A RESIDUE OBSTRUCTION FOR THE WHOLE FAMILY F_2 IN THE PLANE JACOBIAN PROBLEM

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Abstract. In the classification of Guccione, Guccione, Horruitiner and Valqui of the possible shapes of a counterexample to the plane Jacobian conjecture, the family F_2 consists of the standard (m,n)-pairs with $n=2m-1$, $m\geq 2$, starting corner $(5,20)$ and final corner $(7/5,2)$; its degrees are $(25m, 25(2m-1))$, that is $(50,75)$, $(75,125)$, $(100,175)$, ... We show that no such pair exists, for every $m\geq 2$. (Other shapes with the same degrees, which occur for $(50,75)$ and $(125,225)$, are not covered.) This extends to the whole family the argument given for $(m,n)=(3,5)$ in [4]: on a generic fibre of P the form $y^2Q\,dQ$ has no residue along the left side of the Newton polygon, while its residues along the right side add up to $-\frac{2m}{2m-1}\sum G^2$, which cannot vanish. All statements are proved relative to the cited results of Guccione–Guccione–Valqui; the exact computations are reproduced by an ancillary script. The argument has not yet been examined by specialists.

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1. Statement

Throughout, K is an algebraically closed field of characteristic zero, $L=K[x,y]$, $[P,Q]=P_xQ_y-P_yQ_x$, and $m\geq 2$, $n:=2m-1$. We use the notation of [1] (Guccione–Guccione–Valqui): for a weight (ρ,σ), $v_{\rho,\sigma}$ is the weighted degree, $\ell_{\rho,\sigma}$ the leading form, and $\mathrm{st}_{\rho,\sigma}$, $\mathrm{en}_{\rho,\sigma}$ its end points. *Standard (m,n)-pairs* and their *starting triple* are defined in [1, Definitions 4.3, 6.2].

Standing data 1.1. (P,Q) is a standard (m,n)-pair in L with $[P,Q]=J\in K^\times$, $\deg P=25m$, $\deg Q=25n$, starting triple $\big((5,20),(1,0),(5,-1)\big)$, and $(-1,1)<\mathrm{Succ}_P(1,0)<(-1,0)$ as in [1, Corollary 5.21(4)]. Moreover the polynomial h of (D1) below has a root of multiplicity exactly two.

The last condition says that the pair belongs to the family F_2 of [3, Section 5]: a root of multiplicity k generates the corner $(1,0)+k(1/5,1)$ [1, Proposition 5.18], and F_2 is the family with final corner $(7/5,2)$. For $(m,n)=(3,5)$ every pair with this starting triple is of this kind [4]. In [3] the shapes are listed up to exchanging P and Q; here $\deg P<\deg Q$, which is the orientation of F_2.

Theorem 1.2. *For every $m\geq 2$ there is no pair (P,Q) as in Standing data 1.1.*

Consequence of the standing data used below. Put $t:=xy^5$. Rescaling x, P, Q we may assume:

(D1) The support of P lies in $\{(i,j): 5i-j\leq 5m\}$, that of Q in $\{5i-j\leq 5n\}$, and

$$\ell_{5,-1}(P)=\big(x\,h(t)\big)^m,\qquad \ell_{5,-1}(Q)=\mu\big(x\,h(t)\big)^n,\qquad \mu\in K^\times,$$

with $h\in K[t]$ monic of degree 4, $h(0)\neq 0$; thus $\mathrm{st}_{5,-1}(P)=(m,0)$ and $\mathrm{en}_{5,-1}(P)=(5m,20m)$ [1, Proposition 6.1, Theorem 7.6(2)]. We write $u(t):=t\,h(t)$.

By the standing data, h is either $(t-c)^2(t-r_1)(t-r_2)$ with c,r_1,r_2 distinct, or $(t-c_1)^2(t-c_2)^2$ with $c_1\neq c_2$.

