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MAXIMIZING THE FIRST POSITIVE NEUMANN EIGENVALUE OF SIMPLE PENTAGONS

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Abstract. We prove that among all simple pentagons of prescribed area—including non-convex pentagons and pentagons with straight interior angles—the regular pentagon uniquely maximizes the first positive Neumann eigenvalue of the Laplacian up to congruence. The proof separates naturally into two regimes. For pentagons having at least one reflex or straight interior angle, the number of strictly convex vertices is at most four; conformal mapping from the unit disk together with a Schwarz–Christoffel and Blaschke-product estimate yields the uniform bound $|P|\mu_1(P) < 10.34$, strictly below the regular-pentagon value $|R|\mu_1(R) > 10.35335$. For strictly convex pentagons, normalizing a maximum-area vertex triangle reduces the problem to a four-dimensional shape chart coupled with a unimodular metric matrix. Combining covariance bounds, explicit polynomial trial functions, conic duality for pairs of trial functions, and Bernstein-basis branch-and-bound, we confine every potential maximizer to an explicit neighborhood of the regular pentagon in vertex Fourier coordinates. Because the first positive eigenspace of the regular pentagon is two-dimensional, we analyze both eigenfunctions simultaneously together with their first-order corrections to show that every nontrivial local deformation of the similarity class strictly decreases $|P|\mu_1(P)$. All finite-dimensional inequalities invoked in the proof are accompanied by independently verifiable rational and interval arithmetic certificates.

1. The Extremal Problem and Main Result

Among all free planar membranes of equal area shaped like a pentagon, which has the highest fundamental tone? Under homogeneous Neumann boundary conditions, the lowest nonzero frequency of vibration is proportional to $\sqrt{\mu_1(P)}$, where $\mu_1(P)$ denotes the first positive Neumann eigenvalue of the Laplacian $-\Delta$ on the bounded planar domain P. Since a simple polygon is a bounded Lipschitz domain, the embedding $H^1(P) \hookrightarrow L^2(P)$ is compact, so the Neumann spectrum is discrete and the variational infimum is attained. Scaling P by a factor $s > 0$ multiplies the area by s^2 and $\mu_1(P)$ by s^{-2}; we therefore compare the scale-invariant functional

$$F(P) = |P|\mu_1(P), \qquad \mu_1(P) = \inf_{\substack{v \in H^1(P),\, v \neq 0 \\ \int_P v = 0}} \frac{\int_P |\nabla v|^2}{\int_P v^2}.$$

Laugesen posed the problem of maximizing $F(P)$ among n-gons for arbitrary $n \geq 3$ [1]. The triangular case was settled by Laugesen and Siudeja, who proved that the equilateral triangle is the unique maximizer [2]. Among convex quadrilaterals, Endo and Osting established the optimality of the square by combining a local variational analysis near the square with a computer-assisted certificate covering the complement [3].

In this paper we resolve the pentagonal case in full generality, without restricting to convex domains. Throughout, a *simple pentagon* means a bounded open set enclosed by a simple closed polygonal curve of five edges; we permit interior angles equal to π (*straight vertices*) as well as interior angles strictly exceeding π (*reflex vertices*). A vertex whose interior angle is strictly less than π is called *strictly convex*. We denote the regular pentagon by R.

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Theorem 1. *For every simple pentagon P of positive area,*

$$|P|\mu_1(P) \le |R|\mu_1(R),$$

with equality if and only if P is similar to R. Consequently, among simple pentagons of prescribed area, the maximizer is unique up to congruence.

Proof. A non-convex pentagon has at least one interior angle strictly greater than π, while a pentagon with a straight vertex has at most four remaining interior angles. In either case, P possesses at most four strictly convex vertices. By Theorem 2 and Proposition 5,

$$F(P) < 10.34 < 10.35335 < F(R).$$

The remaining polygons are convex pentagons with all five interior angles strictly less than π, for which the inequality and the equality characterization are established in Theorem 18. Finally, two similar pentagons of equal area must have similarity ratio 1, yielding uniqueness up to congruence. □

Overview of the Proof and Organization. The proof proceeds along four main stages:

(1) **Pentagons with reflex or straight angles (Section 2).** Reflex or straight angles reduce the number p of strictly convex vertices. Pulling back via a conformal map $f : \mathbb{D} \to P$, we derive a circular-mean lower bound controlled solely by p via the Schwarz–Christoffel formula and Jensen's formula for finite Blaschke products (Lemma 3). Summing the Rayleigh inequalities for a radially weighted pair $g(r)\cos\theta$, $g(r)\sin\theta$ and taking a positive linear combination of circular-mean bounds at three radii yields a uniform upper bound for every simple polygon with $p \le 4$ (Theorem 2).

(2) **Spectral gap and high-precision benchmark for R (Section 3).** Because convex pentagons can approach R arbitrarily closely, no uniform spectral margin exists in the convex class. Using Crouzeix–Raviart nonconforming finite elements on a symmetric triangulation together with exact matrix inertia and an explicit edge-average interpolation bound, we certify lower bounds for $\mu_1(R)$ and $\mu_3(R)$ (Lemma 4), confirming that the first positive eigenspace E of R is two-dimensional and carries the standard irreducible representation of the dihedral group D_5. An equilibrated flux construction combined with a spectral variance bound then encloses $\mu_1(R)$ and $F(R)$ to sixteen decimal digits (Proposition 5).

(3) **Local and annular comparison in vertex Fourier coordinates (Sections 4–5).** Near R, similarity classes of pentagons are parametrized by three complex vertex Fourier coordinates (ξ_2, ξ_3, ξ_4) with radius $r = \left(\sum_{k=2}^{4} |\xi_k|^2\right)^{1/2}$ vanishing precisely at R. Because $\mu_1(R)$ has multiplicity two, the first-order variation operator degenerates along a kernel K. We construct a polynomial approximation to E (Lemma 6) and determine K together with the linear splitting rate on its orthogonal complement (Lemma 7). On $0 < r \le 1/100$, deformations away from K are governed directly by the lower first-order branch (Lemma 8), whereas near K we perturb both basis eigenfunctions simultaneously and bound the sum of their comparison forms, obtaining the quantitative deficit $F(R) - F(P) \ge F(R)r^2/50$ (Theorem 9). On the surrounding annulus $1/100 \le r \le 47/200$, explicit piecewise-linearly transplanted polynomial trial functions and Bernstein certification establish $F(P) < F(R)$ (Theorem 10).

(4) **Global reduction in affine coordinates (Section 6).** To close the argument, every convex pentagon satisfying $F(P) \ge F(R)$ must lie within the outer annular radius $47/200 = 0.235$. Selecting a maximum-area vertex triangle T and mapping it affinely to a reference right triangle moves the affine distortion into a unimodular positive definite

metric H inside the Rayleigh quotient. Recursive subdivision eliminates both the same-arc configuration (Theorem 11) and the outer boundary of the two-ear configuration (Theorem 12). To tighten the remaining parameter box without subdividing the metric space, we prove a fourth-moment and variance–eigenvalue bound for convex polygons (Proposition 13), deduce a covariance condition-number bound (Corollary 14), and apply a conic duality lemma for pairs of trial functions (Lemma 15) to exclude the remaining fringe (Theorem 16). On the compact core that remains, a harmonic-bridge estimate (Proposition 17) forces $r < 231/1000 < 47/200$, linking seamlessly with the annular bound and completing the proof of Theorem 18.

2. Pentagons with Reflex or Straight Angles

Reflex and straight angles decrease the count of strictly convex vertices. The estimate below depends only on this count and therefore holds for simple polygons with any number of sides.

Theorem 2. *Let P be a simple polygon with at most four interior angles strictly less than π. Then*

$$|P|\mu_1(P) < \frac{393329629}{38041250} < \frac{517}{50} = 10.34.$$

Normalize the area of P to $|P| = \pi$, and let $f : \mathbb{D} \to P$ be a conformal bijection rotated so that $f'(0) = \alpha > 0$. Setting

$$q(z) = \frac{f'(z)}{\alpha} = 1 + \sum_{k\geq 1} \gamma_k z^k, \qquad w_k = \frac{|\gamma_k|^2}{k+1},$$

the area formula $|P| = \int_{\mathbb{D}} |f'(z)|^2 \, dA(z)$ and orthogonality of angular Fourier modes yield

$$\alpha^2 \left(1 + \sum_{k\geq 1} w_k\right) = 1.$$

Below we choose the conformal center $f(0)$ so that our Neumann trial functions have zero mean; at this center, the first coefficient γ_1 need not vanish. We therefore formulate Wang's circular-mean estimate [4, Lemma 3.3] while retaining the $k = 1$ mode.

Lemma 3. *Suppose the simple polygon P has exactly p strictly convex vertices. Then for all $0 < \rho < 1$ and $b \geq 0$,*

$$\sum_{k\geq 1} w_k (k+1) \left(\frac{k}{2} - b\right)^2 \rho^k \geq (1+b)^2 \rho^p - b^2. \tag{1}$$

Proof. Discard all straight vertices, leaving N vertices with interior angles $\theta_i \in (0, 2\pi) \setminus \{\pi\}$ and preimages $\xi_i \in \partial\mathbb{D}$. Set $\nu_i = (\pi - \theta_i)/(2\pi)$; then $\nu_i > 0$ precisely at strictly convex vertices, and the exterior-angle sum of a simple polygon gives $\sum_{i=1}^N \nu_i = 1$. By the Schwarz–Christoffel formula on $\mathbb{D}$,

$$q(z) = \prod_{i=1}^N (1 - \bar{\xi}_i z)^{-2\nu_i}.$$

Define the rational functions

$$H(z) = \frac{z q'(z)}{2q(z)} = \sum_{i=1}^N \frac{\nu_i}{1 - \bar{\xi}_i z} - 1, \qquad L(z) = \frac{H(z) - b}{1 + b + H(z)}.$$

For every $z \in \partial\mathbb{D} \setminus \{\xi_1, \dots, \xi_N\}$, we have $\mathrm{Re}((1-\bar{\xi}_i z)^{-1}) = 1/2$, hence

$$\mathrm{Re}\, H(z) = \frac{1}{2}\sum_{i=1}^{N} \nu_i - 1 = -\frac{1}{2}.$$

Thus $H(z) - b$ and $1 + b + H(z)$ have opposite real parts and equal imaginary parts on each boundary arc, so $|L(z)| = 1$ on $\partial\mathbb{D}$.

Moreover, $L(z) = 1$ occurs precisely at the simple poles ξ_i of H. Near ξ_i, $H(z) \sim \nu_i/(1-\bar{\xi}_i z)$, so $L(z) - 1 \sim -(1+2b)(1-\bar{\xi}_i z)/\nu_i$, giving

$$\xi_i L'(\xi_i) = \frac{1+2b}{\nu_i}.$$

Because $b \geq 0$, as z traverses $\partial\mathbb{D}$ counterclockwise, $L(z)$ crosses 1 in the positive direction at each strictly convex vertex (p times) and in the negative direction at each reflex vertex ($N - p$ times). Let Z and B denote the numbers of zeros and poles of L in $\mathbb{D}$, counted with multiplicity. By the argument principle, the winding number of $L(\partial\mathbb{D})$ around 0 yields $Z - B = 2p - N$. On the other hand, Schwarz reflection across $\partial\mathbb{D}$ exchanges zeros and poles of L, while $L(z) = 1$ has exactly N simple solutions $\xi_1, \dots, \xi_N$ on $\widehat{\mathbb{C}}$; hence $\deg L = N$, which gives $Z + B = N$. Adding the two equations yields $Z = p$.

