Non-existence of non-Eulerian irregular Diophantine quadruples

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Abstract

A Diophantine quadruple $\{a, b, c, d\}$ with $a < b < c < d$ is irregular if $d \neq a + b + c + 2abc + 2\sqrt{(ab+1)(ac+1)(bc+1)}$. It is conjectured that there are none, and He and Togbé have recently given a proof that there are only finitely many, with $c < 10^{338}$. We study the non-Eulerian irregular quadruples, those whose triple $\{a, b, c\}$ is not an Euler triple, that is, $c \neq a + b + 2\sqrt{ab+1}$, and reduce them to nine explicit families. The families are defined by the degree of $\{a, b, c\}$ under canonical descent, by the position of the descent partner $d_-(a, b, c)$ relative to a and b, by the shape of the iterated descent, and by explicit windows for the index of $z = \sqrt{cd+1}$ in two Pell sequences. In particular $c < 10^{99}$, and $c < 10^{79.71}$ when z has an odd-index representation; for triples of degree one, $c < 1.81 \cdot 10^{47}$. He and Togbé obtained $c < 10^{162}$ for degree one and $c < 10^{338}$ for higher degree. With the bound $c < 4 \cdot 10^{48}$ for Euler triples of a companion paper, every irregular Diophantine quadruple has $c < 10^{99}$. For each family we give exact finite tests, in integer arithmetic or with certified rational enclosures, whose success shows that a given triple of the family has no irregular extension. Running these tests over the families is a finite computation. The numerical verification of these remaining cases is in progress, and its results will be appended to the final version of the paper.

Contents

1 Introduction **3**
1.1 The main results 4
1.2 Methods and plan of the paper 5

2 Diophantine triples, descent and the Pell equations **7**
2.1 Diophantine triples and their descent 7
2.2 The Pell equations and the two types 11
2.3 Congruences for the Pell sequences 15

3 Chains of pairs and the position of the descent partner **18**

4 The residual families and the Main Theorem **25**

5 Linear forms in logarithms and index caps **29**
5.1 Lower bounds for linear forms in logarithms 30
5.2 The Pell linear form and the a priori cap 33
5.3 Index caps on windows of c 36

6 Degree one: the families A1 and A2 **52**
6.1 Index congruences and the index floor 52
6.2 The bound for degree one and the windows of A1 and A2 58

7 The class bounds of degree at least two **68**
7.1 First-order defects and the two positions 69
7.2 Polynomial index floors and the a priori bounds 74
7.3 Type II: the bound $c < 10^{79.71}$ 102
7.4 Type I: the bound $c < 10^{99}$ 121
7.5 The generic windows 147

8 The windows of the families of degree at least two **148**
8.1 High triples: the family BH 148
8.2 Slot three: the families BM and BL 150
8.3 Deeper slots: the families B3 and B4 153

9 Deciding the families **161**
9.1 Closed cases and certified ranges 162
9.2 The triples to be tested 168
9.3 Small triples and the line 174
9.4 The generic test and high triples 177
9.5 The slot test for type I 179
9.6 Slot three: the ellipse and the tests for BM and BL 183
9.7 Deeper slots 195
9.8 Candidates and failed tests 198

1 Introduction

A set of positive integers is a *Diophantine tuple* if the product of any two of its elements, increased by 1, is a perfect square. Every Diophantine triple $\{a, b, c\}$ with $a < b < c$, where $ab + 1 = r^2$, $ac + 1 = s^2$ and $bc + 1 = t^2$, extends to the Diophantine quadruple $\{a, b, c, d_+\}$, where

$$d_\pm = d_\pm(a, b, c) = a + b + c + 2abc \pm 2rst,$$

as observed by Arkin, Hoggatt and Straus [1] and, independently, by Gibbs (see [7]). An integer $d > c$ such that $\{a, b, c, d\}$ is a Diophantine quadruple is an *extension* of the triple; it is *regular* if $d = d_+(a, b, c)$ and *irregular* otherwise, and the quadruple is called regular or irregular accordingly (Definition 2.3). The regularity conjecture, which goes back to the same authors, asserts that every Diophantine quadruple is regular (see [10]). It implies that there is no Diophantine quintuple: Dujella proved that there are only finitely many [6], and He, Togbé and Ziegler that there are none [11]. For a fixed triple, Fujita and Miyazaki [10] showed that there are at most eleven extensions, and Cipu, Fujita and Miyazaki [4] that there are at most eight. In a recent preprint [13], He and Togbé give a proof that there are only finitely many irregular Diophantine quadruples: each of them has $c < 10^{338}$. We use several results of this preprint; we have checked their proofs in the generality in which we use them (Section 2). The conjecture itself is open.

The proof of He and Togbé sorts the triple by canonical descent. A Diophantine triple $\{a, b, c\}$ with $a < b < c$ is an *Euler triple* if $c = a + b + 2r$; these are the triples that Dujella [7] and Cipu, Fujita and Miyazaki [4] call regular, and in this paper "regular" refers only to extensions and quadruples. Otherwise, by a theorem of He, Togbé and Ziegler [11] (Theorem 2.7), the number

$$e = d_-(a, b, c)$$

is a positive integer, $\{a, b, e\}$ is again a Diophantine triple and $c = d_+(a, b, e)$. Repeating this step on $\{a, b, e\}$ reaches an Euler triple after finitely many steps, and their number is the *degree* $\deg\{a, b, c\}$. He and Togbé obtain $c < 10^{98}$ when $\{a, b, c\}$ is an Euler triple, $c < 10^{162}$ when it has degree one, and $c < 10^{338}$ when it has degree at least two. Irregular extensions of Euler triples are the subject of the companion paper on Euler triples [9], which proves $c < 4 \cdot 10^{48}$ for them. The present paper treats all the other irregular quadruples, those $a < b < c < d$ whose triple $\{a, b, c\}$ is not an Euler triple. For example, the triple $\{1, 8, 120\}$ is not an Euler triple: here $e = d_-(1, 8, 120) = 3$, its descent is the Euler triple $\{1, 3, 8\}$, and it has degree one.

We call e the *descent partner* of the triple. Its position relative to a and b is the first invariant of this paper: the triple is *high* if $b < e < c$, *middle* if $a < e < b$ and *low* if $e < a$ (Definition 2.8), and exactly one of the three occurs (Lemma 3.4). The second invariant is the shape of the iterated descent. A Diophantine pair $\{p, q\}$, with $pq + 1 = \varrho^2$, carries for a sign σ the *chain*

$$x_0 = 0, \qquad x_1 = p + q - 2\sigma\varrho, \qquad x_{j+1} = (4pq + 2)\, x_j - x_{j-1} + 2(p + q) \quad (j \geq 1),$$

of Jones [14] and of Cipu, Fujita and Miyazaki [4], whenever $x_1 \geq 1$; the three triples $\{p, q, x_\nu\}$, $\{p, x_{\nu-1}, x_\nu\}$ and $\{q, x_{\nu-1}, x_\nu\}$ lie at *slot* ν of the chain (Definition 3.2). A triple of degree at least one is *single-chain* if all its descents after the first keep the same pair of elements, and *bent* otherwise (Definition 3.1); triples of degree one and two are single-chain. By Lemma 3.4, the single-chain triples are exactly the triples at a slot of some chain, a single-chain triple of degree $\nu - 1$ lies at slot ν of the chain of the two smaller elements of its descent, and every bent triple has degree at least three.

Write $ad + 1 = x^2$, $bd + 1 = y^2$ and $cd + 1 = z^2$, and let $\alpha = r + \sqrt{ab}$, $\beta = s + \sqrt{ac}$ and $\gamma = t + \sqrt{bc}$. Eliminating d, the number z is a common term $z = z_{m'}^{(a,c)} = z_{n'}^{(b,c)}$ of two Pell

sequences, of the pair $\{a, c\}$ with the unit β and of the pair $\{b, c\}$ with the unit γ (Section 2.1). By the classification of the seeds of Fujita and Miyazaki [10] and a reindexing of Cipu, Fujita and Miyazaki [4], as arranged by He and Togbé [13], the number z of an irregular extension has a *representation* (λ, m, n), with a sign $\lambda \in \{-1, 1\}$, of one of two types (Lemma 2.13): *type I*, with the seeds $(1, \lambda)$ and even indices $m' = 2m$, $n' = 2n$, and *type II*, with the seeds $(r, \lambda t)$, $(r, \lambda s)$ and odd indices $m' = 2m + 1$, $n' = 2n + 1$. We call m the *index* of z; in both types $n \leq m \leq 2n$ (Lemma 2.14). Each bound for c in this paper compares a lower bound for the index, from the arithmetic of the representation, with an upper bound from linear forms in logarithms.

1.1 The main results

Nine families A1, A2, B0, BH, BM, BL, B3, B4 and BX are defined in Definition 4.2 and summarised in Table 1. Each is a set of pairs (T, ρ) of a triple $T = \{a, b, c\}$ that is not an Euler triple and a candidate representation $\rho = (\lambda, m, n)$, described by explicit conditions on T, the *triples* of the family, and for each type by explicit conditions on ρ, the *window* of the family. All conditions are inequalities, divisibilities and polynomial identities in a, b, c, the descent partners, the chain constants of T and λ, m, n, and every window bounds m. The triple and the index of z *lie in* a family if (T, ρ) belongs to it for every representation ρ of z.

Main Theorem (Theorem 4.3). *Let $a < b < c < d$ be an irregular Diophantine quadruple whose triple $\{a, b, c\}$ is not an Euler triple. Then the triple and the index of z lie in one of the families A1, A2, B0, BH, BM, BL, B3, B4, BX of Table 1. In particular $c < 10^{99}$, and $c < 10^{79.71}$ if z admits a type II representation.*

The letter of a family gives the degree (A for degree one, B for degree at least two); the second symbol gives the range of c (1, 2, 0), the position of the descent partner (H, M, L), the degree (3 for degree three, 4 for degree at least four) or the bent shape (X). A1 consists of the triples of degree one with $c < 2.01 \cdot 10^{26}$, A2 of triples of degree one with $2.01 \cdot 10^{26} \leq c < 1.81 \cdot 10^{47}$ and conditions on the root ϱ of its chain pair and on its residue Θ (Definition 3.6), and B0 of the triples of degree at least two with $c < 10^{30}$. The remaining families have degree at least two and $c \geq 10^{30}$: BH consists of the high triples; BM and BL of the single-chain middle and low triples of degree two; B3 and B4 of the single-chain middle or low triples of degree three and of degree at least four; and BX of the bent middle or low triples. A triple of degree one outside A1 and A2 has no irregular extension.

The windows are finite. In degree at least two, the generic type I window is

$$\kappa\sqrt{c/b} < n < m \leq 2n, \qquad m' < Q_{\mathrm{I}}(c), \qquad \kappa = \tfrac{\sqrt{17}-1}{2},$$

with an explicit step function $Q_{\mathrm{I}}(c) \leq 6.5 \cdot 10^{12}$ on $10^{30} \leq c < 10^{99}$ (Definition 4.1); the generic type II window is $4 \leq n \leq m \leq 2n$ and $m' < Q_{\mathrm{II}}(c) \leq 1.03 \cdot 10^{13}$ on $10^{30} \leq c < 10^{79.71}$, and for a middle or low triple moreover $c < 10^{77.28}$ and the condition that the first-order quotient J of the representation (Definition 2.20) be an integer with $J \geq 1$ and $J(4ae + 1) < 2m(m + 1)$. The other rows sharpen the generic windows by the structure of their triples. For instance, a high triple with a type I representation has $c < 5.7 \cdot 10^{45}$ and $n > 2\kappa\sqrt{a}\, c^{1/4}$; a middle triple of degree two with a type II representation either bounds b by a quadratic polynomial in m or has its indices on an explicit ellipse; and in degree one above $2.01 \cdot 10^{26}$ only type I survives, with its indices on a line determined by the triple.

The Main Theorem contains bounds for c in each class, which we state separately.

Degree one (Proposition 6.12). *Let $a < b < c < d$ be an irregular Diophantine quadruple with $\deg\{a, b, c\} = 1$. Then $c < 1.81 \cdot 10^{47}$. If moreover $c \geq 2.01 \cdot 10^{26}$, then z has no type II*

representation, and every type I representation of z lies on the line of the triple (Definition 3.6) and has $m' < 2.0 \cdot 10^{12}$.

The bound of He and Togbé for degree one is $c < 10^{162}$, from the index floor $m > c^{1/8}$; we prove $m' > 3c^{1/4}$ for type I and $m > (c/4)^{1/4} - 2$ for type II (Corollary 6.7).

Degree at least two (Proposition 7.33, Proposition 7.19). *Let $a < b < c < d$ be an irregular Diophantine quadruple with $\deg\{a, b, c\} \geq 2$. If z has a type I representation, then $c < 10^{99}$. If z has a type II representation, then $c < 10^{79.71}$, and $c < 10^{77.28}$ if $c \leq b^2$.*

For these triples He and Togbé obtain $c < 10^{338}$, from the index floor $m > c^{1/16}/\sqrt{3}$. The condition $c \leq b^2$ is the position again: for degree at least two, $c \neq b^2$, and the triple is middle or low exactly when $c < b^2$ (Lemma 3.5).

Together with the bound $c < 4 \cdot 10^{48}$ of the companion paper [9] for Euler triples, the Main Theorem gives $c < 10^{99}$ for every irregular Diophantine quadruple.

Deciding the families. We do not prove that the families are empty. Since c is bounded, each family has finitely many triples. Section 9 lists, for each family, the triples that can have an irregular extension (Lemma 9.11), and gives a test, exact and finite, in integer arithmetic or with certified rational enclosures, whose success shows that a given triple of the family has no irregular extension: Corollary 9.12 for A1 and B0, Corollary 9.15 for A2, Corollary 9.17 for BH, Corollary 9.29 and Corollary 9.32 for BM and BL, Corollary 9.34 for B3, Corollary 9.35 for B4, and Corollary 9.16 for BX. Running the tests over the families is a finite computation; its results will be appended to the final version of this paper. The regularity conjecture predicts that every test succeeds.

1.2 Methods and plan of the paper

The proof of the Main Theorem squeezes the index between a floor and a cap, separately for each degree, position, shape and type. The caps are analytic. Matveev's theorem [16], applied to the linear form

$$\Lambda = m' \log \beta - n' \log \gamma + \log \delta, \qquad 0 < \Lambda < \tfrac{8}{3}\, ac\, \beta^{-2m'}$$

of Lemma 5.7, gives the a priori cap of He and Togbé, $m' - 2 < K_{\mathrm{a}} (\log c)^2 \log(39(m'+1))$ with $K_{\mathrm{a}} = 4.436 \cdot 10^{13}$ (Proposition 5.9). On windows of c we compare the Pell equation of $\{a, b\}$, at the common index of Lemma 2.17, with that of $\{a, c\}$; this gives a linear form in three logarithms with a correction of small height, and we bound it by the theorem of Mignotte and Voutier [17], treating its degenerate alternative by a centred form in two logarithms and Laurent's theorem [15]. This gives the caps $Q_{\mathrm{I}}(c)$ and $Q_{\mathrm{II}}(c)$ (Proposition 5.15).

The floors are arithmetic, and this is where the position of the descent partner enters. Expanding the two Pell sequences at their common term modulo c and c^2, following Dujella [7] and He and Togbé [13], gives congruences for the indices. For a type II representation they control the *first-order defect*

$$H = bN - aM + e\lambda\Delta, \qquad N = 2n(n+1) - \lambda\Delta, \quad M = 2m(m+1) + \lambda\Delta, \quad \Delta = m - n$$

(Definition 2.20). If the triple is middle or low, then $c \mid H$ and the first-order quotient $J = H/c$ is a positive integer with $J(4ae+1) < 2m(m+1)$ (Lemma 7.3). If it is high, either $H \neq 0$, which forces $n > 0.53\sqrt{a}\, c^{1/4} - 0.75$ (Lemma 7.4), or $H = 0$, which gives a linear Vieta identity between b, the indices and the descent partner f of the descent (Lemma 7.5). For a type I representation the second-order defect $k = (bn^2 + \lambda tn - am^2 - \lambda sm)/c$ is an integer divisible by 4, with a cubic

congruence of discriminant type (Lemma 2.22); together with the floor $n > \kappa\sqrt{c/b}$ of He and Togbé (Lemma 2.23) this gives $m' > 6.24\,c^{1/4}$ for a high triple (Lemma 7.1) and, through Vieta defects of the descended triples, $m' > \frac{1}{2}c^{1/9}$ in general (Lemma 7.13).

For the class bounds of degree at least two, in Section 7, the floors have small exponents and the a priori cap is polylogarithmic, so the first squeeze gives only $c < 10^{163}$ for type I and $c < 10^{154}$ for type II (Proposition 7.7). The bounds 10^{99} and $10^{79.71}$ come from sharper floors on windows of c on which the caps are already small. For type II this is a case analysis by position, by the vanishing of H and, for a high triple with $H = 0$, by the quantity $L = 2n(n+1) - (4af+1)\Delta$ attached to f. For type I it is a gap $m' > 2^{-27/13}c^{5/39}$ on $10^{90} \le c < 10^{130}$ (Lemma 7.38), which gives $c < 10^{106}$ (Proposition 7.34), followed on $10^{99} \le c < 10^{106}$ by floors from Pell covers of the seeds of the descended triples (Lemma 7.43, Lemma 7.44), which exceed the caps there.

The windows of the families come from the chains. A single-chain middle or low triple of degree $\nu - 1$ is $\{a, x_{\nu-1}, x_\nu\}$ on the chain of $\{a, e\}$, so b and c are explicit in a, e and the chain sign, through the chain constants of the pair (Definition 3.8). At slot three, that is, in degree two, the type II congruences become the identity $bK = 2am(m+1) + \lambda\Delta(a-e) + JE$, with $K = N - J(4u^2 - 2)$, $u = \sqrt{ae+1}$ and $E = a + e + 2\sigma u$, where σ is the chain sign, so that E is the term x_{-1} of the chain continued to negative indices (Definition 3.2, Lemma 8.4); for a low triple it brings the divisibilities $a \mid \Xi$ and $c \mid Z$ of two explicit integers, and for a middle triple with $K = 0$ it puts (m, n) on an ellipse. At slot four (degree three), a type II representation with m below an explicit floor lies on a conic, the Pell equation of the pair $\{a, E'\}$ of the third descent (Lemma 8.11). For type I the trace identities of the chain pin n to an integer next to $\eta\rho_{\lambda,k}$, where $\eta = u + \sqrt{ae}$ and $\rho_{\lambda,k} = (\sqrt{4k+1} - \lambda)/2$ (Lemma 8.9).

The tests of Section 9 reduce each window to finite checks, chiefly ranges of indices certified by the reduction of Baker and Davenport [2] in the form of Dujella and Pethő [8] or by gap and shift certificates, with every comparison of real numbers reduced to rational enclosures. Several steps of the proof of the Main Theorem are finite computations: numerical inequalities certified in interval arithmetic with outward rounding, and complete enumerations over explicit finite sets, such as Lemma 7.41 and Lemma 7.42. Every certified numerical inequality is checked either exactly, over the rationals, or in interval arithmetic with outward rounding, where $X < Y$ is accepted only if the upper end of the interval enclosing X lies below the lower end of the interval enclosing Y; the working precision, mostly 160 or 200 bits, affects only whether a check succeeds, never the validity of a check that succeeds. Each finite computation used in a proof is stated with its exact claim inside the proof where it occurs.

Section 2 collects the setting: triples and descent, the Pell equations and the two types, and the congruences of the Pell sequences with the defects of a representation. Section 3 introduces chains and proves the classification by position and chain. Section 4 defines the index caps and the nine families (Table 1), and proves the Main Theorem from seven window propositions; the rest of the paper proves them. Section 5 gives the linear forms and the caps, and with them the window of B0 (Proposition 5.10). Section 6 treats degree one, the families A1 and A2. Section 7, the longest and most technical part, proves the class bounds of degree at least two and the generic windows, which are the windows of BX; after its first two subsections, its subsections on type II and on type I can be read independently of each other. Section 8 proves the windows of BH, BM, BL, B3 and B4. Section 9 lists the triples of each family to be tested, gives the tests, and says how to decide the candidates that a test leaves; it is not used in the proof of the Main Theorem. The appendix collects the notation: the letters that keep one meaning throughout the paper, the letters fixed in one part of it, and the conventions for fonts and constants.

2 Diophantine triples, descent and the Pell equations

Most of the material on Diophantine triples, their descent and the Pell equations of a quadruple is known, and the source is named at each result. We have checked the proofs of the results of He and Togbé [13] that we quote, here and in Section 5, in the generality in which we use them; they are correct, and we quote them without proof. One estimate of that preprint, the constant $\frac{8}{3}$ of its Lemma 8.2, is not justified for Euler triples in a type II representation; what we need of it is proved in Lemma 5.7 and Lemma 9.2. The regular extension maps $d_\pm$ (Definition 2.2) take a real third argument, as the descent needs. The paragraphs after Definition 2.3 fix the notation of the whole paper: $a<b<c<d$ is the quadruple, $ab+1=r^2$, $ac+1=s^2$, $bc+1=t^2$, $ad+1=x^2$, $bd+1=y^2$ and $cd+1=z^2$, and α, β, γ are the units of the pairs $\{a,b\}$, $\{a,c\}$ and $\{b,c\}$; they also define Euler triples, the descent and the degree, which are used at once, and the Pell sequences and the two types of representation of z, which are used from Section 2.2 on. The letters that keep one meaning throughout the paper are listed in the appendix (Section **??**).

2.1 Diophantine triples and their descent

Definition 2.1. A **Diophantine tuple** is a finite set E of distinct positive integers such that $xy+1$ is a perfect square for every pair of distinct elements $x,y\in E$. A positive integer $a<\min E$ is a **smaller extension** of E if $E\cup\{a\}$ is a Diophantine tuple.

Definition 2.2. For positive integers $A<B$ with $AB+1$ a perfect square, the **regular extension maps** are the real functions

$$d_\pm(A,B,X)=A+B+(2AB+1)X\pm 2\sqrt{AB+1}\sqrt{AX+1}\sqrt{BX+1}\qquad(X\geq 0).$$

An ordered Diophantine quadruple $\{A,B,C,D\}$ with $A<B<C<D$ is **regular** if $D=d_+(A,B,C)$, and **irregular** otherwise.

Definition 2.3. A Diophantine tuple (Definition 2.1) with three elements is a **Diophantine triple**, and one with four elements is a **Diophantine quadruple**. Let $\{a,b,c\}$ be a Diophantine triple with $a<b<c$. An **extension** of $\{a,b,c\}$ is an integer $d>c$ such that $\{a,b,c,d\}$ is a Diophantine quadruple. The extension is **regular** or **irregular** according as the quadruple $\{a,b,c,d\}$ is regular or irregular (Definition 2.2), that is, according as $d=d_+(a,b,c)$ or $d\neq d_+(a,b,c)$.

Pell notation. Let $0<a<b<c<d$ be a Diophantine quadruple (Definition 2.3), with positive integers

$$ab+1=r^2,\quad ac+1=s^2,\quad bc+1=t^2,\quad ad+1=x^2,\quad bd+1=y^2,\quad cd+1=z^2,$$

and units $\alpha=r+\sqrt{ab}$, $\beta=s+\sqrt{ac}$, $\gamma=t+\sqrt{bc}$.

Euler triple, descent, degree. A Diophantine triple $p<q<w$ is **Euler** if $w=p+q+2\sqrt{pq+1}$. If it is not Euler, its **descent** is the triple $\{p,q,d_-(p,q,w)\}$, reordered increasingly. The **degree** $\deg T$ of a Diophantine triple T is the least $j\geq 0$ such that the j-fold descent of T is Euler. For $\deg\{a,b,c\}\geq 1$ put $e=d_-(a,b,c)$; for $\deg\{a,b,c\}\geq 2$ let $f=d_-(p,q,w)$, where $p<q<w$ are the elements of the descent $\{a,b,e\}$.

Pell sequences. For integers (x_1,z_1) with $az_1^2-cx_1^2=a-c$ and (y_2,z_2) with $bz_2^2-cy_2^2=b-c$, define $z_j^{(a,c)},x_j^{(a,c)}$ and $z_j^{(b,c)},y_j^{(b,c)}$ $(j\geq 0)$ by

$$z_j^{(a,c)}\sqrt{a}+x_j^{(a,c)}\sqrt{c}=(z_1\sqrt{a}+x_1\sqrt{c})\beta^j,\qquad z_j^{(b,c)}\sqrt{b}+y_j^{(b,c)}\sqrt{c}=(z_2\sqrt{b}+y_2\sqrt{c})\gamma^j.$$

Types. A **representation** of z consists of a sign $\lambda \in \{-1, 1\}$ and integers $m, n \geq 0$ such that one of the following holds:

- **type I** (even–even): $(x_1, z_1) = (1, \lambda)$, $(y_2, z_2) = (1, \lambda)$ and $z = z_{2m}^{(a,c)} = z_{2n}^{(b,c)}$; then $m' = 2m$, $n' = 2n$;
- **type II** (odd–odd): $(x_1, z_1) = (r, \lambda t)$, $(y_2, z_2) = (r, \lambda s)$ and $z = z_{2m+1}^{(a,c)} = z_{2n+1}^{(b,c)}$; then $m' = 2m + 1$, $n' = 2n + 1$.

In both types $\Delta = m - n$.

Lemma 2.4. *Let $a < b < c$ be a Diophantine triple and $r = \sqrt{ab+1}$. Then $c = a + b + 2r$ or $c > 4ab$. Moreover*

$$4abc + c < d_+(a, b, c) < 4abc + 4c.$$

Proof. This is [11, Lemmas 1 and 2]. The dichotomy is a consequence of Jones [14, Lemma 4]. □

Definition 2.5. For real numbers x_1, x_2, x_3, x_4, their **symmetric regularity defect** is

$$\mathcal{R}(x_1, x_2, x_3, x_4) = \sum_{i=1}^{4} x_i^2 - 2 \sum_{1 \leq i < j \leq 4} x_i x_j - 4x_1x_2x_3x_4 - 4.$$

Lemma 2.6. *Let $p \neq q$ be positive integers with $pq + 1 = \varrho^2$ for a positive integer ϱ, and for real $X \geq 0$ write $d_\pm(p, q, X) = d_\pm(\min\{p, q\}, \max\{p, q\}, X)$, the regular extension maps of the pair $\{p, q\}$.*

1. *For real $X \geq 0$, the numbers $d_+(p, q, X) \geq d_-(p, q, X)$ are the two roots of*

$$Y^2 - 2(p + q + X + 2pqX)\, Y + (p + q - X)^2 - 4\varrho^2 = 0,$$

and this equation is equivalent to $\mathcal{R}(p, q, X, Y) = 0$, where the symmetric regularity defect $\mathcal{R}$ is symmetric in its four arguments. In particular

$$d_+(p, q, X) + d_-(p, q, X) = 2(p + q + X + 2pqX),$$
$$d_+(p, q, X)\, d_-(p, q, X) = (X - p - q)^2 - 4\varrho^2.$$

2. *For real $X \geq 0$,*

$$p\, d_\pm(p, q, X) + 1 = \big(p\sqrt{qX + 1} \pm \varrho\sqrt{pX + 1}\big)^2,$$
$$q\, d_\pm(p, q, X) + 1 = \big(q\sqrt{pX + 1} \pm \varrho\sqrt{qX + 1}\big)^2,$$
$$X\, d_\pm(p, q, X) + 1 = \big(\varrho X \pm \sqrt{pX + 1}\sqrt{qX + 1}\big)^2.$$

3. *The function $X \mapsto d_+(p, q, X)$ is strictly increasing on $X \geq 0$, and $d_+(p, q, X) > p + q + X + 4pqX$. For real $X > p + q + 2\varrho$,*

$$0 < d_-(p, q, X) < \frac{X}{4pq} \qquad \text{and} \qquad d_+\big(p, q, d_-(p, q, X)\big) = X.$$

4. *A Diophantine triple $p' < q' < w$ is Euler if and only if $d_-(p', q', w) = 0$. If it is not Euler, then $w > p' + q' + 2\sqrt{p'q' + 1}$ and $d_-(p', q', w)$ is a positive integer.*

5. For $\sigma \in \{-1,1\}$ and $w_\sigma = p+q-2\sigma\varrho$,

$$pw_\sigma + 1 = (\varrho - \sigma p)^2, \qquad qw_\sigma + 1 = (\varrho - \sigma q)^2, \qquad d_+(p,q,w_{-1}) = 4\varrho(\varrho+p)(\varrho+q).$$

Proof. These are the classical identities of regular extensions; see Dujella [7, Section 1.4]. The squares of item 2 are due to Arkin, Hoggatt and Straus [1], and item 5 is Euler's construction. They are used below for real $X \geq 0$, so we include the short proofs. Throughout, $\varphi_p = \sqrt{pX+1}$ and $\varphi_q = \sqrt{qX+1}$, so that $d_\pm(p,q,X) = p+q+X+2pqX \pm 2\varrho\varphi_p\varphi_q$ by Definition 2.2; this expression is symmetric in p and q.

Item 1. The reduced discriminant of the quadratic equation is

$$(p+q+X+2pqX)^2 - (p+q-X)^2 + 4\varrho^2 = 4\varrho^2\varphi_p^2\varphi_q^2,$$

so its roots are $p+q+X+2pqX \pm 2\varrho\varphi_p\varphi_q = d_\pm(p,q,X)$. Expanding the definition, $\mathcal{R}(p,q,X,Y) = (p+q-X-Y)^2 - 4(pq+1)(XY+1)$, and this is the left side of the equation because $2(p+q-X)+4\varrho^2X = 2(p+q+X+2pqX)$. The symmetry of $\mathcal{R}$ is visible in Definition 2.5. The sum and the product of the roots are read off the equation.

Item 2. Since $\varrho^2 = pq+1$,

$$\begin{aligned}\big(p\varphi_q \pm \varrho\varphi_p\big)^2 &= p^2(qX+1) + (pq+1)(pX+1) \pm 2p\varrho\varphi_p\varphi_q \\ &= p\big(p+q+X+2pqX \pm 2\varrho\varphi_p\varphi_q\big) + 1 = p\,d_\pm(p,q,X) + 1.\end{aligned}$$

The second identity is the first with p and q exchanged. For the third,

$$\big(\varrho X \pm \varphi_p\varphi_q\big)^2 = (pq+1)X^2 + (pX+1)(qX+1) \pm 2\varrho X\varphi_p\varphi_q = X\big(p+q+X+2pqX \pm 2\varrho\varphi_p\varphi_q\big) + 1.$$

Item 3. Both $(2pq+1)X$ and $\varphi_p\varphi_q$ increase strictly with X, so $d_+(p,q,X)$ does. As $\varrho > \sqrt{pq}$ and $\varphi_p\varphi_q > \sqrt{pq}\,X$, we have $\varrho\varphi_p\varphi_q > pqX$ (for $X=0$ the left side is $\varrho > 0$), hence $d_+(p,q,X) > p+q+X+4pqX$. Now let $X > p+q+2\varrho$. Then $(X-p-q)^2 - 4\varrho^2 > 0$, so by item 1 and $d_+(p,q,X) > 0$ we get $d_-(p,q,X) > 0$; moreover $0 < X-p-q < X$ gives $(X-p-q)^2 - 4\varrho^2 < X^2$, and $d_+(p,q,X) > 4pqX$, so $d_-(p,q,X) < X^2/(4pqX) = X/(4pq)$. Finally, $d_+(p,q,\cdot)$ is continuous and strictly increasing, with value $p+q+2\varrho$ at 0 and unbounded, so there is $Y_0 > 0$ with $d_+(p,q,Y_0) = X$. By item 1, $\mathcal{R}(p,q,Y_0,X) = 0$, hence by symmetry $\mathcal{R}(p,q,X,Y_0) = 0$, and Y_0 is one of $d_\pm(p,q,X)$. Since $Y_0 < d_+(p,q,Y_0) = X < d_+(p,q,X)$, we get $Y_0 = d_-(p,q,X)$, that is, $d_+(p,q,d_-(p,q,X)) = X$.

Item 4. Put $\varrho' = \sqrt{p'q'+1}$. By item 1 for the pair $\{p',q'\}$ at $X = w$,

$$d_+(p',q',w)\,d_-(p',q',w) = (w-p'-q'-2\varrho')(w-p'-q'+2\varrho'),$$

and the second factor is $(w-q')+(2\varrho'-p') > 0$ because $w > q'$ and $\varrho'^2 > p'^2$. As $d_+(p',q',w) > 0$, $d_-(p',q',w) = 0$ exactly when $w = p'+q'+2\varrho'$, that is, when the triple is Euler. If it is not Euler, then $w > 4p'q'$ by Lemma 2.4, and $4p'q' \geq p'+q'+2\varrho'$, because $p'+q' \leq p'q'+1$, $2\varrho' \leq p'q'+2$ and $p'q' \geq 3$. So $w > p'+q'+2\varrho'$, and $d_-(p',q',w) > 0$ by item 3. It is an integer, because ϱ', $\sqrt{p'w+1}$ and $\sqrt{q'w+1}$ are integers.

Item 5. With $\varrho^2 = pq+1$, $pw_\sigma + 1 = p^2+pq+1-2\sigma p\varrho = (\varrho-\sigma p)^2$, and likewise $qw_\sigma+1 = (\varrho-\sigma q)^2$. For $\sigma = -1$ both roots $\varrho+p$, $\varrho+q$ are positive, so with $w = w_{-1} = p+q+2\varrho$,

$$d_+(p,q,w) = p+q+w+2pqw+2\varrho(\varrho+p)(\varrho+q) = 4\varrho(\varrho+p)(\varrho+q). \qquad \square$$

By the identities of Lemma 2.6 and the gap of Lemma 2.4, d_- inverts d_+ on the triples that are not Euler. Iterating it is the canonical descent of He, Togbé and Ziegler, which gives every triple its degree and its descent partner.

Theorem 2.7. *Let $p < q < w$ be a Diophantine triple which is not Euler, that is, $w \neq p + q + 2\sqrt{pq+1}$. Then $d_-(p,q,w)$ is a positive integer distinct from p and q, $\{p,q,d_-(p,q,w)\}$ is a Diophantine triple, $d_+\big(p,q,d_-(p,q,w)\big) = w$, and $pq\,d_-(p,q,w) < pqw/12$. Iterating this replacement of the largest element reaches an Euler triple after fewer than $\log(pqw)/\log 12$ steps.*

Proof. This is [13, Section 2 and Proposition 2.2], after [11, Propositions 1–3]: the descent replaces the largest element w by $d_-(p,q,w)$ and inverts the regular extension d_+. □

Definition 2.8. Let $T = \{a,b,c\}$ with $a < b < c$ be a Diophantine triple (Definition 2.1) which is not an Euler triple.

Descent partner. The **descent partner** of T is

$$e(T) = d_-(a,b,c)$$

(Definition 2.2). We write $e = e(T)$; it is the number e of the Pell notation. The descent of T is written ∂T; it is the set $\{a,b,e\}$.

Iterated descents. Let $\deg T$ be the degree of T. Put $\partial^0 T = T$ and, for $1 \le i \le \deg T$, let $\partial^i T = \partial(\partial^{i-1}T)$ be the descent of $\partial^{i-1}T$. Thus $\partial^i T$ is the i-fold descent of T; by the definition of the degree, $\partial^0 T, \dots, \partial^{\deg T - 1}T$ are not Euler and $\partial^{\deg T}T$ is an Euler triple. If $\deg T \ge 2$, the descent partner of the descent is

$$f = e(\partial T);$$

it is the number f of the Pell notation.

Position. The **position** of the descent partner of T, briefly the position of T, is

- **high** if $b < e < c$;
- **middle** if $a < e < b$;
- **low** if $e < a$.

The triple T is called **high**, **middle** or **low** accordingly.

The descent and the degree are those of He, Togbé and Ziegler [11].

Two results from the literature turn irregularity into the inequality $d > d_+(a,b,c)$ and give $b > 4000$ (Lemma 2.9); these are the standing inputs of every index estimate below. For triples of degree at least two, He and Togbé add $b > 8a$ (Lemma 2.10). In the proof of the Main Theorem, the fourth-power bound of Cipu, Fujita and Miyazaki (Theorem 2.11) is used only for high triples, in Lemma 7.4; Section 9 uses it again to list the triples to be tested (Lemma 9.11).

Lemma 2.9. *Every irregular Diophantine quadruple $0 < a < b < c < d$ satisfies $d > d_+(a,b,c)$ and $b > 4000$.*

Proof. This is [13, Lemma 2.5]. The inequality $d > d_+(a,b,c)$ is [6, Proposition 1] (see [10, p. 3804]), and, given it, $b > 4000$ is [10, Lemma 2.1], from [3, Lemma 3.4]. □

Lemma 2.10. *If $\deg\{a,b,c\} \ge 2$ and the quadruple $\{a,b,c,d\}$ is irregular, then $b > 8a$.*

Proof. This is [13, Lemma 7.1], whose hypothesis $d > d_+(a,b,c)$ holds by Lemma 2.9. □

Theorem 2.11. *Every irregular Diophantine quadruple $0 < a < b < c < d$ satisfies $c < 200\,b^4$.*

Proof. By Lemma 2.9, an irregular quadruple has $d > d_+(a,b,c)$. By [4, Theorem 1.4(4)], a triple with $c \geq 200b^4$ has no extension $d > d_+(a,b,c)$; this is the bound used in [13, Lemma 7.4]. □

2.2 The Pell equations and the two types

Eliminating d from $ad+1 = x^2$, $bd+1 = y^2$ and $cd+1 = z^2$ gives the equations $az^2 - cx^2 = a-c$ and $bz^2 - cy^2 = b-c$, whose solutions form the Pell sequences defined in Section 2.1. Their fundamental solutions are bounded (Lemma 2.12); the bound is [11, Lemma 8], going back to [5], and the argument is the classical one of Nagell [18]. Fujita and Miyazaki classify the seeds of a common term by the parities of its indices, and one reindexing step of Cipu, Fujita and Miyazaki reduces the classes to the two types (Lemma 2.13).

Lemma 2.12. *Let $P < Q$ be positive integers with $PQ + 1 = S^2$ for an integer S, and put $\eta_{PQ} = S + \sqrt{PQ}$. Every solution in positive integers (X, Y) of $PY^2 - QX^2 = P - Q$ satisfies*

$$Y\sqrt{P} + X\sqrt{Q} = (Y_0\sqrt{P} + X_0\sqrt{Q})\,\eta_{PQ}^k$$

for an integer $k \geq 0$ and an integer solution (X_0, Y_0) of the same equation with

$$1 \leq X_0 \leq \sqrt{\frac{P(Q-P)}{2(S-1)}}, \qquad 1 \leq |Y_0| \leq \sqrt{\frac{(S-1)(Q-P)}{2P}}.$$

In particular $Y_0^2 < Q^2/4$.

Proof. This is [11, Lemma 8] (numbering of the arXiv version 2), going back to [5]. For the last inequality, $S^2 - 1 = PQ$ gives $(S-1)/P = Q/(S+1)$, so $Y_0^2 \leq Q(Q-P)/(2(S+1))$; since $S \geq 2$ and $Q - P < Q$, this is less than $Q^2/4$. □

Lemma 2.13. *If $d > d_+(a,b,c)$ then z has a representation of type I or of type II.*

Proof. This is [13, Lemma 3.2], after [10, Theorem 1.3] and [4, Lemma 2.3]. The positive seeds x_1, y_2 are fixed by $az_1^2 - cx_1^2 = a-c$ and $bz_2^2 - cy_2^2 = b-c$, which give $x_1 = y_2 = 1$ for $z_1 = z_2 = \lambda$ and $x_1 = y_2 = r$ for $(z_1, z_2) = (\lambda t, \lambda s)$. □

The two indices of a representation are comparable, and the smallest ones are excluded.

Lemma 2.14. *Let $d > c$ and let z have a representation (λ, m, n) of type I or II. Then*

$$n \leq m \leq 2n.$$

If the representation is of type I and $\{a,b,c\}$ is not an Euler triple, then $n < m$.

Proof. [13, Lemma 4.2] gives $n \leq m \leq 2n$ in type I but only $m \leq 2n+1$ in type II, and [13, Lemma 7.2] gives $n < m$ in type I only in degree at least two; the sharper statement follows from their comparison of recurrences.

Comparison principle. Let $P > Q > 2$, and let $(U_j)_{j\ge0}$, $(Z_j)_{j\ge0}$ be real sequences with $U_{j+1} = PU_j - U_{j-1}$ and $Z_{j+1} = QZ_j - Z_{j-1}$ for $j \ge 1$, and $Z_j > 0$ for $j \ge 1$. Then $\delta_j = U_j - Z_j$ satisfies

$$\delta_{j+1} - \delta_j = (P-2)\delta_j + (\delta_j - \delta_{j-1}) + (P-Q)Z_j \qquad (j \ge 1).$$

Hence, if $\delta_0 \ge 0$ and $\delta_1 \ge \delta_0$, induction gives $\delta_j \ge 0$ and $\delta_{j+1} > \delta_j$ for all $j \ge 1$; in particular $\delta_j > 0$ for $j \ge 2$, and $\delta_j > 0$ for all $j \ge 1$ if $\delta_1 > 0$.

The subsequences of even and of odd index of $z_j^{(a,c)}$ satisfy the recurrence with coefficient $\beta^2 + \beta^{-2} = 4ac + 2$, those of $z_j^{(b,c)}$ the recurrence with coefficient $\gamma^2 + \gamma^{-2} = 4bc + 2$, and the subsequences of index $\equiv i \pmod 4$ of $z_j^{(a,c)}$ the recurrence with coefficient $(4ac+2)^2 - 2$. Moreover $z \ge 2$, since $z^2 = cd + 1 > 1$.

Type II. Put $V_j = z_{2j+1}^{(a,c)}$ and $W_j = z_{2j+1}^{(b,c)}$. With the seeds $(r, \lambda t)$ and $(r, \lambda s)$, $V_0 = W_0 = \varsigma$, where $\varsigma = cr + \lambda st$ is positive because $(cr)^2 - (st)^2 = c(c-a-b) - 1 > 0$. Further $V_1 = (4ac+1)\varsigma + 2cr < W_1 = (4bc+1)\varsigma + 2cr$.

By induction $V_{j+1} - V_j = 4acV_j + (V_j - V_{j-1}) > 0$, so (V_j) is positive and strictly increasing, and so is (W_j). The comparison principle with $U = W$, $Z = V$ gives $W_j > V_j$ for $j \ge 1$. If $m < n$, then $z = V_m < V_n < W_n = z$, which is absurd; so $m \ge n$. For the upper bound put $C_j = V_{2j} = z_{4j+1}^{(a,c)}$, a sequence with coefficient $(4ac+2)^2 - 2 > 4bc + 2$, $C_0 = W_0$, and

$$C_1 - W_1 = \big((4ac+2)(4ac+1) - 4bc - 2\big)\varsigma + 2(4ac+1)cr > 0,$$

because $16a^2c^2 + 12ac > 4bc$. The comparison principle with $U = C$, $Z = W$ gives $C_j > W_j$ for $j \ge 1$. If $n \ge 1$, then $V_m = z = W_n < C_n = V_{2n}$, so $m < 2n$; if $n = 0$, then $V_m = W_0 = V_0$ and $m = 0$. In both cases $m \le 2n$.

Type I. Put $V_j = z_{2j}^{(a,c)}$ and $W_j = z_{2j}^{(b,c)}$. With the seeds $(1, \lambda)$, $V_0 = W_0 = \lambda$, $V_1 = \lambda(2ac+1) + 2cs$ and $W_1 = \lambda(2bc+1) + 2ct$, so

$$W_1 - V_1 = 2c\big((t-s) + \lambda(b-a)\big).$$

Here $V_1 \ge 2c(s-a) - 1 > 0$ and $V_1 - V_0 = 2c(s + \lambda a) > 0$, as $s > a$; by induction $(V_j)_{j\ge1}$ is positive and $(V_j)_{j\ge0}$ is strictly increasing, and in the same way ($t > b$) so is (W_j). We show $W_1 - V_1 \ge 0$, with strict inequality when $\{a, b, c\}$ is not Euler. For $\lambda = 1$ this is clear. For $\lambda = -1$, $t - s = c(b-a)/(s+t)$ gives $W_1 - V_1 = 2c(b-a)\big(c/(s+t) - 1\big)$. By Lemma 2.4, either $c = a + b + 2r$, and then $s = a + r$, $t = b + r$ by item 5 of Lemma 2.6, so $s + t = c$; or $c > 4ab$, and then $(s+t)^2 \le 2(s^2 + t^2) = 2(a+b)c + 4 < c^2$.

The second case is exactly the case that $\{a, b, c\}$ is not Euler, and then $s + t < c$.

The comparison principle with $U = W$, $Z = V$ (coefficients $4bc + 2 > 4ac + 2$) gives $W_j \ge V_j$ for all $j \ge 0$, and $W_j > V_j$ for all $j \ge 1$ when $\{a, b, c\}$ is not Euler. If $m < n$, then $z = V_m < V_n \le W_n = z$, which is absurd; so $m \ge n$. Since $z \ge 2$ and $V_0 = W_0 = \lambda$, we have $n \ge 1$. If $\{a, b, c\}$ is not Euler and $m \le n$, then $z = V_m \le V_n < W_n = z$, which is absurd; so $n < m$. For the upper bound put $C_j = V_{2j} = z_{4j}^{(a,c)}$, with coefficient $(4ac+2)^2 - 2 > 4bc + 2$, $C_0 = W_0$ and $C_1 = V_2 > W_1$.

The comparison principle gives $C_j > W_j$ for $j \ge 1$; as $n \ge 1$, $V_m = z = W_n < C_n = V_{2n}$, so $m < 2n$. □

Lemma 2.15. *Let $d > d_+(a, b, c)$, and let $l' \ge 0$ be the exponent in the representation*

$$y\sqrt{a} + x\sqrt{b} = (Y_0\sqrt{a} + X_0\sqrt{b})\,\alpha^{l'}$$

of the solution $(X,Y)=(x,y)$ *of* $aY^2-bX^2=a-b$ *given by Lemma 2.12 with* $(P,Q,S)=(a,b,r)$. *Then* $l'>0$ *and* $m'>2$. *In type I,* $m\geq 2$ *and* $n\geq 1$. *In type II, if moreover* $b>4000$ *and* $c>4ab+a+b$, *then* $m,n\geq 4$.

Proof. This is [13, Lemma 4.3]; its type II bound is [4, Lemma 2.6], where it reads $\min\{m',n'\}\geq 9$ with $m'=2m+1$, $n'=2n+1$. □

The third Pell equation, that of the pair $\{a,b\}$, is reached through residues modulo b: both types of representation give a common index l in it (Lemma 2.17), which the analysis of degree one uses. When D is one less than a square, the small values of v^2-Dq^2 are squares times powers of the unit (Lemma 2.18); this serves the arguments on seeds and slots.

Lemma 2.16. *Let* $A<B$ *be positive integers with* $AB+1=R^2$. *Let* (X,Y) *and* (S,T) *be positive integer solutions of* $AY^2-BX^2=A-B$, *and let* $\epsilon\in\{0,1\}$. *If* $Y\equiv R^{\epsilon}T \pmod B$, *then for some integer* q,

$$Y\sqrt{A}+X\sqrt{B}=(T\sqrt{A}+S\sqrt{B})(R+\sqrt{AB})^q.$$

If $\gcd(T,B)=1$, *then* $q\equiv\epsilon \pmod 2$. *If* $Y>T$, *then* $q>0$.

Proof. The bounded reduced seeds make the residue determine the orbit. Put $D=B-A$ and $\alpha=R+\sqrt{AB}$. For a positive solution write $W=Y\sqrt{A}+X\sqrt{B}$. Its norm is $-D$ and $W>\sqrt{D}$. Choose $j\geq 0$ such that

$$\frac{\sqrt{D}}{\sqrt{\alpha}}\leq W_0=W\alpha^{-j}<\sqrt{D}\sqrt{\alpha}.$$

The matrices for multiplication by $\alpha^{\pm 1}$ have integer entries, so $W_0=Y_0\sqrt{A}+X_0\sqrt{B}$ has integer coordinates. Moreover

$$X_0=\frac{W_0+D/W_0}{2\sqrt{B}}>0,\qquad Y_0=\frac{W_0-D/W_0}{2\sqrt{A}},$$
$$|Y_0|^2\leq H^2:=\frac{(R-1)D}{2A}=\frac{BD}{2(R+1)}<\frac{B^2}{4}.$$

The equation excludes $Y_0=0$, since then $0<BX_0^2=D<B$.

Reduce both original solutions, denoting their reduced coordinates by (X_0,Y_0) and (X_1,Y_1). Multiplication by α acts on the Y coordinate modulo B as multiplication by R, and $R^2\equiv 1 \pmod B$. Hence the hypothesis gives either $Y_1\equiv Y_0 \pmod B$ or $Y_1\equiv RY_0 \pmod B$. In the first case $|Y_i|<B/2$ forces $Y_1=Y_0$, and then $X_1=X_0$.

In the second case put

$$k_0=(RY_0-Y_1)/B,\qquad k_1=(RY_1-Y_0)/B.$$

These are integers and satisfy $Rk_0+k_1=AY_0$ and $Rk_1+k_0=AY_1$. The norm equations give the same integer

$$h=X_0^2-k_0^2=X_1^2-k_1^2=\frac{BD-Y_0^2+2RY_0Y_1-Y_1^2}{B^2}.$$

Since $2(R+1)H^2=BD$, one has $0\leq h\leq 2D/B<2$. If $h=1$, integer difference-of-squares factorization gives $X_0=X_1=1$ and $k_0=k_1=0$, whence $AY_0=0$, impossible. Thus $h=0$ and $k_0=\sigma X_0$ for $\sigma\in\{-1,1\}$. Consequently

$$Y_1=RY_0-\sigma BX_0,\qquad X_1=RX_0-\sigma AY_0.$$

For the second equality, the proposed right side has square X_1^2 and is positive, because

$$R^2X_0^2 - A^2Y_0^2 = X_0^2 + AD > 0.$$

Thus the reduced solutions differ by multiplication by $\alpha^{-\sigma}$. Reintroducing their reduction powers proves the asserted common-orbit relation.

For any integer orbit exponent, reduction modulo B gives $Y \equiv R^qT$. If $\gcd(T, B) = 1$, comparison with the hypothesis gives $R^{q-\epsilon} \equiv 1 \pmod{B}$. Since $1 < R < B$ and $R^2 \equiv 1 \pmod{B}$, the parity follows. Finally the positive coordinate $Y = (W - D/W)/(2\sqrt{A})$ is strictly increasing in $W > 0$. Therefore $Y > T$ implies $W(X, Y) > W(S, T)$ and hence $q > 0$. □

Lemma 2.17. *A type I representation of z implies*

$$y\sqrt{a} + x\sqrt{b} = (\sqrt{a} + \sqrt{b})\alpha^{2l}$$

for an integer $l \geq 1$. A type II representation of an irregular quadruple implies

$$y\sqrt{a} + x\sqrt{b} = (t\sqrt{a} + s\sqrt{b})\alpha^{2l+1}$$

for an integer $l \geq 1$.

Proof. The b, c Pell matrix on (z, y) is

$$\begin{pmatrix} t & c \\ b & t \end{pmatrix}.$$

Its even powers act as the identity on the y coordinate modulo b. Thus type I gives $y \equiv 1 \pmod{b}$. Apply Lemma 2.16 to the pair a, b, with the reference solution $(S, T) = (1, 1)$ and $\epsilon = 0$. Since $y > 1$ and $\gcd(1, b) = 1$, the orbit exponent is positive and even, giving the first formula.

For type II the initial y coordinate is r, and an odd matrix power gives $y \equiv rt \pmod{b}$. Apply Lemma 2.16 with the reference solution $(S, T) = (s, t)$ and $\epsilon = 1$. Here $\gcd(t, b) = 1$ follows from $t^2 = bc + 1$. Since $y > t$, the exponent is positive and odd, say $2l + 1$ with $l \geq 0$. For $l = 0$, the coordinates are $y = rt + bs$ and $x = rs + at$, and direct substitution gives $d = d_+(a, b, c)$. Irregularity excludes this case. □

Lemma 2.18. *Let $x \geq 2$ be an integer, $D = x^2 - 1$, and let $v, q \in \mathbb{Z}$ not both vanish. If*

$$|E| < x, \qquad E = v^2 - Dq^2,$$

then $E = r^2 > 0$ for an integer $r \geq 1$, and there is an integer $\ell \geq 0$ such that

$$|v| = rT_\ell(x), \qquad |q| = rU_{\ell-1}(x).$$

Here $T_\ell + U_{\ell-1}\sqrt{x^2 - 1} = (x + \sqrt{x^2 - 1})^\ell$, with $U_{-1} = 0$. In particular r divides both coordinates and $r^2 < x$.

Proof. The integer $D = x^2 - 1$ is nonsquare, so $E \neq 0$. Put $\theta = x + \sqrt{D} > 1$ and $w = |v| + |q|\sqrt{D} > 0$. Choose an integer k such that

$$\sqrt{|E|}\,\theta^{-1/2} \leq w_0 = w\theta^{-k} < \sqrt{|E|}\,\theta^{1/2}.$$

Multiplication by $\theta^{\pm 1}$ is an integral matrix of determinant one on the two coordinates. Thus $w_0 = v_0 + q_0\sqrt{D}$ with integers v_0, q_0 and norm E.

If $E > 0$, then

$$q_0 = \frac{w_0 - E/w_0}{2\sqrt{D}}, \qquad q_0^2 \le \frac{E(\theta + \theta^{-1} - 2)}{4D} = \frac{E}{2(x+1)} < 1.$$

If $E < 0$, the corresponding formula gives

$$q_0^2 \le \frac{|E|(\theta + \theta^{-1} + 2)}{4D} = \frac{|E|}{2(x-1)} < \frac{x}{2(x-1)} \le 1.$$

In both cases $q_0 = 0$. The negative case is therefore impossible, and the positive case gives $w_0 = v_0 = r > 0$ and $E = r^2$. Hence $w = r\theta^k$. Its nonnegative original second coordinate forces $k \ge 0$, so $k = \ell$ gives the asserted Chebyshev representation. The divisibility and size statements follow. □

2.3 Congruences for the Pell sequences

Every index floor of the paper starts from the expansions of the Pell sequences at the common term: to second order modulo c^2 (Lemma 2.19), following Dujella [7] and He and Togbé [13], and, for type I, to third order (Lemma 2.22). The Pell sequences are indexed from 0, and the seeds keep the names (x_1, z_1) and (y_2, z_2) although they are the terms of index 0: their subscripts name the equation, not an index (Section **??**). By Lemma 2.22, the second-order defect k of a type I representation (Definition 2.20) is an integer divisible by 4, and it satisfies a discriminant identity, which the type I floors of Section 7 use.

Lemma 2.19. *For every integer $m \ge 0$,*

$$z_{2m}^{(a,c)} \equiv z_1 + 2cm(sx_1 + amz_1) \pmod{8c^2},$$
$$z_{2m+1}^{(a,c)} \equiv sz_1 + c\big((2m+1)x_1 + 2am(m+1)sz_1\big) \pmod{4c^2},$$

and the same holds for $z^{(b,c)}$ with (a, s, m, x_1, z_1) replaced by (b, t, n, y_2, z_2).

Proof. This is [13, Lemma 4.1], from [6, Lemma 9]. □

Definition 2.20. Let $T = \{a, b, c\}$ be a triple which is not Euler, with descent partner e (Definition 2.8), and let (λ, m, n) be a representation of z, with $\Delta = m - n$.

