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CONVEX, TIME-DECREASING FUNCTIONS NEED NOT PRESERVE SEMIMARTINGALES

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Abstract. We construct a continuous function, strictly convex in space and strictly decreasing in time, whose composition with a bounded continuous martingale is not a semimartingale. We also construct a continuous, spatially convex, time-increasing function that is not a difference of spatially convex, time-nonincreasing functions on a rectangle. The construction combines parabolic oscillations, diffusion superposition and martingale concatenation to obtain infinite drift variation in the composition.

1. Introduction

Theorem 1. *There are a continuous martingale X with $X_0 = 0$ and $|X| \le 2$ and a continuous function $G : \mathbb{R} \times [0,\infty) \to \mathbb{R}$, strictly convex in x and strictly decreasing in t, such that $G(X_t, t)$ is not a semimartingale on $[0,1]$ in the usual natural filtration of X. The derivative G_x is continuous; G is smooth for $t < 1$ and equals $x^2/2 - t$ for $t \ge 1$.*

Theorem 2. *There is a continuous $f : \mathbb{R}^2 \to \mathbb{R}$, strictly convex in x and strictly increasing in t, whose restriction to $Q = (-1,1)^2$ is not $g - h$ for any $g, h : Q \to \mathbb{R}$ convex in x and nonincreasing in t. Moreover, $f - (x^2/2 + t)$ is compactly supported in Q, f_x is continuous, and the distributional derivatives satisfy $f_{xx}, f_t \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$ but $f_t \notin L^\infty(Q)$.*

Theorem 1 answers Lowther's question [1] by completing the construction from [2]. Parabolic oscillations produce smooth, spatially convex pairs F, P with $F_t > 0 > P_t$ and arbitrarily large $\int_Q |P_{xx}F_t + F_{xx}P_t|$. Convex combinations give Theorem 2; diffusion superposition and concatenation turn this divergence into infinite drift variation for Theorem 1.

2. Oscillation

Put $F_0(x,t) = x^2/2 + t$, $P_0(x,t) = x^2/2 - t$ and $K = (-1/2, 1/2)^2$. Integrals over rectangles are with respect to $\mathrm{d}x\,\mathrm{d}t$.

Lemma 3. *There are $F_n, P_n \in C^\infty(\mathbb{R}^2)$, equal to F_0, P_0 off a fixed compact $S \subset (-2/3, 2/3)^2$, with $\|F_n - F_0\|_\infty, \|P_n - P_0\|_\infty \le 1/16$ and positive derivatives*

$$(A_n, B_n, C_n, D_n) = ((F_n)_{xx}, (F_n)_t, (P_n)_{xx}, -(P_n)_t).$$

They satisfy

$$J_n := \int_Q |C_nB_n - A_nD_n| \longrightarrow \infty, \qquad I_n := \int_Q A_nD_n = \int_Q C_nB_n \ge \tfrac{1}{2}J_n, \tag{1}$$

and

$$\int_Q A_n = \int_Q B_n = \int_Q C_n = \int_Q D_n = 4, \qquad \|(F_n)_x - x\|_\infty,\ \|(P_n)_x - x\|_\infty \le 2. \tag{2}$$

Proof. Choose $\chi \in C_c^\infty((-2/3, 2/3)^2)$ with $0 \le \chi \le 1$ and $\chi = 1$ on K, and set $S = \operatorname{supp}\chi$. For the current pair F, P, write $(A, B, C, D) = (F_{xx}, F_t, P_{xx}, -P_t)$ and put

$$a = \frac{\chi}{2(A^{-1} + B^{-1} + C^{-1} + D^{-1})}, \qquad \theta = \lambda x + \lambda^2 t.$$

Replace F, P by $F' = F - \lambda^{-2} a\cos\theta$ and $P' = P - \lambda^{-2} a\sin\theta$. For fixed F, P, uniformly as $\lambda \to \infty$,

$$(A', B', C', D') = (A, B, C, D) + a(\cos\theta, \sin\theta, \sin\theta, \cos\theta) + O(\lambda^{-1}).$$

Since $a \le \frac{1}{2}\min(A, B, C, D)$ and the derivatives equal 1 off S, large frequencies preserve positivity. The increments of $A - D$ and $B - C$ tend uniformly to zero.