Since $|L| = 1$ on $\partial\mathbb{D}$, we may write L as a unimodular constant times the quotient of two finite Blaschke products, whose numerator has degree $Z = p$. Set $r = \sqrt{\rho} \in (0,1)$. By Jensen's formula, each Blaschke factor $\psi_a(z) = (z-a)/(1-\bar{a}z)$ ($|a| < 1$) contributes $\log\max(r, |a|) \geq \log r$ to the circular mean of $\log|L|$ on $|z| = r$, and dividing by the denominator Blaschke product only increases this mean. Since q is zero-free in $\mathbb{D}$ with $q(0) = 1$, $\log|q|$ is harmonic with circular mean 0. Finally, $1 + b + H$ is holomorphic in $\mathbb{D}$ with value $1 + b > 0$ at 0, so Jensen's formula gives a circular mean of at least $\log(1+b)$ for $\log|1 + b + H|$. Applying these bounds to the identity

$$q(H - b) = qL(1 + b + H)$$

and invoking the arithmetic–geometric mean inequality (Jensen's inequality for exp), we obtain

$$\frac{1}{2\pi}\int_0^{2\pi} \left|q(re^{it})(H(re^{it}) - b)\right|^2 dt \geq (1+b)^2 r^{2p}.$$

If the circle $|z| = r$ passes through a zero or pole of L, the inequality follows by taking a limit in r. Expanding

$$q(z)(H(z) - b) = \frac{1}{2}zq'(z) - bq(z) = -b + \sum_{k\geq 1} \gamma_k \left(\frac{k}{2} - b\right) z^k$$

and applying Parseval's identity on $|z| = r = \sqrt{\rho}$ with $|\gamma_k|^2 = w_k(k+1)$ gives (1). □

Proof of Theorem 2. Take the real and imaginary parts of $\phi(re^{it}) = g(r)e^{it}$ ($\phi(0) = 0$) as a pair of trial functions on $\mathbb{D}$, where

$$g(r) = r - \frac{2}{5}r^3 + \frac{9}{100}r^5 - \frac{7}{200}r^7 = \sum_{i\in\{1,3,5,7\}} c_i r^i.$$

Pull back the normalized area measure on P via an arbitrary conformal bijection $f_0 : \mathbb{D} \to P$ to a probability measure μ on $\mathbb{D}$. For $a \in \mathbb{D}$, define

$$m(a) = \frac{1}{g(1)}\int_{\mathbb{D}} \phi\Big(\frac{z+a}{1+\bar{a}z}\Big)\, d\mu(z),$$

noting that $g(1) = 131/200 > 0$. By dominated convergence, m extends continuously to $\overline{\mathbb{D}}$ as the identity map $a \mapsto a$ on $\partial\mathbb{D}$. Since this boundary map has topological degree 1, m vanishes at some $a_* \in \mathbb{D}$. Precomposing f_0 with the inverse disk automorphism produces a conformal map

$f : \mathbb{D} \to P$ satisfying $\int_{\mathbb{D}} |f'|^2 \phi = 0$; rotating $\mathbb{D}$ to achieve $f'(0) = \alpha > 0$ preserves this zero-mean condition because $\phi(e^{i\theta} z) = e^{i\theta}\phi(z)$. The transplanted real functions $\mathrm{Re}(\phi \circ f^{-1})$ and $\mathrm{Im}(\phi \circ f^{-1})$ are bounded on P, and conformal invariance of the Dirichlet integral ensures they lie in $H^1(P)$.

Denote by E the total Dirichlet energy of ϕ on $\mathbb{D}$ divided by π, and by M_j its radial mass moments. Termwise integration yields

$$E = 2\int_0^1 \left(g'(r)^2 + \frac{g(r)^2}{r^2} \right) r\, dr = \sum_{i,j} \frac{2(ij+1)c_i c_j}{i+j} = \frac{300481}{300000},$$

$$M_j = 2j \int_0^1 g(r)^2 r^{2j-1}\, dr = \sum_{s,t} \frac{2j c_s c_t}{2j+s+t}, \qquad M_1 = \frac{18893}{64000}.$$

Because $g''(r) = -\frac{r}{100}(240 - 180r^2 + 147r^4) < 0$ on $(0,1]$, g' is strictly decreasing on $[0,1]$ with $g'(1) = 1/200 > 0$; thus g is strictly increasing and positive on $(0,1]$. Substituting $t = r^{2j}$ gives $M_j = \int_0^1 g(t^{1/(2j)})^2\, dt$, so M_j increases strictly with j.

Summing the Rayleigh inequalities for the two transplanted trial functions, integrating $g^2|f'|^2$ first in the angular variable, and substituting the area identity $\alpha^{-2} = 1 + \sum_{k\ge1} w_k$, we arrive at

$$F(P) \le \pi E \, \frac{1 + \sum_{k\ge1} w_k}{M_1 + \sum_{k\ge1} w_k M_{k+1}}. \tag{2}$$

Set $Q = 30433/100000$. It suffices to show that the denominator of (2) strictly exceeds $Q(1 + \sum_{k\ge1} w_k)$. Although $M_1 < Q$, the deficit is compensated by the higher modes. To quantify this, we apply Lemma 3 at the three squared radii ρ_j listed in Table 1 and form a positive linear combination with weights y_j.

TABLE 1. Parameters ρ_j, b_j and positive weights y_j used at three radii.

j	ρ_j	b_j	$10^8 y_j$
1	9/22	1/22	12422833
2	7/10	10/21	1444049
3	15/17	11/5	95966

Set $d_j = (1+b_j)^2\rho_j^4 - b_j^2$ and $a_{jk} = (k+1)(k/2 - b_j)^2\rho_j^k$, and define

$$\delta_0 = M_1 - Q + \sum_{j=1}^{3} y_j d_j, \qquad \delta_k = M_{k+1} - Q - \sum_{j=1}^{3} y_j a_{jk} \quad (k \ge 1).$$

Exact rational evaluation of M_{k+1} and a_{jk} yields the lower bounds reported in Table 2.

TABLE 2. Lower bounds on δ_k for $0 \le k \le 32$, displayed as $\lfloor 10^6 \min \delta_k \rfloor$.

k	0	1–8	9–16	17–24	25–32
$\lfloor 10^6 \min \delta_k \rfloor$	11	11	11	11	550

For $k \ge 32$, the moments M_{k+1} continue to increase while every a_{jk} decreases strictly with k, since

$$\frac{a_{j,k+1}}{a_{jk}} = \rho_j \frac{k+2}{k+1}\left(1 + \frac{1}{k - 2b_j}\right)^2 \le \frac{15}{17} \cdot \frac{34}{33}\left(1 + \frac{1}{32 - 22/5}\right)^2 = \frac{9295}{9522} < 1.$$

Hence $\delta_k > 0$ for all $k \geq 0$. Because $p \leq 4$ and $0 < \rho_j < 1$, we have $\rho_j^p \geq \rho_j^4$, so (1) implies $\sum_{k\geq 1} w_k a_{jk} \geq d_j$. Multiplying by $y_j > 0$ and summing over j shows that

$$M_1 + \sum_{k\geq 1} w_k M_{k+1} - Q\left(1 + \sum_{k\geq 1} w_k\right) \geq \delta_0 + \sum_{k\geq 1} w_k \delta_k > 0.$$

Thus $F(P) < \pi E/Q$. Substituting $\pi < 3927/1250$ (obtained, for instance, by truncating Machin's formula after five terms of $\arctan(1/5)$ and two terms of $\arctan(1/239)$) yields the rational upper bound in the theorem. □

3. The First Positive Eigenspace of the Regular Pentagon

From this point onward we restrict to convex pentagons. Fix R to be the regular pentagon of circumradius 1, and write $A_0 = |R|$ and $\lambda = \mu_1(R)$. We index the Neumann eigenvalues of R starting at $\mu_0(R) = 0$, repeated according to multiplicity. Two distinct spectral estimates are needed: a coarse lower bound on $\mu_3(R)$ to control the spectral gap above $\mu_2(R)$ and quantify polynomial approximation errors, and a high-precision enclosure of λ itself to serve as the benchmark against competing pentagons.

3.1. **A Spectral Gap Bound via Nonconforming Edge Averages.**

Lemma 4. *The regular pentagon R satisfies*

$$A_0\mu_1(R) > \frac{3148984458240}{305231621431} > 10.31670, \qquad A_0\mu_3(R) > \frac{806191150080}{30662594077} > 26.29233.$$

Proof. Set $c = \cos(2\pi/5) = (\sqrt{5}-1)/4$ and $s = \sin(2\pi/5) = \sqrt{(5+\sqrt{5})/8}$, so that $A_0 = 5s/2$. Connect the center of R to its five vertices. Within each sector triangle spanned by adjacent unit radii v, w, draw lines parallel to the three sides through the points $(iv+jw)/n$ $(i, j \geq 0,\, i+j \leq n)$. This produces $5n^2$ congruent triangles; because the apex angle $2\pi/5$ exceeds $\pi/3$, the longest side is the base, so every triangle has diameter $h = \sqrt{2-2c}/n$. Consider the Crouzeix–Raviart nonconforming finite element space on this triangulation, whose degrees of freedom are the edge averages, matched across interior edges and left free on ∂R.

In each small triangle, label the vertex of angle $2\pi/5$ by 0. If λ_i $(i = 0, 1, 2)$ denotes the i-th barycentric coordinate, the basis function associated with the opposite edge is $1 - 2\lambda_i$. Direct integration yields the local mass matrix $\frac{s}{6n^2} I$ and the local stiffness matrix $\frac{2}{s} G$, where

$$z^T G z = (1-c)\left[(z_0 - z_1)^2 + (z_0 - z_2)^2\right] + c(z_1 - z_2)^2.$$

Assembling G over all elements gives a global matrix H, and letting W be the diagonal matrix recording the number of adjacent triangles for each edge, the discrete generalized eigenvalues $\mu_{k,h}$ satisfy $\frac{2}{s} Hx = \mu_{k,h} \frac{s}{6n^2} Wx$. Multiplying by $A_0 = 5s/2$, the scaled discrete eigenvalues $A_0\mu_{k,h}$ are the generalized eigenvalues of the pencil $(30n^2 H/s,\, W)$. To count eigenvalues below a given rational threshold $q > 0$, replace $1-c$ and c by the integer weights 690983 and 309016 to assemble an integer matrix H_*. From $10^6(1-c) > 690983$, $10^6 c > 309016$, and $951056 < 10^6 s < 951057$, we obtain the quadratic-form bound

$$10^6(30n^2 H - qsW) \succeq 30n^2 H_* - 951057qW.$$

Take $n = 16$, giving $5(3n^2+n)/2 = 1960$ edge degrees of freedom. The inertia of the integer matrix on the right-hand side is recorded in Table 3.

The script `certify_pentagon.py` builds the mesh via `pent_mesh.py`, clears the denominator of q, and performs symmetric Gaussian elimination choosing at each step an uneliminated row with minimal fill-in (breaking ties by index), rounding both endpoints of every Schur-complement entry outward to $2^{-192}\mathbb{Z}$. Every pivot interval stays strictly away from zero; since symmetric

TABLE 3. Inertia of $30n^2H_* - 951057qW$ for $n = 16$.

q	Negative	Zero	Positive
10347/1000	1	0	1959
2649/100	3	0	1957

elimination preserves matrix congruence, the pivot signs certify the exact inertia of the integer matrix. Hence $A_0\mu_{1,h} > 10347/1000$ and $A_0\mu_{3,h} > 2649/100$.

We next bound the edge-average interpolation error. Let K be a triangle of diameter h_K and centroid g. If $w \in H^1(K)$ has zero integral on every edge of ∂K, integrating $\operatorname{div}((x-g)w) = 2w + (x-g)\cdot\nabla w$ over K gives $2\int_K w = -\int_K (x-g)\cdot\nabla w$, since the normal component of $x-g$ is constant on each edge. Moreover,

$$\frac{1}{|K|}\int_K |x-g|^2 = \frac{1}{36}\sum_{e\subset\partial K}|e|^2 \le \frac{h_K^2}{12}.$$

By Cauchy–Schwarz, $|K|\bar{w}^2 \le \frac{h_K^2}{48}\|\nabla w\|^2_{L^2(K)}$, where $\bar{w} = |K|^{-1}\int_K w$. Combining this with the Payne–Weinberger Poincaré inequality on convex domains [5], $\|w-\bar{w}\|^2_{L^2(K)} \le \frac{h_K^2}{\pi^2}\|\nabla w\|^2_{L^2(K)}$, and using $1/48 + 1/\pi^2 < 1/8$, we obtain

$$\|w\|^2_{L^2(K)} \le \frac{h_K^2}{8}\|\nabla w\|^2_{L^2(K)}.$$

Let Iu denote the edge-average interpolant of $u \in H^1(R)$ and set $e = u - Iu$. Because $\nabla(Iu)$ is constant on each triangle and $\int_K \nabla e = \int_{\partial K} e\, n = 0$, Iu and e are orthogonal with respect to the broken energy inner product. For $k \in \{1,3\}$, choose a nonzero linear combination u of the first $k+1$ continuous Neumann eigenmodes (including constants) such that Iu is mass-orthogonal to the first k discrete modes (only k linear constraints). Setting $C = h/\sqrt{8}$, orthogonality, the local interpolation bound, and the discrete variational principle give

$$\mu_k\|u\|_2^2 \ge \|\nabla u\|_2^2 = \|\nabla_h(Iu)\|_2^2 + \|\nabla_h e\|_2^2 \ge \mu_{k,h}\|Iu\|_2^2 + C^{-2}\|e\|_2^2 \ge \frac{\mu_{k,h}}{1+C^2\mu_{k,h}}\|u\|_2^2,$$

where the final step applies Cauchy–Schwarz to $\|u\|_2 \le \|Iu\|_2 + \|e\|_2$. Consequently $A_0\mu_k > q/(1 + h^2 q/(8A_0))$. Substituting the two values of q from Table 3, together with $h^2 = (2-2c)/n^2$, $A_0 = 5s/2$, and the rational bounds for c and s, yields both stated fractions. □

We immediately deduce that $\lambda = \mu_1(R)$ has multiplicity exactly 2. Let E be the span of all Neumann eigenfunctions on R with eigenvalues in $(0, 10)$. Because $A_0 = 5s/2 < 2.378$, Lemma 4 implies $\mu_3(R) > 26.29233/2.378 > 11 > 10$, so $\dim E \le 2$. On the other hand, for any unit vector $e \in \mathbb{R}^2$, the linear function $e\cdot x$ has zero mean on R; by rotational symmetry and integration over a single sector, the area average of $(e\cdot x)^2$ is $(2+c)/12$, giving Rayleigh quotient $12/(2+c) < 6$. If $e\cdot x$ projected trivially onto E, its Rayleigh quotient would be at least $\mu_3(R) > 11$, a contradiction. Therefore orthogonal projection maps the two-dimensional space of linear functions injectively, hence isomorphically, onto E, intertwining the action of the dihedral group D_5. Since rotation by $2\pi/5$ admits no real invariant line, the Neumann Laplacian must act as a scalar multiple of the identity on E; thus $\mu_1(R) = \mu_2(R) = \lambda$ has multiplicity 2, and reflection across any symmetry axis splits E into an even line and an odd line.

