hexagon:2611.00693v2

IRRATIONALITY OF ODD q-ZETA VALUES AT REAL QUADRATIC UNITS

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ABSTRACT. For $|q|<1$ and odd $l\geq 3$ let $\zeta_q(l)=\sum_{m\geq 1} m^{l-1}q^m/(1-q^m)$. Let q be a unit of norm $+1$ in a real quadratic field $\mathbb{Q}(\sqrt{d})$ with $0<|q|<1$. We prove that the dimension of the $\mathbb{Q}$-vector space spanned by $1,\sqrt{d}\,\zeta_q(3),\sqrt{d}\,\zeta_q(5),\dots,\sqrt{d}\,\zeta_q(A-1)$ is at least $(\pi/\sqrt{12+\pi^2}+o(1))\sqrt{A}$, and that one of $\sqrt{d}\,\zeta_q(5),\dots,\sqrt{d}\,\zeta_q(11)$ lies outside $\mathbb{Q}+\mathbb{Q}\sqrt{d}\,\zeta_q(3)$; these seem to be the first results on odd q-zeta values of weight at least five at an algebraic irrational q. We also give an elementary proof that $\sqrt{d}\,\zeta_q(3)$ is irrational, formalised in Lean 4 with Mathlib using only the standard axioms. In particular, for a Lucas sequence (U_m) of a real quadratic unit $\eta>1$, $\sum_{m\geq 1} m^2(U_{m+1}/U_m-\eta)\notin\mathbb{Q}$, for instance $\sum_{m\geq 1} m^2(F_{m+1}/F_m-\varphi)\notin\mathbb{Q}$. The proofs use the very-well-poised basic hypergeometric series of Krattenthaler, Rivoal and Zudilin, whose coefficients turn out to be invariant or anti-invariant under $q\mapsto 1/q$, up to a power of q; at a unit with conjugate $1/q$ this makes suitable integer linear forms in the twisted values $\sqrt{d}\,\zeta_q(l)$ small at both archimedean places. A stronger form of the weight-three statement has been claimed concurrently and independently in unverified preprints.

1. INTRODUCTION

For a complex number q with $|q|<1$ and an integer $l\geq 1$ let

$$\zeta_q(l)=\sum_{m\geq 1}\frac{m^{l-1}q^m}{1-q^m}=\sum_{N\geq 1}\sigma_{l-1}(N)\,q^N,$$

the q-analogue of $\zeta(l)$ studied by Krattenthaler, Rivoal and Zudilin [12]. For even l these are, up to constants, Eisenstein series, and Nesterenko's theorem [16] shows that $\zeta_q(2m)$ is transcendental for every algebraic q with $0<|q|<1$ (see [11, Introduction]). The odd values are much less understood. Let $q=1/p$ with $p\in\mathbb{Z}\setminus\{0,\pm 1\}$. For $p\geq 2$, $\zeta_q(1)$ is irrational [2], and $1,\zeta_q(1),\zeta_q(2)$ are linearly independent over $\mathbb{Q}$ [17]. For odd $l\geq 3$ Krattenthaler, Rivoal and Zudilin proved that the dimension of the $\mathbb{Q}$-span of $1,\zeta_q(3),\zeta_q(5),\dots,\zeta_q(A-1)$ grows at least like a constant times $\sqrt{A}$, and that one of $\zeta_q(3),\dots,\zeta_q(11)$ is irrational [12]; Jouhet and Mosaki reduced this to

2020 *Mathematics Subject Classification.* Primary 11J72; Secondary 11B39, 11R11, 33D15, 68V20.

Key words and phrases. q-zeta values, Lambert series, real quadratic units, Lucas sequences, Fibonacci numbers, irrationality, basic hypergeometric series, Lean.

one of $\zeta_q(3), \dots, \zeta_q(9)$ [11], and Fischler and Zudilin refined these results [8]. In October 2026 a proof that $\zeta_{1/p}(3) = \sum_{n\ge1} n^2/(p^n-1)$ is irrational for every integer $p \ge 2$ was claimed in an unrefereed, AI-assisted research draft with a Lean formalisation, posted under the pen name uplift-universal-inc [23, 18].

At algebraic irrational q, apart from the even weights, the known results are of weight one, for example for Lambert series $\sum_n a_n/(b^n-1)$ with periodic coefficients a_n at suitable algebraic integers b with $|b| > 1$ (Tachiya [19], Bundschuh and Väänänen [4], Dixit, Kumar and Pathak [7]; see Section 10). Krattenthaler, Rivoal and Zudilin remark [12, §3.5] that their theorems would extend to algebraic q, via Jadot's linear independence criterion over number fields [10], if a conjectured asymptotic for the coefficients of their linear forms held. Bundschuh and Zudilin list the arithmetic of $\zeta_q(k)$ at reciprocals of Pisot and Salem numbers among their open problems [6, §6]. Before the concurrent work discussed below, we know of no result on $\zeta_q(3)$ at an algebraic irrational q, not even of "one of" type.

A unit q of a real quadratic field K has norm $+1$ exactly when its Galois conjugate is $1/q$. Our results concern such q. Throughout, K is a real quadratic field, σ its nontrivial automorphism, $q \in \mathcal{O}_K^\times$ a unit of norm $+1$ with $0 < |q| < 1$, so that $\sigma q = 1/q$, and $d \ge 1$ an integer with $K = \mathbb{Q}(\sqrt{d})$.

Theorem 1.1. *For every even $A \ge 4$ and every integer r with $1 \le r < A/2$,*

$$\dim_{\mathbb{Q}}\big(\mathbb{Q} + \mathbb{Q}\sqrt{d}\,\zeta_q(3) + \mathbb{Q}\sqrt{d}\,\zeta_q(5) + \cdots + \mathbb{Q}\sqrt{d}\,\zeta_q(A-1)\big) \ge 1 + \tau(A,r),$$

where

$$\tau(A,r) = \frac{4\pi^2 r(A-2r) - 12A}{12A + \pi^2(A+4r^2)}.$$

In particular, as $A \to \infty$ through even integers, this dimension is at least $(\pi/\sqrt{12+\pi^2} + o(1))\sqrt{A} = (0.6717\ldots + o(1))\sqrt{A}$.

Corollary 1.2. *At least one of $\sqrt{d}\,\zeta_q(5)$, $\sqrt{d}\,\zeta_q(7)$, $\sqrt{d}\,\zeta_q(9)$, $\sqrt{d}\,\zeta_q(11)$ lies outside $\mathbb{Q} + \mathbb{Q}\sqrt{d}\,\zeta_q(3)$, and so is irrational. Infinitely many of the numbers $\sqrt{d}\,\zeta_q(l)$, l odd, are irrational.*

The corollary is the case $A = 12$, $r = 2$, where $1 + \tau = 2.16\ldots > 2$. Theorem 1.1 is not formalised. Its case $A = 4$, $r = 1$ gives $1 + \tau = 1.24\ldots > 1$, that is, the following theorem, for which we give a separate, shorter proof that uses only an elementary bound for $\sum_{\ell \le n} \phi_{\mathrm{E}}(\ell)$ and is formalised in Lean (Section 11).

Theorem 1.3. *$\sqrt{d}\,\zeta_q(3)$ is irrational.*

Equivalently: if q is a real number with $0 < |q| < 1$ and $s = q + q^{-1} \in \mathbb{Z}$, then $(q - q^{-1})\zeta_q(3) \notin \mathbb{Q}$. Indeed, such q is a root of $x^2 - sx + 1$ with $|s| \ge 3$, so it is a quadratic unit of norm 1, and $q - q^{-1} = \pm\sqrt{s^2-4}$ is a nonzero rational multiple of $\sqrt{d}$. Neither theorem says anything about the values $\zeta_q(l)$ themselves; see Remark 8.2.

For integers P, Q let $U_m = U_m(P,Q)$ be the Lucas sequence $U_0 = 0$, $U_1 = 1$, $U_{m+1} = PU_m - QU_{m-1}$. Call (P,Q) *admissible* if $Q = -1$ and $P \geq 1$, or $Q = 1$ and $P \geq 3$. Then $D = P^2 - 4Q > 0$ is not a square, $\eta = (P + \sqrt{D})/2$ is a unit of norm Q of the order $\mathbb{Z}[\eta]$, $\eta > 1$, and every unit $\eta > 1$ of a real quadratic order arises in this way [13, §1]. The Fibonacci numbers are $F_m = U_m(1,-1)$, with $\eta = \varphi = (1+\sqrt{5})/2$, and the Pell numbers are $\mathcal{P}_m = U_m(2,-1)$, with $\eta = 1 + \sqrt{2}$.

Corollary 1.4. *For every admissible (P,Q), with $\eta = (P + \sqrt{P^2 - 4Q})/2$,*

$$\sum_{m \geq 1} m^2 \Big(\frac{U_{m+1}}{U_m} - \eta \Big) \notin \mathbb{Q}.$$

Corollary 1.5. *The numbers*

$$\sum_{m \geq 1} m^2 \Big(\frac{F_{m+1}}{F_m} - \varphi \Big) = \sum_{m \geq 1} \frac{(-1)^m m^2}{\varphi^m F_m} \qquad \text{and} \qquad \sum_{m \geq 1} m^2 \Big(\frac{\mathcal{P}_{m+1}}{\mathcal{P}_m} - 1 - \sqrt{2} \Big)$$

are irrational.

