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SCHUBERT COX RINGS AND ROOT OPERATORS

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Abstract. We identify the Cox ring of every Schubert variety in a symmetrizable Kac–Moody partial flag variety with an upper cluster algebra whose frozen variables are not inverted. The construction also applies to Bott–Samelson varieties for arbitrary words and to their normal projective birational contractions that preserve the standard affine chart. A line-bundle recursion determines the boundary coefficients of the exchange relations. In this mutation class, each nonzero positive simple-root operator and each nonzero stabilizing negative simple-root operator acts in a suitable seed as a monomial times a partial derivative in one frozen variable. Consequently, a single theta basis describes the root filtrations on all divisorial section spaces. We compute the facets of its regular cone from the word and the surviving divisors. The leading terms of the root operators yield tropical formulas for finite-type Levi crystals and polyhedral formulas for branching multiplicities. We obtain formulas for adjoint sections and extend the adapted root charts to projective horospherical varieties with free class group under a simply connected semisimple group.

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Introduction

Let Y be a Schubert variety in a symmetrizable Kac–Moody partial flag variety. For a divisor D supported on the boundary, a regular function on the open Schubert cell defines a section of $\mathcal{O}_Y(D)$ precisely when its poles are bounded by D. The boundary classes freely generate $\mathrm{Cl}(Y)$, so these inequalities describe every homogeneous component of $\mathrm{Cox}(Y)$, including the sections of non-Cartier classes. We encode these inequalities in a cluster seed and identify its upper cluster algebra with the full Cox ring. We then realize the geometric simple-root operators in seeds of the same mutation class.

The construction starts on the Bott–Samelson variety of a word $\mathbf{i} = (i_1, \dots, i_N)$, which may be nonreduced. Each position k has a standard boundary divisor with canonical section τ_k. Let x_k be the minimal homogeneous lift of the corresponding word minor. If $s(k)$ is the

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next occurrence of i_k, the exchange at k is

$$x_k x_k' = \tau_{s(k)} M_k^+(x) + \tau_k M_k^-(x),$$

where $M_k^{\pm}$ are the exchange monomials on the affine chart. The same line-bundle recursion gives the two boundary coefficients and the anticanonical class. Theorem 1.2 reduces upper cluster equality to a degree calculation. Under the hypotheses of that theorem, the Cox ring equals the upper algebra if and only if the sum of the frozen degrees equals the degree of a rational top form with divisor zero. This equality also shows that the initial and adjacent charts cover every codimension-one point of the total coordinate space.

These arguments give the Cox ring of the Bott–Samelson variety (Theorem 3.3). Under a birational contraction preserving the affine chart, the exceptional boundary sections specialize to one. Upper cluster equality descends because the pole order along each exceptional frozen divisor is independent of the seed (Theorem 2.2). For Y, a reduced word for its minimal Weyl group representative therefore gives $\mathrm{Cox}(Y) = \mathcal{U}_{\mathrm{pol}}(\Sigma_Y)$, and all reduced expressions give the same mutation class (Theorem 3.4 and Corollary 3.9).

Theorem 4.1 gives an *adapted seed* for each nonzero positive simple-root operator E_i and each nonzero negative simple-root operator F_i whose rank-one group preserves Y. In this seed the operator is

$$D = c_D x^{v_D} \partial_{b_D}, \qquad (v_D)_{b_D} = 0,$$

where b_D is frozen and x^{v_D} is a monomial with nonnegative exponents. For a nonzero section f, the largest r such that $D^r f \neq 0$ is its polynomial degree in b_D. For E_i, one mutates the successive nonterminal occurrences of i. For F_i, the construction depends on whether i is a left descent. A reduced word starting with i gives an adapted chart in the descent case. In the other case, reversing the word presentation identifies the terminal minor on which F_i acts.

For every divisor class D, the regular theta basis of $H^0(Y, \mathcal{O}_Y(D))$ is indexed by the lattice points of a bounded slice of one rational polyhedral cone (Theorem 5.4). The word seeds of Shen and Weng admit maximal green sequences [SW], so the theta basis is available after inverting the frozen variables [GHKK]. To restore the boundary, we prove Laurent finiteness on the chiral dual by a finite monomial cover and apply theta reciprocity and valuative independence [CMMM]. Theorem 5.6 gives the facets in the initial word coordinates. Each terminal minor contributes the suffix sums along its letter. One mutation sequence computes the orders at all surviving standard boundary divisors; its construction uses Qin's interval variables for the reversed word and transposed Cartan matrix [Qin1]. Each bound on the nilpotence degree of a simple-root operator then cuts out a subspace spanned by a subset of this same basis, even when several of the operators do not commute.

For a finite-type Levi subgroup acting on Y and its graded Cox ring, the leading root arrows give the normal crystal of each section space. To calculate an arrow, carry its tropical index to an adapted seed by tropical mutation, translate the index there, and transport it back (Corollary 5.16). Slicing the common positive-root kernel by divisor class and weight counts branching multiplicities. Section 6 also identifies adjoint sections with interior lattice points and determines the canonical modules of the associated toric degenerations.

Two extensions use the geometry of the boundary. An adapted mutation sequence lifts through an equivariant compactification when the lifted seed gives its Cox ring, since differentiation preserves the filtration of each invariant boundary divisor (Theorem 4.4). For projective horospherical varieties with free class group, the equivariant description of the Cox ring gives adapted charts and crystals from the flag variety of the open orbit (Corollary 5.17). The examples illustrate the Cox grading, the root charts, and the polyhedral section-space basis in both smooth and singular settings.

0.1. **Relation to earlier work.** The cell seeds come from the generalized minors of Berenstein, Fomin, and Zelevinsky [BFZ]. Geiß, Leclerc, and Schröer lifted cluster structures to multihomogeneous coordinate rings of partial flags [GLS]. Demonet extended the categorical construction to nonsimply-laced types [Dem], and Kadhem gave a lifting construction in

arbitrary finite semisimple type [Kad]. Shen and Weng established braid compatibility and maximal green sequences for positive braid words in symmetrizable Kac–Moody type [SW]. Bao and Ye constructed upper cluster structures on open Richardson varieties and twisted products of flag varieties [BY, Theorems 5.13 and 6.5].

Mandel realizes upper cluster algebras as Cox rings of cluster varieties and their partial compactifications [Man, Theorem 3.6 and Corollary 3.7]. For a prescribed compactification, Francone's minimal monomial lifting gives a Cox-ring inclusion and equality after the boundary sections are inverted [Fra1] and [Fra2, Theorem 3.6]. He proves equality before localization for complete flag varieties [Fra2, Theorem 4.4]. Here the boundary recursion and the logarithmic cluster volume form [BMRS, Appendix A] give that equality for arbitrary-word Bott–Samelson varieties and all Schubert varieties, with their full Weil-class grading. Enwright, Francone, Moraga, and Spink show that finite-type Schubert and Bott–Samelson varieties are of cluster type [EFMS]. Our result identifies the upper algebra of a specified seed on each projective variety.

The author constructs root derivations from optimized frozen variables and proves perfectness when generic characters form a basis [Fei]. Here we start with the geometric simple-root operators and find a reachable chart for each of them. We use the interval mutations of Goodearl and Yakimov [GY], the freezing results of Qin [Qin2], and Tang's results on monomial derivatives and opposite seeds [Tan2]. Tang's comparison uses the twist of Kimura, Qin, and Wei [KQW]. The perfect-basis theorem of Berenstein and Kazhdan identifies the resulting Levi crystals [BK]. In the horospherical case the underlying description of the Cox ring as a polynomial extension is due to Brion and Gagliardi [Bri3, Gag].

Earlier polyhedral models include Caldero's toric degenerations [Cal], generalized string polytopes for Bott–Samelson varieties [Fuj], global value semigroups [MSS], cluster valuations on Schubert varieties [FO], and superpotential polytopes for Grassmannian Schubert varieties [RW]. Boundary theta cones also occur in plethysm algebras [FF]. The cone here retains the full divisor class and the simple-root filtrations simultaneously. Its semigroup algebras give toric degenerations by the finite Khovanskii-basis construction [KM].

We work over $\mathbb{C}$ and write $\mathbb{N} = \mathbb{Z}_{\geq 0}$. Frozen variables are polynomial unless explicitly inverted. The rows of an exchange matrix are indexed by mutable variables and its columns by all variables. All Schubert and Bott–Samelson varieties in this paper are finite-dimensional and projective.

1. A criterion for upper cluster equality

1.1. **Seeds with polynomial frozen variables.** A seed $\Sigma = (\mathbf{x}, B)$ consists of a tuple of algebraically independent coordinates $\mathbf{x} = (x_j)_{j \in I}$ and an integer matrix $B = (B_{kj})$, whose rows are indexed by $I_{\mathrm{mut}} \subset I$ and whose columns are indexed by I. Its principal part is skew-symmetrizable. The other indices form $I_{\mathrm{fr}} = I \setminus I_{\mathrm{mut}}$. Mutation at a mutable index k replaces x_k by x'_k, where

$$x_k x'_k = \prod_{j \in I} x_j^{[B_{kj}]_+} + \prod_{j \in I} x_j^{[-B_{kj}]_+}, \qquad [a]_+ = \max(a, 0). \tag{1.1}$$

We use the standard matrix mutation rule in this row convention.

The *mixed Laurent ring* of a seed is $\mathcal{L}_{\mathrm{pol}}(\Sigma) = \mathbb{C}[x_j^{\pm 1} : j \in I_{\mathrm{mut}}][x_f : f \in I_{\mathrm{fr}}]$. Its *upper cluster algebra with polynomial frozen variables* is

$$\mathcal{U}_{\mathrm{pol}}(\Sigma) = \bigcap_{\Sigma' \sim \Sigma} \mathcal{L}_{\mathrm{pol}}(\Sigma').$$

All intersections are taken in the common rational function field. The *cluster algebra* $\mathcal{A}(\Sigma)$ is the subalgebra generated by all coordinates of all reachable seeds, with the frozen variables polynomial. The Laurent phenomenon gives $\mathcal{A}(\Sigma) \subseteq \mathcal{U}_{\mathrm{pol}}(\Sigma)$. We write $\mathcal{L}_{\mathrm{inv}}(\Sigma) = \mathbb{C}[x_j^{\pm 1} : j \in I]$ and $\mathcal{U}_{\mathrm{inv}}(\Sigma)$ for the corresponding rings with every frozen variable inverted. We call the intersection of the initial mixed Laurent ring and those of its one-step mutations

the *upper bound*. A grading is homogeneous if the two monomials in (1.1) have the same degree. Equivalently, each exchange row has degree zero. For a grading by a lattice Γ, a *positive integral coarsening* is a homomorphism $h : \Gamma \to \mathbb{Z}$ for which the induced grading is nonnegative and its degree-zero part is $\mathbb{C}$.

Lemma 1.1. *For each frozen index f, let v_f be the valuation given by the least exponent of x_f in the initial Laurent chart. It has the same coordinate description in every seed. If $R = \mathcal{U}_{\mathrm{pol}}(\Sigma)$ and $b = \prod_f x_f$, then*

$$R = \mathcal{U}_{\mathrm{inv}}(\Sigma) \cap \bigcap_f \mathcal{O}_{v_f}, \qquad R[b^{-1}] = \mathcal{U}_{\mathrm{inv}}(\Sigma).$$

Every frozen coordinate is prime in R, with $x_f R = \{a \in R : v_f(a) \geq 1\}$.

Proof. The remaining seed coordinates have algebraically independent residues for v_f. Reducing an exchange relation modulo x_f gives a nonzero monomial if x_f occurs in the exchange, and a nonzero binomial otherwise. The mutated coordinate is a unit for v_f, and the change of residue coordinates is birational. Induction therefore proves the first assertion. In every seed, its mixed Laurent ring is its Laurent ring intersected with the finitely many valuation rings $\mathcal{O}_{v_f}$. Intersecting over all seeds gives the formula for R.

For $a \in \mathcal{U}_{\mathrm{inv}}(\Sigma)$, a sufficiently large power of b clears all these frozen valuations, proving the localization identity. Dividing an element of R by x_f decreases only its v_f-value, by one. This gives the formula for $x_f R$. Since the other values remain nonnegative and v_f is additive on products, this ideal is prime. □

1.2. **The canonical degree.** The logarithmic cluster volume form detects whether the initial chart and its neighbors cover all prime divisors. The product of the frozen variables cancels its poles along the frozen divisors. At a mutable divisor, regularity of the first mutation controls the remaining pole. Comparison with a generator of the canonical sheaf in codimension one yields the following degree criterion.

Theorem 1.2. *Let $R = \bigoplus_{d \geq 0} R_d$ be a finitely generated normal $\mathbb{C}$-domain with $R_0 = \mathbb{C}$. Suppose a homogeneous rational top form Ω has divisor zero and degree κ. Let $z_1, \ldots, z_N$ be homogeneous pairwise nonassociated prime elements that are algebraically independent and satisfy* $\operatorname{Frac} R = \mathbb{C}(z_1, \ldots, z_N)$.

Equip this tuple with a seed whose first ℓ variables are mutable. Suppose the extended exchange matrix has rank ℓ, every row is primitive, and every first exchange quotient belongs to R. Then

$$\sum_{j=\ell+1}^{N} \deg z_j \geq \kappa.$$

The following conditions are equivalent:

(i) $\sum_{j=\ell+1}^{N} \deg z_j = \kappa$;
(ii) $R = \mathcal{L}_{\mathrm{pol}}(\Sigma) \cap \bigcap_{i=1}^{\ell} \mathcal{L}_{\mathrm{pol}}(\mu_i \Sigma)$;
(iii) $R = \mathcal{U}_{\mathrm{pol}}(\Sigma)$.

When they hold, R is factorial, and the initial chart and its adjacent charts form an open subset of $\operatorname{Spec} R$ *whose complement has codimension at least two. Each chart is a product of an algebraic torus and an affine space.*

Proof. Write $b = \prod_{j>\ell} z_j$ and consider $\eta = b\, d\log z_1 \wedge \cdots \wedge d\log z_N$. We first prove that η is regular in codimension one. It suffices to check the coordinate divisors. At a frozen divisor, the factor b cancels the logarithmic pole. At a mutable divisor (z_i), write $P_i = M_+ + M_- = z_i z_i'$ and $z_i' = z_i^a v$, where $a \geq 0$ and $v \in R^{\times}_{(z_i)}$. Choose j with $B_{ij} \neq 0$. All other initial coordinates are units at (z_i), and $M_- \equiv -M_+$ there. Consequently the coefficient of $d\log z_j$ in dP_i is a

unit, with residue $B_{ij}M_+$. Wedge $dP_i = d(z_i^{a+1}v)$ with dz_i and the $d\log z_k$, $k \neq i,j$, to obtain

$$\eta = uz_i^a\, dz_i \wedge dv \wedge \bigwedge_{k\neq i,j} d\log z_k, \qquad u \in R^\times_{(z_i)}. \tag{1.2}$$

This expression is regular, since a normal variety is smooth at its codimension-one points.

The ratio $F = \eta/\Omega$ therefore belongs to R. It is nonzero and homogeneous of degree $\sum_{j>\ell} \deg z_j - \kappa$, which proves the inequality. Under condition (i), positive grading forces F to be constant:

$$\eta = c\Omega \qquad (c \in \mathbb{C}^\times). \tag{1.3}$$

Set $T = \mathbb{C}[z_1^{\pm1}, \dots, z_N^{\pm1}]$ and $R^\circ = R[(z_1 \cdots z_N)^{-1}]$. The coordinate morphism from the smooth locus of $\operatorname{Spec} R^\circ$ to $\operatorname{Spec} T$ is birational, and (1.3) gives a nonvanishing Jacobian. It is therefore étale, hence an open immersion by Zariski's Main Theorem [Stacks, Tags 05K0 and 0AB1]. Denote its image by U. Normality gives $R^\circ = \Gamma(U, \mathcal{O}_U)$.

Since $R^\times = \mathbb{C}^\times$ and the z_j are pairwise nonassociated primes, every unit of R° is a scalar Laurent monomial in them. If the complement of U contained a prime divisor of the torus, its Laurent irreducible equation would be a further unit on U, a contradiction. That complement has codimension at least two, and normality of the torus gives

$$R^\circ = \Gamma(U, \mathcal{O}_U) = T. \tag{1.4}$$

Nagata's localization theorem now shows that $\operatorname{Cl}(R)$ is generated by the principal coordinate divisors. Thus R is factorial [GELS, Theorem 1.19].

We next recover regularity along each mutable divisor. Equations (1.2) and (1.3) imply $a = 0$, and show that the residues of $z_i' = v$ and z_k, $k \neq i,j$, are algebraically independent. The valuation ord_{z_i} has a height-one center on $T_i = \mathcal{L}_{\mathrm{inv}}(\mu_i\Sigma)$, and this center contains P_i. Primitivity of the exchange row makes P_i irreducible in T_i: a unimodular change of Laurent coordinates takes it to a unit times $1+w$. Hence its center is (P_i). The two discrete valuation rings have the same fraction field, so their local inclusion is an equality:

$$R_{(z_i)} = (T_i)_{(P_i)}. \tag{1.5}$$

By normality and (1.4), $R[b^{-1}] = T \cap \bigcap_{i=1}^{\ell} R_{(z_i)}$. Moreover, $T[P_i^{-1}] = T_i[P_i^{-1}]$, so an element of T can have a pole on $\operatorname{Spec} T_i$ only along (P_i). Equation (1.5) therefore gives

$$R[b^{-1}] = T \cap \bigcap_{i=1}^{\ell} T_i = \mathcal{U}_{\mathrm{inv}}(\Sigma).$$

The last equality is the full-rank upper-bound theorem [BFZ, Proposition 1.8 and Corollary 1.9]; over $\mathbb{C}$, its proof is unchanged, or follows by flat scalar extension of the finite upper bounds.

At a frozen divisor (z_j), nonvanishing of η shows that the residues of all other coordinates are algebraically independent. Thus ord_{z_j} is the usual frozen-coordinate valuation. Lemma 1.1 identifies $\mathcal{L}_{\mathrm{pol}}(\Sigma')$, for every reachable seed, with $\mathcal{L}_{\mathrm{inv}}(\Sigma') \cap \bigcap_{j>\ell} R_{(z_j)}$. Intersecting over the initial and adjacent seeds, or over all seeds, now gives R. This proves (ii) and (iii).

Conversely, assume (ii) or (iii). For each initial or adjacent seed, localizing R at its mutable coordinates gives its mixed Laurent ring. These are smooth open subsets of $\operatorname{Spec} R$. The initial chart misses only the prime divisors (z_i), $i \leq \ell$. The adjacent i-chart contains the generic point of (z_i): the other initial coordinates are units there, and $z_i \nmid z_i'$. Indeed, divisibility would place $z_i'/z_i = (z_i')^2/P_i$ in R, although this expression does not belong to the adjacent mixed Laurent ring. The nonmonomial polynomial P_i has no coordinate factor, so its denominator cannot cancel. These charts therefore cover every codimension-one point.

Mutation preserves η up to sign. On each mixed Laurent chart it is the product of the logarithmic differentials of the mutable coordinates and the ordinary differentials of the frozen coordinates. Hence $\operatorname{div}(\eta) = 0$. The ratio $F = \eta/\Omega$ is now a unit of R, so it is constant and has degree zero. This proves (i), the asserted chart cover, and all remaining claims. □

The primitivity hypothesis is essential, even when R is a polynomial ring. Let $R = \mathbb{C}[x, u, y]$, with $\deg x = \deg u = 1$ and $\deg y = 2$, and set $z = (xu - y)/\mathrm{i}$, where $\mathrm{i}^2 = -1$. Use x as the sole mutable variable and y, z as frozen variables, with exchange $xx' = y^2 + z^2$. The initial coordinates are pairwise nonassociated primes and form a transcendence basis. The exchange quotient is $x' = u(y - \mathrm{i}z) \in R$, and both the sum of the frozen degrees and the degree of $dx \wedge du \wedge dy$ are four. Nevertheless, $u = (y + \mathrm{i}z)/x = x'/(y - \mathrm{i}z)$ is not in the adjacent mixed Laurent ring. The exchange row $(0, 2, -2)$ is not primitive.

2. Cox rings and birational morphisms

2.1. **Boundary divisors and minimal lifts.** Let Y be a normal projective variety with an affine open subset C such that $\mathrm{Cl}(C) = 0$ and $\Gamma(C, \mathcal{O}_C)^\times = \mathbb{C}^\times$. Let $D_1, \dots, D_d$ be the prime divisors in $Y \setminus C$. The divisor-class localization sequence gives $\mathrm{Cl}(Y) = \bigoplus_{a=1}^{d} \mathbb{Z}[D_a]$. They generate because $\mathrm{Cl}(C) = 0$, and any relation would arise from a unit on C. We use this basis to define the *Cox ring*

$$R(Y) = \bigoplus_{\mathbf{m} \in \mathbb{Z}^d} H^0\left(Y, \mathcal{O}_Y\left(\sum_a m_a D_a\right)\right) \boldsymbol{\tau}^{\mathbf{m}} \ \subset \ \mathbb{C}[C][\tau_1^{\pm 1}, \dots, \tau_d^{\pm 1}].$$

Here $\mathcal{O}_Y(D)$ is the rank-one reflexive sheaf of a Weil divisor, and multiplication is multiplication of rational functions. In this realization, a homogeneous expression $f\boldsymbol{\tau}^{\mathbf{m}}$, with $f \in \mathbb{C}[C]$, belongs to $R(Y)$ precisely when $\mathrm{ord}_{D_a}(f) + m_a \geq 0$ for every a. The element τ_a is the canonical section of D_a. This description agrees with the usual construction of Cox rings [ADHL]. For a class $\lambda \in \mathrm{Cl}(Y)$, the notation $\mathcal{O}_Y(\lambda)$ refers to the representative in this boundary basis. Thus $R(Y)_\lambda = H^0(Y, \mathcal{O}_Y(\lambda))$, with the formal degree factor understood.