3.2. **High-Precision Benchmark for the Regular Pentagon.**

Proposition 5. *For the regular pentagon R of circumradius* 1*,*

$$4.35446325047710240 < \mu_1(R) < 4.35446325047710250,$$
$$10.35335162333505933 < F(R) < 10.35335162333505954.$$

Proof. Place one vertex of R on the positive horizontal axis and let Ω be the upper half of R. Impose homogeneous Dirichlet boundary conditions on the horizontal diameter and homogeneous Neumann boundary conditions on the remaining three edges. Odd reflection across the horizontal axis identifies the variational space V of this mixed boundary-value problem isometrically with the odd subspace of $H^1(R)$. Since E carries the standard two-dimensional representation of D_5, it contains a unique odd line; by Lemma 4, every higher eigenvalue in the odd subspace is at least $\Lambda := 11$.

Let $v \in V$ be a nonzero trial function, and set $m = \|v\|_2^2$ and $\rho = a(v,v)/m < \Lambda$, where $a(\cdot,\cdot)$ denotes the Dirichlet energy inner product. Suppose the residual functional $g \mapsto a(v,g) - \rho(v,g)$ has energy dual norm at most $\sqrt{m}\eta$. Define the compact self-adjoint operator T on (V,a) by $a(Tf,g) = (f,g)$, so that $\|v - \rho Tv\|_a \le \sqrt{m}\eta$. Set $M = 1/\mu_1(R)$, $b = 1/\Lambda$, and $t = 1/\rho$. In the energy-normalized state $\tilde{v} = v/\sqrt{m\rho}$, the spectral distribution of T has mean $a(T\tilde{v},\tilde{v}) = t$, variance $\|(T-t)\tilde{v}\|_a^2 \le \eta^2/\rho^3$, and support contained in $[0,b] \cup \{M\}$. Since $(M-T)(T-b) \le 0$ as an operator on V, taking expectations in state $\tilde{v}$ gives

$$(M-t)(t-b) \le \mathrm{Var}_{\tilde{v}}(T) \le \frac{\eta^2}{\rho^3},$$

which rearranges to

$$\frac{\rho}{1 + \eta^2\Lambda/(\rho(\Lambda-\rho))} \le \mu_1(R) \le \rho.$$

Write $c = (\sqrt{5}-1)/4$, $s = \sin(2\pi/5)$, and $a_0 = (1+\sqrt{5})/4$. In scaled coordinates $x = X$, $y = sY$, the four vertices of Ω are $(1,0)$, $(c,1)$, $(-a_0,2c)$, and $(-a_0,0)$. Dropping the common factor s from both the area and energy inner products replaces the Euclidean gradient metric by $D = \mathrm{diag}(1, s^{-2})$. Connecting the average of the four vertices to each vertex and edge midpoint yields eight triangles, each containing one polygon vertex. Around each polygon vertex we construct nested similar triangles of scale 2^{-j} $(1 \le j \le 20)$ and divide each trapezoidal layer into two triangles, obtaining 328 elements in total. We represent the trial function v and a stream function w by continuous piecewise Bernstein polynomials of degrees 14 and 16, respectively, setting the trace coefficients of v to zero on the horizontal Dirichlet edge and those of w to zero on the Neumann edges.

On each element K, let the orientation-preserving affine map from the unit reference triangle Δ have linear part B, and set $J = \det B$ and $H = JB^{-1}DB^{-T}$. Hats denote pullbacks to Δ, and fluxes follow the contravariant Piola convention $q = B\hat{q}/J$. If an equilibrated flux $\hat{q}$ satisfies $\mathrm{div}\,\hat{q} = f := -\rho J\hat{v}$ on each element, has zero normal trace on the Neumann boundary, and has continuous normal trace across interior edges, then integration by parts and Cauchy–Schwarz give, for all $g \in V$,

$$a(v,g) - \rho(v,g) = \sum_K \int_\Delta (H\nabla\hat{v} - \hat{q})^T \nabla\hat{g} \le \left[\sum_K \int_\Delta (H\nabla\hat{v} - \hat{q})^T H^{-1}(H\nabla\hat{v} - \hat{q})\right]^{1/2} \|g\|_a,$$

so we may take

$$m\eta^2 = \sum_K \int_\Delta (H\nabla\hat{v} - \hat{q})^T H^{-1}(H\nabla\hat{v} - \hat{q}).$$

To construct $\hat{q}$, we first assign a constant outward normal flux density F_e (per unit reference parameter) to each edge: zero on Neumann edges and on interior edges outside a spanning forest of the element adjacency graph rooted at elements touching the horizontal cut. Each element

routes the cumulative load $\int_\Delta f$ of itself and all its descendants through the edge leading to its parent (with opposite signs on the two sides of a shared edge), while root elements discharge the remaining load across the Dirichlet edge. Thus $\sum_{e\subset\partial K} F_e = \int_\Delta f$ on every element. In reference coordinates (ξ,ζ), start with the polynomial vector field $q^0 = \big(0, \int_0^\zeta f(\xi,t)\,dt\big)$ and prescribe the tangential derivative of a scalar polynomial ψ along $\partial\Delta$ (counterclockwise) to equal F_e minus the existing outward normal flux density of q^0. By load balance, this prescribed derivative integrates to zero around $\partial\Delta$; integrating from zero along the three edges produces a degree-16 boundary trace, which we extend into Δ by setting interior Bernstein coefficients to zero. Defining $\operatorname{curl} h = (h_\zeta, -h_\xi)$ and setting $\hat q = q^0 + \operatorname{curl}(\psi + \hat w)$ gives a flux with the required divergence, Neumann boundary condition, and inter-element normal continuity, since the outward normal flux density of $\operatorname{curl} h$ equals the tangential derivative of h.

The archived coefficients of v and w are integer multiples of 2^{-60}. On Δ, the Bernstein basis satisfies the exact integration identity

$$\int_\Delta B^p_\alpha B^p_\beta = \frac{(p!)^2}{(2p+2)!}\prod_{i=0}^{2}\binom{\alpha_i+\beta_i}{\alpha_i}.$$

The script `check_flux.py` evaluates this identity using the Python standard library. It reads the witness in `flux-witness.json.gz` and uses `flux_mesh.py` for the mesh. Integer interval arithmetic on the grid $2^{-150}\mathbb{Z}$ gives

$$4.354463250477102490208369 < \rho < 4.354463250477102490208370, \qquad \eta^2 < 5.1679254\times10^{-17}.$$

Substituting the unrounded intervals into the inverse-operator bound and multiplying by $A_0 = 5s/2$ yields the proposition. If P denotes the spectral projection onto the first positive eigenspace, the same computation also gives $\|\nabla(I-P)v\|/\sqrt{m} < 1.1900\times10^{-8}$ and $\|(I-P)v\|/\sqrt{m} < 3.588\times10^{-9}$, because every complementary eigenvalue $\gamma \ge \Lambda$ satisfies $(\gamma-\rho)^2/\gamma \ge \gamma(1-\rho/\Lambda)^2$. □

4. First-Order Splitting of the Double Eigenvalue

Set $\omega = \exp(2\pi i/5)$ and label the vertices of R by $a_j = \omega^j$ $(0 \le j < 5)$. For counterclockwise-ordered vertices $z_0,\dots,z_4 \in \mathbb{C}$ of a pentagon P, define the discrete Fourier coefficients

$$c_k = \frac{1}{5}\sum_{j=0}^{4} z_j \omega^{-jk}.$$

Near R, $c_1 \neq 0$, and the similarity transformation $z \mapsto (z-c_0)/c_1$ yields the unique normalized representation (for this vertex labeling)

$$z_j = a_j + \xi_2 a_j^2 + \xi_3 a_j^3 + \xi_4 a_j^4, \qquad \xi_k = \frac{c_k}{c_1}, \qquad r^2 = |\xi_2|^2 + |\xi_3|^2 + |\xi_4|^2. \tag{3}$$

The three complex coefficients (ξ_2,ξ_3,ξ_4) furnish all six real shape parameters. We pull back the Rayleigh quotient from the deformed pentagon to R; to do so, we first construct an explicit polynomial approximation to E and then identify the directions along which the first variation vanishes.

4.1. Polynomial Approximation of the First Eigenpair.

Henceforth let $\langle f\rangle = A_0^{-1}\int_R f$ denote the area average over R, and normalize norms on R accordingly: $\|f\|_2^2 = \langle |f|^2\rangle$ and $\|\nabla f\|_2^2 = \langle |\nabla f|^2\rangle$. Let P_0 denote the L^2-orthogonal projection onto E, acting separately on real and imaginary parts of complex-valued functions.

Lemma 6. *With $z = x+iy$ and $\rho = |z|$, consider the polynomial*

$$333p(z) = 1000z - 537z\rho^2 + 83z\rho^4 + 89\bar z^4 - 10\bar z^4\rho^2 - 17z^6.$$

Then $\|p\|_2 < 1.415$, $\|\nabla p\|_2 < 2.954$, *and its projection* $u = P_0 p$ *satisfies*

$$\|p - u\|_2 < 0.025308, \qquad \|\nabla(p - u)\|_2 < 0.084159.$$

If either the real or imaginary part of p *is normalized individually to unit* L^2 *norm, the resulting function and its gradient have projection errors strictly below* 0.017906 *and* 0.059544*, respectively.*

Proof. Since $p(\omega z) = \omega p(z)$, we have $\langle p \rangle = 0$. Dihedral symmetry ensures that $\operatorname{Re} p$ and $\operatorname{Im} p$ have equal L^2 norms and equal Dirichlet energies, so both share the Rayleigh quotient $q = \|\nabla p\|_2^2 / \|p\|_2^2$. These quantities follow directly from exact monomial moments: $\langle z^a \bar{z}^d \rangle = 0$ unless $5 \mid (a - d)$, in which case barycentric expansion on the sector $(0, 1, \omega)$ gives

$$\langle z^a \bar{z}^d \rangle = \frac{2}{(a + d + 2)!} \sum_{j=0}^{a} \sum_{k=0}^{d} \binom{a}{j} \binom{d}{k} (a + d - j - k)!(j + k)! \cos \frac{2\pi(j - k)}{5}.$$

Using this formula, `certify_local_radius.py` certifies

$$1.99765839 < \|p\|_2^2 < 1.99765840, \qquad 8.69938239 < \|\nabla p\|_2^2 < 8.69938240, \qquad q < 4.354790.$$

For any such zero-mean trial function v, write $e = v - P_0 v$. Since v is orthogonal to constants and $\mu_3(R) > g := 11$, expanding v in an orthonormal basis of Neumann eigenfunctions yields

$$\|e\|_2^2 \le \|v\|_2^2 \frac{q - \lambda}{g - \lambda}, \qquad \|\nabla e\|_2^2 \le g \|v\|_2^2 \frac{q - \lambda}{g - \lambda},$$

where the second bound uses $\|\nabla e\|_2^2 - \lambda \|e\|_2^2 \ge (1 - \lambda/g) \|\nabla e\|_2^2$. Both fractions decrease with λ; substituting $\lambda > 4.354463$ from Proposition 5 produces all four error bounds. □

4.2. **The Kernel of the First Variation.** For a displacement $\xi = (\xi_2, \xi_3, \xi_4)$ in (3), map each sector triangle $(0, a_j, a_{j+1})$ affinely onto the corresponding perturbed sector while keeping the origin fixed. Along the ray $t\xi$, denote the derivative matrix on a sector by $I + tG$ and put $d = \operatorname{tr} G$. The area average $\langle d \rangle$ is a rotation-invariant real linear functional of (ξ_2, ξ_3, ξ_4); since all three coordinates have nonzero rotational weights, $\langle d \rangle = 0$. Consequently, the area ratio $H(t) = \langle \det(I + tG) \rangle = 1 + O(t^2)$ has zero first derivative at $t = 0$. For $g, h \in E$, multiplying the stiffness form by the area ratio and centering the mass form yields the first-order comparison bilinear form

$$B(\xi)[g, h] = \langle \nabla g^T (dI - G - G^T) \nabla h \rangle - \lambda \langle dgh \rangle. \tag{4}$$

Here the stiffness derivative $dI - G - G^T$ arises from expanding $\det(I + tG)(I + tG)^{-1}(I + tG)^{-T}$ at $t = 0$, and because g, h have zero mean at $t = 0$, recentering affects only $O(t^2)$ and higher terms. We view $B(\xi)$ as a self-adjoint operator on E equipped with the energy inner product $\langle \nabla g \cdot \nabla h \rangle$.

