# A RESIDUE OBSTRUCTION FOR THE FAMILY $F_2$ WITH $(m,n)=(3,5)$: THE DEGREE PAIR $(75,125)$ IN THE PLANE JACOBIAN PROBLEM

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Abstract. Guccione, Guccione and Valqui list the possible shapes of a counterexample to the plane Jacobian conjecture of small degree; the first shape of maximal degree 125 is the family $F_2$ with $(m,n)=(3,5)$, of degrees $(75,125)$. We show that no standard $(3,5)$-pair of this shape exists. The proof combines three results of Guccione–Guccione–Valqui with the residue theorem on a generic fibre of $P$: on such a fibre the form $y^2Q\,dQ$ has no residue along the left side of the Newton polygon, while its residues along the right side add up to a quantity that cannot vanish. All statements are proved relative to the cited results; the exact computations are reproduced by an ancillary script. The argument has not yet been examined by specialists.

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## 1. Statement

Throughout, $K$ is an algebraically closed field of characteristic zero, $L=K[x,y]$, and $[P,Q]=P_xQ_y-P_yQ_x$. We use the notation of [1] (Guccione–Guccione–Valqui): for a weight $(\rho,\sigma)$, $v_{\rho,\sigma}$ is the weighted degree, $\ell_{\rho,\sigma}$ the leading form, and $\mathrm{st}_{\rho,\sigma}$, $\mathrm{en}_{\rho,\sigma}$ its end points. *Standard $(m,n)$-pairs*, their *regular corners* and their *starting triple* are defined in [1, Definitions 4.3, 5.5, 6.2].

**Standing data 1.1.** $(P,Q)$ is a standard $(3,5)$-pair in $L$ with $[P,Q]=J\in K^\times$, $\deg P=75$, $\deg Q=125$, starting triple $\big((5,20),(1,0),(5,-1)\big)$, and $(-1,1)<\mathrm{Succ}_P(1,0)<(-1,0)$ as in [1, Corollary 5.21(4)].

By [2, Theorem 2.20 and Sections 5–6], this is the shape (family $F_2$, $j=1$) that a counterexample of degrees $(75,125)$ produced by [1, Corollary 5.21] must have; see also [3] for the shapes of smaller degree.

**Theorem 1.2.** *There is no pair $(P,Q)$ as in Standing data 1.1.*

**Consequences of the standing data used below.** Put $t:=xy^5$. Adding constants and rescaling $x$, $P$, $Q$ we may assume the following.

(D1) The support of $P$ lies in $\{(i,j):5i-j\le 15,\ i+j\le 75\}$, that of $Q$ in $\{5i-j\le 25,\ i+j\le 125\}$, and

$$\ell_{5,-1}(P)=\big(x\,h(t)\big)^3,\qquad \ell_{5,-1}(Q)=\mu\big(x\,h(t)\big)^5,\qquad \mu\in K^\times,$$

with $h\in K[t]$ monic of degree 4, $h(0)\neq 0$; thus $\mathrm{st}_{5,-1}(P)=(3,0)$ and $\mathrm{en}_{5,-1}(P)=(15,60)$ [1, Proposition 6.1, Theorem 7.6(2)]. We write $u(t):=t\,h(t)$.

(D2) Every root of $h$ has multiplicity 1 or 2, and $h$ has at least one double root. Indeed, write $\ell_{5,-1}(P)=x^3\mathfrak{p}(z)$ with $z=x^{1/5}y$; a root of $h$ of multiplicity $k$ gives roots of $\mathfrak{p}$ of multiplicity $3k$. By [1, Proposition 5.16] some root has multiplicity $\ge\frac{15}{4}$, so $k\ge 2$ occurs. For any root with $k\ge 2$, [1, Proposition 5.18] produces a $(3,5)$-pair in $L^{(5)}$ with a regular corner at $A^{(1)}=(1+k/5,\,k)$. At such a corner, Case II with several factors is excluded for $k\in\{3,4\}$ by [1, Proposition 5.14(4)], because $\gcd(5+k,k)=1$; Case III only translates [1, Proposition 5.17]; and Case I is excluded for $k\in\{3,4\}$ by the end point conditions of [1, Proposition 5.13], which have no solution (ancillary script). The same shapes are recorded in the tables of [2, Section 5], where $F_2$, with final corner $(7/5,2)$, is the only family starting at $(5,20)$ that contains $(m,n)=(3,5)$.

