CHARACTER COEFFICIENTS FOR INTEGRALS OF ODD PRODUCTS OF HYPERBOLIC TANGENTS

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Abstract. For odd $n \geq 3$, we express the integral of $x^{-2}\prod_{k=1}^{n}\tanh(kx)$ on the positive half-line in terms of derivatives of primitive even Dirichlet L-functions at -1 and logarithms of integers. Finite character sums determine all coefficients, including the rational coefficient of each logarithm.

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1. Reduction to primitive characters

Michael Smith asked for the coefficients in a reduction of $I_n = \int_0^\infty x^{-2}\prod_{k=1}^{n}\tanh(kx)\,\mathrm{d}x$ for odd $n \geq 3$. The following finite formula gives every primitive-character and logarithmic coefficient. Smith supplied the Mellin representation, parity observation, and low-order evaluations [1].

Theorem 1.1. *For every odd $n \geq 3$, formula (1) expresses I_n as a sum of primitive even Dirichlet L-derivatives at -1 and logarithms of positive integers. Its logarithmic coefficients are rational. The following finite character sums specify every coefficient.*

Proof. Let $L = 2\,\mathrm{lcm}(1,\dots,n)$ and $P(z) = \prod_{k=1}^{n}(1-z^k)/(1+z^k) = 1 + \sum_{m\geq 1} c(m)z^m$. At a primitive $2j$-th root the excess of denominator zeros over numerator zeros is either zero or one, by counting odd and even multiples of j. Odd-order roots give no poles. Thus $F(z) = (1-z^L)P(z)$ is a polynomial of degree at most L, and $c(m)$ is L-periodic for $m \geq 1$. For odd n, $z^L F(1/z) = F(z)$, so $c(-m) = c(m)$ on residue classes.

For $q \mid L$, let $d = L/q$. For each character $\chi \bmod q$, set

$$b_{q,\chi} = \frac{1}{\varphi(q)} \sum_{u\in(\mathbb{Z}/q\mathbb{Z})^\times} c(du)\overline{\chi(u)}.$$

Use $u = 1$ for $q = 1$. Only even characters survive. Splitting m by $\gcd(m, L)$ and using character orthogonality gives, for $\Re s > 1$,

$$D(s) = \sum_{m\geq 1} c(m)m^{-s} = \sum_{q\mid L}\sum_{\chi \bmod q} b_{q,\chi}(L/q)^{-s}L(s,\chi) = \sum_{\psi} H_\psi(s)L(s,\psi).$$

The final sum runs over primitive even characters, including the trivial conductor-one character. Compute $H_\psi(s) = \sum_t h_{\psi,t}t^{-s}$ by adding, for each $\chi \bmod q$ induced by ψ of conductor f, the polynomial

$$b_{q,\chi}(L/q)^{-s}\prod_{\substack{p\mid q\\ p\nmid f}}(1-\psi(p)p^{-s}).$$

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This also gives a test for which primitive derivative coefficients vanish. The result is

$$\boxed{I_n = -2\sum_{\psi} H_\psi(-1)L'(-1,\psi) + 2\sum_{t} t\log(t)\sum_{\psi} h_{\psi,t}L(-1,\psi).} \tag{1}$$

The coefficient of each $\log t$ is rational. All factors lie in a common cyclotomic field, and its Galois group permutes the terms indexed by ψ, as follows for $L(-1,\psi)$ from $-\frac{f}{2}\sum_{a=1}^{f}\psi(a)B_2(a/f)$ with $B_2(x) = x^2 - x + 1/6$. Derivative coefficients can be algebraic and combine in conjugate pairs.

To justify evaluation at -1, put $G(x) = P(e^{-2x})$. For $\Re s > 1$, the Mellin transform of $G - 1$ is $2^{-s}\Gamma(s)D(s)$. Its continuation to $\Re s > -n$ is $\int_0^1 x^{s-1}G(x)\,\mathrm{d}x + \int_1^\infty x^{s-1}(G(x)-1)\,\mathrm{d}x - 1/s$. At $s = -1$ this equals I_n. The residue of Γ there is -1, so $D(-1) = 0$ and $I_n = -2D'(-1)$, giving the formula. At $n = 3$, $D(s)/\zeta(s) = -2 + 2\,2^{-s} + 2\,3^{-s} + 4\,4^{-s} - 4\,6^{-s}$. This recovers $I_3 = \log(27/4)$. □

References

[1] M. Smith, *Closed form of an integral of products of hyperbolic tangents*, Mathematics Stack Exchange, question 5150767.