Type I. If the representation is of type I, its **second-order defect** is

$$k = \frac{bn^2 + \lambda tn - am^2 - \lambda sm}{c}.$$

Type II. If the representation is of type II, put

$$N = 2n(n+1) - \lambda\Delta, \qquad M = 2m(m+1) + \lambda\Delta,$$

and call

$$H = bN - aM + e\lambda\Delta$$

its **first-order defect**. If T is middle or low (Definition 2.8), put $u = \sqrt{ae+1}$; the **first-order quotient** is $J = H/c$, and

$$K = N - J(4u^2 - 2).$$

If T is high and $\deg T \ge 2$, put

$$v = 2af + 1, \qquad L = 2n(n+1) - (4af+1)\Delta,$$

where f is the descent partner of the descent (Definition 2.8).

Lemma 2.21. *Let P, Q, S be positive integers with $PQ+1 = S^2$, put $\eta_{PQ} = S + \sqrt{PQ}$, and let $u_0, v_0 \in \mathbb{Z}$. For $j \geq 0$ let $u_j, v_j \in \mathbb{Z}$ be defined by*

$$u_j\sqrt{P} + v_j\sqrt{Q} = (u_0\sqrt{P} + v_0\sqrt{Q})\,\eta_{PQ}^{j}.$$

1. *The even subsequences satisfy $X_{i+1} = (4PQ+2)X_i - X_{i-1}$ for $i \geq 1$, where $X_i = u_{2i}$ or $X_i = v_{2i}$.*
2. *For every integer $i \geq 0$,*

$$u_{2i} \equiv u_0 + 2Qi\,(Pi\,u_0 + S\,v_0) \pmod{8PQ^2},$$
$$v_{2i} \equiv v_0 + 2Pi\,(Qi\,v_0 + S\,u_0) \pmod{8P^2Q}.$$

Proof. Item 2 is a form of the second-order congruences of Dujella [5] (see also [7]) that is symmetric in the two square roots and covers both coordinates, with the moduli $8PQ^2$ and $8P^2Q$; we give the proof.

Since $PQ = S^2 - 1$ with $S \geq 2$, PQ is not a square, so $\sqrt{P}$ and $\sqrt{Q}$ are linearly independent over $\mathbb{Q}$ and u_j, v_j are well defined; explicitly $u_{j+1} = Su_j + Qv_j$ and $v_{j+1} = Pu_j + Sv_j$.

Item 1. Since $\eta_{PQ}^{-1} = S - \sqrt{PQ}$, one has $\eta_{PQ}^{2} + \eta_{PQ}^{-2} = 2(2PQ+1) = 4PQ+2$, hence $\eta_{PQ}^{2(i+1)} = (4PQ+2)\eta_{PQ}^{2i} - \eta_{PQ}^{2(i-1)}$. Multiplying by $u_0\sqrt{P} + v_0\sqrt{Q}$ and comparing the coefficients of $\sqrt{P}$ and $\sqrt{Q}$ gives item 1.

Item 2. Put $T = PQ$ and write $\eta_{PQ}^{2i} = U_i + K_i\sqrt{T}$ with $U_i, K_i \in \mathbb{Z}$. Expanding $(u_0\sqrt{P} + v_0\sqrt{Q})(U_i + K_i\sqrt{PQ})$ gives

$$u_{2i} = u_0U_i + Qv_0K_i, \qquad v_{2i} = v_0U_i + Pu_0K_i. \tag{1}$$

From $\eta_{PQ}^{2i+2} = \eta_{PQ}^{2i}\big((2T+1) + 2S\sqrt{T}\big)$ one gets $U_{i+1} = (2T+1)U_i + 2STK_i$ and $K_{i+1} = (2T+1)K_i + 2SU_i$, with $U_0 = 1$, $K_0 = 0$. By induction $K_i = SJ_i$ with $J_i \in \mathbb{Z}$, where $J_0 = 0$ and, using $S^2 = T+1$,

$$J_{i+1} = (2T+1)J_i + 2U_i, \qquad U_{i+1} = (2T+1)U_i + 2T(T+1)J_i.$$

We claim that

$$U_i \equiv 1 + 2i^2T \pmod{8T^2}, \qquad J_i \equiv 2i \pmod{8T}. \tag{2}$$

This holds for $i = 0$. If $U_i = 1 + 2i^2T + 8T^2g$ and $J_i = 2i + 8Th$ with $g, h \in \mathbb{Z}$, then

$$J_{i+1} = 2(i+1) + 4i(i+1)T + 8T\,(h + 2Th + 2Tg),$$

$$U_{i+1} = 1 + 2(i+1)^2T + 4i(i+1)T^2 + 8T^2\,(g + 2Tg + 2h + 2Th),$$

and $8 \mid 4i(i+1)$, which proves (2) for $i+1$.

By (1) and (2),

$$u_{2i} - u_0 - 2Qi(Piu_0 + Sv_0) = u_0\big(U_i - 1 - 2i^2PQ\big) + QSv_0\big(J_i - 2i\big),$$

where the first term is divisible by $8T^2 = 8P^2Q^2$ and the second by $8QT = 8PQ^2$. Likewise

$$v_{2i} - v_0 - 2Pi(Qiv_0 + Su_0) = v_0\big(U_i - 1 - 2i^2PQ\big) + PSu_0\big(J_i - 2i\big) \equiv 0 \pmod{8P^2Q}. \qquad \square$$

Lemma 2.22. *Let $\{a,b,c\}$ be a triple that is not Euler, let z have a type I representation (λ,m,n) with second-order defect k, and put*

$$\delta = m^2 - n^2, \qquad \mathcal{A}_3 = a(m^3-m)(2s+\lambda am) - b(n^3-n)(2t+\lambda bn).$$

1. *$k \in 4\mathbb{Z}$.*
2. *$\mathcal{A}_3 \equiv 3\lambda k \pmod{24c}$.*
3. *There is an integer j with*

$$\delta(a^2m^2 + 2\lambda asm + 1 - n^2) = 3kn^2 + jc.$$

4. *The integers $Y = js - \lambda a^2 m\delta$ and $T_\delta = 2a^2m\delta + jam$ satisfy*

$$4Y^2 - T_\delta^2 = 4ja\delta(1-n^2) - 12ajkn^2 + 4j^2 - j^2a^2m^2.$$

Proof. Items 1–3 are steps in the proofs of [13, Lemmas 7.2 and 7.3], derived there for triples of degree at least two; they are needed here for every triple that is not Euler, and item 4 is new. By Definition 2.20, $kc = bn^2 + \lambda tn - am^2 - \lambda sm$, and $z = z_{2m}^{(a,c)} = z_{2n}^{(b,c)}$ with seeds $(x_1,z_1) = (y_2,z_2) = (1,\lambda)$, by the definition of a type I representation.

Item 1. Item 2 of Lemma 2.21 with $(P,Q,S,u_0,v_0) = (a,c,s,\lambda,1)$, for which $\eta_{PQ} = \beta$, and with $(P,Q,S,u_0,v_0) = (b,c,t,\lambda,1)$, for which $\eta_{PQ} = \gamma$, gives

$$z \equiv \lambda + 2cm(\lambda am + s) \equiv \lambda + 2cn(\lambda bn + t) \pmod{8c^2},$$

which is the even case of Lemma 2.19. Hence $4c$ divides $m(\lambda am+s) - n(\lambda bn+t) = -\lambda kc$, so $k \in 4\mathbb{Z}$.

Item 2. For $q \geq 0$ put

$$F_a(q) = \lambda + 2c(\lambda aq^2 + sq) + \tfrac{2}{3}ac^2(q^3-q)(2s+\lambda aq),$$

an integer since $3 \mid q^3 - q$, and define F_b with (b,t) in place of (a,s). Then $F_a(0) = \lambda = z_0^{(a,c)}$ and $F_a(1) = \lambda(2ac+1) + 2cs = z_2^{(a,c)}$, and, using $s^2 = ac+1$,

$$F_a(q+2) - (4ac+2)F_a(q+1) + F_a(q) = -\tfrac{8}{3}a^2c^3\,q(q+1)(q+2)\big(2s+\lambda a(q+1)\big) \equiv 0 \pmod{16c^3},$$

because $6 \mid q(q+1)(q+2)$. By item 1 of Lemma 2.21, $q \mapsto z_{2q}^{(a,c)}$ satisfies the recurrence $X_{q+2} = (4ac+2)X_{q+1} - X_q$, so induction on q gives $z_{2q}^{(a,c)} \equiv F_a(q) \pmod{16c^3}$, and likewise $z_{2q}^{(b,c)} \equiv F_b(q) \pmod{16c^3}$. Since $z_{2m}^{(a,c)} = z_{2n}^{(b,c)}$,

$$0 \equiv F_a(m) - F_b(n) = 2c\big(\lambda(am^2 - bn^2) + sm - tn\big) + \tfrac{2}{3}c^2\mathcal{A}_3 = -2\lambda kc^2 + \tfrac{2}{3}c^2\mathcal{A}_3 \pmod{16c^3}.$$

As $3 \mid \mathcal{A}_3$, dividing by $2c^2$ gives $\mathcal{A}_3/3 - \lambda k \equiv 0 \pmod{8c}$, that is item 2.

Item 3. Put $P_a = am + \lambda s$ and $P_b = bn + \lambda t$. Then $nP_b - mP_a = kc$, so $mP_a \equiv nP_b \pmod c$. Since $s^2 \equiv t^2 \equiv 1 \pmod c$, we have $P_a^2 \equiv a^2m^2 + 2\lambda asm + 1$ and

$$\lambda\mathcal{A}_3 = am(m^2-1)(am+2\lambda s) - bn(n^2-1)(bn+2\lambda t) \equiv (m^2-1)P_a^2 - (n^2-1)P_b^2 - \delta \pmod c.$$

Multiplying by n^2 and using $n^2P_b^2 \equiv m^2P_a^2$ gives $n^2\lambda\mathcal{A}_3 \equiv \delta(P_a^2 - n^2) \pmod c$. By item 2, $\lambda\mathcal{A}_3 \equiv 3k \pmod c$. Hence

$$3kn^2 \equiv \delta(a^2m^2 + 2\lambda asm + 1 - n^2) \pmod c,$$

which defines the integer j of item 3.

Item 4. Expanding,

$$4Y^2 - T_\delta^2 - \big(4ja\delta(1-n^2) - 12ajkn^2 + 4j^2 - j^2a^2m^2\big) \\ = 4j\Big(j(s^2-1) - a\big(\delta(a^2m^2 + 2\lambda asm + 1 - n^2) - 3kn^2\big)\Big),$$

and the right side vanishes, since $s^2-1 = ac$ and, by item 3, $a\big(\delta(a^2m^2+2\lambda asm+1-n^2)-3kn^2\big) = ajc$. □

The first consequence is the type I floor of He and Togbé in degree at least two, the floor of the generic type I window.

Lemma 2.23. *Let the quadruple $\{a,b,c,d\}$ be irregular, $\deg\{a,b,c\} \geq 2$, and let z have a type I representation. Then, with $\kappa = (\sqrt{17}-1)/2$,*

$$m > n > \kappa\sqrt{c/b}.$$

Proof. This is [13, Lemma 7.2]. □

3 Chains of pairs and the position of the descent partner

The descent of a triple that is not Euler keeps two of its elements, and the kept pairs give the second invariant of the paper: following them from the first descent on, a triple either keeps one pair, and then lies on the chain of that pair, or changes pair, and is bent. The chains are those of Jones [14] and of Cipu, Fujita and Miyazaki [4]. By the classification Lemma 3.4, the descent partner has exactly one position, and a single-chain triple of degree $\nu - 1$ is one of the three triples at slot ν of the chain of its chain pair.

Definition 3.1. Let T be a Diophantine triple of degree $\deg T \geq 1$, with iterated descents $\partial^i T$, $0 \leq i \leq \deg T$ (Definition 2.8).

Kept pairs. For $0 \leq i \leq \deg T$ let $\pi_i(T)$ be the set of the two smaller elements of $\partial^i T$. For $i < \deg T$ it is the pair that the descent of $\partial^i T$ keeps: $\partial^{i+1}T$ consists of $\pi_i(T)$ and the descent partner of $\partial^i T$ (Definition 2.8).

Chain pair. The **chain pair** of T is $\pi_1(T)$, the set of the two smaller elements of the descent ∂T.

Single-chain and bent. The triple T is **single-chain** if

$$\pi_1(T) = \pi_2(T) = \cdots = \pi_{\deg T - 1}(T),$$

a condition that is empty when $\deg T \leq 2$. Otherwise T is **bent**.

Definition 3.2. Let $p \neq q$ be positive integers with $pq + 1 = \varrho^2$, ϱ a positive integer.

The two-sided sequences. For $\sigma \in \{-1, 1\}$ let $(x_j)_{j\in\mathbb{Z}} = (x_j(p,q;\sigma))_{j\in\mathbb{Z}}$ be the sequence with

$$x_0 = 0, \qquad x_1 = p + q - 2\sigma\varrho, \qquad x_{-1} = p + q + 2\sigma\varrho,$$
$$x_{j+1} + x_{j-1} = (4pq + 2)\, x_j + 2(p + q) \qquad (j \in \mathbb{Z},\ j \neq 0);$$

the relation determines x_{j+1} for $j \geq 1$ and x_{j-1} for $j \leq -1$.

Admitted signs and chains. The sign σ is **admitted** for $\{p,q\}$ if $x_1(p,q;\sigma)\geq 1$. For an admitted sign σ, the **chain** of the pair $\{p,q\}$ with sign σ is the sequence $(x_j(p,q;\sigma))_{j\in\mathbb{Z}}$. A sign that is not admitted defines no chain.

Slots. Let $\nu\geq 2$. A Diophantine triple T (Definition 2.1) **lies at slot** ν of the chain if

$$T=\{p,q,x_\nu\},\qquad T=\{p,x_{\nu-1},x_\nu\}\qquad\text{or}\qquad T=\{q,x_{\nu-1},x_\nu\}.$$

Chain sign. Let T be a single-chain triple (Definition 3.1) with chain pair $\pi_1(T)=\{p,q\}$; as p and q belong to the Diophantine triple ∂T (Definition 2.8), $pq+1$ is a square. A **chain sign** of T is an admitted sign σ of $\{p,q\}$ such that T lies at slot $\deg T+1$ of the chain of $\{p,q\}$ with sign σ.

For $j\geq 0$ the numbers $x_j(p,q;\sigma)$ are the numbers $c_j^{-\sigma}$ that Cipu, Fujita and Miyazaki [4, Proposition 1.5] attach to the pair $\{p,q\}$; for $\sigma=-1$ the chain is the sequence of Jones [14, (2.8)].

Lemma 3.3. *Let $p\neq q$ be positive integers with $pq+1=\varrho^2$, ϱ a positive integer, and let $x_j(\sigma)=x_j(p,q;\sigma)$ ($j\in\mathbb{Z}$, $\sigma\in\{-1,1\}$) be the two-sided sequences of the pair $\{p,q\}$; $d_\pm(p,q,X)$ is written as in Lemma 2.6.*

1. *For both signs and every $j\in\mathbb{Z}$,*

$$x_{j+1}(\sigma)+x_{j-1}(\sigma)=(4pq+2)\,x_j(\sigma)+2(p+q),$$

$$4pq\,x_j(\sigma)=\bigl(\sqrt{p}-\sigma\sqrt{q}\bigr)^2\bigl(\varrho+\sqrt{pq}\bigr)^{2j}+\bigl(\sqrt{p}+\sigma\sqrt{q}\bigr)^2\bigl(\varrho+\sqrt{pq}\bigr)^{-2j}-2(p+q),$$

   and $x_{-j}(\sigma)=x_j(-\sigma)$.

2. *$x_{\pm1}(\sigma)=p+q\mp 2\sigma\varrho=d_{\mp\sigma}(p,q,0)$. Both numbers are non-negative; $x_1(\sigma)=0$ exactly when $\sigma=1$ and $|p-q|=2$, and $x_{-1}(\sigma)=0$ exactly when $\sigma=-1$ and $|p-q|=2$. In particular the sign -1 is always admitted, and the sign 1 is admitted unless $|p-q|=2$.*

3. *Let σ be admitted and $x_j=x_j(\sigma)$. Then $0=x_0<x_1<x_2<\cdots$, $x_1\notin\{p,q\}$ and $x_2>\max\{p,q\}$. For $j\geq 1$ the sets $\{p,q,x_j\}$, $\{p,x_j,x_{j+1}\}$ and $\{q,x_j,x_{j+1}\}$ are Diophantine triples, $x_{j+1}=d_+(p,q,x_j)$ and $x_{j-1}=d_-(p,q,x_j)$. The triple $\{p,q,x_1\}$ is an Euler triple.*

4. *Let σ be admitted, $x_j=x_j(\sigma)$ and $\nu\geq 2$. Each of the sets $\{p,q,x_\nu\}$, $\{p,x_{\nu-1},x_\nu\}$ and $\{q,x_{\nu-1},x_\nu\}$ is a Diophantine triple which is not Euler, is single-chain, has degree $\nu-1$, has descent partner $x_{\nu-1}$, q, p respectively, and has $(\nu-1)$-fold descent $\{p,q,x_1\}$. If $\nu\geq 3$, each of them has chain pair $\{p,q\}$ and chain sign σ, the triple $\{p,q,x_\nu\}$ is high, and $\{p,x_{\nu-1},x_\nu\}$ is middle if $p<q$ and low if $p>q$.*

Proof. Item 1. Put $\eta=\varrho+\sqrt{pq}$, so $\eta^{-1}=\varrho-\sqrt{pq}$ and $\eta^{\pm2}=2pq+1\pm2\varrho\sqrt{pq}$, hence $\eta^2+\eta^{-2}=4pq+2$. With $A=p+q$ and $B=2\sigma\sqrt{pq}$, so that $(\sqrt{p}\mp\sigma\sqrt{q})^2=A\mp B$, let

$$\tilde{x}_j=\frac{(A-B)\eta^{2j}+(A+B)\eta^{-2j}-2A}{4pq}\qquad(j\in\mathbb{Z}).$$

Then $\tilde{x}_0=0$ and $\tilde{x}_{\pm1}=A\mp2\sigma\varrho$, which are x_0 and $x_{\pm1}(\sigma)$.

Since $\eta^{2(j+1)}+\eta^{2(j-1)}=(4pq+2)\eta^{2j}$ and likewise for η^{-2j}, we get $\tilde{x}_{j+1}+\tilde{x}_{j-1}=(4pq+2)\tilde{x}_j+2A$ for every $j\in\mathbb{Z}$.

The sequence $x_j(\sigma)$ is determined by x_0, $x_{\pm1}(\sigma)$ and the relation for $j\neq 0$ (Definition 3.2), so induction on $j\geq 1$ and on $j\leq -1$ gives $x_j(\sigma)=\tilde{x}_j$ for all j. This is the closed form, and the

relation holds at $j = 0$ as well. Replacing σ by $-\sigma$ exchanges $A - B$ and $A + B$, which in the closed form amounts to replacing j by $-j$; hence $x_{-j}(\sigma) = x_j(-\sigma)$.

Item 2. By Definition 2.2 at $X = 0$, $d_\pm(p, q, 0) = p + q \pm 2\varrho$, which gives the first assertion. Since $p(p + 1) + 1$ lies strictly between p^2 and $(p + 1)^2$, $|p - q| \neq 1$; so $|p - q| \geq 2$ and $(p + q)^2 - 4\varrho^2 = (p - q)^2 - 4 \geq 0$. Thus $p + q \geq 2\varrho$, with equality exactly when $|p - q| = 2$, and both $p + q \pm 2\varrho$ are non-negative, $p + q - 2\varrho$ vanishing exactly when $|p - q| = 2$. The statements on admitted signs follow from the definition.

Item 3. Let σ be admitted, so $x_1 \geq 1$. If $x_1 = p$, then $q = 2\sigma\varrho$ and $q^2 = 4pq + 4$, so $q(q - 4p) = 4$, whence $q \mid 4$ and $q > 4p \geq 4$, which is absurd; likewise $x_1 \neq q$. By item 1, $x_{j+1} - x_j = (4pq + 1)x_j - x_{j-1} + 2(p + q) > 0$ whenever $x_j \geq x_{j-1} \geq 0$, so by induction $0 = x_0 < x_1 < x_2 < \cdots$, and $x_2 = (4pq + 2)x_1 + 2(p + q) > \max\{p, q\}$.

Next, $\mathcal{R}(p, q, x_j, x_{j+1}) = 0$ for all $j \geq 0$ (Definition 2.5). For $j = 0$ this says, by item 1 of Lemma 2.6 at $X = 0$, that x_1 is one of $d_\pm(p, q, 0)$, which is item 2. If $\mathcal{R}(p, q, x_{j-1}, x_j) = 0$ with $j \geq 1$, then by symmetry x_{j-1} is a root of the equation of item 1 of Lemma 2.6 at $X = x_j$, and by Vieta the other root is $2(p+q+x_j+2pqx_j)-x_{j-1} = x_{j+1}$; so $\mathcal{R}(p, q, x_j, x_{j+1}) = 0$. Consequently, for $j \geq 1$ the two roots of that equation at $X = x_j$ are $x_{j-1} < x_{j+1}$, so $x_{j+1} = d_+(p, q, x_j)$ and $x_{j-1} = d_-(p, q, x_j)$.

The numbers px_j+1 and qx_j+1 are squares of integers for every $j \geq 0$: for $j = 0$ trivially, for $j = 1$ by item 5 of Lemma 2.6 ($px_1+1 = (\varrho-\sigma p)^2$, $qx_1+1 = (\varrho-\sigma q)^2$), and for $j \geq 1$ the case $j+1$ follows from the case j by item 2 of Lemma 2.6, as $x_{j+1} = d_+(p, q, x_j)$. By item 2 of Lemma 2.6 again, $x_jx_{j+1} + 1$ is the square of an integer for $j \geq 1$. Together with $x_1 \notin \{p, q\}$ and the monotonicity, this shows that $\{p, q, x_j\}$, $\{p, x_j, x_{j+1}\}$ and $\{q, x_j, x_{j+1}\}$ ($j \geq 1$) are Diophantine triples. Finally, write $\{p, q, x_1\}$ as $p' < q' < w$. Since $\mathcal{R}(p, q, x_1, 0) = 0$ and $\mathcal{R}$ is symmetric, $\mathcal{R}(p', q', w, 0) = 0$, so 0 is one of $d_\pm(p', q', w)$ by item 1 of Lemma 2.6; as $d_+(p', q', w) > 0$, $d_-(p', q', w) = 0$, and the triple is Euler by item 4 of Lemma 2.6.

Item 4. Let $\nu \geq 2$; all sets below are Diophantine triples by item 3. Consider first $S_\mu = \{p, q, x_\mu\}$ for $\mu \geq 2$. For $0 \leq i \leq \mu - 2$, the triple $\{p, q, x_{\mu-i}\}$ has largest element $x_{\mu-i} \geq x_2$ and $d_-(p, q, x_{\mu-i}) = x_{\mu-i-1} \geq x_1 \geq 1$; by item 4 of Lemma 2.6 it is not Euler, its descent is $\{p, q, x_{\mu-i-1}\}$ and its descent partner is $x_{\mu-i-1}$, and its two smaller elements are p and q. As $\{p, q, x_1\}$ is Euler, the i-fold descent of S_μ is $\{p, q, x_{\mu-i}\}$ for $0 \leq i \leq \mu - 1$, $\deg S_\mu = \mu - 1$, the descent partner of S_μ is $x_{\mu-1}$, its $(\mu - 1)$-fold descent is $\{p, q, x_1\}$, and $\pi_i(S_\mu) = \{p, q\}$ for $0 \leq i \leq \mu - 2$; in particular S_μ is single-chain. This is the claim for $\{p, q, x_\nu\}$. If $\nu \geq 3$, its descent partner $x_{\nu-1} \geq x_2$ exceeds its middle element $\max\{p, q\}$, so it is high, and its chain pair is $\{p, q\}$.

Consider next $L = \{p, x_{\nu-1}, x_\nu\}$; the set $\{q, x_{\nu-1}, x_\nu\}$ is treated in the same way with p and q exchanged. Its largest element is x_ν, since $x_\nu > x_{\nu-1}$ and $x_\nu \geq x_2 > p$. As $\mathcal{R}(p, q, x_{\nu-1}, x_\nu) = 0$, symmetry and item 1 of Lemma 2.6 for the pair $\{p, x_{\nu-1}\}$ at $X = x_\nu$ show that q is $d_+(p, x_{\nu-1}, x_\nu)$ or $d_-(p, x_{\nu-1}, x_\nu)$; since $d_+(p, x_{\nu-1}, x_\nu) > x_\nu > q$ (item 3 of Lemma 2.6), $q = d_-(p, x_{\nu-1}, x_\nu)$. So L is not Euler by item 4 of Lemma 2.6, its descent partner is q, and its descent is $\{p, q, x_{\nu-1}\}$. If $\nu = 2$, this is the Euler triple $\{p, q, x_1\}$ and $\deg L = 1$; triples of degree one are single-chain because the defining condition is empty. If $\nu \geq 3$, the descent is $S_{\nu-1}$, so $\deg L = \nu - 1$, the $(\nu - 1)$-fold descent of L is $\{p, q, x_1\}$, and $\pi_i(L) = \pi_{i-1}(S_{\nu-1}) = \{p, q\}$ for $1 \leq i \leq \nu - 2$; thus L is single-chain with chain pair $\{p, q\}$. Moreover $p < x_2 \leq x_{\nu-1}$, so the ordered triple is $p < x_{\nu-1} < x_\nu$, with descent partner $q \notin \{p, x_{\nu-1}\}$ and $q < x_{\nu-1}$: it is middle if $p < q$ and low if $p > q$.

Finally, for $\nu \geq 3$ each of the three triples is single-chain of degree $\nu - 1$ with chain pair $\{p, q\}$ and lies at slot ν of the chain of $\{p, q\}$ with the admitted sign σ, so σ is a chain sign of it. □

By Lemma 3.3, the chain of a pair is a sequence of regular extensions, $x_{j+1} = d_+(p, q, x_j)$, starting from the Euler triple $\{p, q, x_1\}$, and descent runs it backwards. This gives the classification.

Lemma 3.4. *Let $T = \{a, b, c\}$ with $a < b < c$ be a Diophantine triple which is not an Euler triple, with descent partner e, iterated descents $\partial^i T$, position and kept pairs $\pi_i(T)$; $d_\pm$ of an unordered pair is written as in Lemma 2.6.*

(i) e is a positive integer, $e \notin \{a, b\}$, $e < c$, $\partial T = \{a, b, e\}$ is a Diophantine triple, and $d_+(a, b, e) = c$.

(ii) Exactly one of the positions high, middle, low occurs. If T is high, then $\pi_1(T) = \pi_0(T) = \{a, b\}$. If T is middle or low, then b is the largest element of ∂T, $\pi_1(T) = \{a, e\} \neq \pi_0(T)$ and $c = d_+(a, e, b)$. If $\deg T \geq 2$, the descent partner of the descent is $f = d_-(a, b, e)$ if T is high and $f = d_-(a, e, b)$ if T is middle or low.

(iii) Let T be single-chain, with chain pair $\pi_1(T) = \{p, q\}$, $\varrho = \sqrt{pq + 1}$ and $\nu = \deg T + 1$. Then T has exactly one chain sign σ. With (x_j) the chain of $\{p, q\}$ with sign σ, $x_1 = p + q - 2\sigma\varrho$ is the element of the Euler triple $\partial^{\deg T} T$ outside $\{p, q\}$, so $\partial^{\deg T} T = \{p, q, x_1\}$, and:

- *if T is high, then $T = \{a, b, x_\nu\}$ and $e = x_{\nu-1}$, on the chain of $\{a, b\}$;*
- *if T is middle or low, then $T = \{a, x_{\nu-1}, x_\nu\}$, on the chain of $\{a, e\}$.*

(iv) T is single-chain if and only if it lies at a slot of some chain, and then it lies only at slot $\deg T + 1$. Let $\nu \geq 3$, and consider the data $(W, \{p, q\}, T')$ in which W is an Euler triple, $\{p, q\} \subset W$, and T' is one of the three sets at slot ν of the chain of $\{p, q\}$ whose term x_1 is the element of W outside $\{p, q\}$; this chain exists and is unique. The map $(W, \{p, q\}, T') \mapsto T'$ is a bijection from the set of these data onto the set of single-chain triples of degree $\nu - 1$.

(v) Every bent triple has degree at least 3.

Proof. The proof rests on the symmetry of $\mathcal{R}$ (Definition 2.5) and on item 1 of Lemma 2.6, by which $\mathcal{R}(p, q, X, Y) = 0$ exactly when Y is one of $d_\pm(p, q, X)$. If the descent keeps the pair $\{p, q\}$ twice in a row, then the third elements before and after the two steps are the two roots of that quadratic equation at X equal to the middle one, so by Vieta's formula they obey the recursion of the chains (Definition 3.2). Followed down to the Euler triple, the descents of a single-chain triple are therefore a chain run backwards; this is item (iii).

Item (i). By Theorem 2.7 applied to $a < b < c$, $e = d_-(a, b, c)$ is a positive integer distinct from a and b, $\{a, b, e\}$ is a Diophantine triple, $d_+(a, b, e) = c$, and $abe < abc/12$, so $e < c$.

Item (ii). By (i), $e \notin \{a, b\}$ and $1 \leq e < c$, so exactly one of $e < a$, $a < e < b$, $b < e < c$ holds. If T is high, then $\partial T = \{a, b, e\}$ with $a < b < e$, so $\pi_1(T) = \{a, b\} = \pi_0(T)$. If T is middle or low, then $b > \max\{a, e\}$ is the largest element of ∂T, so $\pi_1(T) = \{a, e\} \neq \{a, b\} = \pi_0(T)$. Since $e = d_-(a, b, c)$, $\mathcal{R}(a, b, c, e) = 0$, hence by symmetry $\mathcal{R}(a, e, b, c) = 0$, and c is one of $d_\pm(a, e, b)$. If ∂T is not Euler, then $d_-(a, e, b) < b < c$ by Theorem 2.7; if ∂T is Euler, then $d_-(a, e, b) = 0$ by item 4 of Lemma 2.6. In both cases $c = d_+(a, e, b)$. If $\deg T \geq 2$, then ∂T is not Euler and $f = e(\partial T)$ is d_- of its two smaller elements and its largest element (Definition 2.8); this is $d_-(a, b, e)$ if T is high and $d_-(a, e, b)$ if T is middle or low.

Item (iii). Let T be single-chain, so that $\deg T = \nu - 1$, and $\{p, q\} = \pi_1(T)$; then $pq + 1 = \varrho^2$ because $\{p, q\} \subset \partial T$. For $1 \leq i \leq \nu - 2$, $\partial^{i+1} T$ consists of $\pi_i(T) = \{p, q\}$ and the descent

partner of $\partial^i T$ (Definition 3.1). Hence $\partial^i T = \{p, q, g_i\}$ for $1 \le i \le \nu - 1$, with $g_i \notin \{p, q\}$; for $1 \le i \le \nu - 2$, g_i is the largest element of $\partial^i T$ (as $\{p, q\} = \pi_i(T)$ consists of its two smaller elements) and $g_{i+1} = d_-(p, q, g_i)$, so $g_{i+1} < g_i$ by Theorem 2.7 and $\mathcal{R}(p, q, g_i, g_{i+1}) = 0$. As $\partial^{\nu-1} T$ is Euler, item 4 of Lemma 2.6 gives $d_- = 0$ for its ordered elements, so $\mathcal{R}(p, q, g_{\nu-1}, 0) = 0$ by symmetry, that is, $g_{\nu-1}$ is one of $d_\pm(p, q, 0) = p+q \pm 2\varrho$. Thus $g_{\nu-1} = p+q-2\sigma\varrho = x_1(p, q; \sigma)$ for exactly one sign σ (the two values differ as $\varrho > 0$), and σ is admitted because $g_{\nu-1} \ge 1$.

Let (x_j) be the chain of $\{p, q\}$ with this sign, and put $\tilde{x}_0 = 0$ and $\tilde{x}_j = g_{\nu-j}$ for $1 \le j \le \nu-1$. Then $0 = \tilde{x}_0 < \tilde{x}_1 < \cdots < \tilde{x}_{\nu-1}$ and $\mathcal{R}(p, q, \tilde{x}_j, \tilde{x}_{j+1}) = 0$ for $0 \le j \le \nu - 2$. For $1 \le j \le \nu - 2$, $\tilde{x}_{j-1}$ and $\tilde{x}_{j+1}$ are two distinct roots of the equation of item 1 of Lemma 2.6 at $X = \tilde{x}_j$, so by Vieta $\tilde{x}_{j+1} = (4pq + 2)\tilde{x}_j - \tilde{x}_{j-1} + 2(p + q)$. Since $\tilde{x}_0 = x_0$ and $\tilde{x}_1 = g_{\nu-1} = x_1$, induction gives $g_{\nu-j} = x_j$ for $1 \le j \le \nu - 1$; in particular $g_1 = x_{\nu-1}$, and $\partial^{\nu-1} T = \{p, q, x_1\}$.

If T is high, then $\{p, q\} = \{a, b\}$ and $e = g_1 = x_{\nu-1}$; by (i), $c = d_+(a, b, x_{\nu-1})$, which is x_ν by item 3 of Lemma 3.3. Hence $T = \{a, b, x_\nu\}$ and $e = x_{\nu-1}$. If T is middle or low, then $\{p, q\} = \{a, e\}$ and $b = g_1 = x_{\nu-1}$. By (ii), $\mathcal{R}(a, e, b, c) = 0$, so c is a root of the equation of item 1 of Lemma 2.6 for the pair $\{a, e\}$ at $X = x_{\nu-1}$; these roots are $x_{\nu-2}$ and x_ν by item 3 of Lemma 3.3, and $c > b = x_{\nu-1} > x_{\nu-2}$, so $c = x_\nu$ and $T = \{a, x_{\nu-1}, x_\nu\}$. In both cases T lies at slot ν of the chain of $\{p, q\}$ with the admitted sign σ, so σ is a chain sign of T.

It is the only one. If σ' is a chain sign of T, then T lies at slot ν of the chain of $\{p, q\}$ with sign σ', so by item 4 of Lemma 3.3 its $(\nu - 1)$-fold descent is $\{p, q, x_1(p, q; \sigma')\}$. This descent is $\partial^{\nu-1} T = \{p, q, x_1(p, q; \sigma)\}$, so $x_1(p, q; \sigma') = x_1(p, q; \sigma)$, that is, $\sigma' = \sigma$.

Item (iv). A triple at slot ν of a chain is single-chain of degree $\nu - 1$ by item 4 of Lemma 3.3; conversely, a single-chain triple lies at slot $\deg T + 1$ of the chain of its chain pair by (iii). So a triple is single-chain if and only if it lies at some slot, and every slot at which it lies equals $\deg T + 1$.

For the bijection, admitted chains correspond to pairs in Euler triples. If (x_j) is the chain of $\{p, q\}$ with an admitted sign, then $\{p, q, x_1\}$ is Euler by item 3 of Lemma 3.3. Conversely, if $\{p, q\}$ is contained in an Euler triple with third element w, then $d_- = 0$ for its ordered elements (item 4 of Lemma 2.6), so $\mathcal{R}(p, q, w, 0) = 0$ and $w = p + q - 2\sigma\varrho$ for exactly one sign, which is admitted as $w \ge 1$; so the chain in the statement exists and is unique. The map is therefore well defined, and its values are single-chain triples of degree $\nu - 1$ by item 4 of Lemma 3.3. It is surjective by (iii): a single-chain triple T of degree $\nu - 1 \ge 2$ is one of the three sets at slot ν of the chain of $\pi_1(T)$ whose x_1 is the third element of the Euler triple $\partial^{\nu-1} T$. It is injective because the data are recovered from the value T: by item 4 of Lemma 3.3 (with $\nu \ge 3$), the pair is $\pi_1(T)$ and the Euler triple is $\partial^{\nu-1} T$, and the set is T. The bijection refines the observation of He, Togbé and Ziegler [11] that the triples of a given degree generated by an Euler triple form a ternary tree.

Item (v). If T is bent, then $\pi_i(T) \ne \pi_j(T)$ for some $1 \le i < j \le \deg T - 1$, so $\deg T - 1 \ge 2$. □

In degree at least two the position can be read off by comparing c with b^2: the triple is middle or low exactly when $c < b^2$. The class bounds of Section 7 treat the two sides of b^2 by different means.

Lemma 3.5. *Let $T = \{a, b, c\}$ with $a < b < c$ be a Diophantine triple of degree at least two, with descent partner e. If $e > b$, then $c > b^2$. If $e < b$, then $c < b^2$; more precisely, the descent partner of the descent $f = e(\partial T)$ satisfies*

$$f \ge 1, \qquad b > 4aef + a + e + f, \qquad c = (4ae + 2)\, b + 2(a + e) - f < b^2.$$

In particular T is middle or low if and only if $c < b^2$.

Proof. By Lemma 3.4, item (i), e is a positive integer, $e \notin \{a, b\}$, $\{a, b, e\}$ is a Diophantine triple and $c = d_+(a, b, e)$.

If $e > b$, then $c = d_+(a, b, e) > a + b + e + 4abe > 4ab \cdot b \geq b^2$ by item 3 of Lemma 2.6.

Let $e < b$. By Lemma 3.4, item (ii), b is the largest element of the descent $\partial T = \{a, b, e\}$, $c = d_+(a, e, b)$ and $f = d_-(a, e, b)$, where here and below the pair $\{a, e\}$ is written in increasing order. Since $\deg T \geq 2$, ∂T is not Euler, so by Theorem 2.7 applied to ∂T the number f is a positive integer, $\{a, e, f\}$ is a Diophantine triple and $d_+(a, e, f) = b$. Hence, by item 3 of Lemma 2.6,

$$b = d_+(a, e, f) > a + e + f + 4aef \geq 4ae + a + e + 1.$$

Adding the two values $d_\pm(a, e, b)$ in Definition 2.2 gives $c + f = 2\big(a + e + (2ae + 1)b\big)$, that is

$$c = (4ae + 2)\, b + 2(a + e) - f < (4ae + 2)\, b + 2(a + e).$$

Finally $b^2 > (4ae+a+e+1)\, b = (4ae+2)\, b+(a+e-1)\, b$, and $(a+e-1)\, b \geq 5(a+e-1) \geq 2(a+e)$ because $b > 4ae + a + e + 1 \geq 4$ and $a + e \geq 3$ ($a \neq e$). So $c < b^2$.

Since $e \neq b$, exactly one of $e < b$, $e > b$ holds, and the last sentence follows from the definition of the positions (Definition 2.8). □

A triple of degree one is a child of an Euler triple: it lies at slot two of the chain of a pair with sign -1 (Lemma 3.7), and its arithmetic is explicit. For example, $\{1, 8, 120\}$ has descent partner $e = 3$, descent $\{1, 3, 8\}$, and the sign ε of Definition 3.6 is -1, as the triple is middle. One residue Θ modulo r^2 governs the type I indices of such a triple; the line of the triple, defined next, is the only place where they can be smaller than $r^2/12$ (Corollary 6.7).

Definition 3.6. Let $T = \{a, b, c\}$ with $a < b < c$ be a Diophantine triple of degree one, with descent partner e (Definition 2.8) and $r = \sqrt{ab + 1}$.

Sign and base point. Put

$$\varepsilon = \frac{e - a - b}{2r}, \qquad m_0 = b + \varepsilon r, \qquad n_0 = r + \varepsilon a,$$

and let $\varrho = \sqrt{pq + 1}$, where $\{p, q\} = \pi_1(T)$ is the chain pair of T (Definition 3.1).

Residue. The **residue** of T is the integer Θ with $-r^2/2 < \Theta \leq r^2/2$ and

$$\Theta \equiv 3(b - a) + 4\varepsilon r(b^2 - a^2) \pmod{r^2};$$

here $4\varepsilon r(b^2 - a^2) = 2(e - a - b)(b^2 - a^2)$ is an integer.

The line. The **line** of T is the set of pairs of integers (m, n) with

$$m - \varepsilon n = \frac{|\Theta|}{3}.$$

Floors. Put

$$\ell_\varepsilon = \begin{cases} 4\kappa\varrho = 2(\sqrt{17} - 1)\, \varrho & \text{if } \varepsilon = 1, \\ 4.34\, \varrho & \text{if } \varepsilon = -1, \end{cases}$$

where $\kappa = (\sqrt{17} - 1)/2$.

Lemma 3.7. *Let $T = \{a, b, c\}$ with $a < b < c$ be a Diophantine triple of degree one, with descent partner e, and let ε, m_0, n_0, ϱ and Θ be as in Definition 3.6.*

1. *$\varepsilon \in \{-1, 1\}$ and $e = a+b+2\varepsilon r$, and the descent $\partial T = \{a, b, e\}$ is an Euler triple. Moreover $\varepsilon = 1$ if T is high, and $\varepsilon = -1$ if T is middle or low.*

2. *m_0 and n_0 are positive integers with*

$$ae+1=n_0^2,\qquad be+1=m_0^2,\qquad e=m_0+\varepsilon n_0,\qquad m_0-\varepsilon n_0=b-a,$$

$$rn_0-am_0=1,\qquad rm_0-bn_0=\varepsilon,\qquad c=4rm_0n_0,$$

$$s=2rn_0-1,\qquad t=2rm_0-\varepsilon,\qquad t\sqrt{a}+s\sqrt{b}=(\varepsilon\sqrt{a}+\sqrt{b})\,\alpha^2;$$

moreover $4\mid c$, and r, m_0, n_0 are pairwise coprime.

3. *Write ∂T as $p<q<w$. Then $\pi_1(T)=\{p,q\}$, $\varrho=\sqrt{pq+1}\geq 2$ and $w=p+q+2\varrho$; $\varrho=r$ if $\varepsilon=1$, and $\varrho=n_0=r-a$ if $\varepsilon=-1$. The triple T is one of $\{p,q,c\}$, $\{p,w,c\}$, $\{q,w,c\}$; it lies at slot 2 of the chain of $\{p,q\}$ with sign -1, and -1 is its chain sign. Finally*

$$c=4\varrho(\varrho+p)(\varrho+q)\geq 16\varrho^3-4\varrho.$$

4. *$c>4ab+a+b$, $b^2<c<\min\{16b^3,32ab^2\}$, and $c<4(\varrho+1)^4$.*

5. *$\Theta\equiv(b-a)(3+4\varepsilon er)\pmod{r^2}$.*

Proof. Item 1. By Theorem 2.7, $c=d_+(a,b,e)$, and by [13, Proposition 2.4] the descent $\{a,b,e\}$ is Euler and $e=a+b\pm 2r$. As $a<r$, $a+b-2r<b<a+b+2r$; so $\varepsilon=1$ if $e>b$, that is, if T is high, and $\varepsilon=-1$ if $e<b$, that is, if T is middle or low.

Item 2: the first six identities. As $\varepsilon=\pm1$, $m_0=b+\varepsilon r$ and $n_0=r+\varepsilon a$ are integers; they are positive because $a<r<b$ (from $a^2<ab+1=r^2\leq b^2-b+1<b^2$). With $e=a+b+2\varepsilon r$ and $ab=r^2-1$, these six identities are polynomial identities in a,b,r.

Item 3. By definition $\pi_1(T)=\{p,q\}$ (Definition 3.1), and $w=p+q+2\varrho$ since ∂T is Euler. As $p\geq 1$ and $q\geq 2$, $\varrho^2\geq 3$, so $\varrho\geq 2$. If $\varepsilon=1$, then $e>b$, $\{p,q\}=\{a,b\}$ and $\varrho=r$; if $\varepsilon=-1$, then b is the largest element of ∂T, $\{p,q\}=\{a,e\}$ and $\varrho=\sqrt{ae+1}=n_0=r-a$. As $\{a,b\}\subset\{p,q,w\}$, T is one of the three listed sets. The sign -1 is admitted for $\{p,q\}$ by item 2 of Lemma 3.3, with $x_1(p,q;-1)=p+q+2\varrho=w$ and, by item 3 there, $x_2=d_+(p,q,w)$. The function d_+ of Definition 2.2 is symmetric in its three arguments, so $c=d_+(a,b,e)=d_+(p,q,w)=x_2$. Hence T is $\{p,q,x_2\}$, $\{p,x_1,x_2\}$ or $\{q,x_1,x_2\}$, so it lies at slot 2 of the chain of $\{p,q\}$ with sign -1. A triple of degree one is single-chain (the defining condition is empty), so -1 is a chain sign of T, and it is the only one by item (iii) of Lemma 3.4. By item 5 of Lemma 2.6 with $\sigma=-1$, $c=d_+(p,q,w)=4\varrho(\varrho+p)(\varrho+q)=4\varrho\big(2\varrho^2-1+\varrho(p+q)\big)$, using $pq=\varrho^2-1$. Since $(p+q)^2=(q-p)^2+4pq\geq 4\varrho^2-3>(2\varrho-1)^2$, we have $p+q\geq 2\varrho$ and $c\geq 4\varrho(4\varrho^2-1)=16\varrho^3-4\varrho$.

Item 2: the remaining assertions. By item 3, $c=4\varrho(\varrho+p)(\varrho+q)$. If $\varepsilon=1$, this is $4r(r+a)(r+b)=4rm_0n_0$. If $\varepsilon=-1$, it is $4n_0(n_0+a)(n_0+e)=4n_0\,r\,m_0$, because $n_0+a=r$ and $n_0+e=r-a+a+b-2r=b-r=m_0$. So $c=4rm_0n_0$ and $4\mid c$. Then $ac+1=4arm_0n_0+1=(2rn_0-1)^2$, because $am_0=rn_0-1$; and $bc+1=4brm_0n_0+1=(2rm_0-\varepsilon)^2$, because $bn_0=rm_0-\varepsilon$. As $2rn_0-1$ and $2rm_0-\varepsilon$ are positive, $s=2rn_0-1$ and $t=2rm_0-\varepsilon$. With $\alpha^2=2r^2-1+2r\sqrt{ab}$,

$$(\varepsilon\sqrt{a}+\sqrt{b})\,\alpha^2=\big(\varepsilon(2r^2-1)+2rb\big)\sqrt{a}+\big(2r^2-1+2\varepsilon ar\big)\sqrt{b}=t\sqrt{a}+s\sqrt{b},$$

since $t=2r(b+\varepsilon r)-\varepsilon$ and $s=2r(r+\varepsilon a)-1$. Finally, $rn_0-am_0=1$ shows that m_0 is prime to n_0 and to r, and $rm_0-bn_0=\varepsilon$ shows that n_0 is prime to r.

Item 4. By item 3 of Lemma 2.6, $c=d_+(a,b,e)>a+b+e+4abe\geq 4ab+a+b+1$. By item 2, $c=4rm_0n_0$ with $m_0<2b$ and $n_0<2r$ (and $m_0<b$, $n_0<r$ if $\varepsilon=-1$), so $c<16r^2b=16(ab+1)b$; as $ab+1<b^2$ and $ab+1<2ab$, $c<\min\{16b^3,32ab^2\}$. For $c>b^2$

note $c > 4abe$. If $\varepsilon = 1$, then $e > b$ and $4abe > b^2$. If $\varepsilon = -1$, then $\varrho^2 = ae + 1$ and $b = a + e + 2\varrho \le \varrho^2 + 2\varrho < 4(\varrho^2 - 1) = 4ae$, using $a + e \le ae + 1$ and $3\varrho^2 - 2\varrho - 4 > 0$ for $\varrho \ge 2$; so $4abe > b^2$. Finally, by item 3 and $p + q \le pq + 1 = \varrho^2$,

$$c = 4\varrho\big(2\varrho^2 - 1 + \varrho(p+q)\big) \le 4\varrho(\varrho^3 + 2\varrho^2 - 1) < 4(\varrho+1)^4.$$

Item 5. With $e = a + b + 2\varepsilon r$ and $\varepsilon^2 = 1$,

$$(b-a)(3 + 4\varepsilon e r) = 3(b-a) + 4\varepsilon r(b^2 - a^2) + 8r^2(b-a) \equiv 3(b-a) + 4\varepsilon r(b^2 - a^2) \pmod{r^2},$$

which is the congruence that defines Θ (Definition 3.6). □

For a single-chain triple that is middle or low, the chain pair is $\{a, e\}$, and b and c are consecutive terms of its chain. The windows of the families of Section 4 (Table 1) are written in the chain constants of this pair, defined next; the floors F_ν and the integers m_ν among them enter at slots four and beyond.

Definition 3.8. Let $a \ne e$ be positive integers with $ae + 1 = u^2$, u a positive integer, let σ be an admitted sign of $\{a, e\}$ and let $(x_j)_{j\in\mathbb{Z}}$ be the chain of $\{a, e\}$ with sign σ (Definition 3.2).

Constants of a pair. The **chain constants** of $(a, e; \sigma)$ are

$$D = ae, \qquad \eta = u + \sqrt{D}, \qquad \chi = \sqrt{a} - \sigma\sqrt{e}, \qquad \chi' = \sqrt{a} + \sigma\sqrt{e}, \qquad \tau = \operatorname{sgn}\chi,$$

$$\vartheta = \frac{\sqrt{a}}{|\chi|}, \qquad G = \frac{a+e}{\chi^2}, \qquad E' = x_1, \qquad E = x_{-1}, \qquad P = 4u^2 - 2, \qquad \psi_u = \vartheta^2 + \frac{2}{u}$$

($\chi \ne 0$ as $a \ne e$); the **floors**

$$F_4 = 1.15\, u^{5/4}\psi_u^{-1/4}, \qquad F_\nu = 1.63\, u^{7/4}\psi_u^{-1/4} \quad (\nu \ge 5);$$

and, for an integer $\nu \ge 4$ and real $m \ge 0$,

$$\Omega_\nu(m) = \frac{12D\vartheta^2 m^2 + 4DGm + 6Gm^2}{\eta^{2\nu-2}} + \frac{3G^2m^2}{\eta^{4\nu-4}},$$

together with the integer m_ν, the largest integer m with $0 \le m \le 2u^2$, $\Omega_\nu(m) < \frac{1}{2}$ and $3m^2\,\Omega_\nu(m) < u$.

Constants of a triple. Let $T = \{a, b, c\}$ with $a < b < c$ be a single-chain triple of degree at least 2 (Definition 3.1) which is middle or low (Definition 2.8), with descent partner e; since $a < b$ and $e < b$, the two smaller elements of the descent $\partial T = \{a, b, e\}$ are a and e, so its chain pair $\pi_1(T)$ is $\{a, e\}$. For a chain sign σ of T (Definition 3.2), the **chain constants of** T are those of $(a, e; \sigma)$.

4 The residual families and the Main Theorem

The Main Theorem places the quadruple in one of nine families (Definition 4.2, summarised in Table 1), whose windows are written in the defects and quotients of a representation (Definition 2.20) and in the index caps (Definition 4.1), defined first. Throughout, a *floor* is a lower bound for an index, a *cap* is an upper bound for m', and a *window* is the set of candidate representations that a family allows for a given triple.

Definition 4.1. A priori cap. The **a priori constant** is $K_{\mathrm{a}} = 4.436 \cdot 10^{13}$. For real $c > 1$, the **a priori cap** $Q_0(c)$ is the least integer $Q \geq K_{\mathrm{a}}(\log c)^2 - 1$ with

$$Q - 2 \geq K_{\mathrm{a}}(\log c)^2 \log\bigl(39(Q+1)\bigr).$$

Index caps. The **index caps** are the step functions $Q_{\mathrm{I}}(c)$ for $10^{30} \leq c < 10^{99}$ and $Q_{\mathrm{II}}(c)$ for $10^{30} \leq c < 10^{79.71}$ given by

$$Q_{\mathrm{I}}(c) = \begin{cases} 8.49 \cdot 10^{11}, & 10^{30} \leq c < 10^{35}, \\ 1.1 \cdot 10^{12}, & 10^{35} \leq c < 10^{40}, \\ 1.39 \cdot 10^{12}, & 10^{40} \leq c < 10^{45}, \\ 1.71 \cdot 10^{12}, & 10^{45} \leq c < 10^{50}, \\ 2.05 \cdot 10^{12}, & 10^{50} \leq c < 10^{55}, \\ 2.41 \cdot 10^{12}, & 10^{55} \leq c < 10^{60}, \\ 2.8 \cdot 10^{12}, & 10^{60} \leq c < 10^{65}, \\ 3.45 \cdot 10^{12}, & 10^{65} \leq c < 10^{70}, \\ 3.9 \cdot 10^{12}, & 10^{70} \leq c < 10^{75}, \\ 4.4 \cdot 10^{12}, & 10^{75} \leq c < 10^{80}, \\ 4.9 \cdot 10^{12}, & 10^{80} \leq c < 10^{85}, \\ 5.45 \cdot 10^{12}, & 10^{85} \leq c < 10^{90}, \\ 6.05 \cdot 10^{12}, & 10^{90} \leq c < 10^{95}, \\ 6.5 \cdot 10^{12}, & 10^{95} \leq c < 10^{99}, \end{cases} \qquad Q_{\mathrm{II}}(c) = \begin{cases} 1.74 \cdot 10^{12}, & 10^{30} \leq c < 10^{35}, \\ 2.29 \cdot 10^{12}, & 10^{35} \leq c < 10^{40}, \\ 2.94 \cdot 10^{12}, & 10^{40} \leq c < 10^{45}, \\ 3.66 \cdot 10^{12}, & 10^{45} \leq c < 10^{50}, \\ 4.49 \cdot 10^{12}, & 10^{50} \leq c < 10^{55}, \\ 5.39 \cdot 10^{12}, & 10^{55} \leq c < 10^{60}, \\ 6.37 \cdot 10^{12}, & 10^{60} \leq c < 10^{65}, \\ 9.1 \cdot 10^{12}, & 10^{65} \leq c < 10^{75}, \\ 1.03 \cdot 10^{13}, & 10^{75} \leq c < 10^{79.71}. \end{cases}$$

Integer forms. For real $c \geq 10^{30}$, let $m_{\mathrm{I}}(c)$ be the largest integer m with $2m < Q_{\mathrm{I}}(c)$ if $c < 10^{99}$, and $m_{\mathrm{I}}(c) = 0$ if $c \geq 10^{99}$; let $m_{\mathrm{II}}(c)$ be the largest integer m with $2m + 1 < Q_{\mathrm{II}}(c)$ if $c < 10^{79.71}$, and $m_{\mathrm{II}}(c) = -1$ if $c \geq 10^{79.71}$.

Each family is given by its triples, described by degree, position, shape and size of c, and by its window for each type. Every window caps m', so for a given triple it admits finitely many candidates, and every condition can be checked exactly.

Definition 4.2. Setting. Throughout, $T = \{a, b, c\}$ with $a < b < c$ is a Diophantine triple which is not an Euler triple, with descent partner e, descent ∂T and position (Definition 2.8); single-chain and bent triples are those of Definition 3.1, chain signs those of Definition 3.2, and $r = \sqrt{ab+1}$. A **candidate representation** is a triple $\rho = (\lambda, m, n)$ of a sign $\lambda \in \{-1, 1\}$ and integers $m, n \geq 0$, of type I or of type II; put $m' = 2m$ for type I and $m' = 2m + 1$ for type II, $\Delta = m - n$ and $S = m + n$, and let k, N, M, H, J, K, v, L be the quantities of Definition 2.20 computed from T and ρ. The constant κ is $(\sqrt{17}-1)/2$; Q_0, Q_{I}, Q_{II} are the caps of Definition 4.1; ε, ϱ, Θ, the line and ℓ_ε are those of Definition 3.6; and the chain constants of T are those of Definition 3.8; for a single-chain T, a condition below that involves σ or a chain constant of T is required for some chain sign σ of T.

Each of the nine families A1, A2, B0, BH, BM, BL, B3, B4, BX below is a set of pairs (T, ρ). The pair (T, ρ) **belongs to** the family F, written $(T, \rho) \in F$, if $b > 4000$, T is one of the triples of F, and ρ satisfies the window of F of its type. An empty window admits no ρ of that type. For an extension d of T (Definition 2.3) with $z = \sqrt{cd+1}$, the triple and the index of z **lie in** F if $(T, \rho) \in F$ for every representation ρ of z. Symbols introduced inside one family are local to it.