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## 2. Homogeneous brackets

Let $N \geq 1$ be an integer and $R_\omega := K[\tau][\omega, \omega^{-1}]$, with the bracket $\{f, g\}_N := \omega^N(f_\tau g_\omega - f_\omega g_\tau)$. For $s \in \mathbb{Q}_{>0}$ put $\nu_s(\tau^a \omega^{-d}) := d - sa$, let $\nu_s(f)$ be the maximum over the monomials of $f$ and $\ell_s(f)$ the sum of the terms realising it. An element with a single $\nu_s$-value $a$ has the form $\omega^{-a} F(\sigma)$ with $\sigma := \tau\omega^{-s}$ and $F \in K[\sigma]$.

**Lemma 2.1.** *(a)* $\{\omega^{-a}F(\sigma), \omega^{-b}G(\sigma)\}_N = \omega^{N-a-b-s-1}\big(aFG' - bF'G\big)(\sigma)$.
*(b)* $\nu_s(\{f, g\}_N) \leq \nu_s(f) + \nu_s(g) + s + 1 - N$, *with equality if and only if* $\{\ell_s f, \ell_s g\}_N \neq 0$.
*(c) If* $\{f, g\}_N = c \in K^\times$, *then* $\Sigma_s := \nu_s(f) + \nu_s(g) + s + 1 - N \geq 0$ *for every* $s$; *if* $\Sigma_s > 0$ *then* $aFG' = bF'G$ *for the leading forms, and if* $\Sigma_s = 0$ *then* $aFG' - bF'G = c$.

*Proof.* (a) is a direct computation; (b) and (c) follow from (a) by comparing $\nu_s$-values. □

The plane enters through $(\tau, W) := (t - \theta,\, y)$ for $\theta \in K$: since $dx \wedge dy = y^{-5}\, dt \wedge dy$, one has $[f, g] = \{f, g\}_5$ with $\omega = W$, and $P = \sum_{d \leq 15} f_d(t)\, W^{-d}$ with $f_{15} = u^3$, $Q = \sum_{d \leq 25} g_d(t) W^{-d}$ with $g_{25} = \mu u^5$.

## 3. The left side

**Lemma 3.1.** *There is* $g \in vK[v]$ *with* $\deg g \leq 3$ *and* $v^2 \mid g$ *such that*

$$\hat{P}(z, v) := P\big(zv^4 + g(v),\, v^{-1}\big), \qquad \hat{Q}(z, v) := Q\big(zv^4 + g(v),\, v^{-1}\big)$$

*are polynomials in* $z, v$. *Moreover* $\hat{P}$ *has total degree* 15, *top form* $H^3$ *with* $H(z, v) := v^5 u(z/v)$, $\hat{P}(z, 0)$ *has degree* 15 *in* $z$, *and*

