an assignment with as many occupied sites as possible. With four or five occupied sites the collision repair of Lemma 3.1 finishes at once. Otherwise the moduli are grouped on at most three sites, and the group sizes give one of the shapes $(3,1,1)$, $(2,2,1)$, $(3,2)$, $(4,1)$ or (5) (Table 1). In the shapes $(3,1,1)$, $(2,2,1)$ and $(3,2)$, each anchor pays the lcm of its group. What remains are residual factors, such as the gcd of two moduli in the same group. These are small enough to have spare multiples, and a few additional sites pay them while keeping the total at six. In the two most concentrated shapes, $(4,1)$ and (5), the shared anchor instead pays one prime from each of its moduli, and the remaining proper divisors are placed elsewhere, as in Section 3. These shapes are the hardest. With all five moduli on one site, a single exceptional family, five of the six pairwise products of four primes, needs separate treatment (Lemma 5.9). With four moduli on one site, a carried gcd may have to pass through several repair sites before it finds a free one (Lemma 5.10).

5.1 Antichain anchor assignments

An *anchor assignment* is a map f from the five moduli to sites with $m_i \mid f(m_i)$. It is *admissible* if the moduli sent to any one site form a divisibility antichain. Incomparable moduli may share a site, but comparable moduli may not.

Lemma 5.1 (A maximal admissible assignment)**.** *Admissible assignments exist. Choose one with the largest possible number of occupied sites. If at least four sites are occupied, five sites pay $\prod_i m_i$. Otherwise the nonempty fibers have one of the shapes*

$$(3,1,1), \quad (2,2,1), \quad (3,2), \quad (4,1), \quad (5).$$

Proof. Assign the moduli in decreasing order. When assigning d, forbid an occupied multiple of d only if it carries an earlier modulus that is a multiple of d. There are at most $\lfloor M/d \rfloor - 1$ such earlier moduli, since they are multiples of d in $(d, M]$, while I contains at least $\lfloor M/d \rfloor$ multiples of d. So an allowed multiple exists. Earlier moduli incomparable with d do not matter, so the result is admissible. There are finitely many admissible assignments, so one maximizes the number of occupied sites. If at least four sites are occupied, four distinct moduli sit at distinct multiples, and Lemma 3.1 with $t = 5$ gives a cover with at most five sites. If at most three sites are occupied, the listed shapes are exactly the partitions of 5 into at most three parts. □

Table 1 summarizes how the remaining shapes are handled.

Lemma 5.2 (Antichain gcd bounds)**.** *If t distinct positive multiples of q lie in $[2, M]$ and form a divisibility antichain, then $M \geq (2t-1)q$. In particular, the gcd of an incomparable pair is at most $M/3$, and the gcd of a pairwise incomparable triple is at most $M/5$. If four distinct originals form two incomparable pairs and q divides all four, then $M \geq 5q$, even if originals from different pairs are comparable.*

Occupied sites	Shape	Treated in	Sites used
≥ 4	—	Lemma 5.1 (one-defect repair)	≤ 5
3	$(3,1,1)$	Lemma 5.4	≤ 6
3	$(2,2,1)$	Lemma 5.5	≤ 6
2	$(3,2)$	Lemma 5.6	≤ 6
2	$(4,1)$	Lemmas 5.10–5.13	≤ 6
1	(5)	Lemma 5.9	≤ 6

Table 1: Case structure of the proof of $g(5) \leq 6$.

Proof. The integers $1, \ldots, N$ split into $\lceil N/2 \rceil$ divisibility chains according to their odd parts. An antichain of size t among the quotients, which are at most $\lfloor M/q \rfloor$, therefore requires $\lfloor M/q \rfloor \geq 2t-1$. For the last assertion, no quotient equals 1, since that original would divide its partner. Four distinct quotients that are at least 2 include one that is at least 5. □

Lemma 5.3 (Paying a triple residual)**.** *Let a, b, c be pairwise incomparable with a common multiple $x \in I$, and let $F \subseteq I$ contain x and have at most three elements. Suppose each of*

$$A = \gcd(a,b), \qquad B = \gcd(a,c), \qquad C = \gcd(b,c)$$

has a multiple in $I \setminus F$. Then at most three distinct sites of $I \setminus F$ pay $abc/\operatorname{lcm}(a,b,c)$.

Proof. Put $g = \gcd(a,b,c)$ and write $A = gu$, $B = gv$, $C = gw$. Comparing prime valuations shows that u, v, w are pairwise coprime and

$$abc = \operatorname{lcm}(a,b,c)\, g^2 uvw. \tag{2}$$

Choose multiples of A, B, C off F. If the three sites are distinct, they pay g^3uvw. If exactly two sites occur, the shared site pays the lcm of two of gu, gv, gw and the other pays the third label, which gives g^2uvw. If all choices coincide at z, that site pays $guvw$. Since $M \geq 5g$ by Lemma 5.2, there is a multiple of g outside $F \cup \{z\}$, which has at most four elements, and it pays the remaining factor. At most three new sites are used in every case. □

5.2 Three occupied sites

Lemma 5.4 (Shape $(3,1,1)$)**.** *Suppose a, b, c are pairwise incomparable and assigned to x, while d and e are assigned to distinct sites $y, z \neq x$. Then at most six sites suffice.*

Proof. Let $A = \gcd(a,b)$, $B = \gcd(a,c)$ and $C = \gcd(b,c)$. If all three have multiples outside $\{x,y,z\}$, apply Lemma 5.3 with $F = \{x,y,z\}$. The three anchors pay $\operatorname{lcm}(a,b,c)$, d and e, and at most three new sites pay the residual $abc/\operatorname{lcm}(a,b,c)$.