Lemma 7. *There exists a real number* $\alpha \in [-0.3, 0]$ *such that the kernel* K *of the linear map* $\xi \mapsto B(\xi)$ *consists precisely of those vectors satisfying* $\bar{\xi}_4 = \alpha \xi_3$*. Its orthogonal complement* Z *with respect to the Euclidean inner product in Fourier coordinates is a real* 2*-plane, and there exists a constant* $b > 1.7$ *such that the two eigenvalues of* $B(\xi)$ *are* $\pm b \operatorname{dist}(\xi, K)$*.*

Proof. Rotating the plane counterclockwise by $2\pi/5$ and cyclically relabeling the vertices sends $\xi_k \mapsto \omega^{1-k} \xi_k$ $(k = 2, 3, 4)$. Because $\operatorname{tr} B(\xi)$ is a rotation-invariant real linear functional, $\operatorname{tr} B(\xi) \equiv 0$. Rotation acts on the two-dimensional space of traceless symmetric forms on E as rotation by $\pm 4\pi/5$ (weights $\pm 2 \bmod 5$), whereas on the ξ_2-plane it acts by $-2\pi/5$ (weight -1). By equivariance, no nonzero intertwining map exists between these representations, so $B(\xi)$ is independent of ξ_2 and depends only on ξ_3 and $\bar{\xi}_4$ (both of weight -2).

Normalize the real and imaginary parts of p from Lemma 6 to unit L^2 norm, denoting them by v_x, v_y, and set $u_x = P_0 v_x$, $u_y = P_0 v_y$. By symmetry, u_x and u_y are orthogonal, share a common squared L^2 norm $m \in (0, 1]$, and are respectively even and odd under complex conjugation.

For $k = 3, 4$, take the real displacement a_j^k and put $B_k = B(\xi)[u_x, u_x]$. Reflection symmetry diagonalizes $B(\xi)$ in the basis $\{u_x, u_y\}$ along both directions. We establish the bounds

$$-2.075 < B_3 < -0.275, \qquad -8.502 < B_4 < -7.493.$$

Specifically, on the j-th sector, G sends a_j, a_{j+1} to a_j^k, a_{j+1}^k. Replacing u_x by v_x in (4) reduces the integral to a polynomial on each sector. Putting $L_1 = dI - G - G^T$, this replacement incurs an error bounded by

$$\|L_1\|_\infty(2\sqrt{q}\, d_1 + d_1^2) + q\|d\|_\infty(2d_0 + d_0^2), \qquad d_0 = 0.017906, \quad d_1 = 0.059544,$$

where $q = 4.354790$ bounds both $\|\nabla v_x\|_2^2$ and λ, and d_0, d_1 come from Lemma 6. The script `certify_vertex_kernel.py` maps each triangle to the unit square via the Duffy transformation $(s, t) \mapsto (s, (1 - s)t)$; after multiplying by the Jacobian determinant, the integrand has degree at most 13 in each variable, so tensor-product quadrature built from integrating the Lagrange basis on $\{0, 1/13, \ldots, 1\}$ is exact. Rounding every interval endpoint outward to $2^{-100}\mathbb{Z}$ certifies the intervals for B_3 and B_4.

Set $\alpha = -B_3/B_4$; the estimates above imply $\alpha \in [-0.3, 0]$. At $(\xi_2, \xi_3, \xi_4) = (0, 1, \alpha)$, one diagonal entry equals $B_3 + \alpha B_4 = 0$ and the trace vanishes, so this displacement lies in K. Its rotations span the real 2-plane $\bar{\xi}_4 = \alpha\xi_3$, $\xi_2 = 0$. Since $B_4 \neq 0$ and rotation on the target plane has no real invariant line, B is surjective onto the two-dimensional space of traceless symmetric forms; thus K is the direct sum of this plane and the ξ_2-plane, i.e., $K = \{\xi : \bar{\xi}_4 = \alpha\xi_3\}$.

Finally, for $\zeta \in Z$, the squared operator norm of $B(\zeta)$ equals $\frac{1}{2}\operatorname{tr}(B(\zeta)^2)$, which is a rotation-invariant positive definite quadratic form on the 2-plane Z and hence takes the form $b^2|\zeta|^2$ with $b > 0$. Along the unit ξ_4-direction, evaluating $B(\xi)$ on the energy-normalized eigenfunction $u_x/\sqrt{\lambda m}$ gives diagonal magnitude $|B_4|/(\lambda m) > 7.493/4.355 > 1.72$ (since $m \leq 1$ and $\lambda < 4.355$). As the orthogonal projection of this unit direction onto Z has length at most 1, we conclude $b > 1.7$. Since $B(\xi)$ is traceless and depends only on the projection of ξ onto Z, its eigenvalues are $\pm b \operatorname{dist}(\xi, K)$. □

5. Comparison in Vertex Fourier Coordinates

Continuing in the normalized coordinates (3) with $z_j = \omega^j + \sum_{k=2}^4 \xi_k \omega^{jk}$ and $r^2 = \sum_{k=2}^4 |\xi_k|^2$, we compare $F(P)$ with $F(R)$ on $r \leq 1/100$ using the first eigenspace E and a first-order correction, and on the outer annulus $1/100 \leq r \leq 47/200$ using explicit polynomial trial functions. In both regimes, all integrals are pulled back sectorwise to R.

5.1. First-Order Eigenvalue Splitting and Remainder Bounds.

Let $I + G$ denote the linear map from a reference sector of R to the corresponding deformed sector. Set $d = \operatorname{tr} G$, $\epsilon = \det G$, $D = \det(I + G) = 1 + d + \epsilon$, and $H = \langle D \rangle$. The real-linear map $z \mapsto Az + B\bar{z}$ carrying $1, \omega$ to $1, \omega^k$ has coefficients

$$A = \frac{\omega^k - \bar{\omega}}{\omega - \bar{\omega}}, \qquad B = \frac{\omega - \omega^k}{\omega - \bar{\omega}}.$$

For $k = 2, 3, 4$, its operator norm $|A| + |B|$ equals $\phi, \phi, 1$, where $\phi = (1 + \sqrt{5})/2$ is the golden ratio. By Cauchy–Schwarz, for all $r > 0$,

$$\|G\| \leq \sqrt{2\phi^2 + 1}\, r = \sqrt{4 + \sqrt{5}}\, r < \frac{5}{2} r.$$

As noted above, orthogonality in the oriented area formula implies $\langle d \rangle = 0$. With $\lambda = \mu_1(R)$, define for zero-mean real functions u, v on R

$$C = D(I + G)^{-1}(I + G)^{-T}, \qquad M(u, v) = \langle Duv \rangle - \frac{\langle Du \rangle \langle Dv \rangle}{H},$$

$$L(u,v) = H\langle \nabla u^T C \nabla v\rangle - \lambda M(u,v).$$

By the centered Rayleigh quotient, every nonzero trial function v satisfies $F(P)/A_0 - \lambda \le L(v,v)/M(v,v)$. Subscripts on bilinear forms indicate their homogeneous degree in ξ; set $S_1 = P_0 L_1 P_0$, with kernel $K = \ker(\xi \mapsto S_1(\xi))$.

Lemma 8. *Let $\delta = \operatorname{dist}(\xi, K)$. For every $0 < r \le 1/100$, there exists a unit-energy trial function $v \in E$ satisfying*

$$L(v,v) \le -1.7\delta + \big[k_2 + 1 + E_* + 2 + \big(a_*(r) + m_*(r)\big)r\big]r^2,$$

where $k_1 = 3.10$, $k_2 = 7.75$, $D_* = 2.36$, $E_* = 2.403$, and

$$a_*(r) = \frac{D_* k_2 + E_* k_1 + k_1 + (E_*+1)k_2 r}{(1 - 5r/2)^2}, \qquad m_*(r) = \frac{2(1.415)E_* + (2 + E_*^2)r}{1 - r^2}. \tag{5}$$

Proof. Restricted to E, S_1 coincides with the bilinear form (4). By Lemma 7, its smaller eigenvalue is $-b\delta$ with $b > 1.7$; let $v \in E$ be a corresponding unit-energy eigenvector. From the defining equation of K, $\delta = |\xi_4 - \alpha\bar{\xi}_3|/\sqrt{1+\alpha^2}$ with $\alpha \in [-0.3, 0]$.

Write $Gz = az + b'\bar{z}$ on each sector. The moduli of the coefficients of ξ_2, ξ_3, ξ_4 are $1, \phi^{-1}, 0$ in a and $\phi^{-1}, 1, 1$ in b'. By Cauchy–Schwarz,

$$|a| \le 1.18r, \qquad |b'| \le 1.55r, \qquad |d| = 2|\operatorname{Re} a| \le D_* r, \qquad |\epsilon| = \big||a|^2 - |b'|^2\big| \le E_* r^2.$$

The discrete Fourier area formula gives $|\langle\epsilon\rangle| \le r^2$, while averaging over the five sectors yields $\langle d^2\rangle = 2(|\xi_2|^2 + \phi^{-2}|\xi_3|^2) \le 2r^2$ because a contains only two frequencies.

In the expansion of $C = D(I+G)^{-1}(I+G)^{-T}$, the linear and quadratic terms following the identity are

$$C_1 = dI - G - G^T, \qquad C_2 = GG^T - \epsilon I,$$

with operator norms $\|C_1\| \le 2|b'| \le k_1 r$ and $\|C_2\| \le 2|b'|(|a| + |b'|) \le k_2 r^2$; the latter bound holds equally for $\operatorname{adj} G \operatorname{adj} G^T - \epsilon I$. The exact remainder identity

$$C - I - C_1 - C_2 = -\frac{dC_2 + \epsilon C_1 + \epsilon C_2}{D},$$

together with $D \ge (1 - 5r/2)^2$ and $\|C - I\| \le (k_1 r + k_2 r^2)/(1 - 5r/2)^2$, implies that after multiplication by $H = 1 + \langle\epsilon\rangle$, the quadratic term of HC has norm at most $(k_2 + 1)r^2$ and the remainder of order ≥ 3 has norm at most $a_*(r)r^3$.

In the mass form M, the quadratic term $\langle \epsilon uv\rangle - \langle du\rangle\langle dv\rangle$ has L^2-bilinear-form norm at most $(E_* + 2)r^2$. Expanding $\langle Du\rangle\langle Dv\rangle/H$ and using $|\langle du\rangle| \le \sqrt{\langle d^2\rangle}\|u\|_2 \le \sqrt{2}\, r\|u\|_2 < 1.415r\|u\|_2$ together with $H \ge 1 - r^2$, the remainder of order ≥ 3 in M has L^2-form norm at most $m_*(r)r^3$. Since $v \in E$ has unit energy norm, $\lambda\|v\|_2^2 = 1$, and summing the stiffness and mass estimates completes the proof. □

5.2. **Quadratic Deficit Near the Regular Pentagon.**

Theorem 9. *In the six-dimensional vertex Fourier coordinates (3), every convex pentagon P with $r \le 1/100$ satisfies*

$$F(R) - F(P) \ge F(R)\frac{r^2}{50}.$$

Proof. Away from K, Lemma 8 already produces a linear decrease of order δ. When $\delta \le 8r^2$, this drop may be too small to absorb quadratic terms; we therefore perturb both real trial functions in E and estimate the sum of their comparison forms, encoding the pair as a single complex-valued function.

Let p be the complex polynomial from Lemma 6 (with denominator 333) and set $u = P_0 p$, acting componentwise (the L^2 and energy projections onto E coincide). By Lemma 6, $\|p - u\|_2 \le$

$d_0 := 0.025308$, $\|\nabla(p-u)\|_2 \le d_1 := 0.084159$, $\|p\|_2 < 1.415$, and $\|\nabla p\|_2 < 2.954$. We construct a complex correction w that is $\mathbb{R}$-linear in (ξ_2, ξ_3) as follows. For each $k \in \{2,3\}$, list all pairs of nonnegative integers (a,b) satisfying $1 \le a+b \le 9$ and $a-b \equiv k,\ 2-k \pmod 5$, ordered first by increasing total degree $a+b$ and then by increasing a. In this order, the coefficients of $z^a\bar{z}^b$ are given by dividing the following integers by 10^4:

$$\begin{aligned} k=2: {} & 14354,\ 74843,\ 65622,\ -7531,\ -181800,\ 6668,\ -163133,\\ & -16820,\ -1566,\ 206183,\ -12458,\ 177226,\ 29357,\ 2416,\ 416,\\ & 4096,\ -90246,\ 5006,\ -67660,\ -15379,\ -1413;\\ k=3: {} & -1580,\ 101627,\ -546,\ 58102,\ -193837,\ -22619,\ 23956,\\ & -154971,\ 2657,\ 213330,\ 27945,\ 2768,\ -33552,\ 161428,\ -5112,\\ & -94028,\ -11530,\ 1603,\ -3896,\ 17645,\ -57858,\ -435. \end{aligned}$$

Multiply monomials of frequency $a-b \equiv k \pmod 5$ by ξ_k, multiply those of frequency $a-b \equiv 2-k \pmod 5$ by $\bar{\xi}_k$, sum over $k=2,3$, and subtract the mean over R to obtain w. Under $\xi_k \mapsto \omega^{1-k}\xi_k$, this satisfies the rotational equivariance $w(\omega^{1-k}\xi_k, \omega z) = \omega w(\xi_k, z)$, and reality of the coefficients yields reflection equivariance. Complexifying all bilinear forms to sesquilinear forms, the real quantity $T = L(u+w, u+w)$ equals the sum of the comparison forms of the two real trial functions. By the eigenvalue equation on E and $\operatorname{tr} S_1 = 0$, both the zeroth- and first-order terms of T in ξ vanish identically.

