ROOT SEPARATION IN ONE-PARAMETER FAMILIES OF INTEGER POLYNOMIALS

ABSTRACT. For every $n \geq 2$ we construct irreducible integer polynomials of degree n and arbitrarily large height H with two roots at distance $\ll_n H^{-(4n-3)/6}$. All the roots can be chosen positive, or the two close roots can be chosen nonreal with all the other roots positive. We also construct monic irreducible examples. For a squarefree family $F(T, X) \in \mathbb{Z}[T, X]$ of positive degree in T, let $e(F)$ denote the exponent for which the least distance between two roots of $F(t, X)$ is $H(F(t, X))^{-e(F)+o(1)}$ as $t \to +\infty$. Among families of degree n in X and degree three in T, the maximum of $e(F)$ is $(2n-1)/3$. It is $(4n-3)/6$ for families irreducible over $\mathbb{Q}(T)$, and $(2n-2)/3$ for monic families of odd degree $n \geq 3$. For arbitrary degree in T, we prove $e(F) \leq (3n-1)/4$. Two roots in different irreducible factors over $\mathbb{C}(T)$ have separation exponent strictly less than $2n/3$. If $E(n)$ denotes the supremum of $e(F)$ over squarefree families of degree n in X, then $E(n)/n$ tends to $\sup_{d \geq 2} E(d)/d \in [2/3, 3/4]$.

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1. INTRODUCTION

Let $P \in \mathbb{Z}[X]$ be squarefree of degree $n \geq 2$, with complex roots $\alpha_1, \ldots, \alpha_n$. Its height $H(P)$ is the largest absolute value of its coefficients, and its root separation is

$$\operatorname{sep}(P) = \min_{i<j} |\alpha_i - \alpha_j|.$$

For $H(P) > 1$ define $e(P)$ by $\operatorname{sep}(P) = H(P)^{-e(P)}$. Let $e(n)$ be the upper limit of $e(P)$ as $H(P) \to \infty$ over squarefree integer polynomials of degree n. Write $e_{\mathrm{irr}}(n)$ for the corresponding upper limit over irreducible polynomials and $e^*_{\mathrm{irr}}(n)$ for the upper limit over monic irreducible polynomials.

Mahler's inequality gives $e(n) \leq n-1$ [11]. For $n \geq 4$, Bugeaud and Dujella constructed reducible polynomials giving $e(n) \geq (2n-1)/3$ [6, Theorem 1]. The two close roots lie in different factors, one of which is linear. For irreducible polynomials they obtained $e_{\mathrm{irr}}(n) \geq n/2 + (n-2)/(4(n-1))$ for $n \geq 4$ [5, Theorem 1]. Our construction gives

$$e_{\mathrm{irr}}(n) \geq \frac{4n-3}{6} \qquad (n \geq 2).$$

For $n \geq 5$ this improves their bound for irreducible polynomials. It falls short of their reducible bound by $1/6$.

The construction in Section 2 starts with the cubic $X = z(z-1)^2$. For small positive X it has two inverse branches near $z = 1$. We construct a rational function $T = B(z)/C(z)$

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so that the two values of $1/T$ on these branches agree to high order. At a fixed large real value $T = t$, the corresponding two values of X are at distance of order $|t|^{-(2n-3/2)}$, while the height of the polynomial is of order $|t|^3$. One sign of t gives n positive roots. The other gives $n-2$ positive roots and a pair of nonreal conjugates.

Using $X = z^3 - z$ in Section 3, we obtain

$$e^*_{\mathrm{irr}}(n) \geq \begin{cases} (2n-2)/3 & \text{for odd } n \geq 3, \\ (2n-4)/3 & \text{for even } n \geq 4. \end{cases}$$

These bounds improve Bugeaud and Dujella's bound $e^*_{\mathrm{irr}}(n) \geq n/2 - 1/4$ for odd $n \geq 5$ and even $n \geq 8$ [6, Theorem 3]. For $n = 3$, Bugeaud and Mignotte proved $e^*_{\mathrm{irr}}(3) \geq 3/2$ [7, Theorem 4]. For reducible monic polynomials, Bugeaud and Dujella obtained $(2n-3)/3$ in even degrees $n \geq 6$ and $(2n-5)/3$ in odd degrees $n \geq 7$ [6, Theorem 2]. Our irreducible construction falls short of the former by $1/3$ and exceeds the latter by one. In degrees $5, 7, 9$, the reducible monic examples of Dujella and Pejković give $7/3, 17/5, 31/7$, respectively [8, (3.2)]. Our monic irreducible examples give $8/3, 4, 16/3$ in those degrees.

We now consider $F \in \mathbb{Z}[T, X]$, squarefree in $\mathbb{Q}(T)[X]$, of degree $h \geq 1$ in T and degree $n \geq 2$ in X. Define its separation exponent by

$$e(F) = \lim_{t \to +\infty} \frac{-\log \operatorname{sep}(F(t, X))}{\log H(F(t, X))},$$

where t runs through the positive integers. For all sufficiently large t, the specialization is squarefree of degree n and has height $H(F(t, X)) \asymp_F t^h$. Here we use the height of the specialization itself. If the coefficients of F in X are coprime in $\mathbb{Q}[T]$, Bézout's identity bounds their common divisor at every integer t by a fixed integer, so passing to the primitive part gives the same exponent. To see that the limit exists, put $u = 1/T$. By the Newton–Puiseux theorem [15, §3.1], the roots have convergent Laurent expansions in $u^{1/q}$ near zero, with coefficients in a number field, for some positive integer q. The difference of any two distinct roots has a first nonzero term cu^s. Their distance at $T = t$ is asymptotic to $|c|t^{-s}$, so their separation exponent is s/h. The number $he(F)$ is the largest such order s.

Let $E(n)$ be the supremum of $e(F)$ over these families, and write $E_{\mathrm{red}}(n)$ and $E_{\mathrm{irr}}(n)$ for the suprema over reducible and irreducible families in $\mathbb{Q}(T)[X]$, respectively. We call F absolutely irreducible when it is irreducible in $\mathbb{C}[T, X]$. If F is irreducible over $\mathbb{Q}(T)$, Hilbert's irreducibility theorem, applied to its primitive part in $\mathbb{Q}[T][X]$, gives arbitrarily large positive integers t for which $F(t, X)$ is irreducible over $\mathbb{Q}$ [18, Lemma 4]. Replacing T by $-T$ gives negative integers of arbitrarily large absolute value.

In the notation of Pejković [14], $E(n) = e^{\mathrm{poly}}(n)$. His results give $E_{\mathrm{irr}}(3) = E(3) = 2$ and $E_{\mathrm{red}}(4) = E(4) = 5/2$ [14, Theorem 2.1 and Corollary 3.6]. In degree four he constructs an irreducible family with exponent $9/4$, and proves $e(F) \leq 5/2 - 1/(2h)$ for every squarefree family [14, Proposition 3.1 and Theorem 3.5]. In particular, his irreducible quartics improve on the value $13/6$ given by Theorem 2.1 in that degree.

When $h = 3$, Theorem 4.3 gives the exact maxima $(2n-1)/3$ for all squarefree families and $(4n-3)/6$ for irreducible families. For odd $n \geq 3$, Proposition 4.4 gives the monic maximum $(2n-2)/3$, attained by an absolutely irreducible family. The main estimate comes from an auxiliary polynomial whose first coefficients vanish when evaluated at one root. The order of the difference between two roots controls how many of the same coefficients vanish at the second root. The degree of the resultant limits the total order of vanishing.

For arbitrary h, Zhuang's bound over $\mathbb{C}(T)$ gives $e(F) \leq n - 1 - n/(40n+2)$ for $n \geq 4$ [20, Corollary 8.1.2]. Take the place at infinity and replace T by T^m. This multiplies both

the order of the root difference and the degree in T by m. Taking mth roots and letting $m \to \infty$ removes the multiplicative constant. Theorem 5.1 gives

$$e(F) \le \frac{3n-1}{4} - \frac{n-3}{4h} \qquad (n \ge 3).$$

If the two roots lie in different irreducible factors over $\mathbb{C}(T)$, Theorem 5.3 gives $s/h \le 2n/3 - 1/h$ for $n \ge 3$. In the Bugeaud–Dujella examples $h = 3$ and equality holds. The two proofs use different properties of the curve $F(T, X) = 0$. The first bounds how long two branches can agree in terms of the two degrees. For roots of different factors we use the number of intersections of the two curves, together with a theorem of Ruberman.

