hexagon:2611.00679v2

SOLUTIONS TO BIANCHI'S COVARIOGRAM PROBLEM AND KOBAYASHI'S FOURIER ZERO-SET PROBLEM FOR C^2_+ CONVEX BODIES

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Abstract. In each dimension $n \geq 2$, the covariogram of a convex body with absolutely continuous surface area measure determines it among all convex bodies, up to translation and point reflection. This includes all bodies of constant width and all C^2_+ bodies. For a general convex body it suffices that the absolutely continuous part of its surface area measure fail to be even. The proof uses averaged stationary phase and a half-plane Jensen inequality. If both antipodal densities are positive on a set of positive spherical measure, the complex zero set of the Fourier–Laplace transform determines the body up to translation within that class without multiplicity data. In dimension three, width and the ordered antipodal curvature ratio also give uniqueness up to translation for $C^{1,1}$ support functions with uniformly positive radii.

1. Introduction

For a convex body $X \subset \mathbb{R}^n$, $n \geq 2$, define

$$g_X(x) = \mathrm{vol}_n(X \cap (X + x)), \qquad F_X(\zeta) = \int_X \mathrm{e}^{\mathrm{i}\zeta\cdot x}\,\mathrm{d}x, \qquad \zeta \in \mathbb{C}^n.$$

A convex body is a compact convex set with nonempty interior. The dot product in the transform is complex bilinear. The convolution identity $g_X = \mathbf{1}_X * \mathbf{1}_{-X}$ gives, by Fubini on complex frequencies, the identity below whenever $g_K = g_L$.

$$F_K(\zeta)F_K(-\zeta) = F_L(\zeta)F_L(-\zeta) \qquad (\zeta \in \mathbb{C}^n). \tag{1}$$

On real frequencies the product is $|F_X|^2$. Both translations and point reflections preserve g_X.

Matheron's planar conjecture [24] was proved for C^2_+ bodies by Bianchi, Segala and Volčič [8] and in full by Averkov and Bianchi [1]. Here C^2_+ means that the boundary is C^2 and its Gauss curvature, the product of its principal curvatures, is positive everywhere. Bianchi proved uniqueness for

2020 *Mathematics Subject Classification.* Primary 52A20, 42B10; Secondary 32A15, 35J15.

Key words and phrases. Bianchi's covariogram problem, Kobayashi's Fourier zero-set problem, phase retrieval, convex bodies, Fourier–Laplace transform, antipodal curvature ratio, surface area measure, stationary phase.

three-dimensional convex polytopes and constructed direct-product counterexamples in dimensions at least four [3, 5]. He asked whether uniqueness holds for three-dimensional C^2_+ bodies [4, Problem 5]. His Fourier–Laplace method gives a uniqueness theorem for pairs of positively curved bodies with a dimension-dependent smoothness assumption, which is C^9 in dimension three [6, Theorem 1.1]. Averkov and Bianchi [2, Theorem 4(i)] proved generic uniqueness in the Baire category sense in each dimension, using covariogram data near the boundary of the difference body.

Let $\sigma = \mathcal{H}^{n-1}|_{\mathbb{S}^{n-1}}$ be spherical area measure. Write S_X for the pushforward of boundary area by the almost everywhere defined outer unit normal. In the Lebesgue decomposition $S_X = s_X\,\sigma + (S_X)_{\mathrm{s}}$, choose a finite nonnegative Borel representative of s_X. Set

$$T_X = \{u \in \mathbb{S}^{n-1} : s_X(u) > 0,\ s_X(-u) > 0\},$$
$$U_X = \{u \in \mathbb{S}^{n-1} : s_X(u) \neq s_X(-u)\}.$$

The sets are independent of this choice up to spherical null sets.

Theorem 1.1. *Let $K, L \subset \mathbb{R}^n$ be convex bodies with $g_K = g_L$. If $\sigma(U_K) > 0$, then L is a translate of K or of $-K$.*

Corollary 1.2. *If S_K is absolutely continuous, its covariogram determines K among all convex bodies up to translation and point reflection. In particular this holds when the support function of $K - K$ is $C^{1,1}$, hence for every body of constant width. It also holds for every C^2_+ convex body.*

Corollary 1.2 answers Bianchi's three-dimensional C^2_+ question and allows an arbitrary convex competitor. The case of arbitrary three-dimensional convex bodies is still open. Theorem 1.1 and Corollary 1.2 reduce that case to nonsymmetric bodies for which the asymmetry of the surface area measure is entirely singular. This can occur even with positive density: for a noncentrally symmetric polytope P and the unit ball B^n, the density of S_{P+rB^n} is r^{n-1} almost everywhere. Indeed, off the boundaries of the vertex normal cones of P the support of P is linear, so the curvature-radius matrix of $P + rB^n$ equals rI.

The direct-product examples have singular surface area measure. For a product $X = X_1 \times X_2$, the points with both coordinates in the respective factor boundaries form a set of boundary area zero. At almost every other boundary point the normal lies in a factor subspace, and the two subspheres of factor normals are spherical null sets. Under an invertible linear map B, the normals change by the sphere diffeomorphism $u \mapsto B^{-\mathsf{T}}u/|B^{-\mathsf{T}}u|$. Thus singularity persists under the linear images used in these examples.

Kobayashi asked whether the complex zero set of F_X determines X up to translation [20, 21]. Bianchi states the convex-body problem in [7, Problem 3.24]. Kobayashi proved injectivity in the smooth positively curved planar class [21, Corollaries 1.12 and 2.3.11].

Theorem 1.3. *Let $K, L \subset \mathbb{R}^n$ be convex bodies. Suppose T_K and T_L each have positive spherical measure. If*

$$\{\zeta \in \mathbb{C}^n : F_K(\zeta) = 0\} = \{\zeta \in \mathbb{C}^n : F_L(\zeta) = 0\}$$

as sets, then L is a translate of K.

Theorem 1.3 applies in particular to any two C^2_+ bodies. Kobayashi also asked about determination by width and the ordered antipodal Gauss-curvature ratio [21, Problem 1.13]. Bianchi restated this question in [7, Problem 3.27]. Write h_X for the support function and $w_X(u) = h_X(u) + h_X(-u)$ for the width. The curvature-radius matrix is $A_X(u) = \nabla^2_{\mathbb{S}^{n-1}} h_X(u) + h_X(u) I$ almost everywhere for a $C^{1,1}$ support function. For a smooth positively curved boundary, let $\kappa_X(u)$ be the Gauss curvature at the point with outer normal u. Then $s_X(u)/s_X(-u) = \kappa_X(-u)/\kappa_X(u)$.

Theorem 1.4. *Let $K, L \subset \mathbb{R}^3$ be convex bodies whose support functions are $C^{1,1}$ and whose curvature-radius matrices are bounded below almost everywhere by a positive multiple of the identity on the tangent plane. If*

$$w_K = w_L, \qquad \frac{s_K(u)}{s_K(-u)} = \frac{s_L(u)}{s_L(-u)} \quad \text{for almost every } u \in \mathbb{S}^2,$$

then K and L are translates. In particular, the ordered data determine any C^2_+ body in $\mathbb{R}^3$ within that class.

The large radial zeros for smooth bodies were studied by Kobayashi [21, Theorem 2.3.6] and Bianchi [6, Theorems 4.1 and 4.2]. In Sections 2 and 3, we identify the endpoint coefficients from the density s_X and prove an averaged formula in a complex strip. The estimate permits an angular exceptional set of arbitrarily small measure, which accommodates concentration at singular parts of the surface area measure. It gives simple radial zeros whose angular count exhausts the leading term in Jensen's bound (Section 4). Thus any part of the divisor that misses these simple zeros has sublinear average radial count on the chosen directions. When one endpoint contribution dominates the other, Lemma 6.2 gives a sublinear count in a half-plane by bounding the boundary logarithmic integral. Section 5 uses a Remez inequality on the Taylor coefficients of the even product of a factor and its reflection. This product has minimal exponential type, and the factor contributes a linear functional to the growth along purely imaginary rays. Since that growth recovers the support function of each body, the divisor comparisons in Section 6 yield translation rigidity. Section 7 proves the width and ratio result by a separate elliptic argument in dimension three.

2. Support functions and section asymptotics

Write $H_X(x) = \max_{p \in X} p \cdot x$ for the homogeneous support function and set

$$a_X(u) = -h_X(-u), \qquad b_X(u) = h_X(u), \qquad d = n - 1, \qquad \beta = d/2.$$

Thus $w_X = b_X - a_X$. An inscribed ball gives a positive lower bound on w_X, and the diameter of X gives an upper bound. Alexandrov's theorem [15, Section 6.4] gives a second-order expansion of H_X at Lebesgue-almost every point of a Euclidean annulus. Since H_X is positively homogeneous of degree one, the existence of this expansion is invariant along each positive ray. It therefore holds at σ-almost every $u \in \mathbb{S}^{n-1}$. At such a direction, $p_X(u) = DH_X(u)$ is the unique supporting point, and

$$A_X(u) = D^2 H_X(u)|_{u^\perp} = \nabla^2_{\mathbb{S}^{n-1}} h_X(u) + h_X(u) I$$

is positive semidefinite. The last equality follows by expanding $H_X(u+v) = |u+v| h_X((u+v)/|u+v|)$ for $v \in u^\perp$. The surface area density satisfies

$$s_X(u) = \det A_X(u) \quad \text{for almost every } u. \tag{2}$$

For the normalization in (2), let $F_1(H_X, \cdot)$ denote the Hessian measure with index 1 in Colesanti and Hug's local Steiner formula. In that formula the coefficient of ρ^{n-1} is $nF_1(H_X, \cdot)$. The homogeneity $\partial H_X(ru) = \partial H_X(u)$ for $r > 0$ implies $F_1(H_X, bG) = bF_1(H_X, G)$ for $b > 0$. Indeed, the parallel set $\{x + \rho p : x \in bG,\ p \in \partial H_X(x)\}$ is b times the corresponding parallel set of G at scale ρ/b. Comparing the coefficients of ρ^{n-1} gives the scaling. In particular, $F_1(H_X, \cdot)$ gives zero mass to the origin. For any Borel $E \subset \mathbb{S}^{n-1}$, [12, Corollary 5.10] now gives

$$nF_1(H_X, \{ru : 0 < r \le 1,\ u \in E\}) = S_X(E).$$

Scaling and subtracting give, for $0 < a < b$,

$$F_1(H_X, \{ru : a < r \le b,\ u \in E\}) = \frac{b-a}{n} S_X(E).$$

The annular sectors determine the measure off the origin. In polar coordinates the preceding identity gives $n^{-1}(\mathrm{d}r) \otimes S_X$. The singular angular part is carried by a Lebesgue-null cone, and Euclidean volume is $r^{n-1}\,\mathrm{d}r\,\mathrm{d}\sigma(u)$. The Lebesgue density at ru is thus $r^{1-n} s_X(u)/n$. By [11, Theorem 2], the same density is $\sigma_{n-1}(D^2 H_X(ru))/n$, where σ_{n-1} is the elementary symmetric function of degree $n-1$. The identity $H_X((1+t)u+v) = (1+t)H_X(u+v/(1+t))$ for $v \perp u$ and $|t| < 1$ shows that the radial row and column of the Alexandrov Hessian at u are zero. Scaling the expansion to ru gives the tangent block $r^{-1}A_X(u)$. Equating the two expressions gives (2).