Generic windows. For $c \geq 10^{30}$:

- the **generic type I window** is: $c < 10^{99}$, $\kappa\sqrt{c/b} < n < m \le 2n$ and $m' < Q_{\mathrm{I}}(c)$;
- the **generic type II window** is: $c < 10^{79.71}$, $4 \le n \le m \le 2n$ and $m' < Q_{\mathrm{II}}(c)$, and, if T is middle or low, moreover $c < 10^{77.28}$, $J \in \mathbb{Z}$, $J \ge 1$ and $J(4ae+1) < 2m(m+1)$.

The families.

A1 Triples: $\deg T = 1$ and $c < 2.01 \cdot 10^{26}$. **Window of either type:** $\frac{2}{5}c^{1/4} < m$, $n \le m \le 2n$ and $m' < 7.7 \cdot 10^{18}$; for type II moreover $c < 10^{26}$.

A2 Triples: $\deg T = 1$, $2.01 \cdot 10^{26} \le c < 1.81 \cdot 10^{47}$, $\varepsilon \in \{-1, 1\}$, $\varrho < 3.21 \cdot 10^{11}$ if $\varepsilon = 1$ and $\varrho < 4.61 \cdot 10^{11}$ if $\varepsilon = -1$, $\Theta \ne 0$ and $3 \mid \Theta$. **Type I window:** $\lambda = -\operatorname{sgn}\Theta$, (m, n) lies on the line of T, $n < m \le 2n$,

$$\max\Big\{\frac{|\Theta|}{3},\ \ell_\varepsilon\Big\} < m' < 2.0 \cdot 10^{12},$$

and, if $\varepsilon = -1$, $m' \le \frac{4}{9}|\Theta|$. **Type II window:** empty.

B0 Triples: $\deg T \ge 2$ and $c < 10^{30}$. **Window of either type:** $n \le m \le 2n$ and $m' < Q_0(c)$.

BH Triples: $\deg T \ge 2$, T high, $c \ge 10^{30}$. Let $f = e(\partial T)$ be the descent partner of the descent. **Type I window:** the generic type I window, $c < 5.7 \cdot 10^{45}$, $c > 16a^2b^2$ and $n > 2\kappa\sqrt{a}\,c^{1/4}$. **Type II window:** the generic type II window and:

- if $H \ne 0$: $n > 0.53\sqrt{a}\,c^{1/4} - 0.75$, $c < 3.22 \cdot 10^{50}$, and $c < 10^{50}$ if $a \ge 2$;
- if $H = 0$: $\lambda = -1$, $\Delta \ge 1$, $L \ge 0$ and $m \ge \min\{fb,\ n_0 + \Delta_0\}$, where $T_0 = 2(e-a-b)(a+b) + 12ab + 6$, $\Delta_0 = \lfloor c/T_0 \rfloor + 1$ and n_0 is the least integer $n_0 \ge 0$ with $2n_0(n_0+1) \ge (4af+1)\Delta_0$;
- if $c \ge 10^{75}$: $H = 0$ and $m > 8.4 \cdot 10^{11}$.

BM Triples: T single-chain, $\deg T = 2$, T middle, $c \ge 10^{30}$. Let σ be a chain sign of T, with chain constants E, P of T. **Type I window:** the generic type I window. **Type II window:** the generic type II window, $bK = 2am(m+1) + \lambda\Delta(a-e) + JE$, and:

- if $K \ne 0$: $b \le 2m(m+1)\big(a + \frac{E}{4ae+1}\big) + \frac{m(e-a)}{2}$;
- if $K = 0$, the **ellipse**: $e \ge a+3$, $\lambda = 1$, $\Delta \ge 1$, $4a(m+1) < e - a$ and

$$2En(n+1) + 2aPm(m+1) = \Delta\big[P(e-a) + E\big].$$

BL Triples: T single-chain, $\deg T = 2$, T low, $c \ge 10^{30}$. Let σ, E, P be as for BM, $u = \sqrt{ae+1}$, $\Gamma = 4aE + 4P$, and put

$$B(m) = 2m(m+1)\Big(a + \frac{E}{4ae+1}\Big) + \frac{m(a-e)}{2}, \qquad \Xi = 4(e - \sigma u)K - J(e + 2\sigma u) + \lambda\Delta e,$$

$$V = -12aE + 24u^2 - 12 + 2\big[E(b+2a-e) - Pa(a-e)\big], \qquad Z = \Delta\big[\lambda V - \Gamma(S+1)\big] - 6JP.$$

Type I window: the generic type I window. **Type II window:** the generic type II window, $K \ge 1$, $bK = 2am(m+1) + \lambda\Delta(a-e) + JE$, $b \le B(m)$, and:

- $a \mid \Xi$; if $\Xi = 0$ and $\sigma = 1$, then $\lambda = 1$, $e \ge 12$ and $m > \sqrt{6}\,(u-e) - \frac{2}{3}$; if $\Xi = 0$, $\sigma = -1$ and $u \ge 5e$, then $\lambda = -1$, $e \ge 10$ and $m > 2(u+e) - \frac{1}{2}$;
- $c \mid Z$; if $Z = 0$, then $\Delta \ge 1$, $\lambda V > 0$ and $|V| - 6JP \le \Gamma(S+1) \le |V|$.

B3 Triples: T single-chain, $\deg T = 3$, T middle or low, $c \geq 10^{30}$. Let σ be a chain sign of T, $u = \sqrt{ae+1}$, and ϑ, E', F_4 the chain constants of T. **Type I window:** the generic type I window. **Type II window:** the generic type II window and at least one of:

- $m > F_4$;
- the **conic**: $K = 0$, $J^2 < u$, $1 - 1.5/a - 12/u \leq \vartheta^{-2} \leq 4 + 1.5/a + 12/u$ and $a(m^2+m) - E'(n^2+n) + \lambda(a - \sigma u)(m-n) = 0$.

B4 Triples: T single-chain, $\deg T \geq 4$, T middle or low, $c \geq 10^{30}$. Let $\nu = \deg T + 1$, $u = \sqrt{ae+1}$, and ϑ, F_ν the chain constants of T for a chain sign σ. **Type I window:** the generic type I window and, if $u \geq 10^3$, $n \geq \min\{u^{3/2}\vartheta^{-1/2}, u^{5/3}\}$. **Type II window:** the generic type II window and, if $u \geq 10^3$, $m > F_\nu$.

BX Triples: T bent, T middle or low, $c \geq 10^{30}$. **Type I window:** the generic type I window. **Type II window:** the generic type II window.

Family	Triples T	Type I window	Type II window
A1	$\deg T = 1$, $c < 2.01 \cdot 10^{26}$	$\frac{2}{5}c^{1/4} < m$, $m' < 7.7 \cdot 10^{18}$	as for type I, and $c < 10^{26}$
A2	$\deg T = 1$, $2.01 \cdot 10^{26} \leq c < 1.81 \cdot 10^{47}$, $\varrho < 4.61 \cdot 10^{11}$, $\Theta \neq 0$, $3 \mid \Theta$	(m,n) on the line of T, $\frac{1}{3}\lvert\Theta\rvert < m' < 2.0 \cdot 10^{12}$	empty
B0	$\deg T \geq 2$, $c < 10^{30}$	$m' < Q_0(c)$	$m' < Q_0(c)$
BH	$\deg T \geq 2$, high	generic, $c < 5.7 \cdot 10^{45}$, $n > 2\kappa\sqrt{a}\,c^{1/4}$	generic; if $H \neq 0$, $n > 0.53\sqrt{a}\,c^{1/4} - 0.75$; if $H = 0$, $\lambda = -1$, $L \geq 0$ and a floor for m
BM	single-chain, $\deg T = 2$, middle	generic	generic, the identity for bK; a bound for b if $K \neq 0$, the ellipse if $K = 0$
BL	single-chain, $\deg T = 2$, low	generic	generic, the identity for bK, $b \leq B(m)$, $a \mid \Xi$, $c \mid Z$
B3	single-chain, $\deg T = 3$, middle or low	generic	generic, and $m > F_4$ or the conic
B4	single-chain, $\deg T \geq 4$, middle or low	generic; if $u \geq 10^3$, a floor for n	generic; if $u \geq 10^3$, $m > F_\nu$
BX	bent, middle or low	generic	generic

Table 1: The nine residual families of Definition 4.2, abbreviated. In every family $b > 4000$, every window requires $n \leq m \leq 2n$, and the families BH to BX require $c \geq 10^{30}$. "Generic" stands for the generic window of the type. The full conditions are those of Definition 4.2.

The generic windows are those of the class bounds of degree at least two (Proposition 7.45), and BX consists of the generic windows alone. The other rows of degree at least two add conditions from the structure of the triple: for a high triple, the first-order defect H and, when $H = 0$, the descent partner f of the descent; at slot three, the identity for bK and, in BM, the

ellipse; at slot four, the conic; and at deeper slots, floors in $u = \sqrt{ae+1}$. The rows A1 and A2 are those of degree one; A2 confines the type I indices to the line of the triple, and its type II window is empty.

Theorem 4.3. *Let $a < b < c < d$ be an irregular Diophantine quadruple whose triple $\{a, b, c\}$ is not an Euler triple. Then the triple and the index of z lie in one of the families A1, A2, B0, BH, BM, BL, B3, B4, BX of Definition 4.2. In particular $c < 10^{99}$, and $c < 10^{79.71}$ if z admits a type II representation.*

Proof. Let $T = \{a, b, c\}$. The proof is a case split on $\deg T$, on the size of c, on the position of T and on whether T is single-chain or bent; each case is settled by one of the window propositions. By Lemma 2.9, $d > d_+(a, b, c)$ and $b > 4000$, so z has a representation of type I or of type II (Lemma 2.13). Since T is not an Euler triple, $\deg T \geq 1$. By Lemma 3.4, item (ii), T has exactly one position, and by Definition 3.1 it is single-chain or bent. Let ρ be a representation of z; we show case by case that (T, ρ) lies in the family of T.

- If $\deg T = 1$, then $(T, \rho) \in$ A1 if $c < 2.01 \cdot 10^{26}$ and $(T, \rho) \in$ A2 otherwise, by Proposition 6.13.
- If $\deg T \geq 2$ and $c < 10^{30}$, then T is a triple of B0, and Proposition 5.10 gives $b > 4000$, $n \leq m \leq 2n$ and $m' < Q_0(c)$, which is the window of B0; so $(T, \rho) \in$ B0.
- If $\deg T \geq 2$, $c \geq 10^{30}$ and T is high, then $(T, \rho) \in$ BH by Proposition 8.3.
- If $\deg T \geq 2$, $c \geq 10^{30}$, T is middle or low and bent, then T is a triple of BX, and Proposition 7.45 gives $b > 4000$ and the generic windows, which are the windows of BX; so $(T, \rho) \in$ BX.
- If $\deg T \geq 2$, $c \geq 10^{30}$ and T is middle or low and single-chain, then (T, ρ) lies in BM or BL if $\deg T = 2$ (Proposition 8.5), in B3 if $\deg T = 3$ (Proposition 8.13), and in B4 if $\deg T \geq 4$ (Proposition 8.14).

So the triple and the index of z lie in the family of T.

For the bounds on c, let F be the family of T and ρ a representation of z. If $\deg T = 1$, then $c < 1.81 \cdot 10^{47}$ by Proposition 6.12; the triples of B0 have $c < 10^{30}$; and every window of BH, BM, BL, B3, B4 and BX includes the conditions of the generic window of its type, hence $c < 10^{99}$ for type I and $c < 10^{79.71}$ for type II. So $c < 10^{99}$. If ρ is of type II, then $F \neq$ A2, since A2 has an empty type II window; the type II window of A1 requires $c < 10^{26}$, the triples of B0 have $c < 10^{30}$, and the other families have $c < 10^{79.71}$. So $c < 10^{79.71}$. □

The bound $c < 10^{99}$ comes only from the type I windows of middle and low triples of degree at least two; the other classes satisfy $c < 1.81 \cdot 10^{47}$ (degree one), $c < 10^{30}$ (B0), $c < 5.7 \cdot 10^{45}$ (type I in BH) and $c < 10^{79.71}$ (type II).

5 Linear forms in logarithms and index caps

The caps bound m' from above through the linear form of Lemma 5.7. The a priori cap of He and Togbé (Proposition 5.9) holds for every triple that is not Euler and gives, below $c = 10^{30}$, the window of B0 (Proposition 5.10); above 10^{30} the caps of Proposition 5.15, from the theorem of Mignotte and Voutier on windows of c, replace it. Every cap reduces to explicit inequalities between real numbers; where these are not checked by hand they are certified in interval arithmetic with outward rounding, and the proof says so.

5.1 Lower bounds for linear forms in logarithms

Definition 5.1. For a nonzero algebraic number η of degree $\mathcal{D} = [\mathbb{Q}(\eta) : \mathbb{Q}]$, let $a_\eta \prod_{j=1}^{\mathcal{D}} (X - \eta^{(j)})$, with $a_\eta > 0$, be its primitive irreducible polynomial over $\mathbb{Z}$. Its **absolute logarithmic height** is

$$\mathrm{h}(\eta) = \frac{1}{\mathcal{D}} \Big(\log a_\eta + \sum_{j=1}^{\mathcal{D}} \log \max\big\{1, |\eta^{(j)}|\big\} \Big).$$

Theorem 5.2. *Let $\mathcal{L}$ be a real number field of degree $\mathcal{D}$, let $\eta_1, \eta_2, \eta_3 \in \mathcal{L}$ be positive and multiplicatively independent, and let $b_1, b_2, b_3 \in \mathbb{Z}$ with $b_3 \neq 0$ and*

$$\Lambda_M = b_1 \log \eta_1 + b_2 \log \eta_2 + b_3 \log \eta_3.$$

Let A_1, A_2, A_3 be real numbers with $A_i \geq \max\{\mathcal{D}\,\mathrm{h}(\eta_i), |\log \eta_i|, 0.16\}$, where h *is the absolute logarithmic height, and put $B_M = \max\{1, |b_1|A_1/A_3, |b_2|A_2/A_3, |b_3|\}$. Then*

$$\log |\Lambda_M| > -C_M(3)\, C_0\, W_0\, \mathcal{D}^2\, A_1 A_2 A_3,$$

where $C_M(3) = \frac{16}{3!}\mathrm{e}^3 \cdot 9 \cdot 5 \cdot 16^4 \cdot \frac{3\mathrm{e}}{2} < 6.441 \cdot 10^8$, $C_0 = \log\big(\mathrm{e}^{20.2} 3^{5.5} \mathcal{D}^2 \log(\mathrm{e}\mathcal{D})\big)$ and $W_0 = \log\big(1.5\mathrm{e}\, B_M \mathcal{D} \log(\mathrm{e}\mathcal{D})\big)$.

Proof. This is Matveev's theorem [16] for three logarithms in a real field; the condition $A_i \geq 0.16$ only restricts the admissible choices of A_i. With $A_j = \max\{\mathcal{D}\,\mathrm{h}(\eta_j), |\log \eta_j|\}$ it is stated in [10, Proposition 6.2], with the constant $11796480\,\mathrm{e}^4 \mathcal{D}^2 \log\big(3^{5.5}\mathrm{e}^{20.2}\mathcal{D}^2 \log(\mathrm{e}\mathcal{D})\big)$, which is $C_M(3) C_0 \mathcal{D}^2$ because $11796480 = \frac{16}{6} \cdot 9 \cdot 5 \cdot 16^4 \cdot \frac{3}{2}$. □

Theorem 5.3. *Let $\eta_1, \eta_2 > 1$ be real algebraic numbers that are multiplicatively independent, put $\mathcal{D} = [\mathbb{Q}(\eta_1, \eta_2) : \mathbb{Q}]$, let b_1, b_2 be positive integers, and*

$$\Lambda_L = b_2 \log \eta_2 - b_1 \log \eta_1.$$

Let $\rho > 1$ and $\frac{1}{3} \leq \mu \leq 1$ be real, and put $\sigma_L = \frac{1+2\mu-\mu^2}{2}$ and $\lambda_L = \sigma_L \log \rho$. With the absolute logarithmic height h*, let A_1, A_2 be real numbers with*

$$A_i \geq \max\big\{1,\ (\rho - 1)\log \eta_i + 2\mathcal{D}\,\mathrm{h}(\eta_i)\big\} \quad (i = 1, 2), \qquad A_1 A_2 \geq \lambda_L^2,$$

and let h_L be a real number with

$$h_L \geq \max\Big\{ \mathcal{D}\Big(\log\Big(\frac{b_1}{A_2} + \frac{b_2}{A_1} \Big) + \log \lambda_L + 1.75 \Big) + 0.06,\ \lambda_L,\ \frac{\mathcal{D} \log 2}{2} \Big\}.$$

Put

$$\mathcal{H} = \frac{h_L}{\lambda_L} + \frac{1}{\sigma_L}, \qquad \omega_L = 2\Big(1 + \sqrt{1 + \frac{1}{4\mathcal{H}^2}}\Big), \qquad \theta_L = \sqrt{1 + \frac{1}{4\mathcal{H}^2}} + \frac{1}{2\mathcal{H}},$$

$$\mathcal{C} = \frac{\mu}{\lambda_L^3 \sigma_L} \Big(\frac{\omega_L}{6} + \frac{1}{2} \sqrt{ \frac{\omega_L^2}{9} + \frac{8 \lambda_L \omega_L^{5/4} \theta_L^{1/4}}{3\sqrt{A_1 A_2}\sqrt{\mathcal{H}}} + \frac{4}{3}\Big(\frac{1}{A_1} + \frac{1}{A_2} \Big) \frac{\lambda_L \omega_L}{\mathcal{H}} } \Big)^2, \qquad \mathcal{C}' = \sqrt{ \frac{\mathcal{C} \sigma_L \omega_L \theta_L}{\lambda_L^3 \mu} }.$$

Then

$$\log |\Lambda_L| \geq -\mathcal{C}\,(h_L + \log \rho)^2 A_1 A_2 - \sqrt{\omega_L \theta_L}\,(h_L + \log \rho) - \log\big(\mathcal{C}'(h_L + \log \rho)^2 A_1 A_2\big).$$

Proof. This is [15, Theorem 2, p. 327], in the case of two real numbers greater than 1 with their real logarithms; there $\mathcal{D} = [\mathbb{Q}(\eta_1,\eta_2):\mathbb{Q}]/[\mathbb{R}(\eta_1,\eta_2):\mathbb{R}]$, which is the degree in this case, and $\lambda_L/\sigma_L = \log\rho$. □

Theorem 5.4. *Let $\eta_1,\eta_2,\eta_3 > 1$ be real algebraic numbers that are multiplicatively independent, put $\mathcal{D} = [\mathbb{Q}(\eta_1,\eta_2,\eta_3):\mathbb{Q}]$, let b_1,b_2,b_3 be positive integers with $\gcd(b_1,b_2,b_3) = 1$, and*

$$\Lambda_{\mathrm{MV}} = b_1\log\eta_1 + b_2\log\eta_2 - b_3\log\eta_3, \qquad 0 < |\Lambda_{\mathrm{MV}}| < 2\pi/w,$$

where w is the largest order of a root of unity in $\mathbb{Q}(\eta_1,\eta_2,\eta_3)$. Let $K_0 \ge 3$, $\ell \ge 5$, $R_{\mathrm{MV}}, S_{\mathrm{MV}}, T_{\mathrm{MV}}$ be positive integers, put $N_* = K_0(K_0+1)\ell/2$, assume $R_{\mathrm{MV}}S_{\mathrm{MV}}T_{\mathrm{MV}} \ge N_*$, and put $g = \frac{1}{4} - \frac{N_*}{12R_{\mathrm{MV}}S_{\mathrm{MV}}T_{\mathrm{MV}}}$. Put $\delta_1 = \gcd(b_1,b_3)$, $\delta_2 = \gcd(b_2,b_3)$ and

$$\mathfrak{b} = \frac{(R_{\mathrm{MV}}-1)b_3 + (T_{\mathrm{MV}}-1)b_1}{2\delta_1} \cdot \frac{(S_{\mathrm{MV}}-1)b_3 + (T_{\mathrm{MV}}-1)b_2}{2\delta_2} \cdot \Big(\prod_{j=1}^{K_0-1} (j!)^{K_0-j}\Big)^{-\frac{12}{K_0(K_0-1)(K_0+1)}}.$$

Let $R_1,R_2,R_3,S_1,S_2,S_3,T_1,T_2,T_3$ be positive integers with $R_{\mathrm{MV}} > R_1+R_2+R_3$, $S_{\mathrm{MV}} > S_1+S_2+S_3$ and $T_{\mathrm{MV}} > T_1+T_2+T_3$. Let $\rho_{\mathrm{MV}} \ge 2$ be real, and let A_1,A_2,A_3 be real numbers with $A_i \ge (\rho_{\mathrm{MV}}-1)\log\eta_i + 2\mathcal{D}\,\mathrm{h}(\eta_i)$, where h *is the absolute logarithmic height, such that*

$$\Big(\frac{K_0\ell}{2} + \frac{\ell}{2} - 0.37K_0 - 2\Big)\log\rho_{\mathrm{MV}} \\ \ge (\mathcal{D}+1)\log N_* + g\ell\,(A_1R_{\mathrm{MV}} + A_2S_{\mathrm{MV}} + A_3T_{\mathrm{MV}}) + \frac{2\mathcal{D}(K_0-1)}{3}\log\mathfrak{b}.$$

Put $\mathcal{V} = \sqrt{(R_1+1)(S_1+1)(T_1+1)}$, and suppose that for some real $\chi_{\mathrm{MV}} > 0$

$$(R_1+1)(S_1+1)(T_1+1) > K_0\,\mathcal{M},$$
$$\mathcal{M} = \max\{R_1+S_1+1,\ S_1+T_1+1,\ R_1+T_1+1,\ \chi_{\mathrm{MV}}\mathcal{V}\},$$
$$\#\{\eta_1^{j_1}\eta_2^{j_2}\eta_3^{j_3} : 0\le j_1\le R_1,\ 0\le j_2\le S_1,\ 0\le j_3\le T_1\} > \ell,$$
$$\#\{\eta_1^{j_1}\eta_2^{j_2}\eta_3^{j_3} : 0\le j_1\le R_2,\ 0\le j_2\le S_2,\ 0\le j_3\le T_2\} > 2K_0\ell,$$
$$(R_2+1)(S_2+1)(T_2+1) > K_0^2, \qquad (R_3+1)(S_3+1)(T_3+1) > 3K_0^2\ell.$$

Then the ordinary alternative

$$\Lambda'_{\mathrm{MV}} = |\Lambda_{\mathrm{MV}}| \cdot \frac{\ell T_{\mathrm{MV}}\,\mathrm{e}^{\ell T_{\mathrm{MV}}|\Lambda_{\mathrm{MV}}|/(2b_3)}}{2b_3} > \rho_{\mathrm{MV}}^{-K_0\ell}$$

holds, or one of the following holds:

(C1) $b_1 \le \max\{R_1,R_2\}$, $b_2 \le \max\{S_1,S_2\}$ and $b_3 \le \max\{T_1,T_2\}$;

(C2) there are integers u_1,u_2,u_3 with $\gcd(u_1,u_2,u_3) = 1$ and $u_1b_1+u_2b_2+u_3b_3 = 0$, such that

$$|u_1| \le \frac{(S_1+1)(T_1+1)}{\mathcal{M}-\max\{S_1,T_1\}}, \qquad |u_2| \le \frac{(R_1+1)(T_1+1)}{\mathcal{M}-\max\{R_1,T_1\}}, \qquad |u_3| \le \frac{(R_1+1)(S_1+1)}{\mathcal{M}-\max\{R_1,S_1\}}.$$

Proof. This is [17, Theorem 2.1] in its real case (numbering of the arXiv version 3), with the conventions and notation of its Sections 2.1 and 2.2. The parameters written here K_0, ℓ, N_*, R_{MV}, S_{MV}, T_{MV}, ρ_{MV}, χ_{MV}, A_i, δ_1, δ_2 are K, L, N, R, S, T, ρ, χ, a_i, d_1, d_2 there, and $\mathcal{D}$ is the

quotient $[\mathbb{Q}(\eta_1,\eta_2,\eta_3):\mathbb{Q}]/[\mathbb{R}(\eta_1,\eta_2,\eta_3):\mathbb{R}]$ of its Liouville inequality, which in the real case is the degree. The number $\mathfrak{b}$ is the quantity b of its definition (2.5), written out by means of (2.3) and (2.4), since $b_3'\eta_0 = \frac{(R_{\mathrm{MV}}-1)b_3+(T_{\mathrm{MV}}-1)b_1}{2\delta_1}$ and $b_3''\zeta_0 = \frac{(S_{\mathrm{MV}}-1)b_3+(T_{\mathrm{MV}}-1)b_2}{2\delta_2}$. For real $\eta_i > 1$ the lower bound for a_i in the source, $\rho|\log\eta_i| - \log|\eta_i| + 2\mathcal{D}\,\mathrm{h}(\eta_i)$ with $\rho = \rho_{\mathrm{MV}}$, is the one stated here. Two remarks on the source. First, $\mathfrak{b}$ is to be read from (2.5): the displayed statement of its Lemma 3.5(b) transposes b_2 and b_3 in the second fraction, a misprint, as the last display of the proof of that lemma and the entries $\binom{s_jb_3''+t_jb_2''}{m_i}$ of its interpolation determinant show. Second, since $\mathbb{Q}(\eta_1,\eta_2,\eta_3)$ is real, $w = 2$, and the condition on Λ_{MV} reads $0 < |\Lambda_{\mathrm{MV}}| < \pi$. □

Lemma 5.5. *Let $\eta \neq 1$ be a positive real algebraic number with $[\mathbb{Q}(\eta):\mathbb{Q}] \leq 4$. Then*

$$\log|\log\eta| > -4\,\mathrm{h}(\eta) - 4.$$

Proof. A lower bound for $|\eta - 1|$ comes from the primitive minimal polynomial of η, evaluated at 1. Let $\mathcal{D} = [\mathbb{Q}(\eta):\mathbb{Q}] \leq 4$ and let $P(X) = a_\eta\prod_{i=1}^{\mathcal{D}}(X-\eta_i) \in \mathbb{Z}[X]$ be the primitive minimal polynomial of η, with $a_\eta > 0$ and $\eta_1 = \eta$. By Definition 5.1, $a_\eta\prod_{i=1}^{\mathcal{D}}\max(1,|\eta_i|) = \mathrm{e}^{\mathcal{D}\,\mathrm{h}(\eta)}$. Since P is irreducible and $\eta \neq 1$, the integer $P(1)$ is nonzero, so

$$1 \leq |P(1)| = a_\eta\,|1-\eta|\prod_{i=2}^{\mathcal{D}}|1-\eta_i| \leq |1-\eta|\cdot a_\eta\prod_{i=2}^{\mathcal{D}}2\max(1,|\eta_i|) \leq 2^{\mathcal{D}-1}\mathrm{e}^{\mathcal{D}\,\mathrm{h}(\eta)}\,|\eta-1|,$$

using $|1-\eta_i| \leq 2\max(1,|\eta_i|)$ and $\max(1,|\eta|) \geq 1$. As $\mathcal{D} \leq 4$ and $\mathrm{h}(\eta) \geq 0$, this gives

$$\log|\eta-1| \geq -4\,\mathrm{h}(\eta) - 3\log 2.$$

This bound passes to $|\log\eta|$ in three cases. If $1 < \eta \leq 2$, then $\log\eta \geq \frac{\eta-1}{\eta} \geq \frac{\eta-1}{2}$, so $\log\log\eta \geq \log(\eta-1) - \log 2 \geq -4\,\mathrm{h}(\eta) - 4\log 2 > -4\,\mathrm{h}(\eta) - 4$. If $0 < \eta < 1$, then $|\log\eta| = -\log\eta \geq 1-\eta = |\eta-1|$, so $\log|\log\eta| \geq -4\,\mathrm{h}(\eta) - 3\log 2 > -4\,\mathrm{h}(\eta) - 4$. If $\eta > 2$, then $\log|\log\eta| > \log\log 2 > -0.37 > -4 \geq -4\,\mathrm{h}(\eta) - 4$. □

An elementary comparison lemma turns an inequality of the shape $w-2 < C(\log c)^2\log(\theta(w+1))$, together with a lower bound $w > Y\,c^{1/\kappa_1}$, into an upper bound for c. The degenerate alternative of the theorem of Mignotte and Voutier is excluded in the proofs of the caps, by a centred form in two logarithms and Laurent's theorem.

Lemma 5.6. *Let $C,\kappa_1,Y,\Lambda_0 > 0$ and $\theta \geq 3.3$, put $w_0 = Y\,10^{\Lambda_0/\kappa_1}$ and*

$$G_{Y,\theta}(w) = \frac{w-2}{\big(\log(w/Y)\big)^2\log\big(\theta(w+1)\big)} \qquad (w > Y),$$

and assume $10^{\Lambda_0/\kappa_1} \geq 21$, $w_0 \geq 50$ and $G_{Y,\theta}(w_0) \geq C\kappa_1^2$. If real numbers $w,\ell > 0$ satisfy

$$w > Y\,\mathrm{e}^{\ell/\kappa_1} \qquad \text{and} \qquad w-2 < C\ell^2\log\big(\theta(w+1)\big),$$

then $\ell < \Lambda_0\log 10$.

Proof. The function $G_{Y,\theta}$ is strictly increasing on $[w_0,\infty)$: for $w \geq w_0$ we have $\log(w/Y) \geq \log 21 > 3$ and $\log(\theta(w+1)) \geq \log(3.3\cdot 51) > 5$, so

$$\frac{G_{Y,\theta}'(w)}{G_{Y,\theta}(w)} = \frac{1}{w-2} - \frac{2}{w\log(w/Y)} - \frac{1}{(w+1)\log(\theta(w+1))} > \frac{1}{w} - \frac{2}{3w} - \frac{1}{5w} > 0.$$

Suppose $\ell \ge \Lambda_0 \log 10$. Then $w > Y\,\mathrm{e}^{\ell/\kappa_1} \ge w_0$ and $\ell < \kappa_1 \log(w/Y)$, so the second hypothesis gives

$$w - 2 < C\kappa_1^2 \big(\log(w/Y)\big)^2 \log\big(\theta(w+1)\big), \qquad \text{that is} \qquad G_{Y,\theta}(w) < C\kappa_1^2 \le G_{Y,\theta}(w_0),$$

which contradicts the monotonicity, since $w > w_0$. Hence $\ell < \Lambda_0 \log 10$. □

5.2 The Pell linear form and the a priori cap

Lemma 5.7. *Let the triple $\{a,b,c\}$ not be an Euler triple, and let z have a representation of type I or II with $m', n' > 0$. Put*

$$\delta = \frac{b}{a} \cdot \frac{x_1\sqrt{ac} + az_1}{y_2\sqrt{bc} + bz_2}, \qquad \Lambda = m'\log\beta - n'\log\gamma + \log\delta.$$

Then

$$0 < \Lambda < \frac{8}{3}\, ac\, \beta^{-2m'}.$$

Proof. The estimate goes back to Dujella [5, Lemma 5]. He and Togbé [13, Lemma 8.2] state it for every triple, but for Euler triples in a type II representation its constant $\frac{8}{3}$ is not justified, so we prove it here for the triples we need. As $\{a,b,c\}$ is not an Euler triple, $c > 4ab$ by Lemma 2.4.

Put $\Pi_1 = x_1\sqrt{c} + z_1\sqrt{a}$, $\Pi_1^* = x_1\sqrt{c} - z_1\sqrt{a}$, $\Pi_2 = y_2\sqrt{c} + z_2\sqrt{b}$ and $\Pi_2^* = y_2\sqrt{c} - z_2\sqrt{b}$. Then $\Pi_1\Pi_1^* = cx_1^2 - az_1^2 = c - a > 0$ and $\Pi_2\Pi_2^* = c - b > 0$, and $\Pi_1, \Pi_2 > 0$ (type I: $\sqrt{c} > \sqrt{a}$, $\sqrt{c} > \sqrt{b}$; type II: $r^2c - t^2a = c - a > 0$ gives $r\sqrt{c} > t\sqrt{a}$, and likewise $r\sqrt{c} > s\sqrt{b}$), so $\Pi_1^*, \Pi_2^* > 0$. The recurrences of the Pell sequences give $x_j^{(a,c)}\sqrt{c} + z_j^{(a,c)}\sqrt{a} = \Pi_1\beta^j$ and $x_j^{(a,c)}\sqrt{c} - z_j^{(a,c)}\sqrt{a} = \Pi_1^*\beta^{-j}$, and likewise for the pair $\{b,c\}$. Therefore

$$2z = \frac{\Pi_1\beta^{m'} - \Pi_1^*\beta^{-m'}}{\sqrt{a}} = \frac{\Pi_2\gamma^{n'} - \Pi_2^*\gamma^{-n'}}{\sqrt{b}}.$$

Put $U = \Pi_1\beta^{m'}/\sqrt{a}$ and $V = \Pi_2\gamma^{n'}/\sqrt{b}$. Since $\delta = \sqrt{b/a}\,\Pi_1/\Pi_2$, we have $\Lambda = \log(U/V)$ and

$$U - V = \frac{\Pi_1^*\beta^{-m'}}{\sqrt{a}} - \frac{\Pi_2^*\gamma^{-n'}}{\sqrt{b}} = \frac{c-a}{aU} - \frac{c-b}{bV}.$$

If $U \le V$, the right side is at least $\frac{c-a}{aV} - \frac{c-b}{bV} = \frac{c(b-a)}{abV} > 0$, a contradiction; so $U > V$ and $\Lambda > 0$. Put $\theta = (U-V)/U$. Then $\theta < \frac{c-a}{aU^2} = \frac{c-a}{\Pi_1^2\beta^{2m'}}$ and $\Lambda = -\log(1-\theta) \le \theta/(1-\theta)$. Note also $\beta^{2m'} \ge \beta^2 > 4ac$.

Type I. Here $\Pi_1 \ge \sqrt{c} - \sqrt{a} = (c-a)/(\sqrt{c} + \sqrt{a})$, so

$$\theta < \frac{(\sqrt{c}+\sqrt{a})^2}{c-a}\,\beta^{-2m'} < \frac{4c}{c-a}\,\beta^{-2m'} < \frac{16}{3}\,\beta^{-2m'},$$

using $c > 4ab \ge 4a$. As $4ac \ge 48$, $\theta < 1/6$, and $\Lambda < \frac{6}{5}\cdot\frac{16}{3}\,\beta^{-2m'} < \frac{8}{3}ac\,\beta^{-2m'}$, since $ac \ge 12$.

Type II. Here $\Pi_1 \ge r\sqrt{c} - t\sqrt{a} = (c-a)/(r\sqrt{c} + t\sqrt{a})$ and

$$(r\sqrt{c} + t\sqrt{a})^2 = 2abc + a + c + 2rt\sqrt{ac} < 2abc + a + c + 2.35\,abc < 4.72\,abc.$$

Hence $\theta < \frac{4.72\,abc}{c-a}\,\beta^{-2m'} < 5.15\,ab\,\beta^{-2m'}$, as $c/(c-a) < 12/11$, and $\theta < 5.15ab/(4ac) < 0.33$ because $b/c < 1/(4a)$. So $\Lambda < \frac{5.15}{0.67}\,ab\,\beta^{-2m'} < 7.7\,ab\,\beta^{-2m'} < \frac{8}{3}ac\,\beta^{-2m'}$, since $c > 4ab \ge 4b$. □

Lemma 5.8. *In the notation of Lemma 5.7, let* $\delta = \frac{b}{a}\cdot\frac{x_1\sqrt{ac}+az_1}{y_2\sqrt{bc}+bz_2}$ *be the seed quotient of a type I or type II representation (type I seeds* $(x_1,z_1)=(y_2,z_2)=(1,\lambda)$*, or type II seeds* $(x_1,z_1)=(r,\lambda t)$*,* $(y_2,z_2)=(r,\lambda s)$*), and assume* $b>4000$*.*

1. *Type I seeds:* $\delta = \dfrac{\sqrt{bc}+\lambda\sqrt{ab}}{\sqrt{ac}+\lambda\sqrt{ab}}$ *and* $\mathrm{h}(\delta)\le\frac{1}{2}\log(bc)+\log\bigl(1+\sqrt{a/c}\bigr)$*; if moreover* $c>4ab$*, then* $4\,\mathrm{h}(\delta)<4\log c$*.*

2. *Type II seeds with* $c>4ab$*:* $\delta = \dfrac{r\sqrt{bc}+\lambda t\sqrt{ab}}{r\sqrt{ac}+\lambda s\sqrt{ab}}$ *and* $\mathrm{h}(\delta)\le\frac{7}{8}\log c+\frac{1}{4}\log b$*, so* $4\,\mathrm{h}(\delta)<4.5\log c$*.*

In both cases (in item 2 with $c>4ab$*),* $4\,\mathrm{h}(\delta)<4.5\log c$*; always* $|\log\delta|\le 4\,\mathrm{h}(\delta)$*.*

Proof. The quotient is rewritten as a ratio of two algebraic integers of the real quartic field $\mathcal{K}=\mathbb{Q}(\sqrt{ab},\sqrt{ac})$ and the height is bounded by the product formula.

Height of a quotient. The field $\mathcal{K}$ has degree 4 and four real embeddings σ. For nonzero algebraic integers $\eta_1,\eta_2\in\mathcal{K}$ the product formula gives

$$\mathrm{h}(\eta_1/\eta_2)=\frac{1}{4}\sum_v \log\max\{|\eta_1|_v,|\eta_2|_v\}\le\frac{1}{4}\sum_\sigma\log\max\{|\sigma\eta_1|,|\sigma\eta_2|\},$$

since every non-archimedean term is ≤ 0 for algebraic integers. The embeddings are $\sigma_{\varepsilon_1\varepsilon_2}$: $\sqrt{ab}\mapsto\varepsilon_1\sqrt{ab}$, $\sqrt{ac}\mapsto\varepsilon_2\sqrt{ac}$, hence $\sqrt{bc}=\sqrt{ab}\sqrt{ac}/a\mapsto\varepsilon_1\varepsilon_2\sqrt{bc}$. Also $|\log|\theta||\le[\mathbb{Q}(\theta):\mathbb{Q}]\,\mathrm{h}(\theta)\le 4\,\mathrm{h}(\theta)$ for $\theta\in\mathcal{K}^\times$.

Type I seeds. $\delta=\frac{b}{a}\cdot\frac{\sqrt{a}(\sqrt{c}+\lambda\sqrt{a})}{\sqrt{b}(\sqrt{c}+\lambda\sqrt{b})}=\frac{\sqrt{b}(\sqrt{c}+\lambda\sqrt{a})}{\sqrt{a}(\sqrt{c}+\lambda\sqrt{b})}=\frac{\sqrt{bc}+\lambda\sqrt{ab}}{\sqrt{ac}+\lambda\sqrt{ab}}$. Every conjugate of $\eta_1=\sqrt{bc}+\lambda\sqrt{ab}$ and of $\eta_2=\sqrt{ac}+\lambda\sqrt{ab}$ has absolute value at most $\sqrt{bc}+\sqrt{ab}=\sqrt{bc}\,(1+\sqrt{a/c})$, so $\mathrm{h}(\delta)\le\frac{1}{2}\log(bc)+\log(1+\sqrt{a/c})$. If $c>4ab$, then $a/c<1/(4b)<1/16000$, $\log(1+\sqrt{a/c})<0.008$, and $4\,\mathrm{h}(\delta)<2\log(bc)+0.032=4\log c-2\log(c/b)+0.032<4\log c$ because $c/b>4a\ge 4$. In general $\log(1+\sqrt{a/c})\le\log 2$ $(a<c)$, so $4\,\mathrm{h}(\delta)\le 2\log(bc)+4\log 2=4\log c-2\log(c/b)+2.773<4\log c+2.773<4.5\log c$, because $0.5\log c>2.773$ as soon as $c>257$, and $c>b>4000$.

Type II seeds. $\delta=\frac{b}{a}\cdot\frac{\sqrt{a}(r\sqrt{c}+\lambda t\sqrt{a})}{\sqrt{b}(r\sqrt{c}+\lambda s\sqrt{b})}=\frac{r\sqrt{bc}+\lambda t\sqrt{ab}}{r\sqrt{ac}+\lambda s\sqrt{ab}}$, with $\eta_1=r\sqrt{bc}+\lambda t\sqrt{ab}$, $\eta_2=r\sqrt{ac}+\lambda s\sqrt{ab}$. Under $\sigma_{\varepsilon_1\varepsilon_2}$, $\eta_1\mapsto\varepsilon_1(\varepsilon_2 r\sqrt{bc}+\lambda t\sqrt{ab})$ and $\eta_2\mapsto\varepsilon_2 r\sqrt{ac}+\lambda\varepsilon_1 s\sqrt{ab}$. Large values: $r\sqrt{bc}+t\sqrt{ab}=b\sqrt{ac}\bigl(\sqrt{1+1/(ab)}+\sqrt{1+1/(bc)}\bigr)<2.0002\,b\sqrt{ac}$ and likewise $r\sqrt{ac}+s\sqrt{ab}<2.0002\,a\sqrt{bc}\le 2.0002\,b\sqrt{ac}$ (as $ab\ge 4001$). Small values: $|r\sqrt{bc}-t\sqrt{ab}|=\frac{|r^2bc-t^2ab|}{r\sqrt{bc}+t\sqrt{ab}}=\frac{b(c-a)}{r\sqrt{bc}+t\sqrt{ab}}<\frac{b(c-a)}{2b\sqrt{ac}}<\frac{1}{2}\sqrt{c/a}$, since both terms of the denominator exceed $b\sqrt{ac}$, and $|r\sqrt{ac}-s\sqrt{ab}|=\frac{a(c-b)}{r\sqrt{ac}+s\sqrt{ab}}<\frac{a(c-b)}{2a\sqrt{bc}}<\frac{1}{2}\sqrt{c/b}$. The conjugate of η_1 is large exactly when $\varepsilon_2=\lambda$, that of η_2 exactly when $\varepsilon_2=\lambda\varepsilon_1$; so among the four embeddings one has both large, one both small, one only η_1 large and one only η_2 large. These small conjugates are what bring the bound below $\log c$, up to the term $\frac{1}{4}\log b$. Hence

$$\begin{aligned}4\,\mathrm{h}(\delta)&\le 2\log(2.0002\,b\sqrt{ac})+\log\bigl(\tfrac{1}{2}\sqrt{c/a}\bigr)+\log(2.0002\,a\sqrt{bc})\\&=3\log 2.0002-\log 2+\tfrac{5}{2}\log b+\tfrac{3}{2}\log a+2\log c.\end{aligned}$$

With $\log a+\log b<\log c-\log 4$ (from $c>4ab$): $\frac{3}{2}(\log a+\log b)+\log b<\frac{3}{2}\log c-2.079+\log b$, and $3\log 2.0002-\log 2<1.387$, so $4\,\mathrm{h}(\delta)<\frac{7}{2}\log c+\log b$, that is, $\mathrm{h}(\delta)<\frac{7}{8}\log c+\frac{1}{4}\log b<\frac{9}{8}\log c$.

□

Proposition 5.9. *Let the quadruple $\{a,b,c,d\}$ be irregular and let $\{a,b,c\}$ not be an Euler triple. Then every representation of z satisfies*

$$m' - 2 < K_{\mathrm{a}} (\log c)^2 \log\bigl(39(m'+1)\bigr),$$

where $K_{\mathrm{a}} = 4.436 \cdot 10^{13}$ is the a priori constant.

Proof. This is [13, Proposition 8.3]: by Lemma 2.9 an irregular quadruple has $d > d_+(a,b,c)$, and the input [13, Lemma 8.2] of that proof holds here as Lemma 5.7, since the triple is not an Euler triple. □

Solving the a priori inequality for m' gives the a priori cap $Q_0(c)$ of Definition 4.1, and with it the window of B0.

Proposition 5.10. *Let $0 < a < b < c < d$ be an irregular Diophantine quadruple whose triple $\{a,b,c\}$ is not an Euler triple, and let $Q_0(c)$ be the a priori cap. Then $b > 4000$, and every representation (λ, m, n) of z, of type I or of type II, satisfies*

$$n \le m \le 2n, \qquad m' < Q_0(c).$$

Proof. By Lemma 2.9, $d > d_+(a,b,c)$ and $b > 4000$; in particular $d > c$, so Lemma 2.14 gives $n \le m \le 2n$ for every representation.

The cap $Q_0(c)$ exists and bounds m'. Put $K_c = K_{\mathrm{a}}(\log c)^2$ and $g(x) = x - 2 - K_c \log\bigl(39(x+1)\bigr)$ for real $x \ge 0$. Since $c > 1$, $K_c > 0$, and $g'(x) = 1 - K_c/(x+1) \ge 0$ for $x \ge K_c - 1$, so g is non-decreasing on $[K_c - 1, \infty)$; as $g(x) \to \infty$, integers $x \ge K_c - 1$ with $g(x) \ge 0$ exist, and $Q_0(c)$ is the least of them. Let (λ, m, n) be a representation of z. As the triple is not an Euler triple, Proposition 5.9 gives $m' - 2 < K_c \log\bigl(39(m'+1)\bigr)$, that is $g(m') < 0$. If $m' \ge Q_0(c)$, then $m' \ge K_c - 1$ and $g(m') \ge g(Q_0(c)) \ge 0$, a contradiction. Hence $m' < Q_0(c)$.

□

With the sharper height of the seed quotient (Lemma 5.8), Matveev's theorem gives a refined cap for large c; it is the cap of the a priori bounds of Section 7.

Proposition 5.11. *Let $0 < a < b < c < d$ be an irregular Diophantine quadruple whose triple $\{a,b,c\}$ is not an Euler triple, and suppose $c \ge 10^{80}$. Then every representation of z satisfies*

$$m' - 2 < 2.79 \cdot 10^{12} (\log c)^2 \log\bigl(17.4(m'+1)\bigr).$$

If moreover the representation is of type I, then

$$m' - 2 < 2.48 \cdot 10^{12} (\log c)^2 \log\bigl(19.6(m'+1)\bigr).$$

Proof. We apply Matveev's theorem (Theorem 5.2) to the linear form of Lemma 5.7, with the height bound of Lemma 5.8 for its third number.

Parameters. By Lemma 2.9, $d > d_+(a,b,c)$ and $b > 4000$, as Lemma 5.8 requires, and $m', n' > 0$: in type II they are odd, and in type I $z > 1$ while $z_0^{(a,c)} = z_0^{(b,c)} = \lambda$. As the triple is not an Euler triple, $c > 4ab$ by Lemma 2.4. The numbers $ab = r^2 - 1$, $ac = s^2 - 1$, $bc = t^2 - 1$ are not squares, and $ab \cdot ac = a^2 bc$, so $\mathcal{K} = \mathbb{Q}(\sqrt{ab}, \sqrt{ac})$ is a real quartic field containing β, γ, δ, and

Theorem 5.2 applies with $\mathcal{L} = \mathcal{K}$ and $\mathcal{D} = 4$. The automorphism of $\mathcal{K}$ that fixes $\sqrt{ac}$ and negates $\sqrt{ab}$ fixes β and maps γ to γ^{-1}; β, γ have norm 1 and $\mathrm{N}_{\mathcal{K}/\mathbb{Q}}(\delta) = \big(b(c-a)/(a(c-b))\big)^2 > 1$, so a relation $\beta^{k_1}\gamma^{k_2}\delta^{k_3} = 1$ forces $k_3 = 0$, then $k_2 = k_1 = 0$. Also $\delta > 0$, since $x_1, y_2 > 0$, $cx_1^2 - az_1^2 = c - a > 0$ and $cy_2^2 - bz_2^2 = c - b > 0$, and $\mathrm{h}(\beta) = \frac{1}{2}\log\beta$, $\mathrm{h}(\gamma) = \frac{1}{2}\log\gamma$. Since $c > 4ab$, Lemma 5.8 gives $4\,\mathrm{h}(\delta) < 4.5\log c$ for both types and $4\,\mathrm{h}(\delta) < 4\log c$ for type I representations, and $|\log\delta| \le 4\,\mathrm{h}(\delta)$. Hence, for $(\eta_1, \eta_2, \eta_3) = (\beta, \gamma, \delta)$, the numbers $A_1 = 2\log\beta$, $A_2 = 2\log\gamma$ and $A_3 = 4.5\log c$ (resp. $4\log c$ for type I) satisfy $A_i \ge \max\{4\,\mathrm{h}(\eta_i), |\log\eta_i|, 0.16\}$. With $\beta < 2c$, $\gamma < 2c$, $n' \le m' + 1$ (Lemma 2.14) and $\log 2/\log c \le 0.0038$,

$$\begin{aligned} B_M &= \max\Big\{1, \frac{2m'\log\beta}{A_3}, \frac{2n'\log\gamma}{A_3}\Big\} \\ &\le \max\Big\{1, \frac{2\cdot 1.0038\,(m'+1)\log c}{A_3}\Big\} \\ &= \max\{1,\ 0.4462(m'+1)\} \quad (\text{resp. } \max\{1, 0.5019(m'+1)\}). \end{aligned}$$

Comparison of the bounds. Theorem 5.2 gives $\log\Lambda > -C_M(3)C_0\mathcal{D}^2\,A_1A_2A_3W_0$ with $C_M(3)C_0\mathcal{D}^2 < 3.0797\cdot 10^{11}$ and $W_0 = \log(38.92\,B_M)$, and Lemma 5.7 gives $\log\Lambda < \log(8ac/3) - 2m'\log\beta < (4 - 2m')\log\beta$. With $A_3 = 4.5\log c$:

$$C_M(3)C_0\mathcal{D}^2A_1A_2A_3 < 5.5435\cdot 10^{12}\log\beta\log\gamma\log c;$$

dividing by $2\log\beta$ and using $\log\gamma < 1.0038\log c$,

$$m' - 2 < 2.7823\cdot 10^{12}(\log c)^2W_0.$$

For $m' \ge 2$ one has $0.4462(m'+1) \ge 1$, so $W_0 \le \log(38.92\cdot 0.4462(m'+1)) < \log(17.4(m'+1))$, and the first claim follows; for $m' \le 1$ it is trivial. With $A_3 = 4\log c$ the same computation gives $m' - 2 < 2.4732\cdot 10^{12}(\log c)^2W_0$ and $W_0 \le \log(38.92\cdot 0.5019(m'+1)) < \log(19.6(m'+1))$ for $m' \ge 1$. □

5.3 Index caps on windows of c

On windows of c we use the Pell equation of $\{a, b\}$ at the common index of Lemma 2.17: the representation gives a linear form in three logarithms whose third term is a correction of smaller height than the seed quotient, Θ_λ for type II (Lemma 5.12) and D_λ for type I (Lemma 5.13); the correction Θ_λ is unrelated to the residue Θ of Definition 3.6. The theorem of Mignotte and Voutier then gives the caps row by row (Proposition 5.14), and Proposition 5.15 records the resulting step functions Q_{I} and Q_{II}.

Lemma 5.12. *Let $0 < a < b < c$ be a positive integer Diophantine triple with $b > 4000$, $b > 8a$, $c > 4ab$. Let r, s, t be its roots and α, β, γ its units, and put $A = \log\alpha$, $B = \log\beta$, $G = \log\gamma$, $\ell_c = \log c$. Define*

$$\Theta_\lambda = \frac{s\sqrt{bc} + t\sqrt{ac}}{r\sqrt{bc} + \lambda t\sqrt{ab}}\,\frac{\alpha^\lambda}{\beta}, \qquad \lambda \in \{-1, 1\}.$$

Then, with the real quartic field $\mathcal{K} = \mathbb{Q}(\sqrt{ab}, \sqrt{ac})$, which contains Θ_λ,

$$0 < \log\Theta_\lambda < 0.000063, \qquad N_{\mathcal{K}/\mathbb{Q}}(\Theta_\lambda) = \left(\frac{c(b-a)}{b(c-a)}\right)^2 < 1,$$

and

$$\mathrm{h}(\Theta_+) < \tfrac{3}{2}\ell_c - \tfrac{1}{4}\log a - 0.3463, \qquad \mathrm{h}(\Theta_-) < \tfrac{3}{4}\ell_c + \tfrac{1}{4}\log(b+1) < \ell_c.$$

Let d be an irregular extension of the triple such that z has a type II representation (λ, m, n), let $j = 2l+1$ be the exponent of Lemma 2.17, so that $y\sqrt{a} + x\sqrt{b} = (t\sqrt{a} + s\sqrt{b})\alpha^j$, and put

$$X_1 = \big(s + t\sqrt{a/b}\big)\alpha^j, \qquad X_2 = \big(r + \lambda t\sqrt{a/c}\big)\beta^{m'}.$$

Then

$$\log X_1 - \log X_2 = (j-\lambda)A - (m'-1)B + \log \Theta_\lambda, \qquad j - \lambda > 0.$$

Proof. Put

$$P_\pm = s\sqrt{bc} \pm t\sqrt{ac}, \quad N_\pm = r\sqrt{bc} \pm t\sqrt{ab},$$

and

$$U = \sqrt{1 + 1/(ab)}, \quad v = \sqrt{1 + 1/(ac)}, \quad w = \sqrt{1 + 1/(bc)}.$$

Then $U > v > w > 1$, and all $P_\pm, N_\pm$ are positive algebraic integers, with

$$P_+P_- = c(b-a), \qquad N_+N_- = b(c-a).$$

Also

$$P_+ = c\sqrt{ab}(v+w), \quad N_+ = b\sqrt{ac}(U+w), \quad \alpha = \sqrt{ab}(1+U), \quad \beta = \sqrt{ac}(1+v).$$

The hyperbolic addition formulas give

$$P_+ = \sinh(B+G), \qquad N_\lambda = \sinh(G + \lambda A),$$

and hence

$$\Theta_\lambda = \frac{1 - \mathrm{e}^{-2(B+G)}}{1 - \mathrm{e}^{-2(G+\lambda A)}} > 1.$$

Since $B > A$, the numerator is larger than the positive denominator. Moreover

$$\log \Theta_\lambda < -\log(1 - (\alpha/\gamma)^2).$$

The unit bounds $\alpha^2 < 4ab + 2$, $\gamma^2 > 4bc$, together with $b \geq 4001$, $c > 4ab$, give

$$(\alpha/\gamma)^2 < \frac{1}{16004} + \frac{1}{2 \cdot 4001 \cdot 16004},$$

and $-\log\big(1-(\alpha/\gamma)^2\big) < 0.000063$. The norm follows from the two displayed product differences; multiplication by units leaves it unchanged. The strict inequality of the norm follows from $c > b > a$.

For the height bounds use the four real embeddings $\sigma_{\omega_1\omega_2}$ of $\mathcal{K}$, $\omega_1, \omega_2 \in \{\pm 1\}$, with $\sqrt{ac} \mapsto \omega_1\sqrt{ac}$ and $\sqrt{bc} \mapsto \omega_2\sqrt{bc}$; they send β, γ, α to $\beta^{\omega_1}, \gamma^{\omega_2}, \alpha^{\omega_1\omega_2}$. The four projective numerator-denominator pairs of Θ_+, in the order $(\omega_1, \omega_2) = (1,1), (1,-1), (-1,1), (-1,-1)$, are

$$(P_+\alpha, N_+\beta), \quad (P_-/\alpha, N_+\beta), \quad (P_-/\alpha, N_-/\beta), \quad (P_+\alpha, N_-/\beta).$$

The elementary square-root upper bound gives

$$P_+\alpha < 4.001abc, \qquad N_+\beta < 4.001abc.$$

Their product differences give

$$P_-/\alpha = \frac{c(b-a)}{P_+\alpha} < 1/(4a), \qquad N_-/\beta < 1/(4a),$$

since both displayed large products exceed $4abc$. Thus three projective maxima are smaller than $4.001abc$, and the remaining one is smaller than $1/(4a)$. The full projective height formula for a quotient of algebraic integers gives

$$\mathrm{h}(\Theta_+) < \tfrac{3}{4}\log(4.001abc) - \tfrac{1}{4}\log(4a) < \tfrac{3}{2}\ell_c - \tfrac{1}{4}\log a - 0.3463.$$

The finite-place summands are nonpositive. The last step uses $ab < c/4$ and $(3/4)\log 4.001 - \log 4 < -0.3463$.