The ratios U_{m+1}/U_m converge to η geometrically, so the series converge absolutely. Corollary 1.4 is Theorem 1.3 at $q = \bar{\eta}/\eta = Q\eta^{-2}$, where $\bar{\eta} = Q/\eta$ (Section 8). It covers only the units of the form $q = \pm\eta^{-2}$, so it is weaker than Theorem 1.3: for example $q = 2 - \sqrt{3}$ is not of this form, since the units of every order of $\mathbb{Q}(\sqrt{3})$ are $\pm(2+\sqrt{3})^k$. In the same way, $\sum_{m \geq 1} m^{l-1}(U_{m+1}/U_m - \eta) = \sqrt{P^2 - 4Q}\,\zeta_q(l)$ for every odd $l \geq 3$, so Theorem 1.1 and Corollary 1.2 hold for these sums; for instance, one of $\sum_{m \geq 1} m^{l-1}(F_{m+1}/F_m - \varphi)$, $l \in \{5,7,9,11\}$, is irrational.

Concurrent work. While this work was being completed, the contributor twinheterodox submitted the Hexagon preprint [22] (submitted 18 November, published 19 November 2026). It is labelled by its author as primarily AI-generated, and its proof is not formally verified. It claims that $Z(b) = \sum_{n \geq 1} n^2/(b^n - 1) = \zeta_{1/b}(3)$ does not lie in $\mathbb{Q}(b)$ for every algebraic integer b with $|b| > 1$ all of whose conjugates outside the closed unit disk belong to $\{b, \bar{b}\}$; this includes all Pisot and Salem numbers. For $b = 1/q$ with q as in Theorem 1.3, this is the statement $\zeta_q(3) \notin K$, which is stronger than Theorem 1.3: it implies both $\sqrt{d}\,\zeta_q(3) \notin \mathbb{Q}$ and $\zeta_q(3) \notin \mathbb{Q}$, and the second does not follow from our method. A companion preprint by the same contributor [21] claims that 1 and the three numbers $\sum_{n \geq 1}(b^n - 1)^{-j}$, $j = 1,2,3$, are linearly independent over $\mathbb{Q}(b)$ for every algebraic integer b with $|b| > 1$ all of whose conjugates other than b and $\bar{b}$ lie in the closed unit disk (by its abstract, a slightly different wording of the same class); since $\zeta_{1/b}(3) = M_1 + 3M_2 + 2M_3$ with $M_j = \sum_{n \geq 1}(b^n - 1)^{-j}$ (expand $(b^n - 1)^{-j}$ in powers of b^{-n} and use $k^2 = 2\binom{k-1}{2} + 3(k-1) + 1$), this also contains Theorem 1.3, as $b = 1/q$ is in that class. The proof in [22] builds on the moving-denominator Hankel construction of [23]. We have not checked their

arguments, and our work is independent of them, and we claim no priority for Theorem 1.3. Relative to [22, 21], the contributions of the present paper are:

- Theorem 1.1 and Corollary 1.2, which concern weights above three, where the claims of [22, 21] say nothing;
- an independent proof of Theorem 1.3, which is elementary and short: it uses no estimate on the coefficients of the linear forms and no input from analytic number theory;
- a complete formal verification of Theorem 1.3 and Corollaries 1.4 and 1.5 in Lean 4 with Mathlib, using only the standard axioms (Section 11);
- the Lucas-sequence form of these results; for Theorem 1.3 it follows equally from [22].

The method. We use a variant of the very-well-poised basic hypergeometric series of [12, §3] (with weight T^e instead of T^{e+1}, summed bilaterally), in the case $A = 4$, $r = 1$. For each n it gives a linear form $S_n(q) = P_0(q) + P_3(q)\zeta_q(3)$ with $P_0, P_3 \in \mathbb{Q}(q)$ (Lemma 4.1). The new observation is a reciprocity under $q \mapsto 1/q$ (Lemma 5.1):

$$P_3(1/q) = q^{-n} P_3(q), \qquad P_0(1/q) = -q^{-n} P_0(q).$$

At a unit with $\sigma q = 1/q$, suppose that $\zeta_q(3) \in \mathbb{Q}\sqrt{d}$. Then $S_n \in K$, and Galois conjugation gives $\sigma(S_n) = -q^{-n} S_n$, so the linear form is small at both archimedean places of K, and the product formula gives a contradiction: after clearing the cyclotomic denominators $d_n(q)^4$ (Lemma 3.2), the norm is about

$$|\operatorname{N} d_n(q)|^4 \, |S_n(q)|^2 \le |q|^{-(\frac{3}{2} - 2 + o(1))n^2},$$

since $\log|S_n| \sim -n^2 \log(1/|q|)$ (Lemma 6.1) and $\log|\operatorname{N} d_n(q)| \le (\frac{3}{8} + o(1))n^2 \log(1/|q|)$ (Lemma 7.1). No upper bound on the coefficients P_0, P_3 is needed. The same construction gives nothing for $\zeta_q(3)$ itself (Remark 8.2), nor for the Fibonacci zeta value $\sum_m F_m^{-3}$ (Remark 8.3). Our earlier proofs that $\sum_m F_m^{-3}$ and $\sum_m U_m^{-3}$ are irrational [14, 13] used a different construction, moving-denominator Hankel determinants.

For Theorem 1.1 we take general (A, r), where $S_n = P_0 + \sum_l P_l \zeta_q(l)$ involves the odd weights $3 \le l \le A - 1$. The same reciprocity shows that the integer forms $\operatorname{Tr}_{K/\mathbb{Q}}(\delta P_0) + \sum_l \operatorname{Tr}_{K/\mathbb{Q}}(\delta P_l/\sqrt{d})\,\sqrt{d}\,\zeta_q(l)$, for a suitable denominator $\delta \in \mathcal{O}_K$, equal $(\delta - q^{-e}\sigma\delta)S_n$. Multiplying δ by a power of the unit q splits the cost of the denominator evenly between the two real places, a standard balancing step, and Nesterenko's linear independence criterion then gives Theorem 1.1. Here we need upper bounds for the coefficients, as in [12], and the asymptotic $\sum_{\ell \le n} \phi_{\mathrm{E}}(\ell) \sim 3n^2/\pi^2$. With the same leading constants (decay, coefficient size and denominator, which our variant of the series shares with that of [12]) and the same single-form criterion, the bound $1+\tau(A, r)$ is exactly twice the bound that Krattenthaler, Rivoal and Zudilin obtain at $q = 1/p$ (Remark 9.4); this compares one method at two kinds of

q, not with the best results known at $q = 1/p$, which are better than those of [12] [11, 8].

Sections 2–6 hold for every even $A \geq 4$ and every $1 \leq r < A/2$; only Sections 7–8 specialise to $A = 4$, $r = 1$, and Section 9 returns to general (A, r). The identities of Section 2 are those of [12] in our normalisation; we give proofs to keep the paper self-contained.

Formal verification. Theorem 1.3, Corollaries 1.4 and 1.5, and every step of their proofs (for $A = 4$, $r = 1$) are formalised in Lean 4 with Mathlib (`https://github.com/telaviv/qzeta3-quadratic-units`; Section 11). Theorem 1.1 and Corollary 1.2 are not formalised.

2. The rational function and its symmetries

Write $(x;q)_m = \prod_{i=0}^{m-1}(1-xq^i)$. Fix an even integer $A \geq 4$ and an integer r with $1 \leq r < A/2$, and put $e = (A-2r)n/2 \in \mathbb{Z}$. For $n \geq 1$ let

$$R(T) = R_n(T;q) = \frac{(q;q)_n^{A-2r}\,(q^{-rn}T;q)_{rn}\,(q^{n+1}T;q)_{rn}\,T^e}{(T;q)_{n+1}^{A}}, \tag{2.1}$$

a rational function of T over the field $\mathbb{Q}(q)$, q an indeterminate. This is the rational function $R_n(T;q)$ of [12, §3.3] without its factor $q^{-(A-2r)n/4}$. Krattenthaler, Rivoal and Zudilin sum $q^k R_n(q^k)$; we sum $R_n(q^k)$. Its numerator has degree $2rn + e = (A+2r)n/2 < A(n+1)$ in T, so R is proper.

Lemma 2.1 (well-poised symmetry)**.** $R(q^{-n}/T) = q^{An/2}\,T^A\,R(T)$.

Proof. Substitute $T \mapsto q^{-n}/T$. The factors $(T;q)_{n+1}$, $(q^{-rn}T;q)_{rn}$ and $(q^{n+1}T;q)_{rn}$ become $\prod_{i=0}^{n}(1-q^{i-n}/T)$, $\prod_{m=n+1}^{n+rn}(1-q^{-m}/T)$ and $\prod_{m=1}^{rn}(1-q^m/T)$. Taking out of each linear factor its term in $1/T$ gives

$$\begin{aligned}
(T;q)_{n+1} &\mapsto (-1)^{n+1}q^{-n(n+1)/2}\,T^{-(n+1)}\,(T;q)_{n+1},\\
(q^{-rn}T;q)_{rn} &\mapsto (-1)^{rn}q^{-rn^2-rn(rn+1)/2}\,T^{-rn}\,(q^{n+1}T;q)_{rn},\\
(q^{n+1}T;q)_{rn} &\mapsto (-1)^{rn}q^{rn(rn+1)/2}\,T^{-rn}\,(q^{-rn}T;q)_{rn},\\
T^e &\mapsto q^{-ne}\,T^{-e}.
\end{aligned}$$

The two numerator blocks trade places. Relative to $R(T)$, the power of T is $A(n+1) - 2rn - 2e = A$, the sign is $(-1)^{2rn+A(n+1)} = 1$, and the power of q is $-rn^2 - ne + An(n+1)/2 = An/2$. □

Lemma 2.2 ($q \mapsto 1/q$)**.** $R(T;1/q) = q^{-e}\,R(q^{-n}T;q)$.