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2. Homogeneous brackets

Let $N \geq 1$ be an integer and $R_\omega := K[\tau][\omega, \omega^{-1}]$, with the bracket $\{f, g\}_N := \omega^N(f_\tau g_\omega - f_\omega g_\tau)$. For $s \in \mathbb{Q}_{>0}$ put $\nu_s(\tau^a \omega^{-d}) := d - sa$, let $\nu_s(f)$ be the maximum over the monomials of f and $\ell_s(f)$ the sum of the terms realising it. An element with a single ν_s-value a has the form $\omega^{-a} F(\sigma)$ with $\sigma := \tau\omega^{-s}$ and $F \in K[\sigma]$.

Lemma 2.1. *(a)* $\{\omega^{-a}F(\sigma), \omega^{-b}G(\sigma)\}_N = \omega^{N-a-b-s-1}\big(aFG' - bF'G\big)(\sigma)$.
(b) $\nu_s(\{f, g\}_N) \leq \nu_s(f) + \nu_s(g) + s + 1 - N$, *with equality if and only if* $\{\ell_s f, \ell_s g\}_N \neq 0$.
(c) If $\{f, g\}_N = c \in K^\times$, *then* $\Sigma_s := \nu_s(f) + \nu_s(g) + s + 1 - N \geq 0$ *for every* s; *if* $\Sigma_s > 0$ *then* $aFG' = bF'G$ *for the leading forms, and if* $\Sigma_s = 0$ *then* $aFG' - bF'G = c$.

Proof. (a) is a direct computation; (b) and (c) follow from (a) by comparing ν_s-values. □

The plane enters through $(\tau, W) := (t - \theta,\ y)$ for $\theta \in K$: since $dx \wedge dy = y^{-5}\, dt \wedge dy$, one has $[f, g] = \{f, g\}_5$ with $\omega = W$, and $P = \sum_{d \leq 5m} f_d(t)\, W^{-d}$ with $f_{5m} = u^m$, $Q = \sum_{d \leq 5n} g_d(t) W^{-d}$ with $g_{5n} = \mu u^n$.

3. The left side

Lemma 3.1. *There is* $g \in vK[v]$ *with* $\deg g \leq 3$ *and* $v^2 \mid g$ *such that*

$$\hat{P}(z, v) := P\big(zv^4 + g(v),\ v^{-1}\big), \qquad \hat{Q}(z, v) := Q\big(zv^4 + g(v),\ v^{-1}\big)$$

are polynomials in z, v. *Moreover* $\hat{P}$ *has total degree* $5m$, *top form* H^m *with* $H(z, v) := v^5 u(z/v)$, $\hat{P}(z, 0)$ *has degree* $5m$ *in* z, *and*

$$\hat{P}_z \hat{Q}_v - \hat{P}_v \hat{Q}_z = -J\, v^2.$$

Proof. Apply the flip ψ_1 of [1, Section 8], so that the weight $(5, -1)$ becomes $(\rho_0, \sigma_0) = (-1, 5)$, with $\mathrm{st}_{\rho_0,\sigma_0} = (20m, 5m)$ and $\mathrm{en}_{\rho_0,\sigma_0} = (0, m)$ for the flipped P. Let F be the element of [1, Theorem 2.6] for (ρ_0, σ_0). As in the proof of [1, Proposition 8.1]: if $\mathrm{st}(F) = (1, 1)$ then $\mathrm{en}(F) \neq (1, 1)$, since F is not a monomial [1, Remark 2.5], so $\mathrm{en}(F) \sim \mathrm{en}(P) \sim (0, 1)$ and $v_{-1,5}(F) = 4$ would give $\mathrm{en}(F) = (0, 4/5)$, which is impossible; hence $\mathrm{st}(F) \sim (20, 5)$, so $\mathrm{st}(F) = (16, 4) = \frac{4}{5} \cdot \frac{1}{m}\,\mathrm{st}(P)$ and $q = 5$ in [1, Corollary 7.4]. That corollary applies to the flipped pair and to every pair obtained from it by a translation of the second variable by a Laurent monomial in the first (the form ℓ_{ρ_0,σ_0}, the degrees $v_{1,1}$, $v_{0,1}$ and F are unchanged). It yields: for every weight between $(1, -4)$ and (ρ_0, σ_0) at which the pair has an edge, the edge form of P is a power R^{5m}. In the original variables: for a weight $(-s, 1)$ with $s < 4$ carrying an edge with end point $(5m, 20m)$, the form is $(xy^4 - \gamma y^{4-s})^{5m}$, so $s \in \mathbb{Z}$, and $s \geq 2$ by the condition on $\mathrm{Succ}_P(1, 0)$. The translation $x \mapsto x + \gamma y^{-s}$ removes this edge and leaves the forms of smaller s unchanged [1, Proposition 3.9]. After at most two such translations ($s = 2, 3$) no edge of weight $(-s, 1)$, $s < 4$, has end point $(5m, 20m)$. Since the weight $(-4, 1)$ lies within a half-turn of $(5, -1)$, the point $(5m, 20m)$ is then still leading for $(-4, 1)$ [1, Remark 2.10, Proposition 3.7]; hence every monomial $x^i y^j$ of the translated P satisfies $j \leq 4i$, that is, $x^i y^j = z^i v^{4i-j}$ is a monomial in $z = xy^4$, $v = y^{-1}$. The same holds for Q: at a weight where P has an edge, [1, Proposition 7.3] with $T_0 = Q$ gives $[\ell(Q), \ell(P)] = 0$ and proportional values, so $\ell(Q) \sim R^{5n}$ and the same translation applies; at the remaining weights $(-s, 1)$, $s < 4$, the values of P and Q are positive, so the leading forms commute [1, Proposition 1.13] and $\ell(Q)$ is the monomial at $(5n, 20n)$; hence $j \leq 4i$ on the support of Q by continuity. The degree statements follow from $5i - j \leq 5m$ in (D1), and the Jacobian from $dx \wedge dy = -v^2\, dz \wedge dv$. □

