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The equilateral triangle is the extremal triangle for the lowest Robin eigenvalue, for every boundary parameter

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Abstract

Let $\lambda_1(T,\alpha)$ be the lowest eigenvalue of the Laplacian on a triangle T with the Robin condition $\partial_\nu u + \alpha u = 0$. Lotoreichik and Lu conjectured that, among triangles of given area, the equilateral triangle minimises $\lambda_1(T,\alpha)$ when $\alpha > 0$ and maximises it when $\alpha < 0$. We prove both halves, with the equilateral triangle as the unique extremiser; the same holds at fixed perimeter. The proof is computer-assisted.

For $\alpha > 0$ the result is new; previously the inequality was known for all triangles only when $\alpha\sqrt{|T|}$ is smaller than an absolute but non-explicit constant (Ferudun). Thin triangles are handled by slicing, and $\alpha\sqrt{|T|} \le 1/100$ by an explicit vector field and a Schur-complement criterion. Near the equilateral triangle E, the second-order perturbation constant of E is bounded from above through certified *lower* bounds for two perturbed eigenvalue problems, uniformly in α. The remaining compact set of shapes is covered by 100 cells; an exact identity in the Klein model of the hyperbolic plane of shapes, together with concavity, reduces each cell to certified Crouzeix–Raviart lower bounds at its vertices, which are checked in exact integer and ball arithmetic.

For $\alpha < 0$ the first proof is due to Ma (computer-assisted, 2026); ours was found independently. Write $\beta = -\alpha\sqrt{|T|}$. For $\beta \le 6/5$ two trial functions suffice. For $\beta \ge 1.1531$ an exact corner bound disposes of all triangles with a small angle, and on the remaining near-equilateral triangles three transplanted corner waves, with a one-step perturbation estimate, reduce the problem to a one-variable inequality, which is verified in ball arithmetic.

AI transparency statement. The results, their proofs, the code for the certified computations and this manuscript were produced by artificial intelligence. The human author initiated the broader research and directed the AI systems, but did not write the text and has not checked the arguments. Details are given at the end of the paper.

1 Introduction

For a bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$ and $\alpha \in \mathbb{R}$ let

$$\lambda_1(\Omega,\alpha) = \inf_{0\neq u\in H^1(\Omega)} \frac{\int_\Omega |\nabla u|^2 + \alpha \int_{\partial\Omega} u^2}{\int_\Omega u^2} \tag{1}$$

be the lowest eigenvalue of $-\Delta$ with the Robin condition $\partial_\nu u + \alpha u = 0$, where ν is the outer normal. For $\alpha > 0$ the Bossel–Daners inequality [5, 12] states that the disc minimises $\lambda_1(\cdot,\alpha)$ among domains of given area. For $\alpha < 0$ we have $\lambda_1 < 0$, and Bareket's conjecture [3] that the disc maximises $\lambda_1(\cdot,\alpha)$ among simply connected planar domains of given area is open (see [2, 21] for its history). The polygonal analogue for triangles is the following.

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Conjecture (Lotoreichik–Lu; AIM problem list [1], Conjecture 1.3; see also [24, 21]). Among all triangles of a given area, $\lambda_1(T,\alpha)$ is minimised by the equilateral triangle when $\alpha > 0$ and maximised by it when $\alpha < 0$.

We prove the conjecture. By scaling, $\lambda_1(sT,\alpha) = s^{-2}\lambda_1(T,s\alpha)$ for $s > 0$, so only the dimensionless coupling $\alpha\sqrt{|T|}$ matters.

Theorem 1.1. *Let T be a triangle and let E_T be the equilateral triangle with $|E_T| = |T|$.*

(1) *If $\alpha > 0$, then $\lambda_1(T,\alpha) \geq \lambda_1(E_T,\alpha)$.*

(2) *If $\alpha < 0$, then $\lambda_1(T,\alpha) \leq \lambda_1(E_T,\alpha)$.*

In both cases equality holds only if T is equilateral.

Corollary 1.2 (fixed perimeter)**.** *Let T be a triangle and let E^T be the equilateral triangle with the same perimeter as T. Then $\lambda_1(T,\alpha) \geq \lambda_1(E^T,\alpha)$ if $\alpha > 0$ and $\lambda_1(T,\alpha) \leq \lambda_1(E^T,\alpha)$ if $\alpha < 0$. In both cases equality holds only if T is equilateral.*

As $\alpha \to +\infty$, Theorem 1.1(1) becomes the Pólya–Szegő theorem for the Dirichlet eigenvalue [32], which is proved by Steiner symmetrisation; as $\alpha \to 0^{\pm}$ both parts reduce to the isoperimetric inequality for triangles, because $\lambda_1(T,\alpha) = \alpha P/|T| + O(\alpha^2)$, with P the perimeter. Steiner symmetrisation does not control the boundary term in (1).

Priority and related work. The case $\alpha < 0$ was first proved by Yicen Ma [27], in a computer-assisted preprint deposited on Zenodo on 6 October 2026, which has not been refereed. The code and certificate files that the preprint describes are not part of the deposit; an AI referee working for us re-implemented its middle-range criterion in Arb ball arithmetic and confirmed it on $1 \leq \beta \leq 10$ (1146 boxes; the script is in our repository, labelled as a check of prior work). We found the proof in Part I independently and learned of [27] only after it was complete; Part I is therefore an independent second proof of Theorem 1.1(2). The two proofs share several ingredients: the trial functions for small coupling and the identity of Lemma 2.5 (which is (5) of [27]), the exact corner bound of Lemma 4.1 (which is (7)–(8) of [27]), the same threshold angle separating near-equilateral from far triangles, and the convexity bound for the strain (Lemma 4.7). They differ in the treatment of near-equilateral triangles. Ma uses the span of u, xu, yu (u the equilateral ground state), the trial function $(1 - \tau\, v \cdot x)u$ and Taylor bounds in the strain, certifies $1 \leq \beta \leq 10$ by interval arithmetic and treats $\beta \leq 1$ and $\beta \geq 10$ analytically. We use the three transplanted corner waves, exact side-ratio identities and an estimate of the tunnelling gap, certify a single one-variable inequality for all $\beta \geq 1.1531$, and use the two-function argument of Theorem 1.3 for $\beta \leq 6/5$. The perimeter version of the case $\alpha < 0$ (Corollary 1.2), a second, independently written certificate and public code are specific to Part I. Ma does not treat $\alpha > 0$.

For $\alpha > 0$, Ferudun [14] proved, in an unrefereed AI-assisted preprint, the comparison of Theorem 1.1 for $0 < |\alpha|\sqrt{|T|} < \varepsilon$, with an absolute but non-explicit ε; for $\alpha > 0$ this is the only earlier result covering all triangles that we know of. Krejčiřík, Lotoreichik and Vu [21] proved, for $\alpha < 0$, that the equilateral triangle is a strict local maximiser for $\beta \leq 0.92$ (numerically up to about 1.63), and that the comparison holds for each fixed triangle when β is sufficiently small or sufficiently large, with thresholds depending on the triangle, and uniformly for small β on compact sets of shapes. Carbotti, Cito and Pallara [8] prove the existence of optimal generalised polygons without identifying them. For the Robin *torsional rigidity*, Gui, Hu, Li and Yang [16] proved that the equilateral triangle is the maximiser among triangles of given area for every positive boundary

parameter, by cutting the polygon from its incentre into cells with Neumann conditions on the cuts; torsional rigidity is additive over the cells, whereas for the eigenvalue this relaxation gives only the smallest cell eigenvalue, which is not enough. For the Dirichlet problem, Endo, Liu and Mariano [13] proved sharp inequalities on triangles with certified finite elements; part of our strategy for $\alpha > 0$ (Crouzeix–Raviart lower bounds [11, 26, 10]) is of the same type.

The case $\alpha < 0$ (Part I). Write $\beta = -\alpha\sqrt{|T|} > 0$, let E be the equilateral triangle of area 1 and let $\beta(M)$ be the increasing parametrisation of the coupling introduced in §2; put $\beta_C = \beta(13/10)$, so that $1.1530164635 \le \beta_C \le 1.1530164636$ (Lemma 4.8). Theorem 1.1(2) follows from two overlapping results.

Theorem 1.3. *If $0 < \beta \le 6/5$, then $\lambda_1(T, -\beta) < \lambda_1(E, -\beta)$ for every non-equilateral triangle T of area 1.*

Theorem 1.4. *If $\beta \ge \beta_C$, then $\lambda_1(T, -\beta) < \lambda_1(E, -\beta)$ for every non-equilateral triangle T of area 1.*

The proof of Theorem 1.3 (Section 3) uses only the constant function and the affinely transplanted ground state of E, together with a two-variable inequality checked in interval arithmetic. The transplanted ground state alone cannot work for large β: its second variation at E changes sign at $\beta \approx 1.63$ (Remark 3.3). The proof of Theorem 1.4 (Section 4) has two parts.

- *A corner bound.* For every triangle, every angle θ and every $\beta > 0$, $\lambda_1(T, -\beta) < -\beta^2/\sin^2(\theta/2)$ (Lemma 4.1). Since $\lambda_1(E, -\beta) = -4\beta^2/\tanh^2 M$, this settles every triangle whose smallest angle θ satisfies $\sin(\theta/2) \le \frac{1}{2}\tanh M$. What remains is a neighbourhood of E of size about $1 - \tanh M \approx 6e^{-2\cdot 3^{1/4}\beta}$.
- *Three corner waves.* The ground state of E is a sum of three functions φ_i, each a plane wave concentrated at one corner plus its reflection in the opposite side. On a near-equilateral triangle we use the span of their affine transplants. A deformation couples the transplanted ground state to the two-dimensional complement, which lies only a tunnelling splitting of order $1 - \tanh M$ above the ground state. A single step of second-order perturbation theory then gains more than the deformation costs, by a factor of order $(1-\tanh M)^{-1}$. The resulting criterion reduces to a one-variable inequality in M, which we verify in ball arithmetic for all $M \ge 13/10$ (Lemma 4.8).

The case $\alpha > 0$ (Part II). Here we need *lower* bounds for $\lambda_1(T, \alpha)$, which no trial function provides. With $|T| = 1$, the shapes of triangles form the hyperbolic plane, and we use its Klein model: a point w of the unit disc, with $w = 0$ for E. The parameter set $\{(w, \alpha)\}$ is covered by four regions (Section 7): thin triangles, for all α (Theorem 7.1, by slicing); weak coupling $\alpha \le 1/100$, for all shapes (Theorem 7.3); a neighbourhood $|w| \le 0.1194$ of E, for all $\alpha \ge 1/100$ (Proposition 14.4); and the remaining compact middle region, for all $\alpha \ge 1/100$ (Proposition 15.2). Near E a Schur-complement criterion needs an *upper* bound for a second-order perturbation constant $R_1(\alpha)$ of E, accurate to a few per cent and uniform in α; we obtain it from certified lower bounds for the eigenvalues of two perturbed problems at distance $1/10$ from E in opposite directions, for which the first-order errors cancel (Lemma 14.2). In the middle region, the lowest eigenvalue is a concave function of the coefficients of the quadratic form pulled back to E; convex combinations of these coefficients are exactly the Euclidean triangles of the Klein model, up to an explicit factor (Lemma 11.3). Certified Crouzeix–Raviart lower bounds at the vertices of 100 cells and at grid points in α then cover the whole region.

Organisation. Part I (Sections 2–6) proves Theorem 1.1(2) and Corollary 1.2 for $\alpha < 0$, and Part II (Sections 7–16) proves Theorem 1.1(1) and Corollary 1.2 for $\alpha > 0$. The two parts are independent of each other, and each has its own notation. Section 17 describes the implementation, the verification and the remaining caveats. Appendix A contains a strong-coupling bound for individual triangles that is not used in the proof.

Part I
The case $\alpha < 0$

2 Preliminaries

In this part $\alpha < 0$, and the notation is independent of Part II. Throughout, $|T| = 1$ by scaling, and $\alpha = -\beta < 0$. Each map $\alpha \mapsto \int |\nabla u|^2 + \alpha \int_{\partial T} u^2$ is affine. By (1) we therefore have the following.

Lemma 2.1. *For fixed T, the function $\alpha \mapsto \lambda_1(T,\alpha)$ is concave and nondecreasing on $\mathbb{R}$, and $\lambda_1(T,0) = 0$. Consequently $\beta \mapsto |\lambda_1(T,-\beta)|/\beta$ is nondecreasing on $(0,\infty)$. For each fixed $u \in H^1(T)\setminus\{0\}$, the Rayleigh quotient $R_\alpha[u]$ is affine in α, and $R_\alpha[u] \geq \lambda_1(T,\alpha)$, with equality only if u is a ground state.*

Proof. An infimum of nondecreasing affine functions is concave and nondecreasing. If f is concave with $f(0) = 0$ and $\alpha < \alpha' < 0$, then $f(\alpha') \geq (\alpha'/\alpha) f(\alpha)$, so $f(\alpha')/\alpha' \leq f(\alpha)/\alpha$. The last statement is the variational principle. □

2.1 The equilateral triangle

Let E be the equilateral triangle of area 1. Its height is $H = 3^{1/4}$, its side length $\ell = 2/H$, its inradius $r = H/3$ and its perimeter $P_* = 6/H = \sqrt{12\sqrt{3}}$. Let $d_i(x)$ be the distance from x to the ith side. For $M > 0$ put

$$t = \tanh M, \quad L = -\operatorname{artanh}(t/2), \quad K = M - L, \quad k = \frac{2K}{H}, \quad \beta(M) = \frac{tK}{H}, \quad \lambda_0(M) = -\frac{4K^2}{H^2}. \tag{2}$$

Proposition 2.2 (cf. [29, 21])**.** *The map $M \mapsto \beta(M)$ is a smooth increasing bijection $(0,\infty) \to (0,\infty)$. For $\beta = \beta(M)$, the function*

$$\psi_0 = \sum_{i=0}^{2} \varphi_i, \qquad \varphi_i = \cosh(L + k\, d_i),$$

is a positive ground state of the Robin Laplacian on E with $\alpha = -\beta$, and $\lambda_1(E,-\beta) = \lambda_0(M) = -4\beta^2/\tanh^2 M$. If n_i is the inward unit normal of side i, then ψ_0 is a combination of the six plane waves $e^{\pm k n_i \cdot x}$.

Proof. The functions t, K, and hence $\beta = tK/H$, are increasing in M, with limits 0 and ∞. Since $|\nabla d_i| = 1$, each φ_i satisfies $\Delta\varphi_i = k^2\varphi_i$, so $-\Delta\psi_0 = \lambda_0\psi_0$. On side 0 we have $d_0 = 0$, $\partial_\nu d_0 = -1$, $\partial_\nu d_1 = \partial_\nu d_2 = \frac{1}{2}$ (the normals meet at 120°) and $d_1 + d_2 = H$. Write $y = kd_1 - K$, so that $L + kd_1 = M + y$ and $L + kd_2 = M - y$. On side 0,

$$\psi_0 = \cosh L + 2\cosh M \cosh y, \qquad \partial_\nu\psi_0 = k\big(-\sinh L + \sinh M \cosh y\big).$$

Hence $\partial_\nu\psi_0-\beta\psi_0=0$ there, for all y, if and only if $k\sinh L=-\beta\cosh L$ and $k\sinh M=2\beta\cosh M$, that is, $\tanh L=-\beta/k$ and $\tanh M=2\beta/k$. Both hold by (2). By symmetry the same holds on the other sides. A positive eigenfunction is a ground state, because eigenfunctions with different eigenvalues are L^2-orthogonal and the ground state does not change sign. Finally $\lambda_0=-k^2=-4\beta^2/t^2$. With the centroid at the origin we have $d_i=n_i\cdot x+r$, so φ_i is a combination of $e^{\pm k n_i\cdot x}$. □

Let $N=\int_E\psi_0^2$, $\mathcal{G}=\int_E|\nabla\psi_0|^2$ and $B=\int_{\partial E}\psi_0^2$, so that $\lambda_0=(\mathcal{G}-\beta B)/N$.

Lemma 2.3. *Let $K_M=dK/dM=1+\frac{2(1-t^2)}{4-t^2}$. Then*

$$\frac{B}{N}=\frac{d\lambda_1(E,\alpha)}{d\alpha}=\frac{8KK_M}{H\,[(1-t^2)K+tK_M]},\qquad \frac{\mathcal{G}}{N}=\lambda_0+\beta\frac{B}{N},$$

$$g:=\frac{\mathcal{G}/N}{2\beta B/N}=\frac14-\frac{K}{2\sinh(2M)K_M}\le\frac{\tanh^2 M}{4}.$$

Moreover K is increasing in M and K_M is decreasing.

Proof. For every α' we have $\lambda_1(E,\alpha')\le R_{\alpha'}[\psi_0]=\lambda_0+(\alpha'-\alpha)B/N$. So B/N is a supergradient at α of the concave function $\lambda_1(E,\cdot)$. By Proposition 2.2 that function is a smooth function of M, and $d\beta/dM>0$, so it is differentiable. Hence B/N equals the derivative $(d\lambda_0/dM)/(d\alpha/dM)$, which gives the stated formula. The identity for $\mathcal{G}/N$ is $\lambda_0=(\mathcal{G}-\beta B)/N$.

Substituting gives $g=\frac14-\frac{(1-t^2)K}{4tK_M}$, and $(1-t^2)/t=2/\sinh 2M$. For the bound we use $K\ge M+\frac12\tanh M\ge\frac32\tanh M$ and $K_M\le\frac32$. These give $K/(\sinh 2M\,K_M)\ge 1/(2\cosh^2 M)$, hence $g\le\frac14-\frac{1}{4\cosh^2 M}=\frac14\tanh^2 M$. Finally K_M decreases because $s\mapsto(1-s)/(4-s)$ decreases on $[0,1]$. □

2.2 Affine transplantation

Let E have its centroid at the origin and vertices $V_0=(-\ell/2,-r)$, $V_1=(\ell/2,-r)$, $V_2=(0,2r)$. Side s joins V_s to V_{s+1} and has unit direction e_s. For $a>0$ and $b\in\mathbb{R}$ let

$$D_{a,b}=\begin{pmatrix}a & b\\ 0 & 1/a\end{pmatrix},\qquad T_{a,b}=D_{a,b}E.$$

Lemma 2.4. *Every triangle is similar to exactly one $T_{a,b}$ with (a,b) in*

$$\mathcal{D}=\Big\{a\ge1,\ -\tfrac{a}{\sqrt3}\Big(\sqrt{4-3a^{-4}}-1\Big)\le b\le 0\Big\}.$$

The equilateral triangle corresponds to $(1,0)$. If $T_{a,b}$ has perimeter P, then $P/P_*\ge 2a/3$.

Proof. Normalise a longest side to $[0,1]$ and the apex to (x,y) with $0\le x\le\frac12$ and $(1-x)^2+y^2\le1$, then scale by $\sigma=\sqrt{2/y}$ to area 1. The linear map that takes the edge vectors V_1-V_0, V_2-V_0 of E to $(\sigma,0)$ and $(\sigma x,\sigma y)$ is $D_{a,b}$ with $a=\sigma/\ell$ and $b=\sigma(x-\frac12)/H$. A direct computation turns the constraints on (x,y) into $(a,b)\in\mathcal{D}$. Finally $P=\sigma(1+|(x,y)|+|(1-x,y)|)\ge 2\sigma=2a\ell$ and $P_*=3\ell$. □

Let D be a real 2×2 matrix with $\det D = 1$, and let $T = DE + x_0$. For $u \in H^1(E)$ let $\tilde{u}(x) = u(D^{-1}(x - x_0))$. Then

$$\int_T |\nabla \tilde{u}|^2 = \int_E \nabla u^\top \mathcal{A}\, \nabla u, \qquad \int_T \tilde{u}^2 = \int_E u^2, \qquad \int_{\partial T} \tilde{u}^2 = \sum_{s=0}^{2} \rho_s \int_{\text{side } s} u^2, \tag{3}$$

where $\mathcal{A} = D^{-1}D^{-\top}$ and $\rho_s = |De_s|$ is the ratio of the length of side s of T to ℓ. For $D = D_{a,b}$ we have $\mathcal{A} = \left(\begin{smallmatrix} a^{-2}+b^2 & -ab \\ -ab & a^2 \end{smallmatrix}\right)$.

Lemma 2.5 (transplanted ground state)**.** *Let T be a triangle of area 1 with sides a_0, a_1, a_2. Let $p = (a_0+a_1+a_2)/P_* \geq 1$ and $\tau = (a_0^2+a_1^2+a_2^2)/(2\sqrt{3}) \geq 2$. Then*

$$\lambda_1(T, -\beta) \leq \frac{\mathcal{G}}{N}\cdot\frac{\tau}{2} - \beta\frac{B}{N}\,p = \lambda_0 + \frac{\mathcal{G}}{N}\Big(\frac{\tau}{2} - 1\Big) - \beta\frac{B}{N}(p-1).$$

Proof. Take $u = \psi_0$ in (3). The matrix $\int_E \nabla\psi_0 \nabla\psi_0^\top$ commutes with the rotation by 120°, so it equals $\frac{\mathcal{G}}{2} I$. Hence the gradient term is $\frac{\mathcal{G}}{2}\operatorname{tr}\mathcal{A}$. For a 2×2 matrix of determinant one, $\operatorname{tr}(D^{-1}D^{-\top}) = \operatorname{tr}(D^\top D)$. Since $\sum_s e_s e_s^\top = \frac{3}{2} I$, we get $\sum_s a_s^2 = \ell^2 \sum_s |De_s|^2 = \frac{3}{2}\ell^2 \operatorname{tr}(D^\top D)$. As $\frac{3}{2}\ell^2 = 2\sqrt{3}$, this gives $\operatorname{tr}\mathcal{A} = \tau$. By symmetry each side of E carries $B/3$, so the boundary term is $\frac{B}{3}\sum_s a_s/\ell = Bp$. Finally, $\tau \geq 2$ because $\operatorname{tr}(D^\top D) \geq 2\det D$, and $p \geq 1$ is the isoperimetric inequality for triangles. □

3 Small coupling: proof of Theorem 1.3

Set $\beta_A = 6/5$, $G_A = 0.1425$ and $q_A = 1.384$.