For Θ_-, the corresponding four pairs are

$$(P_+/\alpha, N_-\beta), \quad (P_-\alpha, N_-\beta), \quad (P_-\alpha, N_+/\beta), \quad (P_+/\alpha, N_+/\beta).$$

We have

$$P_+/\alpha = c\frac{v+w}{1+U} < c.$$

Indeed

$$U - v = \frac{c-b}{abc(U+v)} > 1/c > w - 1,$$

using $c - b > 3ab$, $U + v < 3$, and $w - 1 < 1/(2bc)$. Further

$$P_-\alpha < (b-a)\left(1 + \frac{1}{4ab}\right) < b,$$

$$N_-\beta < (c-a)\left(1 + \frac{1}{4ac}\right) < c, \quad N_+/\beta < b + 1.$$

These follow by rationalizing the small factor and bounding its complementary large factor below. The four projective maxima are therefore smaller than $c, c, b+1, c$. It follows that

$$\mathrm{h}(\Theta_-) < \tfrac{3}{4}\ell_c + \tfrac{1}{4}\log(b+1) < \ell_c,$$

because $b + 1 < c$.

Finally, $P_+ = \sqrt{bc}\,(s + t\sqrt{a/b})$ and $N_\lambda = \sqrt{bc}\,(r + \lambda t\sqrt{a/c})$, so

$$\frac{s + t\sqrt{a/b}}{r + \lambda t\sqrt{a/c}} = P_+/N_\lambda, \qquad \log X_1 - \log X_2 = jA - m'B + \log(P_+/N_\lambda).$$

Since $\Theta_\lambda = (P_+/N_\lambda)\,\alpha^\lambda/\beta$, the right side equals $(j-\lambda)A - (m'-1)B + \log\Theta_\lambda$. By Lemma 2.17, $l \geq 1$, so $j \geq 3$ and $j - \lambda > 0$. □

Lemma 5.13. *Let $0 < a < b < c$ be a positive integer Diophantine triple with $b > 4000$, $b > 8a$ and $c > 4ab$. Put $\ell_c = \log c$, $\mathcal{K} = \mathbb{Q}(\sqrt{ab}, \sqrt{ac})$, let* h *denote the absolute logarithmic height, and for $\lambda \in \{-1, 1\}$ define*

$$D_\lambda = \frac{1 + \sqrt{a/b}}{1 + \lambda\sqrt{a/c}} = \frac{\sqrt{bc} + \sqrt{ac}}{\sqrt{bc} + \lambda\sqrt{ab}}, \qquad \psi = \log D_\lambda.$$

Then $\mathcal{K}$ is a real quartic field, $D_\lambda \in \mathcal{K}$, and

$$N_{\mathcal{K}/\mathbb{Q}}(D_\lambda) = \Big(\frac{c(b-a)}{b(c-a)}\Big)^2 < 1, \qquad 0 < \psi < 0.312,$$

$$\mathrm{h}(D_\lambda) \leq \tfrac{1}{2}\log(bc) + \log\big(1 + \sqrt{a/b}\big) < \ell_c - \tfrac{1}{2}\log a - 0.39.$$

Proof. Since $ab+1$, $ac+1$, $bc+1$ are squares and $ab, ac, bc \geq 1$, none of ab, ac, bc is a square; as $ab \cdot ac = a^2bc$, the field $\mathcal{K}$ has degree four, and it is real. Its four embeddings are $\sigma_{\varepsilon_1\varepsilon_2}$, $\varepsilon_1, \varepsilon_2 \in \{-1, 1\}$, with $\sqrt{ab} \mapsto \varepsilon_1\sqrt{ab}$, $\sqrt{ac} \mapsto \varepsilon_2\sqrt{ac}$, hence $\sqrt{bc} = \sqrt{ab}\sqrt{ac}/a \mapsto \varepsilon_1\varepsilon_2\sqrt{bc}$. Multiplying numerator and denominator of the first expression for D_λ by $\sqrt{bc}$ gives the second, so $D_\lambda = \nu/\delta_\lambda \in \mathcal{K}$ with the algebraic integers

$$\nu = \sqrt{bc} + \sqrt{ac}, \qquad \delta_\lambda = \sqrt{bc} + \lambda\sqrt{ab},$$

whose conjugates are

$$\sigma_{\varepsilon_1\varepsilon_2}(\nu) = \varepsilon_2\big(\varepsilon_1\sqrt{bc} + \sqrt{ac}\big), \qquad \sigma_{\varepsilon_1\varepsilon_2}(\delta_\lambda) = \varepsilon_1\big(\varepsilon_2\sqrt{bc} + \lambda\sqrt{ab}\big).$$

All of them are nonzero, because $ab < ac < bc$.

Each sign ε_i takes the value -1 on exactly two of the four embeddings, so in the norm the outer signs contribute 1 and each inner factor occurs twice:

$$N_{\mathcal{K}/\mathbb{Q}}(\nu) = \big((\sqrt{ac} + \sqrt{bc})(\sqrt{ac} - \sqrt{bc})\big)^2 = c^2(b-a)^2,$$
$$N_{\mathcal{K}/\mathbb{Q}}(\delta_\lambda) = \big((\lambda\sqrt{ab} + \sqrt{bc})(\lambda\sqrt{ab} - \sqrt{bc})\big)^2 = b^2(c-a)^2.$$

Hence $N_{\mathcal{K}/\mathbb{Q}}(D_\lambda) = \big(c(b-a)/(b(c-a))\big)^2$. It is positive since $a < b < c$, and it is less than one because $c(b-a) < b(c-a)$ is equivalent to $ab < ac$.

For $x = \eta_1/\eta_2$ with nonzero algebraic integers $\eta_1, \eta_2 \in \mathcal{K}$, the definition of h and the product formula $\sum_v [\mathcal{K}_v : \mathbb{Q}_v] \log |\eta_2|_v = 0$ give

$$\mathrm{h}(x) = \frac{1}{4}\sum_v [\mathcal{K}_v : \mathbb{Q}_v] \log \max\{|\eta_1|_v, |\eta_2|_v\} \leq \frac{1}{4}\sum_\sigma \log \max\{|\sigma\eta_1|, |\sigma\eta_2|\},$$

since $|\eta_i|_v \leq 1$ at every non-archimedean place and $\mathcal{K}$ has the four real embeddings σ as its archimedean places. By the conjugate formulas and $ab < ac$, every conjugate of ν and of δ_λ has absolute value at most $\sqrt{bc} + \sqrt{ac} = \sqrt{bc}\,\big(1 + \sqrt{a/b}\big)$. Therefore

$$\mathrm{h}(D_\lambda) \leq \tfrac{1}{2}\log(bc) + \log\big(1 + \sqrt{a/b}\big).$$

From $c > 4ab$, $\log b < \ell_c - \log a - \log 4$, so $\frac{1}{2}\log(bc) < \ell_c - \frac{1}{2}\log a - \log 2$. From $b > 8a$, $\sqrt{a/b} < 1/\sqrt{8}$, and

$$\log\big(1 + 1/\sqrt{8}\big) - \log 2 < -0.3904139.$$

Together, $\mathrm{h}(D_\lambda) < \ell_c - \frac{1}{2}\log a - 0.39$.

For $\lambda = 1$, $1 + \sqrt{a/b} > 1 + \sqrt{a/c}$ since $b < c$; for $\lambda = -1$, the numerator exceeds 1 and the denominator lies in $(0, 1)$ since $a < c$. So $D_\lambda > 1$ and $\psi > 0$. From $c > 4ab$ and $b \geq 4001$, $\sqrt{a/c} < 1/(2\sqrt{b}) \leq 1/(2\sqrt{4001})$; with $\sqrt{a/b} < 1/\sqrt{8}$ this gives

$$\psi < \log \frac{1 + 1/\sqrt{8}}{1 - 1/(2\sqrt{4001})} < 0.3106694 < 0.312. \qquad \square$$

Proposition 5.14. *Let $0 < a < b < c < d$ be an irregular Diophantine quadruple with $\deg\{a, b, c\} \geq 2$, and let z have a representation of type I or II. If c lies in a range $[c_0, c_1)$ of the following table, and the entry of that range in the column of the type of the representation is a number Q, then $m' < Q$.*

range $[c_0, c_1)$	*type I*	*type II*
$[10^{30}, 10^{35})$	$8.49 \cdot 10^{11}$	$1.74 \cdot 10^{12}$
$[10^{35}, 10^{40})$	$1.1 \cdot 10^{12}$	$2.29 \cdot 10^{12}$
$[10^{40}, 10^{45})$	$1.39 \cdot 10^{12}$	$2.94 \cdot 10^{12}$
$[10^{45}, 10^{50})$	$1.71 \cdot 10^{12}$	$3.66 \cdot 10^{12}$
$[10^{50}, 10^{55})$	$2.05 \cdot 10^{12}$	$4.49 \cdot 10^{12}$
$[10^{55}, 10^{60})$	$2.41 \cdot 10^{12}$	$5.39 \cdot 10^{12}$
$[10^{60}, 10^{65})$	$2.8 \cdot 10^{12}$	$6.37 \cdot 10^{12}$
$[10^{30}, 10^{65})$	$9.03 \cdot 10^{12}$	$1.73 \cdot 10^{13}$
$[10^{65}, 10^{70})$	$3.45 \cdot 10^{12}$	–
$[10^{65}, 10^{75})$	$3.9 \cdot 10^{12}$	$9.1 \cdot 10^{12}$
$[10^{65}, 10^{80})$	$4.4 \cdot 10^{12}$	$1.03 \cdot 10^{13}$
$[10^{65}, 10^{81})$	–	$1.08 \cdot 10^{13}$
$[10^{65}, 10^{85})$	$4.9 \cdot 10^{12}$	–
$[10^{65}, 10^{90})$	$5.45 \cdot 10^{12}$	$1.44 \cdot 10^{13}$
$[10^{65}, 10^{92})$	–	$1.5 \cdot 10^{13}$
$[10^{65}, 10^{95})$	$6.05 \cdot 10^{12}$	–
$[10^{65}, 10^{99})$	$6.5 \cdot 10^{12}$	–
$[10^{65}, 10^{111})$	$8.5 \cdot 10^{12}$	–
$[10^{65}, 10^{154})$	–	$4.3 \cdot 10^{13}$
$[10^{65}, 10^{163})$	$1.75 \cdot 10^{13}$	–

Proof. Each entry of the table is proved by a row of the tables at the end of the proof: on its range of c, a row turns a cap p_* that is already known into the cap Q. Each row applies the theorem of Mignotte and Voutier, Theorem 5.4, to a linear form in three logarithms in the real quartic field $\mathcal{K} = \mathbb{Q}(\sqrt{ab}, \sqrt{ac})$. The form comes from the Pell equation of $\{a, b\}$ at the common index of Lemma 2.17, and we write it in the basis (α, D, η), with $\eta = \beta/\alpha$ and a correction D of small height: $D = \Theta_\lambda$ of Lemma 5.12 in type II and $D = D_\lambda$ of Lemma 5.13 in type I. The ordinary alternative and the alternative (C1) of the theorem contradict $p \geq Q$ by certified inequalities between explicit numbers; the alternative (C2) is excluded by a centred two-logarithm form and Laurent's theorem, Theorem 5.3, applied for each denominator, and this step carries most of the computation.

The parametrisation of the theorem, the reduction of its hypotheses to the conditions (P) and (M), the consequences of its conclusions, the centred form and the bound from Laurent's theorem are common to both types; the types differ only in the correction and in the sizes of the parameters. Both signs $\lambda \in \{-1, 1\}$ are treated together.

Notation and rows. Throughout, $p = m'$, $\ell_c = \log c$, $A = \log \alpha$, $B = \log \beta$ and $\eta = \beta/\alpha$, and in every row

$$\rho_{\mathrm{MV}} = 15, \qquad M_{\mathrm{MV}} = \tfrac{7}{2}, \qquad u_{\mathrm{MV}} = \rho_{\mathrm{MV}} + 3 = 18, \qquad v_{\mathrm{MV}} = \rho_{\mathrm{MV}} - 1 = 14.$$

A row is of type I or of type II and consists of a range $L_0 \leq \ell_c < U$, a starting cap p_*, a target Q, the parameters χ_{MV} and ℓ of Theorem 5.4, a number n_{sub} of subintervals and, in type I, a bound $h_0 \leq \infty$. For each row we assume that z has a representation of the type of the row with $L_0 \leq \ell_c < U$ and $p < p_*$, suppose $p \geq Q$, and derive a contradiction.

The parametrisation. The choices are those of [17, Section 5.2], made once for a general linear form. Let η_1, η_2, η_3 and b_1, b_2, b_3 be as in Theorem 5.4, with $\mathcal{D} = 4$, with ± 1 as the only

roots of unity in $\mathbb{Q}(\eta_1,\eta_2,\eta_3)$, and $0<\Lambda:=\Lambda_{\rm MV}<1$; let $\mathfrak{a}_1,\mathfrak{a}_2,\mathfrak{a}_3$ be admissible for $\rho_{\rm MV}$ as the parameters A_1,A_2,A_3 of Theorem 5.4 (not to be confused with $A=\log\alpha$), and let $\mathfrak{a}_-$, $\Omega_\pm$, Σ_+ be numbers with

$$\min_i \mathfrak{a}_i > \mathfrak{a}_- > 0, \qquad \Omega_- < \Omega := \mathfrak{a}_1\mathfrak{a}_2\mathfrak{a}_3 < \Omega_+, \qquad \mathfrak{a}_1+\mathfrak{a}_2+\mathfrak{a}_3 < \Sigma_+.$$

For real $0<\chi_{\rm MV}\le 1$, $M_{\rm MV}>0$ and an integer $\ell\ge 5$ put $K_0=\lfloor M_{\rm MV}\ell\Omega\rfloor$, $\theta_1=(\chi_{\rm MV}M_{\rm MV}\ell)^{2/3}$, $\theta_2=(M_{\rm MV}\ell)^{2/3}$, $\theta_3=(3M_{\rm MV}^2)^{1/3}\ell$, $R_i=\lfloor\theta_i\mathfrak{a}_2\mathfrak{a}_3\rfloor$, $S_i=\lfloor\theta_i\mathfrak{a}_1\mathfrak{a}_3\rfloor$, $T_i=\lfloor\theta_i\mathfrak{a}_1\mathfrak{a}_2\rfloor$, $R_{\rm MV}=1+\sum_i R_i$, $S_{\rm MV}=1+\sum_i S_i$, $T_{\rm MV}=1+\sum_i T_i$ and $C_*=\theta_1+\theta_2+\theta_3$. Then $R_i+1>\theta_i\mathfrak{a}_2\mathfrak{a}_3$, and similarly for S_i,T_i. Put $K_-=M_{\rm MV}\ell\Omega_- -1$, $K_+=M_{\rm MV}\ell\Omega_+$ and $N_+=K_+(K_++1)\ell/2$, and suppose

$$\theta_1^2 > \frac{2M_{\rm MV}\ell}{\mathfrak{a}_-}, \qquad \chi_{\rm MV}\sqrt{\theta_1}\,\mathfrak{a}_- > 1, \qquad M_{\rm MV}\Omega_- > 2, \qquad K_-\ge 3. \tag{P}$$

Under (P) the side conditions of Theorem 5.4 hold: $\mathcal{V}^2=(R_1+1)(S_1+1)(T_1+1)>\theta_1^3\Omega^2=(\chi_{\rm MV}M_{\rm MV}\ell\Omega)^2\ge(\chi_{\rm MV}K_0)^2$, so $\mathcal{V}^2>K_0\chi_{\rm MV}\mathcal{V}$; and

$$\frac{(R_1+1)(S_1+1)(T_1+1)}{R_1+S_1+1} > \frac{T_1+1}{(R_1+1)^{-1}+(S_1+1)^{-1}} > \frac{\theta_1^2\Omega}{\mathfrak{a}_1^{-1}+\mathfrak{a}_2^{-1}} \ge \frac{\theta_1^2\mathfrak{a}_-\Omega}{2} > M_{\rm MV}\ell\Omega \ge K_0,$$

and likewise for the other two pairs; this is the first side condition. By independence the two cardinalities in the side conditions equal the corresponding products, and $\theta_1^3\Omega^2>\ell$ (indeed $\theta_1^3\Omega^2>\frac{2M_{\rm MV}\ell}{\mathfrak{a}_-}\theta_1\Omega^2>4\ell\theta_1\mathfrak{a}_-^2>4\ell/\chi_{\rm MV}^2>\ell$ by (P), $M_{\rm MV}\Omega>2$, $\Omega>\mathfrak{a}_-^3$ and $\chi_{\rm MV}\le 1$), $\theta_2^3\Omega^2=(M_{\rm MV}\ell\Omega)^2>\max(2K_0\ell,K_0^2)$ as $M_{\rm MV}\Omega>2$, and $\theta_3^3\Omega^2=3M_{\rm MV}^2\ell^3\Omega^2\ge 3K_0^2\ell$ give the remaining four. Finally $R_{\rm MV}S_{\rm MV}T_{\rm MV}>\theta_3^3\Omega^2>N_*=K_0(K_0+1)\ell/2$, and $K_0>K_-\ge 3$.

By [17, Lemma 3.5(a)], the last factor of the number $\mathfrak{b}$ of Theorem 5.4 is at most $\mathrm{e}^{11/3}K_0^{-2}$. Since $R_{\rm MV}-1\le C_*\mathfrak{a}_2\mathfrak{a}_3$, $S_{\rm MV}-1\le C_*\mathfrak{a}_1\mathfrak{a}_3$ and $T_{\rm MV}-1\le C_*\mathfrak{a}_1\mathfrak{a}_2$, the two numerators of $\mathfrak{b}$ are at most $C_*\Omega\varpi_1$ and $C_*\Omega\varpi_2$, where

$$\varpi_1=\frac{b_3}{\mathfrak{a}_1}+\frac{b_1}{\mathfrak{a}_3}, \qquad \varpi_2=\frac{b_3}{\mathfrak{a}_2}+\frac{b_2}{\mathfrak{a}_3}.$$

With $K_0\ge M_{\rm MV}\ell\Omega-1$ and $\Omega>\Omega_-$, if $\varpi_1<\varpi_1^+$ and $\varpi_2<\varpi_2^+$, then

$$\log\mathfrak{b} < \mathcal{B} := 2\log\frac{C_*}{M_{\rm MV}\ell}+\log\frac{\varpi_1^+\varpi_2^+}{4}+\frac{11}{3}-2\log\Bigl(1-\frac{1}{M_{\rm MV}\ell\Omega_-}\Bigr).$$

Divided by K_0, the main condition of Theorem 5.4 has left side at least $(\frac{\ell}{2}-0.37)\log\rho_{\rm MV}$. On the right, $\frac{5\log N_*}{K_0}<\frac{5\log N_+}{K_-}$; next $g\le\frac14$ and $\mathfrak{a}_1R_{\rm MV}+\mathfrak{a}_2S_{\rm MV}+\mathfrak{a}_3T_{\rm MV}\le 3C_*\Omega+\Sigma_+$, and $(3C_*\Omega+\Sigma_+)/(M_{\rm MV}\ell\Omega-1)$ decreases with Ω; finally $\frac{8(K_0-1)}{3K_0}\log\mathfrak{b}<\frac83\mathcal{B}$ if $\mathcal{B}>0$. Hence the main condition follows from

$$\Bigl(\frac{\ell}{2}-0.37\Bigr)\log\rho_{\rm MV} > \frac{5\log N_+}{K_-}+\frac{\ell}{4}\cdot\frac{3C_*\Omega_-+\Sigma_+}{M_{\rm MV}\ell\Omega_- -1}+\frac83\mathcal{B}. \tag{M}$$

If (P) and (M) hold and $\ell T_{\rm MV}<b_3/2$, then, as $\Lambda<1$, $\Lambda'_{\rm MV}=\Lambda\frac{\ell T_{\rm MV}}{2b_3}\mathrm{e}^{\ell T_{\rm MV}\Lambda/(2b_3)}<\frac14\mathrm{e}^{1/4}\Lambda<\Lambda$, and the ordinary alternative $\Lambda'_{\rm MV}>\rho_{\rm MV}^{-K_0\ell}$ gives

$$\log\Lambda > -M_{\rm MV}\ell^2\Omega\log\rho_{\rm MV}. \tag{O}$$

Alternative (C1) gives $b_3 \le \max(T_1,T_2) \le \max(\theta_1,\theta_2)\,\mathfrak{a}_1\mathfrak{a}_2$. Alternative (C2) gives integers w_1,w_2,w_3 (the u_1,u_2,u_3 of the theorem) with $\gcd(w_1,w_2,w_3)=1$, $w_1b_1+w_2b_2+w_3b_3=0$ and $|w_1| \le \frac{(S_1+1)(T_1+1)}{\mathcal{M}-\max(S_1,T_1)}$, and the two analogous bounds. Since $\mathcal{M} \ge \chi_{\mathrm{MV}}\mathcal{V} > \chi_{\mathrm{MV}}\theta_1^{3/2}\Omega$, $(S_1+1)(T_1+1) \le \theta_1^2\mathfrak{a}_1^2\mathfrak{a}_2\mathfrak{a}_3\bigl(1+\frac{1}{\theta_1\mathfrak{a}_-^2}\bigr)^2$ and $\max(S_1,T_1) \le \theta_1\mathfrak{a}_1\max(\mathfrak{a}_2,\mathfrak{a}_3)$, we get $|w_i| \le F\mathfrak{a}_i$ for $i=1,2,3$, with

$$F = \frac{\sqrt{\theta_1}}{\chi_{\mathrm{MV}}}\,\frac{\bigl(1+1/(\theta_1\mathfrak{a}_-^2)\bigr)^2}{1-1/(\chi_{\mathrm{MV}}\sqrt{\theta_1}\,\mathfrak{a}_-)}.$$

The centred form. Let $\Lambda = KA-PB+\psi > 0$ with positive integers K,P, where $\psi=\log D$ for a number $D>1$ of $\mathcal{K}$ with $\mathrm{N}(D)\ne 1$, and let $\mathcal{A},\nu,C$ be integers with

$$\mathcal{A}P-\nu K+C=0, \qquad \mathcal{A},\nu>0, \qquad \gcd(\mathcal{A},\nu)=1, \qquad \nu<2P.$$

Choose $p_0\in\mathbb{Z}$ with $|p_0|\le \nu/2$ and $\mathcal{A}p_0 \equiv -C \pmod{\nu}$, and put $k_0=(\mathcal{A}p_0+C)/\nu$ and $t_0=(P-p_0)/\nu$; these are integers, $t_0>0$ and $K=k_0+\mathcal{A}t_0$. With

$$\zeta=\beta^{\nu}\alpha^{-\mathcal{A}}, \qquad \xi=D^{-1}\beta^{p_0}\alpha^{-k_0},$$

one has exactly

$$-\Lambda = t_0\log\zeta+\log\xi, \qquad \log\zeta=\frac{\nu(\psi-\Lambda)+CA}{P}, \qquad \log\xi=\frac{p_0}{\nu}\log\zeta-\frac{CA}{\nu}-\psi,$$

and $\mathcal{A}A=\nu B-\log\zeta$, $|k_0|\le\frac{\mathcal{A}}{2}+\frac{|C|}{\nu}$. The four real embeddings of $\mathcal{K}$ send (α,β) to $(\alpha^{\pm1},\beta^{\pm1})$ with independent signs. So the unit $\zeta\ne1$ has the four distinct conjugates $\beta^{\pm\nu}\alpha^{\pm\mathcal{A}}$ and generates $\mathcal{K}$, and $\mathrm{h}(\zeta)=\frac12\max(\nu B,\mathcal{A}A)$; and $\mathrm{N}(\xi)=\mathrm{N}(D)^{-1}\ne1$, so ζ,ξ are multiplicatively independent. If $\log\zeta$ and $\log\xi$ had the same sign, then $\Lambda\ge|\log\xi|$, and Lemma 5.5 would give $\log\Lambda > -4\,\mathrm{h}(\xi)-4$. Hence, if $H_x\ge\mathrm{h}(\xi)$ and $\log\Lambda<-4H_x-4$, the logarithms have opposite signs; inverting as needed, $-\Lambda$ becomes a linear form in two positive logarithms with coefficients t_0 and 1.

Laurent's theorem for each denominator. Let z_0, H_z, H_x, X_ξ, t_ν be upper bounds for $|\log\zeta|$, $\mathrm{h}(\zeta)$, $\mathrm{h}(\xi)$, $|\log\xi|$ and t_0 that depend only on ν and on the row (and on a subinterval of B, when there is one). Apply Theorem 5.3 with $\mathcal{D}=4$, $\rho_\nu=\nu+2$ in place of ρ, $\mu=0.63$, $\sigma_L=\frac{1+2\mu-\mu^2}{2}$, $\lambda_{L,\nu}=\sigma_L\log\rho_\nu$, the parameters $A_z\ge 8H_z+(\rho_\nu-1)z_0+1$ and $A_x\ge 8H_x+(\rho_\nu-1)X_\xi+1$, and

$$h_{L,\nu}\ge\max\Bigl\{4\Bigl(\log\Bigl(\frac{1}{A_z}+\frac{t_\nu}{A_x}\Bigr)+\log\lambda_{L,\nu}+1.75\Bigr)+0.06,\ \lambda_{L,\nu},\ 2\log2\Bigr\},$$

each taken as the upper endpoint of its interval enclosure. With $\mathcal{H}$, ω_L, θ_L, $\mathcal{C}$ and $\mathcal{C}'$ formed from $(h_{L,\nu},\lambda_{L,\nu},A_z,A_x)$ as in Theorem 5.3, and if $A_zA_x>\lambda_{L,\nu}^2$, the theorem gives $\log\Lambda\ge -E_\nu$, where

$$E_\nu=\mathcal{C}(h_{L,\nu}+\log\rho_\nu)^2A_zA_x+\sqrt{\omega_L\theta_L}\,(h_{L,\nu}+\log\rho_\nu)+\log\bigl(\mathcal{C}'(h_{L,\nu}+\log\rho_\nu)^2A_zA_x\bigr).$$

When the bounds also depend on a subinterval j, we write $E_{\nu,j}$.

Type II: sizes. Let z have a type II representation, so that $p=m'=2m+1$. By Lemma 2.9, Lemma 2.10 and Theorem 2.7, $b>4000$, $b>8a$, and $c=d_+(a,b,e)>4abe\ge4ab$ with $e=d_-(a,b,c)\ge1$; hence $c>32a^2$. From $\alpha>2\sqrt{ab}>2\sqrt{4000}$ we get $A>4.8$. From $2\sqrt{ac}<\beta<2\sqrt{ac}\,(1+\frac{1}{4ac})$ and $\log a<\frac12(\ell_c-\log32)$,

$$\tfrac{\ell_c}{2}+\log2<B<\log2+\tfrac12\log a+\tfrac{\ell_c}{2}+\tfrac{1}{4ac}<\tfrac34\ell_c-\tfrac14\log2+\tfrac{1}{4ac}<\tfrac34\ell_c.$$

Also $A < B$, and $\beta/\alpha > \sqrt{c/b}\,/(1+\frac{1}{4ab}) > 1.99$ since $c/b > 4a \ge 4$. In summary

$$4.8 < A < B, \qquad B - A > \log 1.99, \qquad \tfrac{\ell_c}{2} + \log 2 < B < \tfrac{3}{4}\ell_c. \tag{II.1}$$

Type II: the exact form and its smallness. By Lemma 2.17 there is an integer $l \ge 1$ with $y\sqrt{a} + x\sqrt{b} = (t\sqrt{a} + s\sqrt{b})\alpha^{2l+1}$, and the type II representation gives $z\sqrt{a} + x\sqrt{c} = (\lambda t\sqrt{a} + r\sqrt{c})\beta^p$. Put $K = 2l + 1 - \lambda > 0$, let Θ_λ be as in Lemma 5.12, and put $\psi = \log\Theta_\lambda$. That lemma gives, in $\mathcal{K}$,

$$0 < \psi < 0.000063, \quad \mathrm{N}(\Theta_\lambda) = \Big(\frac{c(b-a)}{b(c-a)}\Big)^2 < 1, \quad \mathrm{h}(\Theta_+) < \tfrac{3}{2}\ell_c - \tfrac{1}{4}\log a - 0.3463, \quad \mathrm{h}(\Theta_-) < \ell_c. \tag{II.2}$$

Put $X_1 = (s + t\sqrt{a/b})\alpha^{2l+1}$, $X_2 = (r + \lambda t\sqrt{a/c})\beta^p$, $q_1 = 1 - a/b$, $q_2 = 1 - a/c$. The two identities above, together with the same identities with $\sqrt{a}$ replaced by $-\sqrt{a}$ (which hold because x, y, z, r, s, t are integers), give

$$2x = X_1 + q_1/X_1 = X_2 + q_2/X_2,$$

and Lemma 5.12 gives exactly

$$\Lambda := \log X_1 - \log X_2 = KA - 2mB + \psi.$$

For $p \ge Q$ we have $X_2 > 2$, since $r + \lambda t\sqrt{a/c} \ge r - t\sqrt{a/c} = q_2/(r + t\sqrt{a/c}) > q_2/(2r)$ (note $t^2a/c = ab + a/c < r^2$). As $q_1 < q_2 < 1 < X_2$, the numbers X_i are the larger roots of $X^2 - 2xX + q_i$, so $X_1 > X_2$ and $\Lambda > 0$. Since $X_1 - X_2 = q_2/X_2 - q_1/X_1$ and $q_1/X_1 \ge q_1/X_2 - q_1(X_1 - X_2)/X_2^2$, we get $(X_1 - X_2)(1 - q_1/X_2^2) \le (q_2 - q_1)/X_2 < 1/(8X_2)$, so $X_1 - X_2 < 1/(6X_2)$ and

$$0 < \Lambda < \frac{X_1 - X_2}{X_2} < \frac{1}{6X_2^2} < \frac{4r^2}{6q_2^2}\,\beta^{-2p} < c\beta^{-2p},$$

using $r^2 = ab + 1 < c$ and $q_2 > 1 - 1/16000$. Since $\ell_c < 2B$ by (II.1),

$$\log\Lambda < \ell_c - 2pB < -4mB < -2m\ell_c. \tag{II.3}$$

Type II: signs. By (II.1), (II.2) and $\Lambda > 0$, $KA > 2mB - \psi = 2mA + 2m(B - A) - \psi > 2mA$, so $K > 2m$. The number $\Theta_\lambda \ne 1$ is positive real algebraic of degree at most four, so Lemma 5.5 and (II.2) give $\log\psi > -4\,\mathrm{h}(\Theta_\lambda) - 4 > -6\ell_c - 4$. Since $2m \ge Q - 1 > 7$, (II.3) gives $\log\Lambda < -6\ell_c - 4 < \log\psi$. Hence $\Lambda < \psi$, that is

$$KA < 2mB. \tag{II.4}$$

Type II: the three-logarithm form. We have $\Lambda = (K - 2m)\log\alpha + \log\Theta_\lambda - 2m\log\eta$. Apply Theorem 5.4 with $(\eta_1, \eta_2, \eta_3) = (\alpha, \Theta_\lambda, \eta)$ and $(b_1, b_2, b_3) = (K - 2m, 1, 2m)$; the coefficients are positive integers with gcd one, and $\eta_i > 1$. Since the four real embeddings of $\mathcal{K}$ send (α, β) to $(\alpha^{\pm1}, \beta^{\pm1})$ with independent signs, for integers i, j

$$\mathrm{h}(\beta^i\alpha^j) = \tfrac{1}{4}\big(|iB + jA| + |iB - jA|\big) = \tfrac{1}{2}\max(|i|B, |j|A), \tag{II.5}$$

α and β are multiplicatively independent, and a relation $\alpha^i\Theta_\lambda^j\eta^k = 1$ forces $j = 0$ by the norm in (II.2) and then $i = k = 0$. Thus $\alpha, \Theta_\lambda, \eta$ are multiplicatively independent, $\mathcal{D} = 4$, the only roots of unity in the real field $\mathcal{K}$ are ±1, and $0 < \Lambda < 1 < \pi$. Put

$$\mathfrak{a}_1 = u_{\mathrm{MV}}A, \qquad \mathfrak{a}_2 = 12\ell_c, \qquad \mathfrak{a}_3 = u_{\mathrm{MV}}B - v_{\mathrm{MV}}A = 4B + v_{\mathrm{MV}}(B - A).$$

By (II.5), $\mathrm{h}(\alpha) = \frac{A}{2}$ and $\mathrm{h}(\eta) = \frac{B}{2}$, so $\mathfrak{a}_1$ and $\mathfrak{a}_3$ equal $(\rho_{\mathrm{MV}} - 1)\log\eta_i + 8\,\mathrm{h}(\eta_i)$; by (II.2), $(\rho_{\mathrm{MV}} - 1)\psi + 8\,\mathrm{h}(\Theta_\lambda) < 14 \cdot 0.000063 + \max(12\ell_c - 2\log a - 2.77,\, 8\ell_c) < \mathfrak{a}_2$ for both signs. So all $\mathfrak{a}_i$ are admissible as the parameters A_i of Theorem 5.4. By (II.1), $\mathfrak{a}_1 > 4.8u_{\mathrm{MV}} = 86.4 =: \mathfrak{a}_-$, $\mathfrak{a}_3 > 4B > 2\ell_c \geq 2L_0$ and $\mathfrak{a}_2 \geq 12L_0$, so $\min\mathfrak{a}_i > \mathfrak{a}_-$ (in every row $2L_0 \geq 60\log 10 > \mathfrak{a}_-$) and $\Omega := \mathfrak{a}_1\mathfrak{a}_2\mathfrak{a}_3 > \Omega_- := \mathfrak{a}_- \cdot 2L_0 \cdot 12L_0$. With $r_0 = A/B$, $\mathfrak{a}_1\mathfrak{a}_3/B = u_{\mathrm{MV}}r_0(u_{\mathrm{MV}} - v_{\mathrm{MV}}r_0)B \leq C_\rho B$, $C_\rho = u_{\mathrm{MV}}^3/(4v_{\mathrm{MV}}) = \frac{729}{7}$, so by (II.1)

$$\begin{gathered}\frac{\Omega}{B} < 9C_\rho U^2, \qquad \Omega < \Omega_+ := \tfrac{27}{4}C_\rho U^3, \\ \mathfrak{a}_1 + \mathfrak{a}_2 + \mathfrak{a}_3 = 4A + u_{\mathrm{MV}}B + 12\ell_c < \Sigma_+ := \big(\tfrac{3}{4}(\rho_{\mathrm{MV}} + 7) + 12\big)U.\end{gathered} \tag{II.6}$$

Type II: the theorem applied. We use the parametrisation with these $\mathfrak{a}_i$, $\mathfrak{a}_-$, $\Omega_\pm$ and Σ_+; (P) holds in each row (see the rows of type II below). Here $\delta_1 = \gcd(K - 2m, 2m) \geq 1$ and $\delta_2 = 1$. By (II.4), $K - 2m < 2m(B - A)/A$, and $u_{\mathrm{MV}}(B - A) \leq \mathfrak{a}_3$ as $u_{\mathrm{MV}} > v_{\mathrm{MV}}$; so $\frac{K-2m}{\mathfrak{a}_3} < \frac{2m}{u_{\mathrm{MV}}A}$, and by (II.1) $\varpi_1 < \frac{4m}{u_{\mathrm{MV}}A} < \frac{2p_*}{4.8u_{\mathrm{MV}}} =: \varpi_1^+$. Also $\varpi_2 < (\frac{p_*}{12} + \frac{1}{2})/L_0 =: \varpi_2^+$. So

$$\mathcal{B} = 2\log\frac{C_*}{M_{\mathrm{MV}}\ell} + \log\Big[\frac{2p_*}{4.8u_{\mathrm{MV}}} \cdot \frac{p_*/12 + 1/2}{4L_0}\Big] + \frac{11}{3} - 2\log\Big(1 - \frac{1}{M_{\mathrm{MV}}\ell\Omega_-}\Big),$$

and the certificates verify $\mathcal{B} > 0$ and (M) in each row (margins in the tables).

For the ordinary alternative put $T_0 = \ell\big(1 + C_* \cdot \frac{3}{4}u_{\mathrm{MV}}U \cdot 12U\big)$; by (II.1), $\ell T_{\mathrm{MV}} \leq T_0$, and the certificates give $T_0 < (Q - 1)/2 \leq b_3/2$. So (O) holds, and with (II.3), (II.6),

$$p < \frac{\ell_c}{2B} + \frac{M_{\mathrm{MV}}\ell^2\log\rho_{\mathrm{MV}}}{2} \cdot \frac{\Omega}{B} < 1 + \frac{M_{\mathrm{MV}}\ell^2\log\rho_{\mathrm{MV}}}{2} \cdot 9C_\rho U^2 =: \mathrm{cap};$$

the certificates give $\mathrm{cap} < Q$, contradicting $p \geq Q$.

Alternative (C1) gives $2m = b_3 \leq \max(\theta_1, \theta_2)\,\mathfrak{a}_1\mathfrak{a}_2 < \max(\theta_1, \theta_2) \cdot \frac{3}{4}u_{\mathrm{MV}}U \cdot 12U < Q - 1$ (certified), which is impossible.

Alternative (C2) gives w_1, w_2, w_3 with $w_1(K - 2m) + w_2 + 2mw_3 = 0$ and $|w_i| \leq F\mathfrak{a}_i$. If $w_1 = 0$, then $w_3 = \pm 1$ by coprimality and $2m = |w_2| \leq 12F\ell_c < 12FU < Q - 1$ (certified), impossible. So $w_1 \neq 0$; let $g_0 = \gcd(w_1, w_3 - w_1)$, which divides w_2, and put $\nu = |w_1|/g_0$, $\mathcal{A} = -\operatorname{sgn}(w_1)(w_3 - w_1)/g_0$, $C = -\operatorname{sgn}(w_1)w_2/g_0$. Then

$$2m\mathcal{A} - \nu K + C = 0, \qquad \gcd(\mathcal{A}, \nu) = 1, \qquad 1 \leq \nu \leq Fu_{\mathrm{MV}}A, \qquad |C| \leq 12F\ell_c,$$

and $\mathcal{A} > 0$ because $2m\mathcal{A} = \nu K - C \geq K - |C| > 0$. The centred form applies with $P = 2m$ and $D = \Theta_\lambda$, since $\nu \leq \nu_{\max} < Q - 1 \leq 2m$ with $\nu_{\max}$ below (tables).

Let $B^- = \frac{L_0}{2} + \log 2$, $B^+ = \frac{3}{4}U$, $B_j = B^- + (B^+ - B^-)j/n_{\mathrm{sub}}$ for $0 \leq j \leq n_{\mathrm{sub}}$, and choose j with $B \in [B_j, B_{j+1}]$, possible by (II.1). Then $A < B_{j+1}$, $\ell_c < 2B - 2\log 2 < L_j^+ := \min(U, 2B_{j+1})$, $\nu \leq Fu_{\mathrm{MV}}A < Fu_{\mathrm{MV}}B_{j+1}$ and $|C| < C_j := 12FL_j^+$. Using $0 < \Lambda < \psi < 0.000063$ from (II.4) and (II.2), $2m \geq Q - 1$, (II.5), $\mathcal{A}A = \nu B - \log\zeta$ and $|k_0| \leq \frac{\mathcal{A}}{2} + \frac{|C|}{\nu}$, we get

$$|\log\zeta| < z_0 := \frac{1.000063\,\nu + C_jB_{j+1}}{Q - 1}, \qquad \mathrm{h}(\zeta) = \tfrac{1}{2}\max(\nu B, \mathcal{A}A) \leq \tfrac{1}{2}(\nu B_{j+1} + z_0) =: H_z,$$

$$\mathrm{h}(\xi) \leq \tfrac{3}{2}L_j^+ + \frac{\nu B_{j+1} + z_0}{4} + \frac{C_jB_{j+1}}{2\nu} =: H_x, \qquad |\log\xi| < 0.000063 + \frac{z_0}{2} + \frac{C_jB_{j+1}}{\nu} =: X_\xi,$$

$$t_0 < \frac{p_*}{\nu} + \frac{1}{2} =: t_\nu,$$

where $\mathrm{h}(\Theta_\lambda^{-1}) < \frac{3}{2}\ell_c$ by (II.2). The certificates give $4\max H_x + 4 < 2(Q-1)B^-$, and (II.3) gives $\log\Lambda < -4mB \le -2(Q-1)B^-$; so the logarithms of ζ and ξ have opposite signs.

Laurent's theorem excludes every denominator: for every integer $1 \le \nu \le \nu_{\max} := \lfloor \frac{3}{4} F u_{\mathrm{MV}} U \rfloor$ and every j with $\nu \le F u_{\mathrm{MV}} B_{j+1}$ the certificates verify $A_z A_x > \lambda_{L,\nu}^2$ and $E_{\nu,j} < 2(Q-1)B_j$. Since (II.3) gives $\log\Lambda < -4mB \le -2(Q-1)B_j$, this is a contradiction. All alternatives of the theorem are excluded, so $p < Q$ in each type II row.

Type I: sizes. Let z have a type I representation, so that $p = m' = 2m$. By Lemma 2.9, Lemma 2.10 and Theorem 2.7, $b > 4000$, $b > 8a$, $e = d_-(a,b,c) \ge 1$ and $c = d_+(a,b,e) > 4abe+a+b+e > 4(ab+1)$. Hence $c > 32a^2$, and $c > \alpha^2$ since $\alpha < 2r$. From $\alpha > 2\sqrt{ab} > 2\sqrt{4000}$ and $c > \alpha^2$ we get $4.8 < A < \frac{\ell_c}{2}$. As in type II, $\frac{\ell_c}{2} + \log 2 < B < \frac{3}{4}\ell_c$. Since $\beta/\sqrt{ac} < \alpha/\sqrt{ab}$, we have $\eta < \sqrt{c/b}$; and $\eta > \sqrt{c/b}/(1+\frac{1}{4ab}) > 1.99$ since $c/b > 4a \ge 4$. In summary

$$4.8 < A < \tfrac{\ell_c}{2}, \qquad \log 1.99 < B - A < \tfrac{1}{2}\log(c/b), \qquad \tfrac{\ell_c}{2} + \log 2 < B < \tfrac{3}{4}\ell_c. \tag{I.1}$$

Type I: the bound on $B - A$ (rows with $h_0 < \infty$). In these rows h_0 is an upper bound for $\log(p_*/(2\kappa))$, with the constant $\kappa = (\sqrt{17}-1)/2$. By Lemma 2.23, $m > \kappa\sqrt{c/b}$. With (I.1) and $m = p/2 < p_*/2$,

$$B - A < \log(m/\kappa) < h_0, \qquad B < \tfrac{\ell_c}{2} + h_0, \qquad A > A_0 := \tfrac{L_0}{2} + \log 2 - h_0. \tag{I.2}$$

In the rows with $h_0 = \infty$ we put $A_0 = 4.8$; there (I.2) is not used.

Type I: the exact form. By Lemma 2.17 there is an integer $l \ge 1$ with $y\sqrt{a} + x\sqrt{b} = (\sqrt{a}+\sqrt{b})\alpha^{2l}$, and the type I representation gives $z\sqrt{a} + x\sqrt{c} = (\lambda\sqrt{a} + \sqrt{c})\beta^p$. Let D_λ and $\psi = \log D_\lambda$ be as in Lemma 5.13, and put

$$X_1 = (1+\sqrt{a/b})\alpha^{2l}, \quad X_2 = (1+\lambda\sqrt{a/c})\beta^p, \quad q_1 = 1 - \tfrac{a}{b}, \quad q_2 = 1 - \tfrac{a}{c}.$$

The two identities above, together with their images under $\sqrt{a} \mapsto -\sqrt{a}$ (which hold because x, y, z are integers), give $2x = X_1 + q_1/X_1 = X_2 + q_2/X_2$, so $X_i = x + \sqrt{x^2 - q_i}$ and $X_1 > X_2$ since $q_1 < q_2$. As $X_2 \ge (1-\sqrt{a/c})\beta^p > 2$, $\sqrt{x^2 - q_2} = \frac{1}{2}(X_2 - q_2/X_2) > \frac{3}{8}X_2$ and $q_2 - q_1 < a/b < \frac{1}{8}$, we get $X_1 - X_2 < 1/(6X_2)$. With $\sqrt{a/c} < 1/(2\sqrt{4001})$ and $6(1 - 1/(2\sqrt{4001}))^2 > 1$,

$$0 < \Lambda := 2lA - pB + \psi = \log(X_1/X_2) < \frac{1}{6X_2^2} < \beta^{-2p}, \qquad \log\Lambda < -2pB. \tag{I.3}$$

Since $b > 4000$, $b > 8a$ and $c > 4ab$, Lemma 5.13 gives, in $\mathcal{K}$, which contains D_λ,

$$\mathrm{N}(D_\lambda) = \Big(\frac{c(b-a)}{b(c-a)}\Big)^2 < 1, \qquad \mathrm{h}(D_\lambda) < \ell_c - \tfrac{1}{2}\log a - 0.39, \qquad 0 < \psi < 0.312. \tag{I.4}$$

Since $B - A > 0.68 > \psi$, (I.3) gives $2lA > pB - \psi > pA$, so $2l > p$. By Lemma 5.5 and (I.4), $\log\psi > -4\,\mathrm{h}(D_\lambda) - 4 > -4\ell_c - 4$, while (I.3) gives $\log\Lambda < -p\ell_c < -4\ell_c - 4$ as $p \ge Q - 1 > 5$. Hence $\Lambda < \psi$, and

$$p < 2l, \qquad 2lA < pB. \tag{I.5}$$

Type I: the three-logarithm form. We have $\Lambda = (2l-p)\log\alpha + \log D_\lambda - p\log\eta$. Apply Theorem 5.4 with $(\eta_1,\eta_2,\eta_3) = (\alpha, D_\lambda, \eta)$ and $(b_1,b_2,b_3) = (2l-p, 1, p)$; by (I.5) the coefficients are positive integers with gcd one, and $\eta_i > 1$. As in type II, for integers i, j

$$\mathrm{h}(\beta^i\alpha^j) = \tfrac{1}{2}\max(|i|B, |j|A), \tag{I.6}$$

and α, D_λ, η are multiplicatively independent, by the norm in (I.4). Thus $\mathcal{D} = 4$, the only roots of unity in $\mathcal{K}$ are ± 1, and $0 < \Lambda < 1 < \pi$. Put

$$\mathfrak{a}_1 = u_{\mathrm{MV}} A, \qquad \mathfrak{a}_2 = 8\ell_c - 4\log a + c_D, \quad c_D = 0.312 v_{\mathrm{MV}} - 3.12,$$
$$\mathfrak{a}_3 = u_{\mathrm{MV}} B - v_{\mathrm{MV}} A = 4B + v_{\mathrm{MV}}(B - A).$$

By (I.6), $\mathfrak{a}_1$ and $\mathfrak{a}_3$ equal $(\rho_{\mathrm{MV}} - 1)\log\eta_i + 8\,\mathrm{h}(\eta_i)$; by (I.4), $(\rho_{\mathrm{MV}} - 1)\psi + 8\,\mathrm{h}(D_\lambda) < \mathfrak{a}_2$. So all $\mathfrak{a}_i$ are admissible. From (I.1), (I.2) and $\log a < \ell_c/2$: $\mathfrak{a}_1 > u_{\mathrm{MV}} A_0$, $\mathfrak{a}_3 > 4B > 2\ell_c \geq 2L_0$, $\mathfrak{a}_2 > 6\ell_c \geq 6L_0$; so $\min \mathfrak{a}_i > \mathfrak{a}_- := \min(u_{\mathrm{MV}} A_0, 2L_0)$ and $\Omega := \mathfrak{a}_1\mathfrak{a}_2\mathfrak{a}_3 > \Omega_- := u_{\mathrm{MV}} A_0 \cdot 2L_0 \cdot 6L_0$. From $B < \log 2 + \frac{1}{2}\log a + \frac{\ell_c}{2} + \frac{1}{4ac}$ we get $-4\log a < 4\ell_c - 8B + 8\log 2 + \frac{2}{ac}$, hence

$$\mathfrak{a}_2 < 12\ell_c - 8B + \kappa_D, \qquad \kappa_D = 8\log 2 + c_D + 10^{-6}, \qquad \text{and} \qquad \mathfrak{a}_2 \leq 8\ell_c + c_D. \tag{I.7}$$

With $\hat{B} = B/\ell_c$ and $\hat{A} = A/\ell_c$ this gives

$$\frac{\Omega}{B} < \Phi(\ell_c, \hat{B}, \hat{A}) := \frac{u_{\mathrm{MV}} \hat{A}(u_{\mathrm{MV}} \hat{B} - v_{\mathrm{MV}} \hat{A})}{\hat{B}} \big((12 - 8\hat{B})\ell_c^2 + \kappa_D \ell_c\big) \tag{I.8}$$

on the region $\mathcal{G}$: $L_0 \leq \ell_c \leq U$, $\frac{1}{2} + \frac{\log 2}{\ell_c} < \hat{B} < \frac{3}{4}$, $\frac{4.8}{\ell_c} < \hat{A} < \min(\frac{1}{2}, \hat{B} - \frac{\log 1.99}{\ell_c})$, and in the rows with $h_0 < \infty$ also $\hat{B} - \hat{A} < h_0/\ell_c$ and $\hat{B} < \frac{1}{2} + h_0/\ell_c$, by (I.1) and (I.2). The certificates cover $\mathcal{G}$ by boxes (n_L intervals in ℓ_c, n_B in $\hat{B}$ and n_A in $\hat{A}$, with (n_L, n_B, n_A) given with the tables; boxes violating a necessary condition of $\mathcal{G}$ are skipped) and enclose Φ on each box with ℓ_c at the upper endpoint of its interval, as Φ increases in ℓ_c; the maximum Φ^* of the upper endpoints bounds Ω/B, and the same cover bounds $\Omega = (\Omega/B)\,\hat{B}\ell_c$ by Ω_+. Finally $\mathfrak{a}_1 + \mathfrak{a}_2 + \mathfrak{a}_3 = 4A + u_{\mathrm{MV}} B + \mathfrak{a}_2 < v_{\mathrm{MV}} B + 12\ell_c + \kappa_D < \Sigma_+ := v_{\mathrm{MV}} B^+ + 12U + \kappa_D$, with $B^+ = \frac{3}{4}U$ in the rows with $h_0 = \infty$ and $B^+ = \min(\frac{3}{4}U, \frac{U}{2} + h_0)$ in the other rows.

Type I: the theorem applied. We use the parametrisation with these $\mathfrak{a}_i$, $\mathfrak{a}_-$, $\Omega_\pm$ and Σ_+; (P) holds in each row (see the rows of type I below). Here $\delta_1 = \gcd(2l - p, p) \geq 1$ and $\delta_2 = 1$. By (I.5), $2l - p < p(B - A)/A$, and $\mathfrak{a}_3 \geq u_{\mathrm{MV}}(B - A)$, so $\frac{2l-p}{\mathfrak{a}_3} < \frac{p}{u_{\mathrm{MV}} A}$ and $\varpi_1 < \frac{2p_*}{u_{\mathrm{MV}} A_0} =: \varpi_1^+$; also $\varpi_2 < (\frac{p_*}{6} + \frac{1}{2})/L_0 =: \varpi_2^+$. So

$$\mathcal{B} = 2\log\frac{C_*}{M_{\mathrm{MV}}\ell} + \log\Big[\frac{2p_*}{u_{\mathrm{MV}} A_0} \cdot \frac{p_*/6 + 1/2}{4L_0}\Big] + \frac{11}{3} - 2\log\Big(1 - \frac{1}{M_{\mathrm{MV}}\ell\Omega_-}\Big),$$

and the certificates verify $\mathcal{B} > 0$ and (M) in each row.

The ordinary alternative is excluded as in type II, with $T_0 = \ell\big(1 + C_* u_{\mathrm{MV}} \frac{U}{2}(8U + c_D)\big)$; by (I.1) and (I.7), $\ell T_{\mathrm{MV}} \leq T_0$, and the certificates give $T_0 < (Q - 1)/2 < b_3/2$. So (O) holds, and with (I.3) and (I.8)

$$p < \frac{M_{\mathrm{MV}}\ell^2 \log\rho_{\mathrm{MV}}}{2} \cdot \frac{\Omega}{B} < 1 + \frac{M_{\mathrm{MV}}\ell^2 \log\rho_{\mathrm{MV}}}{2}\,\Phi^* =: \mathrm{cap};$$

the certificates give $\mathrm{cap} < Q$, contradicting $p \geq Q$.

Alternative (C1) gives $p = b_3 \leq \max(\theta_1, \theta_2)\,\mathfrak{a}_1\mathfrak{a}_2 < \max(\theta_1, \theta_2)\,u_{\mathrm{MV}} \frac{U}{2}(8U + c_D) < Q - 1$ (certified), which is impossible.

Alternative (C2) gives w_1, w_2, w_3 with $w_1(2l - p) + w_2 + w_3 p = 0$ and $|w_i| \leq F\mathfrak{a}_i$. If $w_1 = 0$, then $w_3 = \pm 1$ and $p = |w_2| \leq F\mathfrak{a}_2 \leq F(8U + c_D) < Q - 1$ (certified), impossible. So $w_1 \neq 0$; define g_0, ν, $\mathcal{A}$ and C from w_1, w_2, w_3 as in type II. Then

$$\mathcal{A}p - 2\nu l + C = 0, \qquad \gcd(\mathcal{A}, \nu) = 1, \qquad 1 \leq \nu \leq F u_{\mathrm{MV}} A, \qquad |C| \leq F\mathfrak{a}_2 \leq F(8\ell_c + c_D),$$

and $\mathcal{A} > 0$ because $\mathcal{A}p = 2\nu l - C \geq 2l - |C| > p - |C| > 0$, as $|C| < Q - 1$. The centred form applies with $P = p$, $K = 2l$ and $D = D_\lambda$, since the certificates give $\nu_{\max} < 2(Q-1)$, with $\nu_{\max}$ below, so that $\nu < 2p$.

Subintervals of B are taken as in type II: let $B^- = \frac{L_0}{2} + \log 2$, $B_j = B^- + (B^+ - B^-)j/n_{\text{sub}}$ for $0 \leq j \leq n_{\text{sub}}$, and choose j with $B \in [B_j, B_{j+1}]$, possible by (I.1) and (I.2). Then $\ell_c < 2B - 2\log 2 < L_j^+ := \min(U, 2B_{j+1})$, $A < \frac{\ell_c}{2} < \frac{1}{2}L_j^+$, $\nu \leq Fu_{\text{MV}}A < \frac{1}{2}Fu_{\text{MV}}L_j^+$ and $|C| < C_j := F(8L_j^+ + c_D)$. Using $0 < \Lambda < \psi < 0.312$ from (I.4) and (I.5), $p \geq Q$, (I.6), $\mathcal{A}A = \nu B - \log\zeta$, $|k_0| \leq \frac{\mathcal{A}}{2} + \frac{|C|}{\nu}$ and $\mathrm{h}(D_\lambda^{-1}) < \ell_c < L_j^+$ from (I.4), we get

$$|\log\zeta| < z_0 := \frac{1.312\,\nu + \frac{1}{2}C_jL_j^+}{Q-1}, \qquad \mathrm{h}(\zeta) = \tfrac{1}{2}\max(\nu B, \mathcal{A}A) \leq \tfrac{1}{2}(\nu B_{j+1} + z_0) =: H_z,$$

$$\mathrm{h}(\xi) \leq L_j^+ + \frac{\nu B_{j+1} + z_0}{4} + \frac{C_jL_j^+}{4\nu} =: H_x, \qquad |\log\xi| < 0.312 + \frac{z_0}{2} + \frac{C_jL_j^+}{2\nu} =: X_\xi,$$

$$t_0 < \frac{p_*}{\nu} + \frac{1}{2} =: t_\nu.$$

The certificates give $4\max H_x + 4 < 2(Q-1)B^-$, and (I.3) gives $\log\Lambda < -2pB \leq -2(Q-1)B^-$; so the logarithms of ζ and ξ have opposite signs.