Finally, Theorem 6.2 shows that $E(n)/n$ and $E_{\mathrm{red}}(n)/n$ have the same limit, and that this limit equals $\limsup_{n\to\infty} E_{\mathrm{irr}}(n)/n$. Thus a single family of degree d with $e(F) > 2d/3$ would improve the coefficient $2/3$ in every sufficiently large degree. The same improvement would hold for absolutely irreducible families in infinitely many degrees.

2. Irreducible polynomials of every degree

Theorem 2.1. *For every $n \ge 2$ there is an absolutely irreducible polynomial $F \in \mathbb{Z}[T, X]$ of degree three in T and degree n in X with the following property. For all real t of sufficiently large absolute value, two roots of $F(t, X)$ are at distance $\asymp_n |t|^{-(2n-3/2)}$. The pair is real for one sign of t and consists of nonreal conjugates for the other. The polynomial $F(t, X)$ is irreducible over $\mathbb{Q}$ for infinitely many integers t of each sign. For integers t of sufficiently large absolute value, its primitive part has height $\asymp_n |t|^3$. In particular,*

$$e_{\mathrm{irr}}(n) \ge (4n-3)/6.$$

Put $f(z) = z(z-1)^2$. For small positive x, the equation $f(\xi) = x$ has two roots u, v near 1 and a third root z near 0. We construct a rational function $S = A/B$, with $B(1) \ne 0$, for which $A(u)B(v) - A(v)B(u) = c(u-v)z^{2n-2}$. Thus $S(u) - S(v)$ vanishes to higher order than $u - v$ as $x \to 0$. The same identity for $f(z) = z(z-1)(z+1)$ will be used in the monic case.

Lemma 2.2. *Let $d \ge 2$, $N = 2d-2$ and $f(z) = z(z-1)(z-\epsilon)$, where $\epsilon \in \{1, -1\}$. For an indeterminate x let*

$$V_d = \left\{ P \in \mathbb{Q}[z] : \deg P \le d, \quad \sum_{f(\xi)=x} \frac{\xi^N P(\xi)}{f'(\xi)} = 0 \right\}.$$

The space V_d has a basis consisting of polynomials of degrees d and $d-1$. Any two polynomials A, B forming a basis are coprime, and for some $c \in \mathbb{Q} \setminus \{0\}$ we have

$$A(u)B(v) - A(v)B(u) = c(u-v)z^N \tag{1}$$

whenever z, u, v are the three roots of $f(\xi) = x$.

Proof. Put $H_j(x) = \sum_{f(\xi)=x} \xi^j / f'(\xi)$. Setting $z = 1/t$ in the partial fraction expansion of $1/(f(z) - x)$ gives

$$\sum_{j \ge 0} H_j(x) t^j = \frac{t^2}{(1-t)(1-\epsilon t) - xt^3}.$$

For $0 \le j \le d$ the degree of $H_{N+j}(x)$ is at most $d-2$. For $P \in V_d$, we show that the coefficients of z^{d-1} and z^d determine the remaining coefficients. Equating the coefficients of $1, x, \dots, x^{d-2}$ in the definition of V_d gives a linear system whose matrix has entries $[x^r]H_{N+j}$ for $0 \le r, j \le d-2$.

For $\epsilon = 1$, expansion of the generating function in powers of x gives $[x^r]H_{N+j} = \binom{2d-3+j-r}{2r+1}$. For $\epsilon = -1$ the entry vanishes unless r and j have the same parity. For

$r = 2i + e$ and $j = 2l + e$, with $e \in \{0, 1\}$, it is $\binom{d-2+l-i}{2i+e}$. Grouping even and odd indices separately in this case, we obtain in both cases square matrices of the form

$$\left(\binom{L+l-i}{2i+e}\right)_{0\le i,l<k}, \qquad 2(k-1)+e \le L, \qquad e \in \{0,1\}.$$

The (i, l) entry counts paths from $(-l, 0)$ to $(L - i, 2i + e)$ with steps $(1, 0)$ and $(1, 1)$. By the Gessel–Viennot lemma [9], the determinant is a signed count of families of disjoint paths. Only the identity permutation of the endpoints can occur. Indeed, of two disjoint paths, the one starting farther left must be above the other wherever both are defined. If it ended at the lower endpoint, which lies farther to the right, this would already fail when the other path reached the higher endpoint. At least one such family exists. The ith path takes $2i + e$ diagonal steps followed by horizontal steps. Thus the determinant is positive. The two highest coefficients can now be prescribed arbitrarily, and every nonzero polynomial in V_d has degree at least $d - 1$.

Choose A, B of degrees $d, d - 1$, respectively. For the three roots z, u, v put

$$D(z) = \frac{A(u)B(v) - A(v)B(u)}{u - v}.$$

This is a symmetric polynomial in u, v of total degree at most N. As $u + v = 1 + \epsilon - z$ and $uv = (z - 1)(z - \epsilon)$, it is a polynomial in z of degree at most N. To prove (1), we first compare D at two roots. In the equation defining V_d, take first $P = A$ and then $P = B$. Multiply the first equation by $B(v)$ and the second by $A(v)$ and subtract. The term indexed by v disappears. The two remaining terms give

$$\frac{z^N(z - v)}{f'(z)}D(u) + \frac{u^N(u - v)}{f'(u)}D(z) = 0.$$

Here $D(u) = (A(z)B(v) - A(v)B(z))/(z - v)$. Cancelling the factors $z - v$ and $u - v$ gives $u^N D(z) = z^N D(u)$.

Let $z \to 0$ and choose another root u of $f(\xi) = f(z)$ tending to 1. The equality $D(z)/z^N = D(u)/u^N$ shows that $D(z)/z^N$ is bounded at zero. As $\deg D \le N$, this gives $D(z) = cz^N$ with $c \in \mathbb{Q}$. To see that $c \neq 0$, take large positive x. The three roots of $f(\xi) = x$, divided by $x^{1/3}$, tend to the three cube roots of unity. Since $A(\xi)/B(\xi) \sim k\xi$ for some $k \neq 0$, its three values are then distinct.

A common zero β of A, B would make $D(z)$ divisible by

$$(u - \beta)(v - \beta) = z^2 + (\beta - 1 - \epsilon)z + (\beta - 1)(\beta - \epsilon).$$

If this monic quadratic divided cz^N, it would equal z^2. Its linear coefficient would then give $\beta = 1 + \epsilon$, for which the constant term is $\epsilon \neq 0$. Thus A, B are coprime. Changing the basis multiplies the left side of (1) by a nonzero rational constant. □

Proof of Theorem 2.1. Apply the lemma with $d = n$ and $\epsilon = 1$. Since the basis polynomials are coprime, we may choose a basis C, B with $C(1) = 0$ and $B(1) \neq 0$. Put $S = C/B$. Near $z = 1$ choose the square root of z with value 1 there. The function $q = (z-1)\sqrt{z}$ has derivative 1 at $z = 1$. By the inverse function theorem we can write $z = z(q) = 1+q+O(q^2)$, and $f(z(q)) = q^2$. Write $s(q) = S(z(q))$. Let $z_0(x)$ be the analytic solution of $f(z_0(x)) = x$ with $z_0(0) = 0$. As $f'(0) = 1$, we have $z_0(x) = x + O(x^2)$. By (1),

$$s(q) - s(-q) = \frac{c\big(z(q) - z(-q)\big)z_0(q^2)^N}{B(z(q))B(z(-q))}, \qquad N = 2n - 2.$$

The right side has a zero of order $2N + 1$ at $q = 0$. Since $s(0) = 0$, the first nonzero term of $s(q)$ cannot have degree one or three. A zero of order at least four would give

$C = (z-1)^4 P$, which is impossible for $n < 4$. For $n \geq 4$ the identity $z^N C = f(z)^2 z^{N-2} P$ would give $P \in V_{n-1}$ with $\deg P \leq n-4$, contrary to the lemma. Hence $s(q) = aq^2 + O(q^3)$ with $a \neq 0$.

We now fix $y = s(q)$ and compare the two corresponding values of x. Put $r = q\sqrt{s(q)/(aq^2)}$, taking the square root with value 1 at $q = 0$. Then $y = ar^2$ and $r = q + O(q^2)$. Denote by $J(q)$ the value of q corresponding to $-r$. Thus $J(q) = -q + O(q^2)$ and $s(J(q)) = s(q)$. For $s(w) = aw^2 + O(w^3)$ the difference quotient at two nearby points w_1, w_2 is $a(w_1 + w_2) + O(\max(|w_1|, |w_2|)^2)$. Applying this with $w_1 = -q$ and $w_2 = J(q)$ gives

$$s(-q) - s(J(q)) = (-q - J(q))\Big(-2aq + O(q^2)\Big).$$

The left side has a zero of order $2N+1$, so $q + J(q)$ has a zero of order $2N$. The difference $q^2 - J(q)^2$ therefore has order $2N+1$ in q. Its absolute value is of order $|y|^{N+1/2}$ because $y \sim aq^2$. For small real y with $y/a > 0$, both values of x are real. For $y/a < 0$, the two values of r are purely imaginary and conjugate. Since the inverse functions have real Taylor coefficients, the two distinct values of x are then nonreal conjugates.