$$\hat{P}_z \hat{Q}_v - \hat{P}_v \hat{Q}_z = -J\, v^2.$$

*Proof.* Apply the flip $\psi_1$ of [1, Section 8], so that the weight $(5, -1)$ becomes $(\rho_0, \sigma_0) = (-1, 5)$, with $\mathrm{st}_{\rho_0,\sigma_0} = (60, 15)$ and $\mathrm{en}_{\rho_0,\sigma_0} = (0, 3)$ for the flipped $P$. Let $F$ be the element of [1, Theorem 2.6] for $(\rho_0, \sigma_0)$. As in the proof of [1, Proposition 8.1]: if $\mathrm{st}(F) = (1, 1)$ then $\mathrm{en}(F) \neq (1, 1)$, since $F$ is not a monomial [1, Remark 2.5], so $\mathrm{en}(F) \sim \mathrm{en}(P) = (0, 3)$ and $v_{-1,5}(F) = 4$ would give $\mathrm{en}(F) = (0, 4/5)$, which is impossible; hence $\mathrm{st}(F) \sim (60, 15)$, so $\mathrm{st}(F) = (16, 4) = \frac{4}{5} \cdot \frac{1}{3}\,\mathrm{st}(P)$ and $q = 5$ in [1, Corollary 7.4]. That corollary applies to the flipped pair and to every pair obtained from it by a translation of the second variable by a Laurent monomial in the first (the form $\ell_{\rho_0,\sigma_0}$, the degrees $v_{1,1}$, $v_{0,1}$ and $F$ are unchanged). It yields: for every weight between $(1, -4)$ and $(\rho_0, \sigma_0)$ at which the pair has an edge, the edge form of $P$ is a 15th power $R^{15}$. In the original variables: for a weight $(-s, 1)$ with $s < 4$ carrying an edge with end point $(15, 60)$, the form is $(xy^4 - \gamma y^{4-s})^{15}$, so $s \in \mathbb{Z}$, and $s \geq 2$ by the condition on $\mathrm{Succ}_P(1, 0)$. The translation $x \mapsto x + \gamma y^{-s}$ removes this edge and leaves the forms of smaller $s$ unchanged [1, Proposition 3.9]. After at most two such translations ($s = 2, 3$) no edge of weight $(-s, 1)$, $s < 4$, has end point $(15, 60)$. Since the weight $(-4, 1)$ lies within a half-turn of $(5, -1)$, the point $(15, 60)$ is then still leading for $(-4, 1)$ [1, Remark 2.10, Proposition 3.7]; hence every monomial $x^i y^j$ of the translated $P$ satisfies $j \leq 4i$, that is, $x^i y^j = z^i v^{4i-j}$ is a monomial in $z = xy^4$, $v = y^{-1}$. The same holds for $Q$: at a weight where $P$ has an edge, [1, Proposition 7.3] with $T_0 = Q$ gives $[\ell(Q), \ell(P)] = 0$ and proportional values, so $\ell(Q) \sim R^{25}$ and the same translation applies; at the remaining weights $(-s, 1)$, $s < 4$, the values of $P$ and $Q$ are positive, so the leading forms commute [1, Proposition 1.13] and $\ell(Q)$ is the monomial at $(25, 100)$; hence $j \leq 4i$ on the support of $Q$ by continuity. The degree statements follow from $5i - j \leq 15$ in (D1), and the Jacobian from $dx \wedge dy = -v^2\, dz \wedge dv$. □

## 4. Places at infinity of a fibre

Fix a generic $p \in K$ and let $\Gamma_p := \{\hat{P} = p\} \subset \mathbb{A}^2_{z,v}$, with $\bar{\Gamma}_p$ the normalisation of its projective closure. Since $H(1, 0) = 1$, every place $\pi$ of $\bar{\Gamma}_p$ at infinity satisfies $v \to \infty$ and $z/v \to \theta$ with $u(\theta) = 0$; then $y = v^{-1} \to 0$ and $t = xy^5 \to \theta$. So $\pi$ is a branch, at $(\tau, W) = (0, 0)$, of $P - p$ regarded in $K[\tau][W, W^{-1}]$, $\tau = t - \theta$.