For every integer $1 \leq \nu \leq \nu_{\max} := \lfloor \frac{1}{2}Fu_{\text{MV}}U \rfloor$ and every j with $\nu \leq \frac{1}{2}Fu_{\text{MV}}L_j^+$ the certificates verify $A_zA_x > \lambda_{L,\nu}^2$ and $E_{\nu,j} < 2(Q-1)B_j$. Since (I.3) gives $\log\Lambda < -2pB \leq -2(Q-1)B_j$, this is a contradiction. All alternatives of the theorem are excluded, so $p < Q$ in each type I row.

Starting caps and chaining. The rows form the sequences II.1 to II.15, II.16 to II.18, II.19 and II.20, II.21 to II.25, I.1 to I.15, I.16 to I.19, I.20 to I.23 and I.24 to I.32. The first row of each sequence takes p_* from the a priori cap: Proposition 5.9 gives $p - 2 < K_{\text{a}}\ell_c^2\log(39(p+1))$ with the a priori constant $K_{\text{a}} = 4.436 \cdot 10^{13}$ of Definition 4.1; the function $x \mapsto x - 2 - K_{\text{a}}U^2\log(39(x+1))$ increases for $x + 1 > K_{\text{a}}U^2$ and is positive at $x = p_*$, as the certificates check, so $p < p_*$. These first rows are II.1, II.16, II.19, II.21, I.1, I.16, I.20 and I.24, and the certified values of $K_{\text{a}}U^2\log(39(p_* + 1))$ are below $4.8693 \cdot 10^{19}$ (II.1 and I.1, $U = 65\log 10$, $p_* = 4.9 \cdot 10^{19}$), $2.8323 \cdot 10^{20}$ (II.16), $9.9157 \cdot 10^{19}$ (II.19), $9.472 \cdot 10^{19}$ (II.21), $3.1801 \cdot 10^{20}$ (I.16), $1.4521 \cdot 10^{20}$ (I.20) and $1.1620 \cdot 10^{20}$ (I.24), with $p_* + 1 > K_{\text{a}}U^2$ in all eight. Every other row takes as p_* the target Q of an earlier row of its sequence whose range of ℓ_c contains its own: in the rows II.2 to II.15 and I.2 to I.15 with $L_0 = X\log 10$ and $U = (X+5)\log 10$, the row "Xa" uses the first row of its sequence and the row "Xb" uses the row "Xa"; the rows I.27 to I.32 all use I.26; every other row uses the row before it.

The rows of type II. Interval arithmetic with 160-bit outward rounding gives the following rows and certified values. In all of them $\Omega_- > 9 \cdot 10^6$ and $\ell \geq 245$, so $K_- \geq 3$; the certificates check the other three inequalities of (P), cap $< Q$, $T_0 < (Q-1)/2$, the (C1) bound below $Q - 1$ (below $4.67 \cdot 10^8$ in II.1 to II.15 and below $2.7 \cdot 10^9$ in II.16 to II.25), $12FU < Q - 1$, the Laurent

checks in all cases, and $4 \max H_x + 4 < 2(Q-1)B^-$; and the tables give $\nu_{\max} < Q - 1$.

row	$L_0/\log 10$	$U/\log 10$	p_*	Q	χ_{MV}	ℓ	n_{sub}
II.1	30	65	$4.9 \cdot 10^{19}$	$1.73 \cdot 10^{13}$	1/2	416	8
II.2 (60a)	60	65	$1.73 \cdot 10^{13}$	$6.78 \cdot 10^{12}$	1/5	261	4
II.3 (60b)	60	65	$6.78 \cdot 10^{12}$	$6.37 \cdot 10^{12}$	1/5	253	4
II.4 (55a)	55	60	$1.73 \cdot 10^{13}$	$5.82 \cdot 10^{12}$	1/5	262	4
II.5 (55b)	55	60	$5.82 \cdot 10^{12}$	$5.39 \cdot 10^{12}$	1/5	252	4
II.6 (50a)	50	55	$1.73 \cdot 10^{13}$	$4.9 \cdot 10^{12}$	1/5	262	4
II.7 (50b)	50	55	$4.9 \cdot 10^{12}$	$4.49 \cdot 10^{12}$	1/5	251	4
II.8 (45a)	45	50	$1.73 \cdot 10^{13}$	$4.08 \cdot 10^{12}$	1/5	263	4
II.9 (45b)	45	50	$4.08 \cdot 10^{12}$	$3.66 \cdot 10^{12}$	1/5	249	4
II.10 (40a)	40	45	$1.73 \cdot 10^{13}$	$3.3 \cdot 10^{12}$	1/5	263	4
II.11 (40b)	40	45	$3.3 \cdot 10^{12}$	$2.94 \cdot 10^{12}$	1/5	248	4
II.12 (35a)	35	40	$1.73 \cdot 10^{13}$	$2.63 \cdot 10^{12}$	1/5	264	4
II.13 (35b)	35	40	$2.63 \cdot 10^{12}$	$2.29 \cdot 10^{12}$	1/5	246	4
II.14 (30a)	30	35	$1.73 \cdot 10^{13}$	$2.03 \cdot 10^{12}$	1/5	265	4
II.15 (30b)	30	35	$2.03 \cdot 10^{12}$	$1.74 \cdot 10^{12}$	1/5	245	4
II.16	65	154	$2.89 \cdot 10^{20}$	$2 \cdot 10^{14}$	1/2	429	8
II.17	65	154	$2 \cdot 10^{14}$	$4.8 \cdot 10^{13}$	3/10	290	4
II.18	65	154	$4.8 \cdot 10^{13}$	$4.3 \cdot 10^{13}$	3/10	276	4
II.19	65	92	$1.1 \cdot 10^{20}$	$4 \cdot 10^{13}$	1/2	425	8
II.20	65	92	$4 \cdot 10^{13}$	$1.5 \cdot 10^{13}$	3/10	274	4
II.21	65	90	10^{20}	$4 \cdot 10^{13}$	1/2	419	8
II.22	65	90	$4 \cdot 10^{13}$	$1.44 \cdot 10^{13}$	3/10	274	4
II.23	65	81	$1.44 \cdot 10^{13}$	$1.08 \cdot 10^{13}$	3/10	264	4
II.24	65	80	$1.08 \cdot 10^{13}$	$1.03 \cdot 10^{13}$	3/10	261	4
II.25	65	75	$1.03 \cdot 10^{13}$	$9.1 \cdot 10^{12}$	3/10	261	4

row	$\mathcal{B}$	margin of (M)	cap	T_0	F	$\nu_{\max}$	$\frac{\max E_{\nu,j}}{2(Q-1)B_j}$	$\max H_x$
II.1	< 82.6503	> 0.6296	$< 1.7220 \cdot 10^{13}$	$< 2.405 \cdot 10^{12}$	< 18.039	36446	< 0.7140	$< 1.819 \cdot 10^{6}$
II.2	< 52.2338	> 0.0888	$< 6.7781 \cdot 10^{12}$	$< 9.416 \cdot 10^{11}$	< 28.665	57916	< 0.5958	$< 2.890 \cdot 10^{6}$
II.3	< 50.3630	> 0.5010	$< 6.3690 \cdot 10^{12}$	$< 8.859 \cdot 10^{11}$	< 28.372	57324	< 0.5102	$< 2.861 \cdot 10^{6}$
II.4	< 52.3205	> 0.4302	$< 5.8198 \cdot 10^{12}$	$< 8.083 \cdot 10^{11}$	< 28.701	53528	< 0.5848	$< 2.466 \cdot 10^{6}$
II.5	< 50.1450	> 0.5107	$< 5.3840 \cdot 10^{12}$	$< 7.491 \cdot 10^{11}$	< 28.335	52845	< 0.4907	$< 2.434 \cdot 10^{6}$
II.6	< 52.4158	> 0.1760	$< 4.8902 \cdot 10^{12}$	$< 6.792 \cdot 10^{11}$	< 28.701	49068	< 0.5736	$< 2.072 \cdot 10^{6}$
II.7	< 49.8966	> 0.6018	$< 4.4882 \cdot 10^{12}$	$< 6.245 \cdot 10^{11}$	< 28.298	48378	< 0.4709	$< 2.043 \cdot 10^{6}$
II.8	< 52.5209	> 0.4685	$< 4.0724 \cdot 10^{12}$	$< 5.656 \cdot 10^{11}$	< 28.737	44663	< 0.5604	$< 1.715 \cdot 10^{6}$
II.9	< 49.6363	> 0.1528	$< 3.6504 \cdot 10^{12}$	$< 5.081 \cdot 10^{11}$	< 28.223	43864	< 0.4523	$< 1.684 \cdot 10^{6}$
II.10	< 52.6387	> 0.1544	$< 3.2987 \cdot 10^{12}$	$< 4.581 \cdot 10^{11}$	< 28.737	40197	< 0.5495	$< 1.389 \cdot 10^{6}$
II.11	< 49.3301	> 0.3981	$< 2.9331 \cdot 10^{12}$	$< 4.084 \cdot 10^{11}$	< 28.186	39425	< 0.4285	$< 1.362 \cdot 10^{6}$
II.12	< 52.7719	> 0.3718	$< 2.6262 \cdot 10^{12}$	$< 3.647 \cdot 10^{11}$	< 28.773	35775	< 0.5336	$< 1.099 \cdot 10^{6}$
II.13	< 49.0105	> 0.1082	$< 2.2803 \cdot 10^{12}$	$< 3.176 \cdot 10^{11}$	< 28.110	34951	< 0.4069	$< 1.074 \cdot 10^{6}$
II.14	< 52.9257	> 0.5343	$< 2.0260 \cdot 10^{12}$	$< 2.813 \cdot 10^{11}$	< 28.809	31342	< 0.5145	$< 8.422 \cdot 10^{5}$
II.15	< 48.6471	> 0.5062	$< 1.7317 \cdot 10^{12}$	$< 2.412 \cdot 10^{11}$	< 28.073	30541	< 0.3813	$< 8.206 \cdot 10^{5}$
II.16	< 85.4237	> 0.6447	$< 1.02791 \cdot 10^{14}$	$< 1.434 \cdot 10^{13}$	< 18.224	87238	< 0.6335	$< 1.0313 \cdot 10^{7}$
II.17	< 57.0596	> 1.5589	$< 4.69717 \cdot 10^{13}$	$< 6.560 \cdot 10^{12}$	< 22.555	107972	< 0.7219	$< 1.2763 \cdot 10^{7}$
II.18	< 54.2098	> 1.1932	$< 4.25460 \cdot 10^{13}$	$< 5.955 \cdot 10^{12}$	< 22.189	106216	< 0.5838	$< 1.2556 \cdot 10^{7}$
II.19	< 83.4926	> 3.5131	$< 3.60041 \cdot 10^{13}$	$< 5.022 \cdot 10^{12}$	< 18.168	51954	< 0.7772	$< 3.6690 \cdot 10^{6}$
II.20	< 53.8458	> 1.0274	$< 1.49650 \cdot 10^{13}$	$< 2.096 \cdot 10^{12}$	< 22.135	63301	< 0.4953	$< 4.4703 \cdot 10^{6}$
II.21	< 83.3032	> 0.5979	$< 3.3490 \cdot 10^{13}$	$< 4.675 \cdot 10^{12}$	< 18.082	50585	< 0.7234	$< 3.495 \cdot 10^{6}$
II.22	< 53.8458	> 1.0274	$< 1.4322 \cdot 10^{13}$	$< 2.005 \cdot 10^{12}$	< 22.135	61924	< 0.4901	$< 4.278 \cdot 10^{6}$
II.23	< 51.8058	> 0.7902	$< 1.0770 \cdot 10^{13}$	$< 1.511 \cdot 10^{12}$	< 21.864	55050	< 0.4074	$< 3.423 \cdot 10^{6}$
II.24	< 51.2315	> 0.6204	$< 1.0268 \cdot 10^{13}$	$< 1.441 \cdot 10^{12}$	< 21.782	54164	< 0.3881	$< 3.327 \cdot 10^{6}$
II.25	< 51.1367	> 0.8732	$< 9.0241 \cdot 10^{12}$	$< 1.267 \cdot 10^{12}$	< 21.782	50779	< 0.3746	$< 2.924 \cdot 10^{6}$

The Laurent checks hold for 479280, 315923, 310786, 320235, 203389, 313664, 199948, 182209, 179827 and 171364 pairs (ν, j) in the rows II.16 to II.25. In the rows II.1 to II.15 they are checked by blocks, as described below (10213 block cases in II.1 and between 5142 and 5671 in each other row, together covering every pair (ν, j)).

The rows of type I. Interval arithmetic with 160-bit outward rounding gives the following rows and certified values; in the rows with $h_0 < \infty$ the certificates take $h_0 = \log(p_*/(2\kappa))$, which (I.2) admits, and the table prints it rounded up. The box cover of $\mathcal{G}$ has $(n_L, n_B, n_A) = (40, 80, 80)$ in I.1 to I.15 and $(60, 120, 120)$ in I.16 to I.32. In all rows $\Omega_- > 4 \cdot 10^6$ and $\ell \geq 238$, so $K_- \geq 3$; the certificates check the other three inequalities of (P), $\mathrm{cap} < Q$, $T_0 < (Q-1)/2$, the (C1) bound below $Q - 1$ (below $2.09 \cdot 10^8$ in I.1 to I.15 and below $1.34 \cdot 10^9$ in I.16 to I.32), $F(8U + c_D) < Q - 1$, $\nu_{\max} < 2(Q-1)$, the Laurent checks in all cases, and $4 \max H_x + 4 <$

$2(Q-1)B^-$; in I.2 to I.15 also $A_0 > 6.53 > 4.8$.

row	$L_0/\log 10$	$U/\log 10$	p_*	Q	χ_{MV}	ℓ	n_{sub}	h_0
I.1	30	65	$4.9 \cdot 10^{19}$	$9.03 \cdot 10^{12}$	1/2	419	8	∞
I.2 (60a)	60	65	$9.03 \cdot 10^{12}$	$3.08 \cdot 10^{12}$	1/5	249	4	28.6928
I.3 (60b)	60	65	$3.08 \cdot 10^{12}$	$2.8 \cdot 10^{12}$	1/5	238	4	27.6172
I.4 (55a)	55	60	$9.03 \cdot 10^{12}$	$2.68 \cdot 10^{12}$	1/5	250	4	28.6928
I.5 (55b)	55	60	$2.68 \cdot 10^{12}$	$2.41 \cdot 10^{12}$	1/5	238	4	27.4781
I.6 (50a)	50	55	$9.03 \cdot 10^{12}$	$2.29 \cdot 10^{12}$	1/5	251	4	28.6928
I.7 (50b)	50	55	$2.29 \cdot 10^{12}$	$2.05 \cdot 10^{12}$	1/5	238	4	27.3208
I.8 (45a)	45	50	$9.03 \cdot 10^{12}$	$1.92 \cdot 10^{12}$	1/5	252	4	28.6928
I.9 (45b)	45	50	$1.92 \cdot 10^{12}$	$1.71 \cdot 10^{12}$	1/5	238	4	27.1446
I.10 (40a)	40	45	$9.03 \cdot 10^{12}$	$1.58 \cdot 10^{12}$	1/5	254	4	28.6928
I.11 (40b)	40	45	$1.58 \cdot 10^{12}$	$1.39 \cdot 10^{12}$	1/5	238	4	26.9497
I.12 (35a)	35	40	$9.03 \cdot 10^{12}$	$1.28 \cdot 10^{12}$	1/5	257	4	28.6928
I.13 (35b)	35	40	$1.28 \cdot 10^{12}$	$1.1 \cdot 10^{12}$	1/5	238	4	26.7391
I.14 (30a)	30	35	$9.03 \cdot 10^{12}$	$1.01 \cdot 10^{12}$	1/5	260	4	28.6928
I.15 (30b)	30	35	$1.01 \cdot 10^{12}$	$8.49 \cdot 10^{11}$	1/5	239	4	26.5022
I.16	65	163	$3.24 \cdot 10^{20}$	$1.2 \cdot 10^{14}$	1/2	433	8	∞
I.17	65	163	$1.2 \cdot 10^{14}$	$2.6 \cdot 10^{13}$	1/5	272	4	31.2797
I.18	65	163	$2.6 \cdot 10^{13}$	$1.8 \cdot 10^{13}$	1/5	258	4	29.7503
I.19	65	163	$1.8 \cdot 10^{13}$	$1.75 \cdot 10^{13}$	1/5	255	4	29.3826
I.20	65	111	$1.48 \cdot 10^{20}$	$5.37 \cdot 10^{13}$	1/2	426	8	∞
I.21	65	111	$5.37 \cdot 10^{13}$	$9.78 \cdot 10^{12}$	1/5	265	4	30.4757
I.22	65	111	$9.78 \cdot 10^{12}$	$8.54 \cdot 10^{12}$	1/5	249	4	28.7726
I.23	65	111	$8.54 \cdot 10^{12}$	$8.46 \cdot 10^{12}$	1/5	248	4	28.6370
I.24	65	99	$2 \cdot 10^{20}$	$2.2 \cdot 10^{13}$	1/2	430	8	∞
I.25	65	99	$2.2 \cdot 10^{13}$	$7.3 \cdot 10^{12}$	1/5	257	4	29.5833
I.26	65	99	$7.3 \cdot 10^{12}$	$6.5 \cdot 10^{12}$	1/5	246	4	28.4801
I.27	65	95	$6.5 \cdot 10^{12}$	$6.05 \cdot 10^{12}$	1/5	245	4	28.3640
I.28	65	90	$6.5 \cdot 10^{12}$	$5.45 \cdot 10^{12}$	1/5	245	4	28.3640
I.29	65	85	$6.5 \cdot 10^{12}$	$4.9 \cdot 10^{12}$	1/5	245	4	28.3640
I.30	65	80	$6.5 \cdot 10^{12}$	$4.4 \cdot 10^{12}$	1/5	245	4	28.3640
I.31	65	75	$6.5 \cdot 10^{12}$	$3.9 \cdot 10^{12}$	1/5	245	4	28.3640
I.32	65	70	$6.5 \cdot 10^{12}$	$3.45 \cdot 10^{12}$	1/5	245	4	28.3640

row	Φ^*/U^2	$\mathcal{B}$	margin of (M)	cap	T_0	F	$\nu_{\max}$	$\frac{\max E_{\nu,j}}{2(Q-1)B_j}$	$\max H_x$
I.1	< 481.845	< 83.3428	> 0.4922	$< 8.9803 \cdot 10^{12}$	$< 1.0848 \cdot 10^{12}$	< 18.082	24355	< 0.6553	$< 8.111 \cdot 10^5$
I.2	< 465.460	< 49.4838	> 0.5595	$< 3.0637 \cdot 10^{12}$	$< 3.821 \cdot 10^{11}$	< 28.021	37744	< 0.5471	$< 1.257 \cdot 10^6$
I.3	< 463.182	< 47.3107	> 0.0757	$< 2.7853 \cdot 10^{12}$	$< 3.498 \cdot 10^{11}$	< 27.604	37181	< 0.4650	$< 1.239 \cdot 10^6$
I.4	< 469.954	< 49.7215	> 0.4971	$< 2.6569 \cdot 10^{12}$	$< 3.282 \cdot 10^{11}$	< 28.067	34897	< 0.5398	$< 1.073 \cdot 10^6$
I.5	< 467.772	< 47.2625	> 0.2042	$< 2.3968 \cdot 10^{12}$	$< 2.981 \cdot 10^{11}$	< 27.612	34332	< 0.4497	$< 1.056 \cdot 10^6$
I.6	< 473.877	< 49.9944	> 0.3409	$< 2.2692 \cdot 10^{12}$	$< 2.779 \cdot 10^{11}$	< 28.114	32043	< 0.5346	$< 9.031 \cdot 10^5$
I.7	< 471.987	< 47.2096	> 0.3452	$< 2.0321 \cdot 10^{12}$	$< 2.505 \cdot 10^{11}$	< 27.622	31482	< 0.4331	$< 8.873 \cdot 10^5$
I.8	< 476.844	< 50.3159	> 0.0550	$< 1.9022 \cdot 10^{12}$	$< 2.315 \cdot 10^{11}$	< 28.163	29181	< 0.5322	$< 7.478 \cdot 10^5$
I.9	< 475.544	< 47.1614	> 0.4737	$< 1.6921 \cdot 10^{12}$	$< 2.070 \cdot 10^{11}$	< 27.634	28632	< 0.4177	$< 7.337 \cdot 10^5$
I.10	< 477.993	< 50.7098	> 0.1481	$< 1.5691 \cdot 10^{12}$	$< 1.905 \cdot 10^{11}$	< 28.253	26346	< 0.5151	$< 6.077 \cdot 10^5$
I.11	< 477.770	< 47.1370	> 0.5388	$< 1.3770 \cdot 10^{12}$	$< 1.677 \cdot 10^{11}$	< 27.649	25783	< 0.4008	$< 5.948 \cdot 10^5$
I.12	< 479.054	< 51.2264	> 0.4864	$< 1.2721 \cdot 10^{12}$	$< 1.541 \cdot 10^{11}$	< 28.382	23526	< 0.4942	$< 4.825 \cdot 10^5$
I.13	< 478.789	< 47.1781	> 0.4291	$< 1.0903 \cdot 10^{12}$	$< 1.326 \cdot 10^{11}$	< 27.669	22935	< 0.3772	$< 4.704 \cdot 10^5$
I.14	< 480.419	< 52.0110	> 0.1108	$< 9.9961 \cdot 10^{11}$	$< 1.207 \cdot 10^{11}$	< 28.550	20707	< 0.4709	$< 3.717 \cdot 10^5$
I.15	< 480.080	< 47.3481	> 0.5462	$< 8.4406 \cdot 10^{11}$	$< 1.024 \cdot 10^{11}$	< 27.734	20115	< 0.3523	$< 3.611 \cdot 10^5$
I.16	< 476.65	< 86.345	> 0.4709	$< 5.96582 \cdot 10^{13}$	$< 7.270 \cdot 10^{12}$	< 18.281	61748	< 0.5861	$< 5.153 \cdot 10^6$
I.17	< 399.08	< 54.496	> 0.3588	$< 1.97104 \cdot 10^{13}$	$< 2.855 \cdot 10^{12}$	< 28.849	97446	< 0.8449	$< 8.132 \cdot 10^6$
I.18	< 395.17	< 51.408	> 0.5754	$< 1.75596 \cdot 10^{13}$	$< 2.574 \cdot 10^{12}$	< 28.347	95750	< 0.8555	$< 7.990 \cdot 10^6$
I.19	< 394.20	< 50.666	> 0.8392	$< 1.71119 \cdot 10^{13}$	$< 2.516 \cdot 10^{12}$	< 28.237	95378	< 0.8017	$< 7.959 \cdot 10^6$
I.20	< 477.26	< 84.779	> 0.6528	$< 2.68126 \cdot 10^{13}$	$< 3.267 \cdot 10^{12}$	< 18.182	41822	< 0.4569	$< 2.378 \cdot 10^6$
I.21	< 428.50	< 52.872	> 0.6778	$< 9.31565 \cdot 10^{12}$	$< 1.258 \cdot 10^{12}$	< 28.600	65786	< 0.8067	$< 3.740 \cdot 10^6$
I.22	< 423.76	< 49.434	> 0.6925	$< 8.13362 \cdot 10^{12}$	$< 1.114 \cdot 10^{12}$	< 28.014	64439	< 0.6217	$< 3.663 \cdot 10^6$
I.23	< 423.37	< 49.161	> 0.8511	$< 8.06099 \cdot 10^{12}$	$< 1.105 \cdot 10^{12}$	< 27.977	64353	< 0.6067	$< 3.658 \cdot 10^6$
I.24	< 477.49	< 85.381	> 1.3306	$< 2.1742 \cdot 10^{13}$	$< 2.647 \cdot 10^{12}$	< 18.238	37416	< 0.8549	$< 1.897 \cdot 10^6$
I.25	< 435.52	< 51.071	> 0.9031	$< 7.0839 \cdot 10^{12}$	$< 9.425 \cdot 10^{11}$	< 28.310	58080	< 0.6906	$< 2.945 \cdot 10^6$
I.26	< 432.43	< 48.844	> 0.5526	$< 6.4443 \cdot 10^{12}$	$< 8.652 \cdot 10^{11}$	< 27.902	57241	< 0.5948	$< 2.902 \cdot 10^6$
I.27	< 435.49	< 48.610	> 0.6063	$< 5.9277 \cdot 10^{12}$	$< 7.904 \cdot 10^{11}$	< 27.864	54854	< 0.5687	$< 2.669 \cdot 10^6$
I.28	< 439.89	< 48.610	> 0.6063	$< 5.3739 \cdot 10^{12}$	$< 7.094 \cdot 10^{11}$	< 27.864	51967	< 0.5629	$< 2.396 \cdot 10^6$
I.29	< 444.45	< 48.610	> 0.6063	$< 4.8431 \cdot 10^{12}$	$< 6.328 \cdot 10^{11}$	< 27.864	49080	< 0.5545	$< 2.137 \cdot 10^6$
I.30	< 449.15	< 48.610	> 0.6063	$< 4.3354 \cdot 10^{12}$	$< 5.606 \cdot 10^{11}$	< 27.864	46193	< 0.5427	$< 1.893 \cdot 10^6$
I.31	< 453.94	< 48.610	> 0.6063	$< 3.8511 \cdot 10^{12}$	$< 4.927 \cdot 10^{11}$	< 27.864	43306	< 0.5335	$< 1.664 \cdot 10^6$
I.32	< 458.76	< 48.610	> 0.6063	$< 3.3903 \cdot 10^{12}$	$< 4.293 \cdot 10^{11}$	< 27.864	40419	< 0.5204	$< 1.450 \cdot 10^6$

The Laurent checks hold for 431109, 326807, 319948, 318426, 311773, 241279, 235694, 235327, 284959, 219133, 215554, 208868, 200859, 191476, 182093, 172710 and 161676 pairs (ν, j) in the rows I.16 to I.32. In the rows I.1 to I.15 they are checked by blocks (10007 block cases in I.1 and between 4952 and 5456 in each other row, together covering every pair (ν, j)).

Laurent's checks by blocks. In the rows II.1 to II.15 and I.1 to I.15 the certificates verify $A_z A_x > \lambda_{L,\nu}^2$ and $E_{\nu,j} < 2(Q-1)B_j$ by blocks: the integers $1 \le \nu \le \nu_{\max}$ are split into blocks $[\nu_1, \nu_2]$ of consecutive integers of length $\max(1, \lfloor \nu_1/200 \rfloor)$; every quantity is evaluated in interval arithmetic with ν replaced by the interval $[\nu_1, \nu_2]$, which encloses its value at each integer ν of the block; A_z, A_x and $h_{L,\nu}$ are taken as the upper endpoints of their enclosures over the block, which satisfy the required lower bounds for each ν of the block; and a block is checked for j whenever ν_1 satisfies the condition on ν for j ($\nu_1 \le F u_{\mathrm{MV}} B_{j+1}$ in type II, $\nu_1 \le \frac{1}{2} F u_{\mathrm{MV}} L_j^+$ in type I). In the other rows each integer ν is checked.

Conclusion. Every row gives $p < Q$ on its range $L_0 \le \ell_c < U$, that is, on $\mathrm{e}^{L_0} \le c < \mathrm{e}^U$. The entries of the statement are the targets Q of the following rows, whose ranges are those of the entries: on $[10^X, 10^{X+5})$ for $X = 60, 55, 50, 45, 40, 35, 30$, the rows II.3, II.5, II.7, II.9, II.11, II.13, II.15 and I.3, I.5, I.7, I.9, I.11, I.13, I.15; on $[10^{30}, 10^{65})$, II.1 and I.1; on $[10^{65}, 10^{70})$, I.32; on $[10^{65}, 10^{75})$, II.25 and I.31; on $[10^{65}, 10^{80})$, II.24 and I.30; on $[10^{65}, 10^{81})$, II.23; on $[10^{65}, 10^{85})$, I.29; on $[10^{65}, 10^{90})$, II.22 and I.28; on $[10^{65}, 10^{92})$, II.20; on $[10^{65}, 10^{95})$, I.27; on $[10^{65}, 10^{99})$,

I.26; on $[10^{65}, 10^{111})$, I.23, whose target $8.46 \cdot 10^{12}$ is below the entry $8.5 \cdot 10^{12}$; on $[10^{65}, 10^{154})$, II.18; on $[10^{65}, 10^{163})$, I.19. □

Proposition 5.15. *Let $0 < a < b < c < d$ be an irregular Diophantine quadruple with $\deg\{a, b, c\} \geq 2$, and let Q_{I}, Q_{II} be the index caps and m_{I}, m_{II} their integer forms.*

1. *A type I representation of z with $10^{30} \leq c < 10^{99}$ has $m' < Q_{\mathrm{I}}(c)$, that is, $m \leq m_{\mathrm{I}}(c)$.*
2. *A type II representation of z with $10^{30} \leq c < 10^{79.71}$ has $m' < Q_{\mathrm{II}}(c)$, that is, $m \leq m_{\mathrm{II}}(c)$.*

Proof. Each value of Q_{I} and Q_{II} is a cap that holds on a range of c containing the step of that value.

Item 1. On $10^X \leq c < 10^{X+5}$, $X = 30, 35, \ldots, 60$, the value of $Q_{\mathrm{I}}(c)$ is the type I cap of Proposition 5.14 on that range. On the steps $[10^{65}, 10^{70})$, $[10^{70}, 10^{75})$, $[10^{75}, 10^{80})$, $[10^{80}, 10^{85})$, $[10^{85}, 10^{90})$, $[10^{90}, 10^{95})$ and $[10^{95}, 10^{99})$ it is the type I cap of Proposition 5.14 on $[10^{65}, 10^{70})$, $[10^{65}, 10^{75})$, $[10^{65}, 10^{80})$, $[10^{65}, 10^{85})$, $[10^{65}, 10^{90})$, $[10^{65}, 10^{95})$ and $[10^{65}, 10^{99})$ respectively, which contain these steps. Since $m' = 2m$, the inequality $m' < Q_{\mathrm{I}}(c)$ is $m \leq m_{\mathrm{I}}(c)$.

Item 2. On $10^X \leq c < 10^{X+5}$, $X = 30, 35, \ldots, 60$, the value of $Q_{\mathrm{II}}(c)$ is the type II cap of Proposition 5.14 on that range. On $[10^{65}, 10^{75})$ it is the type II cap $9.1 \cdot 10^{12}$ of Proposition 5.14 on $[10^{65}, 10^{75})$, and on $[10^{75}, 10^{79.71})$ it is the type II cap $1.03 \cdot 10^{13}$ on $[10^{65}, 10^{80})$, which contains this step. Since $m' = 2m + 1$, the inequality $m' < Q_{\mathrm{II}}(c)$ is $m \leq m_{\mathrm{II}}(c)$. □

6 Degree one: the families A1 and A2

A triple of degree one is explicit (Lemma 3.7), and the index congruences modulo r and r^2 give floors of order $c^{1/4}$ for both types (Corollary 6.7), against the floor $m > c^{1/8}$ of He and Togbé [13, Proposition 6.2]. Type II is excluded above $c = 10^{26}$ (Proposition 6.8). For type I, the caps of Matveev and of Mignotte and Voutier then give $c < 1.81 \cdot 10^{47}$, and above $2.01 \cdot 10^{26}$ they confine the indices to the line of the triple (Proposition 6.12).

6.1 Index congruences and the index floor

Lemma 6.1. *Let $0 < a < b < c < d$ be an irregular Diophantine quadruple with $\deg\{a, b, c\} = 1$, let ε be as in Definition 3.6, and let z have a type II representation (λ, m, n). Then*

$$y\sqrt{a} + x\sqrt{b} = (\varepsilon\sqrt{a} + \sqrt{b})\,\alpha^{2l+3}$$

for an integer $l \geq 1$, and $2m + 1 \equiv 2n + 1 \equiv \lambda(-1)^l \pmod{2r}$. In particular r divides both m and n, or both $m + 1$ and $n + 1$.

Proof. As the quadruple is irregular, Lemma 2.17 gives an integer $l \geq 1$ with $y\sqrt{a} + x\sqrt{b} = (t\sqrt{a} + s\sqrt{b})\alpha^{2l+1}$, and $t\sqrt{a} + s\sqrt{b} = (\varepsilon\sqrt{a} + \sqrt{b})\alpha^2$ by item 2 of Lemma 3.7. This is the first claim.

The congruences come from the three Pell equations modulo $2r$. By item 2 of Lemma 3.7, $c = 4rm_0n_0$, $s = 2rn_0 - 1$ and $t = 2rm_0 - \varepsilon$, so

$$c \equiv 0, \qquad s \equiv -1, \qquad t \equiv -\varepsilon \pmod{2r}.$$

Multiplication by $\alpha = r + \sqrt{ab}$ acts on $Y\sqrt{a} + X\sqrt{b}$ as $(Y, X) \mapsto (rY + bX, aY + rX)$, and the square of its matrix is $\begin{pmatrix} 2r^2 - 1 & 2rb \\ 2ra & 2r^2 - 1 \end{pmatrix} \equiv -I \pmod{2r}$. Since $(\varepsilon\sqrt{a} + \sqrt{b})\alpha = (b + \varepsilon r)\sqrt{a} + (r + \varepsilon a)\sqrt{b}$, writing $\alpha^{2l+3} = \alpha \cdot \alpha^{2(l+1)}$ gives

$$x \equiv (-1)^{l+1}(r + \varepsilon a), \qquad y \equiv (-1)^{l+1}(b + \varepsilon r) \pmod{2r}.$$

In type II, $m' = 2m + 1$, $n' = 2n + 1$ and $z\sqrt{a} + x\sqrt{c} = (\lambda t\sqrt{a} + r\sqrt{c})\beta^{m'}$, and multiplication by β acts on (z, x) by the matrix $\begin{pmatrix} s & c \\ a & s \end{pmatrix} \equiv -I + \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} \pmod{2r}$, whose odd power m' is $-I + m'\begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}$ modulo $2r$; so $x \equiv -r + m'a\lambda t \equiv -r - \lambda\varepsilon a m' \pmod{2r}$. In the same way, from $z\sqrt{b} + y\sqrt{c} = (\lambda s\sqrt{b} + r\sqrt{c})\gamma^{n'}$ and $\begin{pmatrix} t & c \\ b & t \end{pmatrix} \equiv -\varepsilon I + \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix} \pmod{2r}$, $y \equiv -\varepsilon r + n'b\lambda s \equiv -\varepsilon r - \lambda b n' \pmod{2r}$.

Since $r \equiv -r \pmod{2r}$, the two expressions for x give $\varepsilon a\big(\lambda m' + (-1)^{l+1}\big) \equiv 0 \pmod{2r}$, and the two expressions for y give $b\big(\lambda n' + (-1)^{l+1}\big) \equiv 0 \pmod{2r}$. The brackets are even, since m' and n' are odd; dividing them by two and using $\gcd(a, r) = \gcd(b, r) = 1$ (as $r^2 - ab = 1$) yields $\lambda m' \equiv \lambda n' \equiv (-1)^l \pmod{2r}$, that is, $m' \equiv n' \equiv \lambda(-1)^l \pmod{2r}$. If this common residue is 1, then $2r \mid 2m$ and $2r \mid 2n$; if it is -1, then $2r \mid 2m + 2$ and $2r \mid 2n + 2$. □

Proposition 6.2. *Let $0 < a < b < c < d$ be an irregular Diophantine quadruple with $T = \{a, b, c\}$ of degree one, and let ε, m_0, n_0 and Θ be as in Definition 3.6. Let z have a representation (λ, m, n) of type I with second-order defect k, let $l \geq 1$ be the integer with $y\sqrt{a} + x\sqrt{b} = (\sqrt{a} + \sqrt{b})\alpha^{2l}$ (Lemma 2.17), and put $\Delta_\varepsilon = m - \varepsilon n$. Then:*

1. *$m > n$; in particular $\Delta_\varepsilon \geq 1$.*
2. *If l is even, then $r \mid m$, $r \mid n$ and $r^2 \mid 3\Delta_\varepsilon$.*
3. *If l is odd, then $m \equiv -\lambda m_0$ and $n \equiv -\lambda n_0 \pmod{r}$; the integers $m_1 = (m + \lambda m_0)/r$ and $n_1 = (n + \lambda n_0)/r$ satisfy*

$$r(bn_1^2 - am_1^2) + \lambda(\varepsilon n_1 - m_1) = 4km_0n_0,$$

and

$$3\Delta_\varepsilon \equiv -\lambda\Theta \pmod{r^2}.$$

Proof. We pin (m, n) modulo r by comparing the three Pell equations modulo $2r$: it is $(0, 0)$ if l is even and $-\lambda(m_0, n_0)$ if l is odd. The third-order congruence of Lemma 2.22, item 2, read modulo r, then determines $3k$ modulo r, and the defect equation $kc = bn^2 + \lambda tn - am^2 - \lambda sm$, read modulo r^2, pins $3\Delta_\varepsilon$ modulo r^2.

Identities. By item 2 of Lemma 3.7, $c = 4rm_0n_0$, $s = 2rn_0 - 1$, $t = 2rm_0 - \varepsilon$, $rn_0 - am_0 = 1$, $rm_0 - bn_0 = \varepsilon$ and $m_0 - \varepsilon n_0 = b - a$; in particular $4r \mid c$, $s \equiv -1$ and $t \equiv -\varepsilon \pmod{2r}$. Hence

$$am_0 - s = -rn_0, \qquad bn_0 - t = -rm_0, \qquad m_0^2 - n_0^2 = (m_0 - \varepsilon n_0)(m_0 + \varepsilon n_0) = (b - a)e,$$

using $e = m_0 + \varepsilon n_0$. As $ab = r^2 - 1$, $\gcd(r, ab) = 1$, and modulo r we have $m_0 \equiv b$, $n_0 \equiv \varepsilon a$ and $m_0n_0 \equiv \varepsilon ab \equiv -\varepsilon$. The integer k is divisible by 4 by item 1 of Lemma 2.22.

The three coordinates. Write $(Z_j, X_j) = (z_j^{(a,c)}, x_j^{(a,c)})$ for the sequence with seed $(1, \lambda)$, that is, $(Z_0, X_0) = (\lambda, 1)$. Multiplication by $\beta = s + \sqrt{ac}$ acts as $(z, x) \mapsto (sz + cx, az + sx)$;

since $X_0 = 1$, $Z_0 = \lambda$, $X_1 = \lambda a + s \geq s - a > 0$ and $Z_1 = \lambda s + c \geq c - s > 0$, all later coordinates are positive, so $X_j > 0$ for all $j \geq 0$. Every (Z_j, X_j) satisfies $aZ_j^2 - cX_j^2 = a - c$, hence $X_{2m}^2 = (az^2 - a + c)/c = ad + 1 = x^2$ and, both being positive, $x = X_{2m}$. In the same way $y = y_{2n}^{(b,c)}$. Finally, multiplication by $\alpha = r + \sqrt{ab}$ acts on $y'\sqrt{a} + x'\sqrt{b}$ as $(y', x') \mapsto (ry' + bx', ay' + rx')$, so with

$$R = \begin{pmatrix} r & b \\ a & r \end{pmatrix}$$

the relation of Lemma 2.17 reads $(y, x)^T = R^{2l}(1, 1)^T$.

Item 1. The triple T has degree one, so it is not Euler, and Lemma 2.14 gives $n < m$ for the type I representation. Hence $\Delta_\varepsilon = m - \varepsilon n \geq m - n \geq 1$.

Item 2 and the first part of item 3: the coordinates modulo $2r$. One has

$$R^2 = \begin{pmatrix} 2r^2 - 1 & 2rb \\ 2ra & 2r^2 - 1 \end{pmatrix} \equiv -I \pmod{2r},$$

so $(y, x) \equiv (-1)^l (1, 1) \pmod{2r}$. With

$$\begin{pmatrix} s & c \\ a & s \end{pmatrix}^2 = \begin{pmatrix} 2ac + 1 & 2sc \\ 2as & 2ac + 1 \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ -2a & 1 \end{pmatrix} \pmod{2r},$$

using $4r \mid c$ and $s \equiv -1 \pmod{2r}$, the $2m$-th power is congruent to $\begin{pmatrix} 1 & 0 \\ -2am & 1 \end{pmatrix}$ and, applied to $(Z_0, X_0) = (\lambda, 1)$, gives $x = X_{2m} \equiv 1 - 2\lambda am \pmod{2r}$. The same computation for the pair $\{b, c\}$, with $t \equiv -\varepsilon \pmod{2r}$, gives $y \equiv 1 - 2\lambda\varepsilon bn \pmod{2r}$. Comparing with the Pell equation of $\{a, b\}$,

$$2\lambda am \equiv 1 - (-1)^l, \qquad 2\lambda\varepsilon bn \equiv 1 - (-1)^l \pmod{2r}.$$

If l is even, $r \mid am$ and $r \mid bn$, so $r \mid m$ and $r \mid n$ because $\gcd(r, ab) = 1$. If l is odd, $\lambda am \equiv 1$ and $\lambda\varepsilon bn \equiv 1 \pmod{r}$; multiplying by b, respectively a, and using $ab \equiv -1$ gives $m \equiv -\lambda b \equiv -\lambda m_0$ and $n \equiv -\lambda\varepsilon a \equiv -\lambda n_0 \pmod{r}$.

The third-order congruence modulo r. By item 2 of Lemma 2.22, with $\mathcal{A}_3 = a(m^3 - m)(2s + \lambda am) - b(n^3 - n)(2t + \lambda bn)$, and as $r \mid c$,

$$\mathcal{A}_3 \equiv 3\lambda k \pmod{r}.$$

Item 2, second part. Let l be even and write $m = m_2 r$, $n = n_2 r$ with integers m_2, n_2. Then $m^3 - m \equiv n^3 - n \equiv 0 \pmod{r}$, so $\mathcal{A}_3 \equiv 0$ and $r \mid 3k$. Next,

$$kc = bn^2 + \lambda tn - am^2 - \lambda sm = r^2(bn_2^2 - am_2^2) + \lambda r(tn_2 - sm_2), \qquad tn_2 - sm_2 \equiv m_2 - \varepsilon n_2 \pmod{r},$$

and $kc = 4krm_0n_0$; dividing by r gives $4km_0n_0 \equiv \lambda(m_2 - \varepsilon n_2) \pmod{r}$. Multiplying by 3 and using $r \mid 3k$ yields $3(m_2 - \varepsilon n_2) \equiv 0 \pmod{r}$, that is, $3\Delta_\varepsilon = 3r(m_2 - \varepsilon n_2) \equiv 0 \pmod{r^2}$.

Item 3, the identity. Let l be odd, so $m = -\lambda m_0 + m_1 r$ and $n = -\lambda n_0 + n_1 r$ with integers m_1, n_1. By the identities $am_0 - s = -rn_0$ and $bn_0 - t = -rm_0$,

$$am + \lambda s = -\lambda(am_0 - s) + am_1 r = r(\lambda n_0 + am_1), \qquad bn + \lambda t = -\lambda(bn_0 - t) + bn_1 r = r(\lambda m_0 + bn_1).$$

Hence

$$kc = n(bn + \lambda t) - m(am + \lambda s) = r\big[(-\lambda n_0 + n_1 r)(\lambda m_0 + bn_1) - (-\lambda m_0 + m_1 r)(\lambda n_0 + am_1)\big].$$

Since $kc = 4krm_0n_0$, dividing by r gives

$$r(bn_1^2 - am_1^2) + \lambda(\varepsilon n_1 - m_1) = 4km_0n_0.$$

Item 3, the congruence. Modulo r we have $\lambda am \equiv 1$ and $\lambda bn \equiv \varepsilon$, so $2s + \lambda am \equiv -1$ and $2t + \lambda bn \equiv -\varepsilon$; also $m^3 - m \equiv -\lambda m_0(m_0^2 - 1)$ and $n^3 - n \equiv -\lambda n_0(n_0^2 - 1)$, as $\lambda^3 = \lambda$. Therefore, with $a \equiv \varepsilon n_0$ and $b \equiv m_0$,

$$\mathcal{A}_3 \equiv \lambda am_0(m_0^2-1) - \lambda\varepsilon bn_0(n_0^2-1) \equiv \lambda\varepsilon m_0n_0(m_0^2-n_0^2) \equiv -\lambda(m_0^2-n_0^2) = -\lambda e(b-a) \pmod r,$$

and the third-order congruence gives $3k \equiv -e(b-a) \pmod r$. The identity just proved, read modulo r, gives $\lambda(\varepsilon n_1 - m_1) \equiv 4km_0n_0 \equiv -4\varepsilon k \pmod r$; multiplying by 3,

$$3\lambda(\varepsilon n_1 - m_1) \equiv -4\varepsilon \cdot 3k \equiv 4\varepsilon e(b-a) \pmod r.$$

Multiplying by λr gives $3r(\varepsilon n_1 - m_1) \equiv 4\lambda\varepsilon e(b-a)r \pmod{r^2}$. Finally

$$r(\varepsilon n_1 - m_1) = \varepsilon(n + \lambda n_0) - (m + \lambda m_0) = -\Delta_\varepsilon - \lambda(m_0 - \varepsilon n_0) = -\Delta_\varepsilon - \lambda(b-a),$$

so $3\Delta_\varepsilon \equiv -\lambda(b-a)(3 + 4\varepsilon er) \pmod{r^2}$. By item 5 of Lemma 3.7, $(b-a)(3+4\varepsilon er) \equiv \Theta \pmod{r^2}$, which gives $3\Delta_\varepsilon \equiv -\lambda\Theta \pmod{r^2}$. □

The congruence modulo r^2 of Proposition 6.2 leaves small type I indices only on the line of the triple. Off the line it gives $m \geq r^2/12$ (Corollary 6.3). On the line, the base point (m_0, n_0) of Definition 3.6 satisfies the second-order relation but not the third-order one (Lemma 6.4), so the second-order defect does not vanish (Lemma 6.5), and a nonzero defect forces m above a multiple of ϱ (Lemma 6.6).

Corollary 6.3. *In the setting of Proposition 6.2, suppose that l is even, or $\lambda\Theta \geq 0$, or (m,n) does not lie on the line of T (Definition 3.6). Then*

$$m \geq \frac{r^2}{12} > \frac{(ac)^{1/2}}{48}.$$

If l is odd, $\lambda\Theta < 0$ and (m,n) lies on the line, then $m \geq |\Theta|/6$.

Proof. By item 1 of Proposition 6.2, $\Delta_\varepsilon \geq 1$ and $m \geq \Delta_\varepsilon/2$: for $\varepsilon = 1$, $\Delta_\varepsilon = m - n < m$, and for $\varepsilon = -1$, $\Delta_\varepsilon = m + n < 2m$.

If l is even, item 2 of Proposition 6.2 gives $r^2 \mid 3\Delta_\varepsilon$ with $3\Delta_\varepsilon > 0$, so $3\Delta_\varepsilon \geq r^2$ and $m \geq r^2/6$.

If l is odd, item 3 of Proposition 6.2 gives $3\Delta_\varepsilon \equiv -\lambda\Theta \pmod{r^2}$, so $3\Delta_\varepsilon = -\lambda\Theta + jr^2$ for an integer j; recall $-r^2/2 < \Theta \leq r^2/2$. If $\lambda\Theta > 0$, then $-\lambda\Theta \in [-r^2/2, 0)$ and $3\Delta_\varepsilon > 0$ force $j \geq 1$, whence $3\Delta_\varepsilon \geq r^2 - \lambda\Theta \geq r^2/2$ and $m \geq r^2/12$. If $\Theta = 0$, then $3\Delta_\varepsilon = jr^2 \geq r^2$ and $m \geq r^2/6$. If $\lambda\Theta < 0$, then $-\lambda\Theta = |\Theta| \in (0, r^2/2]$ and $j \geq 0$; for $j \geq 1$ we get $3\Delta_\varepsilon > r^2$ and $m \geq r^2/6$, and for $j = 0$ we get $3\Delta_\varepsilon = |\Theta|$, that is, (m,n) lies on the line, and $m \geq \Delta_\varepsilon/2 = |\Theta|/6$.

The lower bound for r^2 comes from item 2 of Lemma 3.7: $c = 4rm_0n_0$ with $m_0 = b + \varepsilon r \leq b + r$ and $n_0 = r + \varepsilon a < 2r$; moreover $b = (r^2-1)/a < r^2/a$ and $r \leq r^2/a$. Hence $c < 8r^2(r^2/a + r) \leq 16r^4/a$, that is, $r^2 > (ac)^{1/2}/4$, and $r^2/12 > (ac)^{1/2}/48$. □

Lemma 6.4. *Let $T = \{a, b, c\}$ with $a < b < c$ be a Diophantine triple of degree one, with descent partner e, and let ε, m_0 and n_0 be as in Definition 3.6.*

1. $bn_0^2 - tn_0 = am_0^2 - sm_0$.

2. If $b > 4000$, then rm_0n_0 does not divide $abe\,(m_0 - \varepsilon n_0)$.

Proof. Item 1. Item 2 of Lemma 3.7 gives $s = 2rn_0 - 1$, $t = 2rm_0 - \varepsilon$, $am_0 = rn_0 - 1$ and $bn_0 = rm_0 - \varepsilon$. Hence

$$am_0 - s = -rn_0, \qquad bn_0 - t = -rm_0,$$

and $bn_0^2 - tn_0 - am_0^2 + sm_0 = n_0(bn_0 - t) - m_0(am_0 - s) = -rm_0n_0 + rm_0n_0 = 0$.

Item 2. By item 2 of Lemma 3.7, r, m_0 and n_0 are pairwise coprime, $be + 1 = m_0^2$ and $ae + 1 = n_0^2$; so m_0 is prime to b and to e, and n_0 is prime to a and to e. As $r^2 - ab = 1$, r is prime to ab.

Let $b > 4000$ and suppose, for a contradiction, that rm_0n_0 divides $abe(m_0 - \varepsilon n_0)$.

Modulo m_0 we have $abe(m_0 - \varepsilon n_0) \equiv -\varepsilon\, abe\, n_0$, and m_0 is prime to b, e and n_0; hence $m_0 \mid a$. If $\varepsilon = 1$, this is impossible, since $m_0 = b + r > a > 0$. So $\varepsilon = -1$, $n_0 = r - a$, $m_0 = b - r$, and $e = m_0 - n_0$ by item 2 of Lemma 3.7. Modulo n_0 we have $abe(m_0 + n_0) \equiv abe\, m_0$, and n_0 is prime to a, e and m_0; hence $n_0 \mid b$. Finally $r \mid abe(m_0 + n_0)$ and $\gcd(r, ab) = 1$ give $r \mid e(m_0 + n_0) = e(b - a)$.

Write $u = n_0 = r - a$; then $u = \sqrt{ae + 1}$, $u \geq 2$ by item 3 of Lemma 3.7, and

$$ae = u^2 - 1, \qquad m_0 = u + e, \qquad b - a = m_0 + n_0 = e + 2u.$$

Suppose $e \geq 2$. From $(u + e) \mid a$ and $ae = u^2 - 1$ we get $(u + e) \mid u^2 - 1$, and $u^2 - 1 \equiv e^2 - 1 \pmod{u+e}$, so $(u+e) \mid e^2-1$. From $u \mid b = a+e+2u$ we get $u \mid a+e$, hence $u \mid ae+e^2 = u^2-1+e^2$, so $u \mid e^2 - 1$. Since $\gcd(u, u + e) = \gcd(u, e) = 1$ (because $ae = u^2 - 1$) and $e^2 - 1 > 0$, it follows that $u(u + e) \leq e^2 - 1 < e^2$, which forces $u < e$, for $u \geq e$ would give $u(u + e) \geq 2e^2$. But then $a = (u^2 - 1)/e < u^2/e < u$, whereas $(u + e) \mid a$ and $a \geq 1$ give $a \geq u + e > u$. This is a contradiction.

Suppose $e = 1$. Then $a = u^2 - 1$, $r = a + u = u^2 + u - 1$, and $r \mid e(b - a) = 2u + 1$. As $2u + 1 > 0$, $u^2 + u - 1 \leq 2u + 1$, so $u \leq 2$; hence $u = 2$, $a = 3$, $r = 5$ and $b = (r^2 - 1)/a = 8$, contrary to $b > 4000$. □

Lemma 6.5. *In the setting of Proposition 6.2, suppose that l is odd, $\lambda\Theta < 0$ and (m, n) lies on the line of T (Definition 3.6). Then $k \neq 0$.*

Proof. We show that $k = 0$ is impossible: for $a \geq 2$ it forces $(m, n, \lambda) = (m_0, n_0, -1)$, which the third-order congruence excludes, and for $a = 1$ it leads to a negative Pell equation without solutions.

An identity for odd l. Let $m_1 = (m + \lambda m_0)/r$ and $n_1 = (n + \lambda n_0)/r$ be the integers of item 3 of Proposition 6.2. Since $m = rm_1 - \lambda m_0$, $n = rn_1 - \lambda n_0$ and $m_0 - \varepsilon n_0 = b - a$ (item 2 of Lemma 3.7),

$$r(m_1 - \varepsilon n_1) = \Delta_\varepsilon + \lambda(b - a).$$

Multiplying the identity $r(bn_1^2 - am_1^2) + \lambda(\varepsilon n_1 - m_1) = 4km_0n_0$ of item 3 by r and using $c = 4rm_0n_0$ gives

$$r^2(bn_1^2 - am_1^2) = kc + \lambda r(m_1 - \varepsilon n_1) = kc + \lambda\Delta_\varepsilon + b - a. \tag{1}$$

The case $k = 0$. Suppose $k = 0$. On the line $3\Delta_\varepsilon = |\Theta| \leq r^2/2$, and $\Delta_\varepsilon \geq 1$ by item 1 of Proposition 6.2, so $1 \leq \Delta_\varepsilon \leq r^2/6$; by (1), r^2 divides $\lambda\Delta_\varepsilon + b - a$.

Let $a \geq 2$. Then $b = (r^2 - 1)/a \leq (r^2 - 1)/2$, so

$$-r^2 < -\Delta_\varepsilon < \lambda\Delta_\varepsilon + b - a < \Delta_\varepsilon + b < \tfrac{r^2}{6} + \tfrac{r^2}{2} < r^2,$$

hence $\lambda\Delta_\varepsilon + b - a = 0$ and, by (1), $bn_1^2 = am_1^2$. If $n_1 \neq 0$, then $ab = (am_1/n_1)^2$ would be the square of a rational number and hence of an integer, whereas $(r-1)^2 < ab = r^2 - 1 < r^2$. So $n_1 = 0$, then $m_1 = 0$, that is, $m = -\lambda m_0$ and $n = -\lambda n_0$. As $m \geq 1$ and $m_0 > 0$, we get $\lambda = -1$ and $(m, n) = (m_0, n_0)$. Thus $(-1, m_0, n_0)$ is a type I representation of z with $k = 0$, and item 2 of Lemma 2.22 gives $24c \mid \mathcal{A}_3$ with

$$\mathcal{A}_3 = a(m_0^3 - m_0)(2s - am_0) - b(n_0^3 - n_0)(2t - bn_0).$$

By item 2 of Lemma 3.7, $2s - am_0 = s + rn_0 = 3rn_0 - 1$, $2t - bn_0 = t + rm_0 = 3rm_0 - \varepsilon$, $m_0^2 - 1 = be$ and $n_0^2 - 1 = ae$, so

$$\mathcal{A}_3 = abe\big(m_0(3rn_0 - 1) - n_0(3rm_0 - \varepsilon)\big) = -abe\,(m_0 - \varepsilon n_0).$$

Since $24c = 96rm_0n_0$, it follows that rm_0n_0 divides $abe(m_0 - \varepsilon n_0)$. By Lemma 2.9, $b > 4000$, and item 2 of Lemma 6.4 gives a contradiction.