Lemma 2.1. *At almost every u, the section at depth s below the upper supporting hyperplane has d-dimensional volume*

$$\mathcal{H}^d(X \cap \{x : u \cdot x = b_X(u) - s\}) = \omega_d (2s)^\beta \sqrt{s_X(u)} + o(s^\beta) \qquad (s \downarrow 0), \tag{3}$$

where ω_d is the volume of the unit ball in $\mathbb{R}^d$. At a direction with an Alexandrov expansion of H_X, the matrix $A_X(u)$ is positive definite if and only if the boundary near the supporting point is a graph over $u^\perp$ and its depth has a positive-definite quadratic expansion there. In that case the depth Hessian is $A_X(u)^{-1}$.

Proof. Translate the supporting point to 0, use $u^\perp$ as the horizontal plane, and let $A = A_X(u)$. In these coordinates

$$\phi(v) := H_X(u+v) = \tfrac{1}{2} v^{\mathsf{T}} A v + o(|v|^2), \qquad v \in u^\perp.$$

If $y - su \in X$, then $y \cdot v \le s + \phi(v)$. The vectors y in these sections tend uniformly to 0 as $s \downarrow 0$, by compactness and uniqueness of the supporting point. For fixed $\delta > 0$ insert $v = (A+\delta I)^{-1} y$ and use $\phi(v) \le \frac{1}{2} v^{\mathsf{T}}(A+\delta I)v$ near 0. The section is contained in

$$y^{\mathsf{T}}(A+\delta I)^{-1} y \le 2s.$$

If A is singular, the ellipsoid gives a bound for the limsup of the section volume divided by s^β. That bound tends to zero as $\delta \downarrow 0$.

Suppose $A > 0$. A second unit normal at 0 would have a nonzero tangential component, since opposite normals at the same point force X into a hyperplane. Adding a small multiple of the second normal to u and dividing by the u-component gives a support normal $u + v$ for arbitrarily small nonzero $v \in u^\perp$. This gives $\phi(v) = 0$ in contradiction to $A > 0$. Thus u is the unique normal at 0. It follows that 0 belongs to the interior of the projection of X onto $u^\perp$, because a supporting hyperplane of the projection at 0 would give a support normal at 0 tangent to $u^\perp$. The projection of the interior of X equals the interior of the projection. Indeed, any point in the latter set lies on an open segment from the projection of a fixed interior point to another projected point, and the corresponding lifted segment contains interior points of X. It follows that $-s_0 u$ is interior to X for some $s_0 > 0$.

Let $f(y)$ be the depth of the upper boundary, extended by $+\infty$ outside the projection. It is closed and convex, with $f(0) = 0$, and $\phi = f^*$ by the definition of support. The unique normal at 0 implies $\partial f(0) = \{0\}$. Subgradients of f at y near zero are uniformly bounded by local Lipschitz continuity and converge to zero as $y \to 0$, by closedness of the subgradient graph. For $0 < \delta < \lambda_{\min}(A)$, the inequalities $\frac{1}{2} v^{\mathsf{T}}(A-\delta I)v \le \phi(v) \le \frac{1}{2} v^{\mathsf{T}}(A+\delta I)v$ near zero yield

$$\tfrac{1}{2} y^{\mathsf{T}}(A+\delta I)^{-1} y \le f(y) \le \tfrac{1}{2} y^{\mathsf{T}}(A-\delta I)^{-1} y.$$

For the lower bound, evaluate the conjugate at $v = (A+\delta I)^{-1} y$. For the upper bound, take $v \in \partial f(y)$, so that $f(y) = y \cdot v - \phi(v)$, and maximize $y \cdot v - \frac{1}{2} v^{\mathsf{T}}(A-\delta I)v$ over v. The two quadratic forms converge to the form with matrix A^{-1} as $\delta \downarrow 0$. This proves the second-order expansion. The interior point $-s_0 u$ supplies a point of depth s_0 in each nearby vertical fibre. For $s < s_0$ in these fibres, $y - su \in X$ is equivalent to $f(y) \le s$. Since the whole section lies in this neighborhood for small s, the quadratic bounds sandwich it between ellipsoids of volumes $\omega_d (2s)^\beta \sqrt{\det(A-\delta I)}$ and $\omega_d (2s)^\beta \sqrt{\det(A+\delta I)}$. Letting $\delta \downarrow 0$ gives (3).

Conversely, a positive-definite tangent depth expansion implies uniqueness of the supporting point, since a second supporting point would give a segment starting there with zero depth. Supporting points for $u+v$ approach

that point as $v \to 0$, by compactness. Thus the maximizers in the conjugate formula for $f^*(v)$ approach zero. If $Q > 0$ is the depth Hessian, use the lower quadratic bound $\frac{1}{2}y^{\mathsf{T}}(Q - \delta I)y$ at the maximizers for an upper bound on $f^*(v)$. Evaluating at $y = (Q + \delta I)^{-1}v$ and using the upper quadratic bound gives the reverse estimate. This yields the support Hessian Q^{-1}. □

Corollary 2.2. *For almost every u,*

$$\lim_{t\to\infty} t^{\beta+1}\mathrm{e}^{-tb_X(u)}F_X(-\mathrm{i}tu) = (2\pi)^{\beta}\sqrt{s_X(u)}. \tag{4}$$

If $g_K = g_L$, then $w_K = w_L$ and

$$s_K(u)s_K(-u) = s_L(u)s_L(-u) \quad \textit{for almost every } u. \tag{5}$$

Moreover,

$$s_K(u) + s_K(-u) = s_L(u) + s_L(-u) \quad \textit{for almost every } u. \tag{6}$$

Proof. Let $V(s)$ be the section volume in (3). By Fubini,

$$\mathrm{e}^{-tb_X(u)}F_X(-\mathrm{i}tu) = \int_0^{w_X(u)} \mathrm{e}^{-ts}V(s)\,\mathrm{d}s.$$

On a fixed small interval at zero one has $V(s) \le Cs^{\beta}$. There the change of variable $v = ts$, followed by dominated convergence, gives the limit. The integral away from zero is exponentially small. The constant is $2^{\beta}\omega_d\Gamma(1 + \beta) = (2\pi)^{\beta}$.

The support of g_X equals $X - X$, since differences of interior points have $g_X > 0$ and are dense in $X - X$. Its support function is w_X, which gives the width identity. Evaluate (1) at $\zeta = -\mathrm{i}tu$ and multiply each side by $t^{2\beta+2}\mathrm{e}^{-tw_X(u)}$, with $X = K$ or L respectively. By (4) the limit on the X-side is $(2\pi)^{2\beta}\sqrt{s_X(u)s_X(-u)}$ for almost every u. Squaring the equality of the limits gives (5). For C^2_+ bodies this recovers the product identity in [6, Proposition 3.1(I)].

The right directional derivative of g_X at the origin in direction u is minus the $(n-1)$-volume of the projection onto $u^{\perp}$. Indeed, on each line parallel to u, the overlap length is $(l - t)_+$ at translation tu, where l is the chord length. Dominated convergence gives the derivative. Cauchy's projection formula identifies the projection function with one half the spherical cosine transform of S_X. This transform is injective on even finite measures [16, Appendix C.2], so $S_K + S_{-K} = S_L + S_{-L}$. Taking absolutely continuous parts gives (6). □

3. Averaged stationary phase

For $\operatorname{Re} z > 0$, use the holomorphic power of z that is positive on the positive real axis. Define

(7)

$$D_X(z,u) = \mathrm{i} z^{\beta+1} F_X(zu),$$

(8)

$$M_X(z,u) = (2\pi)^\beta \left(\mathrm{e}^{-\mathrm{i}\pi\beta/2} \sqrt{s_X(u)}\, \mathrm{e}^{\mathrm{i}zb_X(u)} - \mathrm{e}^{\mathrm{i}\pi\beta/2} \sqrt{s_X(-u)}\, \mathrm{e}^{\mathrm{i}za_X(u)} \right).$$

For a measurable $V \subset \mathbb{S}^{n-1}$, use the normalized norm

$$\|G\|_{R,V}^2 = \frac{1}{R} \int_V \int_R^{2R} |G(r,u)|^2 \,\mathrm{d}r \,\mathrm{d}\sigma(u).$$

Proposition 3.1. *Let X be any convex body. For every $B, \varepsilon, \eta > 0$ there is a measurable $V \subset \mathbb{S}^{n-1}$, independent of R and v, with $\sigma(\mathbb{S}^{n-1} \setminus V) < \varepsilon$ and*

$$\limsup_{R\to\infty} \sup_{|v|\le B} \|D_X(r+\mathrm{i}v,u) - M_X(r+\mathrm{i}v,u)\|_{R,V} \le \eta. \tag{9}$$

Brandolini, Hofmann and Iosevich [9] proved an $O(r^{-(n+1)})$ bound for the spherical mean square of the Fourier transform of a convex body. In (9), the angular exceptional set allows for concentration at facet normals. For example, for $X = [-1,1]^n$ one has $M_X = 0$ almost everywhere and

$$\|D_X(r,u)\|_{R,\mathbb{S}^{n-1}}^2 = \frac{1}{R} \int_{R\le|\xi|\le 2R} |\xi|^2 \prod_{j=1}^n \left| \frac{2\sin\xi_j}{\xi_j} \right|^2 \mathrm{d}\xi.$$

On the region $R+1 \le \xi_1 \le 2R-1$, $|\xi_j| \le 1/2$ for $j \ge 2$, the integrand is bounded below by a positive multiple of $\sin^2 \xi_1$. This region lies in the annulus for large R, so the full-sphere norm has positive liminf.

