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# Reconstructing three-point supports from their ordered convolution squares

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**Abstract**

Given two three-element subsets of an abelian group, we classify when their labelled ordered sum counts agree, with diagonal pairs included. Equality comes either from a translation by an element killed by doubling or from an explicit exceptional configuration involving both doubling and tripling. We give a proof using parity and the doubling fibres, and identify the corresponding Lean theorem registered in Palomar.

| Symbol | Definition or domain | Role |
|---|---|---|
| $G, S, T$ | $G$ abelian, $S, T \subseteq G$, $\lvert S\rvert = \lvert T\rvert = 3$ | Ambient additive group and the two supports. |
| $R_S(z)$ | $\#\{(x,y) \in S \times S : x + y = z\}$ | Ordered coefficient at the group element $z \in G$. |
| $H, Q, \pi$ | $H = \{u \in G : 2u = 0\}$, $Q = G/H$, $\pi : G \to Q$ | Kernel of doubling, quotient and quotient map. |
| $S + u$ | $\{s + u : s \in S\}$ | Translation by $u \in G$. |
| $[z]$ | $[z](w) = \mathbf{1}_{w=z}$ | Point mass at $z$, used to expand the coefficient function. |

**Reconstruction from sums.** The observation records nine sums with multiplicity at their original labels in $G$. A translation by $u \in H$ preserves every label because each sum shifts by $2u$. The question is whether these translations exhaust the ambiguity. Classical fixed-order reconstruction [1] uses distinct summands, and subset-sum reconstruction [2] uses sums of all subset sizes. Here the diagonal terms are part of the observation and their parity drives the proof.

**Theorem 1.** *Let $G$ be any abelian group and let $S, T \subseteq G$ each have exactly three elements. Then $R_S(z) = R_T(z)$ for every $z \in G$ if and only if at least one of the following holds.*

1. *There is $u \in H$ with $T = S + u$.*
2. *There are $a, b, d, u \in G$ with $d \in H \setminus \{0\}$, $u \in H$, $2a \neq 2b$ and $3(b - a) \in \{0, d\}$, such that*

$$S = \{a, b, b + d\}, \qquad T = \{a, 2b - a, 2b - a + d\} + u.$$

*Proof.* Modulo two, every off-diagonal pair cancels with its reverse. Hence $R_S \equiv \sum_{s \in S} [2s] \pmod 2$. The fibres of doubling are cosets of $H$. Three points occupy those cosets with multiplicity pattern $(1, 1, 1)$, $(3)$ or $(1, 2)$. The first pattern has three odd coefficients, whereas each other pattern has one. In pattern $(3)$ all sums belong to a single coset of $H$. In pattern $(1, 2)$ write the support as $\{a, b, b + d\}$ with $d \in H \setminus \{0\}$ and $x = \pi(a) \neq y = \pi(b)$. The sums $2b$ and $a + b$ occur and have distinct quotient labels $2y$ and $x + y$. Thus equal squares have the same pattern.

In patterns $(1, 1, 1)$ and $(3)$ the parity data and the pattern determine the full multiset of diagonal terms. Subtract them and divide by two to recover the three off-diagonal unordered pair sums. These sums are distinct by cancellation. Any bijection between the three edges of two triangles is induced by matching the vertices opposite those edges. Relabel the vertices $s_i, t_i$ accordingly. Writing $u_i = t_i - s_i$ gives $u_i + u_j = 0$ for every $i \neq j$, which forces $u_1 = u_2 = u_3 \in H$.

For pattern $(1,2)$, the expansion is

$$R_S = [2a] + 2([2b] + [2b+d] + [a+b] + [a+b+d]).$$

If $R_T = R_S$, its singleton has double $2a$, so write $T = \{a+u, c, c+e\}$ with $u \in H$ and $e \in H \setminus \{0\}$. Set $z = \pi(c) \neq x$. Removing the singleton coefficient leaves two pairs of distinct labels in distinct cosets for each support. Their matching is either direct or crossed.