Let $a = 1$. Then $b - a = r^2 - 2$ and r^2 divides $\lambda\Delta_\varepsilon - 2$. By Lemma 2.9, $b > 4000$, so $r^2 > 12$. For $\lambda = -1$ this gives $\Delta_\varepsilon \equiv -2 \pmod{r^2}$, impossible for $1 \leq \Delta_\varepsilon \leq r^2/6$. For $\lambda = 1$ it gives $\Delta_\varepsilon = 2$, and (1) becomes $r^2(bn_1^2 - m_1^2) = r^2$, that is,

$$m_1^2 - (r^2 - 1)n_1^2 = -1.$$

If r is odd, then $8 \mid r^2 - 1$ and $m_1^2 \equiv -1 \pmod 8$, which is impossible; if r is even, then $r^2 - 1 \equiv 3 \pmod 4$ has a prime factor congruent to 3 modulo 4, modulo which -1 is not a square. □

Lemma 6.6. *In the setting of Proposition 6.2, let ϱ be as in Definition 3.6. If $k \neq 0$, then*

$$m > (\sqrt{17} - 1)\,\varrho \quad \text{if } \varepsilon = 1, \qquad m > 2.17\,\varrho \quad \text{if } \varepsilon = -1.$$

Proof. We compare the size of kc with $4c$.

A size bound. Since $k \in 4\mathbb{Z}$ (item 1 of Lemma 2.22) and $k \neq 0$, $4c \leq |k|c = |bn^2 + \lambda tn - am^2 - \lambda sm|$. By item 1 of Proposition 6.2, $0 \leq n < m$; with $a < b$ and $s < t$ this gives $|bn^2 - am^2| \leq \max(bn^2, am^2) < bm^2$ and $|tn - sm| \leq \max(tn, sm) < tm$, hence

$$4c < bm^2 + tm. \tag{1}$$

By item 2 of Lemma 3.7, $c = 4rm_0n_0$ and $t = 2rm_0 - \varepsilon$; and $b = m_0 - \varepsilon r$, $n_0 = r + \varepsilon a$ by the definition of m_0 and n_0 (Definition 3.6).

The case $\varepsilon = 1$. Here $\varrho = r$ by item 3 of Lemma 3.7, and (1) gives $16rm_0n_0 < (m_0 - r)m^2 + (2rm_0 - 1)m < m_0(m^2 + 2rm)$, so $16rn_0 < m^2 + 2rm$. As $n_0 = r + a > r$, this yields $(m + r)^2 > 17r^2$, that is, $m > (\sqrt{17} - 1)r$.

The case $\varepsilon = -1$. Here $\varrho = n_0 = r - a$ by item 3 of Lemma 3.7, and $m_0 = b - r$. We have $0 < n_0 < r$, since $a \geq 1$, and $m_0 - n_0 = e > 0$, since $e = m_0 + \varepsilon n_0$ (item 2 of Lemma 3.7). Suppose $m \leq \theta n_0$ with $\theta = 2.17$. Using $n_0 < r$ and $n_0 < m_0$ we get $(m_0 + r)n_0^2 = m_0n_0 \cdot n_0 + rn_0 \cdot n_0 < 2rm_0n_0$, and $(2rm_0 + 1)n_0 \leq 3rm_0n_0$. Since $b = m_0 + r$ and $t = 2rm_0 + 1$, the right side of (1) satisfies

$$bm^2 + tm \leq (m_0 + r)\theta^2 n_0^2 + (2rm_0 + 1)\theta n_0 < (2\theta^2 + 3\theta)\,rm_0n_0 < 16rm_0n_0 = 4c,$$

because $2\theta^2 + 3\theta = 15.9278$. This contradicts (1). Hence $m > 2.17\,n_0 = 2.17\,\varrho$. □

Corollary 6.7. *Let $0 < a < b < c < d$ be an irregular Diophantine quadruple with $T = \{a, b, c\}$ of degree one, and let ε, ϱ, Θ and the line of T be as in Definition 3.6.*

1. *Let (λ, m, n) be a representation of z of type I, and let $l \geq 1$ be the integer with $y\sqrt{a}+x\sqrt{b} = (\sqrt{a}+\sqrt{b})\alpha^{2l}$ (Lemma 2.17). If l is even, or $\lambda\Theta \geq 0$, or (m, n) does not lie on the line, then $m \geq r^2/12 > c^{1/2}/48$.*
2. *If l is odd, $\lambda\Theta < 0$ and (m, n) lies on the line, then the second-order defect k of the representation is nonzero, and $m > (\sqrt{17}-1)\varrho$ if $\varepsilon = 1$, $m > 2.17\varrho$ if $\varepsilon = -1$.*
3. *Every representation of z of type I satisfies $m > 2.17\varrho$; since $c < 4(\varrho+1)^4$ (item 4 of Lemma 3.7), it also satisfies $m' > 3c^{1/4}$.*
4. *Every representation of z of type II satisfies $r \mid m$ and $r \mid n$, or $r \mid m+1$ and $r \mid n+1$; hence $m \geq r - 1 > (c/4)^{1/4} - 2$.*

Proof. Items 1 and 2 restate Corollary 6.3, as $(ac)^{1/2} \geq c^{1/2}$, and Lemma 6.5 together with Lemma 6.6. For items 3 and 4 we use that $b > 4000$ (Lemma 2.9); that $c > 4ab + a + b$, $c > b^2$ and $c < 4(\varrho+1)^4$ (item 4 of Lemma 3.7); and that $\varrho \leq r$, as $\varrho = r$ or $\varrho = r - a$ (item 3 of Lemma 3.7).

Item 3. In the case of item 2, $m > 2.17\varrho$ because $\sqrt{17} - 1 > 2.17$. Otherwise item 1 gives $m \geq r^2/12$; as $r^2 = ab + 1 > 4001$, $r \geq 64$, so $m \geq \frac{64}{12} r > 2.17r \geq 2.17\varrho$. From $c < 4(\varrho+1)^4$ we get $\varrho > (c/4)^{1/4} - 1$, hence

$$m' = 2m > 4.34\big((c/4)^{1/4} - 1\big) > 3.0688\, c^{1/4} - 4.34,$$

since $4.34 \cdot 4^{-1/4} > 3.0688$. As $c^{1/4} > b^{1/2} > 4000^{1/2} > 63.24$, we have $0.0688\, c^{1/4} > 4.35$, and so $m' > 3c^{1/4}$.

Item 4. The divisibility is Lemma 6.1. By Lemma 2.15, applicable as $b > 4000$ and $c > 4ab + a + b$, $m \geq 4$. If $r \mid m$, then $m \geq r$; if $r \mid m+1$, then $m \geq r - 1$. Finally $c < 4(\varrho+1)^4 \leq 4(r+1)^4$ gives $r + 1 > (c/4)^{1/4}$, so $m \geq r - 1 > (c/4)^{1/4} - 2$. □

6.2 The bound for degree one and the windows of A1 and A2

The bound for degree one is a squeeze on the line. The indices m, n of a type II representation satisfy $r \mid m$ and $r \mid n$, or $r \mid m+1$ and $r \mid n+1$ (Lemma 6.1), and Laurent's theorem on two logarithms excludes type II above 10^{26}. For type I, Matveev's theorem gives a first cap (Lemma 6.9), and the theorem of Mignotte and Voutier, applied to the small linear form of Lemma 6.10, gives the caps of Proposition 6.11; set against the floor of Corollary 6.7, these caps give the bound.

Proposition 6.8. *Let $0 < a < b < c < d$ be an irregular Diophantine quadruple with $\deg\{a, b, c\} = 1$. If z has a representation of type II, then $r < 10^6$, and hence $c < 10^{26}$.*

Proof. The divisibility of the indices by r compresses the linear form to one in two logarithms with the large coefficient $2r$; Laurent's theorem then bounds r. This modulus-and-compression mechanism follows He, Pu, Shen and Togbé [12].

Standing inputs. By Lemma 2.9, $b > 4000$ and $d > d_+(a, b, c)$. By item 4 of Lemma 3.7, $c > 4ab + a + b$, $c > b^2$ and $c < 32ab^2 < 32r^4$. Let (λ, m, n) be a representation of z of type II. By Lemma 2.15, $m, n \geq 4$.

Compression. By Lemma 6.1, r divides m and n, or $m+1$ and $n+1$. Put

$$(\hat m,\hat n,\tau)=\begin{cases}(m/r,\,n/r,\,1), & r\mid m \text{ and } r\mid n,\\ ((m+1)/r,\,(n+1)/r,\,-1), & r\mid m+1 \text{ and } r\mid n+1.\end{cases}$$

Then $\hat m,\hat n$ are positive integers, and $m'=2r\hat m+\tau$, $n'=2r\hat n+\tau$. Shifting the seeds $(x_1,z_1)=(r,\lambda t)$ and $(y_2,z_2)=(r,\lambda s)$ by τ steps gives the common seed

$$z_*=\tau cr+\lambda st,\qquad x_*=rs+\tau\lambda at,\qquad y_*=rt+\tau\lambda bs.$$

It belongs to $d_0=d_{\tau\lambda}(a,b,c)\in\{e,d_+(a,b,c)\}$, with $z_*=\tau\sqrt{cd_0+1}$. Writing $d_+=d_+(a,b,c)$, the four cases are

(τ,λ)	$(1,1)$	$(1,-1)$	$(-1,1)$	$(-1,-1)$
d_0	d_+	e	e	d_+
z_*	$\sqrt{cd_++1}$	$\sqrt{ce+1}$	$-\sqrt{ce+1}$	$-\sqrt{cd_++1}$

The coordinates x_*,y_* are positive; for $d_0=e$ they equal $n_0=r+\varepsilon a$ and $m_0=b+\varepsilon r$ (Definition 3.6, item 2 of Lemma 3.7).

With δ and Λ as in Lemma 5.7 (the triple is not Euler, and $m',n'>0$), define

$$\Gamma_1=\frac{\gamma^{\hat n}}{\beta^{\hat m}},\qquad \Gamma_2=\delta\Big(\frac{\beta}{\gamma}\Big)^{\tau}.$$

Then exactly

$$\Lambda=\log\Gamma_2-2r\log\Gamma_1,\qquad \Gamma_2=\frac{z_*+x_*\sqrt{c/a}}{z_*+y_*\sqrt{c/b}}.$$

Both logarithms are positive, including the cases with $z_*<0$. Indeed, set $H_1=c/a-1>H_2=c/b-1>0$ and $F_H(v)=v+\sqrt{v^2+H}$. Then

$$\Gamma_2=\frac{F_{H_1}(z_*)}{F_{H_2}(z_*)}>1,\qquad \mathrm{e}^{\Lambda}=\frac{F_{H_1}(z)}{F_{H_2}(z)},\qquad \frac{d}{dv}\log\frac{F_{H_1}(v)}{F_{H_2}(v)}=\frac{1}{\sqrt{v^2+H_1}}-\frac{1}{\sqrt{v^2+H_2}}<0.$$

Since $d>d_+\ge d_0$, we have $z>z_*$. Hence $0<\Lambda<\log\Gamma_2$, which proves $\Gamma_1>1$.

Field, independence and heights. The field $\mathcal K=\mathbb Q(\sqrt{ab},\sqrt{ac})=\mathbb Q(\sqrt{ac},\sqrt{bc})$ is real quartic: ab,ac,bc are not squares, and their square classes are distinct. The units β,γ belong to distinct quadratic subfields and are multiplicatively independent. Write

$$\ell_c=\log c,\qquad \mathcal P=\hat m\log\beta,\qquad \mathcal Q=\hat n\log\gamma.$$

Then $\mathcal Q>\mathcal P>0$. The four conjugates of Γ_1 are $\mathrm{e}^{\pm\mathcal P\pm\mathcal Q}$ and are distinct, so Γ_1 generates $\mathcal K$ and the degree is exactly four.

The seed norm identities give

$$\mathrm N_{\mathcal K/\mathbb Q}(\delta)=\left(\frac{b(c-a)}{a(c-b)}\right)^2>1.$$

Indeed the numerator $x_1\sqrt{ac}+az_1$ has norm $a^2(c-a)^2$, the denominator has norm $b^2(c-b)^2$, and the rational factor b/a has norm $(b/a)^4$. Multiplication by β^τ/γ^τ preserves this norm. Since Γ_1 is a unit of norm one and Γ_2 has norm greater than one, a relation $\Gamma_1^{w_1}\Gamma_2^{w_2}=1$ forces $w_2=0$ and then $w_1=0$; so Γ_1,Γ_2 are multiplicatively independent.

Every conjugate of the algebraic-integer numerator and denominator of δ has absolute value below $2c^2$, because every seed coordinate is below c. Thus

$$\mathrm{h}(\delta) \le \mathrm{h}(b/a) + \mathrm{h}(x_1\sqrt{ac} + az_1) + \mathrm{h}(y_2\sqrt{bc} + bz_2) < 5\ell_c + 2\log 2 < 6\ell_c,$$

and, since $\beta < \gamma < 2c$, $\mathrm{h}(\beta) = \frac{1}{2}\log\beta$ and $\mathrm{h}(\gamma) = \frac{1}{2}\log\gamma$, we get $\mathrm{h}(\Gamma_2) < 8\ell_c$. Put $X_1 = r\sqrt{c}$, $X_2 = t\sqrt{a}$, $X_3 = s\sqrt{b}$, so $0 < X_2 < X_3 < X_1$ and

$$\delta = \sqrt{\frac{b}{a}}\,\frac{X_1 + \lambda X_2}{X_1 + \lambda X_3}.$$

For $\lambda = 1$ this lies between 1 and $\sqrt{b/a}$. For $\lambda = -1$, rationalisation gives

$$\delta = \sqrt{\frac{b}{a}}\,\frac{c-a}{c-b}\,\frac{X_1 + X_3}{X_1 + X_2} < 3\sqrt{\frac{b}{a}},$$

using $c > b^2$ and $b > 4000$. Therefore

$$|\log\Gamma_2| \le \log\delta + \log(\gamma/\beta) < \frac{5}{4}\ell_c + \log 6 < 2\ell_c.$$

Finally

$$\mathrm{h}(\Gamma_1) = \frac{\mathcal{Q}}{2}, \qquad 0 < \log\Gamma_1 = \frac{\log\Gamma_2 - \Lambda}{2r} < \frac{\ell_c}{r}, \qquad 8\,\mathrm{h}(\Gamma_1) < 4\mathcal{P} + \frac{4\ell_c}{r}.$$

Laurent's theorem. Suppose $r \ge 10^6$. We apply Theorem 5.3 with $\mathcal{D} = 4$, $(\eta_1, \eta_2) = (\Gamma_1, \Gamma_2)$, $(b_1, b_2) = (2r, 1)$, so that $\Lambda_L = \Lambda$, and with

$$\rho = 61, \quad \mu = 0.7, \quad \sigma_L = 0.955, \quad A_1 = 5\mathcal{P}, \quad A_2 = 200\ell_c,$$
$$X = \frac{r}{200\log r} + 1, \qquad h_L = 4\log X + 13.$$

We have $2\log r < \ell_c < 4\log r + \log 32$ and $\mathcal{P} > \ell_c/2$. Thus

$$(\rho-1)\log\Gamma_1 + 8\,\mathrm{h}(\Gamma_1) < 4\mathcal{P} + \frac{66\ell_c}{r} < 5\mathcal{P}, \qquad (\rho-1)\log\Gamma_2 + 8\,\mathrm{h}(\Gamma_2) < 120\ell_c + 64\ell_c < 200\ell_c.$$

Moreover

$$\frac{b_1}{A_2} + \frac{b_2}{A_1} = \frac{r}{100\ell_c} + \frac{1}{5\mathcal{P}} < X,$$

and $4(\log\lambda_L + 1.75) + 0.06 < 13$. The remaining hypotheses are immediate from $A_1 > 69$, $A_2 > 5500$ and $h_L > 36$.

For explicit margins, throughout $r \ge 10^6$,

$$3.925 < \lambda_L < 3.927, \quad \mathcal{H} > 10, \quad \omega_L < 4.003, \quad \theta_L < 1.052, \quad A_1A_2 > 379500.$$

In the square root defining $\mathcal{C}$, the three summands are respectively below 1.781, 0.044 and 0.031, while the prefactor $\mu/(\lambda_L^3\sigma_L)$ is below 0.0122. Consequently

$$\mathcal{C} < 0.0122\Big(\frac{4.003}{6} + \frac{\sqrt{1.856}}{2}\Big)^2 < 0.023 < 0.03, \qquad \mathcal{C}' < 0.06, \qquad \sqrt{\omega_L\theta_L} < 3.$$

Put $Y_0 = \log X + 4.3$ and $q = A_1A_2$. Since $h_L + \log\rho = 4\log X + 13 + \log 61 < 4Y_0$, the theorem gives

$$\log\Lambda > -0.48Y_0^2 q - 12Y_0 - \log(0.96Y_0^2 q).$$

Comparison of the bounds. Lemma 5.7 gives $\log\Lambda < \log(8ac/3) - 2m'\log\beta$. As $\log(8ac/3) < 4\log\beta$, $m' = 2r\hat{m} + \tau$ with $\tau \in \{-1, 1\}$, and $\hat{m} \geq 1$,

$$\log\Lambda < -(4r\hat{m} - 6)\log\beta \leq -(4r - 6)\mathcal{P}.$$

Divide the comparison by $q = 1000\mathcal{P}\ell_c$. Since $q > 1000$, $Y_0 > 1$, and $\log x \leq \sqrt{x}$ for $x > 0$,

$$\frac{4r-6}{1000\ell_c} < 0.48Y_0^2 + \frac{12Y_0}{q} + \frac{\log(0.96Y_0^2 q)}{q} < 0.523Y_0^2.$$

Hence

$$r < 130.75\ell_c Y_0^2 + 1.5 < 132\ell_c Y_0^2 + 2.$$

Set $v = \log r$ and

$$U(v) = 4v + \log 32, \qquad X(v) = \frac{\mathrm{e}^v}{200v} + 1, \qquad Y(v) = \log X(v) + 4.3,$$

so that $\ell_c < U(v)$ and $Y_0 = Y(v)$. At $r = 10^6$ the brackets

$$13.8155 < \log(10^6) < 13.8156, \quad U < 58.729, \quad X < 362.92, \quad Y < 10.1942$$

give $132UY^2 + 2 < 806000 < 10^6$. For larger v,

$$\frac{d}{dv}\log X(v) = \Big(1 - \frac{1}{v}\Big)\frac{X-1}{X} < 1, \qquad \frac{d}{dv}\log\big(U(v)Y(v)^2\big) < \frac{4}{U} + \frac{2}{Y} < 0.27 < 1.$$

Thus $\mathrm{e}^v/(UY^2)$ increases, and the difference $\mathrm{e}^v - 132UY^2$ increases too, because its derivative is positive whenever $\mathrm{e}^v > 132UY^2$. Therefore the necessary inequality $r < 132UY^2 + 2$ fails for every $r \geq 10^6$, and $r < 10^6$.

Finally, for $r < 10^6$, $c < 32r^4 < 32 \cdot 10^{24} < 10^{26}$. □

Lemma 6.9. *Let $0 < a < b < c < d$ be an irregular Diophantine quadruple with $\deg\{a, b, c\} = 1$, and let z have a representation of type I. If $c \geq 10^{26}$, then*

$$m' - 2 < 1.877 \cdot 10^{12}\,(\log c)^2 \log\big(14.83\,m'\big).$$

Moreover $c < 10^{72}$.

Proof. Matveev's theorem, applied to the linear form of the representation, gives the cap; against the floor $m' > 3c^{1/4}$ it gives $c < 10^{72}$.

Standing inputs. By Lemma 2.9, $b > 4000$ and $d > d_+(a, b, c)$. By item 4 of Lemma 3.7, $c > 4ab + a + b$ and $b^2 < c < 32ab^2$; in particular $a < b < c^{1/2}$ and $c > 4ab$. By Lemma 2.14, $n \leq m \leq 2n$, hence $0 < n' \leq m'$. By item 3 of Corollary 6.7, $m' > 3c^{1/4}$. Put $\ell_c = \log c$ and assume $c \geq 10^{26}$, so $\ell_c \geq 26\log 10$ and $m' > 3 \cdot 10^{6.5} > 9 \cdot 10^6$.

Linear form, field and heights. The triple $\{a, b, c\}$ is not Euler. Let δ and $\Lambda = m'\log\beta - n'\log\gamma + \log\delta$ be as in Lemma 5.7; it gives $0 < \Lambda < \frac{8}{3}ac\,\beta^{-2m'}$, and $\frac{8}{3}ac < \beta^4$ because $\beta > 2\sqrt{ac}$. As in the proof of Proposition 5.11, $\mathcal{K} = \mathbb{Q}(\sqrt{ab}, \sqrt{ac})$ is a real quartic field and $\beta, \gamma, \delta \in \mathcal{K}$ are positive and multiplicatively independent; $\mathrm{h}(\beta) = \frac{1}{2}\log\beta$ and $\mathrm{h}(\gamma) = \frac{1}{2}\log\gamma$. For the seeds of type I, item 1 of Lemma 5.8 gives $4\,\mathrm{h}(\delta) < 4\ell_c$, because $c > 4ab$, and $|\log\delta| \leq 4\,\mathrm{h}(\delta)$. Hence in Theorem 5.2 with $\mathcal{L} = \mathcal{K}$, $\mathcal{D} = 4$, $(\eta_1, \eta_2, \eta_3) = (\beta, \gamma, \delta)$ and $(b_1, b_2, b_3) = (m', -n', 1)$ we may take

$$A_1 = 2\log\beta, \qquad A_2 = 2\log\gamma, \qquad A_3 = 4\ell_c.$$

Size of the units. Since $t^2 = bc+1$, $bc < c^{3/2}$, and $\frac12\log(1+c^{-3/2}) \le \frac12 c^{-3/2} < 10^{-39}$ for $c \ge 10^{26}$,

$$\log\beta < \log\gamma < \log(2t) < \log 2 + \tfrac34\ell_c + \tfrac12\log\bigl(1+c^{-3/2}\bigr) < \gamma_0\ell_c,$$

where

$$\gamma_0 = \frac34 + \frac{\log 2 + 10^{-39}}{26\log 10} < 0.761579.$$

With $n' \le m'$ this gives

$$B_M = \max\Bigl\{1, \frac{2m'\log\beta}{4\ell_c}, \frac{2n'\log\gamma}{4\ell_c}\Bigr\} \le \frac{\gamma_0}{2}\,m',$$

the right side exceeding one because $m' > 9\cdot 10^6$, and therefore

$$W_0 = \log\bigl(1.5\mathrm{e}\cdot 4\log(4\mathrm{e})\,B_M\bigr) \le \log\bigl(3\mathrm{e}(1+\log 4)\gamma_0\,m'\bigr) < \log(14.83\,m').$$

Comparison of the bounds. Theorem 5.2 gives $\log\Lambda > -C_M(3)\,C_0\,\mathcal{D}^2 A_1A_2A_3W_0$, and the upper bound above gives $\log\Lambda < (4-2m')\log\beta$. Dividing by $2\log\beta$,

$$m'-2 < C_M(3)C_0\mathcal{D}^2\cdot 2\log\gamma\cdot 4\ell_c\cdot W_0 < 8\gamma_0\,C_M(3)C_0\mathcal{D}^2\,\ell_c^2 W_0 < 1.877\cdot 10^{12}\ell_c^2\log(14.83\,m'),$$

since $8\gamma_0 C_M(3)C_0\mathcal{D}^2 < 1.8764\cdot 10^{12}$ for $\mathcal{D}=4$.

The bound $c < 10^{72}$. Suppose $c \ge 10^{72}$. Then the first claim applies, and $w = m'$ satisfies

$$w > 3\,\mathrm{e}^{\ell_c/4} \qquad\text{and}\qquad w-2 < C\ell_c^2\log\bigl(\theta(w+1)\bigr), \qquad C = 1.877\cdot 10^{12},\ \theta = 14.83,$$

by item 3 of Corollary 6.7 and the first claim. Apply Lemma 5.6 with $Y=3$, $\kappa_1=4$, $\Lambda_0=72$ and $\ell=\ell_c$: $10^{\Lambda_0/\kappa_1} = 10^{18} \ge 21$, $w_0 = 3\cdot 10^{18} \ge 50$, and

$$G_{3,\theta}(w_0) = \frac{w_0-2}{(\log 10^{18})^2\log\bigl(\theta(w_0+1)\bigr)} > 3.85\cdot 10^{13} > 3.004\cdot 10^{13} > C\kappa_1^2.$$

The lemma gives $\ell_c < 72\log 10$, a contradiction. Hence $c < 10^{72}$. □

Lemma 6.10. *Let $0<a<b<c<d$ be an irregular Diophantine quadruple with $\deg\{a,b,c\} = 1$, and let z have a representation of type I with sign λ. Let $l \ge 1$ be the integer with $y\sqrt{a}+x\sqrt{b} = (\sqrt{a}+\sqrt{b})\alpha^{2l}$ (Lemma 2.17), and put $p = m' = 2m$, $\ell_c = \log c$, $A = \log\alpha$, $B = \log\beta$, $\eta = \beta/\alpha$ and $\mathcal{K} = \mathbb{Q}(\sqrt{ab},\sqrt{ac})$, and let D_λ and $\psi = \log D_\lambda$ be defined as in Lemma 5.13.*

1. *$p \ge 4$ and $0 < \Lambda := 2lA - pB + \psi < 1.04\,\beta^{-2p}$; moreover $p < 2l$ and $2lA < pB$.*
2. *$\frac{\ell_c}{4} - 0.2 < A < \frac{\ell_c}{2}$, $\frac{\ell_c}{2} + \log 2 < B < \frac34\ell_c + \log 2 + 10^{-6}$, and $B - A > \frac{\ell_c}{4} - 0.0002$.*
3. *$0 < \psi < 0.7091$, $\mathrm{N}_{\mathcal{K}/\mathbb{Q}}(D_\lambda) = \left(\frac{c(b-a)}{b(c-a)}\right)^2 \in (0,1)$, and $\mathrm{h}(D_\lambda) < \ell_c - B + A + 0.3468$.*
4. *$\mathcal{K}$ is a real quartic field; for $(\omega_1,\omega_2)\in\{\pm1\}^2$ it has an embedding with $\alpha\mapsto\alpha^{\omega_1}$, $\beta\mapsto\beta^{\omega_2}$. For integers i_1, i_2 one has $\mathrm{h}(\beta^{i_1}\alpha^{i_2}) = \frac12\max(|i_1|B, |i_2|A)$; in particular $\mathrm{h}(\alpha) = \frac{A}{2}$ and $\mathrm{h}(\eta) = \frac{B}{2}$. The numbers α, η, D_λ exceed 1 and are multiplicatively independent.*

Proof. We compare the Pell equation of $\{a,c\}$ with the Pell equation of $\{a,b\}$ at the common value x; the explicit form of degree-one triples supplies the sizes.

Standing inputs. By Lemma 2.9, $b > 4000$ and $d > d_+(a,b,c)$. Let ε, m_0 and n_0 be as in Definition 3.6. By item 2 of Lemma 3.7, $c = 4rm_0n_0$, $s = 2rn_0 - 1$ and $t = 2rm_0 - \varepsilon$, with $m_0 = b + \varepsilon r$ and $n_0 = r + \varepsilon a$. By item 4 of Lemma 3.7, $c > 4ab + a + b$ and $b^2 < c < 32ab^2$, so $a < b < c^{1/2}$ and $a/c < 1/b < 1/4001$. By Lemma 2.15, $m \geq 2$ in type I, so $p \geq 4$. Also $r^2 = ab + 1 > 4001$, so $r > 63$.

The exact small form, item 1. Write $\beta^p = S + T\sqrt{ac}$ with integers S, T. The type I representation gives $z\sqrt{a} + x\sqrt{c} = (\lambda\sqrt{a} + \sqrt{c})\beta^p$, hence $x = S + \lambda aT$, and $\beta^{-p} = S - T\sqrt{ac}$ gives

$$2x = (1 + \lambda\sqrt{a/c})\beta^p + (1 - \lambda\sqrt{a/c})\beta^{-p}.$$

In the same way the Pell equation of $\{a,b\}$ gives $2x = (1 + \sqrt{a/b})\alpha^{2l} + (1 - \sqrt{a/b})\alpha^{-2l}$. Put $X_1 = (1 + \sqrt{a/b})\alpha^{2l}$, $X_2 = (1 + \lambda\sqrt{a/c})\beta^p$, $q_1 = 1 - a/b$, $q_2 = 1 - a/c$. Then $2x = X_i + q_i/X_i$ and $X_i > 1 > q_i$, so $X_i = x + \sqrt{x^2 - q_i}$. As $q_1 < q_2$, $X_1 > X_2$, and

$$X_1 - X_2 = \frac{q_2 - q_1}{\sqrt{x^2 - q_1} + \sqrt{x^2 - q_2}} < \frac{1}{2\sqrt{x^2 - q_2}} = \frac{1}{X_2 - q_2/X_2},$$

because $0 < q_2 - q_1 < a/b < 1$. Hence $\Lambda = \log(X_1/X_2)$ satisfies $0 < \Lambda < (X_1 - X_2)/X_2 < 1/(X_2^2 - 1)$. Since $X_2 \geq (1 - \sqrt{a/c})\beta^p > 0.984\,\beta^p > 10^6$, this gives $\Lambda < 1.04\,\beta^{-2p}$, and $\Lambda = 2lA - pB + \psi$ by the definition of D_λ.

From $\Lambda > 0$ and $\psi < 0.71 < 2 < p(B - A)$ (item 2 gives $B - A > \frac{\ell_c}{4} - 0.0002 > 2$, as $c > b^2 > 1.6 \cdot 10^7$) we get $2lA > pB - \psi > pA$, so $2l > p$. For $2lA < pB$ it suffices that $\Lambda < \psi$. Since $D_\lambda \geq D_1$ and $a \geq 1$,

$$D_1 - 1 = \frac{\sqrt{a}\,(b^{-1/2} - c^{-1/2})}{1 + \sqrt{a/c}} > \frac{b^{-1/2}(1 - b^{-1/2})}{1.02} > 0.96\,b^{-1/2} > 0.96\,c^{-1/4},$$

using $c^{-1/2} < b^{-1}$ and $b < c^{1/2}$; so $\psi > 0.9\,c^{-1/4}$, whereas $\Lambda < 1.04(4c)^{-p} < c^{-4}$ because $\beta^2 > 4ac \geq 4c$ and $p \geq 4$.

Unit sizes, item 2. Since $\alpha < 2r$ and $4r^2 = 4ab + 4 < c$, we get $A < \ell_c/2$. Since $c < 32ab^2 \leq 32(ab)^2 < 32r^4$ and $\alpha > 2r - 1$,

$$A > \log(2r) - \frac{1}{2r - 1} > \log 2 + \frac{\ell_c - \log 32}{4} - 0.008 = \frac{\ell_c}{4} - \frac{1}{4}\log 2 - 0.008 > \frac{\ell_c}{4} - 0.2.$$

Next $\beta > 2\sqrt{ac} \geq 2\sqrt{c}$ gives $B > \frac{\ell_c}{2} + \log 2$, and $\beta < 2s$ with $s^2 = ac + 1 < c^{3/2} + 1$ gives $B < \log 2 + \frac{3}{4}\ell_c + \frac{1}{2}\log(1 + c^{-3/2}) < \frac{3}{4}\ell_c + \log 2 + 10^{-6}$. Finally

$$\Big(\frac{\beta}{\alpha}\Big)^4 > \frac{(4ac)^2}{(4r^2)^2} = c \cdot \frac{a^2 c}{(ab + 1)^2} > c \cdot \frac{a^2 b^2}{(ab + 1)^2} > c\,\big(1 + \tfrac{1}{4001}\big)^{-2},$$

so $B - A > \frac{\ell_c}{4} - \frac{1}{2}\log(1 + \frac{1}{4001}) > \frac{\ell_c}{4} - 0.0002$.

The correction, item 3. The two expressions for D_λ in Lemma 5.13 agree after multiplying numerator and denominator by $\sqrt{bc}$, and $D_\lambda > 1$ since $\sqrt{a/b} > \sqrt{a/c}$. Moreover

$$\psi \leq \log\frac{1 + \sqrt{a/b}}{1 - \sqrt{a/c}} < \log 2 - \log\big(1 - 4001^{-1/2}\big) < 0.7091.$$

For the embeddings $\sigma_{\omega_1,\omega_2}$ of item 4 one has $\sqrt{bc} = \sqrt{ab}\sqrt{ac}/a \mapsto \omega_1\omega_2\sqrt{bc}$. The numerator $\nu = \sqrt{bc} + \sqrt{ac}$ and the denominator $\delta_\lambda = \sqrt{bc} + \lambda\sqrt{ab}$ are algebraic integers with $|\sigma\nu| = \sqrt{bc} + \omega_1\sqrt{ac}$

and $|\sigma\delta_\lambda| = \sqrt{bc} + \lambda\omega_2\sqrt{ab}$. Hence $\mathrm{N}_{\mathcal{K}/\mathbb{Q}}(\nu) = (bc-ac)^2$ and $\mathrm{N}_{\mathcal{K}/\mathbb{Q}}(\delta_\lambda) = (bc-ab)^2$, which gives the norm; it lies in $(0,1)$ because $b(c-a) - c(b-a) = a(c-b) > 0$. The four values of $\max(|\sigma\nu|, |\sigma\delta_\lambda|)$ are $\sqrt{bc}+\sqrt{ac}$ (twice), at most $\sqrt{bc}+\sqrt{ab}$, and $\sqrt{bc}-\sqrt{ab}$. By the product formula, with nonpositive contributions of the finite places,

$$4\,\mathrm{h}(D_\lambda) \le 2\log(\sqrt{bc}+\sqrt{ac}) + \log(bc-ab) < 2\log(bc) + 2\log\bigl(1+\sqrt{a/b}\bigr),$$

so $\mathrm{h}(D_\lambda) < \frac{1}{2}\log(bc) + \frac{1}{2}\log 2$. To compare $\frac{1}{2}\log(bc)$ with $\ell_c - B + A$, note $2(s+1) = \beta + \beta^{-1} + 2 > \beta$, $2r = \alpha + \alpha^{-1}$ and, by item 2 of Lemma 3.7,

$$t + \varepsilon = 2rm_0 = \frac{c}{2n_0} = \frac{cr}{s+1} < \frac{c\,\alpha(1+\alpha^{-2})}{\beta}.$$

Thus $\sqrt{bc} < t \le t+\varepsilon+1 < \frac{c\alpha}{\beta}\bigl(1+\alpha^{-2}+\frac{\beta}{c\alpha}\bigr)$. Here $\alpha^2 = 2r^2 - 1 + 2r\sqrt{r^2-1} > 4r^2 - 3 = 4ab+1$, because $r\sqrt{r^2-1} > r^2 - 1$; hence $\alpha > 2\sqrt{ab}$ and $\alpha^{-2} < (4ab)^{-1} < 6.3\cdot 10^{-5}$. Also $\beta < 2s = 2\sqrt{ac+1}$, so $\frac{\beta}{c\alpha} < \frac{2\sqrt{ac+1}}{2c\sqrt{ab}} < \sqrt{2/(bc)} < 2\cdot 10^{-5}$. So $\frac{1}{2}\log(bc) < \ell_c - B + A + 10^{-4}$ and $\mathrm{h}(D_\lambda) < \ell_c - B + A + 0.3468$.

Field, heights and independence, item 4. The integers ab, ac, bc are not squares, being one less than a positive square, and $ab\cdot ac = a^2\cdot bc$; hence $\mathcal{K}$ is real quartic with the four stated embeddings, $\alpha \in \mathbb{Q}(\sqrt{ab})$, $\beta \in \mathbb{Q}(\sqrt{ac})$ and $\alpha\alpha' = \beta\beta' = 1$. The conjugates of $\beta^{i_1}\alpha^{i_2}$ are $\beta^{\omega_2 i_1}\alpha^{\omega_1 i_2}$; as these are units, $4\,\mathrm{h}(\beta^{i_1}\alpha^{i_2})$ is the sum of the positive parts of $\pm i_1 B \pm i_2 A$, which equals $2\max(|i_1|B, |i_2|A)$. The number $\eta = \beta/\alpha$ exceeds 1 since $B > A$, and $\alpha, D_\lambda > 1$. If $\alpha^{i_1}\eta^{i_2}D_\lambda^{i_3} = 1$, taking norms gives $\mathrm{N}(D_\lambda)^{i_3} = 1$, so $i_3 = 0$; then $\beta^{i_2}\alpha^{i_1-i_2} = 1$, and the embedding $(\omega_1,\omega_2) = (1,-1)$ gives $\beta^{-i_2}\alpha^{i_1-i_2} = 1$, whence $\beta^{2i_2} = 1$, $i_2 = 0$ and $i_1 = 0$. □

Proposition 6.11. *Let $0 < a < b < c < d$ be an irregular Diophantine quadruple with $\deg\{a,b,c\} = 1$, and let z have a representation of type I.*

1. *If $10^{50} \le c < 10^{74}$, then $m' < 9.5\cdot 10^{12}$.*
2. *If $10^{26} \le c < 10^{50}$, then $m' < 2.0\cdot 10^{12}$.*
3. *If $10^{26} \le c < 10^{28}$, then $m' < 5.9\cdot 10^{11}$.*

Proof. We apply the theorem of Mignotte and Voutier, Theorem 5.4, to the linear form of Lemma 6.10 in the basis $(\alpha, D_\lambda, \eta)$, with the parametrisation of the proof of Proposition 5.14, and exclude its alternative (C2) by the centred form and Laurent's theorem, Theorem 5.3, applied for each denominator, as in that proof. The argument runs in four rows.

Rows. Let $p = m'$, $\ell_c = \log c$, A, B, η, D_λ, ψ, Λ and $\mathcal{K}$ be as in Lemma 6.10. Put $\rho_{\mathrm{MV}} = 15$, $M_{\mathrm{MV}} = \frac{7}{2}$, $u_{\mathrm{MV}} = \rho_{\mathrm{MV}} + 3 = 18$, $v_{\mathrm{MV}} = \rho_{\mathrm{MV}} - 1 = 14$, and

	$L_0/\log 10$	$U/\log 10$	p_*	Q	χ_{MV}	ℓ
row 1	50	74	$2.5\cdot 10^{18}$	$9.5\cdot 10^{12}$	2/5	378
row 2	26	50	$1.2\cdot 10^{18}$	$4.4\cdot 10^{12}$	2/5	378
row 3	26	50	$4.4\cdot 10^{12}$	$2.0\cdot 10^{12}$	1/5	253
row 4	26	28	$2.0\cdot 10^{12}$	$5.9\cdot 10^{11}$	1/5	246

Row 1 proves item 1, rows 2 and 3 prove item 2, and row 4 proves item 3. In each row we assume $L_0 \le \ell_c < U$ and first show $p < p_*$. In rows 1 and 2, Lemma 6.9 gives $p - 2 < K_{\mathrm{M}}\ell_c^2\log(14.83p) < K_{\mathrm{M}}U^2\log(14.83p)$ with $K_{\mathrm{M}} = 1.877\cdot 10^{12}$; the function $x \mapsto x - 2 - K_{\mathrm{M}}U^2\log(14.83x)$ increases

for $x \ge K_{\mathrm{M}}U^2$ and is positive at $x = p_*$, because $p_* > K_{\mathrm{M}}U^2$ and $K_{\mathrm{M}}U^2\log(14.83p_*)$ is below $2.4556\cdot10^{18}$ in row 1 and below $1.1029\cdot10^{18}$ in row 2; so $p < p_*$. In row 3, $p < p_*$ by row 2; in row 4, $p < p_*$ by row 3, whose range of c contains that of row 4. Suppose $p \ge Q$; we derive a contradiction in each row. By item 1 of Lemma 6.10,

$$\log\Lambda < -2pB + 0.04. \tag{1}$$

The three-logarithm form. By Lemma 6.10, $\Lambda = (2l-p)\log\alpha + \log D_\lambda - p\log\eta$ with $2l > p$. Apply Theorem 5.4 with $(\eta_1,\eta_2,\eta_3) = (\alpha, D_\lambda, \eta)$ and $(b_1,b_2,b_3) = (2l-p, 1, p)$: the coefficients are positive integers with gcd one, the $\eta_i > 1$ are multiplicatively independent and lie in the real quartic field $\mathcal{K}$ (item 4 of Lemma 6.10), so $\mathcal{D} = 4$, the only roots of unity in $\mathcal{K}$ are ±1, and $0 < \Lambda < 1 < \pi$. Put $c_D = 8\cdot0.3468 + 0.7091\,v_{\mathrm{MV}}$ and

$$\mathfrak{a}_1 = u_{\mathrm{MV}}A, \qquad \mathfrak{a}_2 = 8(\ell_c - B + A) + c_D, \qquad \mathfrak{a}_3 = u_{\mathrm{MV}}B - v_{\mathrm{MV}}A.$$

Since $\mathrm{h}(\alpha) = \frac{A}{2}$ and $\mathrm{h}(\eta) = \frac{B}{2}$, $\mathfrak{a}_1$ and $\mathfrak{a}_3$ equal $(\rho_{\mathrm{MV}}-1)\log\eta_i + 8\,\mathrm{h}(\eta_i)$, and $(\rho_{\mathrm{MV}}-1)\psi + 8\,\mathrm{h}(D_\lambda) < \mathfrak{a}_2$ by item 3 of Lemma 6.10; so all $\mathfrak{a}_i$ are admissible as the parameters A_i of Theorem 5.4. Item 2 of Lemma 6.10 gives $\ell_c - B + A > \frac{\ell_c}{2} - 0.8932$ and $\ell_c - B + A < \frac{3}{4}\ell_c + 0.0002$, hence

$$\mathfrak{a}_1 > u_{\mathrm{MV}}\big(\tfrac{L_0}{4} - 0.2\big), \qquad \mathfrak{a}_2 > 8\big(\tfrac{L_0}{2} - 0.8932\big) + c_D,$$
$$\mathfrak{a}_3 = 4B + v_{\mathrm{MV}}(B-A) > 4\big(\tfrac{L_0}{2} + \log 2\big) + v_{\mathrm{MV}}\big(\tfrac{L_0}{4} - 0.0002\big),$$

so $\min\mathfrak{a}_i > \mathfrak{a}_-$, where $\mathfrak{a}_-$, the least of these three lower bounds, exceeds 466 in row 1 and 245 in rows 2 to 4, and $\Omega := \mathfrak{a}_1\mathfrak{a}_2\mathfrak{a}_3 > \Omega_-$, the product of these three lower bounds, with $\Omega_- > 1.5249\cdot10^8$ in row 1 and $\Omega_- > 2.1625\cdot10^7$ in rows 2 to 4. Also $\mathfrak{a}_1 < \frac{u_{\mathrm{MV}}U}{2}$, $\mathfrak{a}_2 < \mathfrak{a}_2^+ := 8(\frac{3}{4}U + 0.0002) + c_D$, and with $B^+ = \frac{3}{4}U + \log 2 + 10^{-6}$,

$$\mathfrak{a}_1 + \mathfrak{a}_2 + \mathfrak{a}_3 = 12A + (\rho_{\mathrm{MV}}-5)B + 8\ell_c + c_D < 6U + (\rho_{\mathrm{MV}}-5)B^+ + 8U + c_D =: \Sigma_+.$$

The joint product. With $\hat{A} = A/\ell_c$ and $\hat{B} = B/\ell_c$,

$$\frac{\Omega}{B\ell_c^2} = u_{\mathrm{MV}}\,\hat{A}\Big(u_{\mathrm{MV}} - v_{\mathrm{MV}}\frac{\hat{A}}{\hat{B}}\Big)\Big(8(1-\hat{B}+\hat{A}) + \frac{c_D}{\ell_c}\Big),$$

which increases with $c_D/\ell_c \le c_D/L_0$. By item 2 of Lemma 6.10, $(\hat{A},\hat{B})$ lies in the polygon $\frac{1}{4} - \frac{0.2}{L_0} \le \hat{A} \le \frac{1}{2}$, $\frac{1}{2} \le \hat{B} \le \frac{3}{4} + \frac{\log 2 + 10^{-6}}{L_0}$, $\hat{B} - \hat{A} \ge \frac{1}{4} - \frac{0.0002}{L_0}$. Interval evaluation on a 500×500 grid of cells covering the bounding box, discarding the cells disjoint from the polygon, gives

$$\frac{\Omega}{B} < C_\Omega\ell_c^2, \qquad C_\Omega = 477.736 \text{ (row 1)}, \qquad C_\Omega = 485.985 \text{ (rows 2 to 4)}. \tag{2}$$

Hence $\Omega < \Omega_+ := C_\Omega U^2 B^+$.

The theorem applied. We use the parametrisation of the proof of Proposition 5.14 with these $\mathfrak{a}_i$, $\mathfrak{a}_-$, $\Omega_\pm$ and Σ_+; the certificates check $\theta_1^2 > 2M_{\mathrm{MV}}\ell/\mathfrak{a}_-$, $\chi_{\mathrm{MV}}\sqrt{\theta_1}\,\mathfrak{a}_- > 1$ and $M_{\mathrm{MV}}\Omega_- > 2$ in every row, and $K_- \ge 3$ since $\Omega_- > 2\cdot10^7$. Here $\delta_1 = \gcd(2l-p,p) \ge 1$ and $\delta_2 = 1$. As $2lA < pB$ (item 1 of Lemma 6.10), $\frac{2l-p}{u_{\mathrm{MV}}B - v_{\mathrm{MV}}A} < \frac{p(B-A)}{A(u_{\mathrm{MV}}B - v_{\mathrm{MV}}A)} \le \frac{p}{u_{\mathrm{MV}}A}$, so $\varpi_1 \le \frac{2p}{u_{\mathrm{MV}}A} < \frac{2p_*}{u_{\mathrm{MV}}(L_0/4-0.2)}$, and $\varpi_2 < \frac{p_*}{\mathfrak{a}_{2,-}} + \frac{1}{\mathfrak{a}_{3,-}}$, with $\mathfrak{a}_{2,-}, \mathfrak{a}_{3,-}$ the displayed lower bounds. So

$$\log\mathfrak{b} < \mathcal{B} := 2\log\frac{C_*}{M_{\mathrm{MV}}\ell} + \log\Big[\frac{p_*}{2u_{\mathrm{MV}}(\frac{L_0}{4}-0.2)}\Big(\frac{p_*}{\mathfrak{a}_{2,-}} + \frac{1}{\mathfrak{a}_{3,-}}\Big)\Big] + \frac{11}{3} - 2\log\Big(1 - \frac{1}{M_{\mathrm{MV}}\ell\Omega_-}\Big),$$

and the certificates verify $\mathcal{B} > 0$ and the reduced main condition (M) of that proof in each row, with the margins in the table below.

The ordinary alternative. Put $T_0 = \ell\big(1 + C_* \frac{u_{\mathrm{MV}} U}{2} \mathfrak{a}_2^+\big)$, so $\ell T_{\mathrm{MV}} \le T_0$; the certificates give $T_0 < Q/2 \le b_3/2$. So $\log\Lambda > -M_{\mathrm{MV}} \ell^2 \Omega \log\rho_{\mathrm{MV}}$, and with (1), (2) and $B > \frac{L_0}{2} > 20$,

$$p < \frac{M_{\mathrm{MV}} \ell^2 \log\rho_{\mathrm{MV}}}{2} \cdot \frac{\Omega}{B} + \frac{0.02}{B} < \frac{M_{\mathrm{MV}} \ell^2 \log\rho_{\mathrm{MV}}}{2} C_\Omega U^2 + 0.001 =: \mathrm{cap};$$

the certificates give $\mathrm{cap} < Q$, contradicting $p \ge Q$.

Alternative (C1). It gives $p = b_3 \le \max(\theta_1, \theta_2)\, \mathfrak{a}_1 \mathfrak{a}_2 = \theta_2 \mathfrak{a}_1 \mathfrak{a}_2 < \theta_2 \frac{u_{\mathrm{MV}} U}{2} \mathfrak{a}_2^+$, which the certificates bound by $2 \cdot 10^8$; this is impossible.

Alternative (C2). It gives w_1, w_2, w_3 with $\gcd(w_1, w_2, w_3) = 1$, $w_1(2l - p) + w_2 + w_3 p = 0$ and $|w_i| \le F\mathfrak{a}_i$. If $w_1 = 0$, then $w_3 \ne 0$ by coprimality and $p \le |w_2| < F\mathfrak{a}_2^+ < 10^5$, impossible. So $w_1 \ne 0$; define g_0, ν, $\mathcal{A}$ and C from w_1, w_2, w_3 as in the type II part of the proof of Proposition 5.14. Then

$$\mathcal{A}p - 2\nu l + C = 0, \qquad \gcd(\mathcal{A}, \nu) = 1,$$
$$1 \le \nu \le \nu_{\max} := \Big\lfloor F \frac{u_{\mathrm{MV}} U}{2} \Big\rfloor, \qquad |C| \le C_{\max} := \lfloor F\mathfrak{a}_2^+ \rfloor,$$

and $\mathcal{A} > 0$ because $\mathcal{A}p = 2\nu l - C \ge 2l - |C| > 0$. The centred form of the proof of Proposition 5.14 applies with $K = 2l$, $P = p$ and $D = D_\lambda$, since $\nu \le \nu_{\max} < Q \le p$ (table). Using $A < \frac{U}{2}$, $0 < \Lambda < \psi < 0.7091$, $p \ge Q$, item 4 of Lemma 6.10 and $\mathrm{h}(D_\lambda) < \frac{3}{4}U + 0.347 =: h_D^+$, we get

$$|\log\zeta| < z_0 := \frac{C_{\max} U/2 + 0.7091\,\nu}{Q}, \qquad \mathrm{h}(\zeta) = \tfrac{1}{2}\max(\nu B, \mathcal{A}A) \le \tfrac{1}{2}(\nu B + z_0) =: H_z,$$

$$\mathrm{h}(\xi) \le h_D^+ + \frac{\nu B + z_0}{4} + \frac{C_{\max} U}{4\nu} =: H_x,$$
$$|\log\xi| < 0.7091 + \frac{z_0}{2} + \frac{C_{\max} U}{2\nu} =: X_\xi, \qquad t_0 < \frac{p_*}{\nu} + \frac{1}{2} =: t_\nu.$$

The certificates give $4\max H_x + 4 < 2Q(\frac{L_0}{2} + \log 2) - 0.04$ in every row, while (1) gives $\log\Lambda < -2Q(\frac{L_0}{2} + \log 2) + 0.04$; so, by Lemma 5.5 as in the centred form, the logarithms of ζ and ξ have opposite signs.

Laurent's theorem for each denominator. Put $B^- = \frac{L_0}{2} + \log 2$ and split $[B^-, B^+]$ into eight equal subintervals $[B_j, B_{j+1}]$, $0 \le j \le 7$. For fixed ν and $B \in [B_j, B_{j+1}]$ apply Laurent's theorem as in the paragraph "Laurent's theorem for each denominator" of the proof of Proposition 5.14, with A_z and A_x evaluated at $B = B_{j+1}$; it gives $\log\Lambda \ge -E_{\nu,j}$. For every integer $1 \le \nu \le \nu_{\max}$ and every j the certificates verify $A_z A_x > \lambda_{L,\nu}^2$ and $E_{\nu,j} < 2QB_j - 0.04$, the largest ratio being attained at $\nu = \nu_{\max}$, $j = 7$. Since $2pB \ge 2QB_j$, this contradicts (1).

Certified values. Interval arithmetic with 160-bit outward rounding gives

	row 1	row 2	row 3	row 4
$\mathcal{B}$	< 75.4849	< 75.3204	< 50.2864	< 48.7119
margin	> 0.3815	> 0.8203	> 0.7053	> 0.9043
cap	$< 9.393 \cdot 10^{12}$	$< 4.3619 \cdot 10^{12}$	$< 1.9541 \cdot 10^{12}$	$< 5.7935 \cdot 10^{11}$
T_0	$< 8.66 \cdot 10^{11}$	$< 3.976 \cdot 10^{11}$	$< 1.780 \cdot 10^{11}$	$< 5.358 \cdot 10^{10}$
(C1) bound	$< 1.92 \cdot 10^8$	$< 8.79 \cdot 10^7$	$< 6.73 \cdot 10^7$	$< 2.10 \cdot 10^7$
F	< 20.235	< 20.2471	< 28.1811	< 27.9197
$(\nu_{\max}, C_{\max})$	$(31030, 20944)$	$(20979, 14243)$	$(29200, 19824)$	$(16200, 11154)$
$\max \frac{E_{\nu,j}}{2QB_j - 0.04}$	< 0.7506	< 0.5874	< 0.4186	< 0.3293

In every row cap $< Q$, $T_0 < Q/2$, the (C1) bound is below $2\cdot 10^8$, $F\mathfrak{a}_2^+ < 10^5$, and $4\max H_x + 4 < 2QB^- - 0.04$.

All alternatives of the theorem are excluded, so $p < Q$ in each row. Row 1 gives item 1. Row 2 gives $m' < 4.4\cdot 10^{12}$ for $10^{26} \le c < 10^{50}$, and row 3 then gives item 2. Row 4, with the starting cap of row 3, gives item 3. □

Proposition 6.12. *Let $0 < a < b < c < d$ be an irregular Diophantine quadruple with $T = \{a,b,c\}$ of degree one, and let ε, ϱ, Θ and the line of T be as in Definition 3.6. Then $c < 1.81\cdot 10^{47}$. If moreover $c \ge 2.01\cdot 10^{26}$, then:*

1. *z has a representation of type I and none of type II;*
2. *the integer $l \ge 1$ with $y\sqrt{a} + x\sqrt{b} = (\sqrt{a}+\sqrt{b})\alpha^{2l}$ (Lemma 2.17) is odd, and every representation (λ,m,n) of z of type I satisfies $\lambda\Theta < 0$, lies on the line, has second-order defect $k \ne 0$, and has $m' < 2.0\cdot 10^{12}$;*
3. *$\varrho < 3.21\cdot 10^{11}$ if $\varepsilon = 1$, and $\varrho < 4.61\cdot 10^{11}$ if $\varepsilon = -1$.*

Proof. The proof plays the floors of Corollary 6.7 against the caps of Proposition 6.11: off the line the floor $m' > c^{1/2}/24$ is incompatible with the caps once $c \ge 2.01\cdot 10^{26}$, so every representation, which is of type I by Proposition 6.8, lies on the line, and there the floor $m' > 4.34\,\varrho$ bounds ϱ, and hence c. If $c < 2.01\cdot 10^{26}$, then $c < 1.81\cdot 10^{47}$; so let $c \ge 2.01\cdot 10^{26}$.

Item 1. By Lemma 2.9 and Lemma 2.13, z has a representation of type I or II. A representation of type II would give $c < 10^{26}$ by Proposition 6.8. Hence z has a representation of type I and none of type II. Let l be the integer of Lemma 2.17; it is unique because $\alpha > 1$ and x, y depend on d only.

The range $c < 10^{50}$. By Lemma 6.9, $c < 10^{72}$. Suppose $c \ge 10^{50}$. Then item 1 of Proposition 6.11 gives $m' < 9.5\cdot 10^{12}$ for a type I representation, and item 3 of Corollary 6.7 gives $m' = 2m > 4.34\,\varrho$ and $c < 4(\varrho+1)^4$. Hence

$$c < 4\Big(\frac{9.5\cdot 10^{12}}{4.34} + 1\Big)^4 < 9.2\cdot 10^{49},$$

a contradiction. So $2.01\cdot 10^{26} \le c < 10^{50}$, and item 2 of Proposition 6.11 gives $m' < 2.0\cdot 10^{12}$ for every representation of type I.