Lemma 3.1. *For $0 < \beta \leq \beta_A$: (i) $g(\beta) < G_A$; (ii) $q(\beta) := |\lambda_0(\beta)|/(\beta P_*) \leq q(\beta_A) < q_A$.*

Proof. (ii) The first inequality is Lemma 2.1. For the second we enclose $M(\beta_A)$ by bisection in ball arithmetic. This gives $1.3835037091 \leq q(\beta_A) \leq 1.3835037092$.

(i) We cover $[0, M(\beta_A)]$ by 2000 intervals $[m_1, m_2]$. On the first interval we use $g \leq \tanh^2(m_2)/4$. On the others we use $g \leq \frac{1}{4} - K(m_1)/(2\sinh(2m_2)K_M(m_1))$, which is valid by the monotonicity in Lemma 2.3. Every bound is below G_A. The largest value of g on this range is $g(\beta_A) = 0.14195\ldots$. □

Lemma 3.2. *Let T be a non-equilateral triangle of area 1 with $p(T) \leq q_A$. Then $\Phi(T) := (p-1) - G_A(\tau - 2) > 0$.*

Proof. We use the apex coordinates (x, y) of the proof of Lemma 2.4; in them p and τ are explicit algebraic functions. If $y < 19/100$, then $p \geq 2\sqrt{2/y}/P_* > q_A$. The equilateral apex $z_0 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$ minimises both p and τ over all apex positions. Hence $\Phi(z_0) = 0$ and $\nabla\Phi(z_0) = 0$. On the box $|x - \frac{1}{2}|, |y - \frac{\sqrt{3}}{2}| \leq \frac{1}{40}$ we evaluate the Hessian of Φ in interval arithmetic, by second-order automatic differentiation in ball arithmetic on 256 sub-boxes. It is positive definite on every sub-box. Hence $\Phi(z) = \int_0^1 (1-s)(z-z_0)^\top \nabla^2\Phi(z_0 + s(z-z_0))(z-z_0)\,\mathrm{d}s > 0$ for $z \neq z_0$ in the box. An adaptive subdivision of $[0, \frac{1}{2}] \times [\frac{19}{100}, \frac{87}{100}]$ ends in 344 boxes. For each box, one of the following holds:

- it lies inside the local box;
- interval arithmetic or a mean-value form shows that it lies outside the fundamental domain;
- it shows that $p > q_A$ on the box;

- it shows that $\Phi > 0$ on the box.

□

Proof of Theorem 1.3. Let $0 < \beta \le 6/5$ and let T, of area 1, be non-equilateral. First suppose $p(T) \ge q(\beta)$. The constant trial function gives $\lambda_1(T, -\beta) < R[1] = -\beta P_* p \le \lambda_0(\beta)$. The first inequality is strict because constants are not eigenfunctions when $\alpha \neq 0$.

Otherwise $p < q(\beta) < q_A$ by Lemma 3.1(ii), so $\Phi(T) > 0$ by Lemma 3.2. Since $\tau > 2$, Lemma 3.1(i) gives $p - 1 > G_A(\tau - 2) > g(\tau - 2)$. Lemma 2.5 then gives $\lambda_1(T, -\beta) - \lambda_0 \le \beta \frac{B}{N}(g(\tau - 2) - (p - 1)) < 0$. □

Remark 3.3. The transplanted ground state alone works near E exactly when $g(\beta)$ is below $\lim_{T \to E}(p-1)/(\tau - 2)$. Write the side ratios as $\rho_i = 1 + \delta_i + \bar{\delta}$ with $\sum \delta_i = 0$. The area constraint (Lemma 4.3(c) below) gives $p - 1 \sim \frac{1}{2}\sum \delta_i^2$ and $\tau - 2 \sim \frac{8}{3}\sum \delta_i^2$, so the limit is $3/16 = 0.1875$ in every direction. In floating point, $g(\beta) = 3/16$ at $\beta \approx 1.63$, and $g(\beta) < 3/16$ below that value, so the local limit of this argument is $\beta \approx 1.63$. This matches the numerical threshold of [21]. Replacing the constant by the disc bound $\lambda_1(T, \alpha) \le \lambda_1(D_{r_T}, \alpha)$, which holds for tangential polygons by the method of parallel coordinates (r_T the inradius), would raise 6/5 to about 1.29. This is a floating-point estimate that we have not certified, and it is not needed.

4 Large coupling: proof of Theorem 1.4

4.1 The corner bound

Lemma 4.1. *Let T be a triangle of any area, let θ be one of its angles, and let $\beta > 0$. Then*

$$\lambda_1(T, -\beta) < -\frac{\beta^2}{\sin^2(\theta/2)}.$$

Proof. Let V be the vertex with angle θ, and let e be the unit vector along the bisector at V, pointing into T. For $\kappa > 0$ put $w = -\kappa e$ and $u(x) = e^{w \cdot (x - V)}$. Then $|\nabla u|^2 = \kappa^2 u^2$ and $\operatorname{div}(w\, u^2) = 2\kappa^2 u^2$. By the divergence theorem,

$$2\kappa^2 \int_T u^2 = \sum_s (w \cdot \nu_s) \int_{\text{side } s} u^2,$$

where ν_s is the outer unit normal of side s. On the two sides through V we have $e \cdot \nu_s = -\sin(\theta/2)$, so $w \cdot \nu_s = \kappa \sin(\theta/2)$. On the opposite side $e \cdot \nu_s > 0$, because the bisector leaves T through that side. Hence $w \cdot \nu_s < 0 < \kappa \sin(\theta/2)$ there. Since $u > 0$,

$$2\kappa^2 \int_T u^2 < \kappa \sin(\theta/2) \int_{\partial T} u^2.$$

So the Rayleigh quotient of u is $\kappa^2 - \beta \int_{\partial T} u^2 / \int_T u^2 < \kappa^2 - 2\beta\kappa / \sin(\theta/2)$. Now take $\kappa = \beta / \sin(\theta/2)$. □

The trial function is the ground state $e^{-\kappa e \cdot (x - V)}$ of the infinite sector [25, Example 2.5, Lemma 2.6], see also [20]. It was used for triangles in [21, Theorem 5.3], where the bound $\lambda_1(T, \alpha) \le -\frac{\alpha^2}{\sin^2(\theta/2)}(1 - 2e^{2\alpha L' \cot(\theta/2)})$ is obtained (L' the shorter adjacent side), with an exponentially small loss; the divergence identity removes that loss. Lemma 4.1 also appears, with the same proof, as (7)–(8) in Ma's preprint [27]. It is the corner analogue of the bound $\lambda_1(\Omega, -\beta) \le -\beta^2$ for smooth domains, which [6] attribute to [15, Theorem 2.1]. The asymptotics

$\lambda_1(T,-\beta) \sim -\beta^2/\sin^2(\theta_{\min}/2)$ as $\beta \to \infty$ go back to Lacey, Ockendon and Sabina [22] and Levitin and Parnovski [25]; exponentially precise asymptotics are in [19, 17]. We could not consult [22] or the survey [7].

Corollary 4.2. *Let $|T| = 1$, $\beta = \beta(M)$ and $t = \tanh M$, and let $\theta_{\min}$ be the smallest angle of T. If $\sin(\theta_{\min}/2) \le t/2$, then $\lambda_1(T,-\beta) < \lambda_0(M)$.*

Proof. By Lemma 4.1 and Proposition 2.2, $\lambda_1(T,-\beta) < -\beta^2/\sin^2(\theta_{\min}/2) \le -4\beta^2/t^2 = \lambda_0(M)$. □

It remains to treat the *near region* $\theta_{\min} > \theta(t)$, where

$$\theta(t) := 2\arcsin(t/2) \in (0,\pi/3), \qquad \pi/3 - \theta(t) \sim \tfrac{2}{\sqrt{3}}(1-t) \quad (t \to 1). \tag{4}$$

4.2 Three trial functions

Let T be a triangle of area 1 with sides a_0, a_1, a_2, numbered counterclockwise like those of E, and let θ_i be the angle opposite side i. Put

$$\rho_i = \frac{a_i}{\ell}, \qquad S = \rho_0^2 + \rho_1^2 + \rho_2^2, \qquad W = \sum_i (\rho_i - \bar\rho)^2, \qquad \bar\rho = \tfrac{1}{3}\sum_i \rho_i,$$

where $\ell = 2/H$ is the side length of E. Let $x \mapsto Dx + x_0$ be the affine map of E onto T that takes side i to side i. Then $\det D = 1$ and $\rho_i = |De_i|$, and (3) holds with $\mathcal{A} = D^{-1}D^{-\top}$. Let $\mu \ge 1$ be the larger eigenvalue of $\mathcal{A}$. Our trial functions are the transplants $\tilde\varphi_i$ of the three summands $\varphi_i = \cosh(L + kd_i)$ of ψ_0. On T they are the functions $\cosh(L + 2K\, b_i)$, where b_0, b_1, b_2 are the barycentric coordinates of T and b_i vanishes on side i. Each φ_i is the sum of the plane wave that decays away from the vertex opposite side i and its reflection in side i. In this section $\{i,j,m\} = \{0,1,2\}$ always denotes the three distinct indices.

Lemma 4.3. *Let n_i be the inward unit normal of side i of E.*

(a) $n_i^\top \mathcal{A} n_i = \rho_i^2$ *and* $n_i^\top \mathcal{A} n_j = \tfrac{1}{2}(\rho_m^2 - \rho_i^2 - \rho_j^2)$.

(b) $\operatorname{tr}\mathcal{A} = \tfrac{2}{3}S$, *so* $\mu = \tfrac{1}{2}(\tau + \sqrt{\tau^2 - 4})$ *with* $\tau = \tfrac{2}{3}S$.

(c) $S^2 - 2\sum_i \rho_i^4 = 3$. *Consequently* $S \ge 3$ *and* $S - 3 = \dfrac{6\mathcal{V}}{S+3} \le \mathcal{V} \le 4\rho_{\max}^2 W$, *where* $\mathcal{V} = \sum_i (\rho_i^2 - S/3)^2$.

(d) $\rho_i^2 = \dfrac{\sqrt{3}}{2}\dfrac{\sin\theta_i}{\sin\theta_j \sin\theta_m} = \dfrac{\sqrt{3}}{2}(\cot\theta_j + \cot\theta_m)$, *and* $S = \sqrt{3}\sum_i \cot\theta_i$.

Proof. (a) For a 2×2 matrix of determinant one, $D^{-\top} = JDJ^{-1}$, where J is the rotation by $90°$. Since $J^{-1}n_i = \pm e_i$, we get $|D^{-\top}n_i| = |De_i| = \rho_i$. The second formula follows from $n_0 + n_1 + n_2 = 0$.

(b) Sum (a) over i and use $\sum_i n_i n_i^\top = \frac{3}{2}I$. The eigenvalues of $\mathcal{A}$ have product 1 and sum τ.

(c) Heron's formula $16|T|^2 = (\sum a_i^2)^2 - 2\sum a_i^4$ and $\ell^4 = 16/3$ give the identity. Since $\sum \rho_i^4 = \mathcal{V} + S^2/3$, it reads $S^2 = 9 + 6\mathcal{V}$. Finally $\mathcal{V} = \frac{1}{3}\sum_{i<j}(\rho_i^2 - \rho_j^2)^2 \le \frac{4\rho_{\max}^2}{3}\sum_{i<j}(\rho_i - \rho_j)^2 = 4\rho_{\max}^2 W$.

(d) By the law of sines, $a_i = 2R\sin\theta_i$ with R the circumradius, and $1 = |T| = 2R^2\prod_j \sin\theta_j$. So $a_i^2 = 2\sin\theta_i/(\sin\theta_j\sin\theta_m)$. Divide by $\ell^2 = 4/\sqrt{3}$ and use $\sin\theta_i = \sin(\theta_j + \theta_m)$. □

By the symmetry of E, all integrals of products of the φ_i, and of their gradients, over E and over its sides reduce to eight numbers:

$$N_d = \int_E \varphi_0^2, \qquad N_o = \int_E \varphi_0\varphi_1,$$
$$\Gamma_d = k^2\int_E \sinh^2(L+kd_0), \qquad \Gamma_o = k^2\int_E \sinh(L+kd_0)\sinh(L+kd_1),$$
$$\Sigma_1 = \int_{\text{side }0} \varphi_0^2, \qquad \Sigma_2 = \int_{\text{side }1} \varphi_0^2,$$
$$\Sigma_3 = \int_{\text{side }0} \varphi_0\varphi_1, \qquad \Sigma_4 = \int_{\text{side }2} \varphi_0\varphi_1.$$

Thus $\int_E |\nabla\varphi_i|^2 = \Gamma_d$ and $\int_E \nabla\varphi_i\,\nabla\varphi_j^\top = \Gamma_o\, n_i n_j^\top$ for $i \neq j$. We also set

$$G_0 = \Gamma_d - \Gamma_o, \qquad G_\perp = \Gamma_d + \tfrac{1}{2}\Gamma_o, \qquad B_\perp = \Sigma_1 + 2\Sigma_2 - 2\Sigma_3 - \Sigma_4, \qquad N_\perp = N_d - N_o,$$
$$b_{\text{own}} = \Sigma_1 + 2\Sigma_3, \qquad b_{\text{adj}} = \Sigma_2 + \Sigma_3 + \Sigma_4, \qquad \chi = b_{\text{adj}} - b_{\text{own}}.$$

Here $b_{\text{own}} = \int_{\text{side }i} \varphi_i\psi_0$ and $b_{\text{adj}} = \int_{\text{side }j} \varphi_i\psi_0$ for $j \neq i$. Hence $B = 3(b_{\text{own}} + 2b_{\text{adj}})$. Also $\mathcal{G} = \sum_{i,j} n_i \cdot n_j\, \Gamma_{ij} = 3G_0$, where $\Gamma_{ii} = \Gamma_d$ and $\Gamma_{ij} = \Gamma_o$; in particular $G_0 > 0$. The subscript $\perp$ refers to the complement of ψ_0 in the span of the φ_i:

Lemma 4.4. *Let $v \in \mathbb{R}^3$ with $v_0 + v_1 + v_2 = 0$, and let $u_v = \sum_i v_i\varphi_i$. Then*

$$\int_E |\nabla u_v|^2 = G_\perp|v|^2, \qquad \int_{\partial E} u_v^2 = B_\perp|v|^2, \qquad \int_E u_v^2 = N_\perp|v|^2.$$

In particular $G_\perp, B_\perp \geq 0$ and $N_\perp > 0$. Moreover $\gamma := G_\perp - \beta B_\perp - \lambda_0 N_\perp > 0$.

Proof. Since $n_i \cdot n_j = -\frac{1}{2}$ for $i \neq j$ and $\sum_{i\neq j} v_i v_j = -|v|^2$, the first integral is $\Gamma_d|v|^2 - \frac{1}{2}\Gamma_o \sum_{i\neq j} v_i v_j = G_\perp|v|^2$. The other two are computed in the same way; for the boundary, $\sum_s \int_{\text{side }s} \varphi_i^2 = \Sigma_1 + 2\Sigma_2$ and $\sum_s \int_{\text{side }s} \varphi_i\varphi_j = 2\Sigma_3 + \Sigma_4$. The φ_i are linearly independent. Indeed, on side 0 they restrict to $\cosh L$, $\cosh(L+y)$ and $\cosh(U-y)$ for a coordinate y, where $U = L+2K$ and $L + U = 2M \neq 0$. Hence $N_\perp > 0$. Finally, $\gamma|v|^2$ is the value at u_v of the quadratic form of $-\Delta - \lambda_0$ on E. This form is nonnegative and vanishes only on multiples of the ground state $\psi_0 = u_{\mathbf{1}}$, $\mathbf{1} = (1,1,1)$; and u_v is not such a multiple when $v \neq 0$. □

For $c \in \mathbb{R}^3$ let $\tilde{u}_c = \sum_i c_i\tilde{\varphi}_i$, and let

$$\mathcal{P}_{ij} = \int_T \nabla\tilde{\varphi}_i \cdot \nabla\tilde{\varphi}_j - \beta\int_{\partial T} \tilde{\varphi}_i\tilde{\varphi}_j - \lambda_0\int_T \tilde{\varphi}_i\tilde{\varphi}_j.$$

Then $\lambda_1(T, -\beta) < \lambda_0$ as soon as $c^\top\mathcal{P}c < 0$ for some c.

Lemma 4.5. *(a)* $(\mathcal{P}\mathbf{1})_i = G_0(\rho_i^2 - 1) + \beta\chi(\rho_i - 1) - \beta b_{\text{adj}}(\sum_j \rho_j - 3)$.

(b) $\mathbf{1}^\top\mathcal{P}\mathbf{1} = G_0(S-3) - \beta\frac{B}{3}(\sum_j \rho_j - 3) \leq G_0(S-3)$.

(c) If $v_0 + v_1 + v_2 = 0$, then $v^\top\mathcal{P}v \leq \Lambda_T|v|^2$, where $\Lambda_T := \mu G_\perp - \beta\rho_{\min}B_\perp - \lambda_0 N_\perp$.

Proof. (a) By (3) and Lemma 4.3(a),

$$\int_T \nabla\tilde{\varphi}_i \cdot \nabla\tilde{\psi}_0 = \rho_i^2\Gamma_d + \Gamma_o\, n_i^\top\mathcal{A}(n_j + n_m) = \rho_i^2(\Gamma_d - \Gamma_o) = \rho_i^2 G_0,$$

and $\int_{\partial T}\tilde{\varphi}_i\tilde{\psi}_0 = \rho_i b_{\text{own}} + (\rho_j+\rho_m)b_{\text{adj}}$. Hence $(\mathcal{P}\mathbf{1})_i = \rho_i^2 G_0 - \beta\rho_i b_{\text{own}} - \beta(\rho_j+\rho_m)b_{\text{adj}} - \lambda_0(N_d+2N_o)$. For $T = E$ all $\rho_i = 1$, and the left side vanishes because ψ_0 is an eigenfunction with eigenvalue λ_0. Subtracting the two identities gives (a).

(b) Sum (a) over i. The inequality holds because $B > 0$ and $\sum_j \rho_j \geq 3$, which is the isoperimetric inequality for triangles.

(c) By (3), $\int_T |\nabla\tilde{u}_v|^2 = \int_E \nabla u_v^\top \mathcal{A}\nabla u_v \leq \mu \int_E |\nabla u_v|^2$ and $\int_{\partial T}\tilde{u}_v^2 = \sum_s \rho_s \int_{\text{side } s} u_v^2 \geq \rho_{\min}\int_{\partial E} u_v^2$. Now use Lemma 4.4. □

Proposition 4.6 (criterion)**.** *Let T be a non-equilateral triangle of area 1. Suppose that $\chi \geq 0$, and that $\Lambda > 0$ satisfies $v^\top\mathcal{P}v \leq \Lambda|v|^2$ for all $v \perp \mathbf{1}$. If*

$$\Lambda < \frac{(2G_0\rho_{\min} + \beta\chi)^2}{4G_0\,\rho_{\max}^2}, \tag{5}$$

then $\lambda_1(T, -\beta) < \lambda_0$.

Proof. It suffices to find c with $c^\top\mathcal{P}c < 0$. Let $q = \mathcal{P}\mathbf{1}$ and let $q^\perp = q - \bar{q}\,\mathbf{1}$ be its component orthogonal to $\mathbf{1}$. For $c = \mathbf{1} - q^\perp/\Lambda$,

$$c^\top\mathcal{P}c = \mathbf{1}^\top\mathcal{P}\mathbf{1} - \frac{2|q^\perp|^2}{\Lambda} + \frac{(q^\perp)^\top\mathcal{P}q^\perp}{\Lambda^2} \leq \mathbf{1}^\top\mathcal{P}\mathbf{1} - \frac{|q^\perp|^2}{\Lambda}.$$

By Lemma 4.5(a), $q_i - q_j = (\rho_i - \rho_j)(G_0(\rho_i+\rho_j) + \beta\chi)$, and the second factor is at least $2G_0\rho_{\min} + \beta\chi > 0$. Hence

$$|q^\perp|^2 = \tfrac{1}{3}\sum_{i<j}(q_i - q_j)^2 \geq (2G_0\rho_{\min} + \beta\chi)^2\, W.$$

By Lemma 4.5(b) and Lemma 4.3(c), $\mathbf{1}^\top\mathcal{P}\mathbf{1} \leq G_0(S-3) \leq 4G_0\rho_{\max}^2 W$. Therefore

$$c^\top\mathcal{P}c \leq W\Big(4G_0\rho_{\max}^2 - \frac{(2G_0\rho_{\min} + \beta\chi)^2}{\Lambda}\Big) < 0,$$

because $W > 0$ for a non-equilateral triangle. □

The mechanism is as follows. For large β, the Rayleigh quotient of the transplanted ground state $\tilde{\psi}_0$ alone exceeds λ_0 by a positive multiple of W. The deformation, however, couples $\tilde{\psi}_0$ to the two-dimensional space $\{\tilde{u}_v : v \perp \mathbf{1}\}$ with strength $|q^\perp| \approx 3G_0\sqrt{W}$. That space lies only $\gamma \approx 3(1-t)G_0$ above λ_0; this is the tunnelling splitting. Second-order perturbation theory then gains $|q^\perp|^2/\Lambda$, which exceeds the loss by a factor of order $(1-t)^{-1}$.