Write $\langle w\rangle = \int_K w$ and $E = \langle A^2 + B^2\rangle$. Integration by parts in x, with boundary terms $O(\lambda^{-1})$, gives

$$\Delta(\langle A\rangle, \langle B\rangle, \langle AB\rangle) = o(1), \qquad \Delta E = \langle a^2\rangle + o(1).$$

Starting from F_0, P_0, choose the frequency at step $n \ge 0$ so that the averaging errors, mismatch increments and function perturbations are each at most $\delta 2^{-n-1}$, where $\delta = 1/16$. Then

$$\begin{gathered} \|A_n - D_n\|_\infty,\ \|B_n - C_n\|_\infty \le \delta, \qquad |\langle A_n\rangle - 1|,\ |\langle B_n\rangle - 1|,\ |\langle A_n B_n\rangle - 1| \le \delta, \\ E_n = 2 + \sum_{j<n} \langle a_j^2\rangle + e_n, \qquad |e_n| \le \delta. \end{gathered} \tag{3}$$

On K, the bounds $C_n \ge B_n - \delta$, $D_n \ge A_n - \delta$ imply

$$4a_n(A_n + B_n) \ge (A_n - \delta)_+(B_n - \delta)_+ \ge A_n B_n - \delta(A_n + B_n).$$

Integrating with (3) gives $4\langle a_n(A_n + B_n)\rangle \ge 1 - 3\delta - 2\delta^2 = 206/256$, so Cauchy–Schwarz gives

$$\frac{1}{5} < \langle a_n(A_n + B_n)\rangle \le \left(2E_n\langle a_n^2\rangle\right)^{1/2}.$$

If $\sum_n \langle a_n^2\rangle < \infty$, (3) bounds E_n, but the last inequality bounds each summand away from zero, a contradiction. Thus $E_n \to \infty$. On K, $|C_n B_n - A_n D_n| \ge |B_n^2 - A_n^2| - \delta(A_n + B_n) \ge (A_n - B_n)^2 - \delta(A_n + B_n)$, so

$$J_n \ge E_n - 2\langle A_n B_n\rangle - \delta\langle A_n + B_n\rangle \ge E_n - 3 \longrightarrow \infty.$$

Boundary agreement gives the integral identities in (2); convexity and the slopes ± 1 at $x = \pm 1$ give the slope bounds. Integration by parts gives $\int_Q A_n D_n = \int_Q C_n B_n$; positivity then gives $2I_n \ge J_n$. □

Proof of Theorem 2. Choose increasing indices n_j with $I_{n_j} \ge 4^j$ and set

$$f = F_0 + \sum_{j\ge 1} 2^{-j}(F_{n_j} - F_0) = \sum_{j\ge 1} 2^{-j} F_{n_j}.$$

The perturbation series converges uniformly and is supported in S. Positivity gives strict convexity and strict time increase. The bounds (2) give continuity of f_x and

$$f_{xx} = \sum_{j\ge 1} 2^{-j} A_{n_j}, \qquad f_t = \sum_{j\ge 1} 2^{-j} B_{n_j} \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^2).$$

In particular,

$$\int_Q D_{n_j} f_{xx} \ge 2^{-j} I_{n_j} \ge 2^j. \tag{4}$$

Suppose $f = g - h$ on Q with g, h convex in x and nonincreasing in t. For $u = g, h$, spatial continuity and time monotonicity give Borel measurability and local boundedness: on $[a, b] \times [c, d] \Subset Q$, $u(x, d) \le u(x, t) \le u(x, c)$. Thus $\mu_u = u_{xx}$ and $\nu_u = -u_t$ are positive Radon measures. Choose an open R with $S \subset R \Subset Q$. Since $P_n - P_0 \in C_c^\infty(R)$, integration by parts gives

$$\int_R D_n \,\mathrm{d}\mu_u + \int_R C_n \,\mathrm{d}\nu_u = \mu_u(R) + \nu_u(R).$$

Since $\mu_g = f_{xx}\,\mathrm{d}x\,\mathrm{d}t + \mu_h$, positivity implies $\int_R D_n f_{xx} \le \mu_g(R) + \nu_g(R) < \infty$, uniformly in n. Since $D_n = f_{xx} = 1$ almost everywhere on $Q \setminus R$, this contradicts (4). For fixed n, integration by parts for the partial sums and L^1 convergence give $\int_Q D_n f_{xx} = \int_Q C_n f_t$. As $\int_Q C_n = 4$, (4) also forces $f_t \notin L^\infty(Q)$. □

3. Martingale blocks

Lemma 4. *There are smooth p_n on $[-1,1]\times[0,1]$, equal to P_0 off a fixed compact subset of $(-1,1)\times(0,1)$, and continuous martingales Z^n such that*

$$(p_n)_{zz}>0,\quad (p_n)_s<0,\quad \|(p_n)_z\|_\infty\le 1,\quad \|p_n-P_0\|_\infty\le 1/64,$$
$$Z^n_0=0,\qquad |Z^n|\le 1,\qquad Z^n_1\in\{-1,1\}.$$

The finite-variation part V^n in

$$p_n(Z^n_s,s)=p_n(0,0)+\int_0^s (p_n)_z(Z^n_r,r)\,\mathrm{d}Z^n_r+V^n_s \tag{5}$$

satisfies $\mathbb{E}W_n<\infty$ and $\mathbb{E}W_n\to\infty$, where $W_n=\operatorname{Var}_{[0,1]}V^n$.