3.1. **Definite Hessians.** Use orthonormal coordinates $x_f(y) = (y, f(y))$ on a graph over an open ball in $\mathbb{R}^d$. For $a \in L^2$ compactly supported in the ball, write

$$I_{f,a}(\eta,\lambda) = \int a(y) \mathrm{e}^{\mathrm{i}(\eta\cdot y + \lambda f(y))} \,\mathrm{d}y.$$

At fixed real λ, Plancherel gives

$$\int_{\mathbb{R}^d} |I_{f,a}(\eta,\lambda)|^2 \,\mathrm{d}\eta = (2\pi)^d \|a\|_2^2. \tag{10}$$

The polar coordinate Jacobian for $(\eta,\lambda) = ru$ is r^d. Using $2\beta = d$ and integrating (10) over $|\lambda| \le 2R$ gives

$$\int_{\mathbb{S}^{n-1}} \int_R^{2R} |r^\beta I_{f,a}(ru)|^2 \,\mathrm{d}r \,\mathrm{d}\sigma(u) \le 4(2\pi)^d R \|a\|_2^2. \tag{11}$$

A fixed translation of the graph multiplies the real-frequency integral by a phase of modulus one.

Use a Borel representative for a and extend it by zero outside its support. For $u = (u', u_n)$, suppose that y_u solves $u' + u_n Df(y_u) = 0$ at a point with definite Hessian. Here $u_n \neq 0$. The stationary term at this point is

$$(\mathcal{T}_{f,r}a)(u) = \frac{(2\pi)^\beta a(y_u)\, \mathrm{e}^{\mathrm{i}ru\cdot x_f(y_u) + \mathrm{i}\pi\ \mathrm{sig}(u_n D^2 f(y_u))/4}}{|u_n|^\beta \sqrt{|\det D^2 f(y_u)|}}. \tag{12}$$

Here sig is the number of positive eigenvalues minus the number of negative eigenvalues. For a convex or concave graph there is at most one such point in a given direction. Two points with the same gradient force affinity along the segment between them, contradicting a definite quadratic expansion at either point. We use Alexandrov Hessians for nonsmooth graphs and set the stationary term to zero when there is no definite point. The area formula below makes the term independent, almost everywhere in u, of the representative chosen for a.

Lemma 3.2. *Suppose $a \in L^2$ has compact support in a ball on which f is C^2 and its Hessian is uniformly positive definite or uniformly negative definite. Compute $\mathcal{T}_{f,r}$ on that ball. Then*

$$\frac{1}{R}\int_{\mathbb{S}^{n-1}} \int_R^{2R} |r^\beta I_{f,a}(ru) - (\mathcal{T}_{f,r}a)(u)|^2 \,\mathrm{d}r\,\mathrm{d}\sigma(u) \longrightarrow 0.$$

Proof. First take a smooth. Choose an open ball W containing $\operatorname{supp} a$ whose closure lies in the ball of uniform definiteness. For a real, smooth, compactly supported even kernel φ of integral one, put $\rho(y) = \frac{4}{3}\varphi(y) - \frac{1}{3}2^{-d}\varphi(y/2)$. This kernel has integral one and vanishing first and second moments. In particular convolution by $\rho_h(y) = h^{-d}\rho(y/h)$ fixes polynomials of degree at most two. Take $h = R^{-1/2}$ and $f_h = f * \rho_h$ on a fixed neighborhood of $\overline{W}$. Uniformly there,

$$\begin{gathered} \|f_h - f\|_\infty = o(h^2), \qquad \|Df_h - Df\|_\infty = O(h), \\ \|D^k f_h\|_\infty = O_k(h^{2-k}) \quad (k \geq 2). \end{gathered} \tag{13}$$

The first estimate follows by integrating the quadratic Taylor expansion and using uniform continuity of the Hessian. The second uses the Lipschitz bound for Df, and the last follows by differentiating the convolution of $D^2 f$. For $|\lambda| \leq 2R$ one has $|\lambda|\|f_h - f\|_\infty = o(1)$, so $\|a(\mathrm{e}^{\mathrm{i}\lambda f_h} - \mathrm{e}^{\mathrm{i}\lambda f})\|_2 = o(1)$ uniformly. Plancherel on each λ-slice, followed by integration over the annulus as in (11), shows that replacing f by f_h costs $o(1)$ in the norm $\|\cdot\|_{R,\mathbb{S}^{n-1}}$.

The gradient map of f is injective on W with open image, and $|Df(y) - Df(y')| \geq c|y - y'|$ there. This follows from the definite Hessian bounds by integrating along the line segment. The inverse function theorem gives openness. When a stationary point y_u lies in W, one has $|u_n| = (1 + |Df(y_u)|^2)^{-1/2} \geq c' > 0$. Use $y = y_u + hq$ and write $\phi(q) = r(u'\cdot(y_u + hq) + u_n f_h(y_u + hq))$. The rescaled gradient is

$$\nabla_q \phi(q) = rh\, u_n (Df_h(y_u + hq) - Df(y_u)).$$

By (13) and the gradient lower bound,

$$|\nabla_q \phi(q)| \geq c_1|q| - c_2, \qquad |\partial_q^\alpha \phi(q)| \leq C_\alpha \quad (|\alpha| \geq 2),$$

for $R \leq r \leq 2R$ and $y_u + hq$ in the amplitude support. The same bounds hold on a neighborhood of that support within W. The prefactor $r^\beta h^d = (r/R)^\beta$ is bounded. To estimate the tail, multiply the rescaled amplitude by a smooth cutoff that vanishes for $|q| \leq A$ and equals one for $|q| \geq 2A$. For large A, the vector field

$$V_\phi = \frac{\nabla_q \phi}{\mathrm{i}|\nabla_q \phi|^2}$$

and all its derivatives of fixed order are $O(|q|^{-1})$ on this region where the amplitude is supported. This follows by the chain rule, since the order-k derivatives of $v \mapsto v/|v|^2$ are homogeneous of degree $-1-k$ and the phase derivatives of order at least two are bounded. The operator $V_\phi \cdot \nabla_q$ fixes $\mathrm{e}^{\mathrm{i}\phi}$. Applying its formal transpose N times to the amplitude with cutoff yields terms each containing N vector-field factors or their derivatives. The remaining derivatives of the amplitude and cutoff are bounded independently of $h \leq 1$ and $A \geq 1$. For $N > d$ the tail is bounded by the integral of $C_N|q|^{-N}$ over $|q| \geq A$, which is $O_N(A^{d-N})$.

On bounded q-sets, the phase relative to $ru \cdot x_f(y_u)$ is

$$\tfrac{1}{2}(r/R)\, u_n q^{\mathsf{T}} D^2 f(y_u) q + o(1).$$

This follows from (13), stationarity, and the quadratic Taylor expansion of f, uniformly in u with $y_u \in W$ and $R \leq r \leq 2R$. The amplitude converges uniformly there to $a(y_u)$ on each bounded q-set. The tail bound just proved also holds for the quadratic phase with a factor $\mathrm{e}^{-\epsilon|q|^2}$, $0 < \epsilon \leq 1$, since its derivatives of fixed order are uniformly bounded. Therefore the quadratic oscillatory integral defined by smooth cutoffs has the same value as the limit of the Gaussian-regularized integrals. For each eigenvalue $\mu \neq 0$ of $u_n D^2 f(y_u)$, the one-dimensional Gaussian factor in the limit is

$$\left(\frac{2\pi}{(r/R)|\mu|}\right)^{1/2} \mathrm{e}^{\mathrm{i}\pi\,\mathrm{sgn}(\mu)/4}.$$

Multiplying these factors and cancelling $(r/R)^{d/2}$ gives (12). Taking $R \to \infty$ with A fixed and then $A \to \infty$ is justified by the uniform tail bounds. The estimate includes stationary points outside the amplitude support, where $a(y_u) = 0$.

For directions with no stationary point in W, the size of $u' + u_n Df(y)$ has a uniformly positive lower bound on the support. Otherwise a convergent subsequence of directions and support points would give a stationary point in W, and openness of the gradient image would give stationary points in W for the nearby directions. In coordinates $y = hq$, the phase gradient for f_h is now bounded below by ch^{-1}. The vector field and its derivatives are $O(h)$ on the scaled support. The same integrations by parts give $O(h^{N-d})$ for the normalized integral, since the scaled support has volume $O(h^{-d})$. Thus the

normalized smoothed integral converges to the model with an error uniform in u and $R \le r \le 2R$. The Plancherel comparison proves the assertion for smooth a.

For the L^2 extension, the two maps

$$y \longmapsto \pm \frac{(-Df(y), 1)}{\sqrt{1 + |Df(y)|^2}}$$

have spherical area Jacobian $|\det D^2 f(y)|(1 + |Df(y)|^2)^{-(d+1)/2}$. The exponent follows by differentiating the map $p \mapsto (-p, 1)/\sqrt{1 + |p|^2}$. Its squared length factors are $(1 + |p|^2)^{-1}$ in the $d - 1$ directions perpendicular to p and $(1 + |p|^2)^{-2}$ in its direction. The area formula [15, Section 3.3] applies to each normal map. Multiplication by the Jacobian cancels the factor $|\det D^2 f|^{-1}$ in the squared modulus of the stationary term and changes $|u_n|^{-2\beta}$ to $(1 + |Df|^2)^{-1/2}$. Summing over the two signs gives

$$\|\mathcal{T}_{f,r} a\|^2_{L^2(\sigma)} = 2(2\pi)^d \int |a(y)|^2 (1 + |Df(y)|^2)^{-1/2} \, \mathrm{d}y. \tag{14}$$

The resulting bound depends only on d and $\|a\|_2$. Approximation by smooth amplitudes in L^2, together with (11), completes the proof. □

3.2. **Localization for a Lipschitz graph.**

Lemma 3.3. *If f is Lipschitz on the graph domain and $a \in L^2$ is compactly supported there, then for $\psi \in C_c^\infty(\mathbb{R}^d)$*

$$\lim_{|\lambda| \to \infty} \int_{\mathbb{R}^d} \psi(\eta/\lambda) |I_{f,a}(\eta, \lambda)|^2 \, \mathrm{d}\eta = (2\pi)^d \int |a(y)|^2 \psi(-Df(y)) \, \mathrm{d}y. \tag{15}$$

Proof. Extend f as a Lipschitz function outside the compact region in use and a by zero. Expand the square and, for $\lambda \neq 0$, set $\eta = \lambda t$ and $p = \lambda(y - y')$. The Jacobians $|\lambda|^d$ and $|\lambda|^{-d}$ cancel. With $\Psi(p) = \int \psi(t) \mathrm{e}^{\mathrm{i}t \cdot p} \, \mathrm{d}t$, the result is the integral against $\Psi(p)$ of

$$\int a(y) \overline{a(y - p/\lambda)} \mathrm{e}^{\mathrm{i}\lambda(f(y) - f(y - p/\lambda))} \, \mathrm{d}y.$$