4. Places at infinity of a fibre

Fix a generic $p \in K$ and let $\Gamma_p := \{\hat{P} = p\} \subset \mathbb{A}^2_{z,v}$, with $\bar{\Gamma}_p$ the normalisation of its projective closure. Since $H(1, 0) = 1$, every place π of $\bar{\Gamma}_p$ at infinity satisfies $v \to \infty$ and $z/v \to \theta$ with $u(\theta) = 0$; then $y = v^{-1} \to 0$ and $t = xy^5 \to \theta$. So π is a branch, at $(\tau, W) = (0, 0)$, of $P - p$ regarded in $K[\tau][W, W^{-1}]$, $\tau = t - \theta$.

Lemma 4.1. *Let $P, Q \in R_\omega$ with $\{P,Q\}_N = c \in K^\times$, let (N,δ) be one of $(5,5)$, $(5,4)$, $(5,3)$, $(9,9)$, $(9,7)$, put $D := \delta m$, and let $0 \le s_* < \delta$. Assume that for all s in some interval $(s_*, s_*+\varepsilon)$ one has $\ell_s(P) = A'\tau^m\omega^{-D}$ and $\ell_s(Q) = B'\tau^n\omega^{-\delta n}$ with $A'B' \neq 0$. Then $\delta = N$ (so the hypotheses cannot hold for the other three pairs), and there is $\kappa \in K[\omega]$ with $\operatorname{ord}_\omega \kappa > s_*$ such that, for $\tau = \kappa(\omega) + \sigma\omega^\delta$, both P and Q are polynomials in σ, ω. Every branch $\tau(\omega)$ of $P = p$ with $\operatorname{ord}_\omega \tau > s_*$ has σ bounded; in particular Q is regular along it.*

Proof. Let s_0 be the supremum of the $s > s_*$ with $\ell_s(P) = A'\tau^m\omega^{-D}$. It is finite, since otherwise $\tau^m \mid P$ and $\{P,Q\}_N$ would vanish at $\tau = 0$. Suppose $s_0 < \delta$ and write $\ell_{s_0}(P) = \omega^{-a}F_P$, $\ell_{s_0}(Q) = \omega^{-b}F_Q$; then $a = m(\delta - s_0) > 0$, $\deg F_P = m$, F_P has at least two terms, $b \ge n(\delta - s_0) = \frac{n}{m}a$ and $\deg F_Q \le n$ (a monomial of P of τ-degree $> m$, or of Q of τ-degree $> n$, is not leading for any $s > s_*$). By Lemma 2.1(c), either F_P^b is proportional to F_Q^a, or $\Sigma_{s_0} = 0$ and $aF_PF_Q' - bF_P'F_Q = c$.