Otherwise, relabel a, b, c so that A has no multiple outside $\{x,y,z\}$. By Lemma 5.2, $M \geq 3A$, so I contains at least three multiples of A. Hence the multiples of A in I are exactly x, y, z, and $M < 4A$. Since a and b are incomparable, both are proper multiples of A that are at most $M < 4A$, so $\{a,b\} = \{2A, 3A\}$. The third original c is not divisible by A: a positive multiple of A at most M is A, $2A$ or $3A$, which would be either comparable with a or b or equal to one of them. Of d and e, at most one is a multiple of A (and then equals A). Relabel d, e so that $A \nmid d$; this is harmless because y and z are both multiples of A.

Put

$$t = \gcd(6A, c), \qquad k = \gcd(A, d).$$

We claim that some multiple of t lies outside $\{x, y, z\}$. Note that $t \mid x$, because $\operatorname{lcm}(a, b) = 6A$ and c divide x, and that $t \le c \le M < 4A$. If $t > M/2$, then $t > 3A/2$ because $M \ge 3A$. The only divisors of $6A$ strictly between $3A/2$ and $4A$ are $2A$ and $3A$. Either would divide $c < 4A$ and force $c \in \{2A, 3A\}$, a contradiction. Hence $t \le M/2$, and I contains at least two multiples of t. If all of them lay among x, y, z, two of them would differ by A or $2A$, so $t \mid 2A$. The values $t = A$ and $t = 2A$ are excluded because $A \nmid c$. If $t \mid A$ and $t \ne A$, then $t \le A/2$. If $t \mid 2A$, $t \nmid A$ and $t \ne 2A$, then $t \le 2A/3$. So $t \le 2A/3$, and $M \ge 3A$ gives at least four multiples of t, again a contradiction. Choose a multiple w of t outside $\{x, y, z\}$.

Since $A \nmid d$, we have $k \le A/2$, so $M \ge 3A \ge 6k$. Choose a multiple v of k outside $\{x, y, z, w\}$. The sites x, y, z, w, v are distinct. Because A divides each of x, y, z, these sites pay $\operatorname{lcm}(6A, c) = 6Ac/t$, $\operatorname{lcm}(A, d) = Ad/k$ and e, respectively; w and v pay t and k. The product is divisible by

$$\frac{6Ac}{t} \cdot \frac{Ad}{k} \cdot e \cdot t \cdot k = 6A^2cde = abcde.$$

So in this blocked case five sites suffice. □

Lemma 5.5 (Shape $(2, 2, 1)$)**.** *Suppose incomparable pairs a, b and c, d are assigned to x and y, respectively, and e is assigned to a third site z. Then at most six sites suffice.*

Proof. Put $g = \gcd(a, b)$ and $h = \gcd(c, d)$; both are at most $M/3$ by Lemma 5.2. Let $F = \{x, y, z\}$. Suppose first that both have multiples outside F, and choose them. If the two sites differ, they pay gh. Together with x, y, z, which pay $\operatorname{lcm}(a, b)$, $\operatorname{lcm}(c, d)$ and e, this covers $abcde$. If both choices coincide at w, that site pays $\operatorname{lcm}(g, h)$. Put $q = \gcd(g, h)$. The four originals a, b, c, d are distinct multiples of q forming two incomparable pairs, so $M \ge 5q$ by Lemma 5.2. A multiple of q outside $F \cup \{w\}$ pays the remaining factor.

Suppose instead that one pair gcd, say g after interchanging the pairs, has no multiple outside F. Then its multiples in I are exactly the three points of F, so $M < 4g$ and, as in the previous lemma, $\{a, b\} = \{2g, 3g\}$. The other gcd h has a multiple outside F. Otherwise its multiples would also be exactly F; two three-term arithmetic progressions with the same terms have the same step, so $h = g$, and the incomparable pair c, d would also be $\{2g, 3g\}$, contradicting distinctness. Choose a multiple $w \notin F$ of h.

First suppose that neither c nor d is divisible by g. Put $k_c = \gcd(g, c)$, $k_d = \gcd(g, d)$ and $K = \operatorname{lcm}(k_c, k_d)$. Both k_c and k_d are at most $g/2$, so $M \ge 3g \ge 6k_c, 6k_d$. Hence we can choose distinct multiples u of k_c and v of k_d outside $\{x, y, z, w\}$: each label has at least six multiples, and at most five sites are forbidden when the second choice is made. Because $g \mid y$, the anchor y pays $\operatorname{lcm}(c, d, g)$. Comparing valuations gives

$$\operatorname{lcm}(c, d, g) = \frac{cd}{h} \cdot \frac{g}{K}.$$

The six sites x, y, z, w, u, v therefore pay

$$6g \cdot \frac{cdg}{hK} \cdot e \cdot h \cdot k_c k_d = 6g^2cde \cdot \frac{k_c k_d}{K},$$

which is divisible by $abcde = 6g^2cde$ because $K \mid k_c k_d$.