For fixed p, the phase converges to $Df(y) \cdot p$ for almost every y. Translation continuity in L^2 replaces $a(y - p/\lambda)$ by $a(y)$ at a cost tending to zero. Dominated convergence first with respect to $|a(y)|^2 \, \mathrm{d}y$ and then against the integrable function $|\Psi|$ is justified by the bound $\|a\|_2^2$ for the inner integral. Fourier inversion gives (15), with the displayed sign. □

3.3. **Convex graph approximation.**

Lemma 3.4. *Let f be a Lipschitz convex or concave function on the graph ball and $a \in L^2$ compactly supported there. Let Ω_{def} be the set of points with definite Alexandrov Hessians on this ball. Then*

$$\|\mathcal{T}_{f,r} a\|^2_{L^2(\sigma)} = 2(2\pi)^d \int_{\Omega_{\mathrm{def}}} \frac{|a(y)|^2}{\sqrt{1 + |Df(y)|^2}} \, \mathrm{d}y. \tag{16}$$

For every $\varepsilon, \gamma > 0$ there is a measurable $V \subset \mathbb{S}^{n-1}$, with $\sigma(\mathbb{S}^{n-1} \setminus V) < \varepsilon$, such that

$$\limsup_{R\to\infty} \|r^\beta I_{f,a}(ru) - (\mathcal{T}_{f,r}a)(u)\|_{R,V} \le \gamma.$$

Proof. On a compact subregion of a Lipschitz convex graph, the points with Alexandrov expansions are a countable union of sets on each of which the gradient is Lipschitz. To see this, fix $C, \delta > 0$, with δ smaller than the distance of the subregion to the graph domain boundary, and consider points x with

$$0 \le f(x+h) - f(x) - Df(x)\cdot h \le C|h|^2 \qquad (|h| < \delta).$$

Every point of quadratic expansion belongs to one of these sets for suitable C, δ drawn from countably many choices. If $r = |x-y| < \delta/3$, the supporting affine functions at x and y give, for every unit e,

$$(Df(y) - Df(x))\cdot e \le \frac{f(y+re) - f(x) - Df(x)\cdot(y+re-x)}{r} \le 4Cr.$$

The Lipschitz bound on f handles pairs separated by at least $\delta/3$. Apply the argument to $-f$ for a concave graph. Consequently the normal maps take parameter null sets of quadratically differentiable points to spherical null sets.

By [14, Proposition A.1], for every $\epsilon > 0$ there is a C^2 function agreeing with f, Df, and the Alexandrov Hessian of f outside a set of Lebesgue measure less than ϵ. On the compact subregion containing $\operatorname{supp} a$, take approximations with $\epsilon \to 0$. Inner regularity gives countably many compact pieces of agreement covering the part of the definite set in use up to a null set. On each piece the normal maps agree with the C^1 maps of the approximant, whose Jacobian is the one used in (14). Make the pieces disjoint by assigning each point to its first piece. Each sign of the normal map is injective on the definite set, including across the pieces, and has image in a distinct hemisphere. The omitted points have spherical null image by the gradient estimate above. The area formula on the pieces therefore gives (16).

Choose one C^2 approximation with sufficiently small exceptional measure. Within its agreement set choose disjoint compact sets P and N of points with Alexandrov expansions, with definite Hessian on P and singular Hessian on N. The exceptional measure and these sets can be chosen so that $\|a\mathbf{1}_{(P\cup N)^c}\|_2$ is arbitrarily small, by absolute continuity and inner regularity of $|a|^2\,\mathrm{d}y$. Cover P by finitely many balls of uniform definiteness of the agreeing function. Partition P into measurable pieces each contained in a compact subball of one of those balls. For the amplitude on each piece, equality of the values of the graph functions gives equality of the integrals. Equality of their first and second derivatives gives equality of the models, since the amplitudes vanish off the piece. Apply Lemma 3.2 to each piece in its ball of uniform definiteness and add the finitely many estimates. This gives a vanishing normalized error for $a\mathbf{1}_P$. By (11) and (16), the normalized error off $P\cup N$ is at most $C_d\|a\mathbf{1}_{(P\cup N)^c}\|_2$ for every R and every angular set.

The two normal images of N are compact spherical null sets. Indeed, the normal maps of the agreeing C^2 function are continuous and have Jacobian zero on N. The area formula gives spherical measure zero for their images. Remove an open neighborhood of these images of spherical measure less than ε, and let V be its closed complement. The stationary model for $a\mathbf{1}_N$ vanishes. For the graph integral with $a\mathbf{1}_N$, the part with $|\lambda| < \delta R$ contributes at most $2(2\pi)^d\delta\|a\mathbf{1}_N\|_2^2$ to the squared norm, by (10). For $|\lambda| \geq \delta R$ on the annulus and $u \in V$, we have $|u_n| \geq \delta/2$. The slopes $\eta/\lambda = u'/u_n$ range over the compact image of $V \cap \{|u_n| \geq \delta/2\}$ under the slope map. This image is disjoint from $-Df(N)$, since each slope in $-Df(N)$ comes from the two removed normal directions above it. Choose a nonnegative smooth compactly supported function equal to one on the slope set and zero near $-Df(N)$. Lemma 3.3, with the amplitude $a\mathbf{1}_N$, bounds the weighted η-integral by $o(1)$ uniformly for $\delta R \leq |\lambda| \leq 2R$. The total length of the λ-range is at most $4R$. Letting $R \to \infty$ and then $\delta \downarrow 0$ shows that the contribution from N tends to zero in $\|\cdot\|_{R,V}$. The estimates on the three amplitudes prove the lemma. □

3.4. **Boundary integral in the strip.**

Proof of Proposition 3.1. The divergence theorem on the Lipschitz boundary gives

$$\mathrm{i}zF_X(zu) = \int_{\partial X} \mathrm{e}^{\mathrm{i}zu\cdot x}\, u \cdot \nu(x)\, \mathrm{d}\mathcal{H}^d(x). \tag{17}$$

Here ν is the outer unit normal, defined for $\mathcal{H}^d$-almost every boundary point. To obtain a graph chart at $p \in \partial X$, take the vertical direction from an interior point p_0 toward p. Their common projection lies in the interior of the projected body. The vertical fibre contains the interior point p_0 and has p as its upper endpoint. On a smaller ball of the projection the two boundary graphs are separated, and the upper graph is Lipschitz and concave. Reversing the vertical axis gives a convex graph. Compactness gives finitely many charts with a partition of unity compactly supported in them. On each graph the area element times the normal is the vector $\pm(-Df, 1)\,\mathrm{d}y$ in chart coordinates. Thus $(u \cdot \nu)\,\mathrm{d}\mathcal{H}^d$ times the partition function is a sum of terms $u_j a_j(y)\,\mathrm{d}y$ in fixed ambient coordinates, with bounded compactly supported amplitudes a_j.

For $z = r + \mathrm{i}v$, the additional weight $\mathrm{e}^{-vu\cdot x_f(y)}$ in (17) has the absolutely convergent expansion

$$\sum_{\alpha\in\mathbb{N}^n} \frac{(-v)^{|\alpha|}u^\alpha}{\alpha!}\, x_f(y)^\alpha.$$

In this expansion $x_f(y)$ is the boundary point in ambient coordinates, including the rotation and translation of the chart. If its coordinates on the amplitude support are bounded by M, then $\|a_j x_f^\alpha\|_2 \leq M^{|\alpha|}\|a_j\|_2$. In the graph estimates, rotate the real frequency variables with the chart and use rotational invariance of σ. By (11) and (16), the sum of the normalized

norms of the terms, for the integrals and their models, is bounded uniformly in R and $|v| \le B$ by a constant times $\sum_\alpha (BM)^{|\alpha|}/\alpha! = \mathrm{e}^{nBM}$. Choose a finite truncation with tail error less than $\eta/2$ in normalized norm, on the full sphere. Apply Lemma 3.4 to each retained amplitude $a_j x_f^\alpha$. Choose the omitted angular measures to sum to less than ε and the individual errors, multiplied by the uniform bounds for their coefficients, to sum to less than $\eta/2$. The intersection of the resulting sets works uniformly for $|v| \le B$. By (11) the norms of the weighted integrals are bounded on the full sphere uniformly in R and v. Hence replacing r^β there by z^β changes the result by $o(1)$ in normalized norm, since $z^\beta/r^\beta = 1 + O_B(R^{-1})$ on the band.

We now compute the two coefficients of the model. At a differentiable point in a graph chart there is a unique outer normal. Indeed, the base point is interior to the projection, the vertical fibre contains interior points, and the slope of the graph is unique. Projection to the tangent plane also gives a graph near this point, by the interior projection argument in the proof of Lemma 2.1. Let H be the graph Hessian at a stationary point y_u. In orthonormal tangent coordinates the graph's tangent map L satisfies $L^{\mathsf{T}} L = I + Df(Df)^{\mathsf{T}}$, where Df is written as a column. Its projection to the tangent plane has the expansion $t = L(y - y_u) + O(|y - y_u|^2)$, so $y - y_u = L^{-1}t + O(|t|^2)$. The quadratic part of the height in direction u is $\frac{1}{2}(u \cdot e_n)(y - y_u)^{\mathsf{T}} H (y - y_u)$, where e_n is the chart's vertical vector. Thus the Hessian in tangent coordinates is $(u \cdot e_n) L^{-\mathsf{T}} H L^{-1}$. The same invertible change of coordinates, applied to the tangent graph, gives the converse when the second-order expansion is given in tangent coordinates. In either case the absolute determinant of the tangent height Hessian is

$$\frac{|\det H|}{(1 + |Df|^2)^{(d+2)/2}},$$

since $|u \cdot e_n| = (1 + |Df|^2)^{-1/2}$. The factor from (12) and the surface element is

$$\frac{\sqrt{1 + |Df|^2}}{|u \cdot e_n|^\beta \sqrt{|\det H|}} = \frac{(1 + |Df|^2)^{(d+2)/4}}{\sqrt{|\det H|}},$$

the reciprocal square root of the displayed determinant.

