hexagon:2611.00670v1

SHARP PACKING BOUNDS FOR POLY-5 AND HANDY-OCTS

WILL HOWE, GRANT HUFFMAN, JENSEN KOHLMEYER, AND LIAM KRUER

Abstract. On the 89-square Poly-5 board, distinct pentominoes can cover at most 26 of the 28 boundary squares and 56 of the 60 exterior unit edges. Both maxima occur in a full tiling. For the five Handy-Octs octiamonds, the least area of a convex triangular-grid enclosure is 43 unit triangles. There is one optimal enclosure and it has precisely five packings, up to lattice symmetries. We prove the bounds by finite enumeration.

1. The boundary of Poly-5

Kate Jones's Poly-5 consists of one copy of each free polyomino of orders one through five [1]. A free polyomino is a side-connected union of unit squares, considered up to translation, rotation, and reflection. There are 21 pieces. In the tray, their 89 squares fill centered rows of widths

$$1,\ 1,\ 3,\ 5,\ 7,\ 9,\ 11,\ 15,\ 11,\ 9,\ 7,\ 5,\ 3,\ 1,\ 1.$$

Kadon records boundary constructions by Sara Hopper and Doug Caine, followed by tilings of Kate Jones and Nathan Gilbert in which smaller pieces meet only four exterior edges [2]. Here we establish the obstruction to improving that record.

Theorem 1. *Any disjoint packing of distinct free pentominoes on the Poly-5 tray misses at least two boundary squares and at least four exterior unit edges. Both bounds are attained by the pentominoes of a single full tiling.*

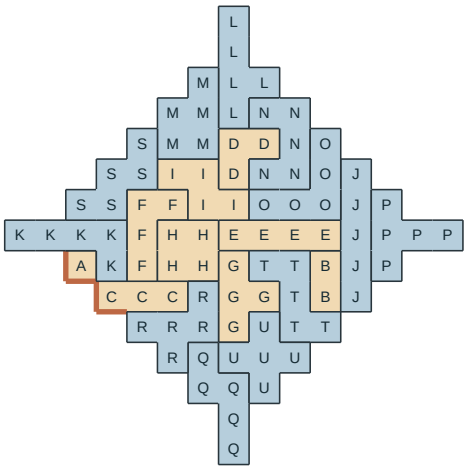


Figure 1. A full tiling attaining both bounds. Each letter labels all the squares of one piece. A and C are the only smaller pieces on the boundary. Each contributes one boundary square with two exposed edges. The other 19 pieces comprise the domino, the other tromino, the five tetrominoes and the twelve pentominoes.

Date: November 19, 2026.

2020 *Mathematics Subject Classification.* 05B45, 52C15.

Key words and phrases. Pentomino, octiamond, lattice packing, exact cover.

The authors acknowledge assistance from OpenAI Codex in mathematical exploration, checking, and manuscript preparation.

2. One possible missing boundary square

Proof of Theorem 1. Center coordinates at the middle of the tray B. Write $\partial_{\mathrm{c}}B$ for its boundary squares and let $w(s)$ count the exterior edges of square s. The four tips have weight three. Each of the other 24 boundary squares has weight two. In particular, a packing that misses at most three exterior edges misses at most one boundary square.

It suffices to test whether a packing can cover $\partial_{\mathrm{c}}B \setminus \{h\}$ for some $h \in \partial_{\mathrm{c}}B$. This includes the case of covering every boundary square, since the test imposes no condition on h. The eight square symmetries give five choices up to congruence (Table 1).

Construct the free pentominoes by starting with one square and adding a side-adjacent square four times. At each step identify translates and the eight square symmetries. Every connected shape has such a growth order, for instance along a rooted spanning tree. The successive numbers of shapes are $1, 1, 2, 5, 12$. For each of the twelve final shapes generate its distinct oriented translates contained in B that meet $\partial_{\mathrm{c}}B$. This gives 1,176 placements. For a fixed h, keep only placements that meet $U_0 = \partial_{\mathrm{c}}B \setminus \{h\}$, obtaining $\mathcal{P}_0$.

The following recurrence specifies the search completely. Its arguments are the squares $U \subseteq U_0$ still to cover and the available placements $\mathcal{P}$. Write $P \sim Q$ if the placements intersect anywhere on the board or have the same free shape. Set $F(\varnothing, \mathcal{P}) = 1$. Otherwise choose $s \in U$ minimizing $\#\{P \in \mathcal{P} : s \in P\}$, breaking ties by lexicographic (x, y). Set

$$F(U, \mathcal{P}) = \sum_{\substack{P \in \mathcal{P} \\ s \in P}} F\big(U \setminus P,\, \{Q \in \mathcal{P} : Q \not\sim P\}\big).$$

An empty sum is zero. Each potential packing selects exactly one placement at the chosen square. Each branch reduces $|U|$. Thus $F(U_0, \mathcal{P}_0) > 0$ precisely when a disjoint packing of distinct shapes covers U_0. Placements covering only squares already covered or outside U_0 are never needed to establish existence.

Table 1. Search for a covering with one permitted missing square. The node count includes the root and all empty sums.

h	Orbit size	$\lvert\mathcal{P}_0\rvert$	Nodes	$F(U_0, \mathcal{P}_0)$
$(7, 0)$	4	1,176	944,160	0
$(6, 0)$	4	1,168	15,893	0
$(5, 1)$	8	1,148	118,978	0
$(4, 2)$	8	1,130	60,023	0
$(3, 3)$	4	1,127	26,437	0

The five zeroes force at least two missing squares, each of weight at least two. Figure 1 supplies equality in both bounds. □

3. Convex enclosures of the Handy-Octs

The five Handy-Octs occupy eight unit triangles each. Kadon illustrates a convex enclosure with three unused triangles and asks if one or two suffice. It credits George Sicherman with ruling out an enclosure with zero unused triangles [3]. An enclosure here is a convex union of closed unit triangles of the fixed triangular lattice. We allow all lattice rotations, reflections, and translations of each piece.