Otherwise one of c, d is a multiple of g. The positive multiples of g that are at most $M < 4g$ are $g, 2g, 3g$, and $2g, 3g$ are already a and b, so we may relabel so that $c = g$. Then neither d nor e is a multiple of g. Put $k_d = \gcd(g, d)$ and $k_e = \gcd(g, e)$; both are at most $g/2$, so each has at least six multiples in I. Choose distinct multiples of k_d and k_e outside F. Since g divides x, y and z, these

anchors pay $6g$, $\operatorname{lcm}(g,d) = gd/k_d$ and $\operatorname{lcm}(g,e) = ge/k_e$. With the two new sites the product is divisible by

$$6g \cdot \frac{gd}{k_d} \cdot \frac{ge}{k_e} \cdot k_d k_e = 6g^3 de = abcde.$$

In this branch five sites suffice. □

5.3 Two occupied sites: the $(3,2)$ shape

Lemma 5.6 (Shape $(3,2)$)**.** *Suppose a,b,c are pairwise incomparable and assigned to x, and d,e are incomparable and assigned to $y \neq x$. Then at most six sites pay $abcde$.*

Proof. Let $A = \gcd(a,b)$, $B = \gcd(a,c)$, $C = \gcd(b,c)$ and $h = \gcd(d,e)$. Each of these four labels is at most $M/3$ by Lemma 5.2, so it has at least three multiples in I and hence at least one outside $\{x,y\}$. (The labels A,B,C divide x and h divides y, but none of them need divide the other anchor.)

Call one of A,B,C *tight* if it has exactly one multiple outside $\{x,y\}$. A tight label R has at least three multiples in I, so it divides both x and y, its multiples in I are exactly x,y,t for some third point t, and $M < 4R$. Thus $M/4 < R \leq M/3$, and the two incomparable originals whose gcd is R are $2R$ and $3R$.

There is at most one tight label. Two tight labels $R < S$ would both divide $x-y$, whose absolute value lies between 1 and $M-1$. But $1 < S/R < 4/3$; writing $S/R = v/u$ in lowest terms gives $v \geq 4$, so $\operatorname{lcm}(R,S) = Rv \geq 4R > M$, a contradiction. Two equal tight labels would force three distinct originals into the two-element set $\{2R, 3R\}$.

Choose a multiple z of h outside $\{x,y\}$ which, if a tight label exists, is different from its point t. If no such z existed, the multiples of h in I, of which there are at least three, would be exactly x,y,t, which are also exactly the multiples of R. Both sets are three-term arithmetic progressions with common difference equal to the label, so $h = R$. The incomparable pair d,e would then be $\{2R,3R\}$, duplicating two of a,b,c. Hence z exists.

Each of A,B,C still has a multiple outside $F = \{x,y,z\}$: a label that is not tight has at least two multiples off $\{x,y\}$, and the tight point t was avoided. By Lemma 5.3, at most three sites outside F pay $abc/\operatorname{lcm}(a,b,c)$. The anchors x and y pay $\operatorname{lcm}(a,b,c)$ and $\operatorname{lcm}(d,e)$, and z pays h. Since $\operatorname{lcm}(d,e)\, h = de$, at most six sites pay $abcde$. □

5.4 Prime units and five proper divisors

For an integer $T \geq 2$, a *prime unit* of T is a pair (p,j) with p prime and $1 \leq j \leq v_p(T)$. There are $\Omega(T) = \sum_p v_p(T)$ prime units.

Lemma 5.7 (Prime-unit matching for four originals)**.** *Let $a_1,\dots,a_4 \geq 2$ be pairwise incomparable and $Q = \operatorname{lcm}(a_1,\dots,a_4)$. Then distinct prime units of Q can be assigned to the four originals so that the unit (p,j) assigned to a_i satisfies $j \leq v_p(a_i)$. If p_i is the prime of the unit assigned to a_i, then $\prod_{i=1}^4 p_i \mid Q$.*

Proof. Join a_i to the units (p,j) with $j \leq v_p(a_i)$. A subfamily S has exactly $\Omega(\operatorname{lcm}(S))$ neighbors. If $|S| = k \leq 4$ and $\Omega(\operatorname{lcm}(S)) < k$, then k pairwise incomparable divisors would lie in the divisor lattice of an integer with at most $k-1 \leq 3$ prime units. That lattice embeds as a subposet of the Boolean lattice on those units, whose width is $1,1,2,3$ for ranks $0,1,2,3$, which is less than k; this is a contradiction. Hall's theorem applies. Distinct units give $\prod_i p_i \mid Q$ even when some of the primes p_i coincide. □

Lemma 5.8 (Five proper divisors off a common anchor)**.** *Let $1 \le t \le 5$, let $a_1, \dots, a_t$ be distinct integers in $[2, M]$, and let r_i be a proper positive divisor of a_i. Suppose $x \in I$ is a multiple of every r_i. Then there are t distinct sites $b_i \in I \setminus \{x\}$ with $r_i \mid b_i$.*

Proof. We verify Hall's condition for the family indexed by $1, \dots, t$; subfamilies satisfy the same hypotheses. Translate x to 0 and reflect if necessary, so that the interval becomes $[-u, v] \cap \mathbb{Z}$ with $u + v = M - 1$ and $v \ge \lfloor M/2 \rfloor$. Divisibility is unchanged because every $r_i \mid x$.

