COUNTING TRIANGLES WITH INTERLACING ROWS

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Abstract. We give an exact formula for the number of ways to arrange the first $n(n+1)/2$ positive integers in n triangular rows so that every parent lies strictly between its two children. The smallest and largest labels occur in the bottom row. Fixing their positions leaves a sequence of row counts with repeated interior labels allowed. Inclusion–exclusion over those labels recovers the desired count and its refinement by the positions of both extrema.

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1. Introduction

Clark Kimberling's Problem 18 [1] asks for the number T_n of fillings of rows of lengths $1, 2, \dots, n$ with the integers $1, \dots, M$, where $M = n(n+1)/2$, each used once. Each parent is required to lie strictly between its two children, in either order. Reflected configurations are counted separately when they differ. James B. Sidoli [3] gives an enumeration in terms of linear extensions of posets. We obtain a formula using integer matrices indexed by rows.

Write $[M] = \{1, \dots, M\}$. For $1 \le r < n$, let H_r have rows indexed by $y \in [M]^{r+1}$, columns indexed by $x \in [M]^r$, and entries

$$H_r(y, x) = \prod_{j=1}^{r} \mathbf{1}\{(x_j - y_j)(x_j - y_{j+1}) < 0\}. \tag{1.1}$$

For $2 \le k \le M$, let b_k be the indicator vector of bottom rows in $[k]^n$ in which 1 and k each occur exactly once, extended by zero to $[M]^n$. Let $\mathbf{1}_M$ be the length-M vector of ones.

Theorem 1.1. *We have $T_1 = 1$. For every $n \ge 2$,*

$$T_n = \sum_{k=2}^{M} (-1)^{M-k} \binom{M-2}{k-2} b_k^{\top} H_{n-1} \cdots H_1 \mathbf{1}_M. \tag{1.2}$$

For fixed distinct bottom positions i, j, restricting b_k to rows with $y_i = 1$ and $y_j = k$ gives the number of fillings with 1 in position i and M in position j.

Proof. First allow repeated labels subject to the strict comparisons in (1.1). For a row y of length r, the y-entry of $H_{r-1} \cdots H_1 \mathbf{1}_M$ counts the admissible fillings above y. At length one there is one empty filling. Each multiplication sums over all possible choices of the row immediately above y, counting each completion once.

A global minimum above the bottom row would have a smaller child, and a global maximum above that row would have a larger child. Consequently both extrema lie in the bottom row, and a bottom row in $[k]^n$ forces the entire triangle to use labels from $[k]$.

Require 1 and M each to occur once in a triangle on $[M]$, and apply inclusion–exclusion to the events that an interior label is absent. Forbidding a fixed set of $M - k$ interior labels leaves a totally ordered alphabet of size k. Its order-preserving bijection with $[k]$ takes the extrema to $1, k$

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and preserves every strict comparison. The number of fillings for this choice is $b_k^\top H_{n-1}\cdots H_1\mathbf{1}_M$. There are $\binom{M-2}{k-2}$ such choices. Every surviving filling uses all M labels in M cells and therefore uses each label once. This proves (1.2). The relabeling preserves the positions of the two extrema, proving the refinement. □

2. Evaluation and checks

Put $F_1(x) = 1$. For each $y \in [M]^{r+1}$, let $\ell_j = \min(y_j, y_{j+1})$ and $u_j = \max(y_j, y_{j+1})$. The row transfer has the form

$$F_{r+1}(y) = \sum_{\ell_1<x_1<u_1} \cdots \sum_{\ell_r<x_r<u_r} F_r(x). \tag{2.1}$$

If some $u_j - \ell_j \le 1$, the sum is zero. Otherwise, tabulate $P_r(t) = \sum_{x\le t} F_r(x)$, where $x \le t$ means $x_j \le t_j$ for every j. The rectangular sum in (2.1) uses 2^r values of P_r. In coordinate j use $u_j - 1$ or ℓ_j, with a minus sign for each lower index chosen.

Tabulate $F_1, \dots, F_{n-1}$ one row length at a time. For the last row, choose the positions of 1 and k, fill its remaining $n-2$ positions from $\{2, \dots, k-1\}$, and evaluate (2.1) at those rows. There are at most $n(n-1)M^{n-1}$ queries over all k, with the empty-tuple convention for $n = 2$. Earlier tables each have at most M^{n-1} entries, and a prefix table takes r one-dimensional passes. The total cost is $O(n^3 2^n M^{n-1})$ integer operations with $O(M^{n-1})$ stored integers. Each integer needs $O(M \log M)$ bits. This gives $\exp(O(\sqrt{M}\log M))$ operations in the number of cells.

2.1. **Small cases.** For $n = 3$, the five scalar counts in (1.2), in order $k = 2, \dots, 6$, are $0, 0, 0, 10, 60$. Thus $T_3 = 60 - 4 \cdot 10 = 20$. For $n = 4$, take the bottom position of 1 as the row index and the bottom position of 10 as the column index. The refined counts are

$$\begin{pmatrix} 0 & 160 & 110 & 68 \\ 160 & 0 & 264 & 110 \\ 110 & 264 & 0 & 160 \\ 68 & 110 & 160 & 0 \end{pmatrix}, \qquad T_4 = 1{,}744.$$

Direct permutation enumeration agrees through $n = 4$. A second check inserts labels in increasing order and marks the occupied cells after each insertion. Whenever both children have the same mark, their parent must have that mark. Imposing this at every step is equivalent to strict interlacing of the final distinct labels. The two methods agree through $n = 6$, giving 1, 2, 20, 1,744, 2,002,568, 42,263,042,752, as recorded in OEIS A347608 [2].

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References

[1] C. Kimberling, *Unsolved problems and rewards*, Problem 18, Triangles with interlacing rows. The listed $50 reward is a donation to the OEIS Foundation in the solver's name.

[2] The OEIS Foundation Inc., *The On-Line Encyclopedia of Integer Sequences*, entry A347608, Number of interlacing triangles of size n.

[3] J. B. Sidoli, *On the number of interlacing triangles of size n*, October 29, 2021.