For $0 \neq f \in \mathbb{C}[C]$, its *minimal homogeneous lift* is $\widetilde{f} = \boldsymbol{\tau}^{\nu(f)} f$, where $\nu(f)_a = -\mathrm{ord}_{D_a}(f)$. It is the unique lift with no canonical boundary section as a factor. We allow negative entries in $\nu(f)$. Minimal lifting and its application to cluster structures were studied by Francone [Fra1, Fra2].

If a cell seed has coordinates $f_1, \dots, f_N$ and exchange matrix B_C, let ν be the matrix with columns $\nu(f_k)$. Its homogeneous lift has coordinates $x_k = \boldsymbol{\tau}^{\nu_{\bullet k}} f_k$ and $\tau_1, \dots, \tau_d$, with the boundary sections frozen and exchange matrix

$$\widetilde{B} = (B_C \mid -B_C \nu^{\mathsf{T}}).$$

Homogeneity determines the added columns. To check regularity, write

$$v_k^{\pm} = \sum_j [\pm B_{kj}]_+ \nu(f_j), \qquad a_k = \max(v_k^+, v_k^-),$$

where the maximum is coordinatewise. The lifted exchange quotient is $\boldsymbol{\tau}^{a_k - \nu(f_k)} f_k'$. If f_k' is regular on C, then

$$\mathrm{ord}_{D_a}(f_k') \geq -(a_k)_a + \nu(f_k)_a$$

for every boundary divisor, by the valuation inequality for its two numerator terms. Thus the lifted quotient belongs to the Cox ring. The inequality may be strict, so the quotient may have a boundary factor. Equality with the Cox ring will show that no such factor occurs anywhere in the mutation pattern.

The divisor-class grading admits a positive integral coarsening. In positive dimension, intersection with $H^{\dim Y - 1}$, for an ample Cartier divisor H, is positive on every nonzero effective divisor class. The resulting degree-zero component is $\mathbb{C}$.

2.2. **Contraction of boundary divisors.** Under a birational contraction, the exceptional divisorial valuations no longer enter the definition of the Cox ring. In the Cox grading this amounts to setting their canonical sections equal to one. For the general contraction formula for finitely generated Cox rings, see [HKL, Proposition 2.2].

Proposition 2.1. *Let $p : X \to Y$ be a proper birational morphism of normal projective varieties over $\mathbb{C}$ identifying an open subvariety $C \subset X$ with an open subvariety of Y. Suppose that $\mathrm{Cl}(C) = 0$ and $\Gamma(C, \mathcal{O}_C)^\times = \mathbb{C}^\times$. Let $D_1, \dots, D_d$ be the prime divisors in $Y \setminus C$, let $\widetilde{D}_a$ be their strict transforms, and let $E_1, \dots, E_e$ be the exceptional prime divisors. Then*

$$\mathrm{Cl}(Y) = \bigoplus_{a=1}^{d} \mathbb{Z}[D_a], \qquad \mathrm{Cl}(X) = \bigoplus_{a=1}^{d} \mathbb{Z}[\widetilde{D}_a] \oplus \bigoplus_{b=1}^{e} \mathbb{Z}[E_b].$$

Write τ_a for the canonical sections of D_a and $\widetilde{D}_a$ in the respective Cox rings, and σ_b for the canonical section of E_b. The map

$$q_p : \mathrm{Cox}(X) \longrightarrow \mathrm{Cox}(Y), \qquad f\boldsymbol{\tau}^{\mathbf{m}}\boldsymbol{\sigma}^{\mathbf{n}} \longmapsto f\boldsymbol{\tau}^{\mathbf{m}}$$

is surjective, respects degrees under $p_* : \mathrm{Cl}(X) \to \mathrm{Cl}(Y)$, and has kernel $(\sigma_1 - 1, \dots, \sigma_e - 1)$. These maps commute with composition of contractions over C. Moreover,

$$\mathrm{Cox}(X)[\sigma_1^{-1}, \dots, \sigma_e^{-1}] = \mathrm{Cox}(Y)[\sigma_1^{\pm 1}, \dots, \sigma_e^{\pm 1}] \tag{2.1}$$

inside $\mathbb{C}(C)(\boldsymbol{\tau}, \boldsymbol{\sigma})$.

Proof. Since Y is normal, p is an isomorphism over the generic point of each D_a. Thus D_a and $\widetilde{D}_a$ define the same valuation, and the remaining boundary divisors of X are the E_b. The divisor-class localization sequence for C gives the asserted bases on X and Y, since a relation among boundary divisors restricts to a unit on C.

For $f \in \Gamma(C, \mathcal{O}_C)$, the homogeneous expression $f\boldsymbol{\tau}^{\mathbf{m}}\boldsymbol{\sigma}^{\mathbf{n}}$ belongs to $\mathrm{Cox}(X)$ precisely when

$$\mathrm{ord}_{D_a}(f) + m_a \geq 0 \quad (1 \leq a \leq d), \qquad \mathrm{ord}_{E_b}(f) + n_b \geq 0 \quad (1 \leq b \leq e).$$

The first set of inequalities characterizes membership of $f\boldsymbol{\tau}^{\mathbf{m}}$ in $\mathrm{Cox}(Y)$. For such a section we can choose $\mathbf{n}$ satisfying the second set. This proves surjectivity and also gives (2.1).

It suffices to compute the kernel in each $\mathrm{Cl}(Y)$-degree. Write an element of the kernel as $\sum_{\mathbf{n}} f_{\mathbf{n}}\boldsymbol{\tau}^{\mathbf{m}}\boldsymbol{\sigma}^{\mathbf{n}}$, so $\sum_{\mathbf{n}} f_{\mathbf{n}} = 0$. Choose $\mathbf{N}$ dominating its finitely many exponents $\mathbf{n}$ coordinatewise. Modulo $(\sigma_b - 1 : 1 \leq b \leq e)$, each summand equals its product with $\boldsymbol{\sigma}^{\mathbf{N}-\mathbf{n}}$. The resulting sum vanishes. The reverse inclusion follows by evaluation. Under a composition the two successive evaluations set exactly the exceptional variables of the composite equal to one. □

Theorem 2.2. *In the setting of Proposition 2.1, let Σ_X and Σ_Y be the minimal homogeneous lifts of a seed of regular functions on C, with all boundary sections frozen. Then*

$$\mathrm{Cox}(X) = \mathcal{U}_{\mathrm{pol}}(\Sigma_X) \quad \Longrightarrow \quad \mathrm{Cox}(Y) = \mathcal{U}_{\mathrm{pol}}(\Sigma_Y).$$

Equality with the initial upper bound also descends. More generally, if the mixed Laurent rings after a nonempty collection of mutation sequences intersect in $\mathrm{Cox}(X)$, then their counterparts on Y intersect in $\mathrm{Cox}(Y)$. The target exchange matrix is obtained by deleting the exceptional frozen columns. Under any of these equalities on X, q_p takes each cluster variable to the corresponding target variable.

Proof. Write B for the exchange matrix on C, and let ν_Y and ν_E record the pole orders of the seed functions at the surviving and exceptional divisors. The lifted matrix on X is $(B \mid -B\nu_Y^{\mathsf{T}} \mid -B\nu_E^{\mathsf{T}})$, whose first two blocks form the lifted matrix on Y. The initial lift on X of each cell coordinate differs from its lift on Y by its exceptional degree monomial. Matrix mutation commutes with deletion of frozen columns, and both terms of each source exchange have the same exceptional degree. Hence the same monomial rescaling, with the new degrees, relates the coordinates after any mutation sequence t. Writing $L_{X,t}$ and $L_{Y,t}$ for the two mixed Laurent rings, and S for the multiplicative system generated by the σ_b, we obtain

$$S^{-1}L_{X,t} = L_{Y,t}[\sigma_1^{\pm 1}, \dots, \sigma_e^{\pm 1}]. \tag{2.2}$$

For any nonempty collection T of mutation sequences,

$$S^{-1}\left(\bigcap_{t\in T} L_{X,t}\right) = \bigcap_{t\in T} S^{-1}L_{X,t}.$$

Indeed, let $h \neq 0$ belong to the right side. By Lemma 1.1, the least exponent $v_b(h)$ of each exceptional variable σ_b is independent of the chart. Thus $\prod_b \sigma_b^{\max(0,-v_b(h))}h$ belongs to every $L_{X,t}$. This proves the reverse inclusion. The forward inclusion holds for every localization.

If $\bigcap_{t\in T} L_{X,t} = \mathrm{Cox}(X)$, apply (2.1) and (2.2) and take the component of exceptional degree zero. The result is $\mathrm{Cox}(Y) = \bigcap_{t\in T} L_{Y,t}$. Take all mutation sequences for the full upper algebra, or the initial and adjacent charts for the upper bound. By the Laurent phenomenon every source cluster variable belongs to the assumed intersection. Evaluation at $\boldsymbol{\sigma} = \mathbf{1}$ removes its exceptional degree monomial and gives the corresponding target variable. □

3. Bott–Samelson and Schubert Cox rings

Let $A = (a_{ij})_{i,j\in I_0}$ be a symmetrizable generalized Cartan matrix, and let G be the associated simply connected Kac–Moody group in a standard realization. Fix opposite Borel subgroups B^+, B^-, a common Cartan subgroup T, and a pinning. We write W for the Weyl group, α_i and $\alpha_i^\vee$ for the simple roots and coroots, and Λ_i for the fundamental weights, and set $\rho = \sum_{i\in I_0} \Lambda_i$. The standard realization includes the additional Cartan directions when A has nonzero corank, so the simple roots are linearly independent. We use the standard facts about the flag variety from [Kum].

For $J \subset I_0$, let P_J be the standard parabolic and let W^J be the minimal representatives of W/W_J. For $w \in W^J$, write $X_w^{P_J} = \overline{B^+wP_J/P_J}$. This is a normal projective variety with open cell $C_w \simeq \mathbb{A}^{\ell(w)}$. The empty word gives a point with Cox ring $\mathbb{C}$.

3.1. **Words and their minors.** Let $\mathbf{i} = (i_1, \ldots, i_N)$ be any word in the simple reflections. Here k denotes a position and i_k its simple-root index. The associated Bott–Samelson variety is

$$Z_{\mathbf{i}} = P_{i_1} \times^{B^+} \cdots \times^{B^+} P_{i_N}/B^+.$$

Let E_j be the divisor on which the j-th factor belongs to B^+. The complement of their union is the affine chart $C_{\mathbf{i}} \simeq \mathbb{A}^N$, with coordinates given by $(z_1, \ldots, z_N) \mapsto [q_{i_1}(z_1), \ldots, q_{i_N}(z_N)]$. Here $q_i(z) = x_i(z)\dot{s}_i$, with the representative $\dot{s}_i = x_i(-1)x_{-i}(1)x_i(-1)$ of [SW, Equation (A.5)]. For each position k, define

$$f_k = \Delta_{\Lambda_{i_k}}\big(q_{i_1}(z_1)\cdots q_{i_k}(z_k)\big).$$

Here Δ_λ is the highest matrix coefficient in the integrable highest-weight module of highest weight λ, normalized to have value one at the identity.

We write $p(k)$ and $s(k)$ for the preceding and following positions bearing the letter i_k, using $-\infty$ and $+\infty$ when the relevant position does not exist. A position is mutable when $s(k) < +\infty$; the last occurrence of each letter is frozen. Define the lower unitriangular matrix $L_{\mathbf{i}}$ and the mutable rows of the word matrix B by

$$(L_{\mathbf{i}})_k = e_k^{\mathsf{T}} + e_{p(k)}^{\mathsf{T}} + \sum_{j<k<s(j)} a_{i_j i_k} e_j^{\mathsf{T}}, \qquad B_k = (L_{\mathbf{i}})_k - (L_{\mathbf{i}})_{s(k)},$$

where a nonexistent predecessor contributes zero. The exchange at k is $f_k f_k' = M_k^+(f) + M_k^-(f)$, with $M_k^\pm(f) = \prod_j f_j^{[\pm B_{kj}]_+}$.

Lemma 3.1. *The functions $f_1, \ldots, f_N$ are pairwise nonassociated prime elements of $\mathbb{C}[C_{\mathbf{i}}]$, and they generate its fraction field. They form a seed with exchange matrix B, and every first exchange quotient f_k' belongs to $\mathbb{C}[C_{\mathbf{i}}]$.*

Proof. Let Z be the Bott–Samelson variety of a prefix, let v be its Demazure product, and let $m : Z \to X_v$ be the multiplication map. The group $U(v)$ acts simply transitively on C_v, so equivariance gives

$$m^{-1}(C_v) \simeq U(v) \times m^{-1}(vB^+).$$

The left side is a nonempty open subset of Z, so it is smooth and irreducible. The same holds for the fiber, and over C_v the morphism m is the first projection.

Suppose that the letter i occurs in the prefix. The zero scheme of the highest section of $\mathcal{L}(\Lambda_i)|_{X_v}$ is the Richardson divisor $X_v \cap X^{s_i}$. This divisor is reduced and irreducible, and its intersection with C_v is dense and nonempty [Kum2, Proposition 6.6]. Its inverse image over C_v is therefore reduced and irreducible. Any other prime component W of the pullback would lie over the Schubert boundary. Choose C_u such that $m^{-1}(C_u) \cap W$ is dense in W. Since W has codimension one, it is a component of the proper closed subset $m^{-1}(X_u)$. The connected group B^+ preserves each component. Thus $m(W)$ contains C_u, and the highest section would vanish on X_u, although it takes the value one at the identity flag. This excludes every further component. The pullback is now generically reduced, and as a Cartier divisor on the smooth variety Z it has no embedded components. It is therefore reduced.

Apply this argument to the prefix ending at k, with $i = i_k$. The pullback section is nonzero at the point whose factors are all the identity. Every standard boundary divisor contains this point, so none is a component of its zero divisor. Its restriction f_k to the affine chart is consequently a nonconstant prime, and remains prime after the later coordinates are adjoined. The rank-one calculation gives

$$f_k = f_{p(k)} z_k + h_k(z_1, \dots, z_{k-1}), \qquad f_{-\infty} = 1.$$

The coefficient is $f_{p(k)}$ because the intervening factors fix the highest-weight vector. These triangular identities recover the z_k successively from the f_k. They also show that f_k depends on z_k, whereas the preceding minors do not. Thus the primes are pairwise nonassociated and generate the fraction field.

To check regularity of the first exchanges, let $C_{\mathbf{i}}^\circ$ be the complement of the terminal minor divisors in $C_{\mathbf{i}}$. Consider the decorated double Bott–Samelson cell for the positive braid $\mathbf{i}$ and the empty braid. Normalize the initial opposite decorated flags to $(U^- h, U^+)$, with $h \in T$. Propagating the positive decorations gives $q_{i_1}(z_1) \cdots q_{i_k}(z_k) U^+$, and the terminal opposition condition is exactly the nonvanishing of the terminal minors. Thus the decorated cell is $T \times C_{\mathbf{i}}^\circ$. In the coordinates of [SW, Definition 3.20], the left open string on level i has coordinate h^{Λ_i}, and the string after position k has coordinate $h^{\Lambda_{i_k}} f_k$. Setting the left open coordinates to one gives the stated matrix B, directly from its string construction. For a Cartan matrix of nonzero corank, the additional Cartan characters are frozen and are also set to one. An index absent from the word contributes the constant coefficient one.

By [SW, Theorem 3.45], the first exchange quotients are regular on the decorated cell. Specializing $h = 1$ gives $f'_k \in \mathbb{C}[C_{\mathbf{i}}^\circ]$. Its exchange formula allows a pole only along $f_k = 0$. The generic point of this divisor lies in $C_{\mathbf{i}}^\circ$, since the mutable prime f_k is distinct from every terminal prime. Thus f'_k is regular on $C_{\mathbf{i}}$. □

For $k \le N$, let $m_k : Z_{\mathbf{i}} \to G/B^+$ multiply the first k factors, and write $\mathcal{L}_k(\lambda) = m_k^* \mathcal{L}(\lambda)$, with $\mathcal{L}_0(\lambda) = \mathcal{O}$. The same recursion on k determines the boundary coefficients of the lifted exchanges and the class appearing in Theorem 1.2.

Lemma 3.2. *The minimal lift x_k of f_k has degree $d_k = [\mathcal{L}_k(\Lambda_{i_k})]$. In the standard boundary basis, its coefficients are*

$$\nu_{jk} = \begin{cases} \left\langle \Lambda_{i_k}, s_{i_k} \cdots s_{i_{j+1}} (\alpha_{i_j}^\vee) \right\rangle, & j \le k, \\ 0, & j > k. \end{cases} \tag{3.1}$$

Furthermore,

$$L_{\mathbf{i}} \nu^{\mathsf{T}} = I_N, \qquad (B\nu^{\mathsf{T}})_k = e_k^{\mathsf{T}} - e_{s(k)}^{\mathsf{T}}. \tag{3.2}$$

The terminal degrees, together with the standard boundary classes, sum to $-K_{Z_{\mathbf{i}}}$.

Proof. The highest section of $\mathcal{L}_k(\Lambda_{i_k})$ restricts to f_k in the standard frame of the affine chart. It has no standard boundary component, as in the previous proof, so it is the minimal lift.

For a general weight λ, the last $\mathbb{P}^1$-bundle in a prefix gives

$$[\mathcal{L}_k(\lambda)] = \langle \lambda, \alpha_{i_k}^\vee \rangle [E_k] + [\mathcal{L}_{k-1}(s_{i_k}\lambda)]. \tag{3.3}$$

Indeed, $\mathcal{L}_k(\lambda)$ has fiber degree $m = \langle \lambda, \alpha_{i_k}^\vee \rangle$, so $\mathcal{L}_k(\lambda)(-mE_k)$ descends to the preceding prefix. Its restriction to the standard section E_k is $\mathcal{L}_{k-1}(\lambda - m\alpha_{i_k})$, since the normal bundle of that section is $\mathcal{L}_{k-1}(\alpha_{i_k})$. Iterating (3.3) proves (3.1).

On a prefix, the fundamental line bundle indexed by a simple root is determined by the last occurrence of that index. Since a weight that pairs trivially with all simple coroots gives a trivial line bundle on a Bott–Samelson variety, the simple-coroot pairings give

$$[\mathcal{L}_{k-1}(\alpha_{i_k})] = 2d_{p(k)} + \sum_{j<k<s(j)} a_{i_j i_k} d_j,$$

with $d_{-\infty} = 0$. Substituting $s_{i_k}\Lambda_{i_k} = \Lambda_{i_k} - \alpha_{i_k}$ into the recursion therefore gives

$$d_k + d_{p(k)} + \sum_{j<k<s(j)} a_{i_j i_k} d_j = [E_k].$$

Writing this relation in the boundary basis gives the first identity in (3.2). Subtracting the relation at $s(k)$ gives the second. In particular, ν is upper unitriangular.

The relative canonical bundle of the last $\mathbb{P}^1$-bundle has class $-2[E_k] + [\mathcal{L}_{k-1}(\alpha_{i_k})]$. Together with (3.3), induction therefore gives

$$-K_{Z_{\mathbf{i}}} = \sum_{j=1}^{N} [E_j] + [\mathcal{L}_N(\rho)].$$

If k is the last occurrence of i, every later simple reflection fixes Λ_i; consequently $\mathcal{L}_N(\Lambda_i) = \mathcal{L}_k(\Lambda_i)$. The fundamental weights indexed by simple roots absent from the word give trivial bundles. Since ρ is the sum of the fundamental weights, the terminal degrees sum to $[\mathcal{L}_N(\rho)]$, as required. □

Theorem 3.3. *For every word in the simple reflections of a symmetrizable Kac–Moody group, minimally lift the seed of Lemma 3.1 to* $\mathrm{Cox}(Z_{\mathbf{i}})$ *and adjoin the standard boundary sections* $\tau_1, \dots, \tau_N$ *as frozen variables. The resulting seed* $\Sigma_{\mathbf{i}}$ *satisfies*

$$\mathrm{Cox}(Z_{\mathbf{i}}) = \mathcal{L}_{\mathrm{pol}}(\Sigma_{\mathbf{i}}) \cap \bigcap_{k \text{ mutable}} \mathcal{L}_{\mathrm{pol}}(\mu_k \Sigma_{\mathbf{i}}) = \mathcal{U}_{\mathrm{pol}}(\Sigma_{\mathbf{i}}).$$

If the word contains s *distinct letters, the seed has* $N - s$ *mutable and* $N + s$ *frozen variables. Its extended exchange matrix* $\widetilde{B} = (B \mid -B\nu^{\mathsf{T}})$ *has full row rank and primitive rows. The exchanges are*

$$x_k x_k' = \tau_{s(k)} M_k^+(x) + \tau_k M_k^-(x). \tag{3.4}$$

Its frozen degrees sum to $-K_{Z_{\mathbf{i}}}$. *The initial and adjacent mixed Laurent charts are smooth and cover the Cox space outside a subset of codimension at least two.*

Proof. The boundary classes freely generate $\mathrm{Pic}(Z_{\mathbf{i}})$, and the Cox ring is factorial [Arz, Theorem 1.1]. Bott–Samelson varieties for arbitrary words are log Fano [And, Theorem A.2], so their Cox rings are finitely generated [GOST, Corollary 5.3]. Choose an ample divisor H on $Z_{\mathbf{i}}$. For $N > 0$, intersection with H^{N-1} makes the Cox ring positively graded with degree-zero part $\mathbb{C}$, since every nonzero effective divisor meets H^{N-1} positively. The case $N = 0$ is immediate.