4.3 The near region

Lemma 4.7. *Let $\pi/4 \leq \theta_* \leq \pi/3$, and let T be a triangle of area 1 all of whose angles are at least θ_*. Then*

$$\rho_{\text{lo}}^2 := \sqrt{3}\tan\frac{\theta_*}{2} \leq \rho_i^2 \leq \sqrt{3}\,\cot\theta_* =: \rho_{\text{hi}}^2, \qquad \mu \leq \mu_{\text{hi}} := \frac{\tau_{\text{hi}} + \sqrt{\tau_{\text{hi}}^2 - 4}}{2}, \quad \tau_{\text{hi}} := \frac{3\cot^2\theta_* + 1}{\sqrt{3}\,\cot\theta_*}.$$

Moreover $\rho_{\text{lo}} \leq 1 \leq \mu_{\text{hi}}$.

Proof. By Lemma 4.3(d), $\rho_i^2 = \frac{\sqrt{3}}{2}(\cot\theta_j + \cot\theta_m) \le \sqrt{3}\cot\theta_*$. Also $\sin\theta_j \sin\theta_m = \frac{1}{2}(\cos(\theta_j - \theta_m) + \cos\theta_i) \le \cos^2(\theta_i/2)$, so $\rho_i^2 \ge \sqrt{3}\tan(\theta_i/2) \ge \sqrt{3}\tan(\theta_*/2)$. For μ, Lemma 4.3(b),(d) give $\tau = \frac{2}{\sqrt{3}}\sum_i \cot\theta_i$. The angles lie in $[\theta_*, \pi - 2\theta_*] \subset (0, \pi/2]$, where cot is convex. So $\sum_i \cot\theta_i$ is a convex function on the triangle $\{\theta_i \ge \theta_*,\ \sum\theta_i = \pi\}$, and its maximum is at a vertex $(\theta_*, \theta_*, \pi - 2\theta_*)$, where it equals $2\cot\theta_* - \cot 2\theta_* = (3\cot^2\theta_* + 1)/(2\cot\theta_*)$. Finally, μ is increasing in τ. Since $\theta_* \le \pi/3$, we have $\rho_{\mathrm{lo}}^2 \le \sqrt{3}\tan(\pi/6) = 1$, and $\mu_{\mathrm{hi}} \ge 1$ because $\tau_{\mathrm{hi}} \ge 2$. □

For $t \in (0,1)$ let $\rho_{\mathrm{lo}}(t)$, $\rho_{\mathrm{hi}}(t)$ and $\mu_{\mathrm{hi}}(t)$ be the quantities of Lemma 4.7 for $\theta_* = \theta(t)$. Since $\sin(\theta(t)/2) = t/2$,

$$\tan\frac{\theta(t)}{2} = \frac{t}{\sqrt{4-t^2}}, \qquad \cot\theta(t) = \frac{2-t^2}{t\sqrt{4-t^2}}, \tag{6}$$

and $\theta(t) \ge \pi/4$ is equivalent to $\cot\theta(t) \le 1$. For $M > 0$ and $t' \in (0,1)$ put

$$\Lambda_{\mathrm{hi}}(M;t') = \mu_{\mathrm{hi}}(t')G_\perp - \beta\rho_{\mathrm{lo}}(t')B_\perp - \lambda_0 N_\perp, \qquad \mathcal{R}(M;t') = \frac{G_0}{\rho_{\mathrm{hi}}(t')^2}\Big(\rho_{\mathrm{lo}}(t') + \frac{\beta\chi}{2G_0}\Big)^2, \tag{7}$$

where $\beta, \lambda_0, G_0, G_\perp, B_\perp, N_\perp, \chi$ are evaluated at M. The geometric quantities depend on a lower bound t' for $t = \tanh M$. This is the form in which the inequality is certified.

Lemma 4.8 (certified)**.** *Let $J_a = [M_a, M_a + 10^{-3}]$ with $M_a = \frac{13}{10} + \frac{a}{1000}$, for $a = 0, \dots, 6700 - 1$, so that the J_a cover $[13/10, 8]$; let $t_a = \tanh M_a$, and let $J_{\mathrm{tail}} = [8, \infty)$ with $t_{\mathrm{tail}} = \tanh 8$. For each of these intervals J, with its value t_J:*

(i) $\cot\theta(t_J) \le 1$;

(ii) for every $M \in J$: $G_0 > 0$, $\chi \ge 0$ and $\Lambda_{\mathrm{hi}}(M;t_J) < \mathcal{R}(M;t_J)$.

Moreover $1.1530164635 \le \beta(13/10) \le 1.1530164636$.

The geometric bounds are monotone in t'. As t' increases, $\theta(t')$ increases, so ρ_{lo} increases while ρ_{hi} and μ_{hi} decrease. Moreover Λ_{hi} is increasing in μ_{hi} and decreasing in ρ_{lo}, because $G_\perp, \beta B_\perp \ge 0$. And $\mathcal{R}$ is decreasing in ρ_{hi} and increasing in ρ_{lo}, because $\chi \ge 0$. Hence (ii) implies the pointwise inequality $\Lambda_{\mathrm{hi}}(M;\tanh M) < \mathcal{R}(M;\tanh M)$ for all $M \ge 13/10$. The proof of Theorem 1.4 below, however, uses (i) and (ii) directly.

Proof. All quantities have closed forms. Put $U = 2M - L = L + 2K$. With $y = kd_i \in [0, 2K]$, the area element of E is $dy_0\,dy_1/(2K^2)$ on $\{y_0, y_1 \ge 0,\ y_0 + y_1 \le 2K\}$, and the arclength on a side is $dy/(HK)$. Elementary integration gives

$$\begin{aligned}
I_1 &:= \int_0^{2K} (2K - y)\cosh^2(L + y)\,\mathrm{d}y = K^2 - \tfrac{K}{2}\sinh 2L + \tfrac{1}{8}(\cosh 2U - \cosh 2L),\\
I_1^s &:= \int_0^{2K} (2K - y)\sinh^2(L + y)\,\mathrm{d}y = -K^2 - \tfrac{K}{2}\sinh 2L + \tfrac{1}{8}(\cosh 2U - \cosh 2L),\\
I_2 &:= \iint \cosh(L + y_0)\cosh(L + y_1) = K\sinh 2M - \sinh L\,(\sinh U - \sinh L),\\
I_2^s &:= \iint \sinh(L + y_0)\sinh(L + y_1) = K\sinh 2M - \cosh L\,(\cosh U - \cosh L),\\
J_2 &:= \int_0^{2K} \cosh^2(L + y)\,\mathrm{d}y = K + \tfrac{1}{4}(\sinh 2U - \sinh 2L),\\
J_3 &:= \int_0^{2K} \cosh(L + y)\,\mathrm{d}y = \sinh U - \sinh L,\\
J_4 &:= \int_0^{2K} \cosh(L + y)\cosh(U - y)\,\mathrm{d}y = K\cosh 2M + \tfrac{1}{2}\sinh 2K,
\end{aligned}$$

and then

$$N_d = \frac{I_1}{2K^2}, \qquad N_o = \frac{I_2}{2K^2}, \qquad \Gamma_d = \frac{2I_1^s}{H^2}, \qquad \Gamma_o = \frac{2I_2^s}{H^2},$$
$$\Sigma_1 = \frac{2\cosh^2 L}{H}, \qquad \Sigma_2 = \frac{J_2}{HK}, \qquad \Sigma_3 = \frac{\cosh L\, J_3}{HK}, \qquad \Sigma_4 = \frac{J_4}{HK}.$$

With $\beta = tK/H$ and $\lambda_0 = -4K^2/H^2$, this gives

$$G_0 = \tfrac{2}{H^2}(I_1^s - I_2^s), \qquad G_\perp = \tfrac{2}{H^2}(I_1^s + \tfrac{1}{2}I_2^s), \qquad -\lambda_0 N_\perp = \tfrac{2}{H^2}(I_1 - I_2),$$
$$\beta B_\perp = \tfrac{2}{H^2}\cdot\tfrac{t}{2}(2K\cosh^2 L + 2J_2 - 2\cosh L\, J_3 - J_4),$$
$$\beta\chi = \tfrac{2}{H^2}\cdot\tfrac{t}{2}(J_2 + J_4 - \cosh L\, J_3 - 2K\cosh^2 L).$$

The inequality $\Lambda_{\mathrm{hi}} < \mathcal{R}$ and the sign conditions are homogeneous of degree one in the five numbers G_0, $G_\perp$, $-\lambda_0 N_\perp$, $\beta B_\perp$ and $\beta\chi$. We may therefore divide them by $\frac{2}{H^2}X$, where $X = e^{2U}/16$. Let $s = e^{-2M}$, $\omega = e^{2L} = (2-t)/(2+t)$ and $u = Ks$. Then $t = (1-s)/(1+s)$, $e^{2U} = 1/(s^2\omega)$ and $e^{2K} = 1/(s\omega)$. The hyperbolic functions of L are rational in t; for example $\cosh^2 L = 4/(4-t^2)$ and $\sinh 2L = -4t/(4-t^2)$. After division by X, each of $I_1, \dots, J_4$ and $K\cosh^2 L$ becomes a polynomial in s, u, ω and these rational functions. For example, $I_2^s/X = 8\omega u(1-s^2) - 4s(\omega(1+s^2) + 1 + \omega^2 s^2) + 64\omega s^2/(4-t^2)$. These expressions remain bounded as $s \to 0$.

We evaluate them in Arb ball arithmetic. On each J_a we enclose s, t, K, u over the whole interval and evaluate the geometric bounds (6) at $t' = t_a$. On the tail we use the single box $s \in [0, e^{-16}]$, $t \in [\tanh 8, 1]$, $u \in [0, K(8)e^{-16}]$, with the geometric bounds at $t' = \tanh 8$. This box contains (s,t,u) for every $M \geq 8$, because $u = Ke^{-2M}$ is decreasing for $M \geq 3/4$ (as $dK/dM \leq \frac{3}{2} \leq 2K$). On every interval the script certifies (i), $G_0 > 0$, $\chi \geq 0$ and $\Lambda_{\mathrm{hi}} < \mathcal{R}$. The ratio $\Lambda_{\mathrm{hi}}/\mathcal{R}$ is at most 0.6013 on $[13/10, 8]$ and at most $2.11 \cdot 10^{-6}$ on the tail. The value $\beta(13/10)$ is enclosed directly. □

Proof of Theorem 1.4. Let $\beta \geq \beta_C$, so that $\beta = \beta(M)$ with $M \geq 13/10$, and put $t = \tanh M$. Let T be a non-equilateral triangle of area 1. If $\sin(\theta_{\min}/2) \leq t/2$, Corollary 4.2 applies.

Otherwise every angle of T exceeds $\theta(t)$. Let J be an interval of Lemma 4.8 containing M. Then $t_J \leq t$, so every angle exceeds $\theta(t_J)$, and $\theta(t_J) \geq \pi/4$ by Lemma 4.8(i). Lemma 4.7 with $\theta_* = \theta(t_J)$ gives $\rho_{\mathrm{lo}}(t_J) \leq \rho_{\min}$, $\rho_{\max} \leq \rho_{\mathrm{hi}}(t_J)$ and $\mu \leq \mu_{\mathrm{hi}}(t_J)$. Since $G_\perp, B_\perp \geq 0$, Lemma 4.5(c) gives $v^\top \mathcal{P} v \leq \Lambda_T |v|^2 \leq \Lambda_{\mathrm{hi}}(M; t_J)|v|^2$ for $v \perp \mathbf{1}$. Moreover

$$\Lambda_{\mathrm{hi}}(M; t_J) - \gamma = (\mu_{\mathrm{hi}}(t_J) - 1)G_\perp + (1 - \rho_{\mathrm{lo}}(t_J))\beta B_\perp \geq 0$$

by Lemma 4.7, so $\Lambda_{\mathrm{hi}}(M; t_J) \geq \gamma > 0$ by Lemma 4.4. By Lemma 4.8(ii) we have $\chi \geq 0$ and

$$\Lambda_{\mathrm{hi}}(M; t_J) < \mathcal{R}(M; t_J) = \frac{(2G_0\rho_{\mathrm{lo}}(t_J) + \beta\chi)^2}{4G_0\rho_{\mathrm{hi}}(t_J)^2} \leq \frac{(2G_0\rho_{\min} + \beta\chi)^2}{4G_0\rho_{\max}^2}.$$

So Proposition 4.6 applies with $\Lambda = \Lambda_{\mathrm{hi}}(M; t_J)$. □

Remark 4.9 (size of the margin). As $M \to \infty$:

- $G_\perp/G_0 \to 1$, $\beta B_\perp/G_0 \to 2$ and $\beta\chi/G_0 \to 1$;
- $\gamma/G_0 \sim 3(1-t)$, $\mu_{\mathrm{hi}}(t) - 1 \sim \frac{8}{3}(1-t)$ and $1 - \rho_{\mathrm{lo}}(t) \sim \frac{2}{3}(1-t)$;
- $1 - t \sim 6e^{-2\cdot 3^{1/4}\beta}$.

Since $\Lambda_{\mathrm{hi}} = \gamma + (\mu_{\mathrm{hi}} - 1)G_\perp + (1 - \rho_{\mathrm{lo}})\beta B_\perp$, it follows that $\Lambda_{\mathrm{hi}} \sim 7(1-t)\,G_0 \approx 42\, e^{-2\cdot 3^{1/4}\beta} G_0$, while $\mathcal{R}$ tends to $\frac{9}{4}G_0$. So the margin of the inequality grows exponentially with β; see Table 1. In floating point, $\Lambda_{\mathrm{hi}}(M; \tanh M) < \mathcal{R}(M; \tanh M)$ holds for $M > 1.103$ (that is, for $\beta > 0.930$) and fails below. So the method alone reaches below 6/5, but not down to small β.

β	M	$\theta(\tanh M)$ (deg)	γ/G_0	$\Lambda_{\rm hi}/G_0$	$\mathcal{R}/G_0$
1.2	1.3429	51.721	0.324	0.832	1.553
1.5	1.6304	55.169	0.202	0.500	1.837
2	2.1695	58.303	0.0751	0.180	2.120
3	3.4090	59.855	0.00655	0.0153	2.246
4	4.7159	59.989	$4.81 \cdot 10^{-4}$	$1.12 \cdot 10^{-3}$	2.250
6	7.3471	59.9999	$2.49 \cdot 10^{-6}$	$5.81 \cdot 10^{-6}$	2.250

Table 1: Lemma 4.8 in floating point, with $t' = \tanh M$; an illustration only (the certified statement is the lemma). The near region is $\theta_{\min} > \theta(\tanh M)$.

5 Proof of Theorem 1.1(2) and of Corollary 1.2 for $\alpha < 0$

Proof of Theorem 1.1(2). By scaling we may take $|T| = 1$, so that $E_T = E$ and $\beta = -\alpha$. If $\beta \le 6/5$, use Theorem 1.3. If $\beta \ge 6/5$, then $\beta \ge \beta_C$ since $\beta_C \le 1.1530164636 < 6/5$, and we use Theorem 1.4. The two ranges overlap on $[\beta_C, 6/5]$. □

Proof of Corollary 1.2 for $\alpha < 0$. By the isoperimetric inequality for triangles, $|T| \le |E^T|$, with equality only for equilateral T. Theorem 1.1(2) gives $\lambda_1(T, \alpha) \le \lambda_1(E_T, \alpha)$, with strict inequality unless T is equilateral. Moreover $\lambda_1(E_T, \alpha) \le \lambda_1(E^T, \alpha)$, because for $\alpha < 0$ the lowest eigenvalue of an equilateral triangle is increasing in its area [21, Prop. 2.3]. This also follows from Proposition 2.2 and scaling: for the equilateral triangle of area A, $\lambda_1 = -4\alpha^2/\tanh^2 M$, where M increases with $\beta = -\alpha\sqrt{A}$. □

6 Independent cross-checks for Part I

None of the checks in this section is used in the proof.

6.1 A second certificate of Lemma 4.8

An independent AI referee wrote a second certificate of Lemma 4.8, sharing no code with the first. It uses the interval arithmetic of mpmath at 150 digits instead of Arb. On $[13/10, 40]$ it evaluates the un-normalised closed forms directly from M and from the definitions of the Σ_k, Γ and N, on 938 adaptively chosen cells. On the tail $M \ge 40$ it uses a normalisation derived separately with a computer algebra system; the resulting expressions have integer coefficients. It computes the geometric bounds from half- and double-angle formulas rather than from (6). Its first cell starts at an exact dyadic number below 13/10. It confirms (i) and (ii) on every cell, with worst ratio $\Lambda_{\rm hi}/\mathcal{R} \le 0.6136$ on the body and $\le 1.1 \cdot 10^{-33}$ on the tail, and it confirms $\beta(13/10) < 6/5$. At $M = 2, 5, 10, 30$ its two forms agree to more than 100 digits. The same referee found no fatal or serious issue in the arguments of Sections 2–5.

6.2 A six-wave certificate for $6/5 \le \beta \le 3$

Before Theorem 1.4 was found, the range $6/5 \le \beta \le 3$ was proved by a much larger computation, which now gives an independent check of Theorem 1.4 there. Let $w_{2s} = kn_s$ and $w_{2s+1} = -kn_s$ $(s = 0, 1, 2)$ be the six wave vectors of Proposition 2.2. Transplanting $u_c = \sum_j c_j e^{w_j \cdot x}$ to $T_{a,b}$ gives

the Rayleigh quotient $c^\top A(a,b;\alpha)c/c^\top Nc$, where

$$A_{jk} = (w_j^\top \mathcal{A}\, w_k)N_{jk} + \alpha \sum_s \rho_s S^s_{jk}, \qquad N_{jk} = \int_E e^{(w_j+w_k)\cdot x}, \quad S^s_{jk} = \int_{\text{side } s} e^{(w_j+w_k)\cdot x}.$$

These are divided differences of exp, which we evaluate in ball arithmetic: by difference quotients when the nodes are separated, and by power series with explicit tail bounds when they are close. The shape domain $\mathcal{D} \setminus \{(1,0)\}$ of Lemma 2.4 is covered by four regions; write $\delta = (a-1, b)$.

(a) *Local boxes* $|\delta|_\infty \le r$. Use the trial vector $c_0(M) + (a-1)c_1 + bc_2$, where c_0 is the coefficient vector of ψ_0 (an exact eigenvector at $\delta = 0$). Then $F = c^\top(A - \lambda_0 N)c$ has a critical point with value 0 at $\delta = 0$, and an interval Hessian in δ, with a mean-value form in M obtained from dual numbers, is negative definite.

(b) *A near annulus* $r < |\delta|_\infty \le R$. The same trial vector, with $F < 0$ checked on cells.

(c) *Far cells in* (a, b, β). Fixed trial data on each cell, with a coefficient vector affine in (a, b). The Rayleigh quotient is affine in α and $\lambda_1(E, \cdot)$ is concave (Lemma 2.1), so it suffices to check the two ends of each β-interval.

(d) *Thin triangles* $a > 6$. The constant trial function and $P/P_* \ge 2a/3$.

The certificate consists of 414 local M-intervals (with r from 1/4000 to 1/60), 4140 near M-intervals (with R from 1/500 to 2/15), and 109 717 far cells in 24 β-ranges that tile $[6/5, 3]$. The local and near intervals cover $[M(6/5), M(3)]$. The cell count grows quickly with β, because the near-equilateral regime shrinks like $e^{-2\cdot 3^{1/4}\beta}$. Removing that obstruction is what Theorem 1.4 achieves.

6.3 Floating-point checks

- *Closed forms.* The closed forms of the eight integrals agree with Gauss quadrature of $\cosh(L + 2Kb_i)$ on actual scalene triangles to 10^{-13}. The normalised polynomial expressions agree with the closed forms to rounding error. The referee's independent quadrature of the pencil on E reproduces the closed forms to 10^{-13}.

- *The chain of inequalities.* Every inequality used in Lemma 4.5, Proposition 4.6 and Lemma 4.7 holds, and $c^\top \mathcal{P} c < 0$, on 400 random triangles of the near region with $M \in [1.3, 5]$. In this test all matrices are computed by quadrature on the triangle itself. On 3000 further near-region triangles (referee's code, $M \in [1.3, 6.5]$), the only failures occur at $W < 10^{-15}$, where $c^\top \mathcal{P} c$ is below the rounding level.

- *The corner bound.* The divergence identity of Lemma 4.1 holds to 10^{-13} on random triangles, and the corner bound holds on 573 (triangle, vertex, β) samples.

- *Finite elements.* On 16 triangles of the near region with $\beta \in \{1.16, 1.5, 2, 3\}$, a P_2 finite element value of $\lambda_1(T, -\beta)$ (an upper bound up to rounding) lies below the three-function Ritz value, to within the finite element error on E, and the Ritz value lies below λ_0. At $\beta = 2$, a P_1 finite element computation with Richardson extrapolation gives $\lambda_1(E, -2) \approx -16.857416$, compared with $\lambda_0 = -16.857437$.

- *Consistency of the two parts.* The eight integrals reproduce g, B/N and λ_0 of Lemma 2.3. Sampling 205 912 random triangles with $p \le q_A$ gives $\min(p-1)/(\tau-2) = 0.1445\ldots > G_A$, in line with Lemma 3.2.

- *Negative tests.* The certificate of Lemma 4.8 fails, as it should, when started at $M = 21/20$, where the inequality is false, or at $M = 1$, where $\cot\theta(t) > 1$. The certificate of Lemma 3.2 fails with $G_A = 0.146$, which exceeds the minimum of $(p-1)/(\tau-2)$.

Part II
The case $\alpha > 0$

7 Overview of Part II

In this part $\alpha > 0$, and the notation is independent of Part I.