Item 2. Let (λ,m,n) be a representation of type I, and suppose that l is even, or $\lambda\Theta \ge 0$, or (m,n) does not lie on the line. Item 1 of Corollary 6.7 gives $m \ge r^2/12 > c^{1/2}/48$, and $m' = 2m$, so

$$c < (24m')^2. \tag{1}$$

With $m' < 2.0\cdot 10^{12}$, (1) gives $c < (4.8\cdot 10^{13})^2 = 2.304\cdot 10^{27} < 10^{28}$. Then item 3 of Proposition 6.11 gives $m' < 5.9\cdot 10^{11}$, and (1) gives $c < (1.416\cdot 10^{13})^2 < 2.006\cdot 10^{26}$, a contradiction. Hence l is odd, $\lambda\Theta < 0$ and (m,n) lies on the line; then $k \ne 0$ by item 2 of Corollary 6.7, and $m' < 2.0\cdot 10^{12}$ was shown above.

Item 3 and the bound on c. By item 2 there is a representation of type I on the line, with $m = m'/2 < 10^{12}$, and item 2 of Corollary 6.7 gives $m > (\sqrt{17}-1)\varrho$ if $\varepsilon = 1$ and $m > 2.17\varrho$ if $\varepsilon = -1$. Hence

$$\varrho < \frac{10^{12}}{\sqrt{17}-1} < 3.2020\cdot 10^{11} \quad (\varepsilon = 1), \qquad \varrho < \frac{10^{12}}{2.17} < 4.6083\cdot 10^{11} \quad (\varepsilon = -1),$$

which is item 3. In both cases $\varrho + 1 < 4.6084\cdot 10^{11}$, and item 3 of Corollary 6.7 gives

$$c < 4(\varrho+1)^4 < 4\,(4.6084\cdot 10^{11})^4 < 1.81\cdot 10^{47}.$$ □

Collecting the floor, the caps and the bound gives the windows of the two families of degree one, A1 and A2.

Proposition 6.13. *Let $0 < a < b < c < d$ be an irregular Diophantine quadruple with $T = \{a, b, c\}$ of degree one, and let ρ be a representation of z. If $c < 2.01 \cdot 10^{26}$, then $(T, \rho) \in$ A1; if $c \geq 2.01 \cdot 10^{26}$, then $(T, \rho) \in$ A2 (Definition 4.2).*

Proof. Write $\rho = (\lambda, m, n)$, and let ε, ϱ (the root $\sqrt{pq+1}$ of the chain pair, not to be confused with the representation ρ), Θ, the line and ℓ_ε be as in Definition 3.6. By Lemma 2.9, $b > 4000$, and by Proposition 6.12, $c < 1.81 \cdot 10^{47}$. By item 4 of Lemma 3.7, $c > b^2$, so $c^{1/4} > b^{1/2} > 63$. By Lemma 2.14, $n \leq m \leq 2n$.

The family A1. Let $c < 2.01 \cdot 10^{26}$; then T is a triple of A1. If ρ is of type I, item 3 of Corollary 6.7 gives $2m = m' > 3c^{1/4}$, so $m > \frac{2}{5}c^{1/4}$. If ρ is of type II, item 4 of Corollary 6.7 gives $m > (c/4)^{1/4} - 2 > \frac{2}{5}c^{1/4}$, because $(4^{-1/4} - \frac{2}{5})c^{1/4} > 0.3 \cdot 63 > 2$; moreover $c < 10^{26}$ by Proposition 6.8.

The cap $m' < 7.7 \cdot 10^{18}$ follows from $m' < Q_0(c)$ (Proposition 5.10). Put $Q_* = 7.7 \cdot 10^{18}$ and $K_c = K_{\mathrm{a}}(\log c)^2$, with $K_{\mathrm{a}} = 4.436 \cdot 10^{13}$ (Definition 4.1). As $c < 2.01 \cdot 10^{26}$,

$$K_c < K_{\mathrm{a}}\big(\log(2.01 \cdot 10^{26})\big)^2 < 1.6272 \cdot 10^{17},$$
$$K_c \log\big(39(Q_* + 1)\big) < 1.6272 \cdot 10^{17} \cdot 47.16 < 7.674 \cdot 10^{18} < Q_* - 2.$$

So the integer Q_* satisfies $Q_* \geq K_c - 1$ and $Q_* - 2 \geq K_c \log(39(Q_* + 1))$. As $Q_0(c)$ is the least such integer (Definition 4.1), $Q_0(c) \leq Q_*$, and $m' < 7.7 \cdot 10^{18}$. Hence ρ satisfies the window of A1 of its type, and $(T, \rho) \in$ A1.

The family A2. Let $c \geq 2.01 \cdot 10^{26}$. By Proposition 6.12, z has no representation of type II, so ρ is of type I; every representation of type I satisfies $\lambda\Theta < 0$, lies on the line, and has $m' < 2.0 \cdot 10^{12}$; and $\varrho < 3.21 \cdot 10^{11}$ if $\varepsilon = 1$, $\varrho < 4.61 \cdot 10^{11}$ if $\varepsilon = -1$. By item 1 of Lemma 3.7, $\varepsilon \in \{-1, 1\}$. As $\lambda\Theta < 0$, $\Theta \neq 0$; as (m, n) lies on the line, $|\Theta|/3 = m - \varepsilon n$ is an integer, so $3 \mid \Theta$. With $2.01 \cdot 10^{26} \leq c < 1.81 \cdot 10^{47}$ and the bounds on ϱ, T is a triple of A2.

As $\lambda\Theta < 0$, $\lambda = -\operatorname{sgn}\Theta$. By item 1 of Proposition 6.2, $n < m$, so $n < m \leq 2n$, and $m' = 2m > m - \varepsilon n = |\Theta|/3$. By item 2 of Corollary 6.7, which applies on the line, $m > (\sqrt{17} - 1)\varrho$ if $\varepsilon = 1$ and $m > 2.17\varrho$ if $\varepsilon = -1$, that is, $m' > \ell_\varepsilon$. Together with $m' < 2.0 \cdot 10^{12}$ this gives

$$\max\Big\{\frac{|\Theta|}{3},\ \ell_\varepsilon\Big\} < m' < 2.0 \cdot 10^{12}.$$

Finally, if $\varepsilon = -1$, then $m + n = |\Theta|/3$ and $m \leq 2n$ give $3m \leq 2(m + n) = \frac{2}{3}|\Theta|$, so $m' = 2m \leq \frac{4}{9}|\Theta|$. Hence $(T, \rho) \in$ A2. □

7 The class bounds of degree at least two

A type I representation over a triple of degree at least two forces $c < 10^{99}$, and a type II representation forces $c < 10^{79.71}$ (Proposition 7.33, Proposition 7.19); the section ends with the generic windows (Proposition 7.45), which are the windows of BX. The first-order defect of type II splits by position, and the type I floor of a high triple has exponent one quarter (Subsection 7.1). Polynomial floors against the refined Matveev cap then give $c < 10^{163}$ for type I and $c < 10^{154}$ for type II (Proposition 7.7). On the windows of c that remain, sharper floors meet the caps of Section 5: for type II through a case analysis by position and by the vanishing of the defects (Subsection 7.3), for type I through a gap of exponent 5/39 and an exclusion of small indices by Pell covers (Subsection 7.4). These two subsections can be read independently of each other.

7.1 First-order defects and the two positions

For a high triple with a type I representation one has $c > 16a^2b^2$, and the floor of He and Togbé (Lemma 2.23) improves to exponent one quarter.

Lemma 7.1. *Let $a < b < c < d$ be an irregular Diophantine quadruple with $\deg\{a,b,c\} \geq 2$ such that $T = \{a,b,c\}$ is high, and let z have a type I representation. Then $c > 16a^2b^2$ and, with $\kappa = (\sqrt{17}-1)/2$,*

$$m > n > 2\kappa\sqrt{a}\,c^{1/4} \geq (\sqrt{17}-1)\,c^{1/4} > 3.12\,c^{1/4}, \qquad m' = 2m > 6.24\,c^{1/4}.$$

Proof. Since Lemma 2.23 gives $n > \kappa\sqrt{c/b}$, it suffices to show that a high descent partner forces $b < \sqrt{c}/(4a)$.

By canonical descent (Theorem 2.7), the descent partner e is a positive integer, $\partial T = \{a,b,e\}$ is a Diophantine triple and $c = d_+(a,b,e) = a+b+e+2abe+2r\sqrt{ae+1}\sqrt{be+1}$ with $r = \sqrt{ab+1}$; as the product of the three roots exceeds abe, $c > 4abe$. Since T is high, $b < e$, so e is the largest element of ∂T; as $\deg \partial T = \deg T - 1 \geq 1$, ∂T is not Euler, and Theorem 2.7 applied to ∂T gives that $f = d_-(a,b,e)$ is a positive integer with $e = d_+(a,b,f) > 4abf \geq 4ab$. Hence

$$c > 4abe > 16a^2b^2, \qquad \text{i.e.} \qquad b < \frac{\sqrt{c}}{4a}.$$

Inserting $c/b > 4a\sqrt{c}$ in $m > n > \kappa\sqrt{c/b}$ (Lemma 2.23) gives $n > \kappa\sqrt{4a}\,c^{1/4} = 2\kappa\sqrt{a}\,c^{1/4} \geq 2\kappa\,c^{1/4} = (\sqrt{17}-1)c^{1/4}$, and $\sqrt{17}-1 > 3.123$; hence $m' = 2m > 2n > 6.24\,c^{1/4}$. □

For type II the first-order defect H of Definition 2.20 decides the route; it is a multiple of c, and the same expansion gives the second-order congruences used in the later cases (Lemma 7.2). When the triple is middle or low, H is a positive multiple of c (Lemma 7.3). When it is high, either $H \neq 0$ and n is at least of order $\sqrt{a}\,c^{1/4}$ (Lemma 7.4), or $H = 0$ and a linear Vieta identity links b, the indices and the descent partner f of the descent (Lemma 7.5). If moreover $L = 0$, two further descent steps follow (Lemma 7.6); they serve the high triples of Subsection 7.3 and, through Lemma 7.9, the type II floor of Subsection 7.2.

Lemma 7.2. *Let $0 < a < b < c < d$ be a Diophantine quadruple such that $\{a,b,c\}$ is not an Euler triple, and let z have a type II representation (λ,m,n) with first-order defect H. Put $C_0 = a+b-e$. Then $c \mid H$, and $J = H/c$ satisfies*

$$c \mid \Delta\big[(8ab - C_0^2)(S+1) + 2\lambda(6ab+3-C_0(a+b))\big] - 6J.$$

If $H = 0$, put $E_1 = e-a-b$. Then

$$c \mid \Delta\big[(4ab-4)(S+1) + 2\lambda(6ab+3+E_1(a+b))\big].$$

If $H = 0$ and $\lambda = -1$, put $N_0 = 2n(n+1)$, $M_0 = 2m(m+1)$ and

$$\mathcal{P} = 4a(b-a)M_0 - 4(ab+1)\Delta(S+1) - 6(2ab+1)\Delta, \qquad \mathcal{Q} = E_1\mathcal{P} - 8(ab+1)\Delta(b+3a).$$

Then $c \mid \mathcal{Q}$.

Proof. Put $p = m(m+1)$, $q = n(n+1)$, $A = ap - bq$,

$$A_2 = a^2p(p-2) - b^2q(q-2), \qquad A_3 = ap(2m+1) - bq(2n+1).$$

Define

$$P_m(x) = \sum_{i=0}^{m} \binom{2m+1}{2i} (1+x)^{m-i}x^i, \quad Q_m(x) = \sum_{i=0}^{m} \binom{2m+1}{2i+1} (1+x)^{m-i}x^i.$$

The defining odd Pell terms are $\lambda stP_m(ac) + rcQ_m(ac)$. Expanding these integer polynomials gives

$$0 = z^{(a,c)}_{2m+1} - z^{(b,c)}_{2n+1} = 2cF + \frac{2}{3}c^2(\lambda stA_2 + rA_3) + c^3R,$$

where $F = r\Delta + \lambda stA$, $R \in \mathbb{Z}$. These formulas follow by expanding

$$P_m(x) = 1 + 2px + \tfrac{2}{3}p(p-2)x^2 + x^3P^*_m(x), \quad Q_m(x) = 2m+1 + \tfrac{2}{3}p(2m+1)x + x^2Q^*_m(x)$$

in $z^{(a,c)}_{2m+1} = \lambda stP_m(ac) + rcQ_m(ac)$, and similarly for b, n. The coefficients displayed with denominator three are integers.

The definition of $e = d_-(a,b,c)$ (Definition 2.2) gives the exact relation

$$2rst = (2ab+1)c + C_0,$$

while $s^2t^2 = 1 + (a+b)c + abc^2$. Since $H = -2A - \lambda C_0\Delta$, one has

$$2stF = c\big[(2ab+1)\Delta + 2\lambda(a+b)A + 2\lambda abcA\big] - \lambda H.$$

The integer first-order expansion gives $c \mid 2F$. Multiplying by st and using the exact displayed identity then gives $c \mid H$ (the congruence of [13, Section 7.2], there under $\deg\{a,b,c\} \geq 2$), so write $H = cJ$. Multiply the displayed equality of common terms by $3st$, divide by c^2, and reduce modulo c. This involves no inversion of two or three modulo c. The result, after multiplication by λ, is

$$3J \equiv 3\lambda(2ab+1)\Delta + 6(a+b)A + 2A_2 + \lambda C_0A_3 \pmod{c}.$$

Put $\Sigma = ap + bq$. Then

$$A_2 = A[\Sigma - 2(a+b)] + 2ab\Delta(S+1), \qquad A_3 = (S+1)A + \Delta\Sigma.$$

Multiply the congruence by two and use $2A \equiv -\lambda C_0\Delta \pmod{c}$. The terms in Σ cancel, leaving exactly

$$6J \equiv \Delta\big[(8ab - C_0^2)(S+1) + 2\lambda(6ab + 3 - C_0(a+b))\big] \pmod{c}.$$

This proves the general assertion.

As $C_0 \equiv 2rst \pmod{c}$ and $s^2t^2 \equiv 1 \pmod{c}$, $C_0^2 \equiv 4(ab+1) \pmod{c}$. This reduction is valid for every value of H. In the case $H = 0$, reduction modulo c therefore replaces $8ab - C_0^2$ by $4ab - 4$, giving the second assertion. In particular, at $\lambda = -1$ the divisible expression is

$$\Delta F', \qquad F' = (4ab-4)(S+1) - 6(2ab+1) - 2E_1(a+b).$$

Let $H = 0$ and $\lambda = -1$. Then $N = N_0 + \Delta$ and $M = M_0 - \Delta$, so $H = 0$ reads $bN_0 - aM_0 = E_1\Delta$; and $E_1^2 = C_0^2 \equiv 4(ab+1) \pmod{c}$. Direct expansion gives the polynomial identity

$$\mathcal{Q} - E_1\Delta F' = 4aE_1\big(bN_0 - aM_0 - E_1\Delta\big) + 2\Delta(3a+b)\big(E_1^2 - 4(ab+1)\big).$$

The first term on the right vanishes and the second is divisible by c. Since $c \mid \Delta F'$, it follows that $c \mid \mathcal{Q}$. □

Lemma 7.3. *Let $a < b < c < d$ be an irregular Diophantine quadruple with $\deg\{a,b,c\} \geq 2$ such that $T = \{a,b,c\}$ is middle or low, and let z have a type II representation (λ, m, n), with first-order defect H and $u = \sqrt{ae+1}$. Then u is an integer, $c \mid H$, the first-order quotient $J = H/c$ is a positive integer, and*

$$J(4ae+1) < 2m(m+1).$$

In particular $u^2 \leq \frac{1}{2}m(m+1)$, so $u < m'/(2\sqrt{2})$.

Proof. For a middle or low triple the first-order defect is a positive multiple of c: $c \mid H$ holds for every type II representation (Lemma 7.2), the inequalities $a < b/8$ and $e < b$ give $H > 0$, and then $c > (4ae+1)b$ turns $H = Jc$ into the bound on J.

By Lemma 2.9, Lemma 2.10, Lemma 2.15 and Lemma 2.14,

$$b > 4000, \qquad b > 8a, \qquad m, n \geq 4, \qquad n \leq m \leq 2n, \qquad 0 \leq \Delta \leq n \leq m.$$

Since $\deg T \geq 2$, T is not Euler, so by canonical descent (Theorem 2.7) e is a positive integer, $\{a,b,e\}$ is a Diophantine triple (so u is an integer) and $c = d_+(a,b,e)$. By Definition 2.2, with $r = \sqrt{ab+1} > \sqrt{ab}$, $u > \sqrt{ae}$ and $\sqrt{be+1} > \sqrt{be}$,

$$c = a + b + (2ab+1)e + 2ru\sqrt{be+1} > 4abe + a + b + e > (4ae+1)b. \tag{1}$$

By the first assertion of Lemma 7.2, $c \mid H$.

Since $|\lambda\Delta| \leq \Delta$, $a < b/8$ and $e < b$,

$$H > b\Big(2n(n+1) - \Delta - \frac{2m(m+1)+\Delta}{8} - \Delta\Big).$$

With $2m(m+1) \leq 4n(2n+1)$ and $\Delta \leq n$ the bracket is at least $2n^2+n-n^2-\frac{5}{8}n-n = n^2-\frac{5}{8}n > 0$. Hence $J = H/c$ is a positive integer.

Since $M \geq 2m(m+1) - \Delta > 0$, we have $aM > 0$, so $H < bN + e\Delta$ when $\lambda = 1$ and $H < bN$ when $\lambda = -1$; in both cases $Jc = H < bN + e\Delta \leq b(N+\Delta)$ because $e < b$. With (1),

$$J(4ae+1) < \frac{Jc}{b} < N + \Delta.$$

If $\Delta = 0$, then $N = 2m(m+1)$ and in fact $J(4ae+1) < N$, as $Jc < bN$ in this case. If $\Delta \geq 1$, then $N \leq 2n(n+1) + \Delta$, and with $n = m - \Delta$,

$$N + \Delta \leq 2(m-\Delta)(m-\Delta+1) + 2\Delta = 2m(m+1) - 2\Delta(2m-\Delta) < 2m(m+1),$$

since $1 \leq \Delta \leq m$. In both cases $J(4ae+1) < 2m(m+1)$.

As $J \geq 1$, $4u^2 - 3 = 4ae + 1 < 2m(m+1)$. As $4u^2 - 3 \equiv 1$ and $2m(m+1) \equiv 0 \pmod 4$, this forces $4u^2 - 3 \leq 2m(m+1) - 3$, that is $u^2 \leq \frac{1}{2}m(m+1) < \frac{1}{8}(2m+1)^2 = \frac{1}{8}m'^2$. □

Lemma 7.4. *Let $a < b < c < d$ be an irregular Diophantine quadruple with $\deg\{a,b,c\} \geq 2$ such that $T = \{a,b,c\}$ is high, and let z have a type II representation (λ, m, n) whose first-order defect H is non-zero. Then*

$$n > 0.53\sqrt{a}\,c^{1/4} - 0.75, \qquad m' = 2m+1 > 1.06\sqrt{a}\,c^{1/4} - 0.5.$$

Proof. The congruence $c \mid H$ holds for every type II representation (Lemma 7.2); when $H \neq 0$ the term $e\lambda\Delta$ or the term bN must be of size c, and both options force an index of order $c^{1/4}$ because $e < c/(4ab)$ and $b < \sqrt{c}/(4a)$.

By Lemma 2.9, $b > 4000$; by Lemma 2.10, $b > 8a$; by Lemma 2.14, $n \le m \le 2n$, so $0 \le \Delta \le n$; by Theorem 2.11, $c < 200b^4$, i.e. $b > (c/200)^{1/4}$. By canonical descent (Theorem 2.7), e is a positive integer and $c = d_+(a,b,e) > 4abe$, so $e < c/(4ab)$. Since T is high and $\deg T \ge 2$, $\partial T = \{a,b,e\}$ has largest element e and is not Euler, so Theorem 2.7 gives $f = d_-(a,b,e) \ge 1$ and $e = d_+(a,b,f) > 4ab$; hence $c > 16a^2b^2$, i.e. $b < \sqrt{c}/(4a)$.

As $\deg T \ge 2$, T is not Euler, so by the first assertion of Lemma 7.2, $c \mid H$; write $H = Jc$, so the first-order quotient J is a nonzero integer.

Since $\Delta \le n$ and $m \le 2n$: $N = 2n(n+1) - \lambda\Delta \ge 2n^2 + n$ and $M = 2m(m+1) + \lambda\Delta \le 2m^2 + 3m \le 8n^2 + 6n$, so $bN > 8a(2n^2+n) \ge a(8n^2+6n) \ge aM$. Hence $0 < bN - aM < bN$.

Now $\lambda e\Delta = Jc - (bN - aM)$. If $bN \ge |J|c/2$, then $N \ge c/(2b) > c/(2\sqrt{c}/(4a)) = 2a\sqrt{c}$. As $N \le 2n(n+1) + \Delta \le 2n^2 + 3n$, we get $(n+\frac{3}{4})^2 > n^2 + \frac{3}{2}n > a\sqrt{c}$, so $n > \sqrt{a}\,c^{1/4} - \frac{3}{4}$. If $bN < |J|c/2$, then $e\Delta = |Jc - (bN - aM)| \ge |J|c - bN > |J|c/2 \ge c/2$, so $\Delta > c/(2e) > 2ab$ (as $e < c/(4ab)$), and $n \ge \Delta > 2ab > 2a(c/200)^{1/4} > 0.531\,a\,c^{1/4} \ge 0.531\sqrt{a}\,c^{1/4}$.

In both cases $n > 0.53\sqrt{a}\,c^{1/4} - 0.75$, and $m' = 2m+1 \ge 2n+1 > 1.06\sqrt{a}\,c^{1/4} - 0.5$. □

Lemma 7.5. *Let $a < b < c < d$ be an irregular Diophantine quadruple with $\deg\{a,b,c\} \ge 2$ such that $T = \{a,b,c\}$ is high, with descent partner e and $f = e(\partial T)$ the descent partner of the descent, and let z have a type II representation (λ, m, n) whose first-order defect vanishes, $H = 0$. Put $N_0 = 2n(n+1)$, $M_0 = 2m(m+1)$, $S = m+n$ and $g = d_-(a,b,f)$. Then $\lambda = -1$, $\Delta \ge 1$,*

$$e\Delta = bN - aM, \qquad N < 2m^2, \qquad M < 4N,$$

the number $L = N - (4af+2)\Delta = N_0 - (4af+1)\Delta$ of Definition 2.20 satisfies $L \ge 0$, and

$$Lb + g\Delta = aM_0 + (a+2f)\Delta, \qquad (bN - aM)g\Delta = \big((b-a-f)^2 - 4(af+1)\big)\Delta^2.$$

If moreover $L \ge 1$, then

$$b < \frac{4am^2}{L}, \qquad a < \frac{m^2}{2f\Delta}, \qquad c < \frac{129\,a^3m^6}{\Delta L^2} < \frac{16.2\,m^{12}}{f^3\Delta^4L^2}.$$

Proof. The relations $\lambda = -1$, $\Delta \ge 1$, $e\Delta = bN - aM$, $N < 2m^2$ and $M < 4N$ are those of [13, Lemma 7.4], which derives $H = 0$ from $c > b^2$ and $m \le c^{1/12}$; here $H = 0$ is assumed and the same argument applies. Multiplying the linear Vieta relation for the triple $\{a,b,f\}$ by Δ and substituting $e\Delta = bN - aM$ eliminates e and gives an exact linear relation between b and g with leading coefficient L; a size bound for g then forces $L \ge 0$, and $L \ge 1$ bounds b and c.

By Lemma 2.9 and Lemma 2.10, $b > 4000$ and $b > 8a$. By canonical descent (Theorem 2.7), $c = d_+(a,b,e)$, and by Lemma 2.6, item 3, $c > a+b+e+4abe > 4ab+a+b$; so Lemma 2.15 and Lemma 2.14 give $4 \le n \le m \le 2n$, and $0 \le \Delta \le m/2 \le n$. Now $H = 0$ reads $e\lambda\Delta = aM - bN$. If $\Delta = 0$, then $N = M = 2m(m+1) > 0$ and $bN = aM$, impossible as $b \neq a$. Moreover $M \le 2m(m+1) + \Delta \le 4n(2n+1) + n = 8n^2 + 5n$ and $8N \ge 16n(n+1) - 8\Delta \ge 16n^2 + 8n$, so $aM < 8aN < bN$. If $\lambda = 1$, then $e\Delta = aM - bN < 0$, which is impossible. Hence $\lambda = -1$, $\Delta \ge 1$ and $e\Delta = bN - aM$. Then $N = N_0 + \Delta$ and $M = M_0 - \Delta$; with $m \le 2n$, $4N - M = 8n(n+1) - 2m(m+1) + 5\Delta \ge 4n + 5\Delta > 0$; and with $n = m - \Delta$, $N = 2m^2 + 2m - \Delta(4m - 2\Delta + 1) \le 2m^2 - 2m + 2\Delta - 1 < 2m^2$, since $1 \le \Delta \le m/2$.

Since T is high and $\deg T \ge 2$, $\partial T = \{a,b,e\}$ has largest element e and is not Euler; by Theorem 2.7 applied to ∂T, $f = d_-(a,b,e)$ is a positive integer, $f \notin \{a,b\}$, $e = d_+(a,b,f) > 4abf$, and $g = d_-(a,b,f) \ge 0$. The expression of Definition 2.2 is symmetric in its three arguments, so $e = d_+(a,f,b)$ and $g = d_-(a,f,b)$, and Lemma 2.6, item 1, with $(p,q,X) = (a,f,b)$ gives

$$e+g = (4af+2)b + 2(a+f), \qquad eg = (b-a-f)^2 - 4(af+1).$$

Multiplying the first relation by Δ and substituting $e\Delta = bN - aM$ gives

$$\big(N - (4af+2)\Delta\big)b + g\Delta = aM + 2(a+f)\Delta = a(M_0 - \Delta) + 2(a+f)\Delta = aM_0 + (a+2f)\Delta,$$

which is $Lb + g\Delta = aM_0 + (a+2f)\Delta$; and $N - (4af+2)\Delta = N_0 - (4af+1)\Delta$ as $N = N_0 + \Delta$. Multiplying $e\Delta = bN - aM$ by $g\Delta$ and substituting the product relation gives $(bN - aM)g\Delta = \big((b-a-f)^2 - 4(af+1)\big)\Delta^2$.

If $f > b$, then $\{a,b,f\}$ is a Diophantine triple with largest element f, so $g = 0$ if it is Euler and $abg < abf/12$ otherwise (Theorem 2.7); in both cases $g < 2f$, hence $Lb = aM_0 + (a + 2f - g)\Delta > 0$ and $L \ge 1$. Suppose now $f < b$. Then $-b < b - a - f < b$, so $e > 4abf$ and $eg = (b-a-f)^2 - 4(af+1) < b^2$ give $0 \le g < \frac{b}{4af}$. If $4af \le \Delta$, then since $\Delta \le n$, $(4af+1)\Delta \le (\Delta+1)\Delta \le n(n+1) = \frac{1}{2}N_0$, so $L \ge \frac{1}{2}N_0 > 0$. If $4af > \Delta$, then $0 \le g\Delta < \frac{b\Delta}{4af} < b$; hence $L \le -1$ would imply $Lb + g\Delta \le -b + g\Delta < 0$, contradicting $Lb + g\Delta = aM_0 + (a+2f)\Delta > 0$. Therefore $L \ge 0$.

Let $L \ge 1$. Since $\Delta \ge 1$, $n \le m-1$ and $N_0 = 2n(n+1) \le 2m^2 - 2m$. From $(4af+1)\Delta = N_0 - L \le N_0 - 1$,

$$4af\Delta \le N_0 - \Delta - 1 < 2m^2 - 2m, \qquad a < \frac{m^2}{2f\Delta}.$$

Because $m \ge 4$, $M_0 = 2m(m+1) \le 2.5\,m^2$, $a\Delta < am \le 0.25\,am^2$, and $2f\Delta < m^2/a \le am^2$. Since $g\Delta \ge 0$,

$$Lb \le aM_0 + (a+2f)\Delta < (2.5 + 0.25 + 1)am^2 < 4am^2, \qquad b < \frac{4am^2}{L}.$$

Finally $e\Delta = bN - aM < bN < 2bm^2$ gives $e < 2bm^2/\Delta$, and $c = d_+(a,b,e) = (4ab+2)e + 2(a+b) - f < 4.01\,abe$ (as $e > b > 4000$) yields

$$c < 4.01\,ab \cdot \frac{2bm^2}{\Delta} = \frac{8.02\,ab^2m^2}{\Delta} < \frac{8.02\,a(4am^2/L)^2m^2}{\Delta} < \frac{129\,a^3m^6}{\Delta L^2} < \frac{129\,m^{12}}{8f^3\Delta^4L^2} < \frac{16.2\,m^{12}}{f^3\Delta^4L^2}.$$

□

Lemma 7.6. *Let $0 < a < b < c < d$ be an irregular Diophantine quadruple with $\deg\{a,b,c\} \ge 2$ and $e > b$, and let z have a type II representation (λ,m,n) with first-order defect $H = 0$. Let f, $v = 2af+1$ and L be as in Definition 2.20, and suppose $L = 0$. Then $1 \le \Delta \le n \le v-1$, $S < 3v$, and $g = d_-(a,b,f) = 2a(v+S) + 2(a+f) > d_+(a,f,0)$. The set $\{a,f,g\}$ is a Diophantine triple with $d_+(a,f,g) = b$; the number $h = d_-(a,f,g)$ is a positive integer, and $\{a,f,h\}$ is a Diophantine triple with $d_+(a,f,h) = g$. Finally, $\ell = d_-(a,f,h)$ is an even integer, and*

$$2v(h-a) = 2aS + \ell, \qquad a+1 \le h \le 4a, \qquad 2 \le \ell < \frac{v}{8a^2}, \qquad f = d_+(a,h,\ell) > 8a(a+1).$$

Proof. When $L = 0$, the factor Δ cancels from the linear Vieta relation of Lemma 7.5, giving an explicit formula for g that forces two further descent steps; the relation $2n(n+1) = (2v-1)\Delta$ bounds $n \le v-1$ and $S < 3v$, which rules out $\ell = 0$ by parity and a square argument modulo 3.

By Lemma 7.5, $\lambda = -1$ and $\Delta \ge 1$; and $\Delta \le n$ by Lemma 2.14. Here $v = 2af + 1 \ge 3$. Since $L = 2n(n+1) - (2v-1)\Delta = 0$ and $\Delta \le n$, we have $(2v-1)\Delta = 2n(n+1) \ge 2\Delta(n+1)$, so $2v - 1 \ge 2n + 2$, and

$$1 \le \Delta \le n \le v-1, \qquad S = m + n = 2n + \Delta \le 3(v-1) < 3v.$$

Moreover $N = 2n(n+1) + \Delta = 2v\Delta$, $M = N + 2\Delta S = 2(v+S)\Delta$, and $M_0 = 2m(m+1) = M + \Delta = (2v + 2S + 1)\Delta$.

Substituting $L = 0$ and $M_0 = (2v+2S+1)\Delta$ into $Lb + g\Delta = aM_0 + (a+2f)\Delta$ (Lemma 7.5) and dividing by $\Delta \ge 1$ gives

$$g = 2a(v+S) + 2(a+f) = (4af+2)a + 2(a+f) + 2aS > d_+(a, f, 0).$$

From $e\Delta = bN - aM = 2\Delta\big(vb - a(v+S)\big)$ and $e = d_+(a,b,f) > 4abf = (2v-2)b$ we get $b > a(v+S) > v > f$. Thus b is the largest element of the Diophantine triple $\{a, f, b\}$, and $d_-(a,f,b) = d_-(a,b,f) = g \ge 1$; so this triple is not Euler, and canonical descent (Theorem 2.7) shows that $\{a, f, g\}$ is a Diophantine triple with $b = d_+(a,f,g) > 4afg = 2(v-1)g > g$. Since $g > d_+(a,f,0)$, the triple $\{a,f,g\}$ is not Euler either, and by Theorem 2.7 the number $h = d_-(a,f,g)$ is a positive integer, $\{a,f,h\}$ is a Diophantine triple, $h < g$ and $g = d_+(a,f,h)$. The sum of the roots $g = d_+(a,f,h)$ and $\ell = d_-(a,f,h) \ge 0$ is $g + \ell = 2vh + 2(a+f)$ (Lemma 2.6, item 1). Subtracting $g = 2va + 2(a+f) + 2aS$ yields

$$2v(h-a) = 2aS + \ell,$$

which shows that ℓ is even and $h \ge a+1$.

Suppose $\ell = 0$. Then $2v(h-a) = 2aS$, and since $\gcd(a,v) = \gcd(a, 2af+1) = 1$, v divides S. Because $0 < S < 3v$, we have $S = qv$ with $q \in \{1,2\}$. Substituting $\Delta = S - 2n = qv - 2n$ into $2n(n+1) = (2v-1)\Delta$ gives

$$2n^2 + 4vn - qv(2v-1) = 0.$$

If $q = 1$, then $2n^2 + 4vn$ is even whereas $v(2v-1)$ is odd, impossible. If $q = 2$, dividing by 2 and completing the square gives $(n+v)^2 = v(3v-1)$. The positive factors v and $3v-1$ are coprime, so both are squares, whereas $3v - 1 \equiv 2 \pmod 3$ is not a square, a contradiction. Therefore $\ell \ge 2$.

Since $\ell = d_-(a,f,h) \ge 2$ and $g = d_+(a,f,h)$, we have $g\ell = (f-a-h)^2 - 4(ah+1) < (\max\{f,h\})^2$. If $h \ge f$, then $h = d_+(a,f,\ell) > 4af\ell \ge 8af = 4(v-1)$, while $g > 4afh$ gives $\ell < \frac{h}{4af} < h$, so $2v(h-a) = 2aS + \ell < 6av + h$ and $(2v-1)h < 8av$, i.e. $h < 5a$, contradicting $h > 8af \ge 8a$. Thus $f > h$, so $f = d_+(a,h,\ell) > 4ah\ell \ge 8a(a+1)$ and $\ell < \frac{f}{4ah} < \frac{v}{8a^2} < v$. Finally, $2v(h-a) = 2aS + \ell < (6a+1)v$ gives $h - a \le 3a$, so $a+1 \le h \le 4a$. □

7.2 Polynomial index floors and the a priori bounds

The first squeeze in degree at least two combines polynomial floors with the refined Matveev cap of Proposition 5.11. The proposition below rests on two floors, $m' > c^{1/8}$ for type II when $c \ge 10^{130}$ (Proposition 7.12) and $m' > 2^{-23/42}c^{5/42}$ for type I when $c \ge 2^{540}$ (Lemma 7.18), which are proved after it.

Proposition 7.7. *Let $0 < a < b < c < d$ be an irregular Diophantine quadruple with $\deg\{a,b,c\} \geq 2$.*

1. *If z has a type I representation, then $c < 10^{163}$.*
2. *If z has a type II representation, then $c < 10^{154}$.*

Proof. Both items follow from a gap between m' and a power of c, Matveev's upper bound for m', and the comparison lemma 5.6, applied with $w = m'$ and $\ell = \log c$. All comparisons below are between integers and rationals, using positive partial sums of the exponential series.

Item 1. If $c < 2^{540}$, the conclusion follows from $2^{540} < 10^{163}$. Otherwise Lemma 7.18 gives

$$m' > Y\, c^{5/42}, \qquad Y = 2^{-23/42},$$

and, since $c \geq 2^{540} > 10^{80}$ and $c = d_+(a,b,e) > 4abe \geq 4ab$ by Theorem 2.7, the type I item of Proposition 5.11 gives

$$m' - 2 < 2.48 \cdot 10^{12} (\log c)^2 \log\bigl(19.6(m'+1)\bigr).$$

Apply Lemma 5.6 with this Y, $\kappa_1 = 42/5$, $C = 2.48 \cdot 10^{12}$, $\theta = 19.6$ and $\Lambda_0 = 163$. Here $w_0 = Y\, 10^{815/42}$ and $w_0^{42} = 2^{-23} 10^{815}$. Put

$$q_1 = 171 \cdot 10^{17}, \qquad q_2 = 175 \cdot 10^{17}, \qquad q_3 = 26 \cdot 10^{18}.$$

The exact comparisons

$$2^{23} q_1^{42} < 10^{815} < 2^{23} q_2^{42}, \qquad 10^{815} < q_3^{42}, \qquad \sum_{j=0}^{150} \frac{45^j}{j!} > q_3, \qquad \sum_{j=0}^{170} \frac{48^j}{j!} > \frac{98}{5}(q_2+1)$$

give $q_1 < w_0 < q_2$, $\log(w_0/Y) < 45$ and $\log(19.6(w_0+1)) < 48$. Hence

$$G_{Y,19.6}(w_0) > \frac{q_1 - 2}{45^2 \cdot 48} > 2.48 \cdot 10^{12} \Big(\frac{42}{5}\Big)^2.$$

The remaining hypotheses $w_0 \geq 50$ and $10^{815/42} \geq 21$ are immediate, so Lemma 5.6 gives $\log c < 163 \log 10$.

Item 2. By Proposition 7.12, $m' > c^{1/8}$ whenever $c \geq 10^{130}$; the case $c < 10^{130}$ is immediate. Otherwise Proposition 5.11 gives

$$m' - 2 < 2.79 \cdot 10^{12} (\log c)^2 \log\bigl(17.4(m'+1)\bigr).$$

Apply Lemma 5.6 with $Y = 1$, $\kappa_1 = 8$, $C = 2.79 \cdot 10^{12}$, $\theta = 17.4$ and $\Lambda_0 = 154$. Here $w_0 = 10^{77/4}$. Put $q_1 = 177 \cdot 10^{17}$ and $q_2 = 18 \cdot 10^{18}$. The exact comparisons

$$q_1^4 < 10^{77} < q_2^4, \qquad \sum_{j=0}^{150} \frac{45^j}{j!} > q_2, \qquad \sum_{j=0}^{170} \frac{48^j}{j!} > \frac{87}{5}(q_2+1)$$

give $q_1 < w_0 < q_2$, $\log w_0 < 45$ and $\log(17.4(w_0+1)) < 48$. Consequently

$$G_{1,17.4}(w_0) > \frac{q_1 - 2}{45^2 \cdot 48} = \frac{8849999999999999999}{48600} > 178560000000000 = C \cdot 8^2.$$

The remaining hypotheses $w_0 \geq 50$ and $10^{\Lambda_0/\kappa_1} \geq 21$ are immediate. The lower bound $m' > c^{1/8}$ and Lemma 5.6 therefore give $\log c < 154 \log 10$. □

The type II floor. Below b^2, a type II representation has $(m')^8 > 22c$ (Lemma 7.8); above, the cases with $H = 0$ are treated in Lemma 7.9, Lemma 7.10 and Lemma 7.11, and Proposition 7.12 assembles them.

Lemma 7.8. *Let $0 < a < b < c < d$ be an irregular Diophantine quadruple with $\deg\{a, b, c\} \geq 2$ and $c \leq b^2$. If z has a type II representation, then*

$$(m')^8 > 22c.$$

Proof. The square congruence modulo c below, made an equality by size, is that of the proof of [13, Proposition 7.5] for $c < b^{5/4}$; here it covers all $c \leq b^2$, and the case $E_\mu \geq 1$ is closed by the integral Vieta determinant which that proof uses for $c > b^2$, in place of its congruence modulo b. Put $p = m' = 2m + 1$, $\Delta = m - n$, $N_0 = 2n(n + 1)$, $M_0 = 2m(m + 1)$, $N = N_0 - \lambda\Delta$, $M = M_0 + \lambda\Delta$, and $H = bN - aM + e\lambda\Delta$, where $e = d_-(a, b, c)$. By Lemma 2.9, Lemma 2.10, Lemma 2.15, and Lemma 2.14,

$$b > 4000, \quad b > 8a, \quad m, n \geq 4, \quad n \leq m \leq 2n, \quad 0 \leq \Delta \leq n.$$

Canonical descent (Theorem 2.7) gives $e \geq 1$, $c = d_+(a, b, e) > (4ae + 1)b + a + e$, and $f = d_-(a, b, e) \geq 1$. Indeed $c \leq b^2$ implies $e < b/(4a) < b$, and the degree assumption excludes an Euler triple $\{a, b, e\}$.

In particular T is middle or low, and Lemma 7.3, with $u = \sqrt{ae + 1}$, gives $c \mid H$, a positive integer $J = H/c$, and

$$J(4ae + 1) < 2m(m + 1) < \frac{p^2}{2}, \qquad 4u^2 \leq 2m(m + 1) < \frac{p^2}{2}.$$

Suppose for contradiction that $p^8 \leq 22c$.

Put $E_\pm = a + e \pm 2u$ and $\Pi = aM - e\lambda\Delta$. The roots $c = d_+(a, b, e)$ and $f = d_-(a, b, e)$ satisfy, by Definition 2.2,

$$c + f = (4ae + 2)b + 2(a + e), \qquad cf = (b - E_+)(b - E_-).$$

Also $Jc = bN - \Pi$. Therefore

$$(\Pi - (a + e)N)^2 \equiv 4u^2N^2 \pmod{c}.$$

Both sides are smaller than c: since $4u^2 < p^2/2$ and $N \leq 2m(m + 1) < p^2/2$ (for $\Delta \geq 1$, $n = m - \Delta$ gives $N \leq 2n(n + 1) + \Delta < 2m(m + 1)$),

$$4u^2N^2 < \frac{p^6}{8} < c,$$

where the last inequality follows from $p^8 \leq 22c$ and $p \geq 9$. Next,

$$\Pi - (a + e)N = a(M - N) - 2en(n + 1).$$

Here $0 \leq M - N = 2\Delta(m + n + 1 + \lambda)$ and $M - N < M - \lambda\Delta = 2m(m + 1) < p^2/2$, as $N > \lambda\Delta$; while $ae < p^2/8$ and $\max(a, e) \leq ae$. Thus

$$|\Pi - (a + e)N| < \frac{p^4}{16}, \qquad (\Pi - (a + e)N)^2 < \frac{p^8}{256} \leq \frac{22}{256}c < c.$$

The congruence is therefore an equality, and $\Pi = E_\mu N$ for some $\mu \in \{-1, 1\}$.

The distinct positive Diophantine pair $\{a, e\}$ has $|a - e| \geq 2$, so $E_\pm \geq 0$ and $ae \geq 3$. If $E_\mu = 0$, then $\Pi = 0$, $|a - e| = 2$, and $e \leq 3a$. But $aM = e\lambda\Delta$ forces $\lambda = 1$, $\Delta \geq 1$, and

$$\frac{e}{a} = \frac{M}{\Delta} = \frac{2m(m+1)}{\Delta} + 1 \geq 2(m+1) + 1 \geq 11,$$

a contradiction. Hence $E_\mu \geq 1$.

Now $Jc = (b - E_\mu)N$, so $b - E_\mu > 0$; and $J(4ae+1)b < Jc < bN$, by $c > (4ae+1)b$ and $E_\mu \geq 1$, gives $J(4ae+1) < N$. Eliminating c from the two Vieta identities gives the exact linear system

$$Jb - Nf = JE_{-\mu}, \qquad \big(N - J(4ae+2)\big)b + Jf = E_\mu N + 2J(a+e).$$

Its determinant

$$D = N^2 - J(4ae+2)N + J^2$$

satisfies

$$Db = (N+J)(NE_\mu + JE_{-\mu}) > 0.$$

Since D is an integer, $D \geq 1$.

The elementary estimate $E_+ \leq \frac{8}{13}(4ae+1)$ holds. For $ae = 3$ it is equality at $\{a, e\} = \{1, 3\}$. Otherwise $ae = u^2 - 1 \geq 8$, and $a + e \leq ae + 1$, $2\sqrt{ae+1} < ae - 1$ give $E_+ < 2ae < \frac{8}{13}(4ae+1)$. Therefore

$$J < \frac{N}{13}, \qquad E_\pm < \frac{8}{13}\frac{N}{J},$$

and

$$Db < \frac{8}{13}\frac{N(N+J)^2}{J} < \frac{8}{13}\left(\frac{14}{13}\right)^2 \frac{N^3}{J} < 0.72\frac{N^3}{J}.$$

Finally,

$$c = \frac{(b - E_\mu)N}{J} < \frac{bN}{J} < \frac{0.72N^4}{DJ^2} \leq 0.72N^4 < 0.045p^8 \leq 0.99c,$$

a contradiction. Thus $p^8 > 22c$. □

Lemma 7.9. *Let $0 < a < b < c < d$ be an irregular Diophantine quadruple with $\deg\{a, b, c\} \geq 2$ and $e > b$, and let z have a type II representation (λ, m, n) with $\lambda = -1$, $\Delta > 0$ and first-order defect $H = 0$. Let f, $v = 2af + 1$ and L be as in Definition 2.20. If $L = 0$, then*

$$c < 1.55(m')^8.$$

If moreover $\Delta > n/2$, then $c < 6.6(m')^6$. If $L = 0$ and $\Delta \leq n/2$, then, with $S = m + n$ and $Y = 2a^2(v + S) + v$,

$$Y < \frac{12}{7}\Delta(S+1) \qquad \textit{and} \qquad c < \frac{12.35N^3(S+1)^2}{a\Delta}.$$

Proof. Write $p = m' = 2m + 1$; as $\lambda = -1$, $N = 2n(n+1) + \Delta$ and $M = 2m(m+1) - \Delta$. By Lemma 2.9, Lemma 2.10, Lemma 2.15 and Lemma 2.14,

$$b > 4000, \quad b > 8a, \quad m, n \geq 4, \quad n \leq m \leq 2n.$$

Since $L = 0$ and $H = 0$,

$$N = 2v\Delta, \quad M = 2(v+S)\Delta, \quad 2n(n+1) = (2v-1)\Delta, \quad e\Delta = bN - aM.$$

The two further descent steps are those of Lemma 7.6: $1 \le \Delta \le n \le v-1$, $S < 3v$, $g = d_-(a,b,f) = 2a(v+S)+2(a+f)$ and $b = d_+(a,f,g)$, and the numbers $h = d_-(a,f,g)$ and $\ell = d_-(a,f,h)$ satisfy

$$2v(h-a) = 2aS+\ell, \quad a+1 \le h \le 4a, \quad 2 \le \ell < \frac{v}{8a^2}, \quad f > 8a(a+1).$$

Hence $v = 2af+1 > 16a^2(a+1)+1 \ge 33$; in particular $v > 32a^2$.

Put $B = b-a$ and $Y = 2a^2(v+S)+v$, so $Y\Delta = a^2M + N/2$. From $b+h = 2vg+2(a+f)$ and $2v(h-a) = 2aS+\ell$ one obtains exactly

$$8avB = 16v^2Y + 32a^2v^2 - 8v^2 - 8a^2S - 8v - 4a\ell. \tag{1}$$

Since $S \ge 9$ and $v+S \ge 42$, the negative terms, together with $v > 32a^2$, give

$$8avb < 16v^2Y\left(1+\frac{1}{v+S}\right) < 1.024\cdot 16v^2Y, \qquad b\Delta^2 < \frac{1.024N(Y\Delta)}{a}.$$

Since $c = d_+(a,b,e)$ (Theorem 2.7) and $f = d_-(a,b,e)$, the Vieta sum for c is

$$c = (4ab+2)e + 2(a+b) - f < 4.001abe.$$

Indeed $e > b > 4000$ and $a \ge 1$ give

$$\frac{2}{ab}+\frac{2}{ae}+\frac{2}{be} < \frac{4}{4001}+\frac{2}{4001^2} < 0.001.$$

Since $e\Delta = bN - aM < bN$, this proves

$$c < \frac{4.20N^3(Y\Delta)^2}{a\Delta^5}. \tag{2}$$

The case $\Delta > n/2$. Here $N < 8\Delta^2$. Also

$$\frac{p^2}{4} - N = -n^2 + 2n\Delta + \Delta^2 - n + \frac{1}{4} > \frac{n^2}{4} - n + \frac{1}{4} > 0$$

because $n \ge 4$. Thus $N < p^2/4$, while $M < p^2/2$. The inequality $f > 8a(a+1)$ gives $a^3\Delta < N/32$. Therefore $Y\Delta = a^2M + N/2 < 0.625a^2p^2$, and (2), factored as follows, yields

$$\begin{aligned} c &< 4.20\left(\frac{N}{\Delta^2}\right)^3 a^3\Delta\left(\frac{Y\Delta}{a^2}\right)^2 \\ &< 4.20\cdot 512\cdot\frac{N}{32}\cdot(0.625p^2)^2 < 6.6p^6 < 1.55p^8. \end{aligned}$$

The case $\Delta \le n/2$. We use the second-order divisibility $c \mid \mathcal{Q}$ of Lemma 7.2 (its last item; here $H = 0$ and $\lambda = -1$). Define

$$c = 8avB^2 + c_1B + c_0, \quad c_1 = 8a^2v + 4v - 8a^2S + 2, \quad c_0 = -8a^3S - 4aS + 4a - f,$$
$$E_1 = (2v-1)B - 2a(S+1),$$
$$\mathcal{Q}_v = E_1\big(4aB(2v+2S+1) - 4(ab+1)(S+1) - 6(2ab+1)\big) - 8(ab+1)(B+4a),$$
$$\mathcal{T}_v = (2v+S-4)c - \mathcal{Q}_v.$$

Here $\mathcal{Q} = \Delta\mathcal{Q}_v$, because $M_0 = M + \Delta = (2v+2S+1)\Delta$, $E_1 = e-a-b$ with $e = 2vb - 2a(v+S)$, and $b + 3a = B + 4a$; so $c \mid \Delta(2v\mathcal{T}_v - Sc)$. Using (1) in the quadratic polynomial $2v\mathcal{T}_v - Sc$ yields

$$c \mid \Delta(R_1 B + R_0),$$

where

$$\begin{aligned} R_1 &= 24(S+1)v^2 + 8a^2S(2v+S+1) - 8Sv - 12v - 2S + 4a\ell, \\ R_0 &= -32a^3Sv(v+S) + 8a^3S(S-2v) - 8aSv(2v+3S+2) \\ &\quad + 4a(S^2+4v^2-S-2v) - f(4v^2+2Sv-S-8v). \end{aligned}$$

The coefficient R_1 satisfies

$$24(S+1)v^2 < R_1 < 24(S+1)v^2 + 8a^2S(2v+S). \tag{3}$$

For the lower bound, replace a^2 by 1 and ℓ by 0 in the excess over $24(S+1)v^2$; this gives at least $(8S-12)v + 8S^2 + 6S > 0$. For the upper bound use $8a^2S < vS/4$ and $4a\ell < v/(2a) \le v/2$.

Since $S \ge 9$, $S < 3v$, and $v \ge 33$, one also has

$$R_0 < -24a^3Sv^2 - 92av^2 - 3fv^2 < 0. \tag{4}$$

Indeed $8a^3S(S-2v) \le 8a^3Sv$, the positive $4a$ term is at most $52av^2$, the negative $8aS$ term is at most $-16aSv^2$, and $4v^2 + 2Sv - S - 8v > 3v^2$. In particular $c_0 - R_0 > 0$.

Moreover (1), or the Vieta expression $B = 4av(v+S) + 4v(a+f) + a + 2f - h$, gives $B > 4av(v+S)$. Set $D_0 = 96a(S+1)v^3(v+S) < R_1 B$. Also

$$\begin{aligned} |R_0| &< 32a^3Sv(v+S) + 16a^3Sv \\ &\quad + 8aSv(2v+3S+2) + f(4v^2+2Sv). \end{aligned}$$

The respective ratios of these four summands to D_0 are less than

$$\frac{1}{96v}, \quad \frac{1}{192v(v+S)}, \quad \frac{1}{v^2}, \quad \frac{1}{192v}.$$

Each is less than $1/32$ for $v \ge 33$. Thus $|R_0| < D_0/8 < R_1B/8$, and $R_1B + R_0 > 0$. Its positive Δ-multiple is divisible by c, whence $c \le \Delta(R_1B + R_0)$.

Since $c_0 - \Delta R_0 \ge c_0 - R_0 > 0$, it follows that $8avB + c_1 < \Delta R_1$. Formula (1) also gives $8avB + c_1 > 16v^2Y$: the difference is

$$(32a^2-8)v^2 + 8a^2v - 16a^2S - 4v - 4a\ell + 2 > 24a^2v^2 - 44.5a^2v > 0.$$

Together with (3), this implies

$$32a^2v^2(v+S) + 16v^3 < 24\Delta(S+1)v^2 + 8a^2\Delta S(2v+S). \tag{5}$$

Now $\Delta \le n/2$ and $2n(n+1) = (2v-1)\Delta$ imply $n < v/2$, $\Delta < v/4$, and $S < 5v/4$. The increasing function $x(2+x)/(1+x)$ therefore gives

$$S(2v+S) < \frac{65}{36}v(v+S), \qquad 8a^2\Delta S(2v+S) < \frac{65}{18}a^2v^2(v+S).$$

After subtracting this term in (5), the remaining left side is larger than $14v^2Y$. Thus

$$Y < \frac{12}{7}\Delta(S+1), \qquad Y\Delta < \frac{12}{7}\Delta^2(S+1).$$

Substitution in (2) yields

$$c < \frac{12.35N^3(S+1)^2}{a\Delta} < \frac{12.35}{8}p^8 < 1.55p^8,$$

using $N < 2m(m+1) < p^2/2$ and $S+1 \le 2m < p$. This proves all three claims. □

Lemma 7.10. *Let $0 < a < b < c < d$ be an irregular Diophantine quadruple with $\deg\{a,b,c\} \ge 2$ and $e > b$, and let z have a type II representation (λ, m, n) with $\lambda = -1$, $\Delta > 0$ and first-order defect $H = 0$. Let f be as in Definition 2.20, and put $g = d_-(a,b,f)$. If $g = 0$ or $f > b$, then $c < (m')^6$.*

Proof. By Lemma 2.9, Lemma 2.10, Lemma 2.15 and Lemma 2.14, $b > 4000$, $b > 8a$, $m, n \ge 4$ and $n \le m \le 2n$. Canonical descent (Theorem 2.7) gives $f \ge 1$, $e = d_+(a,b,f) > 4abf$. Put $N = 2n(n+1) + \Delta$ and $M = 2m(m+1) - \Delta$. The identity $H = 0$ gives

$$e\Delta = bN - aM < bN < 2bm^2, \qquad f < \frac{m^2}{2a\Delta}.$$

Also

$$c = (4ab+2)e + 2(a+b) - f < 4.001abe < \frac{8.002ab^2m^2}{\Delta}.$$

For the first strict bound, divide by abe: since $e \ge b+1$ and $a \ge 1$, the excess over 4 is at most

$$\frac{2}{ab} + \frac{2(a+b)}{abe} \le \frac{2}{b} + \frac{2}{e} + \frac{2}{be} \le \frac{4}{b} < 0.001.$$

If $g = 0$, the pair $\{a, f\}$ is Diophantine and $b = a + f \pm 2\sqrt{af+1}$. Since $b > 8a$, $f \le a$ is impossible. Thus $f > a$, $\sqrt{af+1} < f$, and $b < a + 3f < 4f < 2m^2/(a\Delta)$. It follows that

$$c < \frac{32.008m^6}{a\Delta^3} < 33m^6 < (2m+1)^6.$$

If $g \ge 1$ and $f > b$, f is the largest member of $\{a,b,f\}$ and canonical descent gives $f = d_+(a,b,g) > 4abg \ge 4ab$. Hence

$$b < \frac{f}{4a} < \frac{m^2}{8a^2\Delta}, \qquad c < \frac{8.002m^6}{64a^3\Delta^3} < m^6 < (2m+1)^6.$$ □

Lemma 7.11. *Let $0 < a < b < c < d$ be an irregular Diophantine quadruple with $\deg\{a,b,c\} \ge 2$, $e > b$ and $c \ge 10^{80}$, and let z have a type II representation (λ, m, n) with $\lambda = -1$, $\Delta > 0$ and first-order defect $H = 0$. Let f, $v = 2af + 1$ and L be as in Definition 2.20, and put $g = d_-(a,b,f)$. If $b > f$ and $g \ge 1$, then $L \ge 0$. If moreover $L \ge 1$, then $m' > 1.4c^{1/9}$.*

Suppose $b > f$, $g \ge 1$ and $L \ge 1$, and put $S = m + n$, $T_1 = (S+1)\Delta$,

$$\mathcal{T} = 4aSb^2 + \big((4a^2 + 24af + 10)S + 24af + 6\big)b + (10a + 11f - 6g)S + 6a + 14f - 6g,$$

and $\mathcal{U} = 2v\mathcal{T} - Sc$. Then the following hold.