Clear the fixed denominators of B, C. Form the polynomial

$$F(T, X) = \prod_{f(\xi)=X} \Big(B(\xi) - TC(\xi)\Big). \tag{2}$$

The coefficients are symmetric polynomials with integer coefficients in the three roots of the monic cubic $f(z) - X$, so they belong to $\mathbb{Z}[X]$. The coefficient of T^3 is the nonzero product $-\prod_{f(\xi)=X} C(\xi)$. To see the degree in X, factor $B(z) - tC(z) = L \prod_{j=1}^{n} (z - b_j)$ for any t for which the degree is n. Since $\prod_{f(\xi)=X}(\xi - b_j) = X - f(b_j)$, we obtain

$$F(t, X) = L^3 \prod_{j=1}^{n} (X - f(b_j)).$$

This holds for all but at most one t, since $\max(\deg B, \deg C) = n$. In particular, $\deg_X F = n$. Putting $t = 1/y$ in the preceding calculation now gives the asserted distance between two roots of $F(t, X)$.

Suppose that $F = GH$ in $\mathbb{C}[T, X]$. Substituting $X = f(z)$ and $T = B(z)/C(z)$ gives zero, so one factor, say G, vanishes identically after this substitution. For all but finitely many x, the three roots ξ of $f(\xi) = x$ are distinct, and (1) gives three distinct values of $B(\xi)/C(\xi)$. All three are roots of $G(T, x)$. It follows that $\deg_T G \geq 3$. Thus H depends only on X. For each fixed x, the product in (2) is a nonzero polynomial in T, since B and C have no common zero. Hence H has no complex zero and is constant. This proves absolute irreducibility.

In particular the coefficients of F as a polynomial in X are coprime in $\mathbb{Q}[T]$. By the Euclidean algorithm, some $\mathbb{Z}[T]$-linear combination of them is a nonzero integer m. For integer t the greatest common divisor of the coefficients of $F(t, X)$ divides m. Since $H(F(t, X)) \asymp_n |t|^3$, its primitive part has the same order of height.

Hilbert's irreducibility theorem, applied to $F(T, X)$ and $F(-T, X)$, gives infinitely many irreducible specializations of each sign. Dividing $2n - 3/2$ by 3 gives the lower bound for $e_{\mathrm{irr}}(n)$. □

Corollary 2.3. *For all sufficiently large $|t|$, the polynomial in Theorem 2.1 has n positive real roots for one sign of t, and $n-2$ positive real roots and two nonreal conjugate roots for the other.*

Proof. We locate the other $n-2$ roots by showing that C has $n-2$ simple zeros in $(4/3,\infty)$. For $z>4/3$, the other two solutions of $f(\xi)=f(z)$ are u and $\bar u$, where

$$u=1-\frac{z}{2}+\frac{i}{2}\sqrt{z(3z-4)}.$$

As z increases from $4/3$ to infinity, u traces an arc in the upper half-plane starting at $1/3$, with $|u|\to\infty$ and $\arg u\to 2\pi/3$. The polynomial C has no zero on the open arc, since $C(u)=0$ would also give $C(\bar u)=0$, contradicting (1). The defining equation for V_n gives

$$\frac{z^N C(z)}{f'(z)}=-2\,\mathrm{Re}\left(\frac{u^N C(u)}{f'(u)}\right),\qquad N=2n-2.$$

We count zeros of the real part on the right by estimating the change in the argument of $u^N C(u)/f'(u)$ along this arc.

For a real zero b of C, the argument of $u-b$ lies between 0 and π and tends to $2\pi/3$. Thus its change is at least $-\pi/3$, including the case $b=1/3$. For a nonreal conjugate pair $b,\bar b$, the product $(u-b)(u-\bar b)$ starts on the positive real axis and has imaginary part $2\,\mathrm{Im}(u)(\mathrm{Re}(u)-\mathrm{Re}(b))$. Since $\mathrm{Re}(u)$ decreases strictly, this imaginary part can change sign only once, from positive to negative. The product never vanishes. Its continuous argument starts at zero and tends to $-2\pi/3$ or $4\pi/3$. Summing over the zeros of C with their multiplicities, we find that the change in the argument of $C(u)$ is at least $-n\pi/3$.

Finally $f'(u)=(u-z)(u-\bar u)$. The second factor has constant argument $\pi/2$, and the argument of the first changes from π to $5\pi/6$. Hence the net change of the argument of $u^N C(u)/f'(u)$ is at least

$$\frac{2N\pi}{3}-\frac{n\pi}{3}+\frac{\pi}{6}=(n-2)\pi+\frac{5\pi}{6}.$$

By continuity, the argument takes at least $n-2$ distinct values congruent to $\pi/2$ modulo π in the interior of the arc. The real part vanishes at each of these points. It follows that C has at least $n-2$ distinct zeros in $(4/3,\infty)$. Its double zero at 1 accounts for the rest of its degree, so all $n-2$ zeros are simple. For each such zero b, the function C/B has a simple zero at b, since $B(b)\neq 0$. Its local inverse gives a real solution of $C(z)/B(z)=1/t$ tending to b as $|t|\to\infty$. The corresponding roots of $F(t,X)$ tend to the distinct positive numbers $f(b)$, as f is strictly increasing on $(4/3,\infty)$. When the remaining two roots are real, they too are positive, since $x=q^2$ with q real and nonzero. □

An irreducible sextic for every odd parameter. For $v=z-1$ we have $x=v^2(1+v)$. When $n=6$ we may take

$$\begin{aligned}C(v)&=v^2(34-272v+780v^2-938v^3+391v^4),\\ B(v)&=19-171v+572v^2-852v^3+495v^4-7v^5-46v^6.\end{aligned}$$

For the three roots v,u,w of $v^2(1+v)=x$, substitution of $u+w=-1-v$ and $uw=v(v+1)$ gives

$$\frac{C(u)B(w)-C(w)B(u)}{u-w}=-45885(1+v)^{10}.$$

Since $C(v)/B(v)\sim(34/19)v^2$ at zero, the proof of Theorem 2.1 gives a sextic of height $\asymp|t|^3$ with two roots at distance $\asymp|t|^{-21/2}$. For large positive t this pair is positive. The quartic $C(v)/v^2$ has alternating signs at $1/3, 2/5, 1/2, 1, 2$, beginning with a positive value. Its four distinct positive zeros give the four remaining roots as in Corollary 2.3. For odd t the polynomial $B(v)-tC(v)$ reduces modulo two to $g(v)=v^6+(v+1)^5$. Put $b=X^2+X$ and $a=b+1$. At each root of $v^3+v^2=X$ we have $vg(v)=av+b$, by direct substitution.

Taking the product over the three roots and using their elementary symmetric functions gives

$$XF(t,X) \equiv b^3 + ab^2 + Xa^3 = X(X^6 + X^5 + 1) \pmod 2.$$

Thus $F(t,X)$ reduces to $X^6 + X^5 + 1$. It has no root in $\mathbb{F}_2$. The only remaining possible irreducible factors of degree at most three are $X^2 + X + 1$, $X^3 + X + 1$ and $X^3 + X^2 + 1$. Indeed, a cubic over $\mathbb{F}_2$ is irreducible precisely when it has no root there. Division by the three displayed polynomials gives the respective nonzero remainders $X + 1$, $X + 1$ and X^2. Thus the sextic is irreducible modulo two. It follows that $F(t,X)$ is irreducible over $\mathbb{Q}$ for every odd integer t.

3. Monic irreducible polynomials

Theorem 3.1. *For every $n \geq 3$ there is an absolutely irreducible $F \in \mathbb{Z}[T,X]$, monic of degree n in X and of degree three in T, with the following property. For all sufficiently large positive t, all roots of $F(t,X)$ are real and the distance between two of them is $\asymp_n t^{-M_n}$, where*

$$M_n = \begin{cases} 2n-2, & n \text{ odd}, \\ 2n-4, & n \text{ even}. \end{cases}$$

There is also an absolutely irreducible family, again monic in X and of the same degrees, for which a pair of nonreal conjugate roots is at distance $\asymp_n t^{-M_n}$ for all sufficiently large positive t. In both families the specialized polynomial is irreducible over $\mathbb{Q}$ for infinitely many positive integers t. Consequently $e^*_{\mathrm{irr}}(n) \geq M_n/3$.