Proof. First, $R(1/V;1/q) = q^{rn}V^A R(V;q)$. The computation is of the same kind:

$$\begin{aligned}(1/q;1/q)_n &= (-1)^n q^{-n(n+1)/2}\,(q;q)_n,\\ (V^{-1};1/q)_{n+1} &= (-1)^{n+1} q^{-n(n+1)/2}\,V^{-(n+1)}\,(V;q)_{n+1},\\ (q^{rn}V^{-1};1/q)_{rn} &= (-1)^{rn} q^{rn(rn+1)/2}\,V^{-rn}\,(q^{-rn}V;q)_{rn},\\ (q^{-n-1}V^{-1};1/q)_{rn} &= (-1)^{rn} q^{-rn^2-rn(rn+1)/2}\,V^{-rn}\,(q^{n+1}V;q)_{rn}.\end{aligned}$$

The total power of q is $-(A-2r)n(n+1)/2 - rn^2 + An(n+1)/2 = rn$, the power of V is A and the sign is $+1$. By Lemma 2.1 with $T = q^{-n}V$, $R(1/V;q) = q^{-An/2}V^A R(q^{-n}V;q)$. Hence $R(V;1/q) = q^{rn}V^{-A}R(1/V;q) = q^{rn-An/2}R(q^{-n}V;q)$, and $rn - An/2 = -e$. □

3. Partial fractions and denominators

Since R is proper, with poles only at $T = q^{-j}$, $0 \le j \le n$, each of order at most A,

$$R(T) = \sum_{j=0}^{n}\sum_{s=1}^{A} \frac{c_{s,j}(q)}{(1-q^jT)^s}, \qquad c_{s,j} \in \mathbb{Q}(q), \tag{3.1}$$

and $\sum_{s,j} c_{s,j} = R(0) = 0$, because $e \ge 1$.

Corollary 3.1. *$c_{s,j}(1/q) = q^{-e}c_{s,n-j}(q)$ for all s, j.*

Proof. Apply the automorphism $q \mapsto 1/q$ of $\mathbb{Q}(q)$ to (3.1): this gives the partial fractions of $R(T;1/q)$, with poles at $T = q^j$. By Lemma 2.2 and (3.1), $R(T;1/q) = q^{-e}\sum_{s,j'} c_{s,j'}(q)(1-q^{j'-n}T)^{-s}$. Partial fractions are unique, and the pole q^j corresponds to $j' = n - j$. □

Let Φ_ℓ be the ℓ-th cyclotomic polynomial and $d_n(q) = \prod_{\ell=1}^{n}\Phi_\ell(q)$. For $1 \le m \le n$, $1 - q^m = -\prod_{\ell|m}\Phi_\ell(q)$ divides $d_n(q)$ in $\mathbb{Z}[q,q^{-1}]$, and so does $1 - q^{-m} = -q^{-m}(1-q^m)$.

Lemma 3.2 (denominators)**.** *$d_n(q)^{A-s}\,c_{s,j}(q) \in \mathbb{Z}[q,q^{-1}]$ for all $1 \le s \le A$ and $0 \le j \le n$.*

Proof. Since

$$(q^{-rn}T;q)_{rn} = \prod_{\rho=1}^{r}(q^{-\rho n}T;q)_n \quad\text{and}\quad (q^{n+1}T;q)_{rn} = \prod_{\rho=0}^{r-1}(q^{n+1+\rho n}T;q)_n,$$

we can write $R(T) = T^e\prod_{i=1}^{A} f_i(T)$, where each f_i is one of

- $(q;q)_n/(T;q)_{n+1}$ ($A-2r$ times);
- $(q^hT;q)_n/(T;q)_{n+1}$ with $h = -\rho n$, $1 \le \rho \le r$;
- $(q^hT;q)_n/(T;q)_{n+1}$ with $h = n+1+\rho n$, $0 \le \rho \le r-1$.

Each $f_i = N_i/(T;q)_{n+1}$ is proper with simple poles, so

$$f_i(T) = \sum_{j'=0}^{n} \frac{a_{i,j'}}{1-q^{j'}T}, \qquad a_{i,j'} = \frac{N_i(q^{-j'})}{\prod_{i'\neq j'}(1-q^{i'-j'})},$$

where i' runs over $0 \le i' \le n$, and

$$\prod_{i'\neq j'} (1-q^{i'-j'}) = \pm q^{-j'(j'+1)/2}(q;q)_{j'}(q;q)_{n-j'}.$$

Hence $a_{i,j'} = \pm q^{j'(j'+1)/2}\binom{n}{j'}_q N_i(q^{-j'})/(q;q)_n$, where $\binom{n}{j'}_q \in \mathbb{Z}[q]$ is the Gaussian binomial coefficient. For the first type $N_i/(q;q)_n = 1$. For the other two, $N_i(q^{-j'}) = (q^b;q)_n$ with $b = h - j'$; for the second type $b \le -n$, for the third $b \ge 1$. If $b \ge 1$, $(q^b;q)_n/(q;q)_n = \binom{b+n-1}{n}_q$; if $b \le -n$, then $(q^b;q)_n = \pm q^{-N}(q^c;q)_n$ with $c = 1-b-n \ge 1$ and some integer N, and the quotient is again $\pm q^{-N}$ times a Gaussian binomial coefficient. So every $a_{i,j'}$ lies in $\mathbb{Z}[q,q^{-1}]$.

Fix j and put $u = 1-q^jT$, so $T = q^{-j}(1-u)$. Then $T^e = q^{-je}(1-u)^e$ has coefficients in $\mathbb{Z}[q^{\pm1}]$ as a polynomial in u, and

$$(1-q^jT)f_i(T) = a_{i,j} + u\sum_{j'\neq j} \frac{a_{i,j'}}{\alpha_{j'} + q^{j'-j}u}, \qquad \alpha_{j'} = 1-q^{j'-j}.$$

The coefficient of u^M, $M \ge 1$, in this power series is

$$\sum_{j'\neq j} a_{i,j'}\,(-q^{j'-j})^{M-1}\,\alpha_{j'}^{-M}.$$

Each $\alpha_{j'}$ is $1-q^{\pm m}$ with $1 \le m \le n$, so $d_n(q)^M$ times this coefficient lies in $\mathbb{Z}[q^{\pm1}]$. Finally $c_{s,j}$ is the coefficient of u^{A-s} in $u^AR = T^e\prod_i(1-q^jT)f_i(T)$. Distributing the u-degree $A-s$ over the factors, with degree M_i on the factor $(1-q^jT)f_i$ and $\sum_i M_i \le A-s$, gives $d_n^{A-s}c_{s,j} \in \mathbb{Z}[q^{\pm1}]$: the constant terms $a_{i,j}$ ($M_i = 0$) and the coefficients of T^e need no denominator, and a factor with $M_i \ge 1$ needs at most $d_n^{M_i}$. □

4. The linear forms

From now on q also denotes a complex number with $0 < |q| < 1$. The identities of Sections 2 and 3, and those of Section 5 below, take place in the ring $\mathbb{Q}[q,q^{-1},d_n(q)^{-1}]$ (by Lemma 3.2, and since $1-q^{\pm m}$ divides $d_n(q)$ for $m \le n$), or in the ring of rational functions of T over it. Since every root of d_n is a root of unity, evaluation at a number q with $q \neq 0$ and $|q| \neq 1$ is a ring homomorphism of $\mathbb{Q}[q,q^{-1},d_n(q)^{-1}]$ to $\mathbb{C}$. This ring is stable under $q \mapsto 1/q$, because $\Phi_1(1/q) = -q^{-1}\Phi_1(q)$ and $\Phi_\ell(1/q) = q^{-\phi_{\mathrm{E}}(\ell)}\Phi_\ell(q)$ for $\ell \ge 2$, and the value at q of $f(1/q)$ is the value of f at $1/q$. So all these identities, including those relating $f(1/q)$ and $f(q)$, may be evaluated at

such q. Define

$$S_n(q) = \sum_{k\geq 1} R(q^k) - \sum_{k\geq 1} R(q^{-n-k}).$$

No argument is a pole. By Lemma 2.1, $R(q^{-n-k}) = q^{A(n/2+k)}R(q^k)$, and $R(q^k) = O(|q|^{ke})$, so both sums converge absolutely.

Put $C_s(x) = \binom{x+s-1}{s-1} \in \mathbb{Q}[x]$ and $\mathcal{Q}(x) = \sum_{s,j} c_{s,j}C_s(x) = \sum_\lambda \mathcal{Q}_\lambda x^\lambda$, a polynomial of degree at most $A-1$.

Lemma 4.1 (expansion)**.** *For $0 < |q| < 1$,*

$$S_n(q) = P_0(q) + \sum_{\substack{3\leq l\leq A-1\\ l \text{ odd}}} P_l(q)\,\zeta_q(l),$$

where $P_l = 2\mathcal{Q}_{l-1}$ and

$$P_0 = -\sum_{s,j} c_{s,j}W_{s,j}, \qquad W_{s,j} = \sum_{m=1}^{j}\big((1-q^m)^{-s}-1\big) - \sum_{m=1}^{n-j}(1-q^{-m})^{-s}.$$

Proof. Positive side. Since $\sum_{s,j} c_{s,j} = 0$, $R(q^k) = \sum_{s,j} c_{s,j}\big[(1-q^{j+k})^{-s}-1\big]$. Summing over $k \geq 1$ and using

$$\sum_{m\geq 1}\big[(1-q^m)^{-s}-1\big] = \sum_{m\geq 1}\sum_{i\geq 1} C_s(i)q^{mi} = \sum_{i\geq 1} C_s(i)\frac{q^i}{1-q^i},$$

we get $\sum_{k\geq 1} R(q^k) = \sum_{i\geq 1}\mathcal{Q}(i)\frac{q^i}{1-q^i} - \sum_{s,j} c_{s,j}\sum_{m=1}^{j}\big((1-q^m)^{-s}-1\big)$.