Proof. On $\overline{Q}$, set

$$\widehat{A}_n=A_n+7-4t,\qquad \widehat{B}_n=B_n+1-2x^2.$$

The support condition gives $\widehat{A}_n>0$ on $\overline{Q}$, $\widehat{B}_n>0$ in Q and $\widehat{B}_n(\pm1,t)=0$. For $-1\le t\le 1$, define the mean-zero probability laws

$$\mu^n_t=\frac{\widehat{A}_n(x,t)}{48}\mathbf{1}_{(-1,1)}(x)\,\mathrm{d}x+\frac{4+t}{12}(\delta_{-1}+\delta_1)$$

and the coefficient $\mathfrak{a}_n=2\widehat{B}_n/\widehat{A}_n$ on $|x|<1$, zero elsewhere. The laws are narrowly continuous and the coefficient is bounded for each n. Since $\partial_t\widehat{A}_n=\partial_{xx}\widehat{B}_n$, $\widehat{B}_n(\pm1,t)=0$ and $\partial_x\widehat{B}_n(\pm1,t)=\mp4$, integration by parts gives, for $\varphi\in C^\infty_c(\mathbb{R})$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\varphi\,\mathrm{d}\mu^n_t=\frac{1}{48}\int_{-1}^1\varphi\,\partial_{xx}\widehat{B}_n\,\mathrm{d}x+\frac{\varphi(-1)+\varphi(1)}{12}=\frac{1}{48}\int_{-1}^1\widehat{B}_n\varphi''\,\mathrm{d}x,$$

that is,

$$\partial_t\mu^n_t=\tfrac12\partial_{xx}(\mathfrak{a}_n\mu^n_t).$$

Trevisan's superposition theorem [3, Theorem 2.5] gives a continuous solution Y^n of the martingale problem for $\frac12\mathfrak{a}_n\partial_{xx}$ with marginals μ^n_t. The marginal supports imply $|Y^n|\le1$. Smooth compactly supported test functions equal to x and x^2 on $[-1,1]$ show that Y^n is a martingale with $\mathrm{d}[Y^n]_t=\mathfrak{a}_n(Y^n_t,t)\,\mathrm{d}t$. If U^n is the finite-variation part of $P_n(Y^n_t,t)$, Itô's formula gives

$$\mathbb{E}\operatorname{Var}_{[-1,1]}U^n\ge\frac{1}{48}\int_Q|C_n\widehat{B}_n-D_n\widehat{A}_n|\ge\frac{J_n}{48}-1\longrightarrow\infty. \tag{6}$$

Here we discard the nonnegative atomic contribution and use (2).

Set

$$p_n(z,s)=P_0(z,s)+\tfrac14(P_n-P_0)(2z,4s-2).$$

Then $(p_n)_{zz}=C_n$ and $(p_n)_s=-D_n$ at the rescaled point; convexity and $(p_n)_z(\pm1,s)=\pm1$ give the slope bound. On $[1/4,3/4]$, take $Z^n_s=Y^n_{4s-2}/2$. Prepend the continuous Brownian martingale $\mathbb{E}[h(B_1)\mid\mathcal{F}^B_{4s}]$, choosing $h(B_1)$ with the law of $Y^n_{-1}/2$. Given $h(B_1)$, attach Y^n using its regular conditional law given $Y^n_{-1}=2h(B_1)$, independently of the prefix. These conditional laws are martingale laws for almost every initial value, so the join preserves the martingale property. Append the current endpoint plus an independent Brownian motion, stopped at $\{-1,1\}$ with clock $(s-3/4)/(1-s)$. It exits before $s=1$ almost surely. Thus Z^n is a continuous martingale in its natural filtration.