Choose the coordinate volume form $\omega_C = dz_1 \wedge \cdots \wedge dz_N$. Its divisor is supported on the standard boundary. Write $-K_{Z_{\mathbf{i}}} = \sum_j \kappa_j [E_j]$, using Lemma 3.2. Independence of the boundary classes gives $\operatorname{div}(\omega_C) = -\sum_j \kappa_j E_j$. Consequently

$$\Omega = \boldsymbol{\tau}^{\boldsymbol{\kappa}}\, \omega_C \wedge d\log \tau_1 \wedge \cdots \wedge d\log \tau_N$$

has degree $-K_{Z_{\mathbf{i}}}$. To compute its divisor, recall that the characteristic space is a torus torsor over the smooth variety $Z_{\mathbf{i}}$, and is an open subset of $\operatorname{Spec}\operatorname{Cox}(Z_{\mathbf{i}})$ with complement of codimension at least two [ADHL, Construction 1.6.3.1]. Near the generic point of E_j, choose a local equation t and torus coordinates u_a such that $\tau_j = t u_j$ and $\tau_a = u_a$ for $a \neq j$. Choose local parameters $t, t_2, \ldots, t_N$ on the base and write $\omega_C = t^{-\kappa_j} v\, dt \wedge dt_2 \wedge \cdots \wedge dt_N$, where v is a unit. In the expression for Ω, the powers of t cancel, and the term dt/t from $d\log \tau_j$ vanishes when wedged with dt. The remaining coefficient $v\boldsymbol{u}^{\boldsymbol{\kappa}}$ is a unit for every integral $\boldsymbol{\kappa}$. Over the affine chart the form is also nowhere vanishing, so its divisor on the Cox space is zero.

The lifted minors and the boundary sections are the canonical sections of distinct prime divisors and are therefore pairwise nonassociated primes in the Cox ring. Together they generate its fraction field and form a transcendence basis. Lemma 3.1 gives regular first exchange quotients on the affine chart. At each boundary prime the initial minimal lift in the denominator has order zero, and both terms of the homogenized numerator have nonnegative order. Thus the lifted quotients are regular on the Cox space by normality.

The second identity in (3.2) proves (3.4) and makes every exchange row primitive. For each fixed letter, the vectors $e_k - e_{s(k)}$ are the successive edge vectors of a path, and different letters have disjoint supports. Hence $\widetilde{B}$ has full row rank. Finally, the frozen-degree identity in Lemma 3.2 is precisely the equality required by the canonical-degree criterion. That criterion proves all the asserted equalities and the statement about mixed Laurent charts. □

For explicit canonical coefficients, define $\gamma_j = s_{i_N} \cdots s_{i_{j+1}} \alpha_{i_j}$. These roots need not be positive. Formula (3.1) gives

$$-K_{Z_{\mathbf{i}}} = \sum_{j=1}^{N} \bigl(1 + \langle \rho, \gamma_j^\vee \rangle\bigr)[E_j].$$

For example, in type C_2 take $a_{12} = -2$, $a_{21} = -1$ and the word $(2,1,2)$. Its minors are $z_1, z_2, z_1 z_3 - z_2^2$, so the first mutant on the cell is z_3. The lifted exchange $x_1 x_1' = \tau_3 x_2^2 + \tau_1 x_3$ records both the Cartan entry and the two boundary positions indexed by the simple root 2.

For a nonreduced example, take the word $\mathbf{i} = (1, \ldots, 1)$ of length n in type A_1. Then $Z_{\mathbf{i}} = (\mathbb{P}^1)^n$ and its Cox ring is polynomial in the homogeneous coordinates $[a_j : b_j]$ of the factors. With $a_0 = 1$, the boundary sections are $\tau_1 = b_1$ and $\tau_j = a_{j-1} b_j - b_{j-1} a_j$ for $j > 1$; the lifted exchanges are

$$a_k a_k' = \tau_{k+1} a_{k-1} + \tau_k a_{k+1} \qquad (1 \leq k < n).$$

Successively mutating at $1, \ldots, n-1$ gives the chart adapted to the diagonal positive-root action. This illustrates the arbitrary-word construction without any reducedness assumption.

3.2. **Contractions and partial flag varieties.**

Theorem 3.4. *Let $p : Z_{\mathbf{i}} \to Y$ be a proper birational morphism to a normal projective variety, and suppose that p identifies $C_{\mathbf{i}}$ with an open subset of Y. Delete from $\Sigma_{\mathbf{i}}$ the frozen columns of the standard boundary divisors contracted by p, and set their canonical sections equal to one. The resulting seed Σ_Y satisfies*

$$\operatorname{Cox}(Y) = \mathcal{L}_{\mathrm{pol}}(\Sigma_Y) \cap \bigcap_{k \text{ mutable}} \mathcal{L}_{\mathrm{pol}}(\mu_k \Sigma_Y) = \mathcal{U}_{\mathrm{pol}}(\Sigma_Y).$$

Every nonboundary seed coordinate is its minimal homogeneous lift from $C_{\mathbf{i}}$. The frozen degrees sum to $-K_Y$, and the initial and adjacent mixed Laurent charts cover the Cox space in codimension one.

In particular, this construction applies to every Schubert variety $Y = X_w^{P_J}$ and every reduced expression $\mathbf{i}$ of w. If $w^{\widehat{j}}$ is the Weyl-group product obtained by deleting position j, the surviving standard boundary sections are exactly those satisfying $\ell(w^{\widehat{j}}) = \ell(w) - 1$ and $w^{\widehat{j}} \in W^J$.

Proof. Apply Theorem 2.2 to the seed in Theorem 3.3. This gives the two ring equalities and identifies the surviving boundary columns. A rational top form has the same order along corresponding nonexceptional prime divisors. Thus $p_* K_{Z_{\mathbf{i}}} = K_Y$, and pushing forward the frozen degrees gives their sum on Y.

The quotient description makes $\mathrm{Cox}(Y)$ finitely generated. Since $\mathrm{Cl}(Y)$ is free, its Cox ring is factorial [Arz]. A minimal lift of a cell prime is the canonical homogeneous section of the closure of its prime zero divisor and is therefore prime in the Cox ring. The same holds for the boundary sections. Each initial or adjacent mixed Laurent ring is a localization of $\mathrm{Cox}(Y)$ at that seed's mutable coordinates. As in the last paragraph of the proof of Theorem 1.2, these charts cover every prime divisor. Proposition 3.7 below gives minimality throughout the mutation pattern.

For a reduced word of $w \in W^J$, multiplication gives a proper birational morphism to $X_w^{P_J}$ and identifies the open cells. The image of the standard divisor at position j is indexed by the minimal representative modulo W_J of the Demazure product of the deleted word. It has dimension $\ell(w) - 1$ exactly when the deleted word is reduced and its product belongs to W^J. This proves the survival criterion. □

For a reduced word the cell functions above are the usual *unipotent minors*. Under $C_w \simeq U(w) = U^+ \cap wU^-w^{-1}$, they are $f_k = \Delta_{\Lambda_{i_k}, w_{\le k}\Lambda_{i_k}}|_{U(w)}$. The valuation matrix for the partial flag variety consists of the surviving rows of ν.

Proposition 3.5. *Let $J \subset K$ be subsets of the simple-root index set, let $w \in W^K$, and let $p : X_w^{P_J} \longrightarrow X_w^{P_K}$ be the standard projection. Construct both lifted seeds from the same reduced expression of w. The map q_p of Proposition 2.1 sets equal to one the boundary frozen variables indexed by*

$$v \in W^J, \qquad v < w, \qquad \ell(v) = \ell(w) - 1, \qquad v \notin W^K,$$

and deletes their columns from the extended exchange matrix. It sends every initial minimal lift to the corresponding minimal lift on $X_w^{P_K}$, and it sends corresponding cluster variables to one another after any fixed sequence of mutations. Consequently,

$$\mathcal{U}_{\mathrm{pol}}(\Sigma_K) \simeq \mathcal{U}_{\mathrm{pol}}(\Sigma_J)/(\tau_v - 1 : v \in W^J \setminus W^K,\ v < w,\ \ell(v) = \ell(w) - 1).$$

For $J \subset K \subset L$ and $w \in W^L$, these specializations commute with composition.

Proof. Since $w \in W^K \subset W^J$, the projection identifies the open cells of the two Schubert varieties. Both cells have dimension $\ell(w)$, so the projection is birational. The boundary prime divisors of $X_w^{P_J}$ are $X_v^{P_J}$, where $v \in W^J$, $v < w$, and $\ell(v) = \ell(w) - 1$. The image of this divisor is $X_{v^K}^{P_K}$, where v^K is the minimal representative of vW_K. Its dimension is $\ell(v^K)$, so it is a divisor precisely when $v = v^K$. This identifies the exceptional columns.

Theorem 2.2 now gives the specialization of both the initial lifts and all their mutations. By Theorem 3.4, each upper algebra equals the corresponding Cox ring. The quotient formula and the composition law follow from Proposition 2.1. □

3.3. **The universal grading.** Let S be the set of simple-root indices occurring in $\mathbf{i}$, and let $Q_S = \bigoplus_{i \in S} \mathbb{Z}\alpha_i$ be the support root lattice. The word seed carries the Q_S-grading

$$\lambda_k = \deg_{Q_S} f_k = \Lambda_{i_k} - w_{\le k}\Lambda_{i_k}.$$

Here $w_{\le k} = s_{i_1} \cdots s_{i_k}$ is the ordinary Weyl-group product, whether or not the prefix is reduced. Giving the canonical boundary sections root degree zero makes $\deg_{Q_S} x_k = \lambda_k$ in every compactification. On cell functions the inverse-pullback Cartan weight is $-\lambda_k$. In a divisorial section space the chosen linearization adds a character depending on the divisor class. The representation-theoretic statements use these shifted Cartan weights.

Proposition 3.6. *Let Y be a Schubert variety, or a normal projective contraction of $Z_{\mathbf{i}}$ that is an isomorphism on the open cell. Write d for the number of boundary prime divisors of Y, and let $\widetilde{B} = (B \mid -B\nu^{\mathsf{T}})$ be its lifted exchange matrix. The homomorphism*

$$\mathcal{D} : \mathbb{Z}^{N+d} \longrightarrow \mathrm{Cl}(Y) \oplus Q_S, \qquad e_k \longmapsto (\nu_{\bullet k}, \lambda_k), \quad e_{N+a} \longmapsto ([D_a], 0),$$

fits into an exact sequence

$$0 \longrightarrow \mathbb{Z}^{N-|S|} \xrightarrow{\widetilde{B}^{\mathsf{T}}} \mathbb{Z}^{N+d} \xrightarrow{\mathcal{D}} \mathrm{Cl}(Y) \oplus Q_S \longrightarrow 0. \tag{3.5}$$

Consequently, the divisor-class and root-lattice gradings together give the universal grading *of the lifted seed: every homogeneous grading of the seed with values in an abelian group factors uniquely through $\mathcal{D}$.*

Proof. The cell exchanges are homogeneous for the root grading, so $B(\lambda_1, \ldots, \lambda_N)^{\mathsf{T}} = 0$. The formula for the added frozen columns also gives $\mathcal{D}\widetilde{B}^{\mathsf{T}} = 0$.

We first show that $\mathcal{D}$ is surjective. The boundary columns generate $\mathrm{Cl}(Y)$. If k is the first occurrence of i_k, then $\lambda_k = w_{\le k-1}\alpha_{i_k}$. The preceding product belongs to the parabolic subgroup generated by the earlier letters, none of which equals i_k. Consequently this root is positive, its α_{i_k}-coefficient is one, and every other nonzero simple-root coefficient is indexed by a letter occurring earlier in the word. Ordering these first occurrences from left to right therefore gives a unitriangular matrix in the simple-root basis of Q_S. The λ_k generate Q_S, proving surjectivity.

The successor columns give a maximal minor of B with determinant ± 1. Indeed, the entry in column $s(k)$ of mutable row k is -1, and that row has no nonzero entry in a column greater than $s(k)$. Order the mutable rows by increasing successor and use the corresponding successor columns in the same order. The resulting square matrix is triangular with diagonal entries -1. Hence $\widetilde{B}^{\mathsf{T}}$ is injective and its image is a saturated sublattice of $\mathbb{Z}^{N+d}$. Its rank is $N - |S|$, which is also the rank of $\ker \mathcal{D}$. The inclusion already proved must therefore be an equality. This gives (3.5) and the universal property. □

In particular, the torus of diagonal rescalings compatible with the exchange relations is the product of the Cox grading torus and the *effective Cartan torus* $\mathrm{Hom}(Q_S, \mathbb{C}^\times)$. A birational projection acts on the universal grading by pushforward on $\mathrm{Cl}(Y)$ and the identity on Q_S.

3.4. **Minimal lifting and changes of reduced expression.** Different root operators may require different reduced expressions of w. The corresponding lifted seeds lie in one mutation class: boundary minimality is preserved by mutation, and braid moves connect the unipotent-minor seeds.

Proposition 3.7. *Let Y be a normal projective variety with an affine open subset C, and suppose that the prime divisors $D_1, \ldots, D_d$ in $Y \setminus C$ freely generate $\mathrm{Cl}(Y)$. Let τ_a be the canonical section of D_a, and realize the Cox ring as $R \subset \mathbb{C}[C][\tau_1^{\pm 1}, \ldots, \tau_d^{\pm 1}]$. Suppose that the minimal homogeneous lift $\widetilde{\Sigma}$ of a seed Σ on C, with the τ_a adjoined as polynomial frozen variables, satisfies $R = \mathcal{U}_{\mathrm{pol}}(\widetilde{\Sigma})$. Then every seed obtained from $\widetilde{\Sigma}$ by mutation is the minimal homogeneous lift of the corresponding seed obtained from Σ.*

In particular, minimal lifting commutes with mutation and relabeling, and it identifies the labeled exchange graphs of these two cluster patterns, with the boundary variables fixed.

Proof. The lifted exchanges are homogeneous and specialize, after setting $\tau_a = 1$, to the cell exchanges at every seed. Hence every lifted cluster coordinate has the form $X = \tau^m f$, where

f is the corresponding cell coordinate. The Laurent phenomenon with polynomial frozen variables gives $X \in R$.

In its own mixed Laurent chart a mutable coordinate X is invertible, while each τ_a cuts out a divisor. Thus no τ_a divides X in R. A homogeneous section $\tau^m f$ is divisible by τ_a precisely when $m_a + \operatorname{ord}_{D_a}(f) > 0$. Therefore

$$m_a = -\operatorname{ord}_{D_a}(f) \qquad (1 \le a \le d).$$

This is exactly the minimal lift. The original frozen coordinates do not change under mutation, so their lifts remain minimal as well.

The boundary part of the exchange matrix is also determined by these degrees. If B' is the cell exchange matrix and ν' records the degrees of its minimal lifts, homogeneity forces the lifted matrix to be $\big(B' \mid -B'(\nu')^{\mathsf{T}}\big)$. Consequently two mutation paths with the same terminal labeled seed on C have the same lifted terminal seed. Specialization gives the inverse correspondence. □

Monomial lifting and its compatibility with mutation were developed by Francone [Fra1, Proposition 1.0.3 and Lemma 3.0.8]. The point of Proposition 3.7 is that the equality with the Cox ring forces every coordinate in the lifted pattern to remain minimal.

Lemma 3.8. *Let w belong to the Weyl group of a symmetrizable Kac–Moody group. The normalized unipotent-minor seeds of $\mathbb{C}[U(w)]$ associated with all reduced expressions of w belong to one mutation class.*

Proof. We compare these seeds with the double Bruhat cell $G^{e,w^{-1}}$. For a reduced expression $\mathbf{i} = (i_1, \dots, i_N)$ of w, use the reversed positive double word $\mathbf{i}^{\mathrm{op}} = (i_N, \dots, i_1)$. The chamber minors of this double word consist of the functions

$$\Delta_{\Lambda_{i_k}, w_{\le k}\Lambda_{i_k}} \quad (1 \le k \le N), \qquad h_i = \Delta_{\Lambda_i,\Lambda_i} \quad (i \in I_0).$$

The chamber minor at the position corresponding to k is the prefix minor at the preceding occurrence of i_k, or h_{i_k} when there is no preceding occurrence. The initial frozen minor indexed by i is the terminal prefix minor for that index. This follows directly from the conventions for double-word minors in [BFZ, Equations (2.9)–(2.12)].

The torus factor in $G^{e,w^{-1}} \subset B^+$ is specified by the h_i. For a realization with additional torus directions, we also adjoin the corresponding character functions as frozen coordinates and specialize them to one. Write $U = U^+$. After setting the torus coordinates to one, the chamber minors restrict to $U \cap B^- w^{-1} B^-$. To compare them with functions on $U(w)$, write $U = U(w)U'$, where $U' = U \cap wUw^{-1}$, and let q be the projection to $U(w)$. If u is a prefix of a reduced expression of w, then $U' \subset uUu^{-1}$. Hence U' fixes the extremal vector of weight $u\Lambda_i$, and the minor $\Delta_{\Lambda_i, u\Lambda_i}$ is right U'-invariant. Its restriction is therefore the pullback under q of the same unipotent minor on $U(w)$.

These restricted chamber minors are algebraically independent: the chamber minors and the h_i form a torus chart on the double Bruhat cell, and the torus factor splits off. It follows that the restriction of q to $U \cap B^- w^{-1} B^-$ is dominant. Thus identities among these minors can be checked on that double Bruhat cell.

Shen and Weng prove that braid moves act by the corresponding sequences of mutations, both on the exchange matrices and on the chamber-minor functions, for every symmetrizable Cartan matrix [SW, Propositions 3.7 and 3.25]. Their sequences leave the principal minors h_i frozen. Specializing them to 1 therefore gives the same mutation sequences for our unipotent minors. The exchange matrices specialize to the reduced-word matrices; this also follows directly by reversing the word and replacing each position by its predecessor. The equality of functions follows from the dominance just proved. Finally, any two reduced expressions are joined by braid moves. □

Corollary 3.9. *For every Schubert variety $Y = X_w^{P_J}$, the minimally lifted seeds associated with all reduced expressions of w belong to one mutation class. Every seed in this class is the*

minimal lift of its cell seed, and every sequence of cell mutations lifts uniquely to a sequence of Cox-ring mutations fixing the boundary sections.

Proof. Apply Proposition 3.7 to the Cox-ring equality and then use Lemma 3.8. □

Lemma 3.10. *Let $p : X \to Y$ be a proper birational morphism between normal projective varieties. Every action of a connected algebraic group on X descends uniquely to Y. In particular, the canonical B^+-action on a Bott–Samelson variety descends along every such morphism.*

Proof. Normality of Y gives $p_*\mathcal{O}_X = \mathcal{O}_Y$. Blanchard's lemma therefore gives the first assertion [Bri2, Theorem 7.2.1]. More generally, it gives a homomorphism $\mathrm{Aut}^0(X) \to \mathrm{Aut}^0(Y)$ making p equivariant [Bri2, Corollary 7.2.2]. The canonical Borel action has image in $\mathrm{Aut}^0(X)$, so this also proves the assertion for a Kac–Moody Borel. □

4. Simple-root operators

A seed is *adapted* to a derivation D when $D = cx^v\partial_b$ for a frozen coordinate b, a scalar $c \in \mathbb{C}^\times$, and a vector $v \in \mathbb{N}^I$ with $v_b = 0$. Thus D kills every seed coordinate except the frozen variable b, which we call the *active coordinate*. Choose positive rational numbers t_i satisfying $a_{ij}/t_i = a_{ji}/t_j$, and assign $\delta_k = t_{i_k}$ to each mutable position of a word seed. The symmetrizer weights δ_k remain attached to the corresponding mutable indices under mutation. Equivalently, $t_i = 2/(\alpha_i, \alpha_i)$ for a positive normalization of the invariant form on each connected component.

We use inverse pullback for the action on functions and normalize the infinitesimal operators by $[E_i, F_i] = H_i$ and $[H_i, E_i] = 2E_i$. Scalar factors from the pinning are written explicitly in the monomial formulas.

Let $Y = X_w^{P_J}$, $R = \mathrm{Cox}(Y)$, and let b_i be the minimal lift of the terminal minor indexed by i. The positive simple-root derivations E_i act on R. The negative derivation F_i also acts whenever the rank-one group G_i preserves Y, that is, whenever $i \in I_Y = \{i : s_i w W_J \le w W_J\}$. These actions preserve the divisor-class grading. We use the common mutation class of Corollary 3.9.