In a direct matching $x + z = x + y$, hence $z = y$. Matching the actual pairs in this coset gives $\{a+u+c, a+u+c+e\} = \{a+b, a+b+d\}$. The difference of the two points is $d$ up to sign, and $-d = d$, so $e = d$. Subtracting $a + u$ now gives $\{c, c+e\} = \{b+u, b+d+u\}$, since $-u = u$.

In a crossed matching $x + z = 2y$ and $2z = x + y$. Matching the actual pair in coset $2y$ gives $e = d$ and $\{c, c+e\} = \{2b-a+u, 2b-a+d+u\}$ by the same subtraction. The other pair in $R_T$ is $\{4b-2a, 4b-2a+d\}$. It equals $\{a+b, a+b+d\}$ precisely when $3(b-a) \in \{0, d\}$.

Conversely a translation by $H$ preserves each ordered coefficient. In the exceptional case, first take $u = 0$. The singleton term is unchanged and the pair of mixed sums for $T$ is $\{2b, 2b+d\}$. The other pair is $\{4b-2a, 4b-2a+d\}$, which equals $\{a+b, a+b+d\}$ by the tripling condition. The expansions agree. Translating by $u \in H$ completes the proof. □

**A six-element example.** In $G = \mathbb{Z}/6\mathbb{Z}$, take $S = \{0,1,4\}$ and $T = \{0,2,5\}$. Here $a = 0$, $b = 1$, $d = 3$ and $3(b-a) = d$. In the order of residues $0,1,2,3,4,5$ both coefficient vectors are $(1,2,2,0,2,2)$. Modulo $H = \{0,3\}$ their singleton coset is the same and their double cosets differ. Any ambient translation between the supports would fix the singleton coset and therefore every coset, which is impossible.

**Relation to the cyclic literature.** Yang and Tang [3] study this same ordered coefficient function on cyclic groups under restrictions on the size of the symmetric difference. The example above is covered by Chen and Yan [4, Theorem 1.2(ii)]: the union omits 3, the intersection is $\{0\}$, and the other two antipodal pairs are split between the supports.

In fact, this cyclic example represents every exceptional configuration in Theorem 1. Set $v = b - a$ and take $g = v$ if $3v = d$, or $g = v + d$ if $3v = 0$. In either case $3g = d$ and $2g = 2v \neq 0$, so $g$ has order six. Subtracting $a$ from $S$ and $a + u$ from $T$ gives $\{0, g, 4g\}$ and $\{0, 2g, 5g\}$, respectively. Thus the exceptional forms are embeddings of the same cyclic pattern, with the translations specified in the theorem. The classification also proves necessity for arbitrary abelian ambient groups. This comparison leaves the full bibliographic comparison and priority assessment open.

### Formal statement and provenance

Theorem 1 corresponds to `Couret.Palomar.classification` in the Palomar record PALOMAR-2026-11-08-000010 v1. The formal statement uses finite sets with decidable equality and has no finiteness hypothesis on the ambient group. The immutable Lean statement can be read alongside the proof above. This exposition adapts the earlier public STROBO-2TORS written argument [5]. ChatGPT Codex drafted the exposition from that argument and the registered Lean source. Independent human mathematical review and a full priority assessment remain open.

## References


[1] J. L. Selfridge and E. G. Straus. On the determination of numbers by their sums of a fixed order. *Pacific J. Math.* 8 (1958), 847–856. doi:10.2140/pjm.1958.8.847.

[2] F. Glaudo and N. Kravitz. Reconstructing a set from its subset sums: 2-torsion-free groups. *Discrete Analysis* 2024:14. doi:10.19086/da.125856.

[3] Q.-H. Yang and M. Tang. Representation functions on finite sets with extreme symmetric differences. *J. Number Theory* 180 (2017), 73–85. doi:10.1016/j.jnt.2017.03.024.

[4] S.-Q. Chen and X.-H. Yan. Partitions of $\mathbb{Z}_m$ with identical representation function. *Bull. Aust. Math. Soc.* 103 (2021), 204–209. doi:10.1017/S0004972720000945.

[5] A. Couret. STROBO-2TORS: written classification and proof (2026). Public working note at commit b3e95f8.