Proof. It suffices to arrange that $\deg B = n > \deg C$. Clear the denominators of B and C. The leading coefficient L of $B - TC$ in z is then a nonzero integer independent of T. By (2), the leading coefficient of F in X is $A = L^3$. Thus $A^{n-1}F(T, X/A)$ is monic in X and has integer coefficients.

Take $f(z) = z^3 - z$ and let $d \geq 3$ be odd. By Lemma 2.2, V_d has a basis consisting of an odd polynomial B of degree d and an even polynomial C of degree $d-1$. Indeed, the even and odd parts of each element of V_d belong to V_d, and every nonzero element has degree at least $d-1$. At $x = 0$ the defining equation for C gives $C(1) = C(-1) = 0$. These zeros are simple. Otherwise $C = (z^2-1)^2 P$, which is impossible when $d = 3$. For $d \geq 5$, the identity $z^{2d-2}C = f(z)^2 z^{2d-4} P$ would give $P \in V_{d-1}$ with $\deg P \leq d-5$, contrary to the lemma. Since B and C are coprime and $B(0) = 0$, we have $B(\pm 1) \neq 0$ and $C(0) \neq 0$.

Let z_0, z_+, z_- be the three solutions of $f(z) = x$ near $x = 0$, numbered so that $z_0(0) = 0$, $z_+(0) = 1$ and $z_-(0) = -1$. Put $s_\pm(x) = C(z_\pm(x))/B(z_\pm(x))$. By (1),

$$s_+(x) - s_-(x) = \frac{c(z_+(x) - z_-(x)) z_0(x)^{2d-2}}{B(z_+(x))B(z_-(x))}.$$

Since z_0 has a simple zero, $s_+ - s_-$ vanishes to order $2d-2$ at zero. Both s_+ and s_- have simple zeros there. Their derivatives at zero are therefore equal and nonzero. This common derivative is rational, so we may multiply C by a rational constant to make it one. If $p = s_+^{-1}(y)$ and $q = s_-^{-1}(y)$, then $s_-(p) - s_-(q) = (s_- - s_+)(p)$. As $p, q \to 0$, the quotient $(s_-(p) - s_-(q))/(p-q)$ tends to $s_-'(0) = 1$. Since p has a simple zero in y, $p - q$ vanishes to order $2d-2$ in y. Taking $y = 1/t$ and $d = n$ gives the asserted distance for odd n.

For even $n \geq 4$ take $d = n-1$ and replace C, B by

$$C_1(z) = zC(z), \qquad B_1(z) = zB(z) + \frac{C(z)}{z^2-1}.$$

These polynomials have degrees d and $d+1$ and are coprime. Indeed, $B_1(0) = -C(0) \neq 0$. At $z = \pm 1$, the equalities $s'_\pm(0) = 1$ and $f'(\pm 1) = 2$ give $C'(z) = 2B(z)$ and hence $B_1(z) = 2zB(z) \neq 0$. At every other zero of C_1 we have $B_1(z) = zB(z) \neq 0$. On the two inverse branches the values of C_1/B_1 are $xs_\pm(x)/(x + s_\pm(x))$. Both have derivative $1/2$ at zero, and their difference is

$$\frac{x^2(s_+(x) - s_-(x))}{(x + s_+(x))(x + s_-(x))}.$$

This difference vanishes to order $2d - 2 = 2n - 4$. Since both derivatives are nonzero, the inverse functions give the asserted distance as in the odd case.

We next show that all roots are real for sufficiently large $|t|$. In both constructions $d = 2m + 1$ is odd, so $C(z) = P(z^2)$ for a polynomial P of degree m. For $z > 2/\sqrt{3}$ the other two solutions of $f(u) = f(z)$ are $u, \bar{u}$, where $\operatorname{Im} u > 0$. Set

$$w = \bar{u}^2 = 1 - z^2/2 + iz\sqrt{3z^2 - 4}/2.$$

As z increases, w describes an arc in the upper half-plane from $1/3$ to infinity. Its real part decreases strictly and its argument tends to $2\pi/3$. By (1), $P(w)$ has no zero on the open arc. The estimate in the proof of Corollary 2.3 therefore gives $\Delta \arg P(w) \geq -m\pi/3$. The defining equation for V_d gives

$$\frac{z^{4m}C(z)}{f'(z)} = -2\operatorname{Re}\left(\frac{w^{2m}P(w)}{3w - 1}\right).$$

The argument of $3w - 1$ changes from $\pi/2$ to $2\pi/3$. Hence the change in the argument of the quotient is at least $2m(2\pi/3) - m\pi/3 - \pi/6 = m\pi - \pi/6 > (m-1)\pi$. Its real part vanishes at least $m - 1$ times. Thus C has at least $m - 1$ distinct zeros in $(2/\sqrt{3}, \infty)$. Their negatives and ± 1 account for all $2m$ zeros of C, so each is simple. In even degree $C_1 = zC$ has the additional simple zero 0. Since the denominator has degree one more than the numerator, C/B (or C_1/B_1) also has a simple zero at infinity, as can be seen by putting $z = 1/w$ and expanding at $w = 0$. For sufficiently small real $1/t$, each of these simple real zeros gives one real solution of $C/B = 1/t$ (or $C_1/B_1 = 1/t$). Their images under f are the roots of the polynomial in (2).

Write $S = C/B$ in odd degree and $S = C_1/B_1$ in even degree. In both cases S is odd. Hence $\widetilde{f}(w) = if(iw) = w^3 + w$ and $\widetilde{S}(w) = iS(iw)$ have rational coefficients and the same degrees as f and S. Indeed, every odd rational function can be written as $zR(z^2)$ for some rational function R. The two zeros of $\widetilde{S}$ at $w = \pm i$ give inverse branches interchanged by complex conjugation. For real $1/t$ near zero, their $\widetilde{f}$-values are distinct complex conjugates. The substitutions multiply both coordinates by nonzero constants, so the difference vanishes to the same order as before.

For odd n, identity (1) shows that S takes three distinct values at the three roots of $f(z) = x$ for all but finitely many x. The same holds for even n, since the map $s \mapsto xs/(x + s)$ is one-to-one for $x \neq 0$ wherever it is defined. The substitutions using i preserve this property. Thus the factorization argument following (2) proves absolute irreducibility in all cases. Hilbert's irreducibility theorem gives infinitely many positive integer specializations that are irreducible over $\mathbb{Q}$. The change of scale at the beginning of the proof multiplies all root distances by $|A|$ and preserves the separation exponent. □

4. Sharp bounds for families of degree three in the parameter

Let F have degree h in T and degree n in X. Divide the coefficients of F, considered as a polynomial in X, by their greatest common divisor in $\mathbb{Q}[T]$, and denote the quotient by F_0. Put $a = \deg_T F_0 \leq h$. If $a = 0$, all the roots are constant and $e(F) = 0$. Otherwise,

choose an integer c at which the coefficient of T^a in F_0 does not vanish. With $u = 1/T$ and $v = 1/(X - c)$, the polynomial

$$P(u, v) = u^a v^n F_0(1/u, c + 1/v)$$

has degree a in u, since $F_0(0, X)$ is nonzero by primitivity. The coefficient of v^n in $P(0, v)$ is the nonzero value used to choose c, and $P(u, 0)$ is nonzero because $\deg_X F_0 = n$. Thus P has degree n in v, and neither u nor v divides it. The substitutions are invertible when $uv \neq 0$. Since neither T nor $X - c$ divides F_0, the irreducible factors of F_0 correspond to those of P, over $\mathbb{Q}$ as well as over $\mathbb{C}$. All roots of P stay bounded as $u \to 0$, since $P(0, v)$ has degree n. For two roots we have

$$v_1 - v_2 = \frac{X_2 - X_1}{(X_1 - c)(X_2 - c)}.$$

If $e(F) > 0$, take a pair for which the difference in X has order $he(F)$. The transformed difference has the same order for a finite common limit, and greater order at a common pole. In either case its order M satisfies $M \geq he(F)$.

For later use, let $P(u, v)$ be squarefree of degree d in u and $b \geq 2$ in v, with $\deg_v P(0, v) = b$. Its discriminant with respect to v is a nonzero polynomial of degree at most $2d(b - 1)$ in u, by homogeneity in the coefficients of P. All roots $v_1, \ldots, v_b$ of P are bounded at zero, where the leading coefficient of P is nonzero. Hence

$$\sum_{i<j} \operatorname{ord}_u(v_i - v_j) \leq d(b - 1). \tag{3}$$

We recall the two properties of the resultant used below. If P has leading coefficient $p(u)$ and roots $v_1, \ldots, v_b$, then

$$\operatorname{Res}_v(P, G) = p(u)^{\deg_v G} \prod_{j=1}^{b} G(u, v_j).$$

For nonzero G, the resultant vanishes identically exactly when P, G have a common factor of positive degree in v. Otherwise its degree in u is at most $b \deg_u G + (\deg_v G) \deg_u P$. Indeed, each term of the Sylvester determinant uses $\deg_v G$ coefficients of P and b of G.

Lemma 4.1. *Let $P \in \mathbb{C}[u, v]$ be irreducible, of degree $d \geq 1$ in u and $b \geq 2$ in v, and suppose that $P(0, v)$ has degree b. Let M be the order at $u = 0$ of the difference of two roots. If $M > d(b - 1)/2$, then $2M$ is an integer and*

$$\lfloor M \rfloor \leq (d - 1)(b - 1).$$

Proof. Since $M > d(b - 1)/2$, (3) shows that only one pair has order M. The roots have convergent expansions in $u^{1/q}$ for a common q. Multiplication of $u^{1/q}$ by $e^{2\pi i/q}$ permutes the roots and preserves the order of every difference. Hence it fixes or interchanges the two chosen roots.

For an integer m consider the polynomials $G \in \mathbb{C}[u, v]$ of degree at most m in u and at most $b - 1$ in v. Their coefficients form a vector space of dimension $D = b(m + 1)$. For every nonzero such G, irreducibility of P and $\deg_v G < b$ give

$$R(u) = \operatorname{Res}_v(P, G) \neq 0, \qquad \deg R \leq bm + d(b - 1).$$

If both roots are fixed, only integral powers of u occur, so M is an integer. Choose m so large that $D - 1 \geq M$. We can take a nonzero G so that the first $D - 1$ coefficients of $G(u, v_1)$ vanish. Since $G(u, v_1) - G(u, v_2)$ is divisible by $v_1 - v_2$, the order of $G(u, v_2)$ is at least M. All other factors in the product for R are bounded at zero. Thus $D - 1 + M \leq bm + d(b - 1)$, giving $M \leq (d - 1)(b - 1)$ in this case.

If the roots are interchanged, write them as $v(t)$ and $v(-t)$, where $t^2 = u$. Indeed, two applications of the substitution fix each root, so only powers of $u^{1/2}$ occur. The first odd term of $v(t)$ has degree $2k+1 = 2M$. In every series $G(t^2, v(t))$ the coefficients of $t, t^3, \dots, t^{2k-1}$ vanish. Take m so large that $L = D - 1 + k > 2k$. Requiring the series to vanish to order at least L therefore imposes at most $L - k = D - 1$ homogeneous linear conditions. Choose a nonzero solution G. The product $G(t^2, v(t))G(t^2, v(-t))$ then vanishes to order at least L in u. Hence $D - 1 + k \le bm + d(b-1)$, which gives $k \le (d-1)(b-1)$. □

For every $h \ge 2$, Lemma 4.1 gives the exact maximum $2 - 3/(2h)$ for the separation exponent of an absolutely irreducible cubic family of degree h in the parameter. Indeed, the order M is at most $\max\{h, 2h - 3/2\} = 2h - 3/2$. To attain it, take F_h from Theorem 2.1 with $\deg_X F_h = h$ and put

$$G_h(T, X) = T^h X^3 F_h(1/X, 1/T).$$

The polynomial F_h is absolutely irreducible and divisible by neither variable. Interchanging the variables and taking reciprocals therefore gives an absolutely irreducible G_h of degrees $h, 3$ in T, X, respectively. In the notation of the construction, $1/T = q^2$, and the two roots $s(q), s(-q)$ of G_h have difference of order $4h - 3$ in q. Hence $e(G_h) = (4h-3)/(2h)$.

Lemma 4.2. *Let $F \in \mathbb{Z}[T, X]$ be squarefree over $\mathbb{Q}(T)$, of degree $h \ge 2$ in T and degree $n \ge 2$ in X. Then*

$$he(F) \le (h-1)(n-1) + 1.$$

Proof. We may assume $e(F) > 0$ and use the bounded coordinate v. For two roots of one irreducible factor, Lemma 4.1 bounds the order of their difference by $\max\{h(n-1)/2, (h-1)(n-1) + 1/2\}$. For roots of two distinct factors of degrees (d_1, b_1) and (d_2, b_2), the resultant gives $M \le d_1 b_2 + d_2 b_1$. If one factor is independent of u, it is linear in v and $M \le h$. Otherwise all four degrees are positive and

$$\begin{aligned}(d_1 + d_2 - 1)(b_1 + b_2 - 1) + 1 - (d_1 b_2 + d_2 b_1) \\ = (d_1 - 1)(b_1 - 1) + (d_2 - 1)(b_2 - 1) \ge 0.\end{aligned}$$

Since $d_1 + d_2 \le h$ and $b_1 + b_2 \le n$, both cases give the asserted bound. □

Theorem 4.3. *Let $F \in \mathbb{Z}[T, X]$ be squarefree over $\mathbb{Q}(T)$, with $\deg_T F = 3$ and $\deg_X F = n \ge 2$. Then*

$$e(F) \le \frac{2n-1}{3}.$$

If F is irreducible over $\mathbb{Q}(T)$, then

$$e(F) \le \frac{4n-3}{6}.$$

Both upper bounds are attained for every $n \ge 2$.

Proof. The first bound is Lemma 4.2 with $h = 3$. For the second assume that F is irreducible over $\mathbb{Q}(T)$ and $e(F) > 0$. Choose two roots in the bounded coordinate v whose difference has order $M \ge 3e(F)$. If they lie in the same irreducible factor over $\mathbb{C}$ of degrees (d, b), Lemma 4.1 gives

$$M \le \max\{d(b-1)/2, (d-1)(b-1) + 1/2\} \le 2n - 3/2.$$

If they lie in distinct factors, recall that the primitive polynomial P in u, v is irreducible over $\mathbb{Q}$. Its irreducible factors over $\mathbb{C}$ are conjugate over $\mathbb{Q}$. To see this, substitute $v = u^{a+1}$, where $a = \deg_u P \le 3$. The monomials in each factor give distinct powers of u, so the factor can be recovered from the resulting univariate polynomial. Normalize the latter to

be monic. Its coefficients are algebraic numbers, being symmetric functions of roots of a polynomial in $\mathbb{Q}[u]$. Thus each normalized bivariate factor has algebraic coefficients. The product of the distinct Galois conjugates of such a factor is rational and hence exhausts the factorization of P. If there are $r > 1$ factors, each has degrees $(a/r, n/r)$. Since r divides $a \leq 3$, we have $a/r = 1$. The resultant then gives $M \leq 2n/r \leq n$.

The general bound is attained by the Bugeaud–Dujella families for $n \geq 4$ [6, proof of Theorem 1]. Their construction also covers $n = 2, 3$. Put $D = T^2 + 3T + 1$ and $x = (T + 2)/D$. For $j = 1, 2$ the family of degree $j + 1$ is $(DX - T - 2)p_j(X)$, where

$$p_1(X) = (T + 1)X - 1, \qquad p_2(X) = TX^2 + TX - 1.$$

In both cases $p_j(x) = (-1)^{j-1}D^{-j}$. The positive root y_j of p_j tends to zero, and $p_j'(w) \sim T$ uniformly between x and y_j. Thus $|x - y_j| \sim T^{-(2j+1)}$, as required. Theorem 2.1 gives equality in the irreducible case for all $n \geq 2$. □

Proposition 4.4. *Let $F \in \mathbb{Z}[T, X]$ be squarefree over $\mathbb{Q}(T)$, monic of degree $n \geq 3$ in X, and of degree three in T. Then*

$$e(F) \leq \frac{2n - 2}{3}.$$

For every odd $n \geq 3$, equality is attained by an absolutely irreducible family.

Proof. Put $N = 2n - 2$. Suppose that two roots have difference of order greater than N in $u = 1/T$. If they lie in one irreducible factor over $\mathbb{C}[T, X]$ of degrees (d, b), Lemma 4.1 gives the bound $\max\{d(b - 1)/2, (d - 1)(b - 1) + 1/2\}$ after the change to v. It exceeds N only if $(d, b) = (3, n)$. For roots in different factors the resultant bound is at most three if one factor is independent of T. Otherwise their degrees can be ordered so that $d_1 = 1$ and $d_2 \leq 2$. Then $b_2 + d_2b_1 \leq 2n - b_2$, which can exceed N only for $(d_1, b_1) = (1, n - 1)$ and $(d_2, b_2) = (2, 1)$. Both factors can be taken monic in X. The second then has a polynomial root of degree two in T, whereas every root of the first is $O(|T|)$ by Cauchy's bound. It remains to consider an absolutely irreducible F.