For almost every u the supporting points in directions u and $-u$ are both unique. By Lemma 2.1 and (2), their reciprocal square roots of curvature are $\sqrt{s_X(u)}$ and $\sqrt{s_X(-u)}$ at definite jets. If $A_X(u)$ or $A_X(-u)$ is singular, there is no definite jet at that endpoint. The phase Hessian is negative definite at the upper supporting point and positive definite at the lower one. The values of $u \cdot \nu$ are $+1$ and -1, respectively. The phase factors in (12) are $\mathrm{e}^{-\mathrm{i}\pi d/4}$ at the upper point and $\mathrm{e}^{\mathrm{i}\pi d/4}$ at the lower point. Reinstating the complex weight and summing the partition functions therefore gives precisely (8). The estimates on the graph integrals give (9). □

4. Simple zeros and radial counting

For $u \in T_X$, the zeros of the model (8) are simple:

$$q_j^X(u) = \frac{2\pi j + \pi\beta}{w_X(u)} + \frac{\mathrm{i}}{2w_X(u)} \log \frac{s_X(u)}{s_X(-u)}, \qquad j \in \mathbb{Z}.$$

On a direction set $E \subset T_X$ where both endpoint densities are bounded above and bounded away from zero, the heights are bounded. The real spacing of the model zeros is bounded above and below by positive constants. On any bounded horizontal strip the model has a uniform positive lower bound in modulus at any prescribed positive distance from its zeros. In fact, for every j,

$$M_X(z,u) = (2\pi)^\beta \mathrm{e}^{\mathrm{i}\pi\beta/2} \sqrt{s_X(-u)}\, \mathrm{e}^{\mathrm{i}z a_X(u)} \big(\mathrm{e}^{\mathrm{i}w_X(u)(z-q_j^X(u))} - 1\big).$$

The modulus of the prefactor is bounded away from zero in the strip. Choose j with the closest real part, so that $\mathrm{Re}(w_X(u)(z - q_j^X(u))) \in [-\pi, \pi]$. The values of $w_X(u)(z - q_j^X(u))$ under consideration lie in a compact rectangle and stay a positive distance from the zeros of $\mathrm{e}^{\mathrm{i}z} - 1$.

Lemma 4.1. *Let $E \subset T_X$ be measurable and satisfy the preceding endpoint density bounds. Fix a radius $\rho > 0$ less than one quarter of the infimum of the model spacing on E, and a threshold $\tau > 0$. For $u \in E$, consider closed disks $B_j(u)$ of radius ρ centered at the $q_j^X(u)$ with $R \le \mathrm{Re}\, q_j^X(u) \le 2R$. Call a disk bad if*

$$\sup_{z \in B_j(u)} |D_X(z,u) - M_X(z,u)| > \tau.$$

The integral over E of the number of bad disks is $o(R)$.

Proof. The disks with the same centers and twice the radius are disjoint for each direction by the choice of ρ. They lie in a fixed horizontal strip $|\mathrm{Im}\, z| \le B$ and, for large R, in $R/2 \le \mathrm{Re}\, z \le 4R$. For each point of the original disk, the disk of radius ρ centered at that point lies in the enlarged disk. The submean inequality for the squared modulus of the holomorphic error gives a bound for the supremum of its square by $1/(\pi\rho^2)$ times the integral of the square over the enlarged disk. If $b_{\mathrm{bad}}(R,u)$ is the bad count, summing over the disjoint enlarged disks gives

$$\pi\rho^2\tau^2 b_{\mathrm{bad}}(R,u) \le \int_{-B}^{B} \int_{R/2}^{4R} |D_X(r + \mathrm{i}v, u) - M_X(r + \mathrm{i}v, u)|^2 \,\mathrm{d}r\, \mathrm{d}v.$$

Let V_* be the set from Proposition 3.1 with strip bound B and tolerances ε, η. Integrate this bound over $E \cap V_*$. Apply (9) to the intervals $[R/2, R]$, $[R, 2R]$ and $[2R, 4R]$, and integrate in v. The resulting limsup of the integrated bad count divided by R is $O_{B,\rho}(\eta^2/\tau^2)$. On $E \setminus V_*$ the total count is $O(R)$ per direction and the angular measure is at most ε. Sending ε, η to zero proves the bound. The counts are measurable, since each disk supremum can be taken on a fixed countable dense subset translated by its measurable center. □

Take τ smaller than the uniform lower bound for $|M_X|$ on the disk boundaries. Call the disks meeting the error threshold good. Rouché's theorem gives one zero of $D_X(z,u)$, counted with multiplicity, in each good disk. The normalizing factor in (7) is nonzero there, so this is a simple zero of $z \mapsto F_X(zu)$. The symmetry

$$F_X(-\bar{\zeta}) = \overline{F_X(\zeta)} \tag{18}$$

gives the corresponding simple zero in the reflected disk across the imaginary axis. Fix a positive lower bound for the real parts of the original centers, greater than 2ρ, so that the original disks and the reflected disks are disjoint. Let $b(t,u)$ count the good disks and their reflections that lie entirely in $|z| \le t$. A disk with center q lies inside this circle exactly when $|q| + \rho \le t$. Since the heights are bounded, the upper bound for the real part of q is $t - \rho + O(t^{-1})$, uniformly in u. The real spacing $2\pi/w_X(u)$ gives $w_X(u)t/\pi + O(1)$ disks counting both signs before subtracting the bad ones. The loss is $o(t)$ in integral over E by summing Lemma 4.1 over dyadic intervals. Indeed for every $\delta > 0$, the loss on $[R, 2R]$ is at most δR in integral for all sufficiently large R. The bounded range contributes $O(1)$, and the other ranges contribute $O(\delta t)$. It follows that

$$\int_E b(t,u)\,\mathrm{d}\sigma(u) = \frac{t}{\pi}\int_E w_X(u)\,\mathrm{d}\sigma(u) + o(t). \tag{19}$$

Let $n_X(t,u)$ count the zeros of $z \mapsto F_X(zu)$ in $|z| \le t$ with multiplicity. Bound the modulus by $F_X(0)\exp(-a_X(u)\operatorname{Im} z)$ in the upper half-plane and by $F_X(0)\exp(-b_X(u)\operatorname{Im} z)$ in the lower one. On the circle $z = te^{i\theta}$, the upper and lower semicircles contribute $-ta_X(u)/\pi$ and $tb_X(u)/\pi$, respectively, to the mean of the logarithmic bound after subtracting $\log F_X(0)$. Jensen's formula therefore gives

$$\int_0^t n_X(s,u)\frac{\mathrm{d}s}{s} \le \frac{w_X(u)t}{\pi}. \tag{20}$$

In particular $n_X(t,u) \le C_X t$ uniformly in u, by using (20) at $2t$ and monotonicity.

4.1. **Effective subdivisors.** An effective subdivisor $\mathcal{D} \le \operatorname{div} F_X$ assigns to each irreducible hypersurface of $\operatorname{div} F_X$ a nonnegative integer at most equal to the multiplicity of that hypersurface. We use the finite-order divisor theorem of Lelong and Stoll [22, 26] in the form recorded in [13, Section 1(d)]. An effective divisor of finite order in $\mathbb{C}^n$ is the zero divisor of an entire function of order at most that of the divisor. In particular, an integrated count $O(R)$ gives a representative of order at most one.

Since $F_X(0) > 0$, every subdivisor avoids a neighborhood of zero. Its integrated count, up to a dimensional normalization, is

$$N_{\mathcal{D}}(R) = \int_0^R \frac{\operatorname{vol}_{2n-2}(\mathcal{D} \cap B_{\mathbb{C}^n}(0,t))}{t^{2n-1}}\,\mathrm{d}t,$$

where the hypersurface volume includes multiplicities. For $\operatorname{div} F_X$, Jensen's formula in several variables [13, Section 1(c)–(d)] identifies this count, with the appropriate normalization, with the spherical mean of $\log|F_X|$ at radius R minus $\log F_X(0)$. It is $O(R)$ since

$$|F_X(\zeta)| \le F_X(0)\exp(R_X|\zeta|), \qquad R_X = \max_{x\in X}|x|.$$

The bound $\mathcal{D} \le \operatorname{div} F_X$ gives $N_{\mathcal{D}}(R) = O(R)$. The divisor theorem gives an entire A of order at most one with $\operatorname{div} A = \mathcal{D}$ and $A(0) \neq 0$. If $\mathcal{D} = 0$, take $A = 1$.

Write $n_{\mathcal{D}}(t,u)$ for the number of zeros of $A(zu)$ in $|z| \le t$, counted with multiplicity. At any point of a complex line through the origin, local defining functions for the same divisor differ by a holomorphic unit, whose restriction stays a unit. Also F_X is nonzero at the origin of every such line. Factoring F_X locally as A times a defining function of the complementary divisor, up to a unit, and restricting this equality to the line gives additivity of orders even at tangent intersections. Thus the count is intrinsic to $\mathcal{D}$, with $n_{\mathcal{D}}(t,u) \le n_X(t,u)$. The counts are upper semicontinuous in (t,u). At a given (t,u), choose a slightly larger circle enclosing the same zeros and avoiding zeros on its boundary. Its count persists for nearby directions by the argument principle and bounds the count in nearby smaller closed disks. In particular the counts are measurable.

Lemma 4.2. *Suppose E has positive measure and all the simple zeros counted by $b(t,u)$ in (19) lie outside $\mathcal{D}$ on their radial lines. Then*

$$\int_E n_{\mathcal{D}}(t,u)\,\mathrm{d}\sigma(u) = o(t). \tag{21}$$

Proof. The count $n_X(s,u)$ dominates $n_{\mathcal{D}}(s,u) + b(s,u)$. Integrate (20) over E and use (19). In the latter formula, integrating the $o(s)$ error against $\mathrm{d}s/s$ gives $o(t)$: for any $\epsilon > 0$ it is bounded in modulus by ϵs beyond a fixed cutoff, and the integral below the cutoff is finite. Consequently

$$\int_0^t \int_E b(s,u)\,\mathrm{d}\sigma(u)\frac{\mathrm{d}s}{s} = \frac{t}{\pi}\int_E w_X\,\mathrm{d}\sigma + o(t).$$

The leading terms cancel and give

$$0 \le \int_0^t \int_E n_{\mathcal{D}}(s,u)\,\mathrm{d}\sigma(u)\frac{\mathrm{d}s}{s} = o(t).$$

The estimate at $2t$ bounds $(\log 2)\int_E n_{\mathcal{D}}(t,u)\,\mathrm{d}\sigma(u)$ by $o(t)$. □

5. Growth of factors with sparse radial zeros

Proposition 5.1. *Let $\mathcal{D} \le \operatorname{div} F_X$ be effective and invariant, including multiplicities, under $\zeta \mapsto -\overline{\zeta}$. Suppose (21) holds on a set E of positive spherical measure. For any entire representative A of order at most one there is a real linear functional ℓ_A on $\mathbb{R}^n$ such that*

$$\lim_{r\to\infty} \frac{\log|A(\mathrm{i}ru)|}{r} = \ell_A(u) \qquad (u \in \mathbb{S}^{n-1}). \tag{22}$$

5.1. **Propagation by a polynomial inequality.**

Lemma 5.2. *Suppose $\mathcal{D} \le \operatorname{div} F_X$ satisfies* (21)*. Let A represent $\mathcal{D}$, with order at most one and $A(0) = 1$. Put $P(\zeta) = A(\zeta)A(-\zeta)$. Then*

$$\log \max_{|\zeta|\le R} |P(\zeta)| = o(R) \tag{23}$$

and

$$\sup_{u\in\mathbb{S}^{n-1}} n_{\mathcal{D}}(R,u) = o(R). \tag{24}$$

Proof. For each real unit u, $P(zu)$ is an even entire function of order at most one, with zero count $n_P(t,u) = 2n_{\mathcal{D}}(t,u) \le Ct$. In the one-variable Hadamard factorization [23, Chapter I], pair the roots $\lambda, -\lambda$ with multiplicities. The genus-one exponential factors cancel in each pair. The remaining exponential is constant by evenness, and the normalization at zero gives

$$P(zu) = \prod_{\{\lambda,-\lambda\}} (1 - z^2/\lambda^2).$$

The zero count $O(t)$ gives $\sum_\lambda |\lambda|^{-2} < \infty$, so the product converges locally uniformly.