Notation. T is a triangle with area A, perimeter P, side lengths a_1, a_2, a_3, inradius $r = 2A/P$, incentre I, altitudes $h_j = 2A/a_j$ and $R_I = \max_k |v_k - I|$ over the vertices v_k. E is the equilateral triangle of area A; its perimeter, inradius and altitude are $P_E = \sqrt{12\sqrt{3}\,A}$, $r_E = 2A/P_E$ and $H = 3r_E$. We write $\lambda_0(\alpha) = \lambda_1(E,\alpha)$ and denote by $\Lambda_1(T) < \Lambda_2(T)$ the first two Dirichlet eigenvalues of T. Norms and inner products without subscript are those of L^2 of the domain in question.

Three explicit results. The following statements are proved without computation, apart from an interval check of an elementary inequality in two variables in Theorem 7.3. They are used in the proof of Theorem 1.1(1).

Theorem 7.1 (thin triangles)**.** *If the longest side of T satisfies $a_{\max} \geq P_E$, that is $a_{\max}^2 \geq 12\sqrt{3}\,A$, then $\lambda_1(T,\alpha) > \lambda_0(\alpha)$ for every $\alpha > 0$.*

Theorem 7.2 (weak coupling)**.** *For every triangle T and every $\alpha > 0$,*

$$\lambda_1(T,\alpha) \;\geq\; \alpha\frac{P}{A} - \alpha^2\frac{R_I^2}{r^2}. \tag{8}$$

Consequently $\lambda_1(T,\alpha) > \lambda_0(\alpha)$ whenever T is not equilateral and $0 < \alpha \leq (P - P_E)\,r^2/(A\,R_I^2)$.

Theorem 7.3 (uniform weak coupling)**.** *If $0 < \alpha\sqrt{A} \leq 1/100$, then $\lambda_1(T,\alpha) > \lambda_0(\alpha)$ for every non-equilateral triangle T of area A.*

Structure of the proof of Theorem 1.1(1). By scaling we may take $A = 1$. The shapes of triangles of area 1 are parametrised by the hyperbolic plane; we use its Klein model, a point w of the unit disc, with $w = 0$ for E and $|w| = \tanh 2t$, where e^t is the stretch factor of the symmetric linear map taking E to T (Section 11). The parameter set $\{(w,\alpha)\}$ is covered by four regions.

(A) *Thin triangles* ($a_{\max} \geq P_E$, all α): Theorem 7.1, by slicing perpendicular to the longest side.

(S) *Weak coupling* ($\alpha \leq 1/100$, all shapes): Theorem 7.3.

(C) *Near E* ($|w| \leq 0.1194$, which is slightly less than $\tanh(2\cdot\frac{3}{50})$; $\alpha \geq 1/100$): Proposition 14.4. A Schur-complement criterion (Lemma 13.1) reduces the inequality to $D[\psi_0] > \mathcal{R}/(1-\eta')$, where $D[\psi_0] > 0$ is the explicit second-order gain of the frozen ground state ψ_0 of E and $\mathcal{R}$ is a second-order perturbation sum. The difficulty is an *upper* bound for $\mathcal{R}$, uniform in

$\alpha \in [1/100, \infty)$; in floating point the gain exceeds it only by a factor 1.72 in the worst case (Remark 14.6). We obtain it from certified *lower* bounds for the lowest eigenvalue of two perturbed problems at distance $s = 1/10$ from E in opposite directions (Lemma 14.2); by symmetry one number $R_1(\alpha)$ suffices, and the resulting bound is within a few per cent of the true value.

(B) *Middle region* ($|w| \geq 0.1194$, not thin, $\alpha \geq 1/100$): Proposition 15.2. The lowest eigenvalue is a concave function of the coefficients (G, m) of the quadratic form pulled back to E. Positive combinations of the matrices G are exactly the points of Euclidean triangles in the Klein model, and for such a cell (Lemma 11.3) $\lambda_1(T_w, \alpha) \geq c \sum_i \theta_i \lambda_1(T_{w_i}, \alpha)$ with the explicit constant $c = (\theta^{\mathsf{T}} \Gamma \theta)^{-1/2}$, $\Gamma_{ik} = \cosh 2d(w_i, w_k)$. Together with concavity in α this turns certified lower bounds at the vertices of a prism into a lower bound on the whole prism, with an error of second order in its size.

The certified lower bounds in (B) and (C) are Crouzeix–Raviart bounds (Lemma 12.2); the discrete eigenvalue is bounded from below by a Collatz–Wielandt quotient evaluated in exact integer arithmetic (Lemma 12.3), and the interpolation constant is proved by hand (Lemma 12.1). No verified matrix factorisation is needed, apart from a small congruence test for the second eigenvalue of E (Lemma 12.4).

8 The equilateral triangle

Let d_1, d_2, d_3 be the distances to the sides of E; then $d_1 + d_2 + d_3 = H$ in E.

Proposition 8.1. *For $\alpha > 0$ let $\kappa = \kappa(\alpha)$ be the unique root in $(0, 2\pi/H)$ of*

$$\frac{\kappa H}{2} = \arctan \frac{\alpha}{\kappa} + \arctan \frac{2\alpha}{\kappa}, \tag{9}$$

and let $\delta = \arctan(\alpha/\kappa)$. Then $\lambda_0(\alpha) = \kappa^2$, with eigenfunction $\psi_0 = \sum_{i=1}^{3} \cos(\kappa d_i - \delta)$. Moreover $\lambda_0(\alpha) < \alpha P_E/A$ and $\lambda_0(\alpha) < \Lambda_1(E) = 4\pi^2/H^2$.

Proof. The left side of (9) increases from 0 to π on $(0, 2\pi/H]$ and the right side decreases from π to a value below π, so the root exists and is unique; it depends continuously on α. Since $|\nabla d_i| = 1$, $-\Delta\psi_0 = \kappa^2\psi_0$. The inward unit normals satisfy $\nabla d_i \cdot \nabla d_j = -\frac{1}{2}$ for $i \neq j$. On the side $d_1 = 0$, with outer normal $\nu = -\nabla d_1$, write $d_2 = H/2 + \tau$, $d_3 = H/2 - \tau$. Then

$$\partial_\nu \psi_0 + \alpha\psi_0 = (\alpha\cos\delta - \kappa\sin\delta) + \cos(\kappa\tau)\Big(2\alpha\cos\big(\tfrac{\kappa H}{2} - \delta\big) - \kappa\sin\big(\tfrac{\kappa H}{2} - \delta\big)\Big),$$

and both brackets vanish because $\tan\delta = \alpha/\kappa$ and $\tan(\kappa H/2 - \delta) = 2\alpha/\kappa$ by (9). By symmetry the boundary condition holds on all sides, so κ^2 is a Robin eigenvalue of E with eigenfunction ψ_0; here $\psi_0 \not\equiv 0$ because, in the coordinates (d_1, d_2), $\partial^2\psi_0/\partial d_1 \partial d_2 = -\kappa^2 \cos(\kappa d_3 - \delta) \not\equiv 0$. As $\alpha \to 0$ we have $\kappa \to 0$ (the right side of (9) is at most $3\alpha/\kappa$) and $\alpha/\kappa \to 0$ (otherwise the right side of (9) would stay bounded away from 0 along a sequence while the left side tends to 0), hence $\delta \to 0$ and $\psi_0 \to 3$ uniformly; so $\psi_0 > 0$ and $\kappa(\alpha)^2 = \lambda_0(\alpha)$ for small α. The set of $\alpha > 0$ with $\kappa(\alpha)^2 = \lambda_0(\alpha)$ is closed by continuity. It is open: the first two eigenvalues $\lambda_0(\alpha) < \lambda_2(\alpha)$ are continuous in α by the min–max principle, so for α near a point α^* of the set the only eigenvalue below $\frac{1}{2}(\lambda_0(\alpha^*) + \lambda_2(\alpha^*))$ is $\lambda_0(\alpha)$, and the eigenvalue $\kappa(\alpha)^2$, being continuous, lies below that number. Hence the set is all of $(0, \infty)$. (The formula is a special case of McCartin's [29].) The bound $\lambda_0 < \alpha P_E/A$ follows from the constant trial function, which is not an eigenfunction, and $\lambda_0 < \Lambda_1(E)$ from $\kappa < 2\pi/H$. □

Lemma 8.2. *The function λ_0 is positive, strictly increasing, concave and continuously differentiable on $(0,\infty)$, with*

$$\lambda_0'(\alpha) = \frac{\int_{\partial E}\psi_0^2}{\int_E \psi_0^2} = 2\kappa\,\frac{\dfrac{\kappa}{\kappa^2+\alpha^2}+\dfrac{2\kappa}{\kappa^2+4\alpha^2}}{\dfrac{H}{2}+\dfrac{\alpha}{\kappa^2+\alpha^2}+\dfrac{2\alpha}{\kappa^2+4\alpha^2}}, \qquad \kappa=\kappa(\alpha), \tag{10}$$

and $\int_E|\nabla\psi_0|^2 = (\lambda_0-\alpha\lambda_0')\int_E\psi_0^2$. Moreover $\alpha\lambda_0'(\alpha)\le 2(\lambda_0(\alpha)-\lambda_0(\alpha/2))$, and for every $\alpha^*>0$ the tangent $\alpha\mapsto\lambda_0(\alpha^*)+\lambda_0'(\alpha^*)(\alpha-\alpha^*)$ lies above λ_0 on $(0,\infty)$. If $\beta_1<\dots<\beta_M$ are positive numbers and U is the minimum of the tangents $t_1,\dots,t_M$ at these points, then $U\ge\lambda_0$, and on each interval $[\beta_m,\beta_{m+1}]$ we have $U=\min(t_m,t_{m+1})$, a piecewise linear function whose only possible kink is the point of $[\beta_m,\beta_{m+1}]$ where $t_m=t_{m+1}$.

Proof. λ_0 is an infimum of affine functions of α by (1), hence concave. The implicit function theorem applied to (9) (whose left side minus right side has a positive derivative in κ) gives $\kappa\in C^1$ and the second expression in (10) for $(\kappa^2)'$; it is positive, so λ_0 is strictly increasing. For the first expression let $\alpha,\beta>0$ and insert the normalised ground state for α into the quotient for β, and conversely: $(\beta-\alpha)\int_{\partial E}\psi_{0,\beta}^2\le\lambda_0(\beta)-\lambda_0(\alpha)\le(\beta-\alpha)\int_{\partial E}\psi_{0,\alpha}^2$ for $\|\psi_{0,\cdot}\|=1$; since $\psi_{0,\beta}\to\psi_{0,\alpha}$ uniformly as $\beta\to\alpha$ by the explicit formula, the derivative is $\int_{\partial E}\psi_{0,\alpha}^2$. The formula for $\int|\nabla\psi_0|^2$ is $\lambda_0=\int|\nabla\psi_0|^2+\alpha\int_{\partial E}\psi_0^2$. By concavity the slope of the chord over $[\alpha/2,\alpha]$ is at least $\lambda_0'(\alpha)$, which is the next claim, and the claim on tangents is concavity again. For the last claim let $j<m$: then $t_j(\beta_m)\ge\lambda_0(\beta_m)=t_m(\beta_m)$ and the slope of t_j is at least that of t_m, so $t_j\ge t_m$ on $[\beta_m,\infty)$; similarly $t_j\ge t_{m+1}$ on $(0,\beta_{m+1}]$ for $j>m+1$. Finally $t_m\le t_{m+1}$ at β_m and $t_m\ge t_{m+1}$ at β_{m+1}. □

9 Thin triangles: proof of Theorem 7.1

For $\ell>0$ let $\mu(\ell,\alpha)=k^2$, where $k\in(0,\pi/\ell)$ is the root of $k\tan(k\ell/2)=\alpha$; this is the lowest eigenvalue of $-w''=\mu w$ on $(0,\ell)$ with Robin parameter α at both ends, and it is decreasing in ℓ.

Lemma 9.1. *For every triangle T and $\alpha>0$, $\lambda_1(T,\alpha)\ge\mu(h_{\min},\alpha)$, where $h_{\min}=2A/a_{\max}$ is the altitude on the longest side.*

Proof. Place the longest side on $\{y=0\}$, so that $T=\{0<x<b,\ 0<y<\ell(x)\}$ with ℓ piecewise linear and $0<\ell\le h_{\min}$ (the angles adjacent to the longest side are acute, so the foot of the altitude lies on it). On the two other sides $\mathrm{d}s\ge\mathrm{d}x$. Hence for $u\in C^1(\overline{T})$, and then for $u\in H^1(T)$ by density,

$$\begin{aligned}\int_T|\nabla u|^2+\alpha\int_{\partial T}u^2 &\ge \int_0^b\Big(\int_0^{\ell(x)}|\partial_y u|^2\,\mathrm{d}y+\alpha\,u(x,0)^2+\alpha\,u(x,\ell(x))^2\Big)\,\mathrm{d}x\\ &\ge\int_0^b\mu(\ell(x),\alpha)\int_0^{\ell(x)}u^2\,\mathrm{d}y\,\mathrm{d}x,\end{aligned}$$

and $\mu(\ell(x),\alpha)\ge\mu(h_{\min},\alpha)$. □

Proof of Theorem 7.1. The hypothesis is $h_{\min}\le r_E$, so by Lemma 9.1 it suffices to show $\mu(r_E,\alpha)>\lambda_0(\alpha)$. Let k be the root of $k\tan(kr_E/2)=\alpha$, that is $kr_E/2=\arctan(\alpha/k)$. The function $g(\kappa)=\kappa H/2-\arctan(\alpha/\kappa)-\arctan(2\alpha/\kappa)$ is increasing and vanishes at $\kappa(\alpha)$. Since $H=3r_E$,

$$g(k)=3\arctan\frac{\alpha}{k}-\arctan\frac{\alpha}{k}-\arctan\frac{2\alpha}{k}=2\arctan\frac{\alpha}{k}-\arctan\frac{2\alpha}{k}>0,$$

because arctan is strictly concave on $[0,\infty)$ and vanishes at 0. Hence $k>\kappa(\alpha)$, that is $\mu(r_E,\alpha)>\lambda_0(\alpha)$ by Proposition 8.1. □

10 Weak coupling: proof of Theorem 7.2

Lemma 10.1 (dual principle)**.** *Let Ω be a bounded Lipschitz domain and $V \in C^1(\overline{\Omega};\mathbb{R}^2)$ with $V\cdot\nu \le \alpha$ on $\partial\Omega$. Then $\lambda_1(\Omega,\alpha) \ge \inf_\Omega(\operatorname{div} V - |V|^2)$.*

Proof. For $u \in H^1(\Omega)$, $0 \le |\nabla u + uV|^2$ gives $|\nabla u|^2 \ge -V\cdot\nabla(u^2) - |V|^2u^2$. Integrating by parts, $\int_\Omega |\nabla u|^2 \ge \int_\Omega (\operatorname{div} V - |V|^2)u^2 - \int_{\partial\Omega}(V\cdot\nu)u^2$, and the last term is at least $-\alpha\int_{\partial\Omega} u^2$. □

Proof of Theorem 7.2. Every side of T is tangent to the incircle, so $(x-I)\cdot\nu = r$ on ∂T. Take $V(x) = \alpha(x-I)/r$. Then $V\cdot\nu = \alpha$ on ∂T, $\operatorname{div} V = 2\alpha/r = \alpha P/A$ and $|V|^2 \le \alpha^2 R_I^2/r^2$ on T, since $|x-I|$ is maximal at a vertex. Lemma 10.1 gives (8). If $\alpha R_I^2/r^2 \le (P-P_E)/A$ then (8) gives $\lambda_1(T,\alpha) \ge \alpha P_E/A > \lambda_0(\alpha)$ by Proposition 8.1; here $P > P_E$ by the isoperimetric inequality for triangles. □

11 Shapes, concavity and cells

From now on $A = 1$ (this is no restriction, since $\lambda_1(sT,\alpha) = s^{-2}\lambda_1(T,s\alpha)$). Fix the equilateral triangle E of area 1 with centroid 0, unit side directions $\tau_j = (\cos\vartheta_j, \sin\vartheta_j)$, $\vartheta_j \in \{0, 2\pi/3, 4\pi/3\}$, and sides $\Gamma_1,\Gamma_2,\Gamma_3$ of length $s_E = 2/H$, $H = 3^{1/4}$. For a symmetric positive definite $G \in \mathbb{R}^{2\times2}$ and $m \in [0,\infty)^3$, $m \ne 0$, put

$$q_{G,m}[v] = \int_E \nabla v\cdot G\nabla v + \sum_{j=1}^3 m_j \int_{\Gamma_j} v^2, \qquad \Lambda(G,m) = \inf_{0\ne v\in H^1(E)} \frac{q_{G,m}[v]}{\int_E v^2}.$$

The infimum is attained. If B is an invertible matrix and $u = v\circ B^{-1}$ on $B(E)$, then

$$q_{G,m}[v] = \frac{1}{|\det B|}\Big(\int_{B(E)} \nabla u\cdot BGB^{\mathsf T}\nabla u + \sum_j \frac{m_j|\det B|}{|B\tau_j|}\int_{B\Gamma_j} u^2\Big), \qquad \int_E v^2 = \frac{1}{|\det B|}\int_{B(E)} u^2. \tag{11}$$

With $B = G^{-1/2}$ this shows that $\Lambda(G,m)$ is the lowest eigenvalue of the Laplacian on the triangle $G^{-1/2}(E)$ with the Robin coefficient $\beta_j = m_j(\det G)^{-1/2}/|G^{-1/2}\tau_j|$ on its j-th side. In particular, let $\mathcal{P}$ be the set of symmetric positive definite matrices of determinant one and, for $G \in \mathcal{P}$,

$$T_G = G^{-1/2}(E), \qquad \ell_j(G) = |G^{-1/2}\tau_j| = (\tau_j^{\mathsf T}G^{-1}\tau_j)^{1/2}; \qquad \text{then} \quad \lambda_1(T_G,\alpha) = \Lambda(G,\alpha\,\ell(G)),$$

$|T_G| = 1$, and the sides of T_G have lengths $s_E\ell_j(G)$.

Lemma 11.1 (concavity)**.** *Λ is concave, positively homogeneous of degree one, and nondecreasing with respect to the Loewner order in G and the componentwise order in m. Consequently, if $\theta_i \ge 0$, $\sum_i\theta_i = 1$, $c > 0$ and $G \succeq c\sum_i\theta_i G_i$, $m \ge c\sum_i\theta_i m_i$, then $\Lambda(G,m) \ge c\sum_i\theta_i\Lambda(G_i,m_i)$.*

Proof. For fixed v the quotient is linear in (G,m) and nondecreasing; an infimum of such functions is concave, homogeneous and nondecreasing. □

In particular $\alpha \mapsto \lambda_1(T,\alpha)$ is concave and nondecreasing for every triangle T.

Shape coordinates. For $w = (w_1, w_2)$ in the open unit disc $\mathbb{D}$ let

$$S(w) = \begin{pmatrix} w_1 & w_2 \\ w_2 & -w_1 \end{pmatrix}, \qquad G(w) = \frac{I - S(w)}{\sqrt{1-|w|^2}}, \qquad T_w = T_{G(w)}, \qquad e_j = (\cos 2\vartheta_j, \sin 2\vartheta_j).$$

Lemma 11.2. (a) *$w \mapsto G(w)$ is a bijection from $\mathbb{D}$ onto $\mathcal{P}$, and $G(w)^{-1} = (I+S(w))/\sqrt{1-|w|^2}$. Hence $\ell_j(G(w))^2 = (1 + w \cdot e_j)/\sqrt{1-|w|^2}$, and the angle ω_j of T_w opposite to its j-th side satisfies $\cot \omega_j = (1 - 2\, w \cdot e_j)/\sqrt{3(1-|w|^2)}$.*

(b) *Every triangle of area 1 is congruent to T_w for some w in the closed sector $\Sigma = \{w \in \mathbb{D} : 0 \le \arg w \le \pi/3\} \cup \{0\}$; T_w is equilateral only for $w = 0$.*

(c) *If $(1 + w_1)^2 \ge 81(1 - |w|^2)$, then the longest side of T_w is at least P_E.*

(d) *If $L = \exp(tS_\phi)$ with $t \ge 0$ and $S_\phi = S((\cos 2\phi, \sin 2\phi))$, then $L(E) = T_w$ with $w = \tanh(2t)(\cos 2\phi, \sin 2\phi)$; moreover $G(w) = \cosh(2t)I - \sinh(2t)S_\phi$ and $\ell_j(G(w))^2 = \cosh 2t + \sinh 2t \, \cos(2\phi - 2\vartheta_j)$.*

(e) *For $w, w' \in \mathbb{D}$, $\Gamma(w, w') := \frac{1}{2}\operatorname{tr}\big(G(w)G(w')^{-1}\big) = (1 - w \cdot w')/\sqrt{(1-|w|^2)(1-|w'|^2)} \ge 1$.*

Proof. (a) $S(w)^2 = |w|^2 I$, so $(I - S(w))(I + S(w)) = (1 - |w|^2)I$ and $\det(I - S(w)) = 1 - |w|^2$; thus $G(w) \in \mathcal{P}$ with the stated inverse. Conversely a matrix in $\mathcal{P}$ is $aI + bS((1,0)) + cS((0,1))$ with $a > 0$, $a^2 - b^2 - c^2 = 1$, which is $G(w)$ exactly for $w = -(b,c)/a$. Next $\tau_j^{\mathsf{T}} S(w) \tau_j = w_1 \cos 2\vartheta_j + w_2 \sin 2\vartheta_j = w \cdot e_j$. For the angle use $\cot \omega_j = (\sum_k a_k^2 - 2a_j^2)/(4|T_w|)$ with $a_k^2 = s_E^2 \ell_k^2$, $s_E^2 = 4/\sqrt{3}$ and $\sum_k e_k = 0$. (b) Write the triangle as $M(E) + b$; composing with a reflection of E we may assume $\det M = 1$. Then $M = Q(M^{\mathsf{T}}M)^{1/2}$ with Q a rotation, so the triangle is congruent to T_G with $G = (M^{\mathsf{T}}M)^{-1} \in \mathcal{P}$. If g is one of the six orthogonal symmetries of E, then $T_{g^{\mathsf{T}}Gg} = g^{\mathsf{T}} G^{-1/2} g(E) = g^{\mathsf{T}}(T_G)$ is congruent to T_G, and $g^{\mathsf{T}} S(w) g = S(w')$ where w' is obtained from w by a rotation through a multiple of $2\pi/3$, possibly composed with $w_2 \mapsto -w_2$ (rotating E by ϑ rotates w by -2ϑ). These maps form the dihedral group of order six acting on $\mathbb{D}$, for which Σ is a fundamental domain. If T_w is equilateral then $\ell_1 = \ell_2 = \ell_3$, so $w \cdot e_j$ does not depend on j, hence $w = 0$. (c) The hypothesis is $\ell_1^2 \ge 9$, and the sides of T_w are $s_E \ell_j$, $P_E = 3 s_E$. (d) $\exp(tS_\phi) = \cosh t \, I + \sinh t \, S_\phi$ is symmetric with determinant one, so $L(E) = T_G$ with $G = L^{-2} = \exp(-2tS_\phi) = \cosh 2t \, I - \sinh 2t \, S_\phi = G(w)$ for the stated w, because $\cosh 2t = (1 - \tanh^2 2t)^{-1/2}$. The formula for ℓ_j follows from (a). (e) $\operatorname{tr}(S(w)S(w')) = 2w \cdot w'$ and $\operatorname{tr} S(w) = 0$ give the formula. The matrix $G(w)G(w')^{-1}$ is similar to a positive definite matrix of determinant one, so its eigenvalues are $\mu, 1/\mu$ with $\mu > 0$, and $\frac{1}{2}(\mu + 1/\mu) \ge 1$. □

By (a), $\mathcal{P}$ with the distance $d(G, G') = \frac{1}{2}\operatorname{arcosh}\Gamma$ is the hyperbolic plane and w is its Klein (projective) model; we do not use this, but it explains the next lemma: Euclidean segments in $\mathbb{D}$ are the hyperbolic geodesics, and the constant c below is $1 - O(\mathrm{diam}^2)$ for a cell of hyperbolic diameter diam.

Lemma 11.3 (cells)**.** *Let $w_1, \dots, w_N \in \mathbb{D}$, $G_i = G(w_i)$ and $\Gamma_{ik} = \Gamma(w_i, w_k)$. Let w be a point of the convex hull of $w_1, \dots, w_N$. Then there are $\theta_i \ge 0$ with $\sum_i \theta_i = 1$ such that, with $c = (\theta^{\mathsf{T}}\Gamma\theta)^{-1/2} = \big(\sum_{i,k} \theta_i \theta_k \Gamma_{ik}\big)^{-1/2} \in (0, 1]$,*

$$G(w) = c \sum_i \theta_i G_i \qquad \text{and} \qquad \lambda_1(T_w, \alpha) \;\ge\; c \sum_i \theta_i \, \lambda_1(T_{w_i}, \alpha) \quad \text{for every } \alpha > 0. \tag{12}$$

Proof. Write $w = \sum_i s_i w_i$ with $s_i \ge 0$, $\sum s_i = 1$, put $a_i = (1-|w_i|^2)^{-1/2}$ and $\theta_i = (s_i/a_i)/\sum_k (s_k/a_k)$. Then $\bar G := \sum_i \theta_i G_i = \sum_i \theta_i a_i (I - S(w_i)) = (\sum_i \theta_i a_i)(I - S(w))$, because S is linear and $\theta_i a_i$ is proportional to s_i. Thus $\bar G$ is a positive multiple of $G(w)$, and since $\det G(w) = 1$, $G(w) = c\,\bar G$ with $c = (\det \bar G)^{-1/2}$. For 2×2 matrices $\det X = \tfrac12 \operatorname{tr}(X \operatorname{adj} X)$ and adj is linear, so $\det \bar G = \tfrac12 \sum_{i,k} \theta_i\theta_k \operatorname{tr}(G_i \operatorname{adj} G_k) = \theta^{\mathsf T} \Gamma \theta$, as $\operatorname{adj} G_k = G_k^{-1}$. Since $\Gamma_{ik} \ge 1$ (Lemma 11.2(e)), $\theta^{\mathsf T}\Gamma\theta \ge (\sum \theta_i)^2 = 1$ and $c \le 1$.

Next, for $G \in \mathcal P$ we have $G^{-1} = \operatorname{adj} G = J^{\mathsf T} G J$ with J the rotation through $\pi/2$; hence $\ell_j(G)^2 = (J\tau_j)^{\mathsf T} G (J\tau_j)$ is the restriction to $\mathcal P$ of a *linear* function of G. Therefore $\sum_i \theta_i \ell_j(G_i)^2 = (J\tau_j)^{\mathsf T} \bar G (J\tau_j) = \ell_j(G(w))^2 / c$, and by the Cauchy–Schwarz inequality $\sum_i \theta_i \ell_j(G_i) \le \big(\sum_i \theta_i \ell_j(G_i)^2\big)^{1/2} = c^{-1/2} \ell_j(G(w))$. So $\ell_j(G(w)) \ge c^{1/2} \sum_i \theta_i \ell_j(G_i) \ge c \sum_i \theta_i \ell_j(G_i)$. Now Lemma 11.1, applied to the points $(G_i, \alpha \ell(G_i))$, gives (12). $\square$

Corollary 11.4 (prisms). *Let $w_1, w_2, w_3 \in \mathbb D$, $\Gamma_{ik} = \Gamma(w_i, w_k)$, $0 < \alpha_0 < \alpha_1$, and let $b_i^0, b_i^1 > 0$ satisfy $b_i^0 \le \lambda_1(T_{w_i}, \alpha_0)$, $b_i^1 \le \lambda_1(T_{w_i}, \alpha_1)$ $(i = 1,2,3)$. Let $U : [\alpha_0, \alpha_1] \to \mathbb R$ satisfy $U \ge \lambda_0$.*

(a) *If for all $1 \le i \le k \le 3$ and all $\alpha = (1-u)\alpha_0 + u\alpha_1$, $u \in [0,1]$,*

$$(1-u)\sqrt{b_i^0 b_k^0} + u\sqrt{b_i^1 b_k^1} \;>\; \sqrt{\Gamma_{ik}}\, U(\alpha), \tag{13}$$

then $\lambda_1(T_w, \alpha) > \lambda_0(\alpha)$ for every w in the closed triangle with vertices w_1, w_2, w_3 and every $\alpha \in [\alpha_0, \alpha_1]$.

(b) *If $\sqrt{b_i^0 b_k^0} > \sqrt{\Gamma_{ik}}\, \Lambda_1(E)$ for all $1 \le i \le k \le 3$, then $\lambda_1(T_w, \alpha) > \lambda_0(\alpha)$ for every w in that triangle and every $\alpha \ge \alpha_0$.*

Proof. (a) Fix α and u. By concavity in α, $\lambda_1(T_{w_i}, \alpha) \ge b_i := (1-u)b_i^0 + u b_i^1$. The geometric mean of two positive affine functions of u is concave, so $\sqrt{b_i b_k}$ is at least the left side of (13); hence $b_i b_k > \Gamma_{ik} \lambda_0(\alpha)^2$ for all i, k (for $i > k$ by symmetry). For w in the triangle take θ and c from Lemma 11.3. Then $(\sum_i \theta_i b_i)^2 = \sum_{i,k} \theta_i\theta_k b_i b_k > \lambda_0(\alpha)^2\, \theta^{\mathsf T}\Gamma\theta$, that is $c \sum_i \theta_i b_i > \lambda_0(\alpha)$, and (12) gives the claim. (b) For $\alpha \ge \alpha_0$, $\lambda_1(T_{w_i}, \alpha) \ge \lambda_1(T_{w_i}, \alpha_0) \ge b_i^0$, and the same computation gives $\lambda_1(T_w, \alpha) > \Lambda_1(E) > \lambda_0(\alpha)$ by Proposition 8.1. $\square$

12 Certified lower bounds

Lemma 12.1 (interpolation constant). *Let K be a triangle with longest side h and let $f \in H^1(K)$ have mean value zero on each of the three sides of K. Then*

$$\|f\|_{L^2(K)} \le \kappa_0\, h\, \|\nabla f\|_{L^2(K)}, \qquad \kappa_0 = \Big(\frac{1}{48} + \frac{1}{j_{1,1}^2}\Big)^{1/2} < 0.2983,$$

where $j_{1,1} = 3.8317\ldots$ is the first positive zero of the Bessel function J_1. The same holds with $j_{1,1}$ replaced by π, that is with the constant $(1/48 + 1/\pi^2)^{1/2} < 0.3496$.

Proof. (This is the argument of [9]; see also [10].) Let $\bar f$ be the mean of f over K. Then $\|f\|^2 = \|f - \bar f\|^2 + |K|\bar f^2$, and $\|f - \bar f\| \le (h/j_{1,1})\|\nabla f\|$ by the optimal Poincaré inequality for triangles of Laugesen and Siudeja [23] (with π in place of $j_{1,1}$ this is the Payne–Weinberger inequality for convex domains [31, 4]). Let P_i be the vertex opposite to the side e_i. The field $x - P_i$ is tangent to the two sides through P_i, and $(x - P_i)\cdot \nu = 2|K|/|e_i|$ on e_i. The divergence theorem for $f\,(x - P_i)$ gives $2\int_K f + \int_K (x - P_i)\cdot\nabla f = (2|K|/|e_i|)\int_{e_i} f = 0$. Averaging over $i = 1,2,3$, $2\int_K f = -\int_K (x - c)\cdot \nabla f$

with c the centroid, so $|K|\,|\bar f| \le \tfrac12 \|x-c\|_{L^2(K)} \|\nabla f\|$. The polar moment of inertia of a triangle with sides a, b, c' about its centroid is $\|x-c\|^2_{L^2(K)} = |K|(a^2+b^2+c'^2)/36 \le |K|h^2/12$. Hence $|K|\bar f^2 \le (h^2/48)\|\nabla f\|^2$. □

Let $\mathcal{T}_h$ be a triangulation of a triangle Ω (conforming: two elements meet in a common side, a common vertex, or not at all), V_h the Crouzeix–Raviart space [11] on $\mathcal{T}_h$, i.e. the piecewise affine functions that are continuous at the midpoints of the interior edges, with no boundary condition, and $\Pi_h : H^1(\Omega) \to V_h$ the interpolation operator defined by $\int_e \Pi_h u = \int_e u$ for every edge e. Let $\beta : \partial\Omega \to [0,\infty)$ be constant on each side of Ω and not identically zero, and

$$a_h(u,v) = \sum_{K\in\mathcal{T}_h} \int_K \nabla u \cdot \nabla v + \sum_{e\subset\partial\Omega} \beta_e\, |e|\, \bar u_e\, \bar v_e,$$

where e runs over the boundary edges and $\bar u_e$ is the mean of u on e. Let $\lambda_{1,h} \le \lambda_{2,h} \le \dots$ be the eigenvalues of a_h on V_h with respect to the $L^2(\Omega)$ inner product, and $\lambda_1 \le \lambda_2 \le \dots$ the eigenvalues of the form $q_\beta[u] = \int_\Omega |\nabla u|^2 + \int_{\partial\Omega} \beta u^2$ on $H^1(\Omega)$.

Lemma 12.2 (Crouzeix–Raviart lower bound)**.** *Let $C_h = \kappa_0 h$, where h is the largest diameter of an element of $\mathcal{T}_h$ and κ_0 is a constant for which Lemma 12.1 holds. Then for $k = 1$ and $k = 2$, and every σ with $0 < \sigma \le \lambda_{k,h}$,*

$$\lambda_k \;\ge\; \frac{\lambda_{k,h}}{1 + C_h^2 \lambda_{k,h}} \;\ge\; \frac{\sigma}{1 + C_h^2 \sigma}.$$

Proof. (For $\beta = 0$ on a part of the boundary and Dirichlet conditions on the rest this is due to [10, 26]; the Robin term requires only the observation that it is diagonal in the edge means.) First, a_h is positive definite on V_h: if $a_h(w,w) = 0$ then w is constant on each element, continuous at the midpoints, hence constant, and a boundary edge with $\beta_e > 0$ forces $w = 0$. Extend a_h to $H^1(\Omega) + V_h$ by the same formula. For $v_h \in V_h$, $\partial_\nu v_h$ is constant on each edge of each element, so $\sum_K \int_K \nabla(u - \Pi_h u)\cdot\nabla v_h = 0$; the boundary term of $a_h(u - \Pi_h u, v_h)$ vanishes because the edge means of $u - \Pi_h u$ vanish. Thus

$$a_h(u,u) = a_h(\Pi_h u, \Pi_h u) + a_h(u - \Pi_h u, u - \Pi_h u) \qquad (u \in H^1(\Omega)). \tag{14}$$

By Lemma 12.1 on each element, $\|u - \Pi_h u\| \le C_h\, a_h(u-\Pi_h u, u-\Pi_h u)^{1/2}$. Moreover $|e|\,\bar u_e^2 \le \int_e u^2$, so $a_h(u,u) \le q_\beta[u]$.

Let $U \subset H^1(\Omega)$ be spanned by eigenfunctions for $\lambda_1, \dots, \lambda_k$, so that $q_\beta[u] \le \lambda_k \|u\|^2$ on U. If Π_h is not injective on U, take $u \in U$ with $\|u\| = 1$ and $\Pi_h u = 0$; then $1 = \|u - \Pi_h u\| \le C_h a_h(u,u)^{1/2} \le C_h \lambda_k^{1/2}$, so $\lambda_k \ge C_h^{-2}$, which is larger than the asserted bound. Otherwise $\Pi_h U$ has dimension k and by the min–max principle in V_h there is $u \in U \setminus \{0\}$ with $a_h(\Pi_h u, \Pi_h u) \ge \lambda_{k,h} \|\Pi_h u\|^2$. For this u, by (14) and the Cauchy–Schwarz inequality,

$$\begin{aligned} \|u\| \le \|\Pi_h u\| + \|u - \Pi_h u\| &\le \lambda_{k,h}^{-1/2} a_h(\Pi_h u, \Pi_h u)^{1/2} + C_h\, a_h(u - \Pi_h u, u - \Pi_h u)^{1/2} \\ &\le (\lambda_{k,h}^{-1} + C_h^2)^{1/2} a_h(u,u)^{1/2} \le (\lambda_{k,h}^{-1} + C_h^2)^{1/2} \lambda_k^{1/2} \|u\|. \end{aligned}$$

The second inequality of the lemma holds because $x \mapsto x/(1 + C_h^2 x)$ is increasing. □

The matrices. In the Crouzeix–Raviart basis (one function per edge, equal to 1 at its midpoint; on an element it is $1-2\mu$, with μ the barycentric coordinate of the opposite vertex) the mass matrix M_h is diagonal, with entries $\frac{1}{3}$ of the total area of the (one or two) elements containing the edge; the boundary term of a_h is diagonal with entries $\beta_e|e|$; and the element stiffness matrix has off-diagonal entries $-2\cot\omega$, where ω is the angle of the element opposite to its third edge, and zero row sums. Hence, on a triangulation without obtuse angles, the matrix K_h of a_h has nonpositive off-diagonal entries. We use two triangulations of a triangle Ω, depending on an integer n:

(U) if Ω is acute: the subdivision into n^2 congruent triangles similar to Ω;

(A) in general: Ω is cut along the altitude on its longest side into two right triangles, and each of them is subdivided into n^2 congruent right triangles similar to it (the two subdivisions induce the same n equal segments on the altitude, so the union is conforming).

In both cases all data of K_h and M_h are determined by n and by the squared side lengths, the area and the Robin coefficients of Ω: the cotangent of the angle of Ω opposite to the side a_j is $(\sum_k a_k^2 - 2a_j^2)/(4|\Omega|)$; in (A) the two right triangles have the acute angles $\omega_B, \pi/2-\omega_B$ and $\omega_C, \pi/2-\omega_C$, where ω_B, ω_C are the angles of Ω adjacent to its longest side, their legs on that side have lengths $h_\Omega\cot\omega_B$ and $h_\Omega\cot\omega_C$ with h_Ω the altitude, and the coupling across a right angle is exactly zero. The largest element diameter is the longest side of Ω divided by n in (U), and the longer of the two other sides divided by n in (A).

Lemma 12.3 (Collatz–Wielandt certificate)**.** *Let K be a real symmetric matrix with nonpositive off-diagonal entries and M a diagonal matrix with positive diagonal. If x has positive entries and $Kx \geq \sigma Mx$ entrywise, then every eigenvalue of the pencil (K, M) is at least σ.*

Proof. Let $B = K - \sigma M$ and $X = \operatorname{diag}(x)$. The symmetric matrix XBX has nonpositive off-diagonal entries and row sums $x_i(Bx)_i \geq 0$, so it is diagonally dominant with nonnegative diagonal, hence positive semidefinite by Gershgorin's theorem. So B is positive semidefinite. □

Lemma 12.4 (second eigenvalue)**.** *Let K be real symmetric, M symmetric positive definite, x a vector, $g \geq 0$ and $\sigma \in \mathbb{R}$. If $K - \sigma M + g\,(Mx)(Mx)^{\mathsf{T}}$ is positive definite, then the second eigenvalue of the pencil (K, M) is at least σ. A symmetric matrix B is positive definite if there is a matrix S such that $S^{\mathsf{T}}BS$ is strictly diagonally dominant with positive diagonal.*

Proof. On the hyperplane $\{y : (Mx)^{\mathsf{T}}y = 0\}$ we have $y^{\mathsf{T}}Ky - \sigma y^{\mathsf{T}}My = y^{\mathsf{T}}By > 0$ for $y \neq 0$, and the second eigenvalue is the maximum over hyperplanes of the minimum of the Rayleigh quotient. If $C = S^{\mathsf{T}}BS$ is strictly diagonally dominant with positive diagonal, it is positive definite by Gershgorin's theorem; then S is injective, hence invertible, and $B = S^{-\mathsf{T}}CS^{-1}$ is positive definite. □

How the bounds are certified. For given $(a_1^2, a_2^2, a_3^2, |\Omega|, \beta, n)$, a floating-point approximation $x > 0$ of the discrete ground state is computed (sparse shift-and-invert Lanczos, followed by damped Jacobi sweeps, which make the small components near the corners accurate); it is not trusted. The vector x is converted *exactly* into a vector of positive integers (a common power of two times x). The real coefficients ($2\cot\omega$ for the angle classes, $\beta_e|e|$, and $\frac{1}{3}$ of the element areas; at most a dozen numbers) are enclosed in Arb balls [18] and replaced by rational lower or upper bounds with denominator 2^{60}, the direction of rounding being chosen for each term according to the sign of its exact integer cofactor. The inequality $Kx \geq \sigma Mx$ of Lemma 12.3 then becomes a sign test on exact integers, and the nonnegativity of the cotangents is checked on their lower bounds. This gives

$\lambda_{1,h} \geq \sigma$ for a dyadic rational σ, and Lemma 12.2 gives a lower bound for $\lambda_1(\Omega, \beta)$, evaluated in ball arithmetic with an upper bound for h. The hypothesis $\sigma > 0$ of Lemma 12.2 is checked explicitly wherever the lemma is applied (for σ below $-1/C_h^2$ the right-hand side would be positive and large), and a re-check tests the stored rational σ itself, without rounding. For the second eigenvalue of E the matrices of Lemma 12.4 are formed in ball arithmetic, with x and S floating-point matrices that are entered exactly (S approximates an inverse Cholesky factor).

13 The Schur-complement criterion and uniform weak coupling

The following criterion is the tool for all shapes close to E: for $\alpha \leq 1/100$ in this section and for $\alpha \geq 1/100$ in Section 14.

Lemma 13.1 (Schur-complement criterion near E)**.** *Fix $\alpha > 0$, let $q_0 = q_{I,\alpha(1,1,1)}$, let ψ_0 be its L^2-normalised ground state and $\lambda_0 < \lambda_2$ its first two eigenvalues (so $\lambda_0 = \lambda_1(E,\alpha)$ and λ_2 is the second eigenvalue). Let (G, m) satisfy $q_{G,m}[w] \geq (1-\eta) q_0[w]$ for all $w \in H^1(E)$, with $0 \leq \eta < 1 - \lambda_0/\lambda_2$, and put $D = q_{G,m} - q_0$ (a symmetric bilinear form), $\eta' = \eta\lambda_2/(\lambda_2 - \lambda_0) < 1$ and*

$$\mathcal{R} = \sup\Big\{ \frac{D(\psi_0, w)^2}{q_0[w] - \lambda_0 \|w\|^2} \,:\; 0 \neq w \in H^1(E),\ \langle w, \psi_0 \rangle = 0 \Big\}.$$

If $D[\psi_0] \geq \mathcal{R}/(1-\eta')$, then $\Lambda(G, m) \geq \lambda_0$; if the inequality is strict, then $\Lambda(G, m) > \lambda_0$.

Proof. Let $v = c\psi_0 + w$ with $w \perp \psi_0$ in $L^2(E)$; then $q_0(\psi_0, w) = 0$ and $q_0[w] \geq \lambda_2 \|w\|^2$. Hence

$$q_{G,m}[v] - \lambda_0 \|v\|^2 = c^2 D[\psi_0] + 2c\, D(\psi_0, w) + q_{G,m}[w] - \lambda_0 \|w\|^2,$$

and $q_{G,m}[w] - \lambda_0\|w\|^2 \geq (1-\eta) q_0[w] - \lambda_0\|w\|^2 \geq (1-\eta')(q_0[w] - \lambda_0\|w\|^2)$, the last step being equivalent to $(\eta' - \eta) q_0[w] \geq \eta' \lambda_0 \|w\|^2$, which holds since $(\eta' - \eta)\lambda_2 = \eta'\lambda_0$. Writing $N = q_0[w] - \lambda_0\|w\|^2$ and using $|D(\psi_0, w)| \leq (\mathcal{R} N)^{1/2}$,

$$q_{G,m}[v] - \lambda_0\|v\|^2 \geq c^2 D[\psi_0] - 2|c|(\mathcal{R}N)^{1/2} + (1-\eta')N \geq c^2 \Big(D[\psi_0] - \frac{\mathcal{R}}{1-\eta'} \Big) \geq 0.$$

If $D[\psi_0] > \mathcal{R}/(1-\eta')$, the right side is positive for $c \neq 0$, and for $c = 0$, $w \neq 0$ the left side is at least $(1-\eta')N > 0$; since the infimum defining $\Lambda(G, m)$ is attained, $\Lambda(G, m) > \lambda_0$. □

Remark 13.2. Lemma 13.1 and its proof remain valid if λ_2 is replaced throughout (in the condition on η and in the definition of η') by a number $\underline{\lambda}_2$ with $\lambda_0 < \underline{\lambda}_2 \leq \lambda_2$, because only the inequality $q_0[w] \geq \underline{\lambda}_2 \|w\|^2$ for $w \perp \psi_0$ is used.

Proof of Theorem 7.3

By the scaling $\lambda_1(sT, \alpha) = s^{-2}\lambda_1(T, s\alpha)$ we may take $A = 1$; then $H = 3^{1/4}$, $P_E = 6/H$, $r_E = H/3$ and the side of E is $2/H$. Put $\alpha_1 = 1/100$. Every triangle of area 1 is congruent to $L(E)$ with $L = \exp(tS_\phi)$, $t \geq 0$, $S_\phi = \left(\begin{smallmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{smallmatrix}\right)$ (write the triangle as $M(E) + b$ with $\det M = 1$, after composing with a reflection of E if necessary, and take the polar decomposition of M). If $\tau_j = (\cos\vartheta_j, \sin\vartheta_j)$ are the side directions of E, $\vartheta_j \in \{0, 2\pi/3, 4\pi/3\}$, then

$$G = \exp(-2tS_\phi), \qquad \ell_j^2 = |L\tau_j|^2 = \cosh 2t + \sinh 2t\, \cos(2\phi - 2\vartheta_j), \tag{15}$$

the sides of T are $a_j = 2\ell_j/H$, and $t = 0$ exactly when T is equilateral. We write $\bar\ell = \frac{1}{3}\sum_j \ell_j = P/P_E$, $\hat p = \bar\ell - 1 \geq 0$, $|z_0| = \big(\frac{2}{3}\sum_j (\ell_j - \bar\ell)^2\big)^{1/2}$ and $\zeta = 2\hat p + |z_0|$.

Lemma 13.3 (the equilateral triangle at weak coupling)**.** *Let $A = 1$, $0 < \alpha \le \alpha_1$, and let ψ_0 be the $L^2(E)$-normalised ground state. Then*

(a) $\alpha P_E - \alpha^2/(1 - 4\alpha/P_E) \le \lambda_0(\alpha) \le \alpha P_E - \alpha^2 + \frac{11}{5}\alpha^3/P_E$;

(b) $\int_{\partial E} \psi_0^2 \ge P_E - g_2\alpha$ *and* $\|\nabla\psi_0\|_{L^2(E)} \le g_1\alpha$, *where, with* $c = 1.18$,

$$g_2 = \frac{1}{c-1}\Big(\frac{c^2}{1 - 4c\alpha_1/P_E} - 1 + \frac{11\alpha_1}{5P_E}\Big) < 2.2878, \qquad g_1 = \Big(g_2 - 1 + \frac{11\alpha_1}{5P_E}\Big)^{1/2} < 1.1370;$$

(c) *the second Robin eigenvalue satisfies* $\lambda_2(E,\alpha) \ge \mu_2 := 4\pi^2/(3\sqrt{3})$.

Proof. (a) With $y = \alpha/\kappa$, (9) reads $\lambda_0 = \kappa^2 = \frac{2\alpha}{H}\,(\arctan y + \arctan 2y)/y$. For $y \ge 0$, $y - y^3/3 \le \arctan y \le y - y^3/3 + y^5/5$ (the differences are $\int_0^y s^4/(1+s^2)\,\mathrm{d}s$ and $\int_0^y s^6/(1+s^2)\,\mathrm{d}s$). Since $6/H = P_E$,

$$\alpha P_E(1 - y^2) \le \lambda_0 \le \alpha P_E(1 - y^2 + \tfrac{11}{5}y^4), \qquad y^2 = \alpha^2/\lambda_0.$$

Theorem 7.2 for E (where $R_I = 2r$) gives $\lambda_0 \ge \alpha P_E - 4\alpha^2$, and $\lambda_0 \le \alpha P_E$; hence $\alpha/P_E \le y^2 \le \alpha/(P_E - 4\alpha)$. The lower bound follows. For the upper bound note that $u \mapsto 1 - u + \frac{11}{5}u^2$ is decreasing on $[0, 5/22]$, which contains the range of y^2, and evaluate at $u = \alpha/P_E$.

(b) λ_0 is concave and, being a simple eigenvalue of an analytic family of forms, differentiable with $\lambda_0'(\alpha) = \int_{\partial E}\psi_0^2$. By concavity and (a) (the lower bound in (a) holds for every α with $4\alpha < P_E$),

$$\lambda_0'(\alpha) \ge \frac{\lambda_0(c\alpha) - \lambda_0(\alpha)}{(c-1)\alpha} \ge P_E - \frac{\alpha}{c-1}\Big(\frac{c^2}{1 - 4c\alpha/P_E} - 1 + \frac{11\alpha}{5P_E}\Big) \ge P_E - g_2\alpha.$$

Then $\|\nabla\psi_0\|^2 = \lambda_0 - \alpha\lambda_0'(\alpha) \le \alpha^2(g_2 - 1 + \frac{11}{5}\alpha/P_E) \le g_1^2\alpha^2$.

(c) By the min–max principle $\lambda_2(E,\alpha) \ge \lambda_2(E,0)$, the first nonzero Neumann eigenvalue of E, which equals $16\pi^2/(9s^2)$ for side s [28]; here $s^2 = 4/\sqrt{3}$. □

Lemma 13.4. *Let ν_j be the outer normal on Γ_j, $\beta = \hat p/r_E$, $z_0 = \frac{2}{3}\sum_k(\ell_k - \bar\ell)\nu_k$ and $Z(x) = \beta x + z_0$. Then $Z\cdot\nu_j = \ell_j - 1$ on Γ_j, $\sup_E|Z| \le \zeta$, and for $f \in W^{1,1}(E)$, $\sum_j(\ell_j - 1)\int_{\Gamma_j} f = \int_E(2\beta f + Z\cdot\nabla f)$.*

Proof. On Γ_j, $x\cdot\nu_j = r_E$ and, since $\nu_k\cdot\nu_j = -\frac12$ for $k \ne j$ and $\sum_k(\ell_k - \bar\ell) = 0$, $z_0\cdot\nu_j = \ell_j - \bar\ell$; so $Z\cdot\nu_j = \hat p + \ell_j - \bar\ell = \ell_j - 1$. The identity is the divergence theorem for fZ. The convex function $|Z|$ attains its maximum over E at a vertex, where $|x| = 2r_E$; and $|z_0|^2 = \frac49\cdot\frac32\sum_k(\ell_k - \bar\ell)^2$ because $|\sum_k x_k\nu_k|^2 = \frac32\sum_k x_k^2$ when $\sum_k x_k = 0$. □

Proposition 13.5. *Let $A = 1$, $0 < \alpha \le \alpha_1$, $T = \exp(tS_\phi)(E)$ with $t > 0$, and $\varepsilon_1 = \alpha_1 P_E/\mu_2$, $\varepsilon_2 = \alpha_1 P_E/(\mu_2 - \alpha_1 P_E)$. If*

$$\hat p\,(P_E - g_2\alpha_1)(e^{-2t}(1+\varepsilon_2) - \varepsilon_2) > \alpha_1\Big(\frac{g_1(e^{2t}-1) + \zeta}{(1-\varepsilon_1)^{1/2}} + \frac{g_1\alpha_1\zeta}{(\mu_2 - \alpha_1 P_E)^{1/2}}\Big)^2, \tag{L}$$

then $\lambda_1(T,\alpha) > \lambda_0(\alpha)$.

Proof. We verify the hypotheses of Lemma 13.1 for $(G,m) = (G,\alpha\ell)$. The eigenvalues of G are $e^{\pm 2t}$ and $\ell_j \ge e^{-t}$ by (15), so $q_{G,\alpha\ell} \ge e^{-2t}q_0$ and we take $\eta = 1 - e^{-2t}$. By Proposition 8.1 and Lemma 13.3(c), $\lambda_0/\lambda_2 \le \varepsilon_1$ and $\lambda_0/(\lambda_2 - \lambda_0) \le \varepsilon_2$, hence

$$1 - \eta' = e^{-2t} - (1 - e^{-2t})\frac{\lambda_0}{\lambda_2 - \lambda_0} \ge e^{-2t}(1+\varepsilon_2) - \varepsilon_2,$$

which is positive by (L); in particular $\eta < 1 - \lambda_0/\lambda_2$. Next, by the D_3-invariance of ψ_0 (so that $\int_{\Gamma_j} \psi_0^2$ does not depend on j and $\int_E \partial_a\psi_0\partial_b\psi_0 = \frac12\delta_{ab}\|\nabla\psi_0\|^2$), $\operatorname{tr} G = 2\cosh 2t$ and Lemma 13.3(b),

$$D[\psi_0] = (\cosh 2t - 1)\|\nabla\psi_0\|^2 + \alpha\hat p \int_{\partial E} \psi_0^2 \ge \alpha\hat p\,(P_E - g_2\alpha).$$

For $w \perp \psi_0$, Lemma 13.4 with $f = \psi_0 w$ and $\int_E \psi_0 w = 0$ gives

$$D(\psi_0, w) = \int_E ((G - I)\nabla\psi_0 + \alpha\psi_0 Z)\cdot\nabla w + \alpha\int_E (Z\cdot\nabla\psi_0)\, w,$$

so $|D(\psi_0,w)| \le X\|\nabla w\| + \alpha Y\|w\|$ with $X \le (e^{2t}-1)\|\nabla\psi_0\| + \alpha\zeta \le \alpha(g_1(e^{2t}-1)+\zeta)$ and $Y \le \zeta\|\nabla\psi_0\| \le g_1\alpha\zeta$. With $N = q_0[w] - \lambda_0\|w\|^2$ we have $N \ge (1-\lambda_0/\lambda_2)\|\nabla w\|^2$ and $N \ge (\lambda_2-\lambda_0)\|w\|^2$, therefore

$$\mathcal{R} \le \alpha^2\Big(\frac{g_1(e^{2t}-1)+\zeta}{(1-\varepsilon_1)^{1/2}} + \frac{g_1\alpha\zeta}{(\mu_2-\alpha_1 P_E)^{1/2}}\Big)^2 \le \alpha\,\alpha_1 B^2,$$

where B denotes the bracket on the right of (L) (we used $\alpha \le \alpha_1$ twice). Since also $D[\psi_0] \ge \alpha\hat p\,(P_E - g_2\alpha_1)$, inequality (L) multiplied by α gives $D[\psi_0](1-\eta') > \mathcal{R}$. Now apply Lemma 13.1. □

Proof of Theorem 7.3. Let $A = 1$, $0 < \alpha \le \alpha_1$ and $T = \exp(tS_\phi)(E)$, $t > 0$. It suffices that one of the following holds:

(W) $\max_j \ell_j \ge 3$; then $a_{\max} \ge P_E$ and Theorem 7.1 applies;

(S) $\alpha_1 R_I^2 \le (P - P_E)r^2$, where $r = 2/P$ and $R_I^2 = \max_k (r^2 + (P/2 - a_k)^2)$; then Theorem 7.2 applies;

(L) inequality (L); then Proposition 13.5 applies.

All three conditions are explicit in (t,ϕ) through (15). Since the largest of the three numbers $\cos(2\phi - 2\vartheta_j)$ is at least $\frac12$ and $\cosh(2t) + \frac12\sinh(2t) \ge 9$ for $t \ge 1.2413$, (W) holds for $t \ge 1.2413$. For $0 < t \le t_0 = 1/100$ we use the elementary bounds $\cosh 2t - 1 \le 2C_c t^2$, $\sinh 2t \le 2C_s t$, $e^{2t} - 1 \le 2C_e t$, $e^t - 1 \le C_1 t$, where C_c, C_s, C_e, C_1 are the values of the corresponding quotients at t_0; with $u_j = \ell_j^2 - 1$, Taylor's formula $\sqrt{1+u} \ge 1 + u/2 - u^2/8 - K|u|^3$, $K = \frac{1}{16}(1-u_0)^{-5/2}$, $u_0 = e^{2t_0} - 1$, and $\frac13\sum u_j = \cosh 2t - 1 \ge 2t^2$, $\frac13\sum u_j^2 = (\cosh 2t - 1)^2 + \frac12\sinh^2 2t$, $|u_j| \le e^{2t} - 1$, $\hat p \le \frac16\sum u_j = \frac12(\cosh 2t - 1)$ (from $\sqrt{1+u} \le 1 + u/2$) and $|z_0|^2 \le \frac23\sum(\ell_j - 1)^2 \le 2(e^t - 1)^2$ (the mean minimises the sum of squares, and $e^{-t} \le \ell_j \le e^t$), they give

$$\hat p \ge c_p t^2, \qquad \zeta \le c_\zeta t, \qquad c_p = 1 - \tfrac18(4t_0^2C_c^2 + 2C_s^2) - 8Kt_0C_e^3, \quad c_\zeta = 2C_c t_0 + \sqrt2\, C_1,$$

so that (L) follows, after division by t^2, from $c_p(P_E - g_2\alpha_1)(e^{-2t_0}(1+\varepsilon_2) - \varepsilon_2) > \alpha_1\big((2g_1C_e + c_\zeta)(1-\varepsilon_1)^{-1/2} + g_1\alpha_1 c_\zeta(\mu_2 - \alpha_1 P_E)^{-1/2}\big)^2$. In ball arithmetic the left side exceeds 3.3098 and the right side is below 0.1411. On the remaining rectangle $[t_0, 1.2413]\times[0,\pi]$ the script `weak_cert.py` verifies in ball arithmetic (Arb, through python-flint, 80 bits), by adaptive bisection starting from a 64×96 grid, that each box of the final subdivision satisfies (W), (S) or (L) in its entirety. The run uses 445 704 boxes, of which 335 814 are leaves (726 of type (W), 6606 of type (S), 328 482 of type (L)), with bisection depth at most 7 and no failure. The same script evaluates the inequality $\cosh(2t) + \frac12\sinh(2t) \ge 9$ at $t = 1.2413$ and the two displayed numbers. □

Remark 13.6. The value $1/100$ is limited by condition (S) for tall isosceles triangles close to the thin regime ($t \approx 1.24$), where the threshold of Theorem 7.2 is about 0.0116; near E inequality (L) holds with a factor 23 to spare (the quotient of the two displayed numbers).

14 A neighbourhood of the equilateral triangle

Throughout this section $A = 1$, $\alpha > 0$ is fixed, $q_0 = q_{I,\alpha(1,1,1)}$, ψ_0 is its L^2-normalised ground state, $\lambda_0 = \lambda_0(\alpha)$, $\lambda_2 = \lambda_2(E,\alpha)$ is the second eigenvalue, $\underline{\lambda}_2$ is a number with $\lambda_0 < \underline{\lambda}_2 \le \lambda_2$, and

$$N[w] = q_0[w] - \lambda_0\|w\|^2, \qquad \gamma = \frac{\underline{\lambda}_2}{\underline{\lambda}_2 - \lambda_0}.$$

For $w \perp \psi_0$ we have $q_0[w] \ge \lambda_2\|w\|^2$, hence

$$N[w] \;\ge\; \gamma^{-1} q_0[w] \qquad \text{and} \qquad N[w] \;\ge\; (\underline{\lambda}_2 - \lambda_0)\|w\|^2 \qquad (w \perp \psi_0). \tag{16}$$

A *direction* is a pair $\theta = (\Theta, \mu)$ of a symmetric 2×2 matrix and a vector $\mu \in \mathbb{R}^3$. For a direction let

$$D_\theta(u,v) = \int_E \nabla u\cdot\Theta\nabla v + \alpha\sum_j \mu_j \int_{\Gamma_j} uv, \qquad R(\theta) = \sup\Big\{\frac{D_\theta(\psi_0,w)^2}{N[w]} :\; 0 \ne w \in H^1(E),\; w\perp\psi_0\Big\}.$$

Thus $q_{G,m} - q_0 = D_\theta$ with $\theta = (G - I, (m_j/\alpha - 1)_j)$, and the quantity $\mathcal{R}$ of Lemma 13.1 is $R(\theta)$. By (16), $N^{1/2}$ is a Hilbert norm on $\{w \perp \psi_0\}$ equivalent to the H^1 norm, and $R(\theta)^{1/2}$ is the norm of the functional $D_\theta(\psi_0,\cdot)$ in the dual space. Since D_θ is linear in θ,

$$\theta \mapsto R(\theta) \text{ is a nonnegative quadratic form, and } R^{1/2} \text{ is a seminorm.} \tag{17}$$

14.1 Symmetry

Let $\mathcal{G}$ be the group of the six orthogonal maps g with $g(E) = E$. For $v \in H^1(E)$ put $v^g = v\circ g$, and for a direction let $g\cdot\theta = (g\Theta g^{\mathsf T}, \mu')$, where $\mu'_k = \mu_j$ if $g\Gamma_j = \Gamma_k$. The substitution $x \mapsto gx$ gives

$$D_\theta(u^g, v^g) = D_{g\cdot\theta}(u,v), \qquad q_0[v^g] = q_0[v], \qquad \|v^g\| = \|v\|. \tag{18}$$

Since λ_0 is simple and $\psi_0 > 0$, $\psi_0^g = \psi_0$. Hence $N[w^g] = N[w]$, $w \mapsto w^g$ preserves $\{w \perp \psi_0\}$, and

$$R(g\cdot\theta) = R(\theta) \qquad (g \in \mathcal{G}). \tag{19}$$

Let $\mathsf{P}w = \frac{1}{6}\sum_{g\in\mathcal{G}} w^g$.

Lemma 14.1. (a) *If $\sum_{g\in\mathcal{G}} g\cdot\theta = 0$, then $D_\theta[\psi_0] = 0$ and $D_\theta(\psi_0, \mathsf{P}w) = 0$ for all $w \in H^1(E)$.*

(b) *If $g\cdot\theta = \theta$ for all $g \in \mathcal{G}$, then $D_\theta(\psi_0, w) = D_\theta(\psi_0, \mathsf{P}w)$ for all $w \in H^1(E)$.*

(c) *If θ_A is as in* (b) *and θ_E as in* (a)*, then $R(\theta_A + \theta_E) \le R(\theta_A) + R(\theta_E)$.*

(d) *For $\phi \in \mathbb{R}$ let $c_j(\phi) = \cos(2\phi - 2\vartheta_j)$ and $\Theta_\phi = (-2S_\phi, (c_j(\phi))_j)$. Then $\sum_g g\cdot\Theta_\phi = 0$, and $R(\Theta_\phi)$ does not depend on ϕ. We denote its value by $R_1 = R_1(\alpha)$.*

Proof. By (18) and $\psi_0^g = \psi_0$, $D_\theta(\psi_0, w^g) = D_\theta(\psi_0^g, w^g) = D_{g\cdot\theta}(\psi_0, w)$; averaging over g gives $D_\theta(\psi_0, \mathsf{P}w) = \frac{1}{6} D_{\sum_g g\cdot\theta}(\psi_0, w)$, which is 0 in case (a) and $D_\theta(\psi_0, w)$ in case (b). With $w = \psi_0 = \mathsf{P}\psi_0$, (a) follows.

(c) Let $w \perp \psi_0$. Since $(w^g)^h = w^{gh}$ and N is invariant, the bilinear form of N satisfies $N(w^g, w^h) = N(w^{gh^{-1}}, w)$, so $N(\mathsf{P}w, \mathsf{P}w) = N(\mathsf{P}w, w)$, that is $N[w] = N[\mathsf{P}w] + N[w - \mathsf{P}w]$; also $\mathsf{P}w \perp \psi_0$. By (a) and (b),

$$\begin{aligned} |D_{\theta_A+\theta_E}(\psi_0, w)| &= |D_{\theta_A}(\psi_0, \mathsf{P}w) + D_{\theta_E}(\psi_0, w - \mathsf{P}w)| \\ &\le R(\theta_A)^{1/2} N[\mathsf{P}w]^{1/2} + R(\theta_E)^{1/2} N[w - \mathsf{P}w]^{1/2} \le \big(R(\theta_A) + R(\theta_E)\big)^{1/2} N[w]^{1/2}. \end{aligned}$$

(d) Let g be the rotation through $2\pi/3$. Then $gS_\phi g^{\mathsf T} = S_{\phi+2\pi/3}$ (the matrix S_ϕ has the eigenvector $(\cos\phi, \sin\phi)$ for the eigenvalue 1), and $g\Gamma_j$ is the side with direction angle $\vartheta_j + 2\pi/3$, on which the new coefficient is $c_j(\phi) = \cos(2(\phi + 2\pi/3) - 2(\vartheta_j + 2\pi/3))$. Hence $g \cdot \Theta_\phi = \Theta_{\phi+2\pi/3}$. Similarly the reflection $x_1 \mapsto -x_1$ maps Θ_ϕ to $\Theta_{-\phi}$. So the orbit of Θ_ϕ consists of $\Theta_{\pm\phi+2\pi k/3}$, $k = 0, 1, 2$, and since Θ_ϕ is a linear combination of $\cos 2\phi$ and $\sin 2\phi$ with fixed coefficients $\Theta_0, \Theta_{\pi/4}$, the sum over the orbit vanishes. By (17), $R(\Theta_\phi) = R(\cos 2\phi\,\Theta_0 + \sin 2\phi\,\Theta_{\pi/4}) = a_0 + a_1 \cos 4\phi + a_2 \sin 4\phi$ with constants a_i, and by (19) this function has period $2\pi/3$. A function $a_1 \cos 4\phi + a_2 \sin 4\phi$ that is not identically zero has the periods $k\pi/2$, $k \in \mathbb{Z}$, only; hence $a_1 = a_2 = 0$. □

14.2 An upper bound for R_1 from lower bounds on a ring

Lemma 14.2 (ring). *Let $0 < s < \frac12$ and, for the two signs,*

$$\Lambda_\pm = \Lambda\big(I \mp 2sS_0,\ \alpha(1 \pm s\cos 2\vartheta_j)_{j=1,2,3}\big),$$

the lowest eigenvalues of the forms $q_0 \pm sD_{\Theta_0}$. Then $\Lambda_\pm \le \lambda_0$, and if $\delta_\pm > 0$ satisfy $\delta_\pm \ge \lambda_0 - \Lambda_\pm$, then

$$R_1 \ \le\ \frac{2 + (\delta_+ + \delta_-)/(\underline{\lambda}_2 - \lambda_0)}{s^2\,(1/\delta_+ + 1/\delta_-)}. \tag{20}$$

As a function of the four independent variables $\delta_+, \delta_-, \lambda_0, \underline{\lambda}_2$ (with $\underline{\lambda}_2 > \lambda_0$), the right side is increasing in δ_+, δ_- and λ_0, and decreasing in $\underline{\lambda}_2$.

Proof. $D_{\Theta_0}[\psi_0] = 0$ by Lemma 14.1(a),(d), so the Rayleigh quotient of ψ_0 for $q_0 \pm sD_{\Theta_0}$ is λ_0. Let $w \perp \psi_0$, $F = D_{\Theta_0}(\psi_0, w)$ and $c \in \mathbb{R}$. Since $q_0(\psi_0, w) = \lambda_0\langle\psi_0, w\rangle = 0$,

$$\begin{aligned} 0 &\le (q_0 \pm sD_{\Theta_0})[c\psi_0 + w] - \Lambda_\pm\|c\psi_0 + w\|^2 \\ &= c^2(\lambda_0 - \Lambda_\pm) \pm 2csF + N[w] \pm sD_{\Theta_0}[w] + (\lambda_0 - \Lambda_\pm)\|w\|^2 \\ &\le c^2\delta_\pm \pm 2csF + N[w] \pm sD_{\Theta_0}[w] + \delta_\pm\|w\|^2. \end{aligned}$$

Choosing $c = \mp sF/\delta_\pm$ gives $s^2F^2/\delta_\pm \le N[w] \pm sD_{\Theta_0}[w] + \delta_\pm\|w\|^2$. Adding the two inequalities, the terms $\pm sD_{\Theta_0}[w]$ cancel, and with (16),

$$s^2F^2\Big(\frac{1}{\delta_+} + \frac{1}{\delta_-}\Big) \le 2N[w] + (\delta_+ + \delta_-)\|w\|^2 \le N[w]\Big(2 + \frac{\delta_+ + \delta_-}{\underline{\lambda}_2 - \lambda_0}\Big).$$

The monotonicity statements are evident. □

The point of using both signs is that the first-order error terms $\pm sD_{\Theta_0}[w]$ cancel exactly; the bound (20) exceeds R_1 only by terms of relative order s^2 and by the defect of the bounds $\delta_\pm$ divided by s^2. In a floating-point comparison with R_1 computed by P_2 finite elements, with $s = 1/10$ and the certified values of $\delta_\pm$ and $\underline{\lambda}_2$ used below, the bound exceeds R_1 by 2 to 4 per cent at the grid points α_k; the interpolation between grid points in Lemma 14.5 raises this to 7 per cent near $\alpha = 2$ (and to at most 14 per cent, for small α, where the margin is large).

14.3 Elementary shape bounds

For T_w with $|w| = \tanh 2t$ and $w = |w|(\cos 2\phi, \sin 2\phi)$ write $\ell_j = \ell_j(G(w))$, $\bar\ell = \frac13\sum_j \ell_j$, $c_j = c_j(\phi)$ and

$$\rho_j = \ell_j - \bar\ell - \tfrac12 \sinh(2t)\, c_j.$$

By Lemma 11.2(d), $u_j := \ell_j^2 - 1 = (\cosh 2t - 1) + \sinh 2t\, c_j$. From $\ell - 1 = (\ell^2-1)/(\ell+1)$ one gets the identity $\ell_j - 1 = \tfrac12 u_j - u_j^2/(2(1+\ell_j)^2)$, and since $\sum_j c_j = 0$,

$$\bar\ell - 1 = \tfrac12(\cosh 2t - 1) - t^2\,\bar v, \qquad \rho_j = -t^2\,(v_j - \bar v), \qquad v_j = \frac{(u_j/t)^2}{2(1+\ell_j)^2}, \quad \bar v = \tfrac13 \sum_j v_j. \tag{21}$$

Lemma 14.3. *Let $t_* = 3/50$. For $0 < t \le t_*$ and all ϕ,*

$$\bar\ell - 1 \;\ge\; c_\ell\, t^2, \qquad \max_j |\rho_j| \;\le\; \hat\rho\, t^2, \qquad c_\ell = 0.7361, \quad \hat\rho = 0.2938.$$

Proof. Replacing ϕ by $\phi + \pi/3$ permutes the numbers c_j, so it suffices to consider $\phi \in [0, \pi/3]$. The functions $(\cosh 2t - 1)/t^2$ and $\sinh(2t)/t$ are power series in t^2 with positive coefficients, hence increasing in $t > 0$ with limit 2 at 0, and $u_j/t = t\,(\cosh 2t - 1)/t^2 + c_j \sinh(2t)/t$. The script `local_check.py` evaluates (21) in ball arithmetic on a subdivision of $(0, t_*] \times [0, \pi/3]$ into 40×60 boxes, using these monotonicity properties on each t-interval, and obtains the bounds $c_\ell \ge 0.73612$ and $\hat\rho \le 0.29376$, which imply the stated ones. (The independent verifier of Section 17, with a finer subdivision, obtains 0.7404 and 0.2893.) □

14.4 The local proposition

Proposition 14.4. *Let $A = 1$, $\alpha \ge 1/100$ and $0 < |w| \le \tanh(2t_*)$, $t_* = 3/50$. Then $\lambda_1(T_w, \alpha) > \lambda_0(\alpha)$.*

Proof. In this proof w is the shape parameter and test functions in $H^1(E)$ are called v. Let $|w| = \tanh 2t$, $0 < t \le t_*$, and $w = |w|(\cos 2\phi, \sin 2\phi)$. We apply Lemma 13.1 (with $\underline{\lambda}_2$ in place of λ_2, see Remark 13.2) to $(G, m) = (G(w), \alpha\ell)$, $\ell = (\ell_j(G(w)))_j$, for which $\Lambda(G, m) = \lambda_1(T_w, \alpha)$. The eigenvalues of $G(w)$ are $e^{\pm 2t}$ and $\ell_j \ge e^{-t}$, so $q_{G,m} \ge (1-\eta) q_0$ with $\eta = 1 - e^{-2t}$, and $\eta' = \eta\gamma$. By Lemma 11.2(d), $D = q_{G,m} - q_0 = D_\theta$ with $\theta = \theta_A + \theta_E$,

$$\theta_A = \big((\cosh 2t - 1)I,\ (\bar\ell - 1)(1,1,1)\big), \qquad \theta_E = \big(-\sinh 2t\, S_\phi,\ (\ell_j - \bar\ell)_j\big) = \tfrac12 \sinh(2t)\,\Theta_\phi + (0, \rho).$$

θ_A is invariant under $\mathcal{G}$; Θ_ϕ has vanishing orbit sum by Lemma 14.1(d), and so has $(0,\rho)$ because $\sum_j \rho_j = 0$.

The gain. By Lemma 14.1(a), $D[\psi_0] = D_{\theta_A}[\psi_0]$. Hence, by Lemma 8.2 (recall $\|\psi_0\| = 1$), $\cosh 2t - 1 \ge 2t^2$ and Lemma 14.3,

$$D[\psi_0] = (\cosh 2t - 1)(\lambda_0 - \alpha\lambda_0') + \alpha(\bar\ell - 1)\lambda_0' \;\ge\; t^2(2\lambda_0 - (2 - c_\ell)\,\alpha\lambda_0'). \tag{22}$$

The invariant part. D_{θ_A} is a nonnegative form, and $D_{\theta_A}[v] \le (\cosh 2t - 1)\, q_0[v]$ because $0 \le \bar\ell - 1 \le (\tfrac13 \sum_j \ell_j^2)^{1/2} - 1 = (\cosh 2t)^{1/2} - 1 \le \cosh 2t - 1$. By the Cauchy–Schwarz inequality for D_{θ_A} and (16), $D_{\theta_A}(\psi_0, v)^2 \le D_{\theta_A}[\psi_0]\,(\cosh 2t - 1)\gamma N[v]$ for $v \perp \psi_0$, that is $R(\theta_A) \le (\cosh 2t - 1)\,\gamma\, D[\psi_0]$.

The remainder $(0,\rho)$. For $v \perp \psi_0$, by (16) and $\int_{\partial E} \psi_0^2 = \lambda_0'$,

$$|D_{(0,\rho)}(\psi_0, v)| \le \max_j |\rho_j| \Big(\alpha \int_{\partial E} \psi_0^2\Big)^{1/2} \Big(\alpha \int_{\partial E} v^2\Big)^{1/2} \le \max_j |\rho_j|\,(\alpha\lambda_0'\,\gamma N[v])^{1/2}.$$

Hence, by (17), Lemma 14.1(d) and Lemma 14.3,

$$R(\theta_E)^{1/2} \le \tfrac12 \sinh(2t) R_1^{1/2} + \hat\rho\, t^2 (\alpha\lambda_0'\gamma)^{1/2} \le t\,(s_* R_1^{1/2} + \hat\rho\, t_* (\alpha\lambda_0'\gamma)^{1/2}), \qquad s_* = \frac{\sinh 2t_*}{2t_*}.$$

Conclusion. By Lemma 14.1(c), $\mathcal{R} = R(\theta) \le R(\theta_A) + R(\theta_E)$. With $\eta_* = 1 - e^{-2t_*}$ and $\varepsilon_* = \cosh 2t_* - 1$, the hypothesis $D[\psi_0](1-\eta') > \mathcal{R}$ of Lemma 13.1 therefore follows from $D[\psi_0](1-(\eta_*+\varepsilon_*)\gamma) > R(\theta_E)$, and by (22) this follows from the inequality

$$(2\lambda_0 - (2-c_\ell)\,\alpha\lambda_0')\,(1-(\eta_*+\varepsilon_*)\gamma) \;>\; \Big(s_*\overline{R}_1^{1/2} + \hat\rho\, t_*(\alpha\lambda_0'\gamma)^{1/2}\Big)^2 \tag{C}$$

for some $\overline{R}_1 \ge R_1(\alpha)$ and some admissible $\underline{\lambda}_2$. (The first factor on the left of (C) is at least $c_\ell\lambda_0 > 0$, because $\alpha\lambda_0' \le \lambda_0$; so (C) forces $(\eta_*+\varepsilon_*)\gamma < 1$, in particular $\eta\gamma < 1$, which is the condition $\eta < 1 - \lambda_0/\underline{\lambda}_2$ of Lemma 13.1.) Inequality (C) involves α only. It is verified for all $\alpha \ge 1/100$ in Lemma 14.5. □

Lemma 14.5. *For every $\alpha \ge 1/100$ there are $\underline{\lambda}_2 \in (\lambda_0(\alpha), \lambda_2(E,\alpha)]$ and $\overline{R}_1 \ge R_1(\alpha)$ such that (C) holds.*

Proof. Let α_k, $k = 0, 1, \dots$, be the grid of rational numbers $\alpha_k = \frac{1}{100}2^{\lfloor k/16\rfloor} r_{k \bmod 16}$, where $r_0 = 1$ and r_i is $2^{i/16}$ rounded to six decimals; so $\alpha_0 = 1/100$ and $\alpha_{k+1}/\alpha_k \approx 1.0443$. Let $K = 336$, so that $\alpha_K = 0.01 \cdot 2^{21}$, and $s = 1/10$.

Certified data. (i) For $k = 0, \dots, K$ and both signs the script `local_ring.py` computes numbers $\mu_\pm^k \le \Lambda_\pm(\alpha_k)$ (notation of Lemma 14.2) by the method of Section 12: by (11), $\Lambda_\pm$ is the lowest eigenvalue of an acute isosceles triangle with side-dependent Robin coefficients, to which Lemmas 12.2 and 12.3 are applied on the mesh (U) with $n = N_{\mathrm{ring}}$ ($N_{\mathrm{ring}} = 252$ and $N_2 = 25$ below for $\kappa_0 = 0.2983$; see Table 2). (ii) For $k \in \{0, 4, 8, \dots, K\}$ it computes numbers $\underline{\lambda}_2^k \le \lambda_2(E, \alpha_k)$ by Lemmas 12.2 ($k = 2$) and 12.4 on the mesh (U) of E with $n = N_2$. (iii) $\lambda_0(\alpha_k)$ and $\lambda_0'(\alpha_k)$ are enclosed using Proposition 8.1 and (10).

An interval $I_k = [\alpha_k, \alpha_{k+1}]$, $0 \le k < K$. The functions $\alpha \mapsto \Lambda_\pm(\alpha)$ are concave (Lemma 11.1), so on I_k they lie above the chord through $(\alpha_k, \mu_\pm^k)$, $(\alpha_{k+1}, \mu_\pm^{k+1})$. By Lemma 8.2, λ_0 lies below the two tangents at α_k and α_{k+1}. The difference between the minimum of the two tangents and the chord is piecewise linear on I_k with one kink, so its maximum $\delta_\pm$ is attained at α_k, α_{k+1} or at the kink, and $\delta_\pm \ge \lambda_0(\alpha) - \Lambda_\pm(\alpha)$ on I_k; in particular $\delta_\pm \ge \lambda_0(\alpha_k) - \Lambda_\pm(\alpha_k) \ge 0$, and the script checks $\delta_\pm > 0$. The second eigenvalue $\lambda_2(E,\cdot)$ is nondecreasing, so $\underline{\lambda}_2 := \underline{\lambda}_2^{4\lfloor k/4\rfloor}$ is admissible on I_k provided it exceeds $\lambda_0(\alpha_{k+1})$. On I_k we further have $\lambda_0(\alpha_k) \le \lambda_0 \le \lambda_0(\alpha_{k+1})$ and $\alpha\lambda_0' \le \alpha_{k+1}\lambda_0'(\alpha_k)$. By the monotonicity statement of Lemma 14.2, (C) holds on I_k if it holds with λ_0 replaced by $\lambda_0(\alpha_k)$ in the first factor, $\alpha\lambda_0'$ by $\alpha_{k+1}\lambda_0'(\alpha_k)$, γ by $\underline{\lambda}_2/(\underline{\lambda}_2 - \lambda_0(\alpha_{k+1}))$ and $\overline{R}_1$ by the right side of (20) evaluated with $\delta_\pm$, $\underline{\lambda}_2$ and $\lambda_0(\alpha_{k+1})$.

The interval $[\alpha_K, \infty)$. Here $\Lambda_\pm(\alpha) \ge \Lambda_\pm(\alpha_K) \ge \mu_\pm^K$ by monotonicity and $\lambda_0(\alpha_K) \le \lambda_0 < \Lambda_1(E) = 4\pi^2/\sqrt{3}$, so we take $\delta_\pm = \Lambda_1(E) - \mu_\pm^K > 0$, $\underline{\lambda}_2 = \underline{\lambda}_2^K$ (which exceeds $\Lambda_1(E)$), replace λ_0 by $\Lambda_1(E)$ in γ and in (20), and use $\alpha\lambda_0' \le 2(\Lambda_1(E) - \lambda_0(\alpha_K/2))$ (Lemma 8.2).

The script `local_check.py` evaluates these $K + 1 = 337$ inequalities in ball arithmetic. All hold; the smallest quotient of the left by the right side of (C) is 1.1758, attained for $k = 123$ ($\alpha \approx 2.06$), and the quotient exceeds 1.48 for $\alpha \ge 30$ and 1.5 for $\alpha \le 0.6$. □

Remark 14.6. Floating-point values from P_2 finite elements on the grid $\alpha_0, \dots, \alpha_K$ (not used in the proof): with $a(\alpha) = 2\lambda_0 - \frac{5}{4}\alpha\lambda_0'$, the leading coefficient of $D[\psi_0]/t^2$ as $t \to 0$, the ratio R_1/a is 0.13 at $\alpha = 0.1$, has its maximum 0.581 near $\alpha \approx 1.9$, and is 0.484 at $\alpha_K \approx 2 \cdot 10^4$; so the criterion has a factor of about 1.72 to spare as $t \to 0$, uniformly in α, of which the factor $1 - (\eta_* + \varepsilon_*)\gamma$ and the over-estimation of R_1 consume a part at $t = t_*$. The Hessian of $T \mapsto \lambda_1(T,\alpha)/\lambda_0(\alpha)$ at E in the variable t is $2(a - R_1)/\lambda_0$, between 0.87 (near $\alpha \approx 1.3$) and 2.07.

15 The middle region

Let $A = 1$, $r_* = 0.1194 < \tanh(2t_*) = 0.11942\ldots$, and

$$F(w) = (1+w_1)^2 - 81\,(1-|w|^2), \qquad \Omega_B = \{w \in \mathbb{D} :\ 0 < \arg w < \pi/3,\ |w| > r_*,\ F(w) < 0\}.$$

By Lemma 11.2(b),(c), Theorem 7.1 and Proposition 14.4, the inequality $\lambda_1(T_w, \alpha) > \lambda_0(\alpha)$ for $\alpha \geq 1/100$ remains to be proved only for w in the closure of Ω_B in $\mathbb{D}$.

Lemma 15.1 (coverage)**.** *Let $\mathcal{T}$ be a finite set of closed, positively oriented, nondegenerate triangles in the plane such that every side $[p,q]$ of a triangle of $\mathcal{T}$, traversed from p to q in its boundary, either occurs exactly once more, as the side $[q,p]$ of another triangle, or does not occur again in either direction (a* boundary side*). Let Ω be a connected open set that is disjoint from all boundary sides and contains a point that belongs to the interior of some triangle of $\mathcal{T}$ and to no side of any triangle of $\mathcal{T}$. Then $\overline{\Omega} \subset \bigcup \mathcal{T}$.*

Proof. For a point x on no side, the number $n(x)$ of triangles of $\mathcal{T}$ containing x is the sum of the winding numbers of their boundaries around x, that is the winding number around x of the cycle formed by all oriented sides. The contributions of the sides that occur in both directions cancel, so $n(x)$ is the winding number of the cycle of boundary sides, which is locally constant on the complement of the boundary sides, in particular on Ω; there it equals its value at the given point, which is at least 1. So every point of Ω on no side lies in $\bigcup \mathcal{T}$; these points are dense in Ω and $\bigcup \mathcal{T}$ is closed. □

The cells. The file `coverB_roots.json` contains a list $\mathcal{T}$ of 100 triangles in $\mathbb{D}$ whose vertices have rational coordinates with denominator 2^{20}: the points $\tanh(2t_i)(\cos\psi, \sin\psi)$, rounded, for thirteen radii $t_i \in [0.057, 1.285]$ and angles $\psi \in \{0, \frac{1}{4}\psi_{\max}, \ldots, \psi_{\max}\}$, $\psi_{\max} = \pi/3 + 10^{-3}$ (with four additional points on the outermost circle), joined to a triangulation. The grid of parameters α_k is the one of Lemma 14.5, continued to $k \leq 400$.

Proposition 15.2. (a) $\overline{\Omega_B} \cap \mathbb{D} \subset \bigcup \mathcal{T}$.

(b) *For every $\Delta \in \mathcal{T}$, either $F \geq 0$ on Δ, or there are indices $0 = k_0 < k_1 < \cdots < k_p \leq 400$ and, for each vertex w_i of Δ and each $k \in \{k_0, \ldots, k_p\}$, a certified number $b_i^k \leq \lambda_1(T_{w_i}, \alpha_k)$, such that the hypothesis of Corollary 11.4*(a) *holds on each $[\alpha_{k_{q-1}}, \alpha_{k_q}]$, with U the minimum of the tangents of λ_0 at the grid points in that interval, and the hypothesis of Corollary 11.4*(b) *holds at α_{k_p}.*

Consequently $\lambda_1(T_w, \alpha) > \lambda_0(\alpha)$ for all $w \in \overline{\Omega_B} \cap \mathbb{D}$ and all $\alpha \geq 1/100$.

Proof. (a) In exact rational arithmetic one checks that $\mathcal{T}$ satisfies the hypotheses of Lemma 15.1 and that each of its 36 boundary sides lies in one of the closed sets $\{|w| \leq r_*\}$, $\{w_2 \leq 0\}$, $\{w_2 \geq \sqrt{3}\,w_1\}$, $\{F \geq 0\}$, all of which are disjoint from Ω_B. (For the first three sets, which are convex, it suffices to test the end points; F is a convex quadratic polynomial, and its minimum on a segment is computed exactly.) The set Ω_B is connected: for $0 < \psi < \pi/3$ the function $r \mapsto F(r\cos\psi, r\sin\psi)$ is increasing on $[0,1]$, negative at r_* and positive at 1, so $\Omega_B = \{r_* < r < r_{\max}(\psi)\}$ in polar coordinates with a continuous $r_{\max} > r_*$. Finally the point $(\frac{1}{2}, \frac{1}{5}) \in \Omega_B$ is interior to a triangle of $\mathcal{T}$ and lies on no line that contains a side of a triangle of $\mathcal{T}$.

(b) is the result of the computation described in Section 12 and below, carried out by `cover.py`: one triangle (adjacent to the point $w = (\tanh 2.57, 0)$) is recorded with $F \geq 0$; for each of the other 99 (four of which also lie in $\{F \geq 0\}$) the certificate lists the indices k_q and the bounds b_i^k. The

vertex bounds are Crouzeix–Raviart bounds on the meshes (U) or (A) (the latter whenever the smallest cotangent of T_{w_i} is not certified to exceed 0.02). The function U is piecewise linear, $U \geq \lambda_0$, and by Lemma 8.2 its kinks are the intersection points of consecutive tangents, one in each grid interval (the script encloses each of them in its grid interval). Both sides of (13) are then piecewise linear in α; so (13) is verified, in ball arithmetic, at the end points and at these kinks, for the six pairs $i \leq k$. The steps $k_{q-1} \to k_q$ were chosen greedily as large as possible, so some of these inequalities hold with a small relative margin (the smallest is $8 \cdot 10^{-6}$); they are decided by 100-bit ball arithmetic applied to the stored exact bounds.

The consequence follows from Corollary 11.4, from Lemma 11.2(c) and Theorem 7.1 for the triangle with $F \geq 0$, and from (a). □

In total the certificate for (b) consists of 1730 prisms and uses 1779 vertex bounds, with mesh parameters $n \leq 184$ (at most 51 060 unknowns); the largest grid index needed is $k_p = 286$ ($\alpha \approx 2404$) for the cells closest to E. No cell had to be subdivided.

16 Proof of Theorem 1.1(1); the computations

Proof of Theorem 1.1(1). By scaling, $A = 1$. Let T be a non-equilateral triangle of area 1 and $\alpha > 0$. If $\alpha \leq 1/100$, Theorem 7.3 applies. Let $\alpha \geq 1/100$. By Lemma 11.2(b), T is congruent to T_w with $w \in \Sigma \setminus \{0\}$. If $|w| \leq r_* < \tanh(2t_*)$, Proposition 14.4 applies. If $F(w) \geq 0$, then $a_{\max} \geq P_E$ by Lemma 11.2(c) and Theorem 7.1 applies. Otherwise $w \in \overline{\Omega_B} \cap \mathbb{D}$ and Proposition 15.2 applies. □

Proof of Corollary 1.2 for $\alpha > 0$. Let E_s be the equilateral triangle of area $s^2|E|$. By the remark after Lemma 11.1, $\alpha \mapsto \lambda_1(E, \alpha)$ is concave, and it vanishes at $\alpha = 0$; so for $s > 1$, $\lambda_1(E_s, \alpha) = s^{-2}\lambda_1(E, s\alpha) \leq s^{-1}\lambda_1(E, \alpha) < \lambda_1(E, \alpha)$: the eigenvalue of the equilateral triangle decreases strictly with its area. By the isoperimetric inequality for triangles, $|T| \leq |E^T|$. Hence $\lambda_1(T, \alpha) \geq \lambda_1(E_T, \alpha) \geq \lambda_1(E^T, \alpha)$, and the first inequality is strict unless T is equilateral, by Theorem 1.1(1). □

What is computed. The proof uses the following machine computations, in `code/positive/` of the repository; Section 17 describes how they are re-run and checked.

1. `weak_cert.py`: the interval check of the two-variable inequality in the proof of Theorem 7.3 (445 704 boxes).
2. `local_ring.py`: the $674 = 2 \cdot 337$ ring bounds $\mu_{\pm}^k$ and the 85 bounds $\underline{\lambda}_2^k$ of Lemma 14.5; `local_check.py`: the constants of Lemma 14.3 and the 337 inequalities (C).
3. `cover.py`: the vertex bounds and the prism inequalities of Proposition 15.2(b). The exact rational checks of Proposition 15.2(a) are done by `coverage_check.py`.

The certified statements are of three kinds: (1) sign tests on exact integers (Lemma 12.3); (2) inequalities between elementary functions of rational arguments, evaluated in Arb ball arithmetic [18] (through python-flint, 100 bits), including Gershgorin tests on small dense ball matrices (Lemma 12.4); (3) exact rational arithmetic (the coverage). Floating-point arithmetic is used only to *propose* objects whose properties are then certified: approximate eigenvectors, an approximate inverse Cholesky factor, the trial values of σ, the positions of the grid points and the choice of the mesh parameters and of the side on which the altitude is dropped. Every certified lower bound

is stored after a further relative reduction of the discrete eigenvalue bound σ by 10^{-7} (by 10^{-6} for the second eigenvalue), which leaves room for an independent re-verification with a different eigenvector. In all, the default run solves 2660 sparse eigenvalue problems, in less than half an hour on one core.

Three interpolation constants. The only external analytic input of the certified lower bounds is the interpolation constant of Lemma 12.1. The whole computation was carried out three times, with three constants κ_0 in Lemma 12.2: 0.2983 (Lemma 12.1 with the Laugesen–Siudeja inequality; this is the default run, described in the text), 0.3496 (Lemma 12.1 with the Payne–Weinberger inequality) and 0.1893 (the value obtained by Liu [26], which is close to optimal; it is not proved by hand but by a verified computation). Each run by itself proves Theorem 1.1(1); they differ only in the mesh parameters. Table 2 summarises them.

constant κ_0	0.2983	0.3496	0.1893
ring bounds: n (unknowns)	252 (95 634)	295 (130 980)	160 (38 640)
$\underline{\lambda}_2$ bounds: n (unknowns)	25 (975)	30 (1395)	16 (408)
min. quotient in (C)	1.1758	1.1759	1.1757
cells by prisms / thin	99 / 1	99 / 1	99 / 1
prisms	1730	1730	1735
vertex bounds used	1779	1779	1741
largest n in the cover (unknowns)	184 (51 060)	216 (70 308)	117 (20 709)
largest k_p (α_{k_p})	286 (2404)	286 (2404)	290 (2859)
time: ring, $\underline{\lambda}_2$, cover (min)	12, 11, 3	17, 24, 5	4, 1, 1

Table 2: The three runs. "min. quotient" is the smallest quotient of the two sides of (C); times are for one core of a loaded server.

17 Implementation, verification and caveats

Implementation. All certified steps use Arb [18] through python-flint 0.9.0: at 128 bits in Part I, and at 100 bits (80 bits in `weak_cert.py`) in Part II; Part II also uses exact integer and rational arithmetic. In Part I, the certificate of Theorem 1.3 consists of the 2000 intervals of Lemma 3.1, 256 Hessian sub-boxes and 344 boxes in Lemma 3.2, and that of Theorem 1.4 of 6700 intervals and one tail box; each takes well under a second. Box endpoints are exact binary or rational numbers, and every certified inequality is strict. The certificates of Part II (the default run and the two other runs of Table 2, about 1.3 MB of JSON each) are stored in the repository and are re-checked as described below; the data format is documented there.

Every number in this paper that a computation checks is kept in a single claims file. The script `scripts/verify.py` reruns the computations, compares each computed enclosure or count with these claims in exact rational arithmetic, and the macros of this paper are generated from the same file. By default it reruns the certificates of Part I, the second certificate of Lemma 4.8, the floating-point checks of §6, and for Part II: `weak_cert.py`, `local_check.py` and `coverage_check.py` from scratch, all prism and closing inequalities from the stored bounds, the negative controls, and the independent verifier described below (with a from-scratch re-certification of a sample of the stored bounds for the default run, and a replay of the other two runs). Optional modes replay the six-wave cells of §6.2 and re-certify every stored bound of all three runs of Part II from scratch with both

programs. Notation in the code of Part I: G_0 is `D3`, $G_\perp$, $B_\perp$, $N_\perp$ are `G1`, `B1`, `N1`, the products $\beta B_\perp$, $\beta\chi$ and $|\lambda_0|N_\perp$ are `bB1`, `bkq` and `lN1`, and the Σ_k are `s1`, ..., `s4`; in the second certificate, χ is `kappa` and $\beta\chi$ is `bk`.

Checks of Part II. (i) *Re-certification.* The script `recheck.py` re-evaluates all prism inequalities and closing tests from the stored bounds and re-certifies every stored bound from scratch: a freshly computed eigenvector must pass the exact test of Lemma 12.3 for the *stored* σ. (ii) *Independent verifier.* A second program was written by a separate AI instance from the text of Part II and a description of the certificate format only, without access to our code. It builds each triangle from coordinates and each mesh as an explicit list of elements, assembles the Crouzeix–Raviart matrices element by element, computes its own eigenvectors, and verifies $Kx \geq \sigma Mx$ for the stored σ in outward-rounded floating-point interval arithmetic; it treats the second eigenvalue by an interval LDL^{T} factorisation and Sylvester's law of inertia, and re-derives the constants of Lemma 14.3, the inequalities (C), the prism inequalities and the coverage with a different interval library (mpmath [30]) and exact rationals. For each of the three runs it confirms every record: all 674 ring bounds, all 85 bounds for λ_2, all stored vertex bounds, all 337 inequalities (C) (smallest quotient 1.1758 at $k = 123$ for the default run, as in our computation), all prisms and the coverage. (iii) *Negative controls.* The two programs reject perturbed statements; between them: a σ larger by a factor $1 + 10^{-6}$ (resp. $1 + 10^{-4}$), mesh (U) on an obtuse triangle, λ_0 replaced by $1.01\lambda_0$ (the independent verifier rejects 11 cells already for $1.0001\lambda_0$), the false bound $\lambda_1(T_w, \alpha) \geq \lambda_0(1+0.6t^2)$ near E, t_* replaced by $3/25$, ring bounds lowered by 0.2 per cent near $\alpha = 2$, a cover with a triangle removed, a nonpositive σ or σ_2 (-10^6, 0, and just below the pole $-1/C_h^2$ of $\sigma/(1 + C_h^2\sigma)$), a vertex bound ≤ 0 in a cell, and, on the one-element mesh (U) of the triangle with sides $5, 5, 6$ and Robin coefficients $6, 6, 5$ (discrete eigenvalue exactly $15/2$), the value $\sigma = 15/2 + 2^{-61}$. (iv) *Sanity checks in floating point*, not used in the proof: the certified vertex bounds lie below conforming P_2 finite element values, by relative amounts between $4 \cdot 10^{-7}$ and $1.3 \cdot 10^{-2}$ on random samples, and the quantities of Remark 14.6 are computed with P_2 elements. (v) An AI referee re-derived the new arguments of Part II line by line and reported no error; the gaps it found (the position of the kinks of U, a point on no side in Lemma 15.1, the use of the rounded constants in (C)) are repaired. A second AI referee pass, made when the two parts were merged, again checked the cell identity (Lemma 11.3), the ring bound (Lemma 14.2), the uniformity in α (Lemma 14.5), the closing test of Corollary 11.4(b), Lemmas 12.2 and 12.3 and the coverage lemma, and the corresponding code, and found no error. (vi) An external AI review (ChatGPT Pro) of the code found four defects, now corrected, none of which affects a stored certificate: both checkers of the bounds for λ_2 (and our re-checks of the other bounds) did not enforce $\sigma > 0$ in Lemma 12.2, so that a forged record with a very negative σ would have been accepted (all stored values of σ and σ_2 are positive); our re-check rounded a given σ down to a multiple of 2^{-60} before the test of Lemma 12.3 but then used the unrounded value (all stored values are such multiples); the first cell of the second certificate of Lemma 4.8 started about $4 \cdot 10^{-17}$ above $13/10$; and the reported lower end of the enclosure of $q(6/5)$ was evaluated at the upper end of the enclosure of $M(6/5)$ (the upper bound $q(6/5) < 1.384$, which is what the proof uses, was correct). A subsequent audit of both programs added explicit checks of further hypotheses: positive vertex bounds in the prism tests (with a negative control), the ring parameter $s = 1/10$ in our re-check, and $g \geq 0$ in Lemma 12.4.

Check of prior work. The repository also contains, clearly separated from our proof, the AI referee's re-implementation of the middle-range criterion of Ma [27, Proposition 2] in Arb, which confirms it on $1 \leq \beta \leq 10$ with 1146 boxes; it is run by `scripts/verify.py`.

Caveats. The proofs are computer-assisted and, at the time of writing, have been checked only by AI systems and by the programs described above; no human expert has reviewed them (see the declaration below). In Part I the main residual risks are the hand derivations, especially the closed forms of the eight integrals of Lemma 4.8 and their transcription into code; they are cross-checked against quadrature on actual triangles by two independently written scripts. In Part II, the local region has a comfortable margin (the smallest quotient in (C) is 1.1758). In the middle region no cell had to be subdivided, but the true gap $\lambda_1(T_w,\alpha)/\lambda_0(\alpha)-1$ is small next to the local disc (between 0.14 and 0.34 per cent at the inner vertices $t = 0.057$, in floating point), and individual prism inequalities hold with relative margins down to $8\cdot 10^{-6}$; that region therefore rests entirely on the exactness of the Collatz–Wielandt test, of the constants in Lemma 12.2 and of the ball arithmetic. The most delicate analytic point is the neighbourhood of E: the criterion of Lemma 13.1 has only a factor of about 1.72 to spare in the worst case, so the upper bound for R_1 has to be accurate to a few per cent; Lemma 14.2 achieves this with two eigenvalue bounds per value of α. We did not find a computer-free bound for $R_1(\alpha)$. (By the ground state transformation $N[\psi_0\chi]=\int_E\psi_0^2|\nabla\chi|^2$, one has $R_1=\min\int_E\psi_0^2|Y|^2$ over the vector fields Y with $\int_E\psi_0^2\,Y\cdot\nabla\chi = D_{\Theta_0}(\psi_0,\psi_0\chi)$ for all χ with $\int_E\psi_0^2\chi=0$, so any explicit admissible field would give such a bound.) The only external analytic inputs of Part II are standard facts about the Robin form and the Poincaré inequality of Laugesen and Siudeja [23] (or of Payne and Weinberger [31] in the second run); the constant of the third run is Liu's [26], which comes from a verified computation rather than a hand proof, whereas the default run uses the constant 0.2983 proved in Lemma 12.1. The remaining dependence is on the correctness of Arb/python-flint, of mpmath's interval arithmetic and of Python's integers.

Availability. The code, the certificates, the logs and the claims file are in the accompanying repository[1]. The single command `scripts/verify.py` reruns the checks described above and writes a PASS/FAIL summary.

A A strong-coupling bound for $\alpha > 0$

This appendix uses the notation of Part II and is independent of the proof of Theorem 1.1. It gives an explicit lower bound for $\lambda_1(T,\alpha)$ of an individual triangle as $\alpha\to\infty$.

Theorem A.1 (strong coupling)**.** *Let ψ_T be the torsion function of T ($-\Delta\psi_T=1$ in T, $\psi_T=0$ on ∂T) and $C_T=\sup|\partial_\nu\psi_T|\le h_{\max}/2$, the supremum being over the open sides. For every $\alpha>0$ the number $\lambda=\lambda_1(T,\alpha)$ satisfies*

$$\alpha \;\le\; \frac{2\lambda^2}{r\,\Lambda_1(\Lambda_1-\lambda)} + C_T\,\frac{\lambda\,\Lambda_2}{\Lambda_2-\lambda}, \qquad \Lambda_k=\Lambda_k(T). \tag{23}$$

In particular, with $\alpha_*=C_T\Lambda_1\Lambda_2/(\Lambda_2-\Lambda_1)$,

$$\lambda_1(T,\alpha) \;\ge\; \Lambda_1(T)\Big(1-\frac{2}{r(\alpha-\alpha_*)}\Big) \qquad (\alpha>\alpha_*). \tag{24}$$

Lemma A.2. *If $h\in H^1(T)$ is harmonic in T, then $\int_T h^2\le C_T\int_{\partial T}h^2$, and $C_T\le h_{\max}/2$.*

Proof. Since $\psi_T\ge 0$ vanishes on ∂T and $\Delta(h^2)=2|\nabla h|^2$,

$$\int_T h^2 = -\int_T h^2\Delta\psi_T = -\int_{\partial T} h^2\,\partial_\nu\psi_T - 2\int_T\psi_T|\nabla h|^2 \le C_T\int_{\partial T}h^2.$$

[1] https://github.com/jasondavies/robin-triangles.

Here C_T is understood as the supremum of $|\partial_\nu \psi_T|$ over the open sides, which is finite by the bound proved below, and $\psi_T \in H^2(T)$ because T is convex. The computation is justified when h is smooth up to the boundary. For general h apply it on the homothetic triangles $T_\varepsilon = I + (1-\varepsilon)(T-I)$, on whose closure h is smooth, with their own torsion functions, for which $C_{T_\varepsilon} = (1-\varepsilon)C_T$ by scaling; then let $\varepsilon \to 0$, using that $x \mapsto h(I+(1-\varepsilon)(x-I))$ converges to h in $H^1(T)$ and therefore its trace converges in $L^2(\partial T)$. For the bound on C_T fix a side, with altitude h_j on it, and let S be the strip of width h_j bounded by the line through that side and the parallel line through the opposite vertex. Its torsion function $\psi_S(y) = y(h_j - y)/2$ satisfies $\psi_S - \psi_T \geq 0$ in T by the maximum principle, and both vanish on the side, so $|\partial_\nu \psi_T| \leq |\partial_\nu \psi_S| = h_j/2$ there. □

Proof of Theorem A.1. Let u be the Robin ground state, $\lambda = \lambda_1(T,\alpha)$, $g = u|_{\partial T}$. Restricting (1) to $H^1_0(T)$ gives $\lambda \leq \Lambda_1$, and equality would make ϕ_1 a Robin eigenfunction, which it is not ($\partial_\nu \phi_1 \not\equiv 0 = -\alpha\phi_1$ on ∂T); so $\lambda < \Lambda_1$, and therefore $g \not\equiv 0$. Write $u = h + v$ with h harmonic, $h = g$ on ∂T, and $v \in H^1_0(T)$. Then $-\Delta v - \lambda v = \lambda h$, so $v = \lambda(-\Delta_D - \lambda)^{-1} h$, where $-\Delta_D$ is the Dirichlet Laplacian with normalised eigenfunctions ϕ_k and eigenvalues Λ_k. By Green's formula and $\int_T \nabla h \cdot \nabla v = 0$,

$$-\alpha \int_{\partial T} g^2 = \int_T |\nabla u|^2 - \lambda \int_T u^2 = \int_T |\nabla h|^2 - \lambda \|h\|^2 - \lambda \langle h, v\rangle \;\geq\; -\lambda \|h\|^2 - \lambda^2 \sum_{k \geq 1} \frac{\langle h, \phi_k\rangle^2}{\Lambda_k - \lambda},$$

where norms and inner products without subscript are in $L^2(T)$. Bounding $\Lambda_k - \lambda \geq \Lambda_2 - \lambda$ for $k \geq 2$ gives

$$\alpha \int_{\partial T} g^2 \;\leq\; \lambda \|h\|^2 \frac{\Lambda_2}{\Lambda_2 - \lambda} + \lambda^2 \frac{\langle h, \phi_1\rangle^2}{\Lambda_1 - \lambda}. \tag{25}$$

Since h is harmonic and ϕ_1 vanishes on ∂T, $\Lambda_1 \langle h, \phi_1\rangle = -\int_T h\, \Delta\phi_1 = -\int_{\partial T} g\, \partial_\nu \phi_1$. The Rellich identity $\int_{\partial T} (\partial_\nu \phi_1)^2\, (x-I)\cdot \nu \, ds = 2\Lambda_1$ (valid on convex polygons, where $\phi_1 \in H^2$) and $(x-I)\cdot\nu = r$ give $\int_{\partial T} (\partial_\nu \phi_1)^2 = 2\Lambda_1/r$, so $\langle h, \phi_1\rangle^2 \leq 2\,\|g\|^2_{L^2(\partial T)}/(r\Lambda_1)$. Inserting this and Lemma A.2 into (25) and dividing by $\|g\|^2_{L^2(\partial T)} > 0$ gives (23).

For (24) use $\lambda < \Lambda_1$: the second term of (23) is at most α_* and the first is at most $2\Lambda_1/(r(\Lambda_1 - \lambda))$. □

Remark A.3. The first-order expansion $\lambda_1(\Omega,\alpha) = \Lambda_1 - \alpha^{-1} \int_{\partial\Omega} (\partial_\nu \phi_1)^2 + o(\alpha^{-1})$ as $\alpha \to \infty$ is classical for smooth domains. If it holds on T (we have not located a reference for polygons and do not use it), then by the same Rellich identity $\lambda_1(T,\alpha) = \Lambda_1(1 - 2/(r\alpha)) + o(\alpha^{-1})$, so that (24) is sharp to first order in $1/\alpha$.

Declaration of AI-generated research and authorship

This manuscript is the product of an AI-driven research process. The human author initiated a broader investigation of open problems from the AIM problem lists, and AI systems worked on this problem within it. AI systems produced all of the following: the results and proofs, the computations and the code for the certificates, and the organisation and writing of the text. The systems were Claude Opus 5.5 and Claude Fable 5.2 (Anthropic), used through Claude Code. The human author directed the AI systems but wrote no part of the manuscript.

Theorem 1.1 is computer-assisted and its proof is reproduced by `scripts/verify.py`. At the time of writing, the mathematics and the code have been checked only by AI systems and by computer verification. These checks include the certificates described above; adversarial reviews by independent AI referees, of Part I, of Part II, and of the merged manuscript; a second certificate of

Lemma 4.8, written by the referee of Part I; an independent verification program for the certificates of Part II, written by a separate AI instance from the text and the certificate format only; and an external AI review of the code (ChatGPT Pro), whose findings are corrected (§17). The comparison with Ma's preprint [27] was also made by an AI referee. No independent human expert has reviewed the manuscript. The human author has not read this manuscript or checked its arguments. The author takes responsibility for the accuracy of this disclosure.

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