Negative side. We have

$$\sum_{k\geq 1}(1-q^{j-n-k})^{-s} = \sum_{m\geq 1}(1-q^{-m})^{-s} - \sum_{m=1}^{n-j}(1-q^{-m})^{-s}.$$

Moreover $(1-q^{-m})^{-s} = (-1)^s q^{ms}(1-q^m)^{-s} = (-1)^s\sum_{i\geq s}\binom{i-1}{s-1}q^{mi}$, and $(-1)^s\binom{i-1}{s-1} = -C_s(-i)$; both sides vanish for $1 \leq i < s$. Hence $\sum_{m\geq 1}(1-q^{-m})^{-s} = -\sum_{i\geq 1} C_s(-i)\frac{q^i}{1-q^i}$ and

$$\sum_{k\geq 1} R(q^{-n-k}) = -\sum_{i\geq 1}\mathcal{Q}(-i)\frac{q^i}{1-q^i} - \sum_{s,j} c_{s,j}\sum_{m=1}^{n-j}(1-q^{-m})^{-s}.$$

Difference. The q-zeta part of S_n is

$$\sum_{i\geq 1}\big[\mathcal{Q}(i)+\mathcal{Q}(-i)\big]\frac{q^i}{1-q^i} = \sum_{\lambda \text{ even}} 2\mathcal{Q}_\lambda\,\zeta_q(\lambda+1).$$

Here $\mathcal{Q}_0 = \mathcal{Q}(0) = \sum_{s,j} c_{s,j} = 0$, and $\lambda \leq A-2$ since λ is even and at most $A-1$. All rearrangements are of absolutely convergent double series. □

The values $\zeta_q(l)$ of even weight are absent from S_n: the minus sign between the two sums keeps only the even part of $\mathcal{Q}$.

Remark 4.2. Not used below. Put $S_n^+(q) = \sum_{k\ge1} R(q^k;q)$. By Lemma 2.2, $R(q^{-n-i};q) = q^e R(q^{-i};1/q)$, so $S_n = S_n^+(q) - q^e S_n^+(1/q)$, where $S_n^+(1/q)$ means $\sum_{i\ge1} R(q^{-i};1/q)$. The combination $S_n^{[1]}$ of [12, §3] has the same shape, but is built from $TR(T)$ in place of $R(T)$; it is not a constant multiple of our S_n.

5. Reciprocity

Lemma 5.1. *As elements of $\mathbb{Q}(q)$: $P_l(1/q) = q^{-e}P_l(q)$ for odd $l \ge 3$, but $P_0(1/q) = -q^{-e}P_0(q)$.*

Proof. By Corollary 3.1,

$$\mathcal{Q}(x;1/q) = \sum_{s,j} c_{s,j}(1/q)C_s(x) = q^{-e}\sum_{s,j} c_{s,n-j}(q)C_s(x) = q^{-e}\mathcal{Q}(x;q),$$

which gives the first claim. For P_0: $W_{s,n-j}$ evaluated at $1/q$ is

$$\sum_{m=1}^{n-j}\big((1-q^{-m})^{-s}-1\big) - \sum_{m=1}^{j}(1-q^m)^{-s} = -W_{s,j}(q) - n.$$

Hence, reindexing $j \mapsto n-j$ and using Corollary 3.1,

$$\begin{aligned} P_0(1/q) &= -\sum_{s,j} c_{s,n-j}(1/q)\, W_{s,n-j}(1/q) \\ &= -q^{-e}\sum_{s,j} c_{s,j}(q)\big(-W_{s,j}(q)-n\big) = -q^{-e}P_0(q), \end{aligned}$$

using $\sum_{s,j} c_{s,j} = 0$ once more. □

Remark 5.2. Heuristically, $q \mapsto 1/q$ sends $\zeta_q(l)$ to $-\zeta_q(l) - \zeta(1-l)$, and $\zeta(1-l) = 0$ for odd $l \ge 3$: so S_n behaves like an eigenvector of $q \mapsto 1/q$ if the q-zeta values are treated as anti-invariant. At a unit with $\sigma q = 1/q$, Galois conjugation realises this for the numbers X with $\sigma X = -X$, that is, for $X \in \mathbb{Q}\sqrt{d}$. This remark is not used in the proofs.

6. Size and nonvanishing

Lemma 6.1. *Fix q with $0 < |q| < 1$. For all sufficiently large n, $S_n(q) \ne 0$ and*

$$\log|S_n(q)| = -(rn+1)e\log(1/|q|) + O_q(1);$$

in particular $\log|S_n(q)| = -\frac{1}{2}r(A-2r)\,n^2\log(1/|q|) + O_q(n)$.

Proof. Let $\kappa = |q|$, $c_- = \prod_{i\ge1}(1-\kappa^i) > 0$ and $c_+ = \prod_{i\ge1}(1+\kappa^i)$. For $a \ge 1$ and $m \ge 0$, $c_- \le |(q^a;q)_m| \le c_+$. For $1 \le k \le rn$, $R(q^k) = 0$, because $(q^{k-rn};q)_{rn}$ contains the factor $1-q^0$. For $k \ge rn+1$ every q-Pochhammer symbol in $R(q^k)$ has base q^a with $a \ge 1$; there are $A-2r+2 \le A$ of them in

the numerator and A in the denominator, so $C^{-1}\kappa^{ke} \le |R(q^k)| \le C\kappa^{ke}$ with $C = (c_+/c_-)^{2A}$, a constant depending only on κ and A. By Lemma 2.1,

$$S_n = \sum_{k \ge rn+1} \bigl(1 - q^{A(n/2+k)}\bigr) R(q^k),$$

and $|1 - q^{A(n/2+k)}| \in [\frac{1}{2}, 2]$ for these k once n is large. So

$$\bigl|S_n - (1 - q^{A(n/2+rn+1)}) R(q^{rn+1})\bigr| \le \frac{2C\kappa^{(rn+2)e}}{1-\kappa^e},$$

while

$$\bigl|(1 - q^{A(n/2+rn+1)}) R(q^{rn+1})\bigr| \ge \frac{\kappa^{(rn+1)e}}{2C}.$$

The ratio of the two bounds is $4C^2\kappa^e/(1-\kappa^e) \to 0$, since $e \to \infty$. Hence $|S_n|$ lies between $\kappa^{(rn+1)e}/(4C)$ and $4C\kappa^{(rn+1)e}$ for large n. □

Krattenthaler, Rivoal and Zudilin obtain the analogous limit for their series by a saddle-point argument [12]; we do not need it.

7. Cyclotomic norms at a unit

From now on q is as in Theorem 1.3, and $\kappa = |q|$. Since d_n has integer coefficients and $\sigma q = 1/q$, the norm of $d_n(q)$ is $\mathrm{N}\, d_n(q) = d_n(q)\, d_n(1/q)$. Write ϕ_{E} for Euler's function (the letter φ denotes the golden ratio) and $\mathcal{T}(n) = \sum_{\ell \le n} \phi_{\mathrm{E}}(\ell)$.

Lemma 7.1. *With $B = \sum_{a \ge 1} -\log(1-\kappa^a) < \infty$,*

$$\bigl|\log|\,\mathrm{N}\, d_n(q)| - \mathcal{T}(n)\log(1/\kappa)\bigr| \le 2Bn, \qquad \textit{and} \qquad \mathcal{T}(n) \le \tfrac{3}{8}(n+1)^2.$$

Proof. From $\Phi_\ell(x) = \prod_{a|\ell}(x^a-1)^{\mu(\ell/a)}$ and $\sum_{a|\ell} a\,\mu(\ell/a) = \phi_{\mathrm{E}}(\ell)$,

$$|\Phi_\ell(q)\Phi_\ell(1/q)| = \kappa^{-\phi_{\mathrm{E}}(\ell)} \prod_{a|\ell} |1-q^a|^{2\mu(\ell/a)}.$$

Since $1-\kappa^a \le |1-q^a| \le 1+\kappa^a$, we have $|\log|1-q^a|| \le -\log(1-\kappa^a)$, so $|\log|\,\mathrm{N}\,\Phi_\ell(q)| - \phi_{\mathrm{E}}(\ell)\log(1/\kappa)| \le 2B$. Sum over $\ell \le n$.

For the second claim, $\phi_{\mathrm{E}}(\ell) \le \ell$ for odd ℓ, and $\phi_{\mathrm{E}}(2m) \le m$ (if m is odd, $\phi_{\mathrm{E}}(2m) = \phi_{\mathrm{E}}(m) \le m$; if m is even, $\phi_{\mathrm{E}}(2m) = 2\phi_{\mathrm{E}}(m) \le m$). Hence

$$\mathcal{T}(n) \le \sum_{\substack{\ell \le n \\ \ell \text{ odd}}} \ell + \sum_{m \le n/2} m \le \frac{(n+1)^2}{4} + \frac{n(n+2)}{8} = \frac{3n^2+6n+2}{8} \le \frac{3}{8}(n+1)^2.$$

□

Remark 7.2. By the classical asymptotic $\mathcal{T}(n) = 3n^2/\pi^2 + O(n\log n)$ [9, Theorem 330], Lemma 7.1 gives $\log|\,\mathrm{N}\, d_n(q)| = (3/\pi^2)n^2\log(1/\kappa) + O(n\log n)$. The proof below only needs $\mathcal{T}(n) \le cn^2 + O(n)$ for some $c < \frac{1}{2}$; the trivial bound $\mathcal{T}(n) \le n(n+1)/2$ would not suffice.

8. Proof of the main results

Proof of Theorem 1.3. Take $A = 4$ and $r = 1$, so $e = n$. By Lemma 4.1, $S_n = P_0 + P_3\zeta_q(3)$, and since $[x^2]C_3 = \frac{1}{2}$, $[x^2]C_4 = 1$ and C_1, C_2 have degree less than 2,

$$P_3 = 2\mathcal{Q}_2 = \sum_{j=0}^{n}(c_{3,j} + 2c_{4,j}).$$

By Lemma 3.2, $d_n(q)P_3 \in \mathbb{Z}[q^{\pm 1}]$. Also $d_n(q)^s W_{s,j} \in \mathbb{Z}[q^{\pm 1}]$, because $1-q^{\pm m}$ divides $d_n(q)$ for $m \le n$; with Lemma 3.2 this gives $d_n(q)^4 P_0 \in \mathbb{Z}[q^{\pm 1}]$. Put $\delta_n = d_n(q)^4$. Since q is a unit, $\mathbb{Z}[q^{\pm 1}] \subset \mathcal{O}_K$, so $\delta_n P_0$ and $\delta_n P_3$ are values at q of Laurent polynomials with integer coefficients, and lie in $\mathcal{O}_K$.

Let $\theta \in \mathcal{O}_K$ be nonzero with $\sigma\theta = -\theta$; then θ^2 is a nonzero rational integer. We show that $\theta\zeta_q(3) \notin \mathbb{Q}$; the theorem follows with $\theta = \sqrt{d}$. Suppose $\theta\zeta_q(3) = a/b$ with $a, b \in \mathbb{Z}$, $b \ge 1$. Then $\zeta_q(3) = a\theta/(b\theta^2)$ and

$$y_n := b\theta^2\,\delta_n S_n = b\theta^2\,\delta_n P_0 + a\theta\,\delta_n P_3 \in \mathcal{O}_K.$$

For large n, $y_n \neq 0$ by Lemma 6.1, since $d_n(q) \neq 0$. Apply σ. For a Laurent polynomial f with integer coefficients, $\sigma(f(q)) = f(1/q)$; with Lemma 5.1, evaluated at q as explained in Section 4, and $\sigma\theta = -\theta$,

$$\sigma(y_n) = d_n(1/q)^4\big[b\theta^2 \cdot (-q^{-n}P_0) - a\theta \cdot q^{-n}P_3\big] = -q^{-n}\,d_n(1/q)^4\,b\theta^2\,S_n.$$

Hence

$$|\,\mathrm{N}\,y_n| = |y_n\,\sigma(y_n)| = b^2\theta^4\,\kappa^{-n}\,|\,\mathrm{N}\,d_n(q)|^4\,|S_n|^2.$$

By Lemmas 6.1 and 7.1,

$$\log|\,\mathrm{N}\,y_n| \le \Big(4\cdot\frac{3}{8} - 2\Big)n^2\log(1/\kappa) + O_q(n) = -\frac{1}{2}\,n^2\log(1/\kappa) + O_q(n),$$

which is negative for large n. This contradicts $0 \neq y_n \in \mathcal{O}_K$, for which $|\,\mathrm{N}\,y_n| \ge 1$. □

Remark 8.1. With Remark 7.2 in place of the elementary bound, the exponent is $12/\pi^2 - 2 = -0.7841\ldots$. Numerically, with the exact integer $\mathrm{N}\,d_n(q)$ and S_n summed directly, the quantity $(\log|\,\mathrm{N}\,\delta_n| + 2\log|S_n| + n\log(1/\kappa))/(n^2\log(1/\kappa))$ stays near this value for $6 \le n \le 30$: it lies in $[-0.82, -0.58]$ at $q = \varphi^{-2}$ and in $[-0.84, -0.72]$ at $q = -\varphi^{-2}$. This is not used in the proof. An irrationality measure for $\sqrt{d}\,\zeta_q(3)$ would in addition need an upper bound for $|P_3(q)|$, as in Lemma 9.2; we do not pursue it.

Proof of Corollary 1.4. Let (P, Q) be admissible, $D = P^2 - 4Q$, $\eta = (P + \sqrt{D})/2$ and $\bar\eta = (P - \sqrt{D})/2 = Q/\eta$. Then $\eta \ge \varphi > 1$ (for $Q = -1$, $\eta \ge (1+\sqrt{5})/2$; for $Q = 1$, $\eta \ge (3+\sqrt{5})/2$) and $|\bar\eta| = 1/\eta < 1$, and $U_m = (\eta^m - \bar\eta^m)/\sqrt{D}$ by Binet's formula, so $U_m \neq 0$ for $m \ge 1$. Moreover

$$U_{m+1} - \eta U_m = \frac{\bar\eta^m(\eta - \bar\eta)}{\sqrt{D}} = \bar\eta^m.$$

Put $q = \bar\eta/\eta = Q\eta^{-2}$. Then $0 < |q| < 1$, q is a unit of $\mathbb{Z}[\eta]$, and $\sigma q = \eta/\bar\eta = 1/q$, where σ is the nontrivial automorphism of $K = \mathbb{Q}(\sqrt{D})$; so q is a unit of norm $+1$ of K. Since $q^m/(1-q^m) = \bar\eta^m/(\eta^m - \bar\eta^m) = \bar\eta^m/(\sqrt{D}\,U_m)$,

$$\sum_{m\ge1} m^2\Big(\frac{U_{m+1}}{U_m} - \eta\Big) = \sum_{m\ge1} \frac{m^2\bar\eta^m}{U_m} = \sqrt{D}\,\zeta_q(3),$$

which is irrational by Theorem 1.3 with $d = D$. □

Proof of Corollary 1.5. Take $(P,Q) = (1,-1)$, where $\eta = \varphi$, $\bar\eta = -\varphi^{-1}$ and $q = -\varphi^{-2}$, and $(P,Q) = (2,-1)$, where $\eta = 1+\sqrt{2}$. The second expression for the Fibonacci sum is $\sum_m m^2\bar\eta^m/F_m$. □

Remark 8.2. The method does not show that $\zeta_q(3) \notin \mathbb{Q}$. If $\zeta_q(3)$ were rational, then by Lemma 5.1 $\sigma(S_n) = q^{-n}(S_n - 2P_0)$, which is not small: it has the size of P_0. Numerically $|P_0|$ is about $|q|^{-n^2}$ (we find $\log|P_0|/(n^2\log(1/|q|)) \in [0.90, 0.95]$ at $n = 12$ for the units $q = \pm\varphi^{-2}$, $\pm(2-\sqrt{3})$, $\pm(3-2\sqrt{2})$ and $(5-\sqrt{21})/2$), which is the size of the upper bound of Lemma 9.2 at $A = 4$, $r = 1$ (a matching lower bound, for the series of [12], is the conjecture of [12, §3.5]); with it the norm estimate fails. Heuristically this is the obstruction met at $q = 1/p$, where clearing the p-power denominators of the coefficients costs as much. The concurrent preprint [22] claims $\zeta_q(3) \notin \mathbb{Q}(q)$, which contains this plain statement (see the introduction).

Remark 8.3. Theorem 1.3 and the irrationality of $\zeta_F(3) = \sum_m F_m^{-3}$ [14] do not imply each other, as far as we can see. Expanding F_m^{-3} in powers of φ^{-m} and splitting the resulting double series by the parity of the outer index expresses $\zeta_F(3)$ as a $\mathbb{Q}(\sqrt{5})$-linear combination of the numbers $\zeta_q(l) - 2^{l-1}\zeta_{q^2}(l)$, $l \in \{1,3\}$, at $q = \varphi^{-2}$, and of values of Eisenstein series of odd weight with the character χ_{-4} (cf. [14, §1]). The linear forms used here see only the q-zeta part, so the method gives no information on $\zeta_F(3)$.

9. Many values

In this section $A \ge 4$ is even and $1 \le r < A/2$ are arbitrary, q is as in Theorem 1.3, $\kappa = |q|$, $\nu = \log(1/\kappa) > 0$ and $e = (A-2r)n/2$. We prove Theorem 1.1. Besides the lemmas of Sections 2–7, which hold for all such (A,r), it uses two classical inputs: the asymptotic $\mathcal{T}(n) = 3n^2/\pi^2 + O(n\log n)$ of Remark 7.2, in both directions, and Nesterenko's linear independence criterion. None of this section is formalised.

Denominators.

Lemma 9.1. *In* $\mathbb{Q}(q)$, $(A-1)!\,d_n(q)^{A-l}P_l \in \mathbb{Z}[q^{\pm1}]$ *for odd* $3 \le l \le A-1$, *and* $d_n(q)^A P_0 \in \mathbb{Z}[q^{\pm1}]$. *Hence* $\Delta_n(q) := (A-1)!\,d_n(q)^A$ *satisfies* $\Delta_n P_0, \Delta_n P_l \in \mathbb{Z}[q^{\pm1}]$.

Proof. Since $(s-1)!\,C_s(x) = (x+1)(x+2)\cdots(x+s-1) \in \mathbb{Z}[x]$, $(A-1)!\,[x^\lambda]C_s \in \mathbb{Z}$ for $s \le A$, and $[x^\lambda]C_s = 0$ unless $\lambda \le s-1$. So only $s \ge l$ contributes to $P_l = 2\sum_{s,j} c_{s,j}[x^{l-1}]C_s$, and for those s,

$$(A-1)!\,d_n^{A-l}c_{s,j}\cdot 2[x^{l-1}]C_s = d_n^{s-l}\cdot\big(d_n^{A-s}c_{s,j}\big)\cdot 2(A-1)!\,[x^{l-1}]C_s \in \mathbb{Z}[q^{\pm1}]$$

by Lemma 3.2. For P_0, $d_n^s W_{s,j} \in \mathbb{Z}[q^{\pm1}]$ because $1-q^{\pm m}$ divides d_n for $m \le n$, and $d_n^A c_{s,j}W_{s,j} = (d_n^{A-s}c_{s,j})(d_n^s W_{s,j})$. □

As in the proof of Theorem 1.3, the values $\Delta_n P_0$ and $\Delta_n P_l$ at q lie in $\mathbb{Z}[q^{\pm1}] \subset \mathcal{O}_K$, and by Lemma 5.1

$$\sigma\big(P_l(q)\big) = q^{-e}P_l(q), \qquad \sigma\big(P_0(q)\big) = -q^{-e}P_0(q), \tag{9.1}$$

since σ maps the value at q of a quotient of integer Laurent polynomials to its value at $1/q$, and the denominators $\Delta_n(q)$, $\Delta_n(1/q)$ do not vanish (q is not a root of unity).

Coefficients.

Lemma 9.2. *Let $q \in \mathbb{C}$ with $0 < \kappa = |q| < 1$ and $\rho = (1-\kappa)/2$. Put $c_+ = \prod_{i\ge1}(1+\kappa^i)$, $\gamma_- = \prod_{i\ge1}(1-(1+\rho)\kappa^i)$ and $\gamma_+ = \prod_{i\ge1}(1+(1+\rho)\kappa^i)$. Then $\gamma_- > 0$, and for $0 \le j \le n$, $1 \le s \le A$,*

$$|c_{s,j}(q)| \le \rho^{s-A(j+1)}\, c_+^{A-2r+1}\,\gamma_+\,\gamma_-^{-A}\,(1+\rho)^{rn+e}\,\kappa^{-\mathcal{E}(j)},$$

where

$$\mathcal{E}(j) = \tfrac{A}{2}j(n-j) - \tfrac{A}{2}j + \tfrac{r^2n^2}{2} + \tfrac{rn}{2}.$$

Consequently $\log\max\big(|P_0(q)|, \max_l|P_l(q)|\big) \le \frac{A+4r^2}{8}\,n^2\log(1/\kappa)+O(n)$, *with the implied constant depending only on A, r and κ.*

Proof. Put $u = 1-q^jT$ and $w = 1-u$, so $T = q^{-j}w$. As a function of u, R has its poles at $u_i = 1-q^{j-i}$, $0 \le i \le n$, and $|u_i| \ge 1-\kappa > \rho$ for $i \ne j$. By (3.1) the principal part of R at $u=0$ is $\sum_s c_{s,j}u^{-s}$, so $c_{s,j} = \frac{1}{2\pi i}\oint_{|u|=\rho} R\,u^{s-1}\,du$ and $|c_{s,j}| \le \rho^s \max_{|u|=\rho}|R|$. On this circle $1-\rho \le |w| \le 1+\rho$; note $1-\rho-\kappa = \rho$ and $(1+\rho)\kappa = \kappa(3-\kappa)/2 < 1$, which gives $\gamma_- > 0$. Bound the factors of (2.1):

- $|(q;q)_n|^{A-2r} \le c_+^{A-2r}$;
- $(q^{-rn}T;q)_{rn} = \prod_{M=j+1}^{j+rn}(1-q^{-M}w)$, and $|1-q^{-M}w| \le \kappa^{-M}(1+\rho)(1+\kappa^M)$, so the modulus is at most $\kappa^{-rnj-rn(rn+1)/2}(1+\rho)^{rn}c_+$;
- $(q^{n+1}T;q)_{rn} = \prod_{i=0}^{rn-1}(1-q^{n+1+i-j}w)$ has distinct exponents ≥ 1, so modulus at most γ_+;
- $|T^e| \le \kappa^{-je}(1+\rho)^e$;
- $(T;q)_{n+1} = \prod_{i=0}^{n}(1-q^{i-j}w)$: for $i<j$, $|1-q^{i-j}w| \ge \kappa^{i-j}(|w|-\kappa^{j-i}) \ge \kappa^{i-j}\rho$; for $i=j$ the factor is u, of modulus ρ; the factors with $i>j$ have product at least γ_-. So $|(T;q)_{n+1}| \ge \kappa^{-j(j+1)/2}\rho^{j+1}\gamma_-$.

Multiplying, $|R| \le c_+^{A-2r+1}\gamma_+(1+\rho)^{rn+e}\kappa^{-\mathcal{E}(j)}\rho^{-A(j+1)}\gamma_-^{-A}$, where $\mathcal{E}(j) = rnj + rn(rn+1)/2 + je - Aj(j+1)/2$; since $rnj + je = Anj/2$, this is the stated $\mathcal{E}(j)$, and $\mathcal{E}(j) \le An^2/8 + r^2n^2/2 + rn/2$. Since $\rho^{s-A(j+1)} \le \rho^{-A(n+1)}$, this gives $\log\max_{s,j}|c_{s,j}| \le \frac{A+4r^2}{8}n^2\nu + O(n)$. The coefficients of C_s are nonnegative with sum $C_s(1) = s$, so $|P_l| \le (n+1)A(A+1)\max|c_{s,j}|$. For $1 \le m \le n$, $|(1-q^m)^{-s} - 1| \le (1-\kappa)^{-A} + 1$ and $|(1-q^{-m})^{-s}| \le \max(1, (\kappa/(1-\kappa))^A)$, so $|W_{s,j}| \le nw_0$ with w_0 depending on A and κ only, and $|P_0| \le n(n+1)Aw_0\max|c_{s,j}|$. □

This is the analogue for our coefficients of [12, Lemma 5], proved by the same Cauchy estimate; we include the proof because our coefficients $c_{s,j}$ are not those of [12], and because we want an explicit bound.

Balanced trace forms. Since $\Phi_1(1/q) = -q^{-1}\Phi_1(q)$ and $\Phi_\ell(1/q) = q^{-\phi_{\mathrm{E}}(\ell)}\Phi_\ell(q)$ for $\ell \ge 2$ (Φ_ℓ is self-reciprocal of degree $\phi_{\mathrm{E}}(\ell)$), we have exactly $\sigma(d_n(q)) = d_n(1/q) = -q^{-\mathcal{T}(n)}d_n(q)$, so $|\sigma\Delta_n(q)| = \kappa^{-A\mathcal{T}(n)}|\Delta_n(q)|$. Moreover $|\log|d_n(q)|| \le \sum_{\ell\le n}\sum_{a|\ell}|\log|1-q^a|| \le nB$, with B as in Lemma 7.1, because each a divides at most n of the $\ell \le n$. With Remark 7.2,

$$\log|\Delta_n(q)| = O(n), \qquad \log|\sigma\Delta_n(q)| = \frac{3A}{\pi^2}n^2\nu + O(n\log n). \tag{9.2}$$

In this subsection d is squarefree; the general case is reduced to it at the end of the proof of Theorem 1.1. The point of the construction is to split the cost $\log|\mathrm{N}\,\Delta_n|$ evenly between the two real places. For n put $\varrho_n = \kappa^{-e}|\sigma\Delta_n|/|\Delta_n|$, and let $m = m_n \in \mathbb{Z}$ be the unique integer with

$$2 \le \kappa^{2m}\varrho_n < 2\kappa^{-2}. \tag{9.3}$$

Put $\Delta_n' = q^{-m}\Delta_n(q) \in \mathcal{O}_K$. Since $\sigma q = 1/q$, $|\Delta_n'| = \kappa^{-m}|\Delta_n|$ and $|\sigma\Delta_n'| = \kappa^m|\sigma\Delta_n|$, so $M_n := \kappa^{-e}|\sigma\Delta_n'|$ satisfies $|\Delta_n'| \le M_n/2$ and $M_n^2 = \kappa^{2m}\varrho_n \cdot \kappa^{-e}|\Delta_n\sigma\Delta_n| \in \kappa^{-e}|\Delta_n\sigma\Delta_n| \cdot [2, 2\kappa^{-2})$. By (9.2),

$$\log M_n = \frac{3A}{2\pi^2}n^2\nu + O(n\log n). \tag{9.4}$$

For odd $3 \le l \le A-1$ define

$$\ell_{0,n} = \mathrm{Tr}_{K/\mathbb{Q}}\big(\Delta_n' P_0(q)\big), \qquad \ell_{l,n} = \mathrm{Tr}_{K/\mathbb{Q}}\big(\Delta_n' P_l(q)/\sqrt{d}\big),$$
$$L_n = \ell_{0,n} + \sum_l \ell_{l,n}\sqrt{d}\,\zeta_q(l),$$

where $K = \mathbb{Q}(\sqrt{d})$ with d squarefree.

Lemma 9.3. (a) $\ell_{0,n}, \ell_{l,n} \in \mathbb{Z}$. (b) $L_n = (\Delta_n' - q^{-e}\sigma\Delta_n')\,S_n(q)$. (c) *For large* n, $L_n \ne 0$ *and* $\log|L_n| = \big(\frac{3A}{2\pi^2} - \frac{r(A-2r)}{2}\big)n^2\nu + O(n\log n)$. (d) $\log\max(|\ell_{0,n}|, \max_l|\ell_{l,n}|) \le \big(\frac{3A}{2\pi^2} + \frac{A+4r^2}{8}\big)n^2\nu + O(n\log n)$.

Proof. (a) $\mathrm{Tr}(\mathcal{O}_K) \subset \mathbb{Z}$, and $\mathrm{Tr}(x/\sqrt{d}) \in \mathbb{Z}$ for $x \in \mathcal{O}_K$: if $d \equiv 2, 3 \pmod 4$ then $x = a + b\sqrt{d}$ with $a, b \in \mathbb{Z}$ and the trace is $2b$; if $d \equiv 1 \pmod 4$ then

$x = (a + b\sqrt{d})/2$ with $a \equiv b \pmod 2$ and the trace is b. Apply this to $x = \Delta'_n P_0(q)$ and $x = \Delta'_n P_l(q)$, which lie in $\mathcal{O}_K$ by Lemma 9.1.

(b) Write $\mathrm{Tr}(z) = z + \sigma z$. The terms z add up to $\Delta'_n(P_0 + \sum_l P_l\zeta_q(l)) = \Delta'_n S_n$ by Lemma 4.1. By (9.1) and $\sigma\sqrt{d} = -\sqrt{d}$, the conjugate terms are $\sigma(\Delta'_n P_0) = -q^{-e}\sigma(\Delta'_n)P_0$ and $\sigma(\Delta'_n P_l/\sqrt{d}) \cdot \sqrt{d}\,\zeta_q(l) = -q^{-e}\sigma(\Delta'_n)P_l\zeta_q(l)$; they add up to $-q^{-e}\sigma(\Delta'_n)S_n$.

(c) $|\Delta'_n - q^{-e}\sigma\Delta'_n| \in [M_n - |\Delta'_n|, M_n + |\Delta'_n|] \subset [M_n/2, 3M_n/2]$. Combine with Lemma 6.1 and (9.4).

(d) $|\ell_{0,n}| \le (|\Delta'_n| + M_n)|P_0(q)| \le \frac{3}{2}M_n|P_0(q)|$, and likewise $|\ell_{l,n}| \le \frac{3}{2}M_n|P_l(q)|/\sqrt{d}$. Apply Lemma 9.2 and (9.4). □

Proof of Theorem 1.1. We use Nesterenko's criterion [15], in the precise form of [8, Theorem A] (under its hypothesis (N)), rescaled as in [12, Proposition 2]: let $\vartheta_1, \dots, \vartheta_N$ be real numbers, and suppose that there are linear forms $\sum_j p_{j,n}\vartheta_j$ with $p_{j,n} \in \mathbb{Z}$ such that, as $n \to \infty$, $\frac{1}{n^2}\log|\sum_j p_{j,n}\vartheta_j| \to \alpha < 0$ and $\limsup \frac{1}{n^2}\log|p_{j,n}| \le \beta$ for every j. Then $\dim_{\mathbb{Q}}(\mathbb{Q}\vartheta_1 + \cdots + \mathbb{Q}\vartheta_N) \ge 1 - \alpha/\beta$. (The usual statement has N' in place of n^2; it applies to the forms indexed by $N' \mapsto \lfloor\sqrt{N'}\rfloor$, since $\lfloor\sqrt{N'}\rfloor^2 = N' + O(\sqrt{N'})$, and it does not require the forms to be distinct.)

Take $\vartheta_1 = 1$ and $\vartheta_2, \dots, \vartheta_{A/2}$ equal to $\sqrt{d}\,\zeta_q(3), \dots, \sqrt{d}\,\zeta_q(A-1)$, with the forms L_n. By Lemma 9.3, $\alpha = (\frac{3A}{2\pi^2} - \frac{r(A-2r)}{2})\nu$ and $\beta = (\frac{3A}{2\pi^2} + \frac{A+4r^2}{8})\nu > 0$. If $\alpha \ge 0$ then $\tau(A,r) \le 0$ and there is nothing to prove; otherwise

$$\dim \ge 1 - \frac{\alpha}{\beta} = 1 + \frac{r(A-2r)/2 - 3A/(2\pi^2)}{3A/(2\pi^2) + (A+4r^2)/8} = 1 + \tau(A,r).$$

The bound does not depend on q or d, and it is the same for every d with $K = \mathbb{Q}(\sqrt{d})$, since $\sqrt{d}\,\zeta_q(l)$ changes by a common rational factor. □

Proof of the asymptotic bound and of Corollary 1.2. Put $r = t\sqrt{A}$. Then $\tau(A, t\sqrt{A})/\sqrt{A} \to 4\pi^2 t/(12 + \pi^2 + 4\pi^2 t^2)$ as $A \to \infty$, which is largest at $t = \sqrt{12+\pi^2}/(2\pi) = 0.744\ldots$, with value $\pi/\sqrt{12+\pi^2} = 0.67178\ldots$; taking $r = \lceil t\sqrt{A}\rceil$, which is admissible for $A \ge 6$, gives the same limit. For Corollary 1.2, $1+\tau(12,2) = 1+(64\pi^2-144)/(144+28\pi^2) = 2.1601\ldots > 2$, so $\dim_{\mathbb{Q}}(\mathbb{Q} + \mathbb{Q}\sqrt{d}\zeta_q(3) + \cdots + \mathbb{Q}\sqrt{d}\zeta_q(11)) \ge 3$, while $\dim_{\mathbb{Q}}(\mathbb{Q} + \mathbb{Q}\sqrt{d}\zeta_q(3)) \le 2$. The Lucas form follows from the identity $\sum_m m^{l-1}(U_{m+1}/U_m - \eta) = \sqrt{D}\,\zeta_q(l)$, proved for $l = 3$ in the proof of Corollary 1.4 and valid for every s by the same computation. □

Remark 9.4. Without the balancing, that is with Δ_n in place of Δ'_n, the form carries the whole cost $\log|\mathrm{N}\,\Delta_n|$ at one place, and the same argument gives only $1 + (4\pi^2 r(A-2r) - 24A)/(24A + \pi^2(A+4r^2))$, which is asymptotically $(\pi/\sqrt{24+\pi^2})\sqrt{A} = 0.5398\ldots\sqrt{A}$ and does not recover Theorem 1.3 at $A = 4$. With the balancing, writing $\mathrm{dec} = r(A-2r)/2$, $\deg = 3A/\pi^2$ and $\mathrm{coef} = (A+4r^2)/8$, we have $1 + \tau = (\mathrm{dec} + \mathrm{coef})/(\deg/2 + \mathrm{coef})$, while the bound of [12, Theorem 2] at $q = 1/p$ is $(\mathrm{dec} + \mathrm{coef})/(\deg + 2\,\mathrm{coef})$, where

the second place is p-adic and the denominator must also clear powers of p of size coef. So $1+\tau(A,r)$ is exactly twice the bound of [12]. At $A=4$, $r=1$ the condition $\tau>0$ reads $2>12/\pi^2$, the inequality behind the proof of Theorem 1.3 with the asymptotic of Remark 7.2.

The balancing (9.3) itself is the standard device of moving an $\mathcal{O}_K$-multiplier by a unit so that its archimedean sizes are equal, as in Siegel's lemma over a number field [1, Chapter 2]. What is specific here is its combination with the Galois-equivariance (9.1) and the twist by $\sqrt{d}$, which make the form small at both real places.

The comparison with [12] is one of the same leading constants (decay, coefficient size and denominator, which our variant of the series shares with that of [12]) and the same single-form criterion, at two kinds of q. At $q=1/p$ the results of [12] have been improved: Jouhet and Mosaki [11] prove that one power of d_n can be saved in the denominators of a variant $\tilde{S}_n$ of the series, and deduce that one of $\zeta_q(3),\zeta_q(5),\zeta_q(7),\zeta_q(9)$ is irrational; Fischler and Zudilin [8, Theorems 4 and 6] refine Nesterenko's criterion by using divisibility properties of the coefficients, and obtain for instance that $1,\zeta_q(i_0),\dots,\zeta_q(i_3)$ are linearly independent for some odd $i_0<\cdots<i_3$ with $i_0\le 9$, $i_1\le 37$, $i_2\le 83$, $i_3\le 145$. Neither refinement changes the asymptotic constant, and we have not adapted either to units. In our setting the denominator of P_0 divides d_n^{A-1} in all cases we computed exactly (31 triples (A,r,n) with $A\le 12$), but we have not proved this for S_n; if it held, $12A$ would become $12(A-1)$ in $\tau(A,r)$.

10. Related work

Even weights and weight one. For even k the values $\zeta_q(k)$ are polynomials in the Eisenstein series E_2,E_4,E_6 at q, and at algebraic q with $0<|q|<1$ the values of these series are algebraically independent by Nesterenko's theorem [16]. At weight one there is a long line of irrationality and linear independence results for Lambert series at algebraic bases: Tachiya [19] for algebraic integers of Pisot type, Bundschuh and Väänänen [4, 5], Bundschuh [3] for integers of $\mathbb{Q}$ or of an imaginary quadratic field, and Dixit, Kumar and Pathak [7], whose results on values of the q-exponential function at Pisot bases include the weight-one value $\zeta_q(1)$. All of these concern weight at most two.

Odd weights at $q=1/p$. For odd $k\ge 3$ the known results concern $q=1/p$ with $p\in\mathbb{Z}\setminus\{0,\pm1\}$: the dimension bound of Krattenthaler, Rivoal and Zudilin [12], with asymptotic constant $\pi/(2\sqrt{\pi^2+12})=0.336\ldots$, and its refinements for fixed A by Jouhet and Mosaki [11] and Fischler and Zudilin [8]; Postelmans and Van Assche [17] and Zudilin [24] treat weights one and two. The extension of [12] to algebraic q sketched in [12, §3.5] is conditional on a conjectured asymptotic for the coefficients, and Bundschuh and Zudilin [6, §6] list the arithmetic of $\zeta_q(k)$ at reciprocals of Pisot and Salem numbers as an open problem. The unrefereed draft [23] claims $\zeta_{1/p}(3)\notin\mathbb{Q}$ for integers

$p \geq 2$, with a Lean formalisation of its main theorem (not checked by us), by a moving-denominator Hankel determinant.

Concurrent work. The preprints [22, 21], discussed in the introduction, claim $\zeta_{1/b}(3) \notin \mathbb{Q}(b)$, and the linear independence of $1, \zeta_{1/b}(1), \zeta_{1/b}(2), \zeta_{1/b}(3)$ over $\mathbb{Q}(b)$, for the class of bases b described in the introduction, which contains the Pisot and Salem numbers; the first builds on [23]. A third preprint by the same contributor [20] extends the first claim to some bases b with several conjugates outside the unit disk, under a condition on the Mahler measure. These claims contain Theorem 1.3 as a special case, and the plain statement $\zeta_q(3) \notin \mathbb{Q}$, which our method does not reach (Remark 8.2). Our proof of Theorem 1.3 was found independently, uses a different construction (the very-well-poised forms of [12] and the reciprocity of Lemma 5.1), and is checked in Lean. The results of [22, 21, 20] stop at weight three; we know of no other lower bound for the dimension spanned by odd q-zeta values at an algebraic irrational q than the one of Section 9.