In the first case degrees give $b = \frac{n}{m}a$ and $F_P^n \sim F_Q^m$; as $\gcd(m,n) = 1$, every root of F_P has multiplicity divisible by m, so $F_P = A'(\sigma - \kappa_0)^m$ with $\kappa_0 \neq 0$ and $s_0 \in \mathbb{Z}$; the translation $\tau \mapsto \tau + \kappa_0\omega^{s_0}$ preserves the bracket and the hypotheses, with s_* replaced by s_0.

The second case is impossible. Put $\eta := \delta - s_0 > 0$. From $\Sigma_{s_0} = 0$ and $b \ge \frac{n}{m}a$ we get $N - 1 - s_0 = a + b \ge (m+n)\eta = (3m-1)\eta$, that is $(3m-2)\eta \le N - 1 - \delta$. For $(N,\delta) = (5,5), (5,4), (9,9)$ the right side is ≤ 0, a contradiction. For $(5,3)$ and $(9,7)$ it equals 1, so $0 < \eta \le \frac{1}{3m-2}$. The equation $aF_PF_Q' - bF_P'F_Q = c$ has a nonzero constant term, which requires a monomial $\tau\omega^{-(a+s_0)}$ in P or a monomial ω^{-a} in P; hence $a + s_0 = \delta + (m-1)\eta \in \mathbb{Z}$ or $a = m\eta \in \mathbb{Z}$. Both are impossible, because $0 < (m-1)\eta < m\eta \le \frac{m}{3m-2} < 1$.

Hence after finitely many translations $s_0 \ge \delta$, so $\nu_\delta(P) = 0$ and $P \in K[\sigma,\omega]$ with $P(\sigma,0)$ of degree $m \ge 2$. Let $b := \nu_\delta(Q) \ge 0$. If $b > 0$, Lemma 2.1(c) gives $F_P'F_Q = 0$ or $-bF_P'F_Q = c$, both impossible; so $b = 0$ and $Q \in K[\sigma,\omega]$. Now $dP \wedge dQ = c\,\omega^{-N}d\tau \wedge d\omega = c\,\omega^{\delta-N}d\sigma \wedge d\omega$ is regular, so $\delta \ge N$; among the five pairs this leaves only $\delta = N$. Finally, for every s with $s_* < s < \delta$ other than the finitely many slopes removed by the translations, $\ell_s(P - p)$ is a monomial, so no root $\tau(\omega)$ of $P - p$ has order s; at a removed slope s_0 the roots of order s_0 are those with $\sigma \to \kappa_0$, and the translation moves them to higher order. Hence every root with $\operatorname{ord}\tau > s_*$ has order $\ge \delta$ in the final coordinate, that is, σ is bounded. □

Corollary 4.2. *At every place of $\bar\Gamma_p$ at infinity lying over a simple root θ of u (including $\theta = 0$), the function $\hat Q$ is regular.*

Proof. For small s, $\ell_s(P) = u'(\theta)^m\tau^mW^{-5m}$ and $\ell_s(Q) = \mu u'(\theta)^n\tau^nW^{-5n}$; apply Lemma 4.1 with $(N,\delta) = (5,5)$, $s_* = 0$. □

Lemma 4.3. *Let $\theta = c$ be a double root of h and $u(c+\tau) = U\tau^2 + O(\tau^3)$, $U \neq 0$. Put $s^* := \frac{5m-3}{2m-1}$. Then exactly one of the following holds.*

(i) $\hat Q$ is regular at every place of $\bar\Gamma_p$ over c.

(ii) After a translation $\tau \mapsto \tau + \kappa_1 W + \kappa_2 W^2$, with $\sigma = \tau W^{-s^*}$,

$$\ell_{s^*}(P) = W^{-m/n}\big(B\sigma + U^m\sigma^{2m}\big),$$
$$\ell_{s^*}(Q) = W^{-1}\,G\Big(1 + \tfrac{2m}{m-1}\varsigma + \tfrac{2m^2}{(m-1)(2m-1)}\varsigma^2\Big), \qquad \varsigma := \tfrac{U^m}{B}\sigma^n,$$

with $BG = -J$ and $G = \mu(m-1)nB^2/(2m^2U)$. There are exactly two places of $\bar\Gamma_p$ over c: a place β_1 with $\operatorname{ord}_{\beta_1}(y) = n$, at which $\hat Q$ has a pole of order n and $y\hat Q \to -G/n$; and a place β_2 with $\operatorname{ord}_{\beta_2}(y) = 1$, at which $\hat Q$ has a simple pole and $y\hat Q \to G$.