1. *$0 < g < \frac{b}{4af}$, $af < \frac{m^2}{2\Delta}$, $v\Delta < m^2$ and $b < \frac{4am^2}{L}$.*
2. *$c = 8avb^2 + (8a^2 + 16af - 4ag + 6)b + 6a + 3f - 2g$, and c divides $\Delta\mathcal{T}$ and $\Delta\mathcal{U}$.*

3. $7.6avb^2 < c < 8.003avb^2$, $0 < \mathcal{T} < 4aSb^2 + (4a^2 + 13v)(S+1)b$ *and* $0 < \mathcal{U} < \big(24(S+1)v^2 + 8a^2(v-1)S + 4agS\big)b$.

4. *If* $ab < \frac{13}{2}T_1$, *then* $c < (m')^8$.

5. *If* $ab \geq \frac{13}{2}T_1$, *then* $v < \frac{45}{56}T_1 < 2m\Delta$ *and* $v^2 < 2m^3$. *If moreover* $b \geq \frac{4}{3}aS\Delta$, *then* $c < 0.49(m')^8$ *when* $(v-1)^2 < 2S\Delta$, *and* $b < 80m^3/a$ *when* $(v-1)^2 \geq 2S\Delta$. *If instead* $b < \frac{4}{3}aS\Delta$, *then* $c < \frac{7.12v^4m^2\Delta^2}{f^3}$.

Proof. We take $L \geq 0$ and the Vieta identity (1) from Lemma 7.5, and for $L \geq 1$ we bound c by a power of m' through the divisibilities $c \mid \Delta\mathcal{T}$ and $c \mid \Delta\mathcal{U}$.

Write $S = m + n$, $M_0 = 2m(m+1)$ and $M = M_0 - \Delta$. By Lemma 2.15, Lemma 2.14, Lemma 2.9 and Lemma 2.10, $m, n \geq 4$, $n \leq m \leq 2n$, $b > 4000$ and $b > 8a$. Canonical descent (Theorem 2.7) gives

$$e = d_+(a,b,f) > 4abf, \quad e + g = 2vb + 2(a+f), \quad eg = (b-a-f)^2 - 4(af+1).$$

Since $b > f$ and $a < b$, $0 < g < b/(4af)$. As T is high and $H = 0$, Lemma 7.5 gives $\lambda = -1$, $L \geq 0$, $N < 2m^2$ and

$$Lb + g\Delta = aM_0 + (a+2f)\Delta. \tag{1}$$

Assume henceforth $L \geq 1$. Then Lemma 7.5 also gives $af < m^2/(2\Delta)$, and, as $L = N - 2v\Delta$,

$$v\Delta = \frac{N-L}{2} < m^2, \qquad b < \frac{4am^2}{L}. \tag{2}$$

With $\mathcal{T}$ as in the statement, the exact identities are

$$c = 8avb^2 + (8a^2 + 16af - 4ag + 6)b + 6a + 3f - 2g, \qquad c \mid \Delta\mathcal{T}. \tag{3}$$

Set $C_0 = 2a(v+S+1) + 2f - g$, so (1) is $L(b-a) = C_0\Delta$. In the integer $\mathcal{Q}$ of the last item of Lemma 7.2 (here $H = 0$ and $\lambda = -1$), substitute

$$M_0 = L + (2v + 2S + 1)\Delta, \qquad e = 2vb + 2(a+f) - g.$$

Eliminating $L(b-a)$ shows

$$(2v + S - 4)c - \mathcal{Q}/\Delta - \mathcal{T} = 4a\big(eg - (b-a-f)^2 + 4(af+1)\big) = 0.$$

Hence the divisibility in (3) follows directly from $c \mid \mathcal{Q}$ (Lemma 7.2).

The following bounds hold:

$$7.6avb^2 < \frac{23}{3}avb^2 < c < 8.003avb^2, \qquad 0 < \mathcal{T} < 4aSb^2 + (4a^2 + 13v)(S+1)b. \tag{4}$$

For the lower bound on c, use $g < b/(4af)$ in (3). This gives

$$c > (8av - 1/f)b^2 + (8a^2 + 16af + 6 - 1/(2af))b + 6a + 3f,$$

and $avf \geq 3$. For the upper bound, discard the negative g terms and use

$$8a^2 + 16af + 6 \leq 10av, \qquad 6a + 3f \leq 3av.$$

The first inequality follows from

$$10av - (8a^2 + 16af + 6) = 6(2a+1)(a-1) + 4a(5a-4)(f-1) \geq 0.$$

Thus $c < av(8b^2 + 10b + 3) < 8.003avb^2$.

For $\mathcal{T} > 0$, its negative term $6g(S+1) < 3(S+1)b/(2af)$ is dominated by the positive linear coefficient. For its upper bound, use $24af + 10 < 12v$ and

$$10a + 14f \leq 8v,$$

whose difference is $16(a-1)(f-1) + 6(a-1) + 2(f-1) \geq 0$, and then $b > 8$.

If $ab < \frac{13}{2}(S+1)\Delta$, then $b < 13m\Delta/a$. Since $e\Delta < bN < 2bm^2$ and $e > b > 4000$, one has $c < 4.001abe < 8.002ab^2m^2/\Delta$, whence

$$c < 1353m^4\Delta/a < 1353m^5 < (2m+1)^8.$$

As $c \geq 10^{80}$, this already implies $m' > 1.4c^{1/9}$.

We may therefore assume $ab \geq \frac{13}{2}(S+1)\Delta$. Since $\Delta\mathcal{T}$ is a positive multiple of c, (4) yields

$$1.9v < S\Delta + \frac{a(S+1)\Delta}{b} + \frac{13v(S+1)\Delta}{4ab}.$$

The last term is at most $v/2$ and $a/b < 1/8$, so

$$v < \frac{45}{56}(S+1)\Delta < 2m\Delta, \qquad v^2 < 2m^3, \tag{5}$$

using (2).

A second cancellation supplies a positive linear polynomial:

$$\mathcal{U} = 2v\mathcal{T} - Sc = R_1 b + R_0,$$

where

$$\begin{aligned} R_1 &= (24v^2 - 12v)(S+1) + (8a^2(v-1) + 4ag + 2)S, \\ R_0 &= (40a^2f + 44af^2 - 24afg + 14a + 19f - 10g)S \\ &\quad + 24a^2f + 56af^2 - 24afg + 12a + 28f - 12g. \end{aligned}$$

Both $c \mid \Delta\mathcal{U}$ and

$$0 < \mathcal{U} < \big(24(S+1)v^2 + 8a^2(v-1)S + 4agS\big)b \tag{6}$$

hold. To verify positivity, the negative terms in R_0 have absolute value at most $12v(S+1)g < 9(S+1)b$, whereas the first term of $R_1 b$ is at least $180(S+1)b$. For the upper bound, canonical inversion gives $b = d_+(a,f,g) > 4afg \geq 4f$. Together with $b > 8a$ this gives

$$R_0 < (19af + 9)(S+1)b < (12v - 2)(S+1)b.$$

The negative $-12v(S+1)b + 2Sb$ in $R_1 b$ absorbs this estimate, proving (6). Thus $c \leq \Delta\mathcal{U}$.

First suppose $b \geq \frac{4}{3}aS\Delta$. If $(v-1)^2 < 2S\Delta$, then $v < 1.67m$, and (2), (4) give

$$c < \frac{128.048a^3vm^4}{L^2} < \frac{16.006v^4m^4}{f^3L^2} < \frac{125m^8}{f^3L^2} < 0.49(m')^8.$$

Otherwise $(v-1)^2 \geq 2S\Delta$. In $c \leq \Delta\mathcal{U}$, the second term on the right of (6) is less than $6avb^2$, because $aS\Delta \leq 3b/4$. The third is less than avb^2, because

$$4ag < b/f, \qquad S\Delta/f \leq 2a^2 f < av.$$

Subtracting these from the lower bound $7.6avb^2 < c$ leaves

$$0.6avb^2 < 24(S+1)\Delta v^2 b, \qquad b < \frac{80m^3}{a}.$$

Combine this with $b < 4am^2/L$ to obtain $b^2 < 320m^5/L$. By (5), $av < v^2/(2f) < m^3/f$, so

$$c < \frac{2561m^8}{fL} < 10.01(m')^8. \tag{7}$$

For $c \geq 10^{80}$, either eighth-power estimate implies $m' > 1.4c^{1/9}$.

Finally suppose $b < \frac{4}{3}aS\Delta < \frac{8}{3}am\Delta$. Equations (2), (4), (5) give

$$c < 56.92a^3vm^2\Delta^2 < \frac{7.12v^4m^2\Delta^2}{f^3} < \frac{14.24m^9}{f^3} < 0.028(m')^9.$$

Since $0.028(1.4)^9 < 0.579 < 1$, this yields $m' > 1.4c^{1/9}$. All cases with $L \geq 1$ are covered.

The numbered items were established along the way: item 1 is $0 < g < b/(4af)$, the bound $af < m^2/(2\Delta)$ and (2); item 2 is (3) together with $c \mid \Delta\mathcal{U}$; item 3 is (4) and (6); item 4 is the case $ab < \frac{13}{2}(S+1)\Delta$; and item 5 is (5) with the three subsequent cases. □

Proposition 7.12. *Let $0 < a < b < c < d$ be an irregular Diophantine quadruple with $\deg\{a,b,c\} \geq 2$ and $c \geq 10^{130}$, and let z have a type II representation (λ, m, n). Then*

$$m' > c^{1/8}.$$

Proof. The proof bounds a by exact integer quotients of two successive congruences, using the integral quadratic norm of a further descended regular quadruple.

Put $p = m' = 2m+1$, $e = d_-(a,b,c)$, $\Delta = m-n$, $S = m+n$, $N = 2n(n+1) - \lambda\Delta$, $M = 2m(m+1)+\lambda\Delta$, and $H = bN - aM + e\lambda\Delta$, as in Definition 2.20. Suppose for contradiction that $p \leq c^{1/8}$. The standing inputs Lemma 2.9, Lemma 2.10, Lemma 2.15, and Lemma 2.14 give

$$b > 4000, \qquad b > 8a, \qquad m, n \geq 4, \qquad n \leq m \leq 2n.$$

Reduction to the binding branch. If $c \leq b^2$, Lemma 7.8 gives $p^8 > 22c$, contradicting $p \leq c^{1/8}$. Thus $c > b^2$. Canonical descent (Theorem 2.7) then gives $e > b$. Indeed if $e < b$, write $f = d_-(a,b,e) \geq 1$; the degree assumption gives $b = d_+(a,e,f) > 4ae + a + e + f$. With $A = a+e$, integrality gives $b - 4ae - 2 \geq A + f - 1 \geq A$, whereas $c = (4ae+2)b + 2A - f$. Hence $b^2 - c \geq A(b-2) + f > 0$, a contradiction. Equality $e = b$ is impossible because b^2+1 is not a square.

If $H \neq 0$, Lemma 7.4 gives $p > 1.06\sqrt{a}\,c^{1/4} - 0.5 > c^{1/8}$. Thus $H = 0$. Since $N \geq 2n^2 + n$, $M \leq 8n^2 + 6n$, and $b > 8a$, one has $bN - aM > 0$. Therefore $\lambda = -1$, $\Delta \geq 1$, and $e\Delta = bN - aM$. Put

$$f = d_-(a,b,e) \geq 1, \quad g = d_-(a,b,f) \geq 0, \quad v = 2af + 1, \quad L = N - 2v\Delta.$$

If $g = 0$ or $f > b$, Lemma 7.10 gives $c < p^6$, a contradiction. Equality $f = b$ is impossible because $\{a, b, f\}$ is Diophantine. Thus $b > f$ and $g \geq 1$. By Lemma 7.11, $L \geq 0$. The case $L = 0$ is excluded by Lemma 7.9.

If $\Delta > n/2$, that lemma gives $c < 6.6p^6$; since $p \leq c^{1/8}$, this implies $c < 6.6^4 < 10^4$. If $\Delta \leq n/2$, put $Y = 2a^2(v + S) + v$. The third claim of that lemma gives the first two relations below, and the third is $L = 0$:

$$Y < \frac{12}{7}\Delta(S+1), \qquad c < \frac{12.35N^3(S+1)^2}{a\Delta}, \qquad N = 2v\Delta.$$

Since $v > n^2/\Delta \geq m^2/(4\Delta)$ and $S + 1 \leq 2m$,

$$\frac{a^2m^2}{4\Delta} < a^2v < \frac{Y}{2} < \frac{12}{7}m\Delta, \qquad \Delta > \sqrt{\frac{7}{48}}a\sqrt{m}.$$

Also $N < 2m^2$, so

$$c < \frac{395.2m^8}{a\Delta} < \frac{395.2\sqrt{48/7}\,m^{15/2}}{a^2} < 1036m^{15/2}.$$

The last constant follows from $395.2^2 \cdot 48 < 1036^2 \cdot 7$. As $p \leq c^{1/8}$ gives $c > 256m^8$, one obtains $m < (1036/256)^2 < 17$ and $c < 1036 \cdot 17^{15/2} < 10^{13}$. Both alternatives contradict $c \geq 10^{130}$. Therefore $L \geq 1$.

Since $b > f$, $g \geq 1$ and $L \geq 1$, items 1 to 3 of Lemma 7.11 hold, with $\mathcal{T}$ and $\mathcal{U} = 2v\mathcal{T} - Sc$ as there. Write $T_1 = (S+1)\Delta \leq 2m\Delta$.

Item 4 of Lemma 7.11 gives $c < p^8$ when $ab < \frac{13}{2}T_1$, which contradicts $p \leq c^{1/8}$. Suppose henceforth $ab \geq \frac{13}{2}T_1$. Item 5 of that lemma gives $v^2 < 2m^3$. If $b \geq \frac{4}{3}aS\Delta$ and $(v-1)^2 < 2S\Delta$, it gives $c < 0.49p^8$, another contradiction. If $b \geq \frac{4}{3}aS\Delta$ and $(v-1)^2 \geq 2S\Delta$, it gives $b < 80m^3/a$. Insert this directly into $c < 8.003avb^2$ to obtain

$$c < 51219.2\frac{v}{a}m^6 < 72500m^{15/2},$$

since $51219.2^2 \cdot 2 < 72500^2$. The contrary inequality $c > 256m^8$ then gives $m < (72500/256)^2 < 10^5$ and $c < 72500 \cdot 10^{75/2} < 10^{43}$, incompatible with $c \geq 10^{130}$. We may therefore assume

$$ab \geq \frac{13}{2}T_1, \qquad b < \frac{4}{3}aS\Delta.$$

Item 5 of Lemma 7.11 gives $v < \frac{45}{56}T_1 < 2m\Delta$, whence $v^2 < 2m^3$, and, since $b < \frac{4}{3}aS\Delta$, it gives

$$c < \frac{7.12v^4m^2\Delta^2}{f^3} < \frac{7.12v^2m^6}{f^3} < \frac{65a^2m^6}{f}, \tag{1}$$

using $v \leq 3af$.

Two integral quotients. Put

$$\zeta = \frac{\Delta\mathcal{T}}{c} \geq 1, \qquad \theta = 2v\zeta - S\Delta = \frac{\Delta\mathcal{U}}{c} \geq 1.$$

Division of the bounds for $\mathcal{T}$ and c, using $a/b < 1/8$ and $ab \geq 6.5T_1$, gives

$$\zeta < \frac{45}{76}\frac{T_1}{v} + \frac{5}{19} \leq \frac{45}{38}\frac{m\Delta}{v} + \frac{5}{19}.$$

Since $\zeta - 5/19 \geq 14\zeta/19$,

$$\zeta < \frac{45}{28}\frac{m\Delta}{v} < \frac{2m^3}{v^2}.$$

Consequently

$$c < \frac{14.24m^9}{\zeta f^3}, \qquad a^2 < \frac{m^3}{2\zeta f^2}. \tag{2}$$

When $a \geq 4f$ and $v \geq 2m$, the bound for $\mathcal{U}$ gives

$$\theta < \frac{60}{19}\frac{vT_1}{ab} + \frac{20}{19}\frac{aT_1}{b} + \frac{10}{19}\frac{gT_1}{vb} < \frac{4aT_1}{b} + 0.4\frac{T_1}{v^2}.$$

Here $v/a \leq 3f \leq 3a/4$ and $g/b < 1/[2(v-1)]$. Since $T_1 < 2m^3/v$ and $v \geq 2m$, the last summand is less than 0.1. As $\theta \geq 1$,

$$\theta b < \frac{40}{9}aT_1 < 10am\Delta < \frac{5m^3}{f}. \tag{3}$$

An integral norm bounds a. Put $h = d_-(a, f, g) \geq 0$. Canonical descent gives $b + h = 2vg + 2(a + f)$ and regularity of $\{f, h, a, g\}$. Define

$$c_0 = 6a + 3f - 2g, \quad q_0 = (10a + 11f - 6g)S + 6a + 14f - 6g, \quad E = \theta g - \zeta a.$$

Substituting $S\Delta = 2v\zeta - \theta$ and $b = 2vg + 2(a + f) - h$ in $\zeta c = \Delta\mathcal{T}$ gives the exact identity

$$\begin{aligned} 8avE = {} & \zeta(24v^2 - 12v + 2 - 8a^2 + 4ag) \\ & + \theta(4ah - 12a^2 - 16v + 6) + \Delta(12v - 6) + \frac{\Delta q_0 - \zeta c_0}{b}. \end{aligned} \tag{4}$$

Regularity gives

$$g^2 - (4fh + 2)ag + a^2 = R, \qquad R = 2(f + h)(a + g) - (f - h)^2 + 4.$$

Thus, for the integer $D = \zeta^2 - (4fh + 2)\zeta\theta + \theta^2$, substitution of $\zeta a = \theta g - E$ yields

$$a^2 D = \theta^2 R + E^2 + \theta E[(4fh + 2)a - 2g]. \tag{5}$$

For $h \geq 1$, the discriminant $16fh(fh + 1)$ is nonsquare, so $D \neq 0$. For $h = 0$, $D = (\zeta - \theta)^2$.

We claim that, provided

$$a \geq 4f, \qquad v \geq 2m, \qquad a^2 f \geq m^2,$$

one has, with $K_0 = \max(\zeta, \theta)$ and $q = \min(\zeta, \theta)$,

$$a < \frac{1000fK_0^3}{q}. \tag{6}$$

First $\zeta < 2m^3/v^2 \leq v/4$, $v \geq 9$, and $\Delta/a < m^2/(2a^2 f) \leq 1/2$. Put $F = |E|$. Canonical descent gives $h < \max(a, f, g) \leq a + g$, and $g \leq (\zeta a + F)/\theta$. In (4), the terms independent of g, h are bounded after division by $8av$ by

$$8f\zeta, \qquad 2\theta, \qquad \frac{1.5\Delta}{a},$$

respectively. The g, h terms satisfy

$$\frac{\zeta g}{2v} \leq \frac{\zeta^2}{4f\theta} + \frac{\zeta F}{2v\theta}, \qquad \frac{\theta h}{2v} \leq \frac{\theta + \zeta}{4f} + \frac{F}{2v}.$$

Finally, $|q_0| \le (10a+14f+6g)(S+1)$ and $|c_0| \le 6a+3f+2g$, together with $g/b < 1/[2(v-1)]$, $b > 8a$, $2m \le v$, and $\zeta < v/4$, give

$$\frac{\Delta|q_0| + \zeta|c_0|}{8avb} < \frac{\Delta}{2a} + 1.$$

Indeed the terms involving T_1 are bounded by $3\Delta/(8a)$ and $3\Delta/[8a(v-1)]$, and the remaining terms total less than 1. Since $1/(2v) + \zeta/(2v\theta) < 1/2$, absorb the F terms to obtain

$$F < 30f\zeta + 14\theta + \frac{\zeta^2}{f\theta}. \tag{7}$$

If $a \le 200fK_0^2$, (6) is immediate. Otherwise (7) gives $F < 45fK_0^2 < a/2$, hence

$$\frac{\zeta}{2\theta} < \frac{g}{a} < \frac{3\zeta}{2\theta}.$$

Canonical regularity gives $h < g/(4af)$ for $g > a$ and $h < a/(4fg)$ for $a > g$, so

$$h < \frac{1}{4f} \max\left(\frac{g}{a}, \frac{a}{g}\right).$$

Equality $a = g$ is impossible because $a^2 + 1$ is not a square.

If $\zeta \ge \theta$, then

$$h < \frac{\zeta}{2f\theta}, \quad a+g < \frac{5a\zeta}{2\theta}, \quad f+h < \frac{2f\zeta}{\theta}, \quad |R| < \frac{16af\zeta^2}{\theta^2}, \quad 4fh+2+\frac{2g}{a} < \frac{9\zeta}{\theta}.$$

For the bound for R, use $|R| \le 2(f+h)(a+g) + (f+h)^2 + 4$ and $a > 200f\zeta^2$. If $D \ne 0$, (5) and $F < a/2$ give

$$a < 16f\zeta^2 + 10F\zeta < 456f\zeta^2 + \frac{10\zeta^3}{f\theta} < \frac{466f\zeta^3}{\theta}.$$

If $\theta \ge \zeta$, the corresponding estimates are

$$h < \frac{\theta}{2f\zeta}, \quad a+g < \frac{5a}{2}, \quad f+h < \frac{2f\theta}{\zeta}, \quad |R| < \frac{16af\theta}{\zeta}, \quad 4fh+2+\frac{2g}{a} < \frac{9\theta}{\zeta}.$$

Equations (5) and (7) then give

$$a < \frac{16f\theta^3}{\zeta} + \frac{10F\theta^2}{\zeta} < \frac{466f\theta^3}{\zeta}.$$

The only remaining case has $h = 0$ and $\zeta = \theta$. Then $g = a + f \pm 2\sqrt{af+1}$; because $a \ge 4f$,

$$F = \theta|g-a| \ge \frac{3}{2}\theta\sqrt{af}.$$

But (7) gives $F < 45f\theta$, so $a < 900f$. This proves (6) in every case.

Consequences of the contrary eighth-power bound. Since $p \le c^{1/8}$, one has $c > 256m^8$. The three extra hypotheses for (6) follow. If $a < 4f$, then $af < m^2/2$ gives $a < \sqrt{2}m$, and (1) gives $c < 260am^6 < 400m^7 < 256m^8$. If $v < 2m$, (1) gives $c < 28.48m^8$. If $a^2 f < m^2$, (1) gives $c < 65m^8$. Each is impossible.

Moreover $a < m^2$ and (1) give $c < 65m^{10}$. As $c \geq 10^{130} > 10^{80}$, this implies $m > 10^7$. Combining $c > 256m^8$ with the second bound of (1) gives

$$v > 5.9f^{3/2}m, \qquad \frac{\zeta}{v} < 0.01f^{-9/2}, \qquad \frac{\Delta}{v} < 0.029f^{-3}. \tag{8}$$

The numerical comparisons are $7.12(5.9)^2 < 256$, $2/(5.9)^3 < 0.01$, and $1/(5.9)^2 < 0.029$. In particular $v > 5.9 \cdot 10^7$, and

$$\frac{v}{a} = \frac{2fv}{v-1} < 2.001f. \tag{9}$$

A further integral remainder when $\zeta \geq \theta$. In (4) all terms except the last are integers. Thus

$$V = \frac{\Delta q_0 - \zeta c_0}{b} \in \mathbb{Z}, \qquad W = V + 6\zeta \in \mathbb{Z}.$$

The earlier rational-term bound gives $|V|/(8av) < \Delta/(2a) + 1$. Using $S\Delta = 2v\zeta - \theta$ and $b = 2vg + 2(a+f) - h$, direct expansion gives

$$\begin{aligned} b(W - 10\theta) = &-20vE + \zeta(22fv + 6a + 9f + 2g - 6h) \\ &+ \Delta(6a + 14f - 6g) + \theta(-30a - 31f + 6g + 10h). \end{aligned} \tag{10}$$

Suppose first $\zeta \geq \theta$. The signed error E is positive: from (4), discarding the positive g, h terms and using the bound for V gives

$$\begin{aligned} 8avE &> \zeta(24v^2 - 12v + 2 - 8a^2) - \theta(12a^2 + 16v - 6) \\ &\quad + \Delta(8v - 6) - 8av \\ &\geq \zeta(24v^2 - 28v + 8 - 20a^2) + \Delta(8v - 6) - 8av \\ &> 15v^2 - 28v + 8 + \Delta(8v - 6) > 0. \end{aligned}$$

Here $a < v/2$, $\zeta \geq 1$, and $v \geq 9$. Hence $g > a\zeta/\theta \geq a$.

Assume $h \geq 1$. Canonical inversion gives $g = d_+(a, f, h) > (4fh + 1)a$. Since $fh + 1$ is a square, $fh \geq 3$. Write $t = \theta/\zeta \leq 1$. Dividing (4) by $8avg$, discarding its negative terms, and using $g > a\zeta/\theta$ and $h/g < 1/(4af) = 1/[2(v-1)]$, one obtains

$$\frac{E}{g} < \frac{3v\theta}{a^2} + \frac{\zeta}{2v} + \frac{\theta}{4v(v-1)} + \frac{2\Delta}{ag} + \frac{1}{g} < \frac{0.12013t}{f^{5/2}} + \frac{0.005}{f^{9/2}} + 10^{-5}. \tag{11}$$

For the last error bound, (8) and (9) give $2\Delta/(ag) < 0.233/v$, $1/g < 1/a < 0.34/m$, and $\theta/[4v(v-1)] < 0.003/v$; their sum is less than 10^{-5}. Initially (11) gives $E/g < 0.13$. Since

$$\frac{\zeta}{\theta} = \frac{g}{a}\left(1 - \frac{E}{\theta g}\right), \qquad \theta \geq 1,$$

this implies $t < 1/[0.87(4fh + 1)] < 0.089$. Substitution in (11) gives $E/g < 0.02$ and therefore

$$t < \frac{1}{0.98(4fh + 1)} < 0.079. \tag{12}$$

The integer $W - 10\theta$ vanishes: since $b > 2vg - h > 1.999vg$, one has $1/b < 0.501/(vg)$. Group the a terms in (10) as $6a(\zeta - 5\theta) > 0$, and its f terms as $f[\zeta(22v + 9) - 31\theta + 14\Delta] > 0$. The remaining possible negative terms give

$$W - 10\theta > -10.02\frac{E}{g} - 3.006\frac{\Delta}{v} - 3.006\frac{\zeta}{v}\frac{h}{g} > -0.2004 - 0.087174 - 10^{-5} > -1.$$

For an upper bound discard all negative terms in (10). The positive contributions satisfy

$$\frac{22fv\zeta}{b} < 0.018, \qquad \frac{6a\zeta}{b} < 0.003, \qquad \frac{2g\zeta + 6g\theta}{b} < 0.013,$$
$$\frac{\Delta(6a+14f)}{b} < 0.011, \qquad \frac{9f\zeta + 10h\theta}{b} < 10^{-5}.$$

The first three follow from (8), (9), (12), and $\zeta/g < \theta/a$. The fourth uses $f \le a/4$, $g > 13a$, and $\Delta/v < 0.029$. The last uses $v > 5.9 \cdot 10^7$ and $h/g < 1/[2(v-1)]$. Their sum is less than 1, so $W = 10\theta$.

Now (10) and $E > 0$ imply

$$(6g - 6a - 14f)\Delta < \zeta(22fv + 6a + 9f + 2g) + \theta(6g + 10h).$$

Since $g > 13a$ and $f \le a/4$, the left coefficient exceeds $5.25g$. Divide by g and use $\zeta/g < \theta/a$, (9), and $h/g < 1/(4af)$ to obtain

$$5.25\Delta < 44.022f^2\theta + 14.875\theta + 2\zeta < 59f^2\theta + 2\zeta, \qquad \Delta < 12f^2\theta + 0.4\zeta.$$

Together with $\zeta < (45/28)m\Delta/v$ and (12), this gives

$$v < \frac{45}{28}m(12f^2t + 0.4) < \frac{3375}{686}\frac{f}{h}m + \frac{9}{14}m \le \frac{1908}{343}fm < 5.9f^{3/2}m,$$

contradicting (8).

If $h = 0$, positivity of E gives $g = a + f + 2\sqrt{af+1}$. Put $\omega = \zeta - \theta \ge 0$. If $\omega \ge 1$, then $\omega a < \theta[f + 2\sqrt{af+1}] < 3\theta\sqrt{af}$, so $a < 9f\theta^2 \le 9f\zeta^2$. If $\omega = 0$, (7) gives $E < 45f\theta$, whereas $E > 2\theta\sqrt{af}$, so $a < 507f$. Thus always $a < 507f\zeta^2$. Since $\zeta < m^3/(2a^2f^2)$,

$$a^5 < \frac{127m^6}{f^3}.$$

Also $g < 2.4a$, $b = 2vg + 2(a+f) < 5.1av$, and $v \le 3af$, so $c < 6000a^6f^3$. Combining these with $c > 256m^8$ gives

$$256^5m^{40} < c^5 < \frac{6000^5 \cdot 127^6m^{36}}{f^3} \le 6000^5 \cdot 127^6m^{36}.$$

The exact comparison $6000^5 \cdot 127^6 < 256^5 \cdot 10^{20}$ implies $m < 10^5$, contradicting $m > 10^7$. This excludes the entire branch $\zeta \ge \theta$.

The symmetric quotient bound when $\theta \ge \zeta$. We prove

$$a < 5000f^2\theta^{5/2}. \tag{13}$$

It is immediate if $a \le 200f\theta^2$. Otherwise (7) gives $F < 45f\theta < a/2$. If $D = 0$, the Euler exception treated above gives $a < 900f$, so assume $D \ne 0$. The estimates leading to (6), before discarding $|D|$, give

$$a|D| < \frac{16f\theta^3}{\zeta} + \frac{10F\theta^2}{\zeta} < \frac{466f\theta^3}{\zeta}.$$

If $\theta \le 2\zeta^2$, this yields $a < 466\sqrt{2}\,f\theta^{5/2} < 660f\theta^{5/2}$. If $\theta > 2\zeta^2$, put $A = 4fh+2$ and $\mu = A\zeta - \theta$, so $D = \zeta^2 - \theta\mu$. For $\mu \ne 0$, integrality gives $|D| > \theta/2$ and hence $a < 932f\theta^2/\zeta < 950f\theta^{5/2}$.

In the case $\mu = 0$, that is, $\theta = A\zeta$: since $\theta > 2\zeta^2$, one has $A > 2\zeta \geq 2$ and $h \geq 1$. Moreover

$$\frac{a}{2A} < g < \frac{3a}{2A} < a, \qquad g > 100fA\zeta^2 > \max(f, h).$$

Thus $a = d_+(f, h, g)$, and the other root $q = d_-(f, h, g)$ is nonnegative. Their sum gives

$$\frac{E}{\zeta} = Ag - a = q - 2(f + h) \geq -2(f + h). \tag{14}$$

Divide (4) by $8av\zeta$ and use $\theta/\zeta = A$. The two terms $-2A/a + 3A/(4av)$ have negative sum. Discard this sum and the other negative terms except $-3Aa/(2v)$, obtaining

$$\frac{E}{\zeta} < \frac{3v}{a} + \frac{1}{4av} + \frac{g}{2v} + \frac{Ah}{2v} + \frac{3\Delta}{2a\zeta} + \frac{|V|}{8av\zeta} - \frac{3Aa}{2v}.$$

The six positive terms are bounded respectively by

$$6f + \frac{3}{4}, \quad \frac{1}{144}, \quad \frac{1}{4}, \quad \frac{1}{1600}, \quad \frac{3}{4}, \quad \frac{5}{4}.$$

For the fourth bound use $h < A/(2f)$, $v > 2af$, and $a > 200fA^2\zeta^2$. For the last two use $\Delta/a < 1/2$, $\zeta \geq 1$, and $|V|/(8av) < \Delta/(2a) + 1$. Their excess over $6f$ is less than 4. Also $a/v \geq 4/(9f)$, since $a \geq 4f$. Hence

$$\frac{E}{\zeta} < 6f + 4 - \frac{2A}{3f} \leq 10f - \frac{8}{3}h.$$

Combining this with (14) gives $h < 18f$, so $A < 74f^2$. Finally

$$a < \frac{466f\theta^3}{\zeta} = 466f\sqrt{\frac{A}{\zeta}}\,\theta^{5/2} < 466\sqrt{74}\,f^2\theta^{5/2} < 4100f^2\theta^{5/2}.$$

This proves (13) in every case.

From $c < 25a^2fb^2$ and (3), one has $\theta < 25am^3/(\sqrt{f}\sqrt{c})$. Insert this in (13), then use $a^2 < m^3/(2\zeta f^2)$ from (2). Raising to the fourth power gives

$$c^5 < \frac{5000^4 \cdot 25^{10}}{8}\frac{m^{39}}{\zeta^3 f^3}.$$

Since $m < p/2 \leq c^{1/8}/2$,

$$c^{1/8} < \frac{5000^4 \cdot 25^{10}}{2^{42}\zeta^3 f^3} \leq \frac{5000^4 \cdot 25^{10}}{2^{42}}.$$

The exact rational comparison

$$\left(\frac{5000^4 \cdot 25^{10}}{2^{42}}\right)^8 < 10^{130}$$

contradicts $c \geq 10^{130}$. All binding cases are excluded, proving $m' > c^{1/8}$. □

The type I floor uses the second-order defect k and the cubic congruence. For a middle or low triple, the descent relation $c + f = (4u^2 - 2)b + 2(a + e)$ and a square-residue gap modulo a give $c < 2^{18}m^9$, and in every position $m' > \frac{1}{2}c^{1/9}$ (Lemma 7.13). Above 10^{90} and at the scale $c \geq m^8/20$ the same relation cuts off $u = \sqrt{ae + 1} < 0.71m^{3/5}$ (Lemma 7.14). The integer $\mu_j = (4j^2 - q^2)/a$ is then positive (Lemma 7.17): it is non-negative by size, and if $\mu_j = 0$, then the seed of the pair $\{e, a\}$ is principal (Lemma 7.15), its exponent is forced to be eight, and the divisibilities of that seed (Lemma 7.16) give $c < Cm^8$, a contradiction. Together these give $c < 2^{13}m^{42/5}$ for $c \geq 2^{540}$, and the exponent $5/42$ (Lemma 7.18).

Lemma 7.13. *Let $0 < a < b < c < d$ be an irregular Diophantine quadruple with $\deg\{a,b,c\} \geq 2$ and a type I representation of z. If $e = d_-(a,b,c) < b$, then*

$$c < 2^{18} m^9.$$

In all cases, including $e > b$, one has

$$m' = 2m > \frac{1}{2} c^{1/9}.$$

Suppose now that $e < b$. Put $u = \sqrt{ae+1}$, $\delta = m^2 - n^2$ and $f = d_-(a,b,e)$, let k be the second-order defect of the representation, and let j, Y and T_δ be the integers of Lemma 2.22. Then the following hold.

(i) $f \geq 1$, $b = d_+(a,e,f)$, $c+f = (4u^2-2)b+2(a+e)$, $2ut = c+b-a-e$, $(4u^2-3)b < c < 4u^2 b$ *and* $0 < f/b < 1/(3u^2)$. *Moreover* $u \geq 2$, $m \geq 7$, $m/2 \leq n < m$, $m > 2u$, $a < m^2/9$ *and* $m \leq \delta < m^2$.

(ii) If $c > 100m^4$, *then* $k \in 4\mathbb{Z}$, $4 \leq k < m^2/u^2$, $k < m^2/a$, $|jc - 2\lambda asm\delta| < 5a^{3/2}m^3\delta$, j *has the sign* λ, $0 < |j| < 6a^{3/2}m\delta/\sqrt{c}$, *and* $\lambda Y > a^2 m\delta(1 - 10m^2/\sqrt{c}) > 0$.

(iii) The integers $\mathcal{E} = 2uk - \lambda n$, $\mathcal{F} = n(2un+\lambda) - (4u^2-2)\mathcal{E}$ *and* $\omega = (n+\lambda\mathcal{F})/(2u)$ *satisfy*

$$4u^2\big(k - \omega(\omega+\lambda)\big) = \mathcal{F}^2 - 2u\mathcal{F}(2\lambda\omega+1) + 2k - \lambda\omega = -\mathcal{F}^2 - 2\lambda n\mathcal{F} - 2u\mathcal{F} + 2k - \lambda\omega.$$

If $c > 100m^4$, *then* $|\mathcal{F}| < m^2/u^3 + 16u^5m^2/c + 16u^4 m/\sqrt{c}$.

(iv) If $\omega \geq 2$ *and* $u > 4\omega$, *then* $k \neq \omega(\omega+\lambda)$.

Proof. For the bound, if $c \geq 2^{18}m^9$, item (iv) forces $u < 4m^{3/5}$, and the third-order congruence then gives an impossible square residue modulo a.

Item (i). Suppose $e < b$, and put $S = a + e$ and $K_0 = 4u^2 - 2$. Canonical descent (Theorem 2.7) gives $f \geq 1$, $b = d_+(a,e,f)$ and $c = d_+(a,b,e)$. By item 1 of Lemma 2.6 for the pair $\{a,e\}$ at $X = b$, the two regular roots c, f satisfy

$$c + f = K_0 b + 2S, \qquad cf = (b-S)^2 - 4u^2.$$

By item 3 of Lemma 2.6, $c > (4ae+1)b = (4u^2-3)b$. Also $b = d_+(a,e,f) > 4aef \geq 4ae$ by the same item, and $S \leq ae + 1 < b$, so the sum identity gives $c < 4u^2 b$. Since $a, e < b$, one has $S < 2b$ and consequently $cf < b^2$. As $u^2 = ae + 1 \geq 2$, $u \geq 2$, and we obtain

$$(4u^2-3)b < c < 4u^2 b, \qquad 0 < \frac{f}{b} < \frac{1}{3u^2}. \tag{1}$$

Item 2 of Lemma 2.6 (for the pair $\{a,e\}$ at $X = b$) also gives

$$2ut = c + b - S. \tag{2}$$

By Lemma 2.23 and Lemma 2.14,

$$m > n \geq 1, \qquad m \leq 2n, \qquad n > \kappa\sqrt{c/b}, \qquad \kappa = (\sqrt{17}-1)/2 > 3/2.$$

Since $ae \geq 3$, (1) yields $n > (3/2)\sqrt{13} > 5$, so $m \geq 7$. Moreover

$$m > 2u, \qquad a < \frac{m^2}{9}, \qquad m \leq \delta < m^2. \tag{3}$$

Here $m^2 > \kappa^2(4u^2-3) > 4u^2$ for $u \ge 2$, $m^2 > \kappa^2 c/b > 4\kappa^2 a > 9a$, and $\delta = (m-n)(m+n) \ge m+n > m$.

Item (ii). The second-order defect k (Definition 2.20) satisfies

$$kc = bn^2 + \lambda tn - am^2 - \lambda sm, \tag{4}$$

and $k \in 4\mathbb{Z}$ by item 1 of Lemma 2.22. Item 3 there gives

$$\delta(a^2m^2 + 2\lambda asm + 1 - n^2) = 3kn^2 + jc. \tag{5}$$

Suppose $c > 100m^4$. Since $u < m/2$ and $a < m^2/9$ by (3),

$$c > 24a/m, \qquad c > 24u^2/m, \qquad 4m^2 + \tfrac{5}{4}m < \sqrt{c}, \qquad c > 64au^2. \tag{6}$$

From $s < t < \sqrt{bc} + 1$, $c > 4ab$, and (3),

$$|k| < \frac{bm^2 + 2tm}{c} < \frac{m^2}{4a} + \frac{m}{\sqrt{a}} + \frac{2m}{c} < \frac{m^2}{a}, \tag{7}$$

where $m > 3\sqrt{a}$ and $c > 24a/m$ justify the last inequality.

Put $U = a^{3/2}m\delta\sqrt{c}$. In (5), the term $2asm\delta$ lies strictly between $2U$ and $4U$. By (7), the other terms in absolute value, divided by U, have sum less than

$$\frac{4m^2 + (5/4)m}{\sqrt{c}} < 1.$$

Thus $|jc - 2\lambda asm\delta| < (4m^2 + \frac{5}{4}m)a^{3/2}m\delta \le 5a^{3/2}m^3\delta$ and $|jc - 2\lambda asm\delta| < U < 2asm\delta$. Hence j has the sign λ, and $|j|c < 5U$ gives

$$0 < |j| < \frac{6a^{3/2}m\delta}{\sqrt{c}}. \tag{8}$$

Since $s^2 = ac+1$, the integer $Y = js - \lambda a^2 m\delta$ satisfies $Yc = \lambda a^2 m\delta c + 2\lambda am\delta + (jc - 2\lambda asm\delta)s$. With $s < \sqrt{2ac}$ this gives

$$\lambda Y > a^2 m\delta\Big(1 - \frac{5\sqrt{2}\,m^2}{\sqrt{c}}\Big) > a^2 m\delta\Big(1 - \frac{10m^2}{\sqrt{c}}\Big) > 0.$$

A bound adapted to u follows from (1), (3) and $c > 24u^2/m$:

$$|k| < \frac{m^2}{3u^2} + \frac{2m}{\sqrt{3}\,u} + \frac{2m}{c} < \left(\frac{1}{3} + \frac{1}{\sqrt{3}} + \frac{1}{12}\right)\frac{m^2}{u^2} < \frac{m^2}{u^2}. \tag{9}$$

In fact $k > 0$. By (1) and (6), $b > c/(4u^2) > 16a$. Hence

$$A := bn^2 - am^2 > \frac{3}{4}bn^2.$$

Also $t > 2s$ and $m \le 2n$, so $B := tn - sm > 0$, whereas

$$B < tn < \frac{2}{3}bn^2 + n < \frac{3}{4}bn^2$$

because $bn > 12$. Equation (4) is $kc = A + \lambda B > 0$. Therefore

$$4 \le k < \frac{m^2}{u^2}. \tag{10}$$

Item (iii). Since $\mathcal{E} \equiv -\lambda n$ and $n(2un+\lambda) \equiv \lambda n \pmod{2u}$, one has $\mathcal{F} \equiv -\lambda n \pmod{2u}$, so $\omega = (n+\lambda\mathcal{F})/(2u)$ is an integer. Substitution of $n = 2u\omega - \lambda\mathcal{F}$ into the definition of $\mathcal{F}$ gives the first identity of (iii); substitution of $2u\omega = n + \lambda\mathcal{F}$ into its middle term gives the second. Now suppose $c > 100m^4$. Multiplying (4) by $2u$, using (2), and substituting $c + f = K_0 b + 2S$ yields the exact identity

$$\mathcal{F}b + \mathcal{E}f = V_0 + 2\lambda usm, \qquad V_0 = 2uam^2 + \lambda nS + 2\mathcal{E}S. \tag{11}$$

Since $S \le u^2$, $a < u^2$ and $m > 2u$, (10) gives

$$|\mathcal{E}| < \frac{3m^2}{u}, \qquad |V_0| < 2u^3m^2 + mu^2 + 6m^2u < 4u^3m^2.$$

Using (1), $s < 2\sqrt{ac} < 2u\sqrt{c}$ and $b > c/(4u^2)$ in (11), we obtain

$$|\mathcal{F}| < \frac{m^2}{u^3} + \frac{16u^5m^2}{c} + \frac{16u^4m}{\sqrt{c}}. \tag{12}$$

Item (iv). Suppose $\omega \ge 2$, $u > 4\omega$ and $k = \omega(\omega+\lambda)$. As $\lambda(2\omega+\lambda) = 2\lambda\omega + 1$, the first identity of (iii) becomes

$$\mathcal{F}^2 + (2\omega+\lambda)(\omega - 2\lambda u\mathcal{F}) = 0. \tag{13}$$

Since $2\omega + \lambda > 0$, this forces $\lambda\mathcal{F} > 0$. Put $d_0 = \lambda\mathcal{F} \ge 1$. As $n = 2u\omega - d_0 > 0$, $d_0 < 2u\omega$. The convex quadratic

$$g(x) = x^2 - (2\omega+\lambda)(2ux - \omega)$$

is negative at $x = 1$, because $u > 4\omega$, and at $x = 2u\omega$, because

$$g(2u\omega) = \omega\big((2\omega+\lambda) - 4u^2(\omega+\lambda)\big) < 0.$$

It is therefore negative throughout $[1, 2u\omega]$, whereas $g(d_0) = 0$ by (13). This contradiction proves (iv).

The bound. Assume for a contradiction that

$$c \ge Cm^9, \qquad C = 2^{18}, \qquad \sqrt{C} = 512. \tag{14}$$

Then $c > 100m^4$, so (ii) and (iii) apply. Suppose $u \ge 4m^{3/5}$. As $u < m/2$, (12) implies

$$|\mathcal{F}| < \frac{m^{1/5}}{64} + \frac{\sqrt{m}}{512} + \frac{1}{2Cm^2} < \frac{m}{8}. \tag{15}$$

From $m/2 \le n < m$, (15), and $u \ge 4m^{3/5}$, the integer ω satisfies

$$1 \le \omega < \frac{m}{u}, \qquad \omega < \frac{u}{4}. \tag{16}$$

Equations (10), (15), (16) and $m > 2u$ bound the absolute value of the right side of the first identity of (iii), divided by $4u^2$, by

$$\frac{3}{2}\frac{|\mathcal{F}|m}{u^2} + \frac{5}{8}\frac{m^2}{u^4} < \frac{3}{2048} + \frac{3}{Cm^3} + \frac{6}{512\sqrt{m}} + \frac{5}{2048m^{2/5}} < 1. \tag{17}$$

To see the middle estimate directly, substituting (12) in its first term gives

$$\frac{3m^3}{2u^5} + \frac{24u^3m^3}{c} + \frac{24u^2m^2}{\sqrt{c}},$$

which are bounded respectively by $3/2048$, $3/(Cm^3)$ and $6/(512\sqrt{m})$. The integer $k-\omega(\omega+\lambda)$ therefore vanishes. Since $k \geq 4$, we have $\omega \geq 2$, and (iv) gives a contradiction. Therefore

$$u < 4m^{3/5}, \qquad a < u^2 < 16m^{6/5}. \tag{18}$$

Put $h = |j|$. Equations (8) and (14) imply $a > 16m$: otherwise

$$1 \leq h < \frac{6a^{3/2}m^3}{512m^{9/2}} = \frac{6}{512}\left(\frac{a}{m}\right)^{3/2} \leq \frac{3}{4}.$$

Combining (8), (18), and (14) gives

$$\frac{4h^2}{a} < \frac{144a^2m^2\delta^2}{c} < \frac{9}{64}m^{-3/5} < 1, \qquad \frac{hm}{\delta} < \frac{6a^{3/2}m^2}{\sqrt{c}} < \frac{3}{4}m^{-7/10} < \frac{1}{4}, \tag{19}$$

where the last inequality uses $m \geq 7$.

Recall $Y = js - \lambda a^2 m\delta$ and $T_\delta = 2a^2m\delta + jam$. Since $h < \sqrt{a}/2 < a\delta$, $T_\delta > a^2m\delta > 0$. Completion of the square in (5), using $s^2 = ac+1$, gives

$$4Y^2 - T_\delta^2 = F, \qquad F = 4ja\delta(1-n^2) - 12ajkn^2 + 4j^2 - j^2a^2m^2. \tag{20}$$

Equations (7), (19), $a > 16m$, and $\delta \geq m$ imply

$$\begin{aligned}\frac{|F|}{hT_\delta} &< \frac{4m}{a} + \frac{12m^2}{a^2} + \frac{2}{a^{3/2}m\delta} + \frac{hm}{\delta} \\ &< \frac{1}{4} + \frac{3}{64} + \frac{1}{16} + \frac{1}{4} = \frac{39}{64} < 1.\end{aligned}$$

Consequently the integer $q = T_\delta - 2|Y|$ satisfies

$$|q| = \frac{|F|}{T_\delta + 2|Y|} < h.$$

But $a \mid T_\delta$, $Y \equiv js \pmod a$ and $s^2 \equiv 1 \pmod a$ give $q^2 \equiv 4j^2 \pmod a$, contradicting

$$0 < 4j^2 - q^2 < 4h^2 < a$$

by (19). This proves $c < 2^{18}m^9$, and hence $m' = 2m > \frac{1}{2}c^{1/9}$ when $e < b$.

Finally $e = b$ is impossible because $\{a, b, e\}$ is a Diophantine triple. If $e > b$, Lemma 7.1 gives $m' > 6.24c^{1/4} > \frac{1}{2}c^{1/9}$. This proves the global conclusion. □

Lemma 7.14. *Let $0 < a < b < c < d$ be an irregular Diophantine quadruple with $\deg\{a,b,c\} \geq 2$ and $e = d_-(a,b,c) < b$, let z have a type I representation (λ, m, n), and put $u = \sqrt{ae+1}$. If*

$$c \geq 10^{90}, \qquad c \geq \frac{m^8}{20},$$

then

$$n < \frac{5}{7}m, \qquad u < \frac{125}{546}m, \qquad u < \frac{71}{100}m^{3/5}, \qquad a < \frac{5041}{10000}m^{6/5}.$$

In particular $m > 4u$.

Proof. Put $\ell_c = \log c$, $w = \log m$, $C = 1/20$, $\rho = 71/100$, $\theta = 5/7$, $g_* = 125/546$ and $m_0 = 2^{156/5}$. By Lemma 7.13, $c < 2^{18} m^9$; with $c \geq 10^{90} > 2^{1494/5}$ this gives $m > m_0$. Hence $c \geq Cm^8 > 100m^4$, and items (i) and (ii) of Lemma 7.13 give

$$m > 2u, \quad a < \frac{m^2}{9}, \quad n \geq \frac{m}{2}, \quad \frac{c}{b} > 4u^2 - 2 - \frac{1}{3u^2}, \quad b > \frac{c}{m^2}, \quad k \in 4\mathbb{Z}, \quad 4 \leq k < \frac{m^2}{u^2}. \tag{1}$$

Here $c/b > 4u^2 - 2 - 1/(3u^2)$ follows from $c + f = (4u^2 - 2)b + 2(a + e)$ and $0 < f/b < 1/(3u^2)$, and $b > c/m^2$ from $c < 4u^2 b < m^2 b$. Also $b > c/m^2 \geq m^6/20 > 8a$ and $c > (4u^2 - 3)b > 4ab$.

The index ratio. Write $B = \log \beta$ and $G = \log \gamma$, and put

$$\delta_\lambda = \sqrt{b/a}\, \frac{1 + \lambda\sqrt{a/c}}{1 + \lambda\sqrt{b/c}}.$$

Since $b > 8a$ and $c > 4ab$, one has $\sqrt{b/c} < 1/2$ and $\sqrt{a/c} < 1/4$, so $\delta_\lambda > \sqrt{8}\,(3/4)/(3/2) > 1$ and $\log \delta_\lambda > 0$; moreover

$$\log \delta_\lambda < \tfrac{1}{2} \log(b/a) + \log \frac{1 + \sqrt{a/c}}{1 - \sqrt{b/c}} < \tfrac{1}{2}\ell_c + 1, \tag{2}$$

since $b < c/(4a)$ makes the extra term after $\frac{1}{2}\ell_c$ at most $-\log 2 + \log(5/2) < 1$. For $\tau > 0$ one has $0 < \log(\sqrt{\tau + 1} + \sqrt{\tau}) - \log 2 - \frac{1}{2}\log \tau < 1/(4\tau)$. With $\beta = s + \sqrt{ac}$, $\gamma = t + \sqrt{bc}$ and (1), this gives

$$B < \tfrac{1}{2}\ell_c + w + \log(2/3) + \frac{1}{4ac}, \qquad G > \ell_c - w + \log 2. \tag{3}$$

By Lemma 5.7, $0 < 2mB - 2nG + \log \delta_\lambda$, so $(n/m)G < B + (\log \delta_\lambda)/(2m)$. By (2) and (3), the difference $\theta G - B - (\log \delta_\lambda)/(2m)$ is greater than

$$\left(\frac{3}{14} - \frac{1}{4m}\right)\ell_c - \frac{12}{7}w + \log 3 - \frac{2}{7}\log 2 - \frac{1}{4ac} - \frac{1}{2m}.$$

Using $\ell_c \geq 8w + \log C$, this is greater than

$$-\frac{2w}{m} + \left(\frac{3}{14} - \frac{1}{4m}\right)\log C + \log 3 - \frac{2}{7}\log 2 - \frac{1}{4 \cdot 10^{90}} - \frac{1}{2m} > 0.258627 > 0.$$

The inequality is uniform: w/m and $1/m$ decrease, and the limiting constant is positive. Therefore $n < \theta m$.

Let $\kappa = (\sqrt{17} - 1)/2$. By Lemma 2.23, $n > \kappa\sqrt{c/b}$. For $u \geq 100$, (1) gives $n > (78/25)u$, since

$$\kappa^2 \left(4u^2 - 2 - \frac{1}{3u^2}\right) - (78/25)^2 u^2 > 0;$$

this expression is positive at $u = 100$ and increases thereafter. Combining the two index comparisons gives $u < g_* m$. For $u < 100$, the same bound follows from $m > m_0$. In particular $m > 4u$, as $g_* < 1/4$.

The cutoff. Let $\mathcal{F}$ and ω be the integers of item (iii) of Lemma 7.13. By that item and $c \geq Cm^8$,

$$|\mathcal{F}| < D_m(u), \qquad D_m(v) = \frac{m^2}{v^3} + \frac{16v^5}{Cm^6} + \frac{16v^4}{\sqrt{C}\, m^3}. \tag{4}$$

Suppose $u \geq \rho m^{3/5}$. Since $u < g_* m$,

$$\frac{|\mathcal{F}|}{m} < \rho^{-3} m^{-4/5} + \frac{16 g_*^5}{Cm^2} + \frac{16 g_*^4}{\sqrt{C}} < 0.196565 < \frac{1}{4}. \tag{5}$$

Since $n \geq m/2$, (5) gives $\omega \geq 1$; also $n < m$ and $|\mathcal{F}| < m/4$ give $\omega < m/u$. The contrary cutoff and $m > m_0$ give $u^2/(4m) > 9.5254$, so $\omega < u/4$. The second identity of item (iii), with $n < \theta m$, $k < m^2/u^2$, $\omega < m/u$ and (4), gives

$$|k - \omega(\omega + \lambda)| < F_m(u), \qquad F_m(v) = \frac{D_m(v)^2 + 2(\theta m + v)D_m(v)}{4v^2} + \frac{m^2}{2v^4} + \frac{m}{4v^3}. \tag{6}$$

On expanding, every term of F_m has a positive coefficient, and its exponent in v belongs to

$$\{-8, -5, -4, -3, -1, 0, 2, 3, 4, 6, 7, 8\}.$$

Each corresponding monomial is convex for $v > 0$, including the constant term. Thus F_m is convex, and its maximum on $[\rho m^{3/5}, g_* m]$ occurs at an endpoint. Direct substitution gives, uniformly for $m \geq m_0$,

$$F_m(\rho m^{3/5}) < 1.980166, \qquad F_m(g_* m) < 1.952996. \tag{7}$$

Every variable endpoint term is a positive constant times a nonpