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HANOI TILINGS WITH A PRESCRIBED NUMBER OF BOTTOM SQUARES

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Abstract. For each fixed integer $k \geq 2$, every sufficiently large integer square admits a Hanoi tiling with exactly k squares on its bottom edge. We build nearly equal columns from two coprime widths and prove a rectangle lemma that fills the remaining height.

1. A tiling for each sufficiently large square

Theorem 1.1. *For every fixed integer $k \geq 2$, every sufficiently large integer square has a tiling by integer-sided squares with exactly k on the bottom. The entire lower edge of every other tile rests on strictly larger tiles.*

Proof. Take relatively prime squarefree D, E with $\sigma(d) > kd$ for $d = D, E$, where σ is the sum of positive divisors. Indeed, the divergence of $\sum_p 1/p$ lets us select two disjoint finite sets of primes with $\prod_p (1 + 1/p) > k$ in each. Their respective prime products work.

For each d list the divisors $d = t_1 > t_2 > \cdots$. Stop immediately before the first partial sum that reaches kd. Call the sum so far S and its last term t. The deficit $R = kd - S$ satisfies $1 \leq R \leq t_{\text{next}} < t$. The first three terms sum to at most $d + d/2 + d/3 < 2d$, so at least three terms have been included and $t \leq d/3$.

Write $n = w_1 + \cdots + w_k$ with every $w_j = n/k + O_k(1)$. Choose the first $k - 2$ widths as nearest multiples of D. For the penultimate width choose a multiple of D congruent to $n - \sum_{i=1}^{k-2} w_i \pmod{E}$, nearest to n/k. The Chinese remainder theorem bounds this error by DE. The remaining width is a multiple of E and has bounded error too. All widths are positive for large n.

In a column of width $w = dq$, stack full rows with square sides $qt_1, \ldots, qt$. Each side divides w and the first row is a single square. The remaining height is

$$H = n - qS = qR + (n - kw).$$

The parenthesis stays bounded, so eventually $3 \leq H < qt$ and $w/H \to d/R > 3$. A Hanoi tiling of the top w-by-H rectangle exists by the lemma below. Each cap tile resting on the last complete row has side at most $H < qt$. Putting the k columns together proves the theorem. The at-least-one version also follows by taking $k = 2$. □

For a small example, split a 60-square into two columns of width 30. In each, use full rows of sides $30, 15, 10, 5$. Their heights sum to 60 and their respective row counts are $1, 2, 3, 6$.

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Lemma 1.2. *A height $h \geq 3$ and width $w \geq \lceil 5h/2 \rceil$ suffice for a rectangular Hanoi tiling in this orientation.*

Proof. A bottom square of width $s \leq h$ makes a usable strip if $h - s < s$ and the s-wide cap of height $h - s$ is tilable. Height zero needs no tiles, height one admits all positive widths, and height two admits even widths. For $3 \leq h < 100$, the accompanying exact recurrence starts at width zero and adds every usable strip width. It checks the h consecutive widths beginning at $\lceil 5h/2 \rceil$. Adding width-h strips extends the result indefinitely. For $h \geq 100$, induction makes every integer strip width in $[a, b] = [\lceil 5h/7 \rceil, h - 3]$ usable. The intervals $[ja, jb]$ cover every integer from $3a$ onward, since $3b + 1 \geq 4a$. Also $3a \leq \lceil 5h/2 \rceil$. Both estimates follow from $a \leq (5h + 6)/7$. □

Erich Friedman posed the fixed-base question. His original page credits Dean Hickerson and describes the stronger wide-rectangle lemma used in their earlier unprescribed-base results [1, 2].

Finite check for the rectangle lemma. For $h \geq 3$ put $T_h(0) = 1$ and let $T_h(w)$ be true when there is a usable $s \leq w$ with $T_h(w - s) = 1$. Here s is usable when $1 \leq s \leq h$, $h - s < s$, and the upper rectangle of width s and height $h - s$ is available from a smaller height. The certificate stores one such s at each certified pair (h, w). Extract `cap_certificates.py` and `hanoi-caps.json` together from the supplementary files to check every decomposition. The second Python program calculates reachability independently.

References

[1] E. Friedman, *Problem of the month, November 2003*, Math Magic, original problem and previous constructions.

[2] E. Friedman, *Unsolved problems from Math Magic*, questions 9 and 30, problem list.