**Lemma 4.1.** *Let $P, Q \in R_\omega$ with $\{P,Q\}_N = c \in K^\times$, let $(N,D)$ be one of $(5,15)$, $(5,12)$, $(5,9)$, $(9,27)$, $(9,21)$ and let $0 \le s_* < D/3$. Assume that for all $s$ in some interval $(s_*, s_*+\varepsilon)$ one has $\ell_s(P) = A'\tau^3\omega^{-D}$ and $\ell_s(Q) = B'\tau^5\omega^{-5D/3}$ with $A'B' \neq 0$. Then $D = 3N$ (so the hypotheses cannot hold for the other three pairs), and there is $\kappa \in K[\omega]$ with $\operatorname{ord}_\omega \kappa > s_*$ such that, for $\tau = \kappa(\omega) + \sigma\omega^{D/3}$, both $P$ and $Q$ are polynomials in $\sigma, \omega$. Every branch $\tau(\omega)$ of $P = p$ with $\operatorname{ord}_\omega \tau > s_*$ has $\sigma$ bounded; in particular $Q$ is regular along it.*

*Proof.* Let $s_0$ be the supremum of the $s > s_*$ with $\ell_s(P) = A'\tau^3\omega^{-D}$. It is finite, since otherwise $\tau^3 \mid P$ and $\{P,Q\}_N$ would vanish at $\tau = 0$. Suppose $s_0 < D/3$ and write $\ell_{s_0}(P) = \omega^{-a}F_P$, $\ell_{s_0}(Q) = \omega^{-b}F_Q$; then $a = D - 3s_0 > 0$, $\deg F_P = 3$, $F_P$ has at least two terms, $b \ge 5D/3 - 5s_0 = \frac{5}{3}a$ and $\deg F_Q \le 5$ (a monomial of $P$ of $\tau$-degree $> 3$, or of $Q$ of $\tau$-degree $> 5$, is not leading for any $s > s_*$). By Lemma 2.1(c), either $F_P^b$ is proportional to $F_Q^a$, or $\Sigma_{s_0} = 0$ and $aF_PF_Q' - bF_P'F_Q = c$. In the first case degrees give $b = \frac{5}{3}a$, $F_P^5 \sim F_Q^3$, so $F_P = A'(\sigma - \kappa_0)^3$ with $\kappa_0 \neq 0$ and $s_0 \in \mathbb{Z}$; the translation $\tau \mapsto \tau + \kappa_0\omega^{s_0}$ preserves the bracket and the hypotheses, with $s_*$ replaced by $s_0$. The second case requires a monomial pair $(\tau^1, \tau^0)$ or $(\tau^0, \tau^1)$ with integer exponents in $(\ell_{s_0}P, \ell_{s_0}Q)$ together with $\Sigma_{s_0} = 0$ and $b \ge \frac{5}{3}a > 0$; for the five pairs $(N,D)$ this has no solution $s_0 < D/3$ (ancillary script). Hence after finitely many translations $s_0 \ge D/3$, so $\nu_{D/3}(P) = 0$ and $P \in K[\sigma,\omega]$ with $P(\sigma,0)$ of degree 3. Let $b := \nu_{D/3}(Q) \ge 0$. If $b > 0$, Lemma 2.1(c) gives $F_P'F_Q = 0$ or $-bF_P'F_Q = c$, both impossible; so $b = 0$ and $Q \in K[\sigma,\omega]$. Now $dP \wedge dQ = c\,\omega^{-N}d\tau \wedge d\omega = c\,\omega^{D/3-N}d\sigma \wedge d\omega$ is regular, so $D \ge 3N$; among the five pairs this leaves only $D = 3N$, the other three being contradictory. Finally, for every $s$ with $s_* < s < D/3$ other than the finitely many slopes removed by the translations, $\ell_s(P - p)$ is a monomial, so no root $\tau(\omega)$ of $P - p$ has order $s$; at a removed slope $s_0$ the roots of order $s_0$ are those with $\sigma \to \kappa_0$, and the translation moves them to higher order. Hence every root with $\operatorname{ord}\tau > s_*$ has order $\ge D/3$ in the final coordinate, that is, $\sigma$ is bounded. □

**Corollary 4.2.** *At every place of $\bar\Gamma_p$ at infinity lying over a simple root $\theta$ of $u$ (including $\theta = 0$), the function $\hat Q$ is regular.*

*Proof.* For small $s$, $\ell_s(P) = u'(\theta)^3\tau^3W^{-15}$ and $\ell_s(Q) = \mu u'(\theta)^5\tau^5W^{-25}$; apply Lemma 4.1 with $(N,D) = (5,15)$, $s_* = 0$. □

**Lemma 4.3.** *Let $\theta = c$ be a double root of $h$ and $u(c+\tau) = U\tau^2 + O(\tau^3)$, $U \neq 0$. Then exactly one of the following holds.*

*(i) $\hat Q$ is regular at every place of $\bar\Gamma_p$ over $c$.*

*(ii) After a translation $\tau \mapsto \tau + \kappa_1 W + \kappa_2 W^2$, with $\sigma = \tau W^{-12/5}$,*

$$\ell_{12/5}(P) = W^{-3/5}\big(B\sigma + U^3\sigma^6\big), \qquad \ell_{12/5}(Q) = W^{-1}\,G\Big(1 + 3\varsigma + \tfrac{9}{5}\varsigma^2\Big), \quad \varsigma := \tfrac{U^3}{B}\sigma^5,$$

*with $BG = -J$ and $G = 5\mu B^2/(9U)$. There are exactly two places of $\bar\Gamma_p$ over $c$: a place $\beta_1$ with $\operatorname{ord}_{\beta_1}(y) = 5$, at which $\hat Q$ has a pole of order 5 and $y\hat Q \to -G/5$; and a place $\beta_2$ with $\operatorname{ord}_{\beta_2}(y) = 1$, at which $\hat Q$ has a simple pole and $y\hat Q \to G$.*

*In case* (ii), *$G^6 = 25J^4\mu^2/(81\,U^2)$, and $\kappa_1, \kappa_2, B, G$ do not depend on $p$.*

*Proof.* Work with $f := P - p$ and $Q$ in $K[\tau][W, W^{-1}]$, $N = 5$. For small $s$, $\ell_s(f) = U^3\tau^6W^{-15}$, $\ell_s(Q) = \mu U^5\tau^{10}W^{-25}$. Let $s_0$ be the first $s$ at which $\ell_s(f)$ is not a monomial; as $f$ has a nonzero constant term, $s_0 \le 5/2$. Put $a = \nu_{s_0}(f) = 15 - 6s_0$ and $b = \nu_{s_0}(Q) \ge \frac{5}{3}a$, so that $\Sigma_{s_0} \ge 36 - 15s_0$.

$s_0 < 12/5$. Then $\Sigma_{s_0} > 0$, so $F_P^b \sim F_Q^a$, whence $b = \frac{5}{3}a$ and $F_P = U^3\mathcal{G}^3$, $F_Q = \mu U^5\mathcal{G}^5$ with $\mathcal{G}$ monic quadratic, $\mathcal{G} \neq \sigma^2$. If $\mathcal{G} = (\sigma - \kappa)^2$, then $s_0 \in \{1,2\}$ and we translate by $\kappa W^{s_0}$ and start again. If $\mathcal{G}$ has two distinct roots and $s_0 \in \{1,2\}$, translating to either root gives the situation of Lemma 4.1 with $N = 5$, $D = 15 - 3s_0 \in \{12, 9\}$, $s_* = s_0$, which is contradictory. Otherwise $\mathcal{G} = \sigma^2 - \kappa$ with $2s_0 \in \{1,3\}$; with $W = \omega^2$ one has $\{\cdot,\cdot\}_5 = \frac{1}{2}\{\cdot,\cdot\}_9$, and translating to $\tau = \sqrt{\kappa}\,\omega^{2s_0}$ gives the situation of Lemma 4.1 with $N = 9$, $D = 30 - 6s_0 \in \{27, 21\}$, $s_* = 2s_0$.

For $D = 21$ this is contradictory; for $D = 27$ it gives (i), the six roots of $f$ with $\tau \to 0$ being those near $\pm\sqrt{\kappa}\,\omega$, exchanged by $\omega \mapsto -\omega$.

$s_0 = 12/5$. The only lattice point on the edge is $\tau W^{-3}$, so $F_P = B\sigma + U^3\sigma^6$, $a = 3/5$. If $b > 1$, proportionality $F_P^b \sim F_Q^a$ is impossible by degrees; so $b = 1$, $\Sigma_{s_0} = 0$ and $\frac{3}{5}F_P F_Q' - F_P' F_Q = J$ with $F_Q = G + E\sigma^5 + D\sigma^{10}$, whose unique solution is $E = 3U^3G/B$, $D = 9U^6G/(5B^2)$, $BG = -J$. Comparing $D$ with the coefficient $\mu U^5$ of $\tau^{10}W^{-25}$ gives $G = 5\mu B^2/(9U)$, hence $B^3 = -9JU/(5\mu)$ and the value of $G^6$. For $s > 12/5$ the leading terms are $B\tau W^{-3}$ and $GW^{-1}$, with $\Sigma_s = 0$; the next edge of $f$ can only end at the constant term, at $s = 3$, where Lemma 2.1(c) forces $\ell_3(Q) = GW^{-1}$. Thus the roots $\tau(W)$ of $f$ with $\tau \to 0$ are five conjugate roots of order $12/5$, forming one place $\beta_1$ on which $\sigma^5 \to -B/U^3$ and $WQ \to G(1 - 3 + \frac{9}{5}) = -G/5$, and one root of order 3, the place $\beta_2$, on which $WQ \to G$.

$12/5 < s_0 \le 5/2$. For $s_0 < 5/2$ there is no lattice point on such an edge. For $s_0 = 5/2$ one has $a = 0$ and $\Sigma_{s_0} = b - \frac{3}{2} \ge 0$, and Lemma 2.1(c) gives $F_P' F_Q = 0$ or $b = 3/2$ and $-\frac{3}{2}F_P' F_Q = J$, both impossible since $\deg F_P = 6$. □

## 5. Two double roots

**Lemma 5.1.** *If $h = (t - c_1)^2(t - c_2)^2$ with $c_1 \neq c_2$, then $c_1^2 - 3c_1c_2 + c_2^2 = 0$. In particular $c_1^2 + c_2^2 \neq 0$.*

*Proof.* In $K(t)((y))$ let $C := uy^{-5}(1 + \cdots)$ be the cube root of $P$, and recall $[a(t)y^\alpha, b(t)y^\beta] = (\beta a'b - \alpha ab')y^{\alpha+\beta+4}$. From $3C^2[C, Q] = J$ we get $[C, Q] = \frac{J}{3u^2}y^{10}(1 + \cdots)$. If $R = Q - \sum \lambda_i C^i$ has leading term $b(t)y^\beta$ with $\beta < 11$, then $\beta u'b + 5ub' = 0$, so $b = \lambda u^{-\beta/5}$; as $u$ has a simple zero at 0 and $b \in K(t)$, $5 \mid \beta$ and the term is removed by subtracting $\lambda C^{-\beta/5}$. Hence there are $\lambda_5, \dots, \lambda_{-2} \in K$ with $Q - \sum_{i=-2}^{5} \lambda_i C^i = \varphi(t)y^{11} + \cdots$, $\varphi \in K(t)$, and

$$5u\varphi' + 11u'\varphi = \frac{J}{3u^2}.$$

A rational solution has poles only at the roots of $u$, of order $3k - 1$ at a root of multiplicity $k \le 2$, and is $O(t^{-11})$ at infinity. So $\varphi = (a_0 + a_1 t)\, t^{-2}(t - c_1)^{-5}(t - c_2)^{-5}$, and eliminating $a_0, a_1$ from the resulting polynomial identity gives $c_1^2 - 3c_1c_2 + c_2^2 = 0$ (ancillary script). If moreover $c_1^2 = -c_2^2$, then $c_1c_2 = 0$, a contradiction. □