Let M be the order after the change to $v = 1/(X - c)$. Lemma 4.1 and its proof show that $M = N + 1/2$ and that the two expansions are $v(t), v(-t)$, with $u = t^2$. The first odd term of $v(t)$ has degree $2N + 1$. To see that t is a local parameter on this branch, let $p(t^2)$ be the Taylor polynomial of $v(t)$ through degree $2N$. Then $(v - p(u))/u^N$ has order one in t. We have $v(0) \neq 0$. Indeed, if $v(t)$ had a zero of order $l \geq 1$ in t, the original difference would have order $M - l < N$ in u.

We show that u and X are rational functions of a single parameter. In the function field $L = \mathbb{C}(u, v)$, a nonconstant function g has $[L : \mathbb{C}(g)]$ zeros and equally many poles, counted with multiplicity on all branches, including those at infinity [19, Theorem 1.4.11]. Thus u and v have n and three poles, respectively, counted with multiplicity. Every nonzero polynomial $G(u, v)$ of degrees at most $m, n - 1$ has at most $K = nm + 3(n - 1)$ zeros, since its total pole order is at most K. The space of these polynomials has dimension $D = n(m + 1)$, and $K = D - 1 + N$. Take m with $K - 1 > 2N$. Requiring $G(t^2, v(t))$ to vanish to order at least $K - 1$ imposes at most $K - 1 - N = D - 2$ conditions, since its coefficients of $t, t^3, \ldots, t^{2N-1}$ already vanish. Choose two independent solutions G_0, G_1. Their ratio $z = G_1/G_0$ is nonconstant because $\deg_v P = n$. For all but finitely many $\lambda \in \mathbb{C}$, the zeros of $z - \lambda$ lie away from the chosen branch and the poles of G_0. They are additional zeros of $G_1 - \lambda G_0$. Hence

$$K - 1 + [L : \mathbb{C}(z)] \leq K,$$

which proves $L = \mathbb{C}(z)$.

Every pole of X lies over $u = 0$, since F is monic in X. If there were two distinct poles, both would give $(u, v) = (0, 0)$. Choose a nonzero linear combination of G_0, G_1 vanishing

at $(0,0)$. It would have at least $K-1$ zeros at the chosen branch and one at each pole, since $v(0) \neq 0$ on that branch. This exceeds K. Take the unique pole of X as $z=\infty$. Since $[L:\mathbb{C}(X)]=3$, X is a cubic polynomial in z. At the chosen branch $X(t)=c+1/v(t)$ has first odd term of degree $2N+1$, and $X(t)-X(0)$ has order at most three. Its order is therefore two. Affine changes of z and X now give $X=f(z)=z(z-1)^2$, with the chosen branch at $z=1$. Write $u=C(z)/B(z)$ in lowest terms. This rational function has degree n, a double zero at 1, and a zero at infinity. In particular $\deg C \leq n-1$.

Take the three solutions z,ξ,η of $f(s)=x$ with $z\to 0$ and $\xi,\eta\to 1$ as $x\to 0$. The two values of u near 1 have difference of order $N+1/2$ in x. Indeed, in the parameter t we have $x=A(t^2)+t^{2N+1}E(t^2)$ with $A'(0)E(0)\neq 0$. At fixed x the two parameter values satisfy $t_2/t_1\to -1$. Subtracting the expressions for x gives $t_1^2-t_2^2\sim -2E(0)t_1^{2N+1}/A'(0)$, as required. As in Lemma 2.2,

$$D(z)=\frac{C(\xi)B(\eta)-C(\eta)B(\xi)}{\xi-\eta}$$

is a polynomial of degree at most N. Since $\xi-\eta$ has order $1/2$ in x, $B(1)\neq 0$ and $z\sim x$, we obtain $D(z)=c_0z^N$ with $c_0\neq 0$. Multiply the numerator identity by $C(z)$ and sum cyclically over z,ξ,η. The left side cancels. Using $f'(z)(\xi-\eta)=f'(\xi)(\eta-z)=f'(\eta)(z-\xi)$ on the right gives

$$\sum_{f(s)=x}\frac{s^NC(s)}{f'(s)}=0.$$

Thus C belongs to the space V_n of Lemma 2.2 with complex coefficients and vanishes at 1. Its basis polynomials are coprime, so this condition defines a one-dimensional subspace. The proof of Corollary 2.3 gives a polynomial of degree n spanning that subspace. This contradicts $\deg C\leq n-1$. Theorem 3.1 gives equality for odd n. □

5. Upper bounds for arbitrary degrees

Theorem 5.1. *Let $F\in\mathbb{Z}[T,X]$ be squarefree over $\mathbb{Q}(T)$, of degree $h\geq 1$ in T and degree $n\geq 2$ in X. Then $e(F)\leq (3n-1)/4$. For $n\geq 3$ we have*

$$e(F)\leq \frac{3n-1}{4}-\frac{n-3}{4h}.$$

A curve with local equation $y^2=z^{j+1}$ has a singularity of type A_j for $j\geq 1$. Let $K(D)$ be the largest j that occurs on a plane curve of total degree D. Gusein-Zade and Nekhoroshev proved $15/28\leq \limsup_{D\to\infty}K(D)/D^2\leq 3/4$ [10]. Orevkov raised the lower bound to $7/12$ [13, §5]. Astashov also constructed examples with prescribed degrees in each variable [1, Theorem 4]. We need an upper bound in terms of those two degrees that also applies when other branches pass through the same point.

Lemma 5.2. *Let $P\in\mathbb{C}[u,v]$ be squarefree, with degrees $a,n\geq 1$ in u,v, respectively. Suppose that in analytic coordinates z,y centred at a point of $P=0$, its equation has a factor y^2-z^k, where $k\geq 2$. Then*

$$k-1\leq\left\lfloor\frac{3an-a-n+1}{2}\right\rfloor.$$

Proof. Make a fractional linear change in each variable, keeping the given point finite. Choose the new fibres at 0 and ∞ in each variable to avoid the singular points, to meet the curve transversely and to have all four of their crossings off the curve. This is possible since a reduced curve has only finitely many exceptional fibres in each projection. After clearing denominators, the two degrees are still a,n. All four corner coefficients of the rectangle $[0,a]\times[0,n]$ are nonzero, and the coefficients of u^a and of v^n have only simple nonzero

roots in the other variable. The polynomial is therefore convenient and nondegenerate at infinity. It follows that it is cohomologically tame [17, p. 199], hence spectrally tame [4, Proposition 3.2.5]. The zero fibre has only isolated singularities because it is reduced. We can thus apply the spectral inequality of [4, Theorem 4.3.1].

For $1 \le i < a$ and $1 \le j < n$ put $w_{ij} = \max\{i/a, j/n\}$. Form a list S containing w_{ij} and $2 - w_{ij}$ for each pair, together with $a + n - 1$ copies of 1. For any irrational $0 < x < 1$, the spectral inequality gives

$$\#\{1 \le j < k : x < 1/2 + j/k < x + 1\} \le \#\{s \in S : x < s < x + 1\},$$

where entries are counted with multiplicity. On the right, S is the spectrum at infinity in the convention with entries in $(0, 2)$, by [12, Theorem 5.16]. The list can also be read directly from [3, Table 1, $e = 0$] by counting the pairs with each given value of $\max\{i/a, j/n\}$.

On the left, $1/2 + j/k$ are the entries for $y^2 - z^k$ [2, p. 252]. If the local equation is $(y^2 - z^k)H$ with $H(0, 0) = 0$, replace H by $H + \varepsilon$. For small $\varepsilon \ne 0$ the equation at the origin has just the indicated factor. Local semicontinuity [4, Theorem 4.1.1] therefore bounds its contribution by that of the original equation.

Take $x = 1/2 + \eta$ irrational, with $\eta > 0$ smaller than $1/k$ and than every nonzero $|w_{ij} - 1/2|$. All $k - 1$ local entries lie in $(x, x + 1)$. The global list has $2an - a - n + 1$ entries. By its symmetry about 1, we must subtract one entry for each $w_{ij} \le 1/2$ and one for each $w_{ij} < 1/2$. The total number subtracted is

$$\left\lfloor \frac{a}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{a-1}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor = \left\lceil \frac{(a-1)(n-1)}{2} \right\rceil .$$

This gives the asserted bound. □

Proof of Theorem 5.1. Suppose $e(F) > 0$. Use the polynomial $P(u, v)$ of Section 4, with $u = 1/T$ and $v = 1/(X - c)$. Its degrees are $1 \le a \le h$ and n, and its roots are bounded at $u = 0$. Let M be the largest order of their pairwise differences. Then $he(F) \le M$. For $n = 2$, the discriminant bound (3) gives $M \le a$ and hence $e(F) \le 1$.