Let A_j and M_j denote the homogeneous degree-j parts of HC and M, respectively, and write $L_j = A_j - \lambda M_j$. For zero-mean complex functions, in addition to the earlier formulas we need

$$A_1 = C_1, \qquad A_2 = C_2 + \langle\epsilon\rangle I, \qquad A_3 = -dC_2 + (\langle\epsilon\rangle - \epsilon)C_1,$$

$$M_3(f,g) = -\langle df\rangle\overline{\langle \epsilon g\rangle} - \langle \epsilon f\rangle\overline{\langle dg\rangle}.$$

For $n = 2, 3$, define $\mathcal{T}_n(v) = L_n(v,v) + 2\operatorname{Re} L_{n-1}(v,w) + L_{n-2}(w,w)$, and abbreviate $a = \xi_2$, $b = \xi_3$, $c = \bar{\xi}_4$. Because (a,b,c) carry rotational weights $(-1,-2,-2) \bmod 5$, rotational and reflection invariance force the monomials listed below (with their conjugates) to exhaust all invariant terms of degrees 2 and 3; hence there exist real coefficients q_i, g_i such that

$$\mathcal{T}_2(u) = q_1|a|^2 + q_2|b|^2 + q_3|c|^2 + 2q_4\operatorname{Re}(b\bar{c}),$$

$$\mathcal{T}_3(p) = \operatorname{Re}(g_1 a^2\bar{b} + g_2 a^2\bar{c} + g_3 ab^2 + g_4 abc + g_5 ac^2).$$

Our computational certificate establishes

$$q_1 < -1.19, \qquad q_2 < -15.45, \qquad q_3 < 17.45, \qquad |q_4| < 1.86,$$

$$(|g_1|, |g_2|, |g_3|, |g_4|, |g_5|) < (14,\ 4.4,\ 0.4,\ 10.5,\ 3.9),$$

$$\|\nabla w\|_2^2 \le 55r^2, \qquad \|w\|_2^2 \le 4.5r^2.$$

The verification proceeds as follows. On the j-th sector of R, the area average of $z^a\bar{z}^b$ is exactly

$$\langle z^a\bar{z}^b\rangle_j = \frac{2}{(a+b+2)!}\sum_{k=0}^{a}\sum_{l=0}^{b}\binom{a}{k}\binom{b}{l}(k+l)!(a+b-k-l)!\,\omega^{j(a-b)+k-l}. \tag{6}$$

For each real shape direction, (6) yields the mass and gradient Gram matrices of $1, p, w_2, w_3$, where w_k is the centered correction associated with a unit real coordinate ξ_k. To control the projection error in $\mathcal{T}_2$, define

$$G_* = (A_2 - c_g I)\nabla p + A_1\nabla w, \qquad V_* = (\epsilon - c_m)p + dw - d\langle dp\rangle$$

for arbitrary scalars $c_g, c_m \in \mathbb{R}$. Writing $u = p-(p-u)$ and invoking mass and energy orthogonality of $p-u$ against E yields

$$|\mathcal{T}_2(u) - \mathcal{T}_2(p)| \le 2\|G_*\|_2 d_1 + (\|A_2\|_\infty + 2|c_g|)d_1^2 + \lambda[2\|V_*\|_2 d_0 + (\|\epsilon\|_\infty + \langle d^2\rangle + 2|c_m|)d_0^2].$$

The program selects c_g, c_m as the nearest multiples of 10^{-3} to the midpoints of the corresponding orthogonal-projection scalar intervals. Evaluating along the three coordinate axes yields q_1, q_2, q_3; subtracting the enclosing intervals at $(a,b,c) = (0,1,1)$ and $(0,1,-1)$ and dividing by 4 yields q_4; evaluating at $(1,\pm1,0)$, $(1,0,\pm1)$, and $(1,1,1)$ yields the cubic coefficients g_i. On the two corrected coordinate planes, $\|\nabla w\|_2^2$ is bounded above by 23.383 and 54.005, and $\|w\|_2^2$ by 1.743 and 4.459; on each plane these forms are scalar multiples of the squared norm, and cross terms vanish because the complex frequency sets of w_2 and w_3 are disjoint. All sector moments and interval bounds are certified by `certify_annular_trials.py` with outward rounding to $2^{-100}\mathbb{Z}$.

Fix $R_* = 0.01$. Whenever $\delta = \operatorname{dist}(\xi, K) \le 8r^2$, the kernel distance formula $\delta = |c-\alpha b|/\sqrt{1+\alpha^2}$ with $\alpha \in [-0.3, 0]$ gives

$$|c| \le 0.3|b| + 8\sqrt{1.09}\, r^2 \le 0.3|b| + 0.084r.$$

Setting $y = |b|/r$ and eliminating $|a|^2 = r^2 - |b|^2 - |c|^2$, we obtain

$$\frac{\mathcal{T}_2(u)}{r^2} \le -1.19 - (15.45 - 1.19)y^2 + (17.45 + 1.19)(0.3y + 0.084)^2 + 3.72y(0.3y + 0.084) < -1.02,$$

the last inequality following from the maximum of this concave quadratic. On the unit sphere, $\max|x^2y| = 2/(3\sqrt{3}) < 0.385$ and $\max|xyz| = 1/(3\sqrt{3}) < 0.193$, whence $|\mathcal{T}_3(p)| < 11r^3$. Using k_1, k_2, D_*, E_* from Lemma 8, put $l_3 = D_*k_2 + E_*k_1$, and denote the mass and gradient bounds for p and w/r by $p_0 = 1.415$, $p_1 = 2.954$, $w_0 = 2.122$, and $w_1 = 7.42$. Since A_3, M_3 have norms bounded (in units of r^3) by $l_3 + k_1$ and $2(1.415)E_*$, while A_2, M_2 have norms bounded (in units of r^2) by $k_2 + 1$ and $E_* + 2$, substituting $u = p - (p-u)$ into the cubic term gives

$$\begin{aligned}\frac{|\mathcal{T}_3(u) - \mathcal{T}_3(p)|}{r^3} &\le (l_3 + k_1)(2p_1d_1 + d_1^2) + 2(k_2+1)w_1d_1 \\ &\quad + \lambda[2(1.415)E_*(2p_0d_0 + d_0^2) + 2(E_*+2)w_0d_0] < 31.\end{aligned}$$

For terms of order four and higher, put $a_3 = a_*(R_*)$, $m_3 = m_*(R_*)$ from (5), and $\Delta = (1 - 5R_*/2)^2$. Setting $C_3 = -dC_2 - \epsilon C_1$, the stiffness remainder of order ≥ 4 prior to multiplying by H is $-(dC_3 + \epsilon C_2 + \epsilon C_3)/D$; after multiplying by $H = 1 + \langle\epsilon\rangle$, an additional term $\langle\epsilon\rangle(C - I - C_1)$ enters. For the mass remainder, we expand the exact product of transplanted means divided by H. Thus the order-≥ 4 stiffness and mass remainders have norms (in units of r^4) bounded respectively by

$$a_4 = \frac{D_*l_3 + E_*k_2 + E_*l_3R_*}{\Delta} + k_2 + \frac{(l_3 + E_*k_2R_*)R_*}{\Delta}, \qquad m_4 = \frac{2 + E_*^2 + 2(1.415)E_*R_*}{1 - R_*^2}.$$

Applying the order-4 bounds to (u,u), the order-3 remainder bounds to (u,w), and the order-2 bounds to (w,w) yields

$$\begin{aligned}\frac{|T - \mathcal{T}_2(u) - \mathcal{T}_3(u)|}{r^4} &\le 2\lambda(a_4 + m_4) + 2(a_3p_1w_1 + \lambda m_3p_0w_0) \\ &\quad + 55(k_2 + 1 + a_3R_*) + 4.5\lambda(E_* + 2 + m_3R_*) < 3100.\end{aligned}$$

Combining all orders, we obtain on $\delta \le 8r^2$

$$T \le -(1.02 - 42R_* - 3100R_*^2)r^2 = -0.29r^2.$$

The summed centered masses of the two real trial functions are at most $(1 + 5R_*/2)^2(p_0 + w_0R_*)^2$, so at least one of them yields a relative deficit $(F(R) - F(P))/F(R) \ge r^2/50$.

In the complementary region $\delta \geq 8r^2$, the uncorrected lower-branch trial function of Lemma 8 (normalized to unit energy on R) satisfies

$$L(v,v) \leq -[1.7(8) - k_2 - 1 - E_* - 2 - (a_3 + m_3)R_*]r^2,$$

while its centered mass multiplied by λ is at most $(1 + 5R_*/2)^2$, again yielding the constant $1/50$. Equality holds trivially when $r = 0$. All endpoint comparisons are checked in exact rational arithmetic by the same verification script. □

5.3. **Polynomial Trial Functions on the Annulus.**

Theorem 10. *In the vertex Fourier coordinates $z_j = \omega^j + \sum_{k=2}^4 \xi_k \omega^{jk}$ with $r^2 = \sum_{k=2}^4 |\xi_k|^2$, every convex pentagon P satisfying $1/100 \leq r \leq 47/200$ obeys*

$$F(P) < 10.35335162333505933 < F(R).$$

Combined with Theorem 9, we have $F(P) < F(R)$ on the entire punctured ball $0 < r \leq 47/200$.

Proof. Set the threshold $L = 4.35446325047710240$. Squaring $A_0^2 = 25(5 + \sqrt{5})/32$ verifies $A_0 L < 10.35335162333505933$, which lies strictly below $F(R)$ by Proposition 5.

The data file `annulus-trials.json` stores seven rational polynomials $f_0, \dots, f_6$ in $z, \bar{z}$, where each triple (a, b, n) encodes the monomial $10^{-8} n z^a \bar{z}^b$. Define on R the shape-dependent complex trial function

$$Y = f_0 + \xi_2 f_1 + \bar{\xi}_2 f_2 + \xi_3 f_3 + \bar{\xi}_3 f_4 + \xi_4 (f_5 + \bar{f}_0) + \bar{\xi}_4 f_6,$$

where $\bar{f}_0$ denotes pointwise complex conjugation. The seven groups have frequencies $a - b \pmod 5$ equal to $1, 2, 0, 3, 4, 4, 3$, respectively, and the term $\xi_4 \bar{f}_0$ mixes the two trial directions under affine stretching. Taking $\operatorname{Re} Y$ and $\operatorname{Im} Y$, transplanting them via the continuous piecewise-linear sector map, and subtracting their means produces two trial functions in $H^1(P)$, since the traces agree along every radial spoke $t\omega^j \mapsto t z_j$.

On the j-th sector, write the real-linear sector map as $w = a_j z + b_j \bar{z}$ (determined by $\omega^j \mapsto z_j$ and $\omega^{j+1} \mapsto z_{j+1}$), and set $D_j = |a_j|^2 - |b_j|^2$, $H = \frac{1}{5} \sum_j D_j$, and $\Pi = \prod_j D_j$. By the sector-map norm bound from the start of Section 5, the perturbation norm is strictly less than $\frac{5}{2} r < 1$ on $r \leq 47/200$, so every $D_j > 0$. Writing $\langle \cdot \rangle_j$ for the area average over the j-th sector of R and $U = Y_z$, $V = Y_{\bar{z}}$ for the Wirtinger derivatives, define

$$E_j = 2\big\langle |\bar{a}_j U - \bar{b}_j V|^2 + |-b_j U + a_j V|^2 \big\rangle_j,$$

$$m = \frac{1}{5} \sum_{j=0}^{4} D_j \langle Y \rangle_j, \qquad M = \frac{1}{5} \sum_{j=0}^{4} D_j \langle |Y|^2 \rangle_j,$$

$$\mathcal{P} = \frac{H^2}{5} \sum_{j=0}^{4} E_j \prod_{i \neq j} D_i - L\Pi(HM - |m|^2).$$

Since the inverse map is $z = (\bar{a}_j w - b_j \bar{w})/D_j$, the chain rule shows that E_j/D_j equals the transplanted Dirichlet energy of Y on the j-th sector divided by the reference sector area. The total centered mass divided by A_0 is $M - |m|^2/H$, and the polygon area ratio is H. Hence $\mathcal{P} < 0$ implies $F(P) < A_0 L$ upon summing the comparison forms of $\operatorname{Re} Y$ and $\operatorname{Im} Y$ (even if one centered component vanishes identically, as $\mathcal{P} < 0$ ensures $HM - |m|^2 > 0$).