Taking moduli and integrating by parts in the zero-counting measure gives

$$\log \max_{|z|\le R} |P(zu)| \le \tfrac{1}{2} \int_0^\infty \log(1 + R^2/t^2)\, \mathrm{d}n_P(t,u) = \int_0^\infty \frac{R^2 n_P(t,u)}{t(t^2+R^2)}\, \mathrm{d}t. \tag{25}$$

There is no boundary contribution at zero, and the bound $n_P(t,u) = O(t)$ gives the vanishing boundary term at infinity. Average over E, divide by R, and substitute $t = Rs$. The resulting integrand is

$$\frac{1}{1+s^2} \frac{\int_E n_P(Rs,u)\, \mathrm{d}\sigma(u)}{Rs}.$$

It tends to zero for each $s > 0$ by (21) and is dominated by $C\sigma(E)/(1+s^2)$. Consequently the average of the log maximum on the left of (25) is $o(R)$.

The log maximum is nonnegative since $P(0) = 1$. For each $\delta > 0$ and all large R, a subset $E_R \subset E$ of measure at least $\sigma(E)/2$ satisfies $\max_{|z|\le R} |P(zu)| \le \mathrm{e}^{\delta R}$. Cauchy's estimate for the degree-m homogeneous Taylor polynomial P_m gives $|P_m(u)| \le \mathrm{e}^{\delta R} R^{-m}$ on E_R. By homogeneity this holds on the cone over E_R inside the real unit ball, of volume at least $\sigma(E)/(2n)$. The multivariate Remez inequality of Brudnyi and Ganzburg [10, Theorem 2 and Eq. (8)] gives the bound $C^m \mathrm{e}^{\delta R} R^{-m}$ on the whole real unit ball. The constant C depends only on n and the lower bound on the cone volume. For a complex polynomial the inequality applies to the real part of a constant phase multiple of that polynomial, with the phase chosen at the evaluation point.

For a complex-valued homogeneous polynomial Q_m of degree $m \ge 1$, write B_m for its symmetric multilinear form. For real unit vectors $x_1, \dots, x_m$,

polarization gives

$$B_m(x_1,\dots,x_m)=\frac{1}{2^m m!}\sum_{\epsilon_j=\pm1}\left(\prod_{j=1}^m\epsilon_j\right)Q_m\left(\sum_{j=1}^m\epsilon_j x_j\right).$$

The summed vector has norm at most m. Homogeneity bounds $|B_m|$ on real unit vectors by $m^m/m!$ times the supremum of $|Q_m|$ on the real unit ball. Expand each of the m arguments of $B_m(\zeta,\dots,\zeta)$ into real and imaginary parts of norm at most one for $|\zeta|\le 1$. Since $m^m/m!\le \mathrm{e}^m$, the resulting complex supremum is at most $(2\mathrm{e})^m$ times the real supremum. Absorbing this factor into C, the estimates on P_m give

$$|P_m(\zeta)|\le \mathrm{e}^{\delta R}(C|\zeta|/R)^m.$$

Summing the Taylor series on $|\zeta|\le R/(2C)$ gives $|P(\zeta)|\le 2\mathrm{e}^{\delta R}$. Rescaling the radius and letting $\delta\downarrow 0$ proves (23). Apply Jensen's formula to $P(zu)$ at $2R$. Then $\log\max_{|\zeta|\le 2R}|P(\zeta)|$ bounds $(\log 2)n_P(R,u)$ uniformly in u. By (23) this is $o(R)$, which proves (24). □

5.2. **Two bounds for the radial zeros.**

Lemma 5.3. *The radial zeros λ of $F_X(zu)$, for $u\in\mathbb{S}^{n-1}$, lie outside a fixed disk at zero and satisfy*

$$|\operatorname{Re}\lambda|\ge c_n|\lambda| \tag{26}$$

for a constant $c_n>0$. Counting with multiplicity,

$$\sup_{u\in\mathbb{S}^{n-1}}\sum_{|\lambda|>s}\frac{|\operatorname{Im}\lambda|}{|\lambda|^2}\le C_X\frac{1+\log(1+s)}{s}\qquad(s\ge 1). \tag{27}$$

Proof. The disk at zero follows from

$$|F_X(zu)/F_X(0)-1|\le\exp(R_X|z|)-1.$$

For the cone use the depth $t=u\cdot p-a_X(u)\ge 0$, $p\in X$, from the lower supporting plane, and let $C(t)$ be the cap volume up to depth t. A homothety of ratio $1/k$ about a point on this plane in X sends the cap of depth kt into the cap of depth t for $k\ge 1$. Convexity puts the image inside X, and the volume scales by k^{-n}. Thus $C(kt)\le k^nC(t)$. At $z=x+\mathrm{i}y$ with $y>0$, remove the exponential of the lower supporting height and normalize by the positive integral of $\mathrm{e}^{-yt}\,\mathrm{d}C(t)$. The resulting value is the expectation of $\mathrm{e}^{\mathrm{i}xt}$ under this normalized measure. The denominator is at least $\mathrm{e}^{-1}C(1/y)>0$. Split yt into the ranges $k\le yt<k+1$ for nonnegative integers k. The numerator for the expectation of yt is at most

$$\sum_{k=0}^{\infty}(k+1)\mathrm{e}^{-k}C((k+1)/y)\le C(1/y)\sum_{k=0}^{\infty}(k+1)^{n+1}\mathrm{e}^{-k}.$$

Therefore the expectation of yt is bounded by a constant depending only on n. Since $|\mathrm{e}^{\mathrm{i}xt}-1|\le|x|t$, the normalized value stays within distance $1/2$ of

one when $|x|/y$ is sufficiently small. Apply the same argument to the other endpoint in the lower half-plane. The two zero-free cones give (26).

For (27), the function

$$G(z) = F_X(zu)\mathrm{e}^{-\mathrm{i}a_X(u)z}/F_X(0)$$

has modulus at most one in the upper half-plane. A homothetic copy of X with ratio $\min(1, (sw_X(u))^{-1})$, centered at a point in the lower support, lies in the cap of depth $1/s$. Integration over that cap gives

$$G(\mathrm{i}s) \geq \mathrm{e}^{-1}\min\{1, (sw_X(u))^{-n}\} \geq c_X(1+s)^{-n} > 0.$$

Map the upper half-plane to the disk by $z \mapsto (z-\mathrm{i}s)/(z+\mathrm{i}s)$ and apply Jensen's formula at its center, letting the disk radius increase to one. The zeros in the upper half-plane satisfy

$$\sum_{\operatorname{Im}\lambda>0} \log\left|\frac{\lambda+\mathrm{i}s}{\lambda-\mathrm{i}s}\right| \leq -\log G(\mathrm{i}s) \leq C_X + n\log(1+s).$$

The inequality $-\log r \geq (1-r^2)/2$ for $0 < r \leq 1$ bounds each logarithm from below by $2s\operatorname{Im}\lambda/|\lambda+\mathrm{i}s|^2$. For $|\lambda| > s$ the denominator is at most $4|\lambda|^2$. Apply the upper half-plane estimate in direction $-u$ for the lower half-plane zeros. This proves (27). □

5.3. **Linearity of the imaginary growth.**

Proof of Proposition 5.1. Normalize $A(0) = 1$ and $DA(0) = 0$ by dividing by its value at zero and multiplying by a linear exponential. This changes the growth in (22) by a real linear functional of u. By Lemma 5.2, (23) and (24) hold. For each u pair the radial roots as $\lambda, -\overline{\lambda}$, including multiplicity. The symmetry is the one in the hypothesis, and there are no fixed roots by (26). Define

$$Q_u(z) = \prod_{\{\lambda,-\overline{\lambda}\}} (1-z/\lambda)(1+z/\overline{\lambda}), \qquad S(u) = \sum_{\{\lambda,-\overline{\lambda}\}} (1/\lambda - 1/\overline{\lambda}).$$

In each pair the linear coefficient has absolute value $2|\operatorname{Im}\lambda|/|\lambda|^2$, and the quadratic coefficient has absolute value $|\lambda|^{-2}$. The sums converge by the tail bound (27), the uniform count $n_{\mathcal{D}}(t,u) \leq C_X t$, and the root-free disk. These bounds are uniform in u and give absolute, locally uniform convergence of the paired product in z. In the genus-one Hadamard product each factor has value one and derivative zero at zero, so the value and derivative normalizations remove the prefactor. The two genus-one factors of a pair multiply to $(1-z/\lambda)(1+z/\overline{\lambda})\exp(z(1/\lambda - 1/\overline{\lambda}))$. This gives

$$A(zu) = \exp(zS(u))Q_u(z). \tag{28}$$

The numbers $S(u)$ are purely imaginary and uniformly bounded by (27) and the count of roots in a fixed disk.

For $|z| \le R$, bound $\log|Q_u(z)|$ above by

$$\sum_{\{\lambda,-\overline{\lambda}\}} \log(1 + 2R|\operatorname{Im}(1/\lambda)| + R^2/|\lambda|^2).$$

Use $\log(1+a+b) \le \log(1+a) + \log(1+b)$ for $a, b \ge 0$. The sum of the quadratic logarithms is $o(R)$ uniformly by the counting integral in (25), with $n_{\mathcal{D}}$ in place of n_P, and (24). For the linear logarithms, the pairs with $|\lambda| > s$ contribute at most $2R\sum_{|\lambda|>s} |\operatorname{Im}(1/\lambda)|$. The remaining pairs cost $O_s(\log(1+R))$ by the zero count and the root-free disk. Letting $s \to \infty$ in (27) proves the uniform upper bound $\log|Q_u(z)| \le o(R)$ for $|z| \le R$.