Theorem 4.1. *Every nonzero E_i, and every nonzero F_i with $i \in I_Y$, has an adapted seed in this mutation class. In such a seed, with active frozen variable $b(D)$,*

$$D = c_D \prod_{x \neq b(D)} x^{m_x(D)}\partial_{b(D)}, \qquad c_D \in \mathbb{C}^\times, \qquad m_x(D) \in \mathbb{Z}_{\ge 0}.$$

For $D = E_i$ or $D = F_i$, its mutable exponents are

$$m_{x_k}(D) = \frac{\delta_k}{t_i}|B^D_{k,b(D)}|.$$

Here the nonzero entries in the active column are positive for E_i and negative for F_i.

The following constructions give the adapted seeds.

1. *For E_i, start from any reduced-word seed and mutate the nonterminal occurrences of i in increasing order. The active variable is b_i.*
2. *For F_i with $s_i w < w$, choose a reduced expression beginning with i. In its initial lift $F_i = c_i x_1 \partial_{\tau_a}$, where τ_a is the canonical section of $X^{P_J}_{s_i w}$.*
3. *For F_i with $s_i w > w$ and $i \in I_Y$, one has $w^{-1}\alpha_i = \alpha_j$ for a unique $j \in J$. Pass to the reverse interval order of the symmetric CGL presentation, then mutate the nonterminal occurrences of j in that order. The active variable is b_j.*

The operator E_i is nonzero exactly when $i \in \mathrm{supp}(w)$. For $i \in I_Y$, the same criterion holds for F_i.

4.1. **A mutation criterion.** Let R be a complex domain with $\mathcal{A}(\Sigma) \subseteq R \subseteq \mathcal{U}_{\mathrm{pol}}(\Sigma)$. Assume that the extended exchange matrix B has full row rank. For mutable positions, choose a positive skew-symmetrizer (δ_j), so that $\delta_j B_{jk} = -\delta_k B_{kj}$.

Lemma 4.2. *A Laurent monomial in any reachable seed belongs to R only if all its exponents are nonnegative. If R is factorial, every coordinate of every reachable seed is prime in R.*

Proof. Frozen exponents are nonnegative by membership in the mixed Laurent ring. A negative mutable exponent introduces a power of the exchange binomial in the denominator after mutation. Its two terms have no common coordinate factor, and the exchange row is nonzero by full rank. Thus this binomial cannot divide the monomial numerator, which contradicts regularity in the adjacent chart.

If a mutable coordinate factors in R, both factors are units in its mixed Laurent ring, hence scalar Laurent monomials in the mutable coordinates. The first assertion makes them ordinary monomials, so one is scalar. If a frozen coordinate factors, its primality in the mixed Laurent ring makes one factor a unit cx^a. Both cx^a and the complementary monomial are regular, so the first assertion gives $a = 0$. Every coordinate is therefore irreducible in R, and factoriality makes it prime. □

A *simple oriented exchange chain* of sign $\varepsilon \in \{1, -1\}$ is a sequence of distinct seed positions $a_1, \dots, a_r$ whose exchange entries satisfy

$$B_{a_p,a_q} = \begin{cases} \varepsilon, & q = p-1, \\ -\varepsilon, & q = p+1, \\ 0, & |p-q| \neq 1, \end{cases} \qquad p < r.$$

Proposition 4.3. *Let D be a locally nilpotent derivation of R, and let $a_1, \dots, a_r = b$ be a simple oriented exchange chain with $a_1, \dots, a_{r-1}$ mutable and b frozen. Assume $D^2 x_{a_p} = 0 \neq Dx_{a_p}$ for all p, that D kills every other initial coordinate, and that Dx_{a_1} is a Laurent monomial in invariant coordinates. Mutation at $a_1, \dots, a_{r-1}$ in that order gives an adapted seed.*

Let h be the indicator vector of the chain, and let $m^{(0)}$ be the exponent vector of Dx_{a_1}. If $\delta_{a_p} = t > 0$ for every mutable chain position and

$$m_j^{(0)} = \varepsilon \frac{\delta_j}{t} (Bh)_j + \mathbf{1}_{j=a_1} \qquad (j \text{ mutable}), \tag{4.1}$$

then the final mutable exponents satisfy

$$m_j = \varepsilon \frac{\delta_j}{t} B^{\mathrm{fin}}_{j,b} \qquad (j \text{ mutable}). \tag{4.2}$$

In particular, $\varepsilon B^{\mathrm{fin}}_{j,b} \geq 0$.

Proof. The degree of a nonzero element under a locally nilpotent derivation is additive on products. This follows from the binomial formula for the highest nonzero derivative, using characteristic zero.

At a stage whose first remaining chain position is a, let s be the next one. Among the remaining chain positions, a is joined only to s, with entry $-\varepsilon$. Mutation at a leaves the chain on the remaining positions unchanged. Its exchange therefore reads

$$x_a x'_a = M + x_s C,$$

where M and C are monomials in invariant coordinates. The numerator has derivation degree one. Since the old coordinate has degree one and the new coordinate belongs to R, additivity gives $Dx'_a = 0$. Differentiation of this exchange yields

$$Dx_s = \frac{x'_a}{C} Dx_a. \tag{4.3}$$

Iterating this recurrence leaves b as the only moving coordinate and expresses Dx_b as a Laurent monomial in the others. Since $Dx_b \in R$, Lemma 4.2 makes its exponents nonnegative.

It remains to track the mutable exponents. At each stage, let h indicate the remaining chain positions, and let m be the exponent vector of Dx_a. We maintain

$$m_j = \varepsilon \frac{\delta_j}{t}(Bh)_j + \mathbf{1}_{j=a} \qquad (j \text{ mutable}). \tag{4.4}$$

Its initial instance is (4.1). After mutation at a, we have $h' = h - e_a$. Since $B_{a,s} = -\varepsilon$ is the only nonzero entry at a remaining chain position, matrix mutation gives

$$(B'h')_j = (Bh)_j - \varepsilon[\varepsilon B_{j,a}]_+ \quad (j \neq a), \qquad (B'h')_a = \varepsilon.$$

The recurrence (4.3) gives

$$m'_j = m_j - [-\varepsilon B_{a,j}]_+ + \mathbf{1}_{j=s} \quad (j \neq a), \qquad m'_a = 1.$$

The identity $\delta_j B_{j,a} = -tB_{a,j}$ shows that these formulas preserve (4.4). At the last stage, $h = e_b$, and the indicator term vanishes at all mutable positions because b is frozen. This proves (4.2). □

4.2. **Equivariant compactifications.** An adapted mutation sequence on an affine open set lifts to an equivariant compactification when the boundary valuations are invariant and the minimally lifted seed describes its Cox ring.

Let Y be a normal projective $\mathbb{G}_a$-variety, and let $C \subset Y$ be an invariant factorial affine open set with $\mathbb{C}[C]^\times = \mathbb{C}^\times$. Write $D_1, \dots, D_d$ for its boundary prime divisors, τ_a for their Cox sections, and $\nu_a(f) = -\operatorname{ord}_{D_a}(f)$. Assume that the minimally lifted seed $\widetilde{\Sigma}$ of a seed Σ on C satisfies $\operatorname{Cox}(Y) = \mathcal{U}_{\mathrm{pol}}(\widetilde{\Sigma})$, with full-rank extended exchange matrix and polynomial frozen variables. Let D be the infinitesimal action on $\mathbb{C}[C]$.

Proposition 4.4. *The action on* $\operatorname{Cox}(Y)$ *fixes the* τ_a *and satisfies, for* $Df \neq 0$,

$$\widetilde{D}(\widetilde{f}) = \boldsymbol{\tau}^{\nu(f)-\nu(Df)}\widetilde{Df}, \qquad \nu(f) - \nu(Df) \in \mathbb{Z}^d_{\geq 0}. \tag{4.5}$$

It preserves the nilpotence degree of every minimal lift.

If a mutation sequence on C *reaches a seed* Σ' *with* $D = c\prod_{j\neq b}(f'_j)^{m_j}\partial_{f_b}$, *where* b *is frozen and* $m_j \in \mathbb{Z}$, *the lifted sequence gives an adapted Cox seed. In that seed,*

$$\widetilde{D} = c\boldsymbol{\tau}^{\nu(f_b)-\sum_{j\neq b} m_j \nu(f'_j)} \prod_{j\neq b} (x'_j)^{m_j}\partial_{x_b}, \tag{4.6}$$

where $x'_j = \widetilde{f'_j}$ *and all displayed exponents are nonnegative. The construction commutes with equivariant projective birational contractions that preserve* C *and satisfy the same Cox-ring hypotheses.*

Proof. Because the action is connected, each boundary prime divisor is invariant. At its generic point the infinitesimal action preserves the local discrete valuation ring and its maximal ideal. It therefore preserves every integral power of that ideal, including negative powers in the fraction field. Hence

$$\operatorname{ord}_{D_a}(Df) \geq \operatorname{ord}_{D_a}(f).$$

The valuation realization of the Cox ring now shows that the rule $\widetilde{D}(f\boldsymbol{\tau}^u) = (Df)\boldsymbol{\tau}^u$ is a regular homogeneous derivation. It is locally nilpotent because D is locally nilpotent on $\mathbb{C}[C]$, and its exponential is the lifted action. This proves (4.5). Iterating the rule gives equality of the two degrees.

By Proposition 3.7, every coordinate reached by the lifted sequence is the minimal lift of the corresponding cell coordinate. The boundary sections and every x'_j with $j \neq b$ are invariant. Applying (4.5) to f_b, and using additivity of every boundary valuation on products, gives (4.6) and its nonnegative boundary exponents. Its coefficient is a Laurent monomial regular in $\operatorname{Cox}(Y) = \mathcal{U}_{\mathrm{pol}}(\widetilde{\Sigma})$; Lemma 4.2 therefore gives $m_j \geq 0$ for every j.

Under a contraction, Proposition 2.1 sets the exceptional canonical sections equal to one. They are invariant, so the quotient commutes with the derivation and with the displayed mutation sequence. □

In particular, the chain construction of Proposition 4.3 lifts from C to Y: minimal lifting preserves invariance, derivation degree one, the first monomial derivative, and the exchange entries along the chain. The mutable exponents are unchanged. If the adapted cell column is sign-coherent and contains every mutable coordinate appearing in the coefficient, the lifted active column retains both properties.

The chain conditions have a geometric interpretation. The degree of a regular function with respect to D is its degree on a general orbit, with the orbit parametrized by $\mathbb{G}_a$. Thus degree one means that its zero divisor meets a general orbit once, with multiplicity one. A prime principal divisor stable under $\mathbb{G}_a$ has invariant defining equation, since the only units on C are constants and $\mathbb{G}_a$ has no nontrivial characters. Finally, if the invariant coordinate divisors are prime and the zero divisor of the first derivative is supported on them, factoriality and the unit hypothesis make that derivative a monomial in their equations.

Remark 4.5. An additive action need not have an adapted seed in a given mutation class. On $\mathrm{Cox}(\mathbb{P}^2) = \mathbb{C}[x, y, z]$, take the rank-zero seed and the homogeneous locally nilpotent derivation $D = x\partial_y + y\partial_z$. Since $D^2 z = x \neq 0$, this seed is not adapted, and no mutation is available. The hypothesis of an adapted cell seed in Proposition 4.4 is therefore substantive.

4.3. **Positive simple roots.** Let Y be a Bott–Samelson variety for an arbitrary word $\mathbf{i} = (i_1, \dots, i_N)$, or a normal projective birational contraction that is an isomorphism on $C_{\mathbf{i}}$. Equip $R = \mathrm{Cox}(Y)$ with its minimally lifted word seed. Write E_i for the derivation induced by the positive simple root subgroup, and b_i for the lifted terminal minor indexed by i. We set $\tau_j = 1$ whenever the corresponding standard boundary divisor is contracted.

Proposition 4.6. *Each E_i is locally nilpotent and preserves the Cox grading. It is nonzero precisely when i occurs in $\mathbf{i}$. For such an i, mutate its nonterminal occurrences from left to right. The resulting seed is adapted to E_i, with active frozen variable b_i, and*

$$E_i = c_i \prod_{z \neq b_i} z^{m_z(i)} \partial_{b_i}, \qquad c_i \in \mathbb{C}^\times, \qquad m_z(i) \in \mathbb{Z}_{\geq 0}.$$

The mutable exponents are

$$m_{x_k}(i) = \frac{\delta_k}{t_i} B^i_{k,b_i}.$$

The operators and adapted seeds commute with every birational contraction preserving the affine chart.

Proof. Every positive root subgroup preserves the boundary divisors. It therefore acts on the valuation realization of the Cox ring through the coefficient functions and fixes all canonical boundary sections. The resulting derivations are regular, locally nilpotent and homogeneous for the Cox grading.

In the fundamental highest-weight module $V(\Lambda_j)$, the highest coordinate functional has e_i-string length one when $i = j$, and zero otherwise. Thus the initial minors indexed by i have E_i-degree one; the others are invariant. The derivative is nonzero whenever i occurs: its section is nonzero at $s_i B$, which belongs to the prefix Schubert image. Invariant boundary sections make the same assertions true for the minimal lifts.

Let a be the first occurrence of i. The preceding factors lie in the derived Levi subgroup generated by indices other than i. The vector of weight $\Lambda_i - \alpha_i$ in the fundamental simple-root string is highest for this Levi, with pairings $-a_{ji}$, for $j \neq i$. Normalized highest matrix coefficients multiply under addition of weights, as their restrictions to the Gauss cell are torus characters. Extra Cartan characters are trivial on the derived Levi. It follows, up to a

nonzero pinning scalar, that $E_i f_a = \prod_{j<a<s(j)} f_j^{-a_{i_j i}}$. The prefix line-bundle recursion then gives

$$E_i x_a = c_i \tau_a \prod_{j<a<s(j)} x_j^{-a_{i_j i}}. \tag{4.7}$$

This proves nonvanishing when i occurs. If i is absent, every initial coordinate is invariant, so $E_i = 0$.

The occurrences of i form a simple oriented chain ending at the frozen terminal minor. Formula (4.7) supplies the monomial hypothesis of Proposition 4.3. Its column hypothesis follows from the word matrix. If h is the indicator of all i-positions, then at a mutable position with $i_j = i$, $(Bh)_j$ equals -1 at the first occurrence and zero elsewhere. At a position of another letter, it equals $-a_{i,i_j}$ when $j < a < s(j)$, and zero otherwise. The identity $t_{i_j} a_{i,i_j} = t_i a_{i_j,i}$ converts these values into the exponents in (4.7). The proposition gives the adapted seed and its exponent formula.

A birational contraction sets the exceptional boundary sections equal to one. This commutes with each E_i, which kills those sections, and with every exchange in the same mutation sequence. The final assertion follows. □

4.4. **Negative roots on Bott–Samelson varieties.** Left multiplication on the first factor of a Bott–Samelson variety gives a negative root operator in the initial Cox seed.

Proposition 4.7. *Let $\mathbf{i} = (i_1, \dots, i_N)$ be any nonempty word, and let $p : Z_{\mathbf{i}} \to Y$ be a proper birational morphism to a normal projective variety that is an isomorphism on $C_{\mathbf{i}}$. The first standard boundary divisor E_1 is not contracted. The rank-one group G_{i_1} acting on the first factor descends to Y and its Cox ring. Its negative simple-root derivation satisfies*

$$F_{i_1} = c\, x_1 \partial_{\tau_1}, \qquad c \in \mathbb{C}^\times,$$

in the initial lifted word seed. Here x_1 is the first lifted minor, and τ_1 is the canonical section of the image of E_1. Its active column is zero except for the entry -1 at position one, when that position is mutable. Thus its mutable coefficient exponents are

$$m_{x_k} = -\frac{\delta_k}{t_{i_1}} B_{k,\tau_1}.$$

These formulas commute with further projective birational contractions preserving the affine chart.

Proof. The degree of the first minor is $[E_1]$, so the nonconstant rational function $f_1 = x_1/\tau_1$ has its only pole along E_1. If E_1 were contracted, this function would have nonnegative order along every prime divisor of Y. Normality would make it regular on Y, contradicting the fact that every regular function on an irreducible projective variety is constant. Hence E_1 survives.

On $Z_{\mathbf{i}}$, the group G_{i_1} acts by left multiplication in the first factor. The line bundle $\mathcal{O}(E_1) = m_1^* \mathcal{O}_{\mathbb{P}^1}(1)$ has its natural rank-one linearization. Every standard boundary divisor E_j, for $j > 1$, is stable under this action, so its line bundle has the canonical divisor linearization. These line bundles form a basis of the Picard group and define an action on the Cox ring. The canonical section of E_j, for $j > 1$, is fixed by G_{i_1}, which has no nontrivial characters. The negative root subgroup also fixes every prefix highest section x_k: a highest coordinate functional is annihilated by negative simple-root multiplication on the left. The first prefix is $\mathbb{P}^1$, and its two fundamental sections give $F_{i_1}\tau_1 = cx_1$, with $c \neq 0$. The resulting linearization of each prefix line bundle agrees with its natural rank-one linearization, since a connected semisimple group has no nontrivial characters. Since the initial coordinates generate the fraction field, this proves the formula on $\mathrm{Cox}(Z_{\mathbf{i}})$.

Every exceptional prime divisor of p is a standard boundary divisor, and the first one is not exceptional. Thus all its exceptional canonical sections are G_{i_1}-fixed. The Cox quotient ideal generated by these sections minus one is stable under G_{i_1}. The action consequently

descends to $\mathrm{Cox}(Y)$, preserving its divisor-class grading. Taking the section ring of an ample Cartier divisor recovers an action on Y. It agrees with the induced rational action on the common function field, so p is equivariant. The operator formula descends at the same time.

Finally, the boundary incidence identity gives entry -1 in the τ_1-column only at the first mutable position; there is no earlier position whose successor is one. Since $\delta_1 = t_{i_1}$, the exponent formula follows. The same quotient argument proves compatibility with further contractions. □

The following criterion extends the corresponding finite-type result for reduced words [CKP, Proposition 6.3] to arbitrary words in a symmetrizable Kac–Moody group.

Proposition 4.8. *Let i occur in $\mathbf{i}$, and let a be its first occurrence. The canonical action of the rank-one Borel $B_i = G_i \cap B^+$ on $Z_{\mathbf{i}}$ extends to G_i if and only if $a_{i,i_k} = 0$ for every $k < a$. Equivalently, commuting adjacent orthogonal letters can move this occurrence of i to the beginning of the word.*

When these conditions hold, the action descends to every contraction in Theorem 3.4, the divisor E_a survives, and in the initial lifted seed its negative root derivation is $F_i = c_i x_a \partial_{\tau_a}$, with $c_i \in \mathbb{C}^\times$.

Proof. Suppose first that the action extends. Each prefix projection $\pi_k : Z_{\mathbf{i}} \to Z_{(i_1,\dots,i_k)}$ is an iterated $\mathbb{P}^1$-bundle, so $(\pi_k)_*\mathcal{O}_{Z_{\mathbf{i}}} = \mathcal{O}_{Z_{(i_1,\dots,i_k)}}$. Blanchard's lemma makes π_k equivariant for a unique G_i-action on the prefix [Bri2, Theorem 7.2.1].

For $k < a$, the positive simple-root subgroup indexed by i acts trivially on the prefix, since the word has no occurrence of i. Hence the relation $[E_i, F_i] = H_i$ forces its coroot torus to act trivially there as well. The standard curve obtained by varying only the factor at position k is P_{i_k}/B^+; the coroot torus acts on its affine coordinate with weight a_{i,i_k}. Thus $a_{i,i_k} = 0$ for every preceding position.

Conversely, under these orthogonality conditions the commuting rank-two factor identifications move the occurrence at position a to the first factor. They are B^+-equivariant and identify the corresponding lifted minors and boundary sections. Left multiplication on that factor gives the G_i-action. Proposition 4.7 then gives its descent, the survival of E_a, and the formula for F_i. □

Corollary 4.9. *Let $w \in W^J$, and suppose that i is a left descent of w. Choose a reduced word beginning with i. The initial lifted seed of $\mathrm{Cox}(X_w^{P_J})$ is adapted to F_i, with active frozen variable the canonical section of $X_{s_i w}^{P_J}$, and coefficient a nonzero scalar multiple of the first lifted minor.*

Proof. Apply Proposition 4.7 to the Bott–Samelson map. The image of E_1 is $X_{s_i w}^{P_J}$. It is a divisor: if $s_i w$ had a right descent in J, then w would also have that right descent, contrary to $w \in W^J$. Thus $s_i w \in W^J$ and its length is $\ell(w) - 1$. □

4.5. **Stabilizing negative simple roots.** Return to $Y = X_w^{P_J}$. For $i \in I_Y$, the action of G_i lifts uniquely to the graded Cox ring because the simply connected rank-one group has trivial character group and Picard group [Vez, Section 2.2 and Remark 2.3.4]. Corollary 4.9 treats the left descents. For the remaining i, a change of coordinates on the open cell turns F_i into a positive root operator on $U(w^{-1})$.