Substituting the sector-moment formula (6) expresses $\mathcal{P}$ as a polynomial in ξ_2, ξ_3, ξ_4 and their conjugates with coefficients in $\mathbb{Q}(\omega)$, having 7,213 nonzero monomials and total degree at most 16. The verification script confirms that every coefficient lies in $\mathbb{Q}(\sqrt{5})$: the constant term is positive and at most $149434352254/2^{55}$, all linear terms vanish identically (so the slight positive residual of the trial function at R relative to L is absorbed directly into the annular test), and the

coefficients of $|\xi_2|^2, |\xi_3|^2, |\xi_4|^2$ lie in $(-2.732, -2.730)$, $(-17.597, -17.594)$, and $(-18.857, -18.853)$, respectively.

To bound $\mathcal{P}$ uniformly over all phases, for each monomial we take an upper bound on the modulus of its coefficient whenever any conjugate pair has unequal exponents, and a signed upper bound whenever all three conjugate pairs have matching exponents (in which case the monomial is nonnegative). Grouping by the total exponents of $(|\xi_2|, |\xi_3|, |\xi_4|)$, bracketing $\sqrt{5}$ to precision 2^{-100} via integer square roots, and rounding each grouped upper bound upward to $2^{-55}\mathbb{Z}$, we divide by r^2. For total degree d, the remaining radial factor r^{d-2} is bounded at the upper or lower endpoint of each radial subinterval according to the signs of $d-2$ and the grouped coefficient (including $d = 0$).

Because the three nonnegative moduli $|\xi_k|/r$ have unit Euclidean norm, selecting the largest component covers the sphere by the three coordinate permutations of $(1, y, z)/\sqrt{q}$ with $q = 1 + y^2 + z^2$ and $0 \le y, z \le 1$. Multiplying by q^8 produces powers $q^{(16-d)/2}$ for even d; every odd-degree grouped coefficient is nonnegative, allowing us to replace $\sqrt{q}$ by $(1+q)/2$ to obtain a polynomial upper bound on $[0,1]^2$ of degree at most 17 in each variable. Clearing positive denominators, we compute exact integer Bernstein coefficients using the univariate conversion factor $\binom{k}{i}/\binom{n}{i}$ in each variable.

Divide $[1/100, 47/200]$ into 45 intervals of length $1/200$. This gives 135 parameter squares. Only one needs to be bisected at its midpoint. Every Bernstein coefficient on each of the resulting 136 squares is strictly negative. The scripts `check_annulus.py` and `annulus_five_sectors.py` use the data in `annulus-trials.json`. Using only the standard library, they reconstruct the five sector maps from the vertex equations, perform the integrations and check all signs. An independent implementation `annulus_polynomial.py` using cyclic-group projection and FLINT produces identical exact coefficients. Prior to summing sectors, the standard-library script also cross-checks the energy identities directly in scaled Cartesian coordinates $x = X$, $y = sY_c$ ($s = \sin(2\pi/5)$): forming B from the two scaled target-sector vertex columns, it verifies $D_j = \det B$ and independently recomputes E_j from reference barycentric gradients and $\operatorname{adj} B \operatorname{diag}(1, s^{-2}) \operatorname{adj} B^T$ as exact polynomial identities in the six real shape variables. □

6. Global Comparison in Affine Coordinates

To show that every convex competitor to R lies within the annular radius of Theorem 10, we select a triangle T of maximum area among the ten vertex triples of a convex pentagon P. The three vertices of T partition ∂P into three boundary arcs, and the remaining two vertices lie either on the same arc or on two distinct arcs (Figure 1). An affine transformation maps T onto a fixed reference triangle, encoding the affine distortion as a unimodular positive definite matrix inside the Rayleigh quotient.

6.1. **Metric Parametrization and Moment Formulas on a Reference Polygon.** Modulo translations, write $P = LD$, put $A = |D|$, and define $H = |\det L|\, L^{-1}L^{-T}$ (so $\det H = 1$) and $S(f,g) = A\int_D fg - \int_D f \int_D g$. Changing variables in the Rayleigh quotient gives, for every nonconstant $f \in H^1(D)$,

$$F(P) \le \frac{A^2 \int_D \nabla f^T H \nabla f}{S(f,f)}, \qquad \det H = 1.$$

If $F(P) \ge C$, testing all linear functions yields the matrix inequality $A^3 H \succeq CG$, where $G_{ij} = S(x_i, x_j)$. Every 2×2 symmetric positive definite matrix of determinant 1 admits the parametrization

$$H = \frac{1}{1-\rho^2}\begin{pmatrix} 1+2z+\rho^2 & 2w \\ 2w & 1-2z+\rho^2 \end{pmatrix}, \qquad \rho^2 = z^2 + w^2 < 1,$$

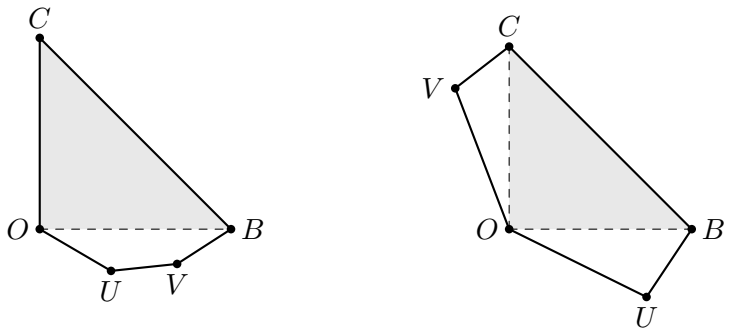


FIGURE 1. The maximum-area vertex triangle OBC is shaded. Left: the remaining vertices U, V lie on the same boundary arc from O to B. Right: U and V lie on distinct boundary arcs, forming the two ear triangles OUB and OCV outside OBC.

with $z = (H_{11}-H_{22})/(\operatorname{tr} H+2)$ and $w = 2H_{12}/(\operatorname{tr} H+2)$, whose smaller eigenvalue is $(1-\rho)/(1+\rho)$. Hence $F(P) < C$ holds whenever some trial function f satisfies

$$J_f = C(1-\rho^2)S(f,f) - A^2 \int_D [(1+2z+\rho^2)f_x^2 + 4w f_x f_y + (1-2z+\rho^2)f_y^2] > 0.$$

For a fixed reference polygon D, the quadratic part of J_f in (z,w) is $-\rho^2(CS(f,f) + A^2 \int_D |\nabla f|^2)$, so J_f is concave in (z,w); positivity on any metric rectangle therefore reduces to checking its four corners.

On a triangle with ordered vertices $0, U, V$, setting $n = i + j$, barycentric integration gives the oriented moment formula

$$\int_{\triangle(0,U,V)} x^i y^j = \det(U,V) \sum_{k=0}^{i} \sum_{\ell=0}^{j} \binom{i}{k}\binom{j}{\ell} \frac{(n-k-\ell)!(k+\ell)!}{(n+2)!} U_x^{i-k} V_x^k U_y^{j-\ell} V_y^\ell. \tag{7}$$

Summing over a triangulation of D evaluates every entry of the mass and stiffness matrices exactly.

6.2. **Two Vertices on the Same Boundary Arc.**

Theorem 11. *Let T be a triangle of maximum area among all vertex triples of a convex pentagon P. If the remaining two vertices lie on the same boundary arc between two adjacent vertices of T, then $F(P) < 103/10$. In particular, whenever $F(P) \geq 103/10$, the remaining two vertices of every maximum-area vertex triangle must lie on distinct boundary arcs.*

Proof. Apply an affine transformation mapping T to $O = (0,0)$, $B = (1,0)$, $C = (0,1)$, and label the remaining vertices U, V so that the counterclockwise vertex order is O, U, V, B, C. Maximality of T implies that every signed barycentric coordinate of U and V relative to T has absolute value at most 1 (replacing the corresponding vertex of T scales its area by that absolute value). Thus points on the arc beyond OB take the form $(x, -y)$ with $0 \leq y \leq x \leq 1$, and the chain O, U, V, B forms a lower convex graph in this strip.

Write $U = (u, -u\alpha)$ and let $v = u + (1-u)r$ be the abscissa of V. Because the downward depth y below OB is a concave function of x, the depth $d = -V_y$ satisfies $u\alpha(1-r) \leq d \leq v\alpha$ (bounded below by the chord UB and above by the ray OU). Every admissible shape is therefore parametrized by $(u, r, \alpha, \gamma) \in [0,1]^4$ via

$$U = (u, -u\alpha), \qquad V = (v, -d), \qquad v = u + (1-u)r, \qquad d = \alpha\big[u(1-r) + \gamma r\big].$$

Before computing moments, apply the area-preserving shear $(x,y) \mapsto (x, x+y)$, placing all vertices inside $[0,1]^2$. Triangulating from O, the non-identically-zero oriented determinants for (U,V),

(V,B), and (B,C) are $u\alpha r(1-\gamma)$, d, and 1. Each vertex coordinate and determinant has degree at most 1 in each shape variable; by (7), a monomial moment of total degree j has degree at most $j+1$ in each variable. Consequently, both J_f for cubic trial functions and $\det(G - A^3 I/24)$ have degree at most 8 in each of the four shape variables.

Evaluating the moments exactly on the tensor grid $\{0, 1/8, \dots, 1\}^4$ and inverting the univariate Bernstein evaluation matrix $W_{ij} = \binom{8}{j}(i/8)^j(1-i/8)^{8-j}$ $(0 \le i, j \le 8)$ along each axis recovers exact Bernstein coefficients. The minimum Bernstein coefficients of $(G - A^3 I/24)_{11}$ and $\det(G - A^3 I/24)$ are $5/576$ and $1/36864$, respectively, so $G \succ A^3 I/24$ throughout $[0,1]^4$. Thus $F(P) \ge C = 103/10$ implies $H \succeq (C/24)I$, and $(1-\rho)/(1+\rho) \ge 103/240$ forces $\rho \le 137/343 < 2/5$.

We start from the box $[0,1]^4 \times [-2/5, 2/5]^2$ and bisect it recursively. The certificate `maxtri_case_b_cover.json` records the tree. At each split both closed child boxes are checked, including their shared face. Each leaf specifies a nonzero integer combination of the nine monomials of degrees 1 through 3 as a trial function f; at each of the four metric corners, all 9^4 tensor-product Bernstein coefficients of J_f are strictly positive. By nonnegativity of the Bernstein basis and concavity of J_f in (z,w), $J_f > 0$ throughout the leaf box. The tree contains 9,447 leaves of maximum depth 41.

Initial matrix coefficients are computed in exact rational arithmetic and rounded downward to multiples of 2^{-34}. Each shape bisection involves at most 8 rounds of integer averaging, so after g bisections the coefficient error in units of 2^{-34} is at most $1+8g$; for trial coefficients c_i, each quadratic form prior to applying metric weights incurs an error of at most $(1+8g)\big(\sum_i |c_i|\big)^2$. The verification code checks overflow bounds on every fixed-width multiplication and accumulation, independently recomputes two leaves in arbitrary-precision integer arithmetic, and certifies 247,927,068 strict positive signs across the full tree. □

6.3. **Boundary Exclusion When Two Vertices Lie on Distinct Arcs.**

Theorem 12. *Suppose the two vertices outside a maximum-area vertex triangle T lie on distinct boundary arcs. Map T affinely to $O = (0,0)$, $B = (1,0)$, $C = (0,1)$ so that the pentagon has counterclockwise order O, U, B, C, V. Then there exist parameters $u, v, \alpha, \beta \in [0,1]$ such that*

$$U = (u, -u\alpha), \qquad V = (-v\beta, v).$$

If $\min(u,v) \le 1/2$ or $\min(\alpha,\beta) \le 1/4$, then $F(P) < 517/50 = 10.34$.

Proof. The barycentric-coordinate argument of Theorem 11 yields $0 \le -U_y \le U_x \le 1$ and $0 \le -V_x \le V_y \le 1$, giving the parametrization above. Apply the area-preserving transformation

$$(X, Y) = (x + y,\ (y - x)/2).$$

The reference pentagon D decomposes into the interior-disjoint union of $T = OBC$ and the two ear triangles $D_1 = OUB$ and $D_2 = OCV$. Reflection across $Y = 0$ exchanges D_1, D_2 and swaps (u, α) with (v, β). The oriented determinants of D_1, T, D_2 are $u\alpha, 1, v\beta$. As in Theorem 11, interpolation on a four-dimensional grid yields minimum Bernstein coefficients $5/576$ for $(G - A^3 I/24)_{11}$ and $5/110592$ for $\det(G - A^3 I/24)$; hence $F(P) \ge C = 517/50$ forces $\rho \le 683/1717 < 2/5$, and in particular $|z|, |w| \le 2/5$.