For $z = \pm \mathrm{i}R$, each individual factor satisfies

$$\left|1 - \frac{z}{\lambda}\right| = \frac{|\lambda - z|}{|\lambda|} \ge \frac{|\operatorname{Re}\lambda|}{|\lambda|} \ge c_n.$$

The pairs with $|\lambda| \le 2R$ therefore contribute at least $-o(R)$ to $\log|Q_u(z)|$. For $|\lambda| > 2R$, apply $\log|1-w| = -\operatorname{Re} w + O(|w|^2)$ when $|w| \le 1/2$. The absolute sum of the paired linear terms is at most $2R\sum_{|\lambda|>2R} |\operatorname{Im}(1/\lambda)| = o(R)$. The remainders are bounded by a constant times

$$R^2 \sum_{|\lambda|>2R} \frac{1}{|\lambda|^2} \;\le\; 2R^2 \int_{2R}^{\infty} \frac{n_{\mathcal{D}}(t,u)}{t^3}\,\mathrm{d}t \;=\; o(R)$$

uniformly, by (24). Thus

$$\log|Q_u(\pm \mathrm{i}R)| = o(R) \qquad \text{uniformly for } u \in \mathbb{S}^{n-1}. \tag{29}$$

In particular the two imaginary limits in each direction exist and are opposite, with the positive one $\operatorname{Re}(\mathrm{i}S(u))$. The upper bound for Q_u and boundedness of $S(u)$ give $\max_{|z|\le R}|A(zu)| \le \exp(CR)$ for all large R, uniformly in u. If A_m is the degree-m Taylor term of A, Cauchy's estimate on the real radial lines gives $\sup_{|u|\le 1,\ u\in\mathbb{R}^n} |A_m(u)| \le \mathrm{e}^{CR}R^{-m}$. The polarization bound from the proof of Lemma 5.2 gives $|A_m(\zeta)| \le \mathrm{e}^{CR}(2\mathrm{e}|\zeta|/R)^m$. Summation on $|\zeta| \le R/(4\mathrm{e})$ gives $|A(\zeta)| \le 2\mathrm{e}^{CR}$ there, establishing finite exponential type on $\mathbb{C}^n$.

Set $u_R(\zeta) = R^{-1}\log|A(R\zeta)|$. These functions are subharmonic on $\mathbb{R}^{2n}$, locally uniformly bounded above, and $u_R(0) = 0$. Subharmonic compactness [18, Theorem 4.1.9] gives a sequence $R_j \to \infty$ for which u_{R_j} converges in L^1_{loc} to a subharmonic function v. The value at zero excludes local uniform convergence to $-\infty$. For every fixed $T > 0$, (23) gives

$$\sup_{|\zeta|\le T} \big(u_R(\zeta) + u_R(-\zeta)\big) \le R^{-1}\log \max_{|\xi|\le RT} |P(\xi)| \longrightarrow 0.$$

Passing to the L^1_{loc} limit shows that $v(\zeta) + v(-\zeta) \le 0$ almost everywhere. The submean inequality for the sum yields the same bound everywhere. At the origin, $u_R(0) = 0$ and the submean inequality give

$$0 \le \frac{1}{|B(0,t)|}\int_{B(0,t)} v(\zeta)\,\mathrm{d}\zeta \qquad (t > 0)$$

on passing to the limit. Upper semicontinuity then gives $v(0) \geq 0$. The sum has maximum zero at the origin, so it vanishes everywhere. Consequently $-v = v(-\cdot)$ is subharmonic, and v is harmonic. The exponential type bound on A implies $v(\zeta) \leq C|\zeta|$ almost everywhere and hence everywhere by continuity. Oddness gives the corresponding lower bound. On balls of radius tending to infinity centered at any fixed point, the interior second-derivative estimate for v tends to zero, by this linear growth bound. Thus v is real linear.

At each ζ, take the submean bound for u_{R_j} on a ball centered at ζ, pass to the limit of the averages, and use the mean value property of v. This gives $\limsup_{j\to\infty} u_{R_j}(\zeta) \leq v(\zeta)$. For $\zeta = \pm \mathrm{i}u$, the limits of $u_R(\pm \mathrm{i}u)$ as $R \to \infty$ are $g(u) = \mathrm{Re}(\mathrm{i}S(u))$ and $-g(u)$ by (28) and (29). Thus $g(u) \leq v(\mathrm{i}u)$ and $-g(u) \leq v(-\mathrm{i}u) = -v(\mathrm{i}u)$, which gives equality. This proves (22) with $\ell_A(u) = v(\mathrm{i}u)$ under the chosen normalization. □

6. Determination of the bodies

For convex bodies K, L, set

$$\Delta = \operatorname{div} F_K - \operatorname{div} F_L, \qquad \mathcal{D}_K = \Delta_+, \qquad \mathcal{D}_L = (-\Delta)_+,$$

where positive parts are taken coefficientwise. By (18), both effective divisors are invariant under $\zeta \mapsto -\overline{\zeta}$.

Lemma 6.1. *If each of $\mathcal{D}_K, \mathcal{D}_L$ satisfies* (21) *on some positive-measure set of real directions, then K and L are translates.*

Proof. Let A_K, A_L be their entire representatives of order at most one, with $A_K(0) = A_L(0) = 1$. The identity $\mathcal{D}_K - \mathcal{D}_L = \Delta$ makes the divisors of $F_K A_L$ and $F_L A_K$ equal. On each complex line through zero the restrictions have the same zeros with multiplicity, by the restriction rule in Section 4. Both restricted functions have order at most one. Their quotient on the line extends through the zeros and is the exponential of an affine function by Hadamard factorization. In a neighborhood of zero in $\mathbb{C}^n$, the quotient is holomorphic and nonvanishing. Its value c and its logarithmic differential Λ at zero fix the constant factor and the linear term on each line. Varying the line gives

$$F_K(\zeta)A_L(\zeta) = c\exp(\Lambda \cdot \zeta)F_L(\zeta)A_K(\zeta) \qquad (\zeta \in \mathbb{C}^n) \tag{30}$$

for $c \neq 0$ and $\Lambda \in \mathbb{C}^n$.

For every real unit u,

$$\lim_{r\to\infty} r^{-1} \log F_X(-\mathrm{i}ru) = h_X(u). \tag{31}$$

The integral is positive and is at most $F_X(0)\mathrm{e}^{rh_X(u)}$. A fixed cap of depth $\delta > 0$ has positive volume and gives a lower bound of that volume times $\mathrm{e}^{r(h_X(u)-\delta)}$. Letting $\delta \downarrow 0$ proves (31). The functions $F_X(\pm \mathrm{i}ru)$ are positive,

so the two subdivisors also avoid these rays. Evaluate (30) at $-\mathrm{i}ru$, take logarithms of moduli, and divide by r. Proposition 5.1 and (31) give

$$h_K(u) - h_L(u) = \operatorname{Re}(-\mathrm{i}\Lambda \cdot u) + \ell_{A_K}(-u) - \ell_{A_L}(-u).$$

The right-hand side is a real linear functional of u. Support functions determine convex bodies, and adding a linear functional to a support function translates the body. □

Lemma 6.2. *Let $E \subset \mathbb{S}^{n-1}$ be measurable and suppose that $0 < m \le s_X(-u) \le M < \infty$ and $s_X(u) \le q_0^2 s_X(-u)$ almost everywhere on E, for some $0 \le q_0 < 1$. The number of zeros of $F_X(zu)$ in $|z| \le R$, $\operatorname{Im} z \ge 0$, integrated over E with multiplicities, is $o(R)$.*

Proof. Set $a = a_X(u)$, $p = \beta + 1$, and $c(u) = (2\pi)^\beta \sqrt{s_X(-u)}$. Choose $B > 0$ so small that $q_0 \exp(B \sup w_X) < 1$. Use the analytic function

$$G_u(z) = c(u)^{-1}(-\mathrm{i}(z + \mathrm{i}(B+1)))^p \mathrm{e}^{-\mathrm{i}az} F_X(zu) \qquad (\operatorname{Im} z \ge -B),$$

where the power is evaluated using the logarithm on the right half-plane. The multiplier is nonzero there. By (4) in direction $-u$, $|G_u(\mathrm{i}R)| \to 1$ for almost every $u \in E$. On the line $\operatorname{Im} z = -B$ define

$$\theta(u) = \mathrm{e}^{-\mathrm{i}\pi\beta + Bw_X(u)} \sqrt{s_X(u)/s_X(-u)}, \qquad H_u(x) = |1 - \theta(u)\mathrm{e}^{\mathrm{i}w_X(u)|x|}|.$$

There is a fixed $q < 1$ with $|\theta(u)| \le q$, hence $1 - q \le H_u \le 1 + q$. By Proposition 3.1, for any $\varepsilon, \eta > 0$ there is $V \subset \mathbb{S}^{n-1}$ with $\sigma(\mathbb{S}^{n-1} \setminus V) < \varepsilon$ such that

$$\limsup_{T\to\infty} \frac{1}{T} \int_{E\cap V} \int_{T<|x|<2T} \big(|G_u(x - \mathrm{i}B)| - H_u(x)\big)^2 \,\mathrm{d}x \,\mathrm{d}\sigma(u) \le C\eta^2. \tag{32}$$

For $x > 0$, the normalized stationary model has modulus $(|x + \mathrm{i}|/|x - \mathrm{i}B|)^p H_u(x)$. The prefactor tends to one uniformly as $x \to \infty$, and the multiplier of the error in (9) is bounded for $x \ge 1$. The negative band has the same modulus by $F_X(-\overline{\zeta}) = \overline{F_X(\zeta)}$.

After deleting a further set of measure at most ε from $E \cap V$, write E' for the remaining set. Egorov's theorem, applied to the limits at integer R, gives $\log|G_u(\mathrm{i}R)| \to 0$ uniformly on E' along those integers. The Poisson–Jensen inequality in the half-plane above $\operatorname{Im} z = -B$, centered at $\mathrm{i}R$, gives

$$\sum_{\operatorname{Im}\lambda > -B} \log\left|\frac{\lambda + \mathrm{i}(R+2B)}{\lambda - \mathrm{i}R}\right| \le \int_{\mathbb{R}} P_{R+B}(x) \log|G_u(x - \mathrm{i}B)| \,\mathrm{d}x - \log|G_u(\mathrm{i}R)|. \tag{33}$$

Here $P_t(x) = t/(\pi(x^2 + t^2))$, and the sum is over the radial zeros with multiplicities. To justify the inequality, G_u is analytic near each finite point of the boundary line and has at most polynomial growth in the closed half-plane: the transform after multiplication by $\mathrm{e}^{-\mathrm{i}az}$ is bounded there by $\operatorname{vol}(X)\exp(Bw_X(u))$. Compose with the disk map sending 0 to $\mathrm{i}R$ and apply the disk Jensen formula. The positive logarithms of the composed

function on concentric circles are bounded by an integrable function of the angle, using the polynomial growth and $|1-\rho\,\mathrm{e}^{\mathrm{i}\theta}| \geq c|\theta|$ for $1/2 \leq \rho < 1$ and $|\theta| \leq \pi$. Fatou's lemma applied to the difference between that upper bound and the integrand gives the inequality as $\rho \uparrow 1$. The image of the angular probability measure is the Poisson kernel on the line.

The Poisson integral of $\log H_u$ tends to zero uniformly in $u \in E$. To see this, expand $\log H_u(x) = -\operatorname{Re}\sum_{k\geq 1} \theta(u)^k \mathrm{e}^{\mathrm{i}k w_X(u)|x|}/k$. Integration by parts on $x > 0$ gives

$$\left| \int_0^\infty P_t(x) \mathrm{e}^{\mathrm{i}k w_X(u) x}\, \mathrm{d}x \right| \leq \frac{2P_t(0)}{k \inf w_X}.$$

The series converges absolutely and its weighted integral is $O(t^{-1})$. Also $\log v \leq \log H + (v-H)/H$ for $v, H > 0$. By (32) and Cauchy–Schwarz, the integral of $\big||G_u| - H_u\big|$ over any sufficiently large dyadic band $T < |x| < 2T$ and E' is bounded by $C\eta T$. Multiplying by the maximum of the Poisson kernel on each band and summing costs at most $C\eta$, since $\sum_{j\geq 0}(R+B)2^j/((2^j)^2+(R+B)^2)$ is bounded. The contribution from a fixed bounded range tends to zero since H_u and F_X are uniformly bounded there and $c(u)$ is uniformly bounded away from zero. Thus after integration over E' the right-hand side of (33) has limsup at most $C\eta$.

For a zero with $|\lambda| \leq R$ and $y = \operatorname{Im}\lambda \geq 0$, the difference of the squared numerator and denominator in its logarithm is $4(R+B)(y+B)$. The denominator is at most $4R^2$, and $y + B \geq B$. Each such zero therefore contributes at least $\frac{1}{2}\log(1 + B/R)$, which is bounded below by c_B/R for $R \geq 1$. Consequently the upper limit of the desired integrated count divided by R, on E', is at most $C_B\eta$. On $E \setminus E'$ the uniform Jensen count costs at most $2C_X\varepsilon R$. Letting η and ε tend to zero proves the claim along integers, and monotonicity of the count extends it to all R. □

Proof of Theorem 1.1. Since U_K has positive measure, a positive-measure set has $s_K(-u) > s_K(u)$. On a positive-measure subset E we have $m \leq s_K(-u) \leq M$ and $s_K(u) \leq q_0^2 s_K(-u)$ for some $0 < m \leq M < \infty$ and $0 \leq q_0 < 1$: the sets given by $m = 1/j$, $M = j$, $q_0 = 1 - 1/j$ exhaust the strict inequality. By (5) and (6), the unordered pairs of densities for K, L coincide almost everywhere. After restricting E to a positive-measure subset and, if necessary, reflecting L, we have $s_L(-u) = s_K(-u)$ and $s_L(u) = s_K(u)$ there.

The divisor identity $\Delta + j^*\Delta = 0$, for $j(\zeta) = -\zeta$, gives $\mathcal{D}_L = j^*\mathcal{D}_K$. In $|z| \leq R$ the zeros of $\mathcal{D}_K$ in the upper half-plane and on the real line are bounded in number by the zeros of $F_K(zu)$ there. Negation maps the zeros in the open lower half-plane with multiplicities to zeros of $\mathcal{D}_L$ in the upper half-plane, where their number is bounded by the zeros of $F_L(zu)$. Lemma 6.2 for both bodies shows that the total count is $o(R)$ in integral over E. The counts for $\mathcal{D}_K, \mathcal{D}_L$ agree in centered disks. Lemma 6.1 now proves that K, L are translates after the possible reflection. □

Proof of Corollary 1.2. If $\sigma(U_K) > 0$, apply Theorem 1.1. Otherwise the definition of U_K gives $s_K(u) = s_K(-u)$ almost everywhere. Absolute continuity then gives $S_K = S_{-K}$, and Minkowski uniqueness [25, Theorem 7.2.1] makes K centrally symmetric. Centrally symmetric convex bodies are determined by their covariograms, as in [7, Theorem 5.3]. For the convex competitors here, equality of covariograms gives $K - K = L - L$ and $\mathrm{vol}_n(K) = \mathrm{vol}_n(L)$. Hence

$$\mathrm{vol}_n(L - L) = \mathrm{vol}_n(K - K) = 2^n \mathrm{vol}_n(K) = 2^n \mathrm{vol}_n(L).$$

This is equality in Brunn–Minkowski for L and $-L$ [25, Section 7.1]. The bodies are homothetic, and equal volume makes them translates, so L is centrally symmetric. For each centrally symmetric body X with center c_X, one has $X - X = 2(X - c_X)$, which identifies L and K up to translation.

If h_{K-K} is $C^{1,1}$, the summand h_K is also $C^{1,1}$ by [19, Proposition 2.5]. The inverse support map $u \mapsto \nabla_{\mathbb{S}^{n-1}} h_K(u) + h_K(u)u$ is then Lipschitz. The boundary points with outer normal in a spherical null set lie in its image of that set, which is null for boundary area. Thus S_K is absolutely continuous. If K has constant width w, then $h_{K-K} = w$ on $\mathbb{S}^{n-1}$.

For C^2_+ bodies every normal has a unique supporting point, since a boundary segment would contradict positive curvature. The Gauss map is a C^1 diffeomorphism onto the sphere, its tangential derivative being the positive-definite curvature operator. Changing variables gives the continuous positive density s_X as the reciprocal of the Gauss curvature. □

Proof of Theorem 1.3. Choose a positive-measure subset of T_K with uniform endpoint density bounds and apply Lemma 4.1. At any resulting simple radial zero $\zeta_0 = zu$, $\partial_u F_K(\zeta_0) \neq 0$. The holomorphic inverse function theorem supplies local coordinates w in which $w_1 = F_K$, and the local divisor of F_K is $w_1 = 0$ with multiplicity one. Equality of the zero sets gives $F_L(0, w_2, \ldots, w_n) = 0$ in these coordinates, so the power series in w_1 shows that w_1 divides F_L. The zeros counted in (19), including the reflected ones, avoid the support of $\mathcal{D}_K$. Lemma 4.2 applies to that divisor. Exchanging K and L proves (21) for $\mathcal{D}_L$ on a positive-measure subset of T_L. Lemma 6.1 uses the two sets of directions independently and yields the claimed translation. □

7. Width and ordered curvature ratio in dimension three

Proof of Theorem 1.4. Let $p = h_K - h_L$ and $P(x) = |x| p(x/|x|)$ for $x \neq 0$, with $P(0) = 0$. The width identity holds everywhere by continuity, so p and P are odd. Identify the tangent planes at u and $-u$ with the same Euclidean plane $u^\perp$, and set $A_p(u) = A_K(u) - A_L(u)$. The homogeneous extension is $C^{1,1}_{\mathrm{loc}}$ off the origin. Its Hessian has radial row and column zero and tangent block $|x|^{-1} A_p(x/|x|)$ almost everywhere, as in Section 2. Apply the chain rule twice to $P(-x) = -P(x)$ to obtain $D^2P(-x) = -D^2P(x)$ at points of twice differentiability in both directions. Restriction to the

tangent plane yields $A_p(-u) = -A_p(u)$ almost everywhere. For $0 \le t \le 1$ let $R_t(u) = (1-t)A_L(u) + tA_K(u)$. Use (2) at u and $-u$ to write the logarithm of the ratio identity as

$$[\log\det R_t(u) - \log\det R_t(-u)]_{t=0}^{t=1} = 0.$$

Jacobi's determinant formula and integration in t give

$$0 = \operatorname{tr}\left[\left(\int_0^1 \left(R_t(u)^{-1} + R_t(-u)^{-1}\right)\,\mathrm{d}t\right) A_p(u)\right].$$

The plus sign comes from $A_p(-u) = -A_p(u)$. The $C^{1,1}$ bound and positivity give $mI \le R_t(\pm u) \le MI$ uniformly almost everywhere, for some $0 < m \le M < \infty$. The matrix in parentheses is bounded between $2M^{-1}I$ and $2m^{-1}I$. For $u = x/|x|$, let $B(x)$ have this matrix on $u^\perp$ and eigenvalue one on the radial direction. Set $B = I$ on the exceptional null set, including at zero. This gives a measurable uniformly elliptic coefficient field and $\operatorname{tr}(B(x)D^2P(x)) = 0$ almost everywhere.

The bounds $|P(x)| \le C|x|$, $|DP(x)| \le C$, and $|D^2P(x)| \le C/|x|$ hold off the origin, with the Hessian bound holding almost everywhere. In dimension three $\int_{|x|\le 1} |x|^{-2}\,\mathrm{d}x < \infty$. To see that these derivatives are also weak derivatives across zero, integrate by parts outside the ball of radius ε. The first-derivative boundary term is $O(\varepsilon^3)$ by the bound on P, and the boundary term for the derivative of DP is $O(\varepsilon^2)$. Both vanish in the limit, so $P \in W^{2,2}_{\mathrm{loc}}(\mathbb{R}^3)$ and is an order-one homogeneous strong solution of the uniformly elliptic equation. Theorem 1.1 of Han, Nadirashvili and Yuan [17] gives that P is linear, which amounts to translating the body.

Finally, for a C^2_+ boundary the inverse Gauss map $p_X(u) = \nabla_{\mathbb{S}^{n-1}} h_X(u) + h_X(u)u$ is C^1. This identity gives $h_X \in C^2$, and its matrix A_X is the inverse of the Gauss map differential. Compactness therefore gives the uniform positive lower bound in the hypotheses. □

AI assistance

OpenAI Codex assisted with literature searches, proof development and checking, and preparation of the manuscript.

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