Lemma 4.10. *Let $\mathbf{i} = (i_1, \dots, i_N)$ be a reduced expression of w, and let $\mathbf{i}^{\mathrm{op}} = (i_N, \dots, i_1)$. The reverse interval-order seed of $\mathbb{C}[U(w)]$ is reachable from the word seed. In reverse order, its coordinates are, up to nonzero pinning scalars,*

$$g_r = \Delta_{w_{<k}\Lambda_{i_k},\, w\Lambda_{i_k}}|_{U(w)}, \qquad k = N + 1 - r.$$

Its full cell exchange matrix and mutable symmetrizer are

$$B^{\mathrm{rev}} = -B_{\mathbf{i}^{\mathrm{op}}}, \qquad \delta_r = t_{i_{N+1-r}}. \tag{4.8}$$

Chevalley transposition gives an isomorphism of varieties

$$\iota_w : U(w) \longrightarrow U(w^{-1}), \qquad n \longmapsto \bar{w}^{-1} n^T \bar{w},$$

which identifies these coordinates, up to the same scalars, with the inverse-word minors. Each terminal minor is the original terminal minor indexed by its letter.

Proof. The interval-order mutations of [GY, Theorems 3.11(c), 4.8(b), and 7.3] reach the reverse order. If $a(k)$ is the last occurrence of i_k, the corresponding prime is the interval prime for $[k, a(k)]$. By [GY, Equation (3.33) and Theorem 7.1(c)], this prime specializes at $q = 1$, up to a nonzero scalar, to $\Delta_{w_{<k}\Lambda_{i_k}, w_{\leq a(k)}\Lambda_{i_k}}$. Since the later reflections fix Λ_{i_k}, this is g_r, with $k = N + 1 - r$. It is terminal in the reverse order precisely when k is the first occurrence of i_k. Then $w_{<k}\Lambda_{i_k} = \Lambda_{i_k}$, so it is the original terminal minor indexed by that simple root.

We check the exchange matrix before specializing q. Write $\beta_k = w_{<k}\alpha_{i_k}$ and $k_r = N+1-r$. The inverse-word root sequence satisfies $\beta_r^{\mathrm{op}} = -w^{-1}\beta_{k_r}$. The elementary commutation parameters are $\nu_{kl} = q^{(\beta_k, \beta_l)/2}$ for $k > l$, with $\nu_{lk} = \nu_{kl}^{-1}$. Reversal changes the order of indices, while Weyl invariance preserves the pairings; hence $\nu_{k_r, k_s} = (\nu_{rs}^{\mathrm{op}})^{-1}$. The prime-coordinate parameters are products of these elementary parameters over intervals joining equal letters [GY, Equation (3.32)]. The reverse toric matrix is therefore the entrywise inverse of the inverse-word toric matrix. Their weight maps have the same kernel: at corresponding positions,

$$w_{<k}\Lambda_{i_k} - w\Lambda_{i_k} = -w\big(\Lambda_{i_k} - w^{-1}w_{<k}\Lambda_{i_k}\big).$$

In the column convention of [GY], an exchange vector is uniquely determined by zero character weight and its commutation pairings with every seed coordinate [GY, Theorem 3.11(a)]. The normalization for the simple-root index i_k is $\lambda^* = q_{i_k}^{-2}$ in both presentations [GY, Equation (7.4)]. Negating an inverse-word exchange vector compensates for inversion of the toric matrix and satisfies exactly these conditions for the reverse presentation. Uniqueness proves (4.8) for the full extended matrix, including its frozen part. The same interval-order mutations transport the symmetrizer weights with the mutable indices.

Chevalley transposition reverses multiplication and sends the root subgroup of β to that of $-\beta$. Conjugation by $\bar{w}^{-1}$ then gives an anti-isomorphism from $U(w)$ to $U(w^{-1})$. The inverse-word prefix at position r is $w^{-1}w_{<k}$. Applying the contravariant form on the highest-weight module shows that its highest minor pulls back to g_r; the extremal-vector normalizations account for the nonzero scalars. □

Proposition 4.11. *Suppose that $i \in I_Y$ is not a left descent of w. Then $w^{-1}\alpha_i = \alpha_j$ for a unique $j \in J$. The derivation F_i is nonzero exactly when $j \in \operatorname{supp}(w)$. In that case, every minimally lifted word mutation class contains a seed adapted to F_i, with active frozen variable b_j. Its mutable exponents satisfy*

$$m_{x_k}(F_i) = -\frac{\delta_k}{t_i} B^i_{k,b_j}.$$

Proof. Since $w < s_i w$ and $i \in I_Y$, the two cosets coincide. Thus $w^{-1}s_i w \in W_J$, and $w^{-1}\alpha_i$ is a positive root of its root system. Write

$$w^{-1}\alpha_i = \sum_{j \in J} c_j \alpha_j, \qquad c_j \in \mathbb{Z}_{\geq 0}.$$

All the $w\alpha_j$ are positive by minimality of $w \in W^J$. Their sum with coefficients c_j is the simple root α_i, so precisely one coefficient is nonzero and it equals one. Thus $w^{-1}\alpha_i = \alpha_j$, and Weyl invariance gives

$$t_i = t_j. \tag{4.9}$$

To identify the action on the cell, take an inversion root $\beta \in \Phi^+ \cap w\Phi^-$. Since $\beta \neq \alpha_i$, some simple-root coefficient other than the ith is positive throughout its α_i-string. Likewise $w^{-1}\beta \neq -\alpha_j$, so its α_j-string retains a negative coefficient away from j. Thus the entire

α_i-string of β lies in the inversion set. The inversion Lie algebra is stable under $\mathfrak{sl}_{2i}$, and G_i normalizes $U(w)$. Since $\bar{w}^{-1}G_i\bar{w} = G_j \subset P_J$, G_i fixes wP_J. It therefore acts by conjugation on the open cell $U(w)wP_J$. Connectedness fixes each boundary prime, and its canonical linearization fixes τ_a. In the Cox-ring realization by boundary valuations, F_i thus acts on the cell functions and kills every τ_a.

For a nonzero pinning constant c_w, transposition gives

$$\iota_w\big(x_{-i}(u)nx_{-i}(-u)\big) = x_j(-c_w u)\iota_w(n)x_j(c_w u).$$

Since $w\alpha_j = \alpha_i > 0$, the subgroup x_j belongs to the complementary positive subgroup for $U(w^{-1})$. The same root-string argument shows that it normalizes $U(w^{-1})$. Its projected left action on the inverse cell is therefore conjugation. With our inverse-pullback convention, the displayed identity identifies F_i with a nonzero scalar multiple of E_j on cell functions.

Now lift the mutation sequence of Lemma 4.10 to R. By Proposition 3.7, the resulting Cox coordinates y_r are the minimal lifts of the reversed minors g_r. Their F_i-degrees are the E_j-degrees of the inverse-word minors, since F_i fixes the boundary sections. Thus the moving coordinates are precisely the positions bearing the letter j, and each has degree one. If j is absent, $F_i = 0$. Otherwise let a be its first occurrence in $\mathbf{i}^{\mathrm{op}}$. The first-occurrence calculation of Proposition 4.6 on the inverse cell gives

$$F_i y_a = c\tau^{\mu} \prod_{r<a<s^{\mathrm{op}}(r)} y_r^{-a_{i_r^{\mathrm{op}},j}}, \qquad c \in \mathbb{C}^{\times}, \quad \mu \in \mathbb{Z}^{\{\text{boundary primes}\}}.$$

Every factor is invariant. The boundary exponent μ records the difference of the lift degrees on Y.

Let h indicate all positions bearing the letter j. By (4.8), their chain has orientation $\varepsilon = -1$. Negating the inverse-word column identity gives

$$m_r^{(0)} = -\frac{\delta_r}{t_j}(\widetilde{B}^{\mathrm{rev}}h)_r + \mathbf{1}_{r=a} \qquad (r \text{ mutable}).$$

Here the boundary entries of h vanish. Proposition 4.3 now applies: mutating the nonterminal occurrences of j gives

$$F_i = c_D \prod_{z \neq b_j} z^{m_z}\partial_{b_j}, \qquad c_D \neq 0, \qquad m_z \in \mathbb{Z}_{\geq 0}, \qquad m_{x_k} = -\frac{\delta_k}{t_j}B^{\mathrm{fin}}_{k,b_j} = -\frac{\delta_k}{t_i}B^{\mathrm{fin}}_{k,b_j}.$$

The final equality uses (4.9). By Lemma 4.10, the terminal minor in this chain lifts to the original frozen coordinate b_j. Its degree-one string gives $F_i \neq 0$. □

Proof of Theorem 4.1. For E_i apply Proposition 4.6. For F_i, Corollary 4.9 treats the left descents and Proposition 4.11 treats the remaining stabilizing simple-root indices. Corollary 3.9 allows us to choose a word beginning with any prescribed left descent. Lemma 4.10 realizes reversal by mutations as well. All three constructions therefore take place in the same Cox mutation class.

For the nonvanishing statement, only the third case requires an additional observation. Write $K = \mathrm{supp}(w)$. The action of $w \in W_K$ is the identity modulo the span of $\{\alpha_k : k \in K\}$. From $w\alpha_j = \alpha_i$, we therefore obtain $j \in K$ if and only if $i \in K$. The nonvanishing criterion in Proposition 4.11 is consequently equivalent to the one asserted here. □

Proposition 4.12. *Let R be a complex upper cluster algebra with polynomial frozen variables and a full-rank extended exchange matrix. Suppose that a reachable seed is adapted to a locally nilpotent derivation D, so that $D = M\partial_b$ with b frozen. Assume that the nonzero entries in the b-column have one sign and that every mutable variable dividing M belongs to the support of this column. If z is a cluster monomial and $d = \max\{j : D^j z \neq 0\}$, then $D^d z/d!$ is a nonzero scalar multiple of a cluster monomial in a reachable seed.*

Proof. Replacing the exchange matrix by its negative leaves every exchange relation unchanged. We may therefore assume that $B_{k,b} \leq 0$ for every mutable index k, and write $J = \{k : B_{k,b} < 0\}$. The separation formula gives $z = x^g F(y)$, where F is a polynomial with constant term one and $y_k = \prod_j x_j^{B_{k,j}}$; see [Qin2, Theorem 2.5]. Every y_k with $k \in J$ lowers the exponent of b, and the remaining y_k have b-exponent zero. Since the exchange matrix has full rank, the constant monomial of F cannot cancel after substitution. Thus the largest exponent of b in z is $g_b = d$, and its terms of that degree form

$$z^{\mathrm{top}} = x^g F(y)\big|_{y_k=0 \ (k\in J)}.$$

By Qin's freezing theorem [Qin2, Theorem 5.11], z^{top} is a localized cluster monomial in a seed of the pattern obtained by freezing J. Its mutation path avoids J and is also a path in the original pattern. Every variable dividing M, and the frozen variable b, is unchanged along this path. Hence

$$\frac{D^d z}{d!} = M^d b^{-d} z^{\mathrm{top}}$$

is a Laurent monomial in that reachable seed.

The derivative belongs to R, so Lemma 4.2 makes this an ordinary monomial in the reachable seed, up to a nonzero scalar. □

Corollary 4.13. *Let D be any nonzero operator in Theorem 4.1, and let z be a cluster monomial. If $d = \max\{q : D^q z \neq 0\}$, then $D^d z/d!$ is a nonzero scalar multiple of a cluster monomial in the same mutation class.*

Proof. The exponent formula shows that the active column has one sign and that its support is exactly the set of mutable variables occurring in the coefficient of D. Apply Proposition 4.12. □

Corollary 4.14. *Let L be a finite-type Levi subgroup generated by rank-one groups preserving Y, and let $V \subset R$ be a simple finite-dimensional L-module. If one extremal vector of V is a scalar multiple of a cluster monomial, then every extremal vector has this property. If the original vector is a cluster variable, every extremal vector is a scalar multiple of a cluster variable.*

Proof. Successive simple reflections connect the extremal weights of V. On an extremal vector, a nontrivial reflection is realized, up to a nonzero scalar, by the maximal power of the corresponding raising or lowering operator. Apply Corollary 4.13 along such a sequence.

For the last assertion, cluster variables are prime in R by Lemma 4.2. A Weyl-group representative in L acts by a ring automorphism, so each resulting extremal vector is prime as well. A nonconstant cluster monomial can be prime only when it is one cluster variable to the first power. This proves the assertion. □

5. Theta bases and root filtrations

A theta function extends across the frozen boundary when its order along each boundary divisor is nonnegative. Theta reciprocity turns these orders into inequalities on its tropical index. In a seed adapted to a simple root, the same basis records both the length of the root string and its leading arrow.

Let Σ be a seed with mutable index set I_{mut}, frozen index set I_{fr}, and character lattice $M = \mathbb{Z}^I$. We index theta functions by tropical points $m \in M$, distinct from the simple-root indices i and the word positions k. We write $\mathcal{U}_{\mathrm{inv}}(\Sigma)$ for the upper cluster algebra in which every frozen variable is inverted. For $n \in N = \mathrm{Hom}(M, \mathbb{Z})$, the *monomial valuation* in this seed is

$$v_n\left(\sum_m c_m x^m\right) = \min_{c_m \neq 0} \langle m, n \rangle.$$

These valuations are the integral tropical points of the cluster variety. Coordinates of a tropical point are transported when the seed changes. We normalize $\vartheta_0 = 1$.

A *reddening sequence* is a mutation sequence whose terminal c-vectors, calculated with principal coefficients, are all nonpositive. Its terminal c-matrix is a negative permutation matrix. The corresponding mutable g-vectors are the negatives of the coordinate vectors after relabeling; in this situation the seed is also called *injective-reachable*. A *maximal green sequence* is a reddening sequence in which each mutation is at a green vertex. The word seeds used here satisfy this stronger condition.

Lemma 5.1. *The principal exchange matrix of every word seed has a maximal green sequence. The same holds for its negative.*

Proof. Shen–Weng construct a maximal green sequence for the one-sided string seed of every positive braid [SW, Theorem 4.1]. Its principal matrix is the word matrix above, up to the global sign determined by the side carrying the word. Reversing the word sends the closed string $(k, s(k))$ to $(N+1-s(k), N+1-k)$ and negates the principal matrix. Apply their theorem to the word and its reversal. □

To apply theta reciprocity over the integral theta-indexing lattice, we use the chiral dual $(Q^\bullet, P^\bullet)$ of the seed datum (P, Q) in Cheung–Magee–Mandel–Muller [CMMM, Section 4.1].

Lemma 5.2. *Let Σ have full-rank extended exchange matrix and a reddening sequence. In the conventions of* [CMMM, Section 4.1]*, let (P, Q) be its seed datum, with character lattice M, and let $(Q^\bullet, P^\bullet)$ be its chiral dual, with character lattice $N^\bullet$. Every theta function of the chiral dual indexed by an integral point is a Laurent polynomial in its positive seed chart.*

For every $n \in N_{\mathbb{Q}}$, there is a finite $A_n \subset N_{\mathbb{Q}}$ such that

$$v_n(\vartheta_m) = \min_{a \in A_n} \langle m, a \rangle \qquad (m \in M).$$

Specifically, choose $q \in \mathbb{Z}_{>0}$ with $qn \in N^\bullet$, and let S_{qn} be the Laurent support of the dual theta function $\vartheta^\vee_{qn,+}$. One may take $A_n = q^{-1} S_{qn}$. If $n \in N$, every vertex of $\operatorname{conv}(A_n)$ belongs to N, so the minimum can be taken over finitely many integral linear forms.

Proof. The maps P and Q are injective over $\mathbb{Q}$: this is full rank for P, whereas the columns of Q are part of the seed basis. The two maps of the chiral-dual datum are therefore injective as well. Write them as integral $d \times r$ matrices (P', Q') in mutually dual bases of the chiral-dual lattices. Complete the columns of Q' to a nonsingular integral matrix $A = (Q' \mid K)$, and define

$$P_0 = A^{\mathsf{T}} P', \qquad Q_0 = \begin{pmatrix} I \\ 0 \end{pmatrix}.$$

The identities $A^{\mathsf{T}} P' = P_0$ and $A Q_0 = Q'$ make A^{T} an injective linear morphism of seed data from (P', Q') to (P_0, Q_0). The latter is an ordinary $\mathcal{A}$-seed with full-rank extended matrix. To check its principal matrix in the nonsimply-laced case, put $C = Q^{\mathsf{T}} P$ and use the symmetrizing matrix D of [CMMM, Section 4.1]. Since $P^\bullet = PD$, $Q^\bullet = QD^{-1}$, and $D^{-1}C$ is skew-symmetric, one obtains

$$Q'^{\mathsf{T}} P' = (P^\bullet)^{\mathsf{T}} Q^\bullet = D C^{\mathsf{T}} D^{-1} = -C.$$

Thus it has the opposite principal matrix. The map on tori is finite; the integral matrix A need not be unimodular. Moreover, $Q_0^{\mathsf{T}} A^{\mathsf{T}} p = Q'^{\mathsf{T}} p$, so the map of exponent spaces takes the positive chamber to the positive chamber.

The negative of a matrix admitting a reddening sequence also admits one: reverse the path between its framed and coframed seeds and negate the exchange matrices, using the terminal negative permutation matrix for the relabeling [SW, Appendix A, equation (A.25)]. The negative chamber therefore belongs to the cluster complex [GHKK, Lemma 5.12 and Lemma 8.29]. Full rank also implies the strict convexity hypothesis of [GHKK, Theorem 0.3(7)]. Theorem 0.3 and Proposition 8.28 of [GHKK] give Laurent finiteness for all target theta

functions. By [CMMM, Theorem 4.9], the injective monomial map $x^m \mapsto x^{A^\intercal m}$ sends each source theta function to the target theta function with the transported index. Its injectivity prevents distinct exponents from merging, so the source support is finite too. Full rank also makes each exchange-direction exponent determine its auxiliary coefficient exponent. Consequently no infinite auxiliary-coefficient support is concealed by specializing those coefficients to one.

For rational tropical points, use the rational broken-line exponent sets of [CMMM, Section 3.1]. Their bending rule is homogeneous: scaling attached exponents by q and reparametrizing the path gives $\Theta^\vee_{qn,+} = q\Theta^\vee_{n,+}$. Corollary 3.2 of [CMMM] therefore identifies their valuation with the minimum over $q^{-1}S_{qn}$. The original theta functions are Laurent finite by the same GHKK criterion. Both sides of theta reciprocity are consequently finite, and [CMMM, Claim 4.16 and Theorem 5.19] gives the stated equality.

Finally, suppose $n \in N$, and write $F(m) = \min_{a\in A_n}\langle m, a\rangle$. The equality already proved makes F integral on M. If a is a vertex of $\operatorname{conv}(A_n)$, the region where it is the unique minimizer has nonempty interior and is rational. Choose an integral point u in this region. For a lattice basis $m_1, \dots, m_d$ and all sufficiently large integers k, both ku and $ku + m_j$ remain in this region. Hence

$$\langle m_j, a\rangle = F(ku + m_j) - F(ku) \in \mathbb{Z}.$$

It follows that $a \in N$. Taking just these vertices proves the integral assertion. □

Proposition 5.3. *Let $R = \mathcal{U}_{\mathrm{pol}}(\Sigma)$, where the extended exchange matrix has full rank and its principal part admits a reddening sequence. Then there is a rational polyhedral cone $\Xi_\Sigma \subset M_{\mathbb{R}}$ such that*

$$R = \bigoplus_{m\in\Xi_\Sigma\cap M} \mathbb{C}\vartheta_m.$$

The basis is homogeneous for every grading for which the seed and all exchange relations are homogeneous, and its multiplication has nonnegative integral structure constants.

If $e_f \in N$ is the coordinate covector of a frozen variable x_f, then

$$\Xi_\Sigma = \{m \in M_{\mathbb{R}} : \langle m, a\rangle \geq 0 \text{ for all } a \in A_{e_f},\ f \in I_{\mathrm{fr}}\}.$$

If a homogeneous lattice grading $\gamma : M \to \Gamma$ has a positive integral coarsening, then every degree slice $\Xi_\Sigma \cap \gamma^{-1}(\lambda)$ is a bounded rational polyhedron, possibly empty.

Proof. The reddening sequence and full rank give a theta basis $\{\vartheta_m : m \in M\}$ of $\mathcal{U}_{\mathrm{inv}}(\Sigma)$, by [GHKK, Proposition 8.28], as in the preceding lemma. Lemma 1.1 gives

$$R = \{g \in \mathcal{U}_{\mathrm{inv}}(\Sigma) : v_{e_f}(g) \geq 0 \text{ for every } f \in I_{\mathrm{fr}}\}.$$

The cluster scattering functions satisfy [CMMM, Assumption 3.9 and Lemma 4.3]; specialization of the auxiliary coefficients to one over $\mathbb{C}$ sends no nonzero integer monomial to zero. Their valuative independence theorem [CMMM, Theorems 3.8 and 3.10] therefore gives

$$v_{e_f}\left(\sum_m c_m\vartheta_m\right) = \min_{c_m\neq 0} v_{e_f}(\vartheta_m).$$

Thus a theta expansion belongs to R exactly when each of its nonzero terms belongs to R. Lemma 5.2 turns the finitely many frozen-valuation conditions into finitely many integral linear inequalities, proving the asserted formula.

Let $\gamma : M \to \Gamma$ describe a homogeneous grading. Every exponent in the seed expansion of ϑ_m differs from m by an integral combination of exchange vectors, all of degree zero. Hence $\deg\vartheta_m = \gamma(m)$. Positivity and integrality of the structure constants follow from the theta multiplication in [GHKK]. Since R is a subalgebra and its theta expansion is unique, multiplication cannot introduce basis elements outside Ξ_Σ.