11. Formal verification

The Lean development is the repository

`https://github.com/telaviv/qzeta3-quadratic-units`

all file names below refer to it. Its Lean sources are those of commit `6318359`; later commits change only documentation and this paper. It (library `QZeta3`, Lean 4.35.0, Mathlib v4.35.0, about 3200 lines) proves the following four theorems, stated in `QZeta3/Final.lean`:

```
theorem QZeta3.irrational_sqrt_mul_zetaq3 {q : ℝ} {s : ℤ}
    (hq0 : q ≠ 0) (hq1 : |q| < 1) (hs : q + q⁻¹ = s)
    {d : ℕ} (hd : 0 < d) (hsq : IsSquare ((d : ℤ) * (s ^ 2 - 4))) :
    Irrational (Real.sqrt d * zetaq3 q)

theorem QZeta3.irrational_sub_inv_mul_zetaq3 {q : ℝ} {s : ℤ}
    (hq0 : q ≠ 0) (hq1 : |q| < 1) (hs : q + q⁻¹ = s) :
    Irrational ((q - q⁻¹) * zetaq3 q)

theorem QZeta3.irrational_lucas_sum {a : ℕ} {E : ℤ}
    (h : (E = 1 ∧ 1 ≤ a) ∨ (E = -1 ∧ 3 ≤ a)) :
    Irrational (∑' m : ℕ, (m : ℝ) ^ 2 *
      ((lucasU a E (m + 1) : ℝ) / lucasU a E m - epsU a E))

theorem QZeta3.irrational_fib_sum :
    Irrational (∑' m : ℕ, (m : ℝ) ^ 2 *
      ((Nat.fib (m + 1) : ℝ) / Nat.fib m - Real.goldenRatio))
```

The definitions they use are, in `QZeta3/Defs.lean`,

```
noncomputable def zetaq3 (q : ℝ) : ℝ :=
  ∑' m : ℕ, (m : ℝ) ^ 2 * q ^ m / (1 - q ^ m)

def lucasU (a : ℕ) (E : ℤ) : ℕ → ℤ
  | 0 => 0
  | 1 => 1
```

```
  | (m + 2) => a * lucasU a E (m + 1) + E * lucasU a E m

noncomputable def epsU (a : ℕ) (E : ℤ) : ℝ :=
  (a + Real.sqrt ((a : ℝ) ^ 2 + 4 * E)) / 2
```

The proofs use only the axioms `propext`, `Classical.choice` and `Quot.sound` (`Axioms.lean`); there is no `sorry`, no `axiom` declaration, no `native_decide` and no `implemented_by`. The definitions and the six ingredient statements, in the files `QZeta3/Defs.lean` and `QZeta3/Statements.lean`, were fixed ("frozen") in the first commit, before any ingredient was proved, and were not changed afterwards. The four theorem statements above were committed with placeholder proofs before the last two ingredients were proved, and their text is unchanged since then.

The statements. In `irrational_sqrt_mul_zetaq3`, the hypotheses $q \neq 0$, $|q| < 1$ and $q+q^{-1} = s \in \mathbb{Z}$ say exactly that q is a real quadratic unit of norm $+1$ with $0 < |q| < 1$ (as noted after Theorem 1.3), and $d > 0$ with $d(s^2-4)$ a square says exactly that $\mathbb{Q}(\sqrt{d}) = \mathbb{Q}(q)$; so it is Theorem 1.3. `irrational_sub_inv_mul_zetaq3` is the form with $\theta = q - q^{-1}$. In `irrational_lucas_sum` the recurrence is $U_{m+2} = aU_{m+1} + EU_m$, so $(a, E) = (P, -Q)$, and the hypothesis is admissibility; `epsU a E` is η. The $m = 0$ terms of the series are 0 in Lean (for `zetaq3` the term is $0 \cdot 1/0 = 0$; in the Lucas series the factor m^2 vanishes). `irrational_fib_sum` is the case $a = E = 1$, with Mathlib's `Nat.fib` and `Real.goldenRatio` (`QZeta3.lucasU_one_one`, `QZeta3.epsU_one_one`).

What has to be trusted. The kernel checks the proofs; to see that the statements mean what is claimed one reads the four statements above, the three definitions, and Mathlib's `tsum`, `Irrational`, `Real.sqrt`, `IsSquare`, `Nat.fib` and `Real.goldenRatio`. A divergent `tsum` has the value 0, which is rational, so none of the theorems can hold for a degenerate reason; in addition `QZeta3.summable_zetaq3` and `QZeta3.summable_lucas` prove that the series converge. A blind back-translation of the statements, by a reviewer (also a Claude model) who saw only the Lean source without comments or intended meaning, found no mismatch with the author's working statement of Theorem 1.3 and Corollaries 1.4–1.5 (the same content as here); it is recorded in `FIDELITY.md`. The author also checked the statements against Theorem 1.3 and the corollaries (`STATEMENT_REVIEW.md`), and `TRUST.md` summarises the trust base.

The chain. `QZeta3/Defs.lean` defines the construction for $A = 4$, $r = 1$ over an arbitrary field: the q-Pochhammer symbol `qPoch`, the rational function `Rfun` of (2.1), the coefficients `cc` of (3.1), the coefficients `P3` and `P0` of Lemma 4.1 (with `Wf` for $W_{s,j}$), and, over $\mathbb{R}$, `dn` for d_n and `Sn` for S_n. `QZeta3/Statements.lean` states six ingredients as propositions, and `QZeta3.irrational_sub_inv_mul_zetaq3_of` (in `QZeta3/Main.lean`) derives the theorem from them by the argument of Section 8; `QZeta3/Reduction.lean` and `QZeta3/Corollary.lean` derive

the $\sqrt{d}$ form and the corollaries. In the following table all Lean names are in the namespace `QZeta3`.

Paper	Lean statement	Lean proof
(3.1)	`PFIdentity`	`PartialFrac.pfIdentity`
Lemma 4.1	`Expansion`	`Expand.expansion`
Lemmas 2.2, 5.1, Cor. 3.1	`Recip`	`Reciprocity.recip`
Lemma 3.2, §8	`Integrality`	`Denom.integrality`
Lemmas 2.1, 6.1	`SizeBound`	`Size.sizeBound`
Lemma 7.1	`CostBound`	`Cost.costBound`
Theorem 1.3, $\theta = q - q^{-1}$		`irrational_sub_inv_mul_zetaq3_of`
Theorem 1.3, $\sqrt{d}$		`irrational_sqrt_mul_zetaq3_ing`
Corollary 1.4		`lucas_sum_eq`
		`irrational_lucas_sum_ing`

The formal statements are the lemmas in the form the assembly needs, for $A = 4$, $r = 1$ and real q. `Expansion` and `Recip` are derived from `PFIdentity`. `Recip` states Lemma 5.1 at every real $q \neq 0$ with $|q| \neq 1$. `Integrality` says that $d_n^4 P_0$ and $d_n^4 P_3$ agree, at every real $x \neq 0$ with $|x| \neq 1$, with one fixed integer Laurent polynomial (`LaurentInt`); this is what allows evaluation at both q and q^{-1}. `SizeBound` gives $S_n \neq 0$ and $|S_n| \leq C|q|^{n(n+1)}$ for large n, and `CostBound` is $|d_n(q)d_n(q^{-1})|^8 \leq C^n|q|^{-3(n+1)^2}$, the bound $\mathcal{T}(n) \leq \frac{3}{8}(n+1)^2$ of Lemma 7.1.

Differences from the written proof.

- *No number fields.* The formal proof does not use Mathlib's number fields. The role of $\sqrt{d}$ is played by $\theta = q - q^{-1}$, as in Section 8; σ is replaced by evaluation of integer Laurent polynomials at q^{-1}, and the norm by the product $p(q)\,p(q^{-1})$, which is an integer for $p \in \mathbb{Z}[X]$ because $q + q^{-1} \in \mathbb{Z}$ (`QZeta3.norm_int`).
- *Partial-fraction coefficients.* The coefficients $c_{s,j}$ are *defined* as the coefficients of the power-series expansion of $u^4 R_n$ at the pole $T = q^{-j}$, in the variable $u = 1 - q^j T$ (`Gser`, `cc`), and (3.1) is then proved. With this definition, Corollary 3.1 follows directly from Lemma 2.2, without appeal to the uniqueness of partial fractions.
- *The cyclotomic bound.* The formal proof uses the elementary bound $\mathcal{T}(n) \leq \frac{3}{8}(n+1)^2$ of Lemma 7.1, with exponent $-\frac{1}{2}$, not the asymptotic of Remark 7.2, which is not in Mathlib. (The author's working notes used the asymptotic, with exponent $12/\pi^2 - 2$.)
- *Generality.* The formal statements fix $A = 4$, $r = 1$; Theorem 1.3 needs nothing else. The Lean form of Theorem 1.3 allows every integer $d > 0$ with $\mathbb{Q}(\sqrt{d}) = \mathbb{Q}(q)$, as does the statement above.

Provenance. The library name `QZeta3` (for the q-analogue of $\zeta(3)$) is also the name of the Lean formalisation [18] of [23]. The two are different developments, of different theorems, and ours is not a fork. One file, `QZeta3/Cost/Cyclo.lean` (cyclotomic size bounds), is adapted from the formalisation accompanying [13], which generalises that of [14], which in

turn adapts part of [18] (Apache License 2.0); the file says so in its header, and the repository's `NOTICE` file records it. The mathematics of the present paper does not depend on [23]: the construction is that of [12].

Use of AI tools. This work was carried out with substantial assistance from AI systems (Anthropic's Claude models), which were used to search the literature, to develop and check the proof, to write the Lean formalisation and to draft this text. The author directed the project and is responsible for its content. The formal proof is checked by the Lean kernel and uses only the standard axioms `propext`, `Classical.choice` and `Quot.sound`.

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