Remove repeated labels, and then remove every label that is a strict multiple of another. The resulting *primitive* set R has the same union of candidates, and every a_i is still a proper multiple of some member of R. For $s \ge 0$ let $B_R(s)$ be the number of positive integers at most s divisible by some $r \in R$, and put

$$N_1 = B_R(M) - |R|, \qquad N_2 = B_R(u) + B_R(v).$$

Because no member of R divides another, N_1 is the number of proper multiples of members of R in $[1, M]$. Hence $N_1 \ge t$, since the a_i are distinct. The number N_2 is the size of the union of nonzero candidates. Suppose Hall's condition fails, so $N_2 < t \le 5$ and $N_2 \le 4$. Every $r \in R$ is at most $M/2$, so its positive point r is a candidate. In particular $|R| \le N_2 \le 4$.

For a positive integer d put

$$\delta_d = \left\lfloor \frac{M}{d} \right\rfloor - \left\lfloor \frac{u}{d} \right\rfloor - \left\lfloor \frac{v}{d} \right\rfloor.$$

Since $M = u + v + 1$, considering residues modulo d shows $\delta_d \in \{0, 1\}$. If $R = \{r\}$, then $N_1 - N_2 = \delta_r - 1 \le 0$. If $R = \{r, s\}$, inclusion–exclusion gives

$$N_1 - N_2 = \delta_r + \delta_s - \delta_{\operatorname{lcm}(r,s)} - 2 \le 0.$$

Either way $N_1 \le N_2$, a contradiction, so $|R| \ge 3$.

If $|R| \in \{3, 4\}$, there is at most one candidate besides the positive points of R. We claim that no candidate is a common nonzero multiple of two distinct $r, s \in R$. If such a point is negative, then $-r$ and $-s$ are two further candidates. If it is positive, write it as $\alpha r = \beta s$. Then $\alpha, \beta \ge 2$, and not both equal 2 since $r \ne s$. If, say, $\alpha \ge 3$, then $2r$ and αr are two further candidates; neither is a member of R, since no member of a primitive set is a proper multiple of another. Both cases contradict $N_2 \le 4$.

So the candidate sets of the members of R are pairwise disjoint, and $N_2 = \sum_{r \in R} (\lfloor u/r \rfloor + \lfloor v/r \rfloor)$. On the other hand $N_1 \le \sum_{r \in R} (\lfloor M/r \rfloor - 1)$, and each term satisfies $\lfloor M/r \rfloor - 1 \le \lfloor u/r \rfloor + \lfloor v/r \rfloor$ because $\delta_r \le 1$. Hence $N_1 \le N_2$, contradicting $N_1 \ge t > N_2$. Hall's condition therefore holds. □

5.5 A single occupied site: the (5) shape

Lemma 5.9 (Five incomparable originals at one anchor)**.** *If five distinct pairwise incomparable originals have a common multiple $x \in I$, then at most six sites pay their product.*

Proof. Let $Q = \operatorname{lcm}(a_1, \dots, a_5)$, so $Q \mid x$, and form the prime-unit graph of Lemma 5.7. Subfamilies of size at most four satisfy Hall's condition, and the full family does unless $\Omega(Q) \le 4$. A divisor lattice with at most three prime units has width at most three, so a failure requires $\Omega(Q) = 4$.

In that case Q is squarefree. If some prime occurs in Q to exponent $e \ge 2$, fixing the exponents of the other primes splits the divisor lattice into at most $2^{4-e} \le 4$ chains, which cannot contain five pairwise incomparable divisors. So the divisor lattice is the Boolean lattice on four primes. A five-element antichain in it consists of five of the six two-element subsets. Indeed, a singleton leaves room for at most three further pairwise incomparable subsets of the other three primes;

by complementation the same holds for a three-element subset; and the empty and full sets are comparable with everything. After renaming the four primes, the five originals are therefore

$$\{pq,\ pr,\ ps,\ qr,\ qs\}. \tag{3}$$

If the family is not of this form, Hall's theorem assigns five distinct prime units with primes p_i, and $\prod_i p_i \mid Q \mid x$. Put $r_i = a_i/p_i$. Lemma 5.8 gives five distinct multiples of the r_i off x. Together with x they pay $\prod_i a_i$ using at most six sites.

For the family (3), $\prod_i a_i = p^3q^3r^2s^2$. Apply Lemma 5.8 to the four distinct originals pq, qr, pr, ps with proper divisors q, q, r, s. This gives four distinct sites off x that pay q^2rs, and x pays $pqrs$. It remains to pay p^2 with at most one more site. If $v_p(x) = 1$, choose a multiple of p^2 in I; one exists since $p^2 < pq \le M$, and it is not x. If $v_p(x) = 2$, choose a multiple of p other than x; there are at least two since $p \le M/2$. If $v_p(x) \ge 3$, nothing more is needed. If the extra site coincides with one of the four repair sites, that site is divisible by both its repair label and the required power of p, which are coprime, so it pays both. Thus at most six distinct sites suffice. □

5.6 Four originals at one anchor: the $(4, 1)$ shape

The last shape requires a controlled collision repair. We first isolate its invariant.

Lemma 5.10 (Gcd/lcm propagation with site accounting)**.** *Let $a_1, \dots, a_4 \in [2, M]$ be distinct, pairwise incomparable, with a common multiple $x \in I$. Suppose factors $p_i \ge 2$ and proper divisors $r_i = a_i/p_i$ are given such that $P = \prod_i p_i$ divides x, and four distinct sites $b_i \in I \setminus \{x\}$ satisfy $r_i \mid b_i$. Let e be a fifth original, distinct from the a_i, and let $y \in I \setminus \{x\}$ be a multiple of e. Assume one of the following:*

(i) $e > M/2$;

(ii) $2e \notin \{a_1, \dots, a_4\}$;

(iii) $2e = a_j$ *for some* j, *and* $r_j < e$;

(iv) e *is even and* $2P \mid x$.

Then x, the four sites b_i, and at most one further site pay $e \prod_i a_i$.

Proof. The sites $x, b_1, \dots, b_4$ pay $P \prod_i r_i = \prod_i a_i$. Let $e' = e$ in cases (i)–(iii) and $e' = e/2$ in case (iv); in case (iv) the anchor x pays the extra factor 2, so it suffices to pay e' in addition. If y is not one of the b_i, add y; it pays e, hence e', and we are done. Otherwise $y = b_j$ carries both r_j and e'. Let that site pay $\operatorname{lcm}(r_j, e')$, and carry $\delta = \gcd(r_j, e')$ as an outstanding factor. This is legitimate because $\operatorname{lcm}(r_j, e') \mid y$ and $\operatorname{lcm}(r_j, e') \gcd(r_j, e') = r_j e'$. Whenever the outstanding factor is placed at an unvisited repair site b_i, let b_i pay $\operatorname{lcm}(\delta, r_i)$ and carry $\gcd(\delta, r_i)$ instead.

After k repair sites have been visited, δ divides e' and each visited r_i, so it divides the k corresponding originals; each of these is a proper multiple of δ, since $a_i = p_i r_i \ge 2r_i$. We claim $M \ge (k + 2)\delta$. If $\delta < e$, the k visited originals and e are $k + 1$ distinct proper multiples of δ, so the largest is at least $(k + 2)\delta$. This covers case (iv), where $\delta \le e/2$. Suppose $\delta = e$, so that $e' = e$. Case (i) is impossible, since $\delta \mid r_i \le M/2 < e$. In case (ii), the visited originals are proper multiples of e different from $2e$, so they have k distinct coefficients at least 3, and the largest is at least $(k + 2)e$. In case (iii), $a_j = 2e$ is never visited, because $\delta = e$ would have to divide $r_j < e$; so again all coefficients are at least 3.

Hence I contains at least $k + 2$ multiples of δ, while only x and the k visited sites are forbidden. Choose another multiple. If it is a new site, it pays δ and we are done. If it is an unvisited b_i, merge there and continue. The visited set grows, so after at most four merges a new site is reached. Each

merge preserves the required product by the gcd/lcm identity. In total we use x, the four b_i and at most one new site. □

Lemma 5.11 (Reserve or odd-unit selection)**.** *Let four distinct originals lie in $(M/2, M]$ (hence form an antichain), and suppose one of them is $a = 2e$. Put $Q = \mathrm{lcm}(a_1, \dots, a_4)$. Then there is an assignment of four distinct prime units of Q to the originals as in Lemma 5.7, with primes p_i and $P = \prod_i p_i$, such that either*

(a) the prime assigned to $a = 2e$ is odd, or

(b) e is even and $2P \mid Q$.

Proof. Case 1: a has an odd prime factor. Apply Hall's theorem as in Lemma 5.7 after deleting the 2-units from the neighborhood of a only. If Hall's condition holds, (a) follows. Otherwise every deficient subfamily contains a. A deficient pair $\{a, b\}$ would force b to be an odd prime dividing a, contradicting incomparability. In a deficient triple $\{a, b, c\}$, the originals b, c have exactly two units in their combined lcm, so they are the two primes of a squarefree integer, and one of them divides the odd part of a, which is again impossible.

It remains to treat a deficient quadruple. The other three originals have exactly three prime units in their combined lcm and form a three-element antichain in that divisor lattice. The lattice is then squarefree, and the three originals are either the three primes or the three pairwise products. The primes are excluded by incomparability with a. The pairwise products cannot involve the prime 2: for odd primes $p < q$, the products $2p, 2q, pq$ have ratio $pq/(2p) = q/2 \geq 5/2$, whereas any two originals have ratio less than 2. So the other three originals are pq, pr, qr for three odd primes p, q, r. The odd part of a divides pqr and cannot contain two of these primes, since then one of the other originals would divide a. After renaming, $a = 2^j p$. If $j = 1$, then $a = 2p > M/2$, so $M < 4p$. But the larger of the two originals divisible by p is $p \max(q, r) \geq 5p$, because q and r are distinct odd primes; this contradicts $M < 4p$. Thus $j \geq 2$. Give one 2-unit to a, and match pq, pr, qr to the distinct units p, r, q. An unused 2-unit remains in Q, and $e = 2^{j-1}p$ is even, proving (b).

Case 2: $a = 2^j$. Each of the other three originals has an odd prime factor, because distinct powers of 2 are comparable. Any two of them have at least two odd prime units in their combined lcm: otherwise both would have the same odd part and differ only by powers of 2, which is impossible in $(M/2, M]$. If the three together have at least three odd units, they can be matched to odd units. If they have only two, then at least one of them is even, since a two-unit odd integer has no three pairwise incomparable divisors, and one 2-unit completes the matching. So these three originals use at most one 2-unit, and we assign another 2-unit to a. If $j \geq 3$, a 2-unit is left over, $e = 2^{j-1}$ is even, and (b) holds. If $j = 1$, then $a = 2$ and four distinct originals cannot lie in $(M/2, M]$. If $j = 2$, four distinct originals in $(M/2, M]$ including 4 force $M = 7$ and the originals $\{4, 5, 6, 7\}$. Choosing the primes $2, 5, 3, 7$ gives $P = 210$ and $Q = 420 = 2P$, and $e = 2$ is even, so (b) holds. □

Lemma 5.12 (Four unique anchors plus one original)**.** *Suppose four pairwise incomparable originals $a_1, \dots, a_4$ lie in $(M/2, M]$ and have the same unique multiple x in I. Let e be a fifth original. Then at most six sites pay all five originals.*

Proof. If e has no multiple other than x, then $e > M/2$, and the five originals are pairwise incomparable upper-half integers with common multiple x; Lemma 5.9 applies. Otherwise choose a multiple $y \neq x$ of e. If $2e$ is one of the a_i, take the prime-unit assignment of Lemma 5.11; otherwise take any assignment from Lemma 5.7. Let p_i be the assigned primes and $r_i = a_i/p_i$. Since $P = \prod_i p_i$ divides Q and hence x, Lemma 5.8 gives four distinct multiples $b_i \neq x$ of the r_i. If $e > M/2$, apply Lemma 5.10(i). If $e \leq M/2$ and $2e$ is not one of the a_i, apply Lemma 5.10(ii). If $2e = a_j$, then in

alternative (a) of Lemma 5.11 the residual $r_j = 2e/p_j \le 2e/3 < e$, and Lemma 5.10(iii) applies; in alternative (b), e is even and $2P \mid Q \mid x$, and Lemma 5.10(iv) applies. □

Lemma 5.13 (Shape $(4,1)$)**.** *Suppose an admissible assignment with the largest number of occupied sites places four incomparable originals at x and the fifth original e at $y \neq x$. Then at most six sites suffice.*

Proof. By maximality, every multiple in I of each of the four originals at x lies in $\{x, y\}$: moving one of them to a third site would give an admissible assignment with three occupied sites. As in Lemma 3.6, at most one of these originals divides both x and y, and such a *bridge* equals $D = |x-y|$.

If there is no bridge, all four originals have the unique multiple x, so they lie in $(M/2, M]$, and Lemma 5.12 applies.

Suppose there is a bridge. The other three originals have the unique multiple x and lie in $(M/2, M]$. If e had a multiple $z \notin \{x, y\}$, moving the bridge from x to y and e from y to z would give an admissible assignment with three occupied sites, contradicting maximality. So the multiples of e lie in $\{x, y\}$. If e divided both x and y, it would also equal D, the bridge, contradicting distinctness. Thus y is the unique multiple of e, and $e > M/2$. Apply Lemma 5.7 to the four incomparable originals at x (the bridge need not lie in the upper half). The product of the assigned primes divides x, and Lemma 5.8 provides four distinct residual sites off x. Lemma 5.10(i) then gives a cover with at most six sites. □

Proof of $g(5) \le 6$. Choose a maximal admissible assignment by Lemma 5.1. If at least four sites are occupied, that lemma gives at most five sites. With three occupied sites apply Lemma 5.4 or Lemma 5.5; with two, Lemma 5.6 or Lemma 5.13. With one occupied site, the five originals are pairwise incomparable with a common multiple, and Lemma 5.9 applies. The shapes are exhaustive, and every construction uses at most six distinct sites. □

6 A sharp five-modulus instance

Proposition 6.1. *Let*

$$A_5 = \{5, 10, 12, 15, 20\}, \qquad I_5 = \{51, 52, \dots, 70\}.$$

Every product-cover of A_5 in I_5 has at least six elements, and six suffice.

Proof. Here $|I_5| = 20 = \max A_5$ and

$$P(A_5) = 5 \cdot 10 \cdot 12 \cdot 15 \cdot 20 = 180000 = 2^5 \cdot 3^2 \cdot 5^4.$$

Exactly four elements of I_5 are divisible by 5, namely $55, 60, 65, 70$, and none is divisible by 25. So all four belong to every cover. Their product $55 \cdot 60 \cdot 65 \cdot 70 = 15015000$ contains $2^3 \cdot 3 \cdot 5^4$ in the primes $2, 3, 5$. A fifth element w would have to supply $2^2 \cdot 3$, that is $12 \mid w$. The only multiple of 12 in I_5 is 60, which is already chosen. Hence no five-element cover exists. On the other hand, $B_5 = \{55, 60, 63, 64, 65, 70\}$ is a cover, since $63 = 3^2 \cdot 7$ and $64 = 2^6$. □

This instance is due to Chen [2]; the general lower bound $g(n) \ge n + 1$ is due to Erdős and Surányi [4].