Let $n \ge 3$. If $2M \le a(n - 1)$, then $4M \le 2a(n - 1) \le 3an - a - n + 3$. Otherwise (3) shows that exactly one unordered pair attains M. Continuation around $u = 0$ preserves this pair. Its mean $m(u)$ and its squared difference are therefore convergent power series in u. Write the latter as $4u^k W(u)$ with $k = 2M$ an integer and $W(0) \ne 0$. Taking a local square root of W, the coordinates $z = u$ and $y = (v - m(u))/\sqrt{W(u)}$ give the factor $y^2 - z^k$. Lemma 5.2 gives $4M \le 3an - a - n + 3$ in this case as well. Since $he(F) \le M$ and $a \le h$, the result follows. □

For roots in different factors, the resultant bounds the order of their difference by the total number of intersections of the two curves, counted with multiplicity. When the order s exceeds $hn/2$, a small translation splits the contact into s simple intersections. A second bound follows by enclosing these intersections in a single disc and applying a theorem of Ruberman.

Theorem 5.3. *Let $F \in \mathbb{Z}[T, X]$ be squarefree over $\mathbb{Q}(T)$, of degree $h \ge 1$ in T and $n \ge 2$ in X. If two roots belong to different irreducible factors over $\mathbb{C}(T)$, the order of their difference in $u = 1/T$ is strictly less than $2hn/3$. For $n \ge 3$ it is at most $2hn/3 - 1$.*

Proof. We may assume that the order is positive. Pass to the bounded coordinate v of Section 4, and denote the order of the transformed difference by s. It is at least the original order. Let p, q be the degrees of the two factors A, B in u, and let m, ℓ be their degrees in v, respectively. Put

$$I = p\ell + qm, \qquad J = pm + q\ell.$$

Their resultant has degree at most I in u. Its order at zero is the sum of the orders of all $m\ell$ differences between a root of A and a root of B, since both leading coefficients are nonzero there. In particular, $s \leq I$ and $I + J = (p+q)(m+\ell) \leq hn$. If $n = 2$, then $m = \ell = 1$ and $I \leq h$. A factor independent of u is linear in v and again $I \leq h$. These cases give both asserted bounds whenever applicable. We may therefore take $p, q \geq 1$ and $n \geq 3$. Then $hn \geq 6$, so $s \leq hn/2$ suffices.

Suppose that $s > hn/2$. Since $2s > I$, the chosen pair is the only one from the two factors whose difference has order at least s. Continuation around zero preserves each factor and hence fixes each root of the pair. Write the two roots as convergent power series $f(u), g(u)$. In particular, s is an integer. Also $I > J$, so we may label the factors so that $p > q$ and $\ell > m$.

We first separate the chosen pair from all the other roots on a disc containing s intersections. Write $f - g = u^s E(u)$ with $E(0) \neq 0$. Choosing a holomorphic s-th root of E near zero gives a local coordinate $z = uE(u)^{1/s}$ in which $f - g = z^s$. Translate the curve B by $\varepsilon > 0$ in the v-coordinate. The graphs of f and $g + \varepsilon$ now meet at the s solutions of $z^s = \varepsilon$. Put $\rho = \varepsilon^{1/s}$. In the annulus $\rho/2 < |z| < 2\rho$, these two graphs are at distance at most $(2^s + 1)\varepsilon$ in each fibre. Every other root of either factor is farther away. Indeed, before translation its difference from f has order $\lambda < s$. For a root of B this follows from uniqueness of the pair. For another root of A apply uniqueness to its difference from g and use $f - g = z^s$. Its distance from f on the annulus is therefore $\gg \rho^\lambda$, uniformly on its Puiseux branches. As $\varepsilon \to 0$, this is arbitrarily large compared with ε, even after the translation.

Choose ε sufficiently small. Cut the annulus along a ray between two successive solutions and take a closed disc D with smooth boundary in the cut annulus containing all s solutions in its interior. Over D the tube $|v - f| \leq (2^s + 2)\varepsilon$ contains just the two chosen graphs, one from each curve. We can now perturb the coefficients of both equations slightly and generically, keeping their bidegrees. In $\mathbf{P}^1(\mathbb{C}) \times \mathbf{P}^1(\mathbb{C})$ the resulting curves are smooth, connected and transverse to each other by Bertini's theorem for the two systems of positive bidegree. Inside the tube each still has one simple root over every point of D. Their difference stays nonzero on ∂D and, by Rouché's theorem, still has s zeros in D. There are I intersections in all, by the intersection formula for curves of the two bidegrees.

Reverse the orientation of the second curve. Remove the two graph discs over D and join their boundaries by the straight segment between the two roots in each fibre over ∂D. These segments form an embedded annulus. The tube is free of other sheets, and the two boundary orientations agree with this gluing. The surface is now connected. At each of the remaining $I - s$ transverse intersections take local complex coordinates x, y in which the original sheets are the axes and the second is the y-axis. After its orientation is reversed, the local smoothing $x\overline{y} = \delta$, for small $\delta \neq 0$, joins the two sheets by an oriented annulus. Replacing two small discs by this annulus adds one handle to the connected surface. A general curve of bidegree (p, m) projects to the u-line with degree m and $2p(m-1)$ simple branch points, counted by its homogeneous discriminant. Riemann–Hurwitz gives its genus $(p-1)(m-1)$, and likewise $(q-1)(\ell-1)$ for the second curve. Thus the resulting closed embedded oriented surface has genus

$$(p-1)(m-1) + (q-1)(\ell-1) + I - s.$$

The first replacement takes place in the contractible tube over D, and each later one in a small ball. Thus the projection degrees of the resulting surface are the differences $m - \ell$ and $p - q$. By Ruberman's theorem [16, Corollary 1.3], an embedded surface with nonzero projection degrees i, j has genus at least $(|i| - 1)(|j| - 1)$. Consequently

$$(p - q - 1)(\ell - m - 1) \leq (p-1)(m-1) + (q-1)(\ell-1) + I - s.$$

This gives $s \le 2J - 2m - 2q + 1 \le 2J - 3$. For example, when $m = q = 1$ the two bounds for a contact of order $s > hn/2$ are $p\ell + 1$ and $2(p + \ell) - 3$. Adding $2s \le 2I$ to $s \le 2J - 3$ yields $3s \le 2(I + J) - 3 \le 2hn - 3$, as required. □

Proposition 5.4. *Let F be as in Theorem 5.1. Suppose two roots belonging to distinct irreducible factors over $\mathbb{C}(T)$ have difference of order greater than $hn/2$ in $u = 1/T$. Both factors can then be chosen over $\mathbb{Q}(T)$, and both roots have convergent Laurent expansions in u with rational coefficients. Consequently a pair in distinct factors with at least one root nonreal for arbitrarily large positive T has difference of order at most $hn/2$.*

If F is irreducible over $\mathbb{Q}(T)$ and has $r \ge 2$ irreducible factors over $\mathbb{C}(T)$, then $e(F) \le 2n/r^2$.

Proof. Use the bounded coordinate v of Section 4 and let $a \le h$ be the parameter degree after removing the common factor of the coefficients. The transformed pair has order $M > hn/2$. The discriminant bound (3) gives a total order at most $a(n - 1) < hn$, so every other pair has smaller order. Continuation around zero fixes the unordered pair and preserves each factor. The two roots therefore have ordinary power series in u.

The coefficients of these series are algebraic numbers by the Newton–Puiseux theorem [15, §3.1]. Every conjugation over $\mathbb{Q}$ preserves the order of each difference and thus fixes the unordered pair. If a conjugation interchanged its roots, the two factors would have the same degrees p, d in u, v. Their resultant would give $M \le 2pd \le an/2$, since $2p \le a$ and $2d \le n$. Thus each root and its factor are fixed. The coefficients of the series and of the factors, after one coefficient of each factor is scaled to one, are rational. The formula $X = c + 1/v$ gives the asserted Laurent expansions, which take real values for all sufficiently large positive T.

For the last assertion the r factors are conjugate over $\mathbb{Q}$, as in the proof of Theorem 4.3, so each has degrees $a/r, n/r$. For roots in different factors the resultant bounds the order by $2an/r^2$. For roots of one factor (3) bounds it by $(a/r)(n/r - 1)$. Division by h gives the result. □

6. From one degree to all large degrees

Orevkov used power substitutions and a linear change of variables to produce A_j-singularities on plane curves of increasing total degree [13, Proposition 5.2]. Here we increase the degree in X while keeping the degree in T fixed. The order of a single root difference gives a lower bound in the new degrees.