In case (ii), *$G^3 = (m-1)n\mu J^2/(2m^2U)$, and κ_1, κ_2, B, G do not depend on p.*

Proof. Work with $f := P - p$ and Q in $K[\tau][W, W^{-1}]$, $N = 5$. For small s, $\ell_s(f) = U^m\tau^{2m}W^{-5m}$, $\ell_s(Q) = \mu U^n\tau^{2n}W^{-5n}$. Let s_0 be the first s at which $\ell_s(f)$ is not a monomial; for generic p the constant term of f is nonzero, also after the translations below (which do not depend on p), so $s_0 \le 5/2$. Put $a = \nu_{s_0}(f) = m(5 - 2s_0)$ and $b = \nu_{s_0}(Q) \ge \frac{n}{m}a$, so that $\Sigma_{s_0} \ge (3m-1)(5-2s_0) + s_0 - 4 = (6m-3)(s^* - s_0)$. Note $2 < s^* < \frac{5}{2}$ and $5 - 2s^* = \frac{1}{n}$.

$s_0 < s^*$. Then $\Sigma_{s_0} > 0$, so $F_P^b \sim F_Q^a$, whence $b = \frac{n}{m}a$, $F_P^n \sim F_Q^m$, and $F_P = U^m\mathcal{G}^m$, $F_Q = \mu U^n\mathcal{G}^n$ with $\mathcal{G}$ monic quadratic, $\mathcal{G} \neq \sigma^2$. If $\mathcal{G} = (\sigma-\kappa)^2$, then $s_0 \in \{1,2\}$ and we translate by κW^{s_0} and start again. If $\mathcal{G}$ has two distinct roots and $s_0 \in \{1,2\}$, translating to either root gives the situation of Lemma 4.1 with $N=5$, $\delta = 5-s_0 \in \{4,3\}$, $s_* = s_0$, which is contradictory. Otherwise $\mathcal{G} = \sigma^2 - \kappa$ with $2s_0 \in \{1,3\}$; with $W = \omega^2$ one has $\{\cdot,\cdot\}_5 = \frac{1}{2}\{\cdot,\cdot\}_9$, and translating to $\tau = \sqrt{\kappa}\,\omega^{2s_0}$ gives the situation of Lemma 4.1 with $N=9$, $\delta = 10-2s_0 \in \{9,7\}$, $s_* = 2s_0$. For $\delta = 7$ this is contradictory; for $\delta = 9$ it gives (i), the $2m$ roots of f with $\tau \to 0$ being those near $\pm\sqrt{\kappa}\,\omega$, exchanged by $\omega \mapsto -\omega$.

$s_0 = s^*$. A lattice point $\tau^{a'}W^{-d}$ on the edge has $(2m-a')\frac{5m-3}{2m-1} \in \mathbb{Z}$; as $\gcd(5m-3, 2m-1) = 1$, this forces $a' = 1$, $d = 3$. So $F_P = B\sigma + U^m\sigma^{2m}$ and $a = m/n$. If $b > 1$, proportionality $F_P^b \sim F_Q^a$ is impossible by degrees; so $b = 1$, $\Sigma_{s_0} = 0$ and $\frac{m}{n}F_PF_Q' - F_P'F_Q = J$, where $F_Q = G + E\sigma^n + D\sigma^{2n}$ (the exponents of σ in F_Q are multiples of n). Its unique solution is

$$BG = -J, \qquad E = \frac{2m\,U^mG}{(m-1)B}, \qquad D = \frac{2m^2\,U^{2m}G}{(m-1)n\,B^2}.$$

Comparing D with the coefficient μU^n of $\tau^{2n}W^{-5n}$ gives $G = \mu(m-1)nB^2/(2m^2U)$, hence $B^3 = -2m^2JU/((m-1)n\mu)$ and the value of G^3. For $s > s^*$ the leading terms are $B\tau W^{-3}$ and GW^{-1}, with $\Sigma_s = 0$; since $\nu_{s^*}(f) = m/n < 1$, f has no monomial W^{-d} with $d \geq 1$, so the next edge of f can only end at the constant term, at $s = 3$, where Lemma 2.1(c) forces $\ell_3(Q) = GW^{-1}$. Thus the roots $\tau(W)$ of f with $\tau \to 0$ are n conjugate roots of order s^*, forming one place β_1 on which $\varsigma \to -1$ and $WQ \to G\big(1 - \frac{2m}{m-1} + \frac{2m^2}{(m-1)(2m-1)}\big) = -G/n$, and one root of order 3, the place β_2, on which $WQ \to G$.

$s^* < s_0 \leq 5/2$. Suppose $s_0 < 5/2$ and put $\eta := 5 - 2s_0 \in (0, \frac{1}{n})$, so $a = m\eta$. Now $\frac{n}{m}a + a + s_0 - 4 < 0$, so $\Sigma_{s_0} \geq 0$ forces $b > \frac{n}{m}a$; then proportionality is impossible by degrees, so $\Sigma_{s_0} = 0$ and the bracket equation has a nonzero constant term, which requires $a \in \mathbb{Z}$ or $a + s_0 \in \mathbb{Z}$. But $0 < a < \frac{m}{n} < 1$, and $a + s_0 = \frac{5}{2} + \frac{n}{2}\eta$ with $0 < n\eta < 1$ is not an integer. For $s_0 = 5/2$ one has $a = 0$ and $\Sigma_{s_0} = b - \frac{3}{2} \geq 0$, and Lemma 2.1(c) gives $F_P'F_Q = 0$ or $b = 3/2$ and $-\frac{3}{2}F_P'F_Q = J$, both impossible since $\deg F_P = 2m$. □

5. Two double roots

Lemma 5.1. *If $h = (t-c_1)^2(t-c_2)^2$ with $c_1 \neq c_2$, then $c_1^2 - 3c_1c_2 + c_2^2 = 0$. In particular $c_1^2 + c_2^2 \neq 0$.*

Proof. In $K(t)((y))$ let $C := uy^{-5}(1+\cdots)$ be the m-th root of P, and recall $[a(t)y^\alpha, b(t)y^\beta] = (\beta a'b - \alpha ab')y^{\alpha+\beta+4}$. From $mC^{m-1}[C,Q] = J$ we get $[C,Q] = \frac{J}{m\,u^{m-1}}y^{5(m-1)}(1+\cdots)$. If $R = Q - \sum \lambda_i C^i$ has leading term $b(t)y^\beta$ with $\beta < 5m-4$, then $\beta u'b + 5ub' = 0$, so $b = \lambda u^{-\beta/5}$; as u has a simple zero at 0 and $b \in K(t)$, $5 \mid \beta$ and the term is removed by subtracting $\lambda C^{-\beta/5}$. Hence there are constants λ_i with $Q - \sum_i \lambda_i C^i = \varphi(t)y^{5m-4} + \cdots$, $\varphi \in K(t)$, and $5u\varphi' + (5m-4)u'\varphi = \frac{J}{m\,u^{m-1}}$. For $\psi := \frac{m}{J}\varphi\,u^m \in K(t)$ this reads

$$5u\psi' - 4u'\psi = u,$$

which does not involve m. A rational solution ψ has no pole: if $\psi \sim c\tau^{-e}$ with $e > 0$ at a root of u of multiplicity k, the left side has the nonzero leading term $-(5e+4k)Uc\tau^{k-e-1}$, of order below that of u; at the other points the equation is regular. It has degree at most 4: if $\psi \sim ct^d$ at infinity with $d > 4$, the left side has the nonzero leading term $(5d-20)ct^{d+4}$. So ψ is a polynomial of degree ≤ 4, and eliminating its coefficients gives $c_1^2 - 3c_1c_2 + c_2^2 = 0$ (ancillary script). If moreover $c_1^2 = -c_2^2$, then $c_1c_2 = 0$, a contradiction. □

6. Proof of Theorem 1.2

On $\bar{\Gamma}_p$ consider the meromorphic forms

$$\omega_1 := v^{-1}\,d\hat{Q}, \qquad \omega_2 := v^{-2}\,\hat{Q}\,d\hat{Q}.$$

Finite points. They are regular where $v \neq 0$. The points of Γ_p with $v = 0$ are the $(z_0, 0)$ with $\hat{P}(z_0, 0) = p$; for generic p, $\hat{P}_z(z_0, 0) \neq 0$, so v is a local parameter and, by Lemma 3.1, $d\hat{Q} = -J\, v^2 \hat{P}_z^{-1}\, dv$ on Γ_p. Hence ω_1, ω_2 are regular at these points.

Places at infinity. There $y = v^{-1}$ vanishes, so $\omega_j = y^j \hat{Q}^{j-1} d\hat{Q}$ is regular wherever $\hat{Q}$ is regular. By Corollary 4.2 and Lemma 4.3, $\hat{Q}$ has poles only at the places β_1, β_2 over the double roots c of h that are in case (ii); there, writing $\omega_2 = (y\hat{Q})^2\, d\hat{Q}/\hat{Q}$,

$$\mathrm{Res}_{\beta_1} \omega_2 + \mathrm{Res}_{\beta_2} \omega_2 = -n\Big(\frac{G}{n}\Big)^2 - G^2 = -\frac{2m}{2m-1}\, G^2,$$

$$\mathrm{Res}_{\beta_1} \omega_1 + \mathrm{Res}_{\beta_2} \omega_1 = -n\Big(-\frac{G}{n}\Big) - G = 0.$$

Conclusion. By the residue theorem, $\sum_c G_c^2 = 0$, the sum being over the double roots in case (ii). If there is no such root, $\hat{Q}$ has no pole on $\bar{\Gamma}_p$ and is constant on each component, whereas dQ has no zero on the affine curve $\{P = p\}$ because $[P, Q] = J \neq 0$. If there is exactly one, $G_c = -J/B \neq 0$, a contradiction. The only remaining possibility is $h = (t - c_1)^2 (t - c_2)^2$ with both roots in case (ii) and $G_{c_1}^2 = -G_{c_2}^2$. Then $G_{c_1}^6 = -G_{c_2}^6$, and Lemma 4.3 gives $U_1^2 = -U_2^2$, where $U_k = c_k(c_1 - c_2)^2$; thus $c_1^2 + c_2^2 = 0$, contradicting Lemma 5.1. □

7. Scope

- **Proved here, relative to the cited results:** Theorem 1.2, for every $m \geq 2$. The inputs are (D1) and [1, Theorem 2.6, Corollary 7.4, Propositions 1.13, 3.7, 3.9, 7.3], as indicated. For $m = 3$ this is the statement of [4]; for $m = 2$ it concerns the part of the case $(50, 75)$ with a double root, a case already discarded in [2], where the reduction step is stated without proof.
- **Verified by exact computation:** the bracket formula; for $2 \leq m \leq 12$, the solution of the terminal bracket equation, the value $-1/n$ and the two residue sums; the closed formulas for general m; the arithmetic facts used in Lemmas 4.1 and 4.3, by exhaustive search for small m; and the elimination in Lemma 5.1 (ancillary file `verify_family.py` in the folder `anc`; SymPy).
- **Not claimed:** a statement on the plane Jacobian conjecture itself, on the pairs with the same starting triple whose polynomial h has no root of multiplicity exactly two, or on the other families of [3]. In particular the degree pairs $(50, 75)$ and $(125, 225)$ also carry a shape of another family (F_3 with a triple root, respectively F_5 with a quadruple root, up to exchanging P and Q), which is not covered; and the two shapes of maximal degree 126 are not covered.
- [1, Proposition 7.3 and Corollary 7.4], on which Lemma 3.1 rests, are stated in [1] with the indication that their proofs mimic those of Proposition 7.1 and Corollary 7.2; they are applied here to Laurent translates, as in [1, Section 8].
- The argument has been checked only by the author with AI assistance (Claude, Anthropic); it has not been validated by professionals.

References

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