Repository	https://github.com/jjtripodo-lgtm/erdos-708-g4-g5
Commit	00fbb23c3084daa88ea420aa9c74177127d5283f
Toolchain	`leanprover/lean4:v4.35.0-rc2` [8]
Mathlib [7]	`065356127b1dc0016f66b7283ce0ce2c4055aa55`
Licence	Apache-2.0
Registry entry	PALOMAR-2026-11-19-000019, version 1 [9]
Permitted axioms	`propext`, `Quot.sound`, `Classical.choice`

Table 2: The Lean artifact. All dependency revisions are pinned in `lake-manifest.json`.

7 Formal verification

7.1 The artifact

The formalization is summarized in Table 2.

The formalization is registered in the Palomar registry as PALOMAR-2026-11-19-000019, version 1, registered 19 November 2026 at 09:09:46 UTC with trust level "high" [9]. The registry entry records the official verification run `44442073024` of the Palomar submission workflow (submission `5h2qqh1ww61l`, verified 02:16:02 UTC). That run checked the four declarations below with the Lean kernel checker and the independent kernels `nanoda` and `con-ron`. The automated editorial review recorded no warnings. The registry preserves the source commit and every dependency commit in archived forks under `PalomarArchive`.

7.2 What is proved

The file `Challenge.lean` states the theorems with `sorry` placeholders and imports only Mathlib. `Solution.lean` proves the same statements, and Palomar's comparator checks that the proved declarations have exactly the Challenge types and depend only on the axioms `propext`, `Quot.sound` and `Classical.choice`. The definitions and statements are reproduced verbatim below, with comments removed:

```
noncomputable def interval (L : ℤ) (M : ℕ) : Finset ℤ := Finset.Icc L (L + (M : ℤ) - 1)

def ExactCover (n k : ℕ) : Prop :=
  ∀ m : Fin n → ℕ, Function.Injective m → (∀ i, 2 ≤ m i) → ∀ L : ℤ,
    ∃ B ⊆ interval L (Finset.univ.sup m), B.card = k ∧
      (∏ i, (m i : ℤ)) ∣ ∏ z ∈ B, z

def HasCoveringSize (n k : ℕ) : Prop :=
  ∀ A : Finset ℕ, A.card = n → (∀ a ∈ A, 2 ≤ a) →
    ∀ m : ℕ, 0 < m →
      ∃ B ⊆ Finset.Ico m (m + A.sup id), B.card ≤ k ∧ (∏ a ∈ A, a) ∣ ∏ b ∈ B, b

def IsMinCoveringSize (g : ℕ → ℕ) : Prop :=
  ∀ n, IsLeast {k | HasCoveringSize n k} (g n)

theorem erdos_708.variants.four : IsLeast {k | ExactCover 4 k} 5
theorem erdos_708.variants.five : IsLeast {k | ExactCover 5 k} 6
theorem erdos_708.variants.g_four (g : ℕ → ℕ) (hg : IsMinCoveringSize g) : g 4 = 5
theorem erdos_708.variants.g_five (g : ℕ → ℕ) (hg : IsMinCoveringSize g) : g 5 = 6
```

All four declarations live in the namespace `Erdos708`.

Fidelity. `ExactCover n k` is $C^{=}(n,k)$ of Definition 1.2, with A given as an injective enumeration and intervals starting at any integer. So `four` and `five` state $g^{=}(4)=5$ and $g^{=}(5)=6$ unconditionally. `HasCoveringSize` and `IsMinCoveringSize` are copied verbatim from the open Formal Conjectures pull request #4388 [6]. They use the at-most reading with windows of positive integers. Since the upper bounds are proved for every integer start and both extremal instances start at positive integers, this window convention does not affect the values. The statements `g_four` and `g_five` are implications whose hypothesis requires $g(n)$ to exist for every n. That hypothesis is satisfiable: Chen's uniform finite bound $g(n)\le 12n$ [2] supplies a covering size for every $n\ge 1$. At $n=0$, the empty original set is covered by the empty subset of the empty window, so `HasCoveringSize 0 0` holds. By well-ordering, every n therefore has a least covering size, defining a function that satisfies `IsMinCoveringSize`. Chen's bound is not part of this artifact. The same development also proves the unconditional statements `Erdos708.isLeast_four` and `Erdos708.isLeast_five` (in `FC4388/Bridge.lean`), namely IsLeast $\{k \mid$ `HasCoveringSize` $4\ k\}\ 5$ and its analogue for five, but only the four declarations above are compared by the comparator.

7.3 Correspondence with the written proof

Table 3 lists the Lean counterparts of the main written results. The Lean development follows the same case structure, but several auxiliary results are packaged differently. For example, the Lean lemma `reserve` returns a general factor split rather than an explicit prime-unit matching, and the Lean $(4,1)$ argument first separates whether $2e$ is an original, rather than whether a bridge exists. Palomar's comparator checks the four final declarations. Their intermediate Lean dependencies are also checked by the kernel, but not every literal written lemma is a separate comparator target or identical in statement to its Lean counterpart.