We use quartic trial functions $f(X,Y)$ with 15 integer coefficients (including the constant term). On each ear D_1 or D_2, monomial moments of total degree at most 8 have degree at most 9 in each of its two parameters by (7), so ten equally spaced nodes per axis recover exact bivariate Bernstein coefficients. Put $A = \int_D 1$, $m = \int_D f$, $M = \int_D f^2$, and at a fixed metric corner set

$$C' = C(1 - \rho^2), \qquad E = \int_D \big[(1 + 2z + \rho^2) f_X^2 + 4w f_X f_Y + (1 - 2z + \rho^2) f_Y^2\big].$$

The desired strict inequality is $C'(M - m^2/A) - AE > 0$. Taking midpoint values A_0, E_0 on each box and defining $L = C'M - A_0 E - E_0 A$, each of L, A, E, m is the sum of contributions

from T, D_1, D_2. Enclosing each ear's contribution by the minimum and maximum of its bivariate Bernstein coefficients and adding the exact contribution of T yields intervals $[X_-, X_+]$; set $m_* = \max(|m_-|, |m_+|)$. Because

$$C'M - AE = L + A_0E_0 - (A - A_0)(E - E_0)$$

and $A_- \geq 1/2$, an explicit sufficient condition for $C'(M - m^2/A) - AE > 0$ is

$$L_- + A_0E_0 - \max_{a\in\{A_-,A_+\},\, e\in\{E_-,E_+\}} (a - A_0)(e - E_0) > C'\frac{m_*^2}{A_-}.$$

Concavity of the original comparison form in (z, w) ensures that positivity at the four metric corners suffices across the entire metric box.

By reflection symmetry, it suffices to cover the regions $u \leq 1/2$ and $\alpha \leq 1/4$ (with all other coordinates ranging over $[0, 1]$). The two trees and their quartic trial functions are archived in `twoear_u_half.json` (32,691 leaves) and `twoear_alpha_quarter.json` (10,931 leaves). After clearing positive denominators, the verifier executes all moment calculations, subdivisions, and sign checks in arbitrary-precision integer arithmetic. □

6.4. **Covariance Bounds for Competitors to the Regular Pentagon.** The vertex parameters above fix only the affine equivalence class of P. To control the linear map from the reference polygon to P, we bound the directional variances and covariance condition number of any convex competitor.

Proposition 13. *Let x be uniformly distributed on a planar convex polygon P, and for any unit vector $e \in \mathbb{R}^2$, set $X = e \cdot x$. Then*

$$\mathbb{E}\big[(X - \mathbb{E}X)^4\big] \leq \frac{12}{5}\operatorname{Var}(X)^2, \qquad \mu_1(P)\operatorname{Var}(X) < \frac{35}{36}.$$

Proof. Rescale X affinely so that its support is $[-1, 1]$. By convexity, cross-sectional lengths orthogonal to e form a piecewise-linear concave function, so the density of the rescaled variable is a nonnegative piecewise-linear concave function on $[-1, 1]$. Every such density is a convex mixture of triangular densities with base $[-1, 1]$, height 1, and apex $T \in [-1, 1]$: subtracting the linear interpolant of the endpoints removes a nonnegative combination of the two endpoint triangles, and subtracting an appropriate positive multiple of the triangle peaked at each interior breakpoint cancels every downward jump in slope; the remaining affine function vanishes at both endpoints and hence vanishes identically. Because each triangle integrates to 1, the mixing weights sum to 1.

Put $m = \mathbb{E}T$, $z = \mathbb{E}T^2$, $w = \mathbb{E}T^3$, and $v = \mathbb{E}T^4$. Integrating over each triangle and averaging over T gives the first four raw moments of the rescaled variable as $m/3$, $(1 + z)/6$, $(m + w)/10$, and $(1 + z + v)/15$. Expanding the central moments yields the identity

$$15\left(\frac{12}{5}\operatorname{Var}(X)^2 - \mathbb{E}\big[(X - \mathbb{E}X)^4\big]\right) = (z - m^2)^2 + \mathbb{E}\big[(1 - T^2)(T - m)^2\big] \geq 0.$$

Since the ratio $\mathbb{E}[(X - \mathbb{E}X)^4]/\operatorname{Var}(X)^2$ is affine invariant, the first inequality follows.

Returning to the original coordinates, put $\sigma^2 = \operatorname{Var}(X)$, $Y = (X - \mathbb{E}X)/\sigma$, and $k = \mathbb{E}Y^4 \leq 12/5$, and test the centered cubic function $g = Y - Y^3/24 - \mathbb{E}(Y - Y^3/24)$. Since $\operatorname{Var}(Y) = 1$, Cauchy–Schwarz gives

$$\operatorname{Var}(g) > \big(\mathbb{E}[Yg]\big)^2 = \left(1 - \frac{k}{24}\right)^2,$$

with strict inequality because a cubic polynomial cannot be affine on a nondegenerate interval. With probability-measure normalization, the Rayleigh numerator multiplied by σ^2 equals

$$\mathbb{E}\left[\left(1 - \frac{Y^2}{8}\right)^2\right] = \frac{3}{4} + \frac{k}{64}.$$

Because $(3/4 + k/64)/(1 - k/24)^2$ is strictly increasing on $k \in [0, 24)$, substituting $k \le 12/5$ gives

$$\mu_1(P)\sigma^2 < \frac{63/80}{(9/10)^2} = \frac{35}{36}. \qquad \square$$

Corollary 14. *Let Σ be the covariance matrix of the uniform probability measure on a convex pentagon P. If $F(P) \ge F(R)$, then*

$$\sqrt{\frac{\lambda_{\max}(\Sigma)}{\lambda_{\min}(\Sigma)}} < \frac{35}{36} \cdot \frac{30\sin(2\pi/5)}{(2 + \cos(2\pi/5))\, 10.34902} < 1.161.$$

Proof. The quotient $|P|/\sqrt{\det \Sigma}$ is affine invariant. Apply a unimodular affine transformation to make the covariance matrix isotropic ($\sqrt{\det \Sigma}\, I$) and translate the centroid to the origin. Let A_j, θ_j denote the area and central angle of the j-th sector triangle, with radial side lengths r_j, r_{j+1}. Using $r_j^2 + r_{j+1}^2 \ge 2r_j r_{j+1}$ and $r_j r_{j+1} = 2A_j/\sin\theta_j$, the polar moment of inertia of the j-th triangle satisfies

$$\frac{A_j}{6}(r_j^2 + r_{j+1}^2 + r_j r_{j+1}\cos\theta_j) \ge \frac{A_j^2(2 + \cos\theta_j)}{3\sin\theta_j}.$$

The function $f(t) = \sin t/(2 + \cos t)$ is concave on $(0, \pi)$, since $f''(t) = -2\sin t(1 - \cos t)/(2 + \cos t)^3 < 0$. Applying Cauchy–Schwarz and then Jensen's inequality (using $\sum_{j=1}^5 \theta_j = 2\pi$) gives

$$\int_P |x|^2\, dx \ge \frac{1}{3}\sum_{j=1}^5 \frac{A_j^2}{f(\theta_j)} \ge \frac{|P|^2}{3\sum_{j=1}^5 f(\theta_j)} \ge \frac{|P|^2}{15 f(2\pi/5)}.$$

In isotropic coordinates this polar moment is $2|P|\sqrt{\det \Sigma}$. We therefore have $|P|/\sqrt{\det \Sigma} \le 30 f(2\pi/5)$. Applying Proposition 13 in the original pentagon P along the maximal-variance eigenvector gives $\mu_1(P)\lambda_{\max}(\Sigma) < 35/36$; multiplying by $|P|/\sqrt{\det \Sigma}$ and dividing by $F(P) \ge F(R) > 10.34902$ establishes the corollary. $\square$

6.5. **Excluding Remaining Fringe Boxes with Pairs of Trial Functions.** Whereas a single trial function produces one scalar inequality for a fixed metric, two trial functions and all their linear combinations produce a 2×2 matrix inequality capable of excluding an entire family of metrics simultaneously. This allows us to subdivide solely in the four-dimensional vertex parameter space.

Lemma 15. *Let D be a planar polygon of area A, let f_1, f_2 be real polynomials, and set $B_{ij} = \int_D f_i f_j - A^{-1}\int_D f_i \int_D f_j$. For a symmetric matrix $H = \left(\begin{smallmatrix} h_1 & h_3 \\ h_3 & h_2 \end{smallmatrix}\right)$, define the symmetric matrix $\Phi(H)_{ij} = \int_D \nabla f_i^T H \nabla f_j$. Suppose a positive definite matrix H_* satisfies*

$$\det H_* > 1, \qquad CB - A\Phi(H_*) \succ 0.$$

For $c = (s, t)^T \in \mathbb{R}^2$, define $e(c) \in \mathbb{R}^3$ by $e(c) \cdot h = c^T \Phi(H) c$, where $h = (h_1, h_2, h_3)^T$. If a unimodular positive definite metric H satisfies that $\nu(H) := (h_2, h_1, -2h_3)^T$ lies in the convex cone generated by finitely many vectors $e(c)$, then the first area-normalized Neumann eigenvalue for the metric H is strictly less than C.

Proof. Suppose to the contrary that the target eigenvalue is at least C. Testing all centered linear combinations of f_1, f_2 yields $A\Phi(H) \succeq CB \succ A\Phi(H_*)$, so $e(c) \cdot (h - h_*) > 0$ for every nonzero c. Taking the nonnegative linear combination representing $\nu(H) \ne 0$ gives $\nu(H) \cdot (h - h_*) > 0$. Yet $\det H = 1$ implies $H^{-1} = \left(\begin{smallmatrix} h_2 & -h_3 \\ -h_3 & h_1 \end{smallmatrix}\right)$, so $\nu(H) \cdot h = 2\det H = 2$, while applying the arithmetic–geometric mean inequality to $H^{-1/2} H_* H^{-1/2}$ gives

$$\nu(H) \cdot h_* = \operatorname{tr}(H^{-1} H_*) \ge 2\sqrt{\det H_*} > 2,$$

a contradiction. $\square$

Theorem 16. *In the two-ear chart of Theorem 12, if* $\min(u,v) \le 3/4$ *or* $\min(\alpha,\beta) \le 7/16$*, then* $F(P) < 10.349$*.*

Proof. Set $C = 10.349$ and retain the coordinates $(X,Y) = (x+y,(y-x)/2)$ from Theorem 12. Put $\mathbf{x} = (X-\bar{X}, Y-\bar{Y})^T$ and $K = \int_D \mathbf{x}\mathbf{x}^T$. Pulling back the Euclidean metric on an affine image TD yields the unimodular metric $H = |\det T|\, T^{-1}T^{-T}$. Applying Proposition 13 in every physical direction shows that $F(TD) \ge C$ requires $H \succeq M := (36C/35)K/A^2$. The polar-moment bound from the proof of Corollary 14 gives $A/\sqrt{\det(K/A)} < 309/25$, whence

$$d := \det M > \left(\frac{36C}{35 \cdot 309/25}\right)^2 > \frac{17}{23}.$$

Existence of H also requires $d \le 1$. Because the two eigenvalues of $M^{-1/2}HM^{-1/2}$ are at least 1 and multiply to $1/d$, we have $M \preceq H \preceq M/d$. Scaling by an appropriate positive constant p yields

$$-\frac{3}{20}K \preceq pH - K \preceq \frac{3}{20}K, \qquad \text{since } \frac{1-d}{1+d} < \frac{3}{20}.$$

Writing $pH - K = K^{1/2}EK^{1/2}$ with $\|E\| \le 3/20$, the diagonal errors satisfy $|pH_{ii} - K_{ii}| \le \frac{3}{20}K_{ii}$ and the off-diagonal error satisfies $|pH_{12} - K_{12}| \le \frac{3}{20}\sqrt{K_{11}K_{22}} \le \frac{3}{40}(K_{11}+K_{22})$. Consequently, the three components of $p\nu(H)$ lie in intervals centered at the corresponding components of $\nu(K)$ with radii $\frac{3}{20}K_{22}$, $\frac{3}{20}K_{11}$, and $\frac{3}{20}(K_{11}+K_{22})$.

At each trial-function leaf, define the 3×3 matrix L by $Lh = (\Phi(H)_{11}, \Phi(H)_{22}, \Phi(H)_{12})^T$; the verifier first certifies $\det L \neq 0$ via interval arithmetic. Choose the eight cyclically ordered directions $c_i = (s_i,t_i)^T$:

$$(1,0),\ (2,1),\ (1,1),\ (1,2),\ (0,1),\ (-1,2),\ (-1,1),\ (-2,1).$$

Setting $r(c) = (s^2,t^2,2st)^T$, we have $e(c) = L^T r(c)$. Normalizing $r(c_i)$ so that $r_1 + r_2 = 1$ places the points $(r_1 - r_2, r_3)$ in cyclic order around the unit circle, so adjacent pairs $(r(c_i), r(c_{i+1}))$ determine all eight supporting hyperplanes of the cone. Checking orientation at $w = (1,1,0)^T$ shows that w lies in the interior of the cone if and only if $\det[r(c_i), r(c_{i+1}), w] < 0$ for every adjacent pair. The script certifies that $\det[e(c_i), e(c_{i+1}), p\nu(H)]$ has constant sign $-\operatorname{sgn}(\det L)$ throughout the component box above; hence $L^{-T}p\nu(H)$ lies inside the cone, and Lemma 15 applies.