For the final assertion, write $h : \Gamma \to \mathbb{Z}$ for the positive coarsening. Nonnegativity of the grading on the theta functions gives $h\gamma \geq 0$ on Ξ_Σ, by rationality. The rational cone

$\Xi_\Sigma \cap \ker(h\gamma)$ is zero: otherwise it would contain a nonzero lattice point, and the theta function with that index would be linearly independent of $\vartheta_0 = 1$ in degree zero. The recession cone of any degree slice is $\Xi_\Sigma \cap \ker \gamma$, so it too is zero. Every slice is therefore bounded. Its lattice points are finite, which also proves that each homogeneous component of R is finite-dimensional. □

Theorem 5.4. *Let Y be a Bott–Samelson variety for any word, or a normal projective birational contraction preserving its standard affine chart. Let Σ be its minimally lifted word seed. The Cox ring of Y has a theta basis indexed by the integral points of a rational polyhedral cone Ξ_Σ. The basis is homogeneous for the divisor-class and Cartan-weight gradings.*

Let $\gamma : M \to \mathrm{Cl}(Y)$ be the degree map. For every divisor class λ, the set $P_\lambda = \Xi_\Sigma \cap \gamma^{-1}(\lambda)$ is a bounded rational polyhedron, possibly empty, and

$$H^0(Y, \mathcal{O}_Y(\lambda)) = \bigoplus_{m \in P_\lambda \cap M} \mathbb{C}\vartheta_m.$$

The analogous formula for a Cartan-weight space is obtained by adding the corresponding linear weight equation.

Proof. Theorem 3.4 gives the upper-algebra equality and full row rank. Minimal lifting and contraction preserve the principal matrix, so Lemma 5.1 supplies a maximal green sequence. The Cox grading has a positive integral coarsening. Proposition 5.3 now gives the basis and the bounded degree slices. □

In a seed adapted to a locally nilpotent derivation D, the valuation at $-e_b$ records the D-degree of each nonzero element of R, where b is the active coordinate.

Proposition 5.5. *Assume the hypotheses of Proposition 5.3. Let $D_1, \dots, D_t$ be locally nilpotent derivations of R. Suppose that each D_j admits a mutation-equivalent seed in which $D_j = M_j \partial_{b_j}$, where $M_j \neq 0$ is a Laurent monomial independent of b_j and all other seed variables are killed by D_j. Let n_j be the cluster tropical point represented by $-e_{b_j}$ in that seed. Then the same theta basis is compatible with every intersection of the kernel filtrations:*

$$\bigcap_{j=1}^{t} \ker D_j^{r_j+1} = \bigoplus_{\substack{m \in \Xi_\Sigma \cap M \\ v_{n_j}(\vartheta_m) \geq -r_j \ (1 \leq j \leq t)}} \mathbb{C}\vartheta_m \qquad (r_j \in \mathbb{N}).$$

Each condition on the right is a finite collection of integral linear inequalities. If the grading has a positive integral coarsening, each homogeneous component of the left side is indexed by a bounded rational polyhedron.

Proof. Fix $D = D_j$, and let K be the field generated by the other coordinates of its adapted seed. The seed expansion embeds R in $K[b, b^{-1}]$, while $D = M\partial_b$ with $M \in K^\times$. Local nilpotence on R rules out negative powers of b, since a nonzero negative-power term survives every iterated derivative in characteristic zero. Thus, for every nonzero $f \in R$,

$$\max\{q : D^q f \neq 0\} = \deg_b f = -v_{n_j}(f).$$

Valuative independence shows that this degree of a finite theta expansion is the maximum of the degrees of its nonzero terms. Hence $\ker D_j^{r_j+1}$ is spanned by precisely the indicated subset of the theta basis. Intersecting subspaces spanned by subsets of one basis intersects those subsets, whether or not the derivations commute. Lemma 5.2 gives the finite linear inequalities, and the boundedness assertion of Proposition 5.3 applies to each homogeneous slice under the stated grading hypothesis. □

By Theorem 4.1, this description applies on every Schubert divisorial section space to all positive simple-root operators and all stabilizing negative simple-root operators.

5.1. **The regular cone in word coordinates.** We give inequalities for the cone of Theorem 5.4 in the initial seed. Use the sign of the word matrix in Section 3, so $B_{k,s(k)} = -1$. Write a theta index as $m = ((m_k)_{k=1}^N, (u_j)_{j\in S})$, where S indexes the standard divisors that survive the contraction and u_j is the coordinate at τ_j. The row matrix of the lifted seed is $\widetilde{B} = (B \mid C)$, where

$$C_{kj} = \mathbf{1}_{s(k)=j} - \mathbf{1}_{k=j}.$$

For a Schubert variety in a partial flag variety, one obtains S from the reduced word for the minimal coset representative: deletion of j must give a reduced word for a minimal representative of length $N-1$.

The following sweep depends only on repetitions of the word letters. Start with the lifted seed and a theta index m. Read $j = N, \dots, 1$. Just before processing j, record the current τ_j-coordinate as $U_j(m)$ if $j \in S$. To process j, write the occurrences of its letter as $a_1 < \cdots < a_d$, with $j = a_h$, and mutate at

$$a_{d-1},\ a_{d-2},\ \dots,\ a_{d-h+1} \qquad \text{in this order.} \tag{5.1}$$

The list is empty for $h = 1$. At every mutation transport the theta index as well. Explicitly, for the current row matrix D, mutation at k sends m_k to $-m_k$ and sends each m_l, $l \neq k$, to

$$m_l + [D_{kl}]_+ m_k - D_{kl} \min(m_k, 0). \tag{5.2}$$

Here $[v]_+ = \max(v, 0)$. The sweep includes every position, even when its boundary section has been set equal to one.

Theorem 5.6. *For each simple-root index i with occurrences $a_1 < \cdots < a_d$, the order of a theta function along its terminal minor is*

$$v_{e_{b_i}}(\vartheta_m) = \min_{1\le h\le d} \sum_{r=h}^{d} m_{a_r}.$$

For every $j \in S$ we have $v_{e_{\tau_j}}(\vartheta_m) = U_j(m)$. Thus Ξ_Σ is cut out by the nonnegativity of all the displayed suffix sums and of the functions U_j. A single sweep computes all $U_j(m)$ using $\sum_i \binom{d_i}{2}$ mutations, where d_i is the number of occurrences of i. The distinct linear forms attained by each frozen order on full-dimensional regions give precisely the primitive facet inequalities of Ξ_Σ.

Proof. Suppose the column of a frozen variable f is nonnegative in a seed with theta index m'. In that seed every monomial in the pointed expansion of ϑ_m has exponent $m' + nD$, $n \ge 0$, and the term of exponent m' has coefficient one. Thus the least f-exponent is m'_f. By Lemma 1.1 this computes the same valuation in every seed.

For $f = b_i$, Proposition 4.6 gives a nonnegative column after mutations at $a_1, \dots, a_{d-1}$. The submatrix on the i-positions is a path with successive entries $B_{a_r,a_{r+1}} = -1$. Mutation at the first remaining vertex leaves the subpath on later vertices unchanged, since that vertex meets only its successor there. Hence, if t_r is the coordinate at a_r just before its mutation and t_d the final terminal coordinate, (5.2) gives

$$t_1 = m_{a_1}, \qquad t_{r+1} = m_{a_{r+1}} + \min(0, t_r).$$

Induction gives the asserted minimum, including the case $d = 1$. On the cell seed these suffix inequalities also follow from Qin's description of the dominant degrees [Qin1, Lemma 8.11].

For the boundary sections, consider the auxiliary word obtained by reversing $\mathbf{i}$ and using the transposed Cartan matrix. Write $j^* = N + 1 - j$. A mutable position k represents the interval $(k, s(k))$, whose reflection starts at $\rho(k) = s(k)^*$. Under this identification the auxiliary *principal* row matrix is $H = B^{\mathsf{T}}_{\mathrm{prin}}$. Indeed, reversal reverses each arrow between adjacent intervals of one letter and interchanges the two crossing orders for intervals of distinct letters. In the latter case the magnitude of B_{kl} is $-a_{i_l i_k}$. Transposing the Cartan matrix

exchanges the two magnitudes. Nested and disjoint intervals of distinct letters contribute zero. This gives the identity on every entry, and matrix mutation preserves transposition.

Fix $j \in S$. On mutable rows its initial column is $c = e_{p(j)} - e_j$, with absent or frozen indices omitted. After identification by ρ, this is $e_{j^*} - e_{p_{\mathrm{rev}}(j^*)}$, again restricted to mutable indices, where p_{rev} is the predecessor in the reversed word. This difference is the mutable part of the initial degree of Qin's singleton interval variable $W_{[j^*,j^*]}$ for the auxiliary word [Qin1, Definition 8.3 and Lemma 8.4]. Here Qin uses mutable columns, so his matrix is the transpose of our auxiliary row matrix; the signs and Cartan entries agree with [Qin1, Equation (6.1)].

This identification persists under mutation. Indeed, the update of a frozen column at k, in the original row convention, is

$$c'_k = -c_k, \qquad c'_l = c_l + [D_{lk}]_+[c_k]_+ - [-D_{lk}]_+[-c_k]_+ \quad (l \neq k).$$

This agrees with (5.2) for the mutable coordinate in the transposed principal matrix. Thus it suffices to reach the seed containing the singleton. Qin's sequence processes the reversed word from the left. At the v-th occurrence of a letter of multiplicity d, it mutates at the first $d - v$ occurrences of that letter in increasing order. Just before the position j^* is processed, $W_{[j^*,j^*]}$ occupies the first slot of its letter [Qin1, Sections 8.1–8.2]. Its mutable degree there is a positive unit vector if the letter is repeated, and zero otherwise. Translating back by ρ gives exactly (5.1): the v-th reversed occurrence is the h-th original one with $v = d + 1 - h$. Consequently the column of τ_j is nonnegative when its coordinate is recorded. Deleting the other contracted columns has no effect on this calculation. The frozen-valuation criterion in Proposition 5.3 now gives the cone. For a letter occurring d times, the block has length $0 + 1 + \cdots + (d-1)$.

We finally identify the facets. Write z for the mutable part of m and v_f for its initial coordinate at a frozen variable f. By (5.2), the computed order at f has the form $v_f + \psi_f(z)$: each increment uses only the current mutable coordinates. By Lemma 5.2, ψ_f is concave, first on rational tropical points by homogeneity and then on real tropical points by continuity. Subdivide the real space of mutable theta indices by the signs of the coordinates at the successive mutations. On each full-dimensional region ψ_f agrees with an integral linear form L. Concavity gives $L \geq \psi_f$ everywhere. The regions cover a dense set, so the minimum of their distinct forms is ψ_f everywhere.

Each inequality $v_f + L(z) \geq 0$ defines a facet. Choose z in an open region attaining L, away from the equality hyperplanes of the other distinct forms for f. Set $v_f = -L(z)$ and make all other frozen coordinates sufficiently large. Only the chosen inequality is tight; all are strict if v_f is then increased. Its normal is primitive because its v_f-coefficient is one. For a terminal minor every suffix sum occurs on an open region: consecutive sums differ by independent mutable coordinates. This proves the facet assertion. To test a single theta index, one needs only (5.2). □

Example 5.7. In type A_4 take the reduced word $2\,1\,2\,3\,2\,4\,3$. In one-line notation this word represents 34521; deleting the third position gives 34512. Thus τ_3 survives on the full-flag Schubert variety. The mutable positions are $1, 3, 4$; in that order the principal matrix and the column of τ_3 are

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \qquad (1, -1, 0)^{\mathsf{T}}.$$

For this single divisor there is a shorter calculation. Mutations at 3 and 4 send its column first to $(0, 1, -1)^{\mathsf{T}}$ and then to $(0, 0, 1)^{\mathsf{T}}$. The resulting coordinate is

$$u_3 + \min(0, m_3) + \min\bigl(0, m_4 + \max(m_3, 0)\bigr) = u_3 + \min(0, m_3, m_3 + m_4).$$

The facet $u_3 + m_3 + m_4 \geq 0$ involves two different letters.

5.2. **Factoriality and two cluster charts.** We use two results on upper cluster algebras in their polynomial-frozen form: a class-group formula and a two-chart criterion.

Proposition 5.8. *Let $R = \mathcal{U}_{\mathrm{pol}}(\Sigma)$ be finitely generated, and suppose that the exchange matrix has full row rank. Then*

$$\mathrm{Cl}(R) \simeq \mathbb{Z}^{\sum_{k\in I_{\mathrm{mut}}}(c_k-1)}, \qquad c_k = \gcd\{|B_{kj}| : j \in I\}.$$

Thus R is factorial precisely when every exchange row is primitive.

Proof. The ring R is normal. By Lemma 1.1, inverting the frozen coordinates gives $\mathcal{U}_{\mathrm{inv}}(\Sigma)$ and removes only principal prime divisors. Nagata's exact sequence gives $\mathrm{Cl}(R) \simeq \mathrm{Cl}(\mathcal{U}_{\mathrm{inv}}(\Sigma))$. Apply the full-rank class-group formula of [Pom, Theorem 4.8]. Over $\mathbb{C}$, the exchange binomial at k has exactly c_k irreducible factors: up to a Laurent monomial, it is $1 + x^{c_k a}$ with a primitive. This gives the formula and the polynomial-frozen form of the factoriality criterion in [CKQ, Theorems 4.9 and 4.13]. □

Corollary 5.9. *Suppose that the exchange matrix has full row rank and primitive rows, and that $\mathcal{U}_{\mathrm{pol}}(\Sigma)$ is finitely generated. If two seeds Σ_1, Σ_2 have no common mutable cluster variable, then*

$$\mathcal{U}_{\mathrm{pol}}(\Sigma) = \mathcal{L}_{\mathrm{pol}}(\Sigma_1) \cap \mathcal{L}_{\mathrm{pol}}(\Sigma_2).$$

Their mixed Laurent charts cover $\operatorname{Spec}\mathcal{U}_{\mathrm{pol}}(\Sigma)$ in codimension one. For every Cox ring in Theorem 3.4, one may take a word seed and the terminal seed of a reddening sequence.

Proof. The ring $R = \mathcal{U}_{\mathrm{pol}}(\Sigma)$ is factorial by Proposition 5.8. Every mutable cluster variable x is prime in $\mathcal{U}_{\mathrm{inv}}(\Sigma)$ [CKQ, Theorem 4.9]. Its frozen valuations are zero. If $a = xh$ with $a \in R$ and $h \in \mathcal{U}_{\mathrm{inv}}(\Sigma)$, then $v_f(h) = v_f(a) \geq 0$ for every frozen index f, so Lemma 1.1 gives $h \in R$. Primality thus descends to R. Distinct cluster variables are nonassociated [CKQ, Lemma 4.22]. Each mixed Laurent ring is the localization of R at its mutable coordinates. A reduced denominator in their intersection could therefore involve only prime coordinates common to the two mutable clusters. There are none. The same argument shows that every height-one prime lies in at least one chart. This is the polynomial-frozen form of [CKQ, Theorem 4.23].

For the Cox rings, the word matrix has full row rank and primitive rows by the successor minor. A reddening sequence has terminal mutable g-vectors equal, after relabeling, to the negative coordinate vectors. None of its mutable variables is initial, so the two clusters are disjoint. □

5.3. **Theta functions and root strings.** We use theta functions for a full-rank classical seed, with rows of the extended exchange matrix B indexed by mutable vertices. For $g \in \mathbb{Z}^I$, the corresponding theta function has Laurent expansion

$$\vartheta_g = x^g \sum_{n\in\mathbb{N}^{I_{\mathrm{mut}}}} c_{g,n} x^{nB}, \qquad c_{g,0} = 1. \tag{5.3}$$

For a full-rank injective-reachable seed, the results of Gross, Hacking, Keel, and Kontsevich [GHKK] give a basis of the upper cluster algebra with all frozen variables inverted, consisting of these Laurent polynomials. A *regular theta basis* of a subalgebra R is a basis consisting of theta functions regular on $\operatorname{Spec} R$.

Proposition 5.10 (Tang)**.** *Let Σ be a full-rank injective-reachable classical seed. Write Σ^{op} for the seed with the same coordinates and exchange matrix $-B$. The two seeds define the same theta functions in their common rational function field. If a subset of the theta functions is a basis of a subalgebra R, it consists of all theta functions in R for either choice of sign.*

Proof. The first assertion is [Tan2, Proposition 4.2], which includes the frozen coordinates. The second follows from linear independence in the common field [Tan2, Lemma 4.3]. □

The tropical index of a theta function depends on the choice of sign for the exchange matrix. In the next proposition we use the index for the chosen orientation. The leading derivative formula follows Tang's freezing argument [Tan2, Proposition 3.2]. Retaining the scalar in the derivation fixes the normalization of the maximal divided derivative.

Proposition 5.11. *Let R be a complex subalgebra of the rational function field of a full-rank injective-reachable seed, and suppose that R has a regular theta basis $\mathcal{B}$. Let D be a locally nilpotent derivation of R which, in this seed, has the form $D = cx^v\partial/\partial x_b$, where $c \in \mathbb{C}^\times$, b is frozen, and $v_b = 0$. Suppose that the b-column of B is sign-coherent and that $v_k = 0$ for each mutable k with $B_{kb} = 0$.*

Choose the sign of B so that $B_{kb} \le 0$ for all mutable k, and write $\mathcal{B} = \{\vartheta_g : g \in S\}$ in this orientation. For every $g \in S$, the integer $d = g_b$ is nonnegative and is the largest integer such that $D^d\vartheta_g \neq 0$. If $d > 0$, then $g + v - e_b \in S$ and

$$D\vartheta_g - cd\,\vartheta_{g+v-e_b} \in \ker D^{d-1}. \tag{5.4}$$

In particular, for every $g \in S$,

$$\frac{D^d\vartheta_g}{d!} = c^d\vartheta_{g+d(v-e_b)}. \tag{5.5}$$

For each integer $r \ge 0$, the kernel of D^r has basis $\{\vartheta_g : g \in S,\ g_b < r\}$. The assignment $\vartheta_g \mapsto \vartheta_{g+v-e_b}$, defined when $g_b > 0$, is injective.

Proof. Proposition 5.10 permits the choice of orientation in the statement without changing $\mathcal{B}$. Let K_0 be the rational function field in the variables other than x_b. Every theta function is in $K_0[x_b^{\pm1}]$, and hence so is every element of R. Since the coefficient cx^v is in $K_0^\times$, a nonzero term of negative x_b-degree would survive every power of D. Local nilpotence therefore implies $R \subset K_0[x_b]$. For nonzero $f \in R$, its D-degree is its degree as a polynomial in x_b.

Let $J = \{k \in I_{\mathrm{mut}} : B_{kb} < 0\}$ and freeze the vertices in J. Denote theta functions for the resulting seed by ϑ'_g. The freezing theorem [Qin2, Theorem 5.2] says that ϑ'_g is obtained from (5.3) by selecting the terms with $n_k = 0$ for all $k \in J$. These are exactly the terms of largest x_b-degree. Thus the highest homogeneous part of ϑ_g is ϑ'_g and has degree g_b.

The remaining rows of B are linearly independent. Choose a real linear functional on the exponent lattice that is strictly positive on each of them. In any finite linear relation among distinct ϑ'_g, an index of least value under this functional contributes its leading monomial with coefficient one, and no other term can cancel it. Hence the functions ϑ'_g, for all integral indices g, are linearly independent. It follows that the degree of a nonzero finite sum $\sum_g a_g\vartheta_g$ is the largest g_b for which $a_g \neq 0$. This proves the assertion about $\ker D^r$.

Suppose $d = g_b > 0$. All variables occurring in x^{v-e_b} are frozen after freezing J. Multiplication by this Laurent monomial therefore translates theta indices, so the highest homogeneous part of $D\vartheta_g$ is $cd\,x^{v-e_b}\vartheta'_g = cd\,\vartheta'_{g+v-e_b}$. Expand $D\vartheta_g$ in the basis $\mathcal{B}$. The independence just proved forces the index $g + v - e_b$ to occur with coefficient cd and excludes every other term of degree $d - 1$. Every remaining term has degree at most $d - 2$. This proves (5.4).

The target index has b-coordinate $d - 1$. Repeatedly applying (5.4), the lower-degree terms vanish after d derivatives, and the leading coefficients multiply to $c^d d!$. This proves (5.5). Injectivity follows because translation by $v - e_b$ is injective. □

Corollary 5.12. *Let Y be as in Theorem 3.4. Its regular theta basis satisfies Proposition 5.11 for every nonzero positive simple-root derivation, using the adapted seeds of Proposition 4.6. For a Schubert variety, the same assertion holds for every nonzero stabilizing negative simple-root derivation by Theorem 4.1. In particular, every maximal divided root derivative of a theta function is a nonzero scalar multiple of a theta function.*

For any finite family of these derivations D_a and nonnegative integers r_a, the intersection $\bigcap_a \ker D_a^{r_a}$ is spanned by the theta functions that it contains.

Proof. The adapted seeds are full-rank and injective-reachable. In each, the active column has one sign, and its nonzero mutable entries are precisely the mutable positions occurring in

the coefficient of the derivation. Thus Proposition 5.11 applies in each chart. Its description of the kernels gives the assertion about their intersections. □

5.4. **Perfect bases and Levi crystals.**

Lemma 5.13. *Let V be a finite-dimensional $\mathfrak{sl}_2$-module, with operators E, F, H, and let $\mathcal{B}$ be a weight basis compatible with the kernels of all powers of E. Suppose that $c \in \mathbb{C}^\times$ and an injective partial map $\widetilde{e} : \mathcal{B} \dashrightarrow \mathcal{B}$, defined exactly on $\{b \in \mathcal{B} : Eb \neq 0\}$, satisfy*

$$Eb - c\varepsilon_E(b)\widetilde{e}(b) \in \ker E^{\varepsilon_E(b)-1}$$

for every b with $Eb \neq 0$, where $\varepsilon_D(b) = \max\{r : D^r b \neq 0\}$. Then $\mathcal{B}$ is also compatible with the kernels of all powers of F. The inverse partial map $\widetilde{f} = \widetilde{e}^{-1}$ is defined precisely when $Fb \neq 0$, and

$$Fb - c^{-1}\varepsilon_F(b)\widetilde{f}(b) \in \ker F^{\varepsilon_F(b)-1}.$$

For a basis vector of weight μ, we have $\varepsilon_F(b) = \varepsilon_E(b) + \mu$.

Proof. The leading relation forces $\widetilde{e}(b)$ to have weight $\mu + 2$ and E-degree $\varepsilon_E(b) - 1$ when b has weight μ. Write $V_{\leq n}$ for the sum of the isotypic components of highest weight at most n. If a nonzero weight-μ vector v has a nonzero component of highest weight n and none of larger highest weight, the rank-one representation formulas give

$$\varepsilon_E(v) = (n - \mu)/2, \qquad \varepsilon_F(v) = (n + \mu)/2.$$

Thus, in each weight space, the filtrations by E-kernels, F-kernels, and $V_{\leq n}$ coincide after reindexing. The basis is consequently compatible with all three.

Fix b of weight μ, and write $p = \varepsilon_E(b)$, $q = p + \mu$, and $n = \mu + 2p$. Its image belongs to the induced basis of $\operatorname{gr}_n V = V_{\leq n}/V_{\leq n-1}$, a direct sum of copies of the irreducible module of highest weight n. The leading E-relation preserves n, and its remainder belongs to $V_{\leq n-2}$. If $q > 0$, the map $E : (\operatorname{gr}_n V)_{\mu-2} \to (\operatorname{gr}_n V)_\mu$ is an isomorphism. Its matrix in the induced bases has exactly one nonzero entry in each column, so the leading arrows give a bijection between those bases. There is therefore a unique b' with $\widetilde{e}(b') = b$, and $Eb' = c(p+1)b$ modulo $V_{\leq n-2}$.

On $(\operatorname{gr}_n V)_\mu$, the operator EF is multiplication by $(p + 1)q$. Applying the preceding isomorphism gives $Fb = c^{-1}qb'$ modulo $V_{\leq n-2}$. At weight $\mu - 2$, every vector in the remainder has F-degree at most $q - 2$, which proves the asserted leading relation. If $q = 1$, this remainder is zero. If $q = 0$, then $Fb = 0$, and the weight-$\mu - 2$ space of $\operatorname{gr}_n V$ is zero, so no leading E-arrow ends at b. Thus the two partial maps are mutually inverse on their stated domains. □

Proposition 5.14. *Let Y be as in Theorem 3.4, and suppose that a connected reductive group L of finite type acts on Y and on $\operatorname{Cox}(Y)$, preserving Cox degrees. Assume that the regular theta basis is homogeneous for a maximal torus T_L of L and satisfies Proposition 5.11 for every nonzero positive simple-root operator E_i, with leading constant c_i. Set $c_i = 1$ when $E_i = 0$. Let $P_L = X^*(T_L)$ be its character lattice, and choose a character $\chi : P_L \to \mathbb{C}^\times$ satisfying $\chi(\alpha_i) = c_i$. Then the basis $\widehat{\vartheta}_m = \chi(\operatorname{wt}\vartheta_m)\vartheta_m$ is perfect for both simple-root directions, with leading coefficient $\varepsilon_D(\vartheta_m)$ for each $D = E_i, F_i$. In particular,*

$$\frac{D^{\varepsilon_D(\vartheta_m)}\widehat{\vartheta}_m}{\varepsilon_D(\vartheta_m)!} = \widehat{\vartheta}_{\widetilde{e}_D^{\max}(m)}.$$

Its multiplication constants agree with those of the original theta basis. The required character exists because the simple roots of L are linearly independent in $(P_L)_\mathbb{Q}$.

Proof. Each Cox-degree piece is finite-dimensional. Apply Lemma 5.13 to the pinned $\mathfrak{sl}_2$ action in each piece. It supplies the negative leading arrows and their constants c_i^{-1}. For an operator of weight $\beta = \alpha_i$ or $-\alpha_i$, the leading target has weight $\operatorname{wt}\vartheta_m + \beta$. Rescaling therefore changes the leading coefficient to

$$\chi(\beta)\varepsilon_D(\vartheta_m)\frac{\chi(\operatorname{wt}\vartheta_m)}{\chi(\operatorname{wt}\vartheta_m + \beta)} = \varepsilon_D(\vartheta_m).$$

Iteration gives the maximal divided-derivative formula. Every term of a theta product has the sum of the input weights, so rescaling leaves all multiplication constants unchanged. Finally, the prescribed values define a character on the root lattice, and this character extends to P_L because $\mathbb{C}^\times$ is divisible. □

Corollary 5.15. *Under the hypotheses of Proposition 5.14, the leading root arrows on the regular theta basis of each section space $H^0(Y,\mathcal{O}_Y(\lambda))$ form the normal Kashiwara crystal of that L-module. More precisely, write $\mathcal{B}_\lambda = \{\vartheta_m : m \in \Xi_\Sigma \cap M,\ \gamma(m) = \lambda\}$. If $H^0(Y,\mathcal{O}_Y(\lambda)) \simeq \bigoplus_\mu V_L(\mu)^{\oplus m_{\lambda,\mu}}$, then*

$$\mathcal{B}_\lambda \simeq \bigsqcup_\mu B_L(\mu)^{\sqcup m_{\lambda,\mu}}.$$

The weight of m is $\eta(m)$, and its string lengths are $\varepsilon_i(\vartheta_m) = \deg_{E_i}\vartheta_m$ and $\varphi_i(\vartheta_m) = \deg_{F_i}\vartheta_m$. The highest vertices are precisely the theta functions in the common kernel of the E_i. Their multiplicities are given by Corollary 6.2. The arrows are unchanged by the weight-character rescaling and are independent of the adapted seed used to calculate them.

Proof. The preceding proposition gives a perfect basis in the sense of Berenstein–Kazhdan [BK, Definition 5.49]. Lemma 5.13 identifies the two arrows on each rank-one string and gives $\varphi_i - \varepsilon_i = \langle \eta(m), \alpha_i^\vee\rangle$. The perfect-basis theorem [BK, Lemma 5.51 and Theorem 5.55] identifies this crystal with that of any other perfect basis. Comparing with the union of global crystal bases in an irreducible decomposition gives the stated disjoint union [BK, Lemma 5.50]. Kernel compatibility identifies the highest vertices with a basis of $H^0(Y,\mathcal{O}_Y(\lambda))^{U_L}$, whose weight-$\mu$ part is the corresponding multiplicity space. Finally, the leading target is uniquely determined modulo the smaller kernel filtration, so it does not depend on the chart. □

For the canonical root actions, the positive hypothesis follows from Proposition 4.6. Hence the corollary applies to every finite-type Levi subgroup whose action extends to Y and its graded Cox ring, including Bott–Samelson varieties and their contractions. Only the positive adapted charts are needed for this conclusion.

For the explicit crystal arrows we negate the word exchange matrix and use this orientation throughout its mutation class. Every positive-root adapted column is then nonpositive. By Proposition 5.10, the indices in this orientation parametrize the same theta basis.

Corollary 5.16. *Let Y be as in Theorem 3.4, and let L be a finite-type Levi whose action extends to Y and its graded Cox ring. For each nonzero E_i, choose the adapted seed Σ_i of Proposition 4.6, with $E_i = c_i x^{v_i}\partial_{b_i}$. Write T_i for the tropical coordinate change from the reference seed to Σ_i, and put $\zeta_i = v_i - e_{b_i}$. Use the opposite exchange orientation fixed above. For $m \in \Xi_\Sigma \cap M$, the crystal string lengths are*

$$\varepsilon_i(m) = (T_i(m))_{b_i}, \qquad \varphi_i(m) = \varepsilon_i(m) + \langle \eta(m), \alpha_i^\vee\rangle.$$

The arrows are given by

$$\widetilde{e}_i(m) = T_i^{-1}\big(T_i(m) + \zeta_i\big), \qquad \widetilde{f}_i(m) = T_i^{-1}\big(T_i(m) - \zeta_i\big),$$

on the respective domains $\varepsilon_i(m) > 0$ and $\varphi_i(m) > 0$. The iterated arrows are obtained by replacing ζ_i by $r\zeta_i$ or $s\zeta_i$, as long as the corresponding string lengths permit the iteration. These formulas are integral piecewise-linear, preserve the divisor-class slices, and do not depend on the adapted chart.

Proof. Theta indices are tropical points, so a mutation sequence induces an integral piecewise-linear bijection between their seed coordinates [GHKK, Section 3], [Qin2, Section 2.4]. For clarity, mutation at a mutable index k has the formula

$$(T_k(m))_k = -m_k, \qquad (T_k(m))_j = m_j + [B_{kj}]_+ m_k - B_{kj}\min(m_k, 0) \quad (j \neq k),$$

where $[a]_+ = \max(a, 0)$ and we use the exchange matrix before the mutation. In the adapted chart, Proposition 5.11 identifies the E_i-degree with the b_i-coordinate and the leading

arrow with translation by ζ_i. Conjugating this translation by T_i gives the formula for $\widetilde{e}_i$. Lemma 5.13 identifies $\widetilde{f}_i$ with its inverse and gives its domain and string length. Iteration gives the remaining formulas. Their degree and independence assertions follow because they calculate the intrinsic leading root arrows on the theta basis. □

5.5. **Horospherical varieties.** The adapted operators also extend to a class of equivariant compactifications through a known description of their Cox rings.

Corollary 5.17. *Let G be a semisimple simply connected complex group, and let X be a normal projective horospherical G-variety with free divisor class group. Write G/H for its open orbit, $P = N_G(H)$, taken standard after conjugation, and $D_1, \dots, D_s$ for its G-stable prime divisors. Adjoin s isolated polynomial frozen variables to a Cox seed for G/P, assigning them the degrees $[D_1], \dots, [D_s]$. Assign each color degree of G/P the class of the corresponding color of X. The resulting graded upper cluster algebra is $\mathrm{Cox}(X)$. Every nonzero geometric simple-root operator E_i or F_i has an adapted seed obtained by the same mutation sequence as for G/P, with the same monomial coefficient.*

Proof. The Cox-ring description of Brion and Gagliardi gives an isomorphism of graded G-algebras

$$\mathrm{Cox}(X) \simeq \mathrm{Cox}(G/P)[z_1, \dots, z_s],$$

with the indicated degrees, where z_j is the canonical section of D_j and is G-invariant [Bri3, Theorem 4.3.2 and Lemma 4.1.2] [Gag, Theorems 3.6 and 3.8]. For the semisimple simply connected group, the equivariant divisor class group is the ordinary divisor class group, so this is the lift of the given geometric action. Polynomial extension by isolated frozen variables commutes with the intersection defining the upper cluster algebra, by comparing coefficients of the z-monomials. Theorem 3.4 identifies the flag factor. Every root operator kills the z_j, and hence the adapted seeds of Theorem 4.1 remain adapted after this extension. □

The corresponding basis is $\{\vartheta_m z^a : a \in \mathbb{N}^s\}$, where ϑ_m runs through the regular theta basis of $\mathrm{Cox}(G/P)$. Isolated frozen variables contribute no scattering walls. The regular theta cone is $\Xi_{G/P} \times \mathbb{R}^s_{\geq 0}$. If $\iota : \mathrm{Pic}(G/P) \to \mathrm{Cl}(X)$ sends flag color classes to their counterparts on X, the Cox degree of the basis element indexed by (m, a) is

$$\gamma_X(m, a) = \iota(\gamma_{G/P}(m)) + \sum_{j=1}^{s} a_j [D_j].$$

Slicing by $\gamma_X(m, a) = \lambda$ gives the theta basis of $H^0(X, \mathcal{O}_X(\lambda))$. Root operators preserve the flag degree and a, so the crystal is the disjoint union of the flag crystals occurring in this slice. Only finitely many occur, because the section space is finite-dimensional.

For a general spherical variety, Brion's description involves monomial base change of invariant boundary parameters [Bri3, Theorem 4.3.2]. Such substitutions can introduce common boundary factors into exchanges. Transferring adapted seeds through that base change requires an additional argument; the polynomial-extension proof above treats the horospherical case.

6. Root invariants and polyhedral applications

For an adapted derivation D, the D-degree of a section is its polynomial degree in the active coordinate. Imposing bounds on several such degrees cuts the regular theta cone by linear inequalities. We use the resulting semigroups to describe the common invariant ring and the Rees algebra of the joint filtration. The corresponding toric degenerations come from finite Khovanskii bases [KM].

For the next theorem, let $R = \mathcal{U}_{\mathrm{pol}}(\Sigma)$ have full-rank extended exchange matrix, a reddening sequence for its principal part, and a positive integral coarsening of its grading. Let $D_1, \dots, D_t$

be nonzero homogeneous locally nilpotent derivations, each adapted to a frozen coordinate b_j. Write $B = \bigcap_j \ker D_j$, and set

$$d_j(f) = \max\{q : D_j^q f \neq 0\}\ (f \neq 0), \qquad F_{\mathbf{r}}R = \bigcap_j \ker D_j^{r_j+1}, \qquad \phi_j(m) = v_{-e_{b_j}}(\vartheta_m).$$

Here $d_j(0) = -\infty$, and the tropical point $-e_{b_j}$ is expressed in the j-th adapted seed.

Theorem 6.1. *The cones*

$$C_{\mathbf{d}} = \{m \in \Xi_\Sigma : \phi_j(m) \geq 0 \text{ for all } j\},$$
$$\widetilde{C}_{\mathbf{d}} = \{(m, \mathbf{r}) : m \in \Xi_\Sigma,\ \phi_j(m) + r_j \geq 0 \text{ for all } j\}$$

are rational polyhedral. Their lattice points index theta bases of B and of $\mathscr{R}_{\mathbf{d}}(R) = \bigoplus_{\mathbf{r}\in\mathbb{N}^t} F_{\mathbf{r}}R\,\mathbf{t}^{\mathbf{r}}$, respectively. The rings R, B, and $\mathscr{R}_{\mathbf{d}}(R)$ are finitely generated normal domains with flat graded degenerations to the normal semigroup algebras of their cones. In each case, the theta functions indexed by a Hilbert basis generate the ring. The operators D_j need not commute.

Proof. Proposition 5.5 identifies the displayed theta bases. Lemma 5.2 expresses each ϕ_j as the minimum of finitely many integral linear forms. Moreover, $\phi_j = -d_j \leq 0$ on the lattice points of Ξ_Σ, hence on the entire cone. The inequalities for $\widetilde{C}_{\mathbf{d}}$ therefore imply $r_j \geq 0$.

For each of the three algebras, write C for its cone and b_m for the basis element indexed by m. Pointed theta multiplication gives

$$b_m b_n = b_{m+n} + \sum_{q\in\mathbb{N}^r\setminus\{0\}} c_q b_{m+n+\sum_i q_i p_i},$$

where the exchange directions p_i are linearly independent and have degree zero. In the Rees algebra the directions are $(p_i, 0)$. Choose an integral linear form ℓ positive on each direction. Adding a multiple of the positive grading makes ℓ positive on the Hilbert basis H of the relevant lattice semigroup. For the Rees algebra use the grading $h(\gamma(m)) + \sum_j r_j$.

Each homogeneous component is finite-dimensional by Proposition 5.3. If $m = \sum_i h_i$ with $h_i \in H$, the product $\prod_i b_{h_i}$ has leading term b_m. Every correction term has the same degree and strictly larger ℓ-value. Descending induction on ℓ within this component therefore expresses b_m as a polynomial in the b_h. The filtration by ℓ has associated graded algebra $\mathbb{C}[C \cap M]$, using $M \oplus \mathbb{Z}^t$ for the Rees cone. The same induction generates its extended Rees algebra by u and $b_h u^{-\ell(h)}$ for $h \in H$. This algebra is torsion-free over $\mathbb{C}[u]$, hence flat. Saturation of the lattice semigroup gives normality of the toric fiber.

Finally, R is normal as an intersection of mixed Laurent rings. If K_j is the field generated by the inactive coordinates of the adapted seed, then $B = R \cap \bigcap_j K_j$, proving its normality. Since $-d_j$ is a valuation, extend it by the Gauss rule $\widetilde{v}_j(t_k) = \delta_{jk}$. Then

$$\mathscr{R}_{\mathbf{d}}(R) = R[t_1, \ldots, t_t] \cap \bigcap_j \mathcal{O}_{\widetilde{v}_j},$$

which proves normality of the Rees algebra. □

Corollary 6.2. *Let a connected reductive group L of finite type act on a Cox ring R from Theorem 3.4, preserving Cox degrees. Assume that the theta basis is homogeneous for a maximal torus and that the positive simple-root operators are adapted. Write γ for the Cox-degree map and η for the weight map. For $\mathbf{d} = (\deg_{E_i})_{E_i\neq 0}$, the ring R^{U_L} has theta basis indexed by $C_{\mathbf{d}} \cap M$ and degenerates to $\mathbb{C}[C_{\mathbf{d}} \cap M]$. The multiplicity of the simple module of highest weight μ in $H^0(Y, \mathcal{O}_Y(\lambda))$ is*

$$\#\{m \in C_{\mathbf{d}} \cap M : \gamma(m) = \lambda,\ \eta(m) = \mu\}.$$

Proof. The U_L-invariants are the common kernels of the positive simple-root operators. Complete reducibility identifies their weight-μ part with the corresponding multiplicity space. Apply Theorem 6.1. □

Finite generation of these invariants is classical [Gro]. The point here is the common system of inequalities for branching spaces and simultaneous root filtrations. For a finite-type Levi, the negative-root inequalities also follow from the positive ones: Lemma 5.13 gives

$$-\deg_{F_i}\vartheta_m = -\deg_{E_i}\vartheta_m - \langle \eta(m), \alpha_i^\vee\rangle.$$

Proposition 6.3. *Under the hypotheses of Theorem 6.1, let g be the theta index of $b = \prod_{f\in I_{\mathrm{fr}}} x_f$. Then*

$$\operatorname{int}(\Xi_\Sigma)\cap M = g + (\Xi_\Sigma \cap M), \qquad \operatorname{int}(\widetilde{C}_{\mathbf{d}})\cap (M\oplus\mathbb{Z}^t) = (g,2,\dots,2) + (\widetilde{C}_{\mathbf{d}}\cap (M\oplus\mathbb{Z}^t)).$$

Consequently the two normal semigroup algebras are Gorenstein, with canonical-module generators in degrees $\gamma(g)$ and $(\gamma(g),2,\dots,2)$, respectively. For the Cox rings of Theorem 3.4, $\gamma(g) = -K_Y$.

Proof. Multiplication by frozen monomials translates theta indices, so $\vartheta_{m+g} = b\vartheta_m$. Thus the frozen valuation functions $\psi_f(m) = v_{e_f}(\vartheta_m)$ satisfy $\psi_f(m+g) = \psi_f(m)+1$. Express each as a minimum of finitely many integral linear forms, keeping only vertices of their convex hull. Each retained form is uniquely minimal on a full-dimensional chamber. Comparing sufficiently deep points in that chamber with their translates by g shows that every retained form takes value one at g. These forms define Ξ_Σ; hence g is interior, and strict positivity at lattice points is equivalent to positivity after subtracting g.

Since the active variable is frozen, $d_j(b) = 1$, and $\phi_j(m+g) = \phi_j(m)-1$. The same argument shows that every retained defining form a of ϕ_j satisfies $a(g) = -1$. The forms defining the Rees cone are therefore $(a_f, 0)$ and (a, e_j), all of which take value one at $(g,2,\dots,2)$. This proves the second translation formula. The canonical-module statement follows from the Danilov–Stanley description by interior lattice points [BH, Theorem 6.3.5]. For Cox rings the frozen-degree identity gives $\gamma(g) = -K_Y$. □

Corollary 6.4. *Let Y be as in Theorem 3.4, and put $P_\lambda = \Xi_\Sigma \cap \gamma^{-1}(\lambda)$. For every integral class λ in the interior of the effective cone,*

$$\operatorname{relint}(P_\lambda)\cap M = g + (P_{\lambda+K_Y}\cap M), \qquad \#(\operatorname{relint}(P_\lambda)\cap M) = h^0(Y,\mathcal{O}_Y(\lambda+K_Y)).$$

In particular, $-K_Y$ is interior to the effective cone and g is the unique relative-interior lattice point of P_{-K_Y}. If $K = \ker\gamma$, the polar of $P_{-K_Y} - g \subset K_{\mathbb{R}}$ is a lattice polytope in K^*.

Proof. The effective cone is $\gamma(\Xi_\Sigma)$. For an interior class its affine fiber meets $\operatorname{int}(\Xi_\Sigma)$, so $\operatorname{relint}(P_\lambda) = \operatorname{int}(\Xi_\Sigma)\cap\gamma^{-1}(\lambda)$. Slice Proposition 6.3 by degree. The anticanonical assertions follow from $\gamma(g) = -K_Y$ and $P_0 = \{0\}$. Finally, if the integral defining forms of Ξ_Σ are a_j, then

$$P_{-K_Y} - g = \{v\in K_{\mathbb{R}} : a_j(v) \ge -1 \text{ for all } j\}.$$

Its polar is $\operatorname{conv}\{-a_j|_K\}$, whose vertices are integral. □

7. Examples

7.1. **The full flag variety of SL_3.** The Cox ring of the incidence variety in $\mathbb{P}^2\times(\mathbb{P}^2)^*$ has the presentation

$$R = \mathbb{C}[x,x',a,b,c,d]/(xx'-ab-cd), \qquad \deg(x,a,c) = (1,0), \quad \deg(x',b,d) = (0,1).$$

Use the seed with mutable coordinate x and frozen coordinates a,b,c,d. To index the theta basis, choose the exchange orientation $p = (0,1,1,-1,-1)$, opposite to the positive-root convention of the word seed. The theta functions themselves are unchanged by this choice.

For $m = (q,A,B,C,D)\in\mathbb{Z}^5$, the regular theta cone is

$$A\ge 0, \quad B\ge 0, \quad C\ge 0, \quad D\ge 0, \quad C+q\ge 0, \quad D+q\ge 0. \tag{7.1}$$

The corresponding basis element is

$$\vartheta_m = \begin{cases} x^q a^A b^B c^C d^D, & q \geq 0, \\ (x')^{-q} a^A b^B c^{C+q} d^{D+q}, & q < 0. \end{cases}$$

These are exactly the standard monomials containing at most one of x, x'. The relation proves that they span. Their leading monomials in the seed (x, a, b, c, d) are distinct, which proves independence. The Cox degree is $\gamma(m) = (q + A + C, B + D)$. The slice of degree (r, s) contains $(r + 1)(s + 1)(r + s + 2)/2$ lattice points, as expected from the dimension formula for the corresponding irreducible SL_3-module.

Two positive simple-root operators are $E_1 = a\partial_x + x'\partial_b$ and $E_2 = x\partial_c + d\partial_{x'}$. Both preserve the incidence relation. The initial seed is adapted to $E_2 = x\partial_c$, and mutation at x gives the seed adapted to $E_1 = x'\partial_b$. Consequently

$$\deg_{E_1} \vartheta_m = B + \max(q, 0), \qquad \deg_{E_2} \vartheta_m = C.$$

The common invariant cone is the face given by $q = B = C = 0$, so $R^{U^+} = \mathbb{C}[a, d]$. In degree (r, s) its unique basis element is $a^r d^s$. For instance, in the adjoint degree $(1, 1)$, six of the eight standard monomials belong to $\ker E_1^2 \cap \ker E_2^2$. The two excluded monomials are xb, of E_1-degree two, and cx', of E_2-degree two.

The frozen product $abcd$ is the theta function of index $g = (0, 1, 1, 1, 1)$, and all primitive defining forms of (7.1) take value one at g. Thus the interior translation is visible in this elementary example.

7.2. **A singular Schubert variety.** Let $Y = X_{3412} \subset \mathrm{SL}_4/B^+$, with reduced word $\mathbf{i} = (2, 1, 3, 2)$, and let $F_\bullet$ be the standard flag in $\mathbb{C}^4$. A flag $L \subset P \subset H$ belongs to Y precisely when $L \subset F_3$ and $F_1 \subset H$. Forgetting P exhibits Y as a $\mathbb{P}^1$-bundle over the incidence hypersurface

$$Q = \{([\ell_1 : \ell_2 : \ell_3], [h_2 : h_3 : h_4]) : \ell_2 h_2 + \ell_3 h_3 = 0\} \subset \mathbb{P}^2 \times \mathbb{P}^2.$$

The singular locus of Q is the point (F_1, F_3), where its local equation is $uv + zt = 0$. The singular locus of Y is the $\mathbb{P}^1$ above this point.

Deleting any position from $\mathbf{i}$ leaves a reduced expression of length three, and the four deletions give distinct elements. The standard boundary divisors of $Z_{\mathbf{i}}$ therefore map to distinct Schubert divisors D_j on Y, and $e_j = [D_j]$ form a basis of $\mathrm{Cl}(Y)$. On the standard affine chart the four prefix minors and the first mutant are

$$(f_1, f_2, f_3, f_4) = (z_1, z_2, z_3, z_1 z_4 - z_2 z_3), \qquad f_1' = z_4.$$

Write x, a, b, c for their four initial lifts. Only x is mutable. Writing y for its mutant, we obtain

$$\mathrm{Cox}(Y) = \mathbb{C}[x, y, a, b, c, \tau_1, \tau_2, \tau_3, \tau_4]/(xy - \tau_4 ab - \tau_1 c).$$

Indeed, an initial Laurent expansion belongs to the y-chart exactly when its coefficient of x^{-k} is divisible by $(\tau_4 ab + \tau_1 c)^k$ for every $k > 0$. Hence the ordinary and upper cluster algebras both have the indicated presentation.

The degrees are

$$\deg x = e_1, \quad \deg a = e_1 + e_2, \quad \deg b = e_1 + e_3, \quad \deg c = \deg y = e_1 + e_2 + e_3 + e_4, \quad \deg \tau_j = e_j.$$

The restrictions of the three fundamental line bundles, of degrees $\deg a, \deg c, \deg b$ respectively, form a basis of $\mathrm{Pic}(Y)$ [Bri1, Proposition 2.2.8]. Thus

$$\mathrm{Pic}(Y) = \{d \in \mathbb{Z}^4 : d_1 - d_2 - d_3 + d_4 = 0\}, \qquad -K_Y = 4e_1 + 3e_2 + 3e_3 + 2e_4,$$

where the anticanonical class is the sum of the frozen degrees.

Choose $B = (0, 1, 1, -1, -1, 0, 0, 1)$ in the coordinate order $(x, a, b, c, \tau_1, \ldots, \tau_4)$. For $m = (p, q, r, s, u_1, u_2, u_3, u_4)$, the regular theta cone is

$$q, r, s, u_1, u_2, u_3, u_4 \geq 0, \qquad p + s \geq 0, \qquad p + u_1 \geq 0. \tag{7.2}$$

If $p \geq 0$, the theta function is $x^p a^q b^r c^s \tau^u$. If $p = -k < 0$, it is $y^k a^q b^r c^{s-k} \tau_1^{u_1-k} \tau_2^{u_2} \tau_3^{u_3} \tau_4^{u_4}$. These are all cluster monomials. The exchange relation eliminates each product xy, so they span. Their distinct leading Laurent monomials prove independence, and nonnegativity of their exponents gives (7.2). Theorem 5.6 gives the same inequalities directly: the suffix sums at c are $s, p+s$, and the order at τ_1 is $u_1 + \min(0,p)$. To obtain the section space of degree d, impose

$$d_1 = p+q+r+s+u_1, \quad d_2 = q+s+u_2, \quad d_3 = r+s+u_3, \quad d_4 = s+u_4.$$

All three positive simple-root operators act nontrivially. Only the rank-one subgroup G_2 preserves Y. With suitable pinning, the four operators and their nilpotence degrees are

D	mutable coordinate	D in this chart	$\varepsilon_D(m)$
E_1	x	$x\tau_2\partial_a$	$q + \max(0,-p)$
E_3	x	$x\tau_3\partial_b$	$r + \max(0,-p)$
F_2	x	$x\partial_{\tau_1}$	u_1
E_2	y	$y\partial_c$	$p+s$

For example, differentiating the exchange gives $E_2 x = \tau_1$ in the y-chart and $F_2 y = c$ in the x-chart. The table also gives the simultaneous filtrations. The bound $\varepsilon_{E_1}(m) \leq r_1$ imposes $q \leq r_1$ and $q - p \leq r_1$, and the common kernel of all three positive operators is $\mathbb{C}[\tau_1, \ldots, \tau_4]$.

The crystal for the simple root 2 has coroot weight $u_1 - p - s$ and arrows

$$\widetilde{e}_2(m) = m - e_x + e_{\tau_1} \quad (p+s > 0), \qquad \widetilde{f}_2(m) = m + e_x - e_{\tau_1} \quad (u_1 > 0).$$

Here e_x, e_{τ_1} are exponent vectors. Both arrows stay in the same Cox-degree slice of the cone. For instance, $D = e_1 + e_2 + e_3$ is non-Cartier, and its slice consists of the four sections $x\tau_2\tau_3$, $\tau_1\tau_2\tau_3$, $a\tau_3$, $b\tau_2$. The first two are a single string of length one. Both root operators kill the last two. Thus $H^0(Y, \mathcal{O}_Y(D)) \simeq V(1) \oplus 2V(0)$ for this SL_2, where $V(n)$ has highest weight n. The frozen product has theta index $g = (0,1,1,1,1,1,1,1)$, at which each primitive facet form in (7.2) has value one. This realizes the interior translation of Proposition 6.3.

Projecting Y to the plane P gives $\mathrm{Gr}(2,4)$. Only D_1 survives as a divisor, so τ_2, τ_3, τ_4 specialize to one. In Plücker coordinates the resulting tuple $(x, a, b, c, \tau_1; y)$ is $(p_{24}, p_{14}, p_{23}, p_{12}, p_{34}; p_{13})$, and the exchange is $p_{24}p_{13} = p_{14}p_{23} + p_{12}p_{34}$. Here $w^{-1}\alpha_1 = \alpha_3$, as in the non-left-descent case of Theorem 4.1. The negative simple root subgroup adds a multiple of row one to row two. Its derivation sends p_{24}, p_{23} to p_{14}, p_{13}, respectively, up to a common pinning scalar. After mutating p_{24} to p_{13}, the only moving coordinate is the terminal minor p_{23} indexed by 3, and F_1 is a scalar multiple of $p_{13}\partial_{p_{23}}$.

7.3. **An equivariant compactification of the affine plane.** Let Y be the blowup of $\mathbb{P}^2$ at $[1:0:0]$ and $[0:1:0]$. Its toric Cox ring is $R = \mathbb{C}[a,b,c,e,f]$ [ADHL]. On the invariant affine plane $C = \{Z \neq 0\}$, set $x = X/Z$ and $y = Y/Z$; then $x = \frac{a}{ce}$ and $y = \frac{b}{cf}$. The action $[X:Y:Z] \mapsto [X:Y+tZ:Z]$ fixes the line at infinity pointwise, so it lifts through both blowups. Its infinitesimal action on C, after choosing the sign of the parameter, is $D = \partial_y$.

Give $\mathbb{C}[C]$ the rank-one seed with mutable coordinate y, frozen coordinate $z_0 = xy - 1$, and exchange $yx = z_0 + 1$. The boundary valuations, in the order (c, e, f), give

$$\nu(y) = (1,0,1), \qquad \nu(x) = (1,1,0), \qquad \nu(z_0) = (2,1,1).$$

Thus the lifted seed is (b, z, c, e, f), where b is mutable and $z = ab - c^2ef$ is frozen. Its exchange is $ba = z + c^2ef$. The two mixed Laurent charts are $R[b^{-1}]$ and $R[a^{-1}]$. Since a and b are relatively prime in the polynomial ring R, their intersection is R. Consequently this lifted seed has polynomial upper cluster algebra exactly $\mathrm{Cox}(Y)$.

The lifted action fixes a, c, e, f and satisfies $\widetilde{D}b = cf$ and $\widetilde{D}z = acf$. The initial moving coordinates (b, z) form a chain. Mutation at b reaches the seed (a, z, c, e, f), in which $\widetilde{D} = acf\,\partial_z$. This is an instance of Theorem 4.4 with a nontrivial mutation. Since Y has dimension two and Picard rank three, it is not a Bott–Samelson variety or a contraction of

one of the same dimension. The calculation is therefore an instance of equivariant adaptation beyond the Schubert setting.

Declaration of generative AI and AI-assisted technologies

During the development of the paper, the author used ChatGPT to assist in exploring and formulating most proof arguments. The author subsequently checked, revised, and independently verified all arguments and takes full responsibility for the mathematical content of the paper.

References

[And] D. Anderson, *Effective divisors on Bott–Samelson varieties*, Transform. Groups **24** (2019), no. 3, 691–711. arXiv:1501.00034.

[ADHL] I. Arzhantsev, U. Derenthal, J. Hausen, and A. Laface, *Cox Rings*, Cambridge Studies in Advanced Mathematics **144**, Cambridge University Press, Cambridge, 2015. arXiv:1003.4229.

[Arz] I. V. Arzhantsev, *On factoriality of Cox rings*, Math. Notes **85** (2009), no. 5, 623–629. arXiv:0802.0763.

[BY] H. Bao and J. Y. Ye, *Upper cluster structure on Kac–Moody Richardson varieties*, Preprint, 2025. arXiv:2506.10382v2.

[BMRS] A. Benito, G. Muller, J. Rajchgot, and K. E. Smith, *Singularities of locally acyclic cluster algebras*, Algebra Number Theory **9** (2015), no. 4, 913–936. doi:10.2140/ant.2015.9.913.

[BFZ] A. Berenstein, S. Fomin, and A. Zelevinsky, *Cluster algebras III: Upper bounds and double Bruhat cells*, Duke Math. J. **126** (2005), no. 1, 1–52. doi:10.1215/S0012-7094-04-12611-9.

[BK] A. Berenstein and D. Kazhdan, *Geometric and unipotent crystals. II. From unipotent bicrystals to crystal bases*, in *Quantum Groups*, Contemp. Math. **433**, Amer. Math. Soc., Providence, RI, 2007, 13–88. doi:10.1090/conm/433/08321.

[Bri1] M. Brion, *Lectures on the geometry of flag varieties*, in *Topics in Cohomological Studies of Algebraic Varieties*, Trends in Mathematics, Birkhäuser, Basel, 2005, 33–85. arXiv:math/0410240.

[Bri2] M. Brion, *Some structure theorems for algebraic groups*, in *Algebraic Groups: Structure and Actions*, Proc. Sympos. Pure Math. **94**, American Mathematical Society, Providence, RI, 2017; arXiv:1509.03059.

[Bri3] M. Brion, *The total coordinate ring of a wonderful variety*, J. Algebra **313** (2007), 61–99. Author's manuscript.

[BH] W. Bruns and J. Herzog, *Cohen–Macaulay Rings*, revised edition, Cambridge Studies in Advanced Mathematics **39**, Cambridge University Press, Cambridge, 1998.

[Cal] P. Caldero, *Toric degenerations of Schubert varieties*, Transform. Groups **7** (2002), no. 1, 51–60. arXiv:math/0012165.

[CKQ] P. Cao, B. Keller, and F. Qin, *The valuation pairing on an upper cluster algebra*, J. Reine Angew. Math. **806** (2024), 71–114. doi:10.1515/crelle-2023-0080. arXiv:2204.09576v2.

[CKP] B. Narasimha Chary, S. Senthamarai Kannan, and A. J. Parameswaran, *Automorphism group of a Bott–Samelson–Demazure–Hansen variety*, Transform. Groups **20** (2015), no. 3, 665–698. doi:10.1007/s00031-015-9327-8.

[CMMM] M.-W. Cheung, T. Magee, T. Mandel, and G. Muller, *Valuative independence and cluster theta reciprocity*, arXiv:2505.09585v2, 2026.

[Dem] L. Demonet, *Categorification of skew-symmetrizable cluster algebras*, Algebr. Represent. Theory **14** (2011), no. 6, 1087–1162. doi:10.1007/s10468-010-9228-4. arXiv:0909.1633.

[EFMS] J. Enwright, L. Francone, J. Moraga, and H. Spink, *Fano compactifications of mutation algebras*, preprint, 2025. arXiv:2512.21839.

[FF] Y. Fan and J. Fei, *Cluster algebras for bosonic plethysm*, Preprint, 2026. arXiv:2608.00963v1.

[Fei] J. Fei, *Crystal structure of upper cluster algebras*, Preprint, 2023, revised 2026. arXiv:2309.08326v4.

[Fra1] L. Francone, *The minimal monomial lifting of cluster algebras I: branching problems*, Adv. Math. **500** (2026), Paper No. 111102. doi:10.1016/j.aim.2026.111102.

[Fra2] L. Francone, *Cluster structures on Cox rings*, Preprint, 2024. arXiv:2412.04173.

[Fuj] N. Fujita, *Newton–Okounkov bodies for Bott–Samelson varieties and string polytopes for generalized Demazure modules*, J. Algebra **515** (2018), 408–447. doi:10.1016/j.jalgebra.2018.08.019.

[FO] N. Fujita and H. Oya, *Newton–Okounkov polytopes of Schubert varieties arising from cluster structures*, Trans. Amer. Math. Soc. Ser. B **12** (2025), 910–973. doi:10.1090/btran/231.

[Gag] G. Gagliardi, *The Cox ring of a spherical embedding*, J. Algebra **397** (2014), 548–569. arXiv:1209.2100.

[GELS] A. Garcia Elsener, P. Lampe, and D. Smertnig, *Factoriality and class groups of cluster algebras*, Adv. Math. **358** (2019), 106858. doi:10.1016/j.aim.2019.106858.

[GLS] C. Geiß, B. Leclerc, and J. Schröer, *Partial flag varieties and preprojective algebras*, Ann. Inst. Fourier (Grenoble) **58** (2008), no. 3, 825–876. doi:10.5802/aif.2371.

[GOST] Y. Gongyo, S. Okawa, A. Sannai, and S. Takagi, *Characterization of varieties of Fano type via singularities of Cox rings*, J. Algebraic Geom. **24** (2015), no. 1, 159–182. arXiv:1201.1133.

[GY] K. R. Goodearl and M. T. Yakimov, *Integral quantum cluster structures*, Duke Math. J. **170** (2021), no. 6, 1137–1200. doi:10.1215/00127094-2020-0061.

[GHKK] M. Gross, P. Hacking, S. Keel, and M. Kontsevich, *Canonical bases for cluster algebras*, J. Amer. Math. Soc. **31** (2018), no. 2, 497–608. The numbering of results used here follows arXiv:1411.1394.

[Gro] F. D. Grosshans, *Algebraic Homogeneous Spaces and Invariant Theory*, Lecture Notes in Mathematics **1673**, Springer-Verlag, Berlin, 1997.

[HKL] J. Hausen, S. Keicher, and A. Laface, *Computing Cox rings*, Math. Comp. **85** (2016), no. 297, 467–502. doi:10.1090/mcom/2989.

[Kad] F. Kadhem, *A cluster structure on the coordinate ring of partial flag varieties*, J. Algebra **628** (2023), 328–349. doi:10.1016/j.jalgebra.2023.03.021.

[KM] K. Kaveh and C. Manon, *Khovanskii bases, higher rank valuations, and tropical geometry*, SIAM J. Appl. Algebra Geom. **3** (2019), no. 2, 292–336.

[KQW] Y. Kimura, F. Qin, and Q. Wei, *Twist automorphisms and Poisson structures*, SIGMA Symmetry Integrability Geom. Methods Appl. **19** (2023), Paper No. 105, 39 pp. doi:10.3842/SIGMA.2023.105.

[Kum] S. Kumar, *Kac–Moody Groups, Their Flag Varieties and Representation Theory*, Progr. Math. **204**, Birkhäuser, Boston, MA, 2002. doi:10.1007/978-1-4612-0105-2.

[Kum2] S. Kumar, with an appendix by M. Kashiwara, *Positivity in T-equivariant K-theory of flag varieties associated to Kac–Moody groups*, J. Eur. Math. Soc. **19** (2017), 2469–2519. doi:10.4171/JEMS/722.

[Man] T. Mandel, *Cluster algebras are Cox rings*, Manuscripta Math. **160** (2019), nos. 1–2, 153–171. doi:10.1007/s00229-018-1054-8. arXiv:1707.05819.

[MSS] G. Merz, D. Schmitz, and H. Seppänen, *On the Mori theory and Newton–Okounkov bodies of Bott–Samelson varieties*, Math. Z. **300** (2022), no. 2, 1203–1240. doi:10.1007/s00209-021-02812-9. arXiv:1709.09910v2.

[Pom] M. Pompili, *On class groups of upper cluster algebras*, Canad. J. Math. **78** (2026), no. 5, 1609–1629. doi:10.4153/S0008414X25000033.

[Qin1] F. Qin, *Analogs of the dual canonical bases for cluster algebras from Lie theory*, arXiv:2407.02480v4, 2025.

[Qin2] F. Qin, *Applications of the freezing operators on cluster algebras*, J. Algebra Appl. **25** (2026), no. 3, Paper No. 2550357, 30 pp. doi:10.1142/S0219498825503578.

[RW] K. Rietsch and L. Williams, *A superpotential for Grassmannian Schubert varieties*, Preprint, 2025. arXiv:2409.00734v2.

[SW] L. Shen and D. Weng, *Cluster structures on double Bott–Samelson cells*, Forum Math. Sigma **9** (2021), Paper No. e66, 89 pp. doi:10.1017/fms.2021.59.

[Stacks] The Stacks Project Authors, *The Stacks Project*, `https://stacks.math.columbia.edu`, 2026.

[Tan2] Q. Tang, *Biperfectness of the theta basis of a maximal unipotent subgroup*, Preprint, 2026. Hexagon:2610.00603v1.

[Vez] A. Vezier, *Equivariant Cox ring*, Transform. Groups **28** (2023), no. 4, 1721–1774. doi:10.1007/s00031-022-09720-0.

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