## 6. Proof of Theorem 1.2

On $\bar\Gamma_p$ consider the meromorphic forms

$$\omega_1 := v^{-1}\, d\hat Q, \qquad \omega_2 := v^{-2}\, \hat Q\, d\hat Q.$$

*Finite points.* They are regular where $v \neq 0$. The points of $\Gamma_p$ with $v = 0$ are the $(z_0, 0)$ with $\hat P(z_0, 0) = p$; for generic $p$, $\hat P_z(z_0, 0) \neq 0$, so $v$ is a local parameter and, by Lemma 3.1, $d\hat Q = -J\, v^2 \hat P_z^{-1}\, dv$ on $\Gamma_p$. Hence $\omega_1, \omega_2$ are regular at these points.

*Places at infinity.* There $y = v^{-1}$ vanishes, so $\omega_j = y^j \hat Q^{j-1} d\hat Q$ is regular wherever $\hat Q$ is regular. By Corollary 4.2 and Lemma 4.3, $\hat Q$ has poles only at the places $\beta_1, \beta_2$ over the double roots $c$ of $h$ that are in case (ii); there, writing $\omega_2 = (y\hat Q)^2\, d\hat Q/\hat Q$,

$$\operatorname{Res}_{\beta_1}\omega_2 + \operatorname{Res}_{\beta_2}\omega_2 = -5\Big(\frac{G}{5}\Big)^2 - G^2 = -\frac{6}{5}G^2, \qquad \operatorname{Res}_{\beta_1}\omega_1 + \operatorname{Res}_{\beta_2}\omega_1 = -5\Big(-\frac{G}{5}\Big) - G = 0.$$

*Conclusion.* By the residue theorem, $\sum_c G_c^2 = 0$, the sum being over the double roots in case (ii). If there is no such root, $\hat Q$ has no pole on $\bar\Gamma_p$ and is constant on each component, whereas $dQ$ has no zero on the affine curve $\{P = p\}$ because $[P, Q] = J \neq 0$. If there is exactly one, $G_c = -J/B \neq 0$, a contradiction. By (D2) the only remaining possibility is $h = (t - c_1)^2(t - c_2)^2$ with both roots in case (ii) and $G_{c_1}^2 = -G_{c_2}^2$. Then $G_{c_1}^6 = -G_{c_2}^6$, and Lemma 4.3 gives $U_1^2 = -U_2^2$, where $U_k = c_k(c_1 - c_2)^2$; thus $c_1^2 + c_2^2 = 0$, contradicting Lemma 5.1. □

## 7. Scope

- **Proved here, relative to the cited results:** Theorem 1.2. The inputs are (D1), (D2) and [1, Theorem 2.6, Corollary 7.4, Propositions 5.13–5.18], as indicated.
- **Verified by exact computation:** the bracket formula, the solution of the terminal bracket equation and the two residue sums, the formula for $G^6$, the arithmetic exclusions in (D2) and in Lemmas 4.1 and 4.3, and the elimination in Lemma 5.1 (ancillary file `verify_residue.py`, in the folder `anc`; it uses SymPy).
- **Not claimed:** a statement on the plane Jacobian conjecture itself, or on other shapes of degree $\geq 125$. Whether a counterexample of degrees $(75, 125)$ is excluded unconditionally depends on the reduction to standard pairs in [1, 2].
- [1, Proposition 7.3 and Corollary 7.4], on which Lemma 3.1 rests, are stated in [1] with the indication that their proofs mimic those of Proposition 7.1 and Corollary 7.2; they are applied here to Laurent translates, as in [1, Section 8].
- The argument has been checked only by the author with AI assistance (Claude, Anthropic); it has not been validated by professionals.

## References

[1] C. Valqui, J. A. Guccione, J. J. Guccione, *On the shape of possible counterexamples to the Jacobian Conjecture*, J. Algebra **471** (2017), 13–74, doi:10.1016/j.jalgebra.2016.08.039; arXiv:1401.1784v3. Statement numbers refer to the arXiv version v3.

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