Theorem 2. *The least area of a convex triangular-grid enclosure of all five Handy-Octs is 43 unit triangles. Up to lattice symmetry there is exactly one enclosure of this area that admits a packing, and it admits exactly five inequivalent packings.*

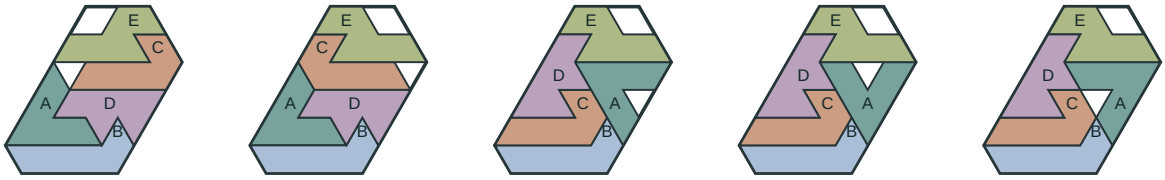


Figure 2. The five optimal packings. The white cells are the three unused unit triangles in each enclosure.

Take axial coordinates in two unit directions at 60°. Write $ij+$ for the triangle with vertices $(i, j), (i + 1, j), (i, j + 1)$ and $ij-$ for the other triangle of the same unit parallelogram. Our labels A through E follow reading order in Kadon's five-piece drawing. The precise inputs are

Piece	Triangles
A	`00+ 01+ 02+ 00- 01- 11+ 10- 20+`
B	`01+ 00- 01- 11+ 11- 21+ 21- 31+`
C	`01+ 00- 01- 10+ 11+ 11- 21+ 21-`
D	`01+ 01- 11+ 10- 11- 21+ 20- 21-`
E	`00- 01- 10+ 11+ 10- 11- 21+ 21-`

Proof of Theorem 2. Every boundary edge of a convex grid enclosure lies in one of the three lattice directions. Intersecting the extreme supporting half-planes, then translating the coordinate minima to zero, gives

$$0 \le x \le a, \qquad 0 \le y \le b, \qquad \ell \le x + y \le a + b - h.$$

Here a, b, ℓ, h are integers. Because the four coordinate extrema are attained, $0 \le \ell, h \le \min(a, b)$. Conversely such bounds describe a grid enclosure whenever its area is positive. Removing the two corner triangles from the parallelogram gives area

$$N = 2ab - \ell^2 - h^2.$$

Every unit-width column and row strip between the extrema contains at least one unit triangle, so $a, b \le N$.

For each $40 \le N \le 43$, enumerate $1 \le a, b \le N$ and $0 \le \ell \le \min(a, b)$. Keep the quadruple if $2ab - \ell^2 - N = h^2$ for an integer $0 \le h \le \min(a, b)$. In cyclic order the six directed sides, including zero lengths, measure

$$(a - \ell,\ b - h,\ h,\ a - h,\ b - \ell,\ \ell).$$

This list determines the polygon by following the six successive lattice directions. Cyclic shift and reversal account for all rotations and reflections. For each equivalence class keep the lexicographically first (a, b, ℓ, h).

For a representative board D, form the placement set $\mathcal{P}_j$ of each piece j. Apply the six powers of $(x, y) \mapsto (-y, x + y)$, with and without reflection $(x, y) \mapsto (x + y, -y)$, to all its triangle vertices. Translate the minima of x, y to zero. Try integer offsets $0 \le i < a$, $0 \le j' < b$ and keep each distinct placement whose vertices satisfy all six board bounds. Convexity gives containment, and two placements have disjoint interiors precisely when they share no triangular cell.

For an occupied cell set M, start with $g_0(\varnothing) = 1$ and zero for all other sets. For each piece in the order A, B, C, D, E, add $g_{j-1}(M)$ to $g_j(M \cup P)$ for each $P \in \mathcal{P}_j$ disjoint from M, starting each new array at zero. Each partial packing has a unique predecessor on removing its last piece. Table 2 sums these counts over all occupied sets and over one representative per board class.

TABLE 2. Packings by prefix of the piece list. The final row gives the contribution of the unique packable area-43 board.

Area or board	Boards	A	AB	ABC	ABCD	ABCDE
40	8	159	1,840	7,622	1,836	0
41	3	63	755	3,228	942	0
42	4	66	806	3,382	1,808	0
43	5	102	1,489	8,005	3,730	5
$(4, 6, 1, 2)$	1	36	535	2,887	1,468	5

The pieces occupy 40 cells, so the first possible area is 40. Only $(4, 6, 1, 2)$ succeeds through area 43. Its side lengths are $(3, 4, 2, 2, 5, 1)$. The unique side of length five has neighbors of different lengths, so the board has no nonidentity symmetry. The five packings counted on it are therefore inequivalent. □

Computation and disclosure. OpenAI Codex drafted the manuscript, wrote and ran the enumerations, and generated the figures from the placements. The ancillary scripts `poly5.py` and `handy_octs.py` reproduce Tables 1 and 2. Separate implementations checked the results using a boundary-weight search and scaled triangle centroids, respectively. This work is licensed under CC BY 4.0.

REFERENCES

[1] K. Jones and Kadon Enterprises, *Poly-5*.
[2] Kadon Enterprises, *Poly-5: rare solutions*.
[3] Kadon Enterprises, *Handy-Octs*.