Proposition 6.1. *Let $F \in \mathbb{Z}[T, X]$ be squarefree over $\mathbb{Q}(T)$, with $\deg_X F = b \ge 2$ and $e(F) > b/2$. For every sufficiently large n there is a reducible squarefree family of degree n in X and degree b in T with exponent at least*

$$\frac{e(F)}{b}\, n - O_F(1).$$

If F is irreducible over $\mathbb{Q}(T)$, the same estimate holds for absolutely irreducible families of degree b in T along an infinite arithmetic progression of degrees n. If $e(F) \ge 2b/3$, then F has an absolutely irreducible factor $A \in \mathbb{Z}[T, X]$ with $e(A) \ge e(F)$.

Proof. Put $h = \deg_T F$ and use the bounded coordinate of Section 4. The resulting polynomial $G(u, v)$ has degrees a, b, where $1 \le a \le h$, and its leading coefficient in v is nonzero at $u = 0$. Its coefficients in v have no common zero in u, since the original equation was divided by the common factor of its coefficients. Choose a pair of roots of F whose difference has order $he(F)$, and denote the corresponding roots of G by v_1, v_2. Then

$$M = \operatorname{ord}_u(v_1 - v_2) \ge he(F) > ab/2.$$

By (3), the sum of the orders of all pairs is at most $a(b-1)$. Every other pair therefore has order strictly less than M. Applying any conjugation over $\mathbb{Q}$ to the algebraic coefficients of the Puiseux expansions preserves all orders and thus fixes the unordered pair $\{v_1, v_2\}$. In particular its common value θ at $u = 0$ is rational. Continuation around $u = 0$ also fixes the pair, so it either fixes or interchanges its two members. Thus v_1 and v_2 have power series in $u^{1/2}$.

If v_1, v_2 belong to the same absolutely irreducible factor, every conjugation fixes that factor once a coefficient is normalized to one. Its coefficients are algebraic, as in the proof of Theorem 4.3, and hence rational. When $e(F) \geq 2b/3$, Theorem 5.3 shows that the two original roots do belong to the same factor. Its primitive equation $A \in \mathbb{Z}[T, X]$ divides F, and $1 \leq \deg_T A \leq h$. The two roots give $e(A) \geq he(F)/\deg_T A \geq e(F)$.

Choose $\lambda \in \mathbb{Q} \setminus \{0\}$ such that $G(\lambda, V)$ has degree b and has only simple roots. There are only finitely many excluded values. Put

$$\widetilde{G}(U, V) = (1 + U)^a G\Big(\lambda U/(1 + U), V\Big).$$

For any even integer $r > b$ with $r \geq 4$, set

$$\phi(x) = \frac{\lambda x^r}{1 + x^r}, \qquad P_r(T, X) = T^b \widetilde{G}(X^r, \theta + 1/T - X).$$

After choosing a square root of $\lambda/(1+x^r)$ near zero, the two functions $f_i(x) = x+v_i(\phi(x))-\theta$ have convergent power series $x + O(x^2)$. Let $x_i(y)$ be their local inverses. Then $x_i(y) \sim y$, and

$$\begin{aligned} x_1(y) - x_2(y) &\sim f_2(x_1(y)) - f_2(x_2(y)) \\ &= (v_2 - v_1)\Big(\phi(x_1(y))\Big). \end{aligned}$$

The last expression has order rM in y. Taking $y = 1/t$ gives two roots of $P_r(t, X)$ at distance $\asymp_{F,r} t^{-rM}$.

To compute the degrees, put $H_r(X, S) = \widetilde{G}(X^r, \theta + S - X)$. The coefficient of U^a in $\widetilde{G}(U, V)$ is $G(\lambda, V)$, of degree b. The highest power of X in H_r is therefore X^{ar+b}, with nonzero constant coefficient. In particular $H_r(X, 0) \neq 0$. Also $\deg_S H_r = b$, by the substitution for V. It follows that P_r has degrees b in T and $ar + b$ in X, and that $P_r(0, X) \neq 0$.

The polynomial $\widetilde{G}(X^r, V)$ is squarefree over $\mathbb{C}(X)$ because the discriminant of $G(z, V)$ is nonzero at the transcendental element $z = \phi(X)$. Its coefficients have no common zero in X. For $1 + X^r \neq 0$ this follows from the corresponding property of G. At $X^r = -1$ the polynomial specializes to $(-\lambda)^a$ times the nonzero coefficient of u^a in $G(u, V)$. Thus it is squarefree in $\mathbb{C}[X, V]$. The substitution $V = \theta + S - X$ is invertible, and replacing S by $1/T$ preserves squarefreeness after denominators are cleared since $P_r(0, X) \neq 0$. After multiplication by a nonzero integer, P_r belongs to $\mathbb{Z}[T, X]$ and has exponent at least rM/b.

Given a sufficiently large n, take the largest even r for which $ar + b < n$. Then $k = n - ar - b$ satisfies $1 \leq k \leq 2a$. Multiplication by k distinct factors $X - c_i$, with integers c_i chosen outside the constant roots of P_r, gives a reducible squarefree family of the required degrees. Its exponent is at least $rM/b \geq (n - b - k)e(F)/b$.

Finally suppose F is irreducible over $\mathbb{Q}(T)$. Since $e(F) > b/2$, Proposition 5.4 shows that G is absolutely irreducible. Take a root v of $G(z, V)$ and put $L = \mathbb{C}(z, v)$. At a simple root of $G(\lambda, V)$ the implicit function theorem gives an embedding $L \subset \mathbb{C}((z - \lambda))$. The polynomial $W^r - (\lambda - z)/z$ is Eisenstein at $z - \lambda$, so it is irreducible over $\mathbb{C}((z - \lambda))$ and hence over L. If $w^r = z/(\lambda - z)$, it follows that $[L(w) : L] = [\mathbb{C}(w) : \mathbb{C}(z)] = r$. As $[L : \mathbb{C}(z)] = b$, we obtain $[L(w) : \mathbb{C}(w)] = b$. Thus $\widetilde{G}(w^r, V)$ is irreducible over $\mathbb{C}(w)$. Its coefficients have no common zero, so it is irreducible in $\mathbb{C}[w, V]$. The same changes of

variables as above now give the absolute irreducibility of P_r. Taking $n = ar + b$ proves the estimate along the stated progression. □

Theorem 6.2. *We have*

$$\lim_{n\to\infty} \frac{E(n)}{n} = \lim_{n\to\infty} \frac{E_{\mathrm{red}}(n)}{n} = \limsup_{n\to\infty} \frac{E_{\mathrm{irr}}(n)}{n} = \sup_{d\geq 2} \frac{E(d)}{d}. \tag{4}$$

The common value belongs to $[2/3, 3/4]$. *It exceeds* $2/3$ *if and only if some absolutely irreducible family* $A \in \mathbb{Z}[T, X]$ *satisfies* $e(A) > 2\deg_X(A)/3$. *Moreover,* $E(n) \leq 2n/3 + O(1)$ *as* $n \to \infty$ *if and only if* $E(d) \leq 2d/3$ *for every* $d \geq 2$.

Proof. Write $C = \sup_{d\geq 2} E(d)/d$. Since $E(3)/3 = 2/3$ and Theorem 5.1 gives $E(d)/d \leq 3/4$, we have $2/3 \leq C \leq 3/4$. If $1/2 < c < C$, choose a family F of some degree d with $e(F) > cd$. Proposition 6.1 gives $\liminf_{n\to\infty} E_{\mathrm{red}}(n)/n \geq e(F)/d > c$. Since $E_{\mathrm{red}}(n) \leq E(n) \leq Cn$, the first two limits equal C.

The absolutely irreducible examples of Theorem 2.1 give $\limsup_{n\to\infty} E_{\mathrm{irr}}(n)/n \geq 2/3$. If $C > 2/3$, choose $2/3 < c < C$ and a family F of degree d with $e(F) > cd$. Proposition 6.1 gives an absolutely irreducible factor A with $e(A) > cd \geq c\deg_X A$. Applying the same proposition to A gives absolutely irreducible families in infinitely many degrees with $\limsup_{n\to\infty} E_{\mathrm{irr}}(n)/n \geq e(A)/\deg_X A > c$. Letting c tend to C proves (4). Conversely, every absolutely irreducible family with $e(A) > 2\deg_X(A)/3$ gives $C > 2/3$ by definition.

If $E(n) \leq 2n/3 + O(1)$, the first limit in (4) gives $C \leq 2/3$. The definition of C then gives $E(d) \leq 2d/3$ for all $d \geq 2$, and the converse follows at once. □

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