On the middle interval, $p_n(Z^n_s,s)=\frac14P_n(Y^n_{4s-2},4s-2)-\frac12$, so (6) gives $\mathbb{E}W_n\to\infty$. Finally, $\mathbb{E}[Z^n]_1=1$ and smoothness give $\mathbb{E}W_n\le\|(p_n)_s\|_\infty+\frac12\|(p_n)_{zz}\|_\infty<\infty$. □

Proof of Theorem 1. For each $m \geq 1$, choose a block with $\mathbb{E}W > 2 \cdot 4^m$. By the weak law, choose increasing integers r_m so that 4^{r_m} independent copies have sample average exceeding 4^m with probability at least $1 - 2^{-m}$. Take the groups independent, scale group m by $\epsilon = \sqrt{3}\, 2^{-m-r_m}$, and enumerate the copies in increasing group order as $(p^k, Z^k, V^k, W^k, \epsilon_k)$. The scales satisfy

$$\sum_k \epsilon_k^2 = 1, \qquad \epsilon_k/\epsilon_{k+1} \in \mathbb{N}, \qquad \sum_k \epsilon_k^2 W^k = \infty \quad \text{a.s.} \tag{7}$$

Each group has total squared scale $3 \cdot 4^{-m}$. By Borel–Cantelli, all but finitely many groups succeed, each contributing more than 3.

Following [2], put $T_0 = 0$, $T_k = \sum_{j \leq k} \epsilon_j^2$ and concatenate:

$$M_t = \sum_{j<k} \epsilon_j Z_1^j + \epsilon_k Z^k_{(t-T_{k-1})/\epsilon_k^2}, \qquad T_{k-1} \leq t \leq T_k.$$

This is a continuous martingale with $\mathbb{E}M_t^2 \leq 1$. The endpoint series converges almost surely and in L^2, and within-block oscillations are at most $2\epsilon_k \to 0$. Thus M extends continuously to 1, with $\mathbb{E}M_1^2 = 1$; hold it constant thereafter.

Put $\Lambda_k = \sum_{j<k} \epsilon_j + 2\epsilon_k \mathbb{Z}$. By (7), $M_{T_{k-1}} \in \Lambda_k$. On each tile $[a - \epsilon_k, a + \epsilon_k] \times [T_{k-1}, T_k]$, $a \in \Lambda_k$, define

$$G(x,t) = ax - \frac{a^2}{2} - T_{k-1} + \epsilon_k^2 p^k \left(\frac{x-a}{\epsilon_k}, \frac{t - T_{k-1}}{\epsilon_k^2} \right). \tag{8}$$

Near tile boundaries, $G = P_0$, so G is smooth for $t < 1$, with $G_{xx} > 0$ and $G_t < 0$. In block k,

$$|G - P_0| \leq \epsilon_k^2/64, \qquad |G_x - x| \leq 2\epsilon_k.$$

Extend $G = P_0$ for $t \geq 1$. Since $\epsilon_k \to 0$, G, G_x are continuous at $t = 1$, and strict convexity and time decrease persist: $G(x,t) > \lim_{s \uparrow 1} G(x,s) = P_0(x,1)$ for $t < 1$.

Stop M at $\tau = \inf\{t : |M_t| \geq 2\}$ and put $X = M^\tau$. The maximal inequality gives $\mathbb{P}(\tau > 1) \geq 3/4$. In the usual natural filtration of X, define on $[0,1]$

$$\mathcal{A}_t = G(X_t, t) - G(0,0) - \int_0^t G_x(X_s, s)\, \mathrm{d}X_s.$$

The integral is a continuous martingale since G_x is bounded on $[-2,2] \times [0,1]$. Itô's formula gives, on each $[0, T_k]$, $\mathcal{A}_t = \int_0^t G_t(X_s, s)\, \mathrm{d}s + \frac{1}{2} \int_0^t G_{xx}(X_s, s)\, \mathrm{d}[X]_s$. Since $|Z^k| \leq 1$, M stays in its starting tile during block k. On $\{\tau > 1\}$, $X = M$ and $\mathrm{d}[M] = \epsilon_k^2\, \mathrm{d}[Z^k]$ in rescaled time, so (8) and (5) give drift increment $\epsilon_k^2 V^k$ in block k. Consequently,

$$\mathrm{Var}_{[0,T_k]}\, \mathcal{A} = \sum_{j \leq k} \epsilon_j^2 W^j \longrightarrow \infty \quad \text{a.s. on } \{\tau > 1\}.$$

If $G(X_t, t)$ were a semimartingale on $[0,1]$, its continuous finite-variation part, starting at zero, would equal $\mathcal{A}$ on each $[0, T_k]$ by uniqueness. Its variation on $[0,1]$ would dominate all these sums, a contradiction. □

References

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[3] D. Trevisan, *Well-posedness of multidimensional diffusion processes with weakly differentiable coefficients*, Electron. J. Probab. **21** (2016), no. 22, 1–41.

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