Each trial-function leaf stores two rational quartic polynomials and 15 integer coefficients (with common denominator 2^{24}) specifying an affine family $H_*(s) = H_0 + \sum_{k=1}^4 H_k s_k$ over $s_k \in [-1,1]$. The script verifies at all 16 corners of $[-1,1]^4$ that $H_*(s) \succ 0$ and $\det H_*(s) > 1$. Since $\sqrt{\det X} = \inf_{Y\succ 0,\,\det Y = 1} \frac{1}{2}\operatorname{tr}(YX)$ is concave on the positive definite cone, its superlevel sets are convex, so both properties hold on the entire box. To verify $W := CB - A\Phi(H_*) \succ 0$, the script certifies the four Gershgorin inequalities $W_{ii} \pm W_{12} > 0$ $(i = 1,2)$. For $j \neq i$, write $M = \int_D f_i(f_i \pm f_j)$, $m = \int_D f_i$, $m' = \int_D (f_i \pm f_j)$, and $Z_k = \int_D \nabla f_i^T H_k \nabla(f_i \pm f_j)$. Choosing rational reference values a, z_k near the box midpoints of A, Z_k with maximum absolute deviations $\delta A, \delta Z_k$, expanding AZ_0 and AZ_k around (a, z_k) shows that each Gershgorin quantity is bounded below by the lower endpoint of

$$CM - aZ_0 - z_0 A + az_0 - a\sum_{k=1}^4 z_k s_k$$

minus the nonnegative error term

$$\delta A\, \delta Z_0 + \sum_{k=1}^4 (\delta A\, |z_k| + A_+\, \delta Z_k) + C\frac{m_* m_*'}{A_-},$$

using the interval notation $+, -, *$ of Theorem 12. Each linear integral decomposes into the fixed triangle T plus the two ears D_1, D_2, controlled by bivariate Bernstein coefficients.

Geometric leaves eliminate entire boxes directly: maximality of OBC requires $|\det(U-B, V-B)| \le 1$ and $|\det(U-C, V-C)| \le 1$, which translate to

$$v(1-u) + u\alpha(1+v\beta) \le 1, \qquad u(1-v) + v\beta(1+u\alpha) \le 1.$$

Both left-hand sides are multiaffine (affine in each variable separately), so their minima on any box are attained at corners; if either exceeds 1 at every corner, the box is infeasible.

By Theorem 12, we may assume $u, v > 1/2$ and $\alpha, \beta > 1/4$. Exploiting the symmetry exchanging the two ears, it suffices to cover two closed boxes in coordinates (u, α, v, β):

$$[1/2, 3/4] \times [1/4, 1] \times [1/2, 1] \times [1/4, 1], \qquad [3/4, 1] \times [1/4, 7/16] \times [3/4, 1] \times [1/4, 1].$$

The verifier `verify_metric_cover.py` obtains initial coefficients on these two boxes by exact interpolation and rounds them downward to multiples of 2^{-45}. Each midpoint subdivision performs nine rounds of integer averaging, so after g bisections on an ear's coordinates the coefficient error is at most $(1+9g)2^{-45}$, which is accounted for in the interval checks. All fixed-width products and sums are checked for overflow, and final scalar comparisons use arbitrary-precision integers. □

6.6. **Returning to Vertex Fourier Coordinates: The Harmonic Bridge.** It remains to compare the actual vertex Fourier radius of any pentagon in the remaining core against the annular outer radius $47/200 = 0.235$. We first control the second harmonic of the reference polygon Q and then use Corollary 14 to bound the affine distortion of the first harmonic.

Proposition 17. *Let Q be the pentagon with counterclockwise vertices $O = (0,0)$, $U = (u, -u\alpha)$, $B = (1,0)$, $C = (0,1)$, and $V = (-v\beta, v)$, where $3/4 \le u, v \le 1$, $7/16 \le \alpha, \beta \le 1$, and*

$$v(1-u) + u\alpha(1+v\beta) \le 1, \qquad u(1-v) + v\beta(1+u\alpha) \le 1.$$

Let $P = TQ$, where T is an orientation-preserving invertible affine map. If the covariance matrix of the uniform area measure on P satisfies $\sqrt{\operatorname{cond} \Sigma_P} \le 1161/1000$, then the discrete Fourier coefficients $c_k = \frac{1}{5}\sum_{j=0}^{4} z_j e^{-2\pi i j k/5}$ of the vertices of P satisfy

$$c_1 \ne 0, \qquad r^2 = \frac{|c_2|^2 + |c_3|^2 + |c_4|^2}{|c_1|^2} \le \frac{11063200009}{207407376400} < \left(\frac{231}{1000}\right)^2.$$

Proof. Write $p_j \in \mathbb{R}^2$ $(0 \le j \le 4)$ for the vertices of Q, and set $t_j = (\cos(2\pi j/5), \sin(2\pi j/5))^T$. The real discrete Fourier expansion reads $p_j = \bar{p} + Lt_j + Kt_{2j}$, where

$$L = \frac{2}{5}\sum_{j=0}^{4} p_j t_j^T, \qquad K = \frac{2}{5}\sum_{j=0}^{4} p_j t_{2j}^T.$$

Expanding $\det L$ in pairwise vertex determinants and setting $\tau = (\sqrt{5}-1)/2$ gives

$$\frac{25\det L}{4\sin(2\pi/5)} = 1 + u\alpha + v\beta + \tau\big(u + v - uv(1-\alpha\beta)\big) > 0,$$

so L is orientation-preserving and invertible. Put $H = L^{-1}K$ and $S = L^{-1}\Sigma_Q L^{-T}$. Below we certify the two uniform bounds

$$\frac{\|H\|_F}{\sqrt{2}} < \frac{7}{40}, \qquad \sqrt{\operatorname{cond} S} < \frac{11}{10}. \tag{8}$$

Let us first deduce the proposition from (8). Denote by M the product of the linear part of T with L, so that $\Sigma_P = MSM^T$. Then $\Sigma_P^{-1/2} M S^{1/2}$ is orthogonal, and the ratio of the largest to

smallest singular values of M satisfies

$$\operatorname{cond} M \le \sqrt{\operatorname{cond}\Sigma_P}\,\sqrt{\operatorname{cond} S} \le \frac{1161}{1000}\cdot\frac{11}{10} =: \kappa.$$

In complex notation, write $M(z) = Az + B'\bar{z}$ and $H(z) = ez + f\bar{z}$. Since $\det M > 0$, we have $|A| > |B'|$ and the singular values of M are $|A| \pm |B'|$. Thus $c_1 = A \neq 0$, $c_4 = B'$, $|B'|/|A| \le s := (\kappa-1)/(\kappa+1)$, and $|e|^2 + |f|^2 = \|H\|_F^2/2$. The second harmonic of P is given by MH, so $(c_2, c_3) = A(e, f) + B'(\bar{f}, \bar{e})$. Applying the triangle inequality in $\mathbb{C}^2$ gives $|c_2|^2 + |c_3|^2 \le (|A| + |B'|)^2(|e|^2 + |f|^2)$, whence by (8),

$$r^2 \le s^2 + (1+s)^2\left(\frac{7}{40}\right)^2 = \frac{11063200009}{207407376400} < \left(\frac{231}{1000}\right)^2.$$

To certify (8), put $G_1 = 25LL^T$, $G_2 = 25KK^T$, $S_1 = \sum_j p_j p_j^T$, $h = \sum_j p_j$, and $E_1 = \sum_j (p_j p_{j+1}^T + p_{j+1}p_j^T)$ (indices modulo 5). Trigonometric sum-and-difference identities give

$$G_1 = A_0 + \sqrt{5}\,A_1, \qquad G_2 = A_0 - \sqrt{5}\,A_1, \qquad A_0 = 5S_1 - hh^T, \qquad A_1 = 2E_1 - hh^T + S_1.$$

For the three triangles with common vertex O and opposite pairs $(X, Y) \in \{(U, B), (B, C), (C, V)\}$, whose doubled areas are $d = u\alpha, 1, v\beta$, define

$$D = \sum d, \qquad m = \sum d(X+Y), \qquad E = \sum d\big[(X+Y)(X+Y)^T + XX^T + YY^T\big],$$

and set $N = 3DE - 4mm^T$. By the first- and second-moment formulas for triangles, $N = 36D^2\Sigma_Q$. Writing $[G, V] = \operatorname{tr}((\operatorname{adj} G)V)$, we have $\|H\|_F^2 = [G_1, G_2]/\det G_1$, and with $k = \sqrt{\operatorname{cond} S}$ we have $k + k^{-1} = [G_1, N]/\sqrt{\det G_1 \det N}$. Since $k + k^{-1}$ increases strictly on $k \ge 1$ and takes the value $221/110$ at $k = 11/10$, (8) is equivalent to strict positivity of

$$P_1 = 49\det G_1 - 800[G_1, G_2], \qquad P_2 = 221^2 \det G_1 \det N - 110^2[G_1, N]^2.$$

Both polynomials have coefficients in $\mathbb{Q}(\sqrt{5})$. Because G_1, G_2 and N have degree at most 2 and 4 in each variable, P_1 and P_2 have degree at most 4 and 12 in each variable, respectively. Taking $n = 4, 12$ and evaluating the two rational components on the $(n+1)$-point uniform tensor grid, axis-by-axis inversion of the univariate Bernstein matrix yields exact Bernstein coefficients up to a common positive integer factor. Bracketing $\sqrt{5}$ between $\lfloor\sqrt{5}\cdot 10^{36}\rfloor/10^{18}$ and the next multiple of 10^{-18}, taking the lower endpoint of each coefficient, rescaling so the maximum absolute value equals 2^{45}, and flooring to integers produces valid lower bounds; midpoint Bernstein subdivision uses floored averages, which remain valid lower bounds by nonnegativity of Bernstein weights.

The script `verify_harmonic_bridge.py` bisects u, α, v, β cyclically: a polynomial is certified positive on a box (and all descendants) once its minimum Bernstein coefficient is positive, and a box is discarded geometrically if either multiaffine feasibility expression exceeds 1 at every corner. Across 83 bisections (maximum depth 12), the 84 leaf boxes split into 40 certified positive and 44 excluded geometrically. □

6.7. **Completion of the Convex Case.**

Theorem 18. *Let P be a convex pentagon of positive area. Then $F(P) \le F(R)$, with equality if and only if P is a regular pentagon.*

Proof. Suppose $F(P) \ge F(R)$. By Proposition 5, $F(R) > 10.35335$. If P had a straight vertex, it would possess at most four strictly convex vertices and Theorem 2 would give $F(P) < 10.34$, a contradiction. Hence all five vertices of P are strictly convex.

Choose a triangle T of maximum area among the vertex triples of P. By Theorem 11 (since $F(P) > 10.3$), the remaining two vertices lie on distinct boundary arcs. In the two-ear coordinates of Theorem 12, Theorem 16 (since $F(P) > 10.349$) further implies $u, v > 3/4$ and $\alpha, \beta > 7/16$.

Maximality of T enforces the two feasibility inequalities of Proposition 17, while Corollary 14 gives $\sqrt{\operatorname{cond}\Sigma_P} < 1.161$. Applying Proposition 17 to P yields $c_1 \neq 0$ and $r < 231/1000$.

The normalized vertices of P satisfy (3), and their Fourier radius satisfies $r < 231/1000 < 47/200$. If $0 < r \leq 1/100$, Theorem 9 gives $F(P) < F(R)$; if $1/100 \leq r < 231/1000$, Theorem 10 likewise gives $F(P) < F(R)$. Both cases contradict $F(P) \geq F(R)$. Hence $r = 0$, so the normalized vertices are $a_0, \dots, a_4$ and P is regular. Conversely, every regular pentagon is similar to R, and similarity preserves F, so equality holds. □

Computational Materials and Manuscript Notes

The accompanying computational archive contains all trial-function coefficients, subdivision trees, and standalone verification programs. The file `verification.md` lists the exact commands to execute and maps each computation to the corresponding step in the proof. The programs verify the specified finite-dimensional inequalities using exact integer or rational interval arithmetic; the deduction of the Neumann eigenvalue bounds from those inequalities rests entirely on the variational and geometric arguments presented in the text.

An earlier preprint by the authors [6] combined the pentagonal result with asymptotic estimates for large numbers of sides; the present manuscript isolates and streamlines the complete proof for pentagons. Generative artificial intelligence tools were used during the derivation of arguments, the implementation of verification scripts, and the drafting and revision of the manuscript.

References

[1] R. S. Laugesen, Problem on polygonal Neumann eigenvalues, in *Low Eigenvalues of Laplace and Schrödinger Operators*, Oberwolfach Report 6/2009, p. 405. Workshop report.

[2] R. S. Laugesen and B. A. Siudeja, Maximizing Neumann fundamental tones of triangles, *J. Math. Phys.* **50** (2009), 112903. doi:10.1063/1.3246834.

[3] R. Endo and B. Osting, Maximizing the fundamental Laplace–Neumann eigenvalue on quadrilaterals, preprint (2026). arXiv:2609.02784.

[4] H. Wang, The Pólya–Szegő inequalities for polygons, preprint (2026). arXiv:2611.22305.

[5] L. E. Payne and H. F. Weinberger, An optimal Poincaré inequality for convex domains, *Arch. Rational Mech. Anal.* **5** (1960), 286–292. doi:10.1007/BF00252910.

[6] L. Dong and H. Wang, The first positive Neumann eigenvalue of simple pentagons and the large-side limit, Zenodo, version 1 (18 November 2026). doi:10.5281/zenodo.24356848.

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