Written result	Lean module and declaration
Theorem 1.6, $n = 4$ upper	`Erdos708/UniversalUpper.lean`: `four_moduli_five_cover`
Lemma 3.1	`Erdos708/OneDefectFinite.lean`: `one_defect_four_moduli`; `Erdos708/Five/MissingOne.lean`: `missing_one`
Lemma 3.2	`Erdos708/ProperDivisorSDR.lean`: `proper_divisor_sdr_fin_four`
Lemma 3.3	`Erdos708/UpperHalf.lean`: `pair_gcd_le_one_third`, `triple_gcd_le_one_fifth`
Lemma 3.4	`Erdos708/OneAnchor.lean`: `one_anchor_four_moduli`
Lemma 3.5	`Erdos708/TripleAnchor.lean`: `triple_anchor_four_moduli`
Lemma 3.6	`Erdos708/TwoAnchorCombined.lean`: `two_anchor_combined_candidate`
Proposition 4.1	`Erdos708/Sharpness.lean`: `sharp_full_no_four_cover`, `sharp_example_exactly_five`
Lemma 5.1	`Erdos708/Five/Assign.lean`: `exists_max_validAssign`
Lemma 5.2	`Erdos708/Five/Antichain.lean`
Lemma 5.3	`Erdos708/Five/Case311.lean`: `case311_free` (corresponding payment assembly)
Lemma 5.4	`Erdos708/Five/Case311.lean`: `case311`
Lemma 5.5	`Erdos708/Five/Case221.lean`: `case221`
Lemma 5.6	`Erdos708/Five/Case32.lean`: `case32`
Lemma 5.7	`Erdos708/SmallAntichainUnion.lean`: `small_pairwise_incomparable_biUnion_card_candidate`; matching in `Erdos708/Five/Case41.lean`
Lemma 5.8	`Erdos708/Five/SDR5.lean`: `proper_divisor_sdr_fin_five`
Lemma 5.9	`Erdos708/Five/Case5.lean`: `case5_holds`
Lemma 5.10	`Erdos708/Five/Case41Core.lean`: `case41_core`
Lemma 5.11	`Erdos708/Five/Case41Reserve.lean`: `reserve`
Lemmas 5.12, 5.13	`Erdos708/Five/Case41.lean`: `case41_holds`
Theorem 1.6, $n = 5$ upper	`Erdos708/Five/Final.lean`: `five_moduli_six_cover`
Proposition 6.1	`Erdos708/Five/Sharpness.lean`: `sharp5_no_five_cover`, `sharp5_six_cover`
Lemma 1.3, Corollary 1.4	`Adapter.lean`: `exact_of_bound`; `Solution.lean`: `least_exact`
At-most statements	`FC4388/Bridge.lean`: `isLeast_four`, `isLeast_five`

Table 3: Written results and structural Lean counterparts at commit `00fbb23c`; rows are not claims of literal theorem-type identity. Declarations in `Erdos708/Five/` are in the namespace `Erdos708.Five`, those in `Adapter.lean` in `Erdos708Finite`, and the others in `Erdos708`.

7.4 Reproduction

With `elan` installed, the following commands from the repository README rebuild the proofs and rerun the comparator (the comparator requires `bubblewrap`):

```
lake exe cache get
lake build Challenge Solution
lake comparator --config comparator.json
lake comparator --config comparator.json --paranoid
```

Before release, the paranoid comparator was also run locally on the same toolchain, where the proofs were accepted by Lean's default and paranoid checks and by the external kernels `lean4lean`, `nanoda`, `con-leche` and `con-ron`. Mutation tests confirmed that altered statements are rejected: changing the value 5 to 4 in `four`, weakening $2 \le m_i$ to $1 \le m_i$ in `ExactCover`, changing `g 4 = 5` to `g 4 = 6`, and changing $0 < m$ to $1 < m$ in `HasCoveringSize`. These checks are documented in the public repository. Preparing this revision of the manuscript did not involve a further Lean build.

8 Limitations and open problems

Theorem 1.6 determines $g(n)$ only for $n = 4, 5$. It does not bear on the questions $g(n) \le 2n$ or $g(n) \le (2 + o(1))n$. The shape analysis does not extend automatically from five to six moduli: a maximal assignment may exhibit new collision patterns and incompatible simultaneous residual allocations. For $n = 6$ the known lower bound is $g(6) \ge 8$ (Remark 1.5), and no matching upper bound is claimed here. Determining $g(6)$ is the natural next case. By Remark 1.5 it must be posed in the at-most reading.

The written proofs in Sections 3–6 have undergone internal, AI-assisted adversarial review, which is not independent human refereeing. The formal verification certifies the four statements of Section 7; it does not certify the exposition, and it says nothing about priority.

Data and code availability

The Lean 4 source, including the statements, proofs, comparator configuration and release notes, is available under Apache-2.0 at `https://github.com/jjtripodo-lgtm/erdos-708-g4-g5`, pinned at commit `00fbb23c3084daa88ea420aa9c74177127d5283f`, and is archived by the Palomar registry as entry PALOMAR-2026-11-19-000019 [9]. The two extremal instances are verified in Lean (Table 3) and can also be checked by direct computation.

Provenance, AI use and responsibility

Most of this paper's prose, the development of the proofs, the Lean formalization and the testing were produced or revised with OpenAI ChatGPT/Codex and Anthropic Claude, under the direction of John Tripodo, who chose the problem and scope and is the sole human contributor. The author does not claim personal understanding or line-by-line verification of the mathematical arguments. The evidence for correctness is the machine-checked Lean formalization of the stated theorems, together with the elementary verifications of the extremal instances. AI systems are not listed as authors. Neither AI review nor formal verification constitutes human peer review. The author is responsible for the content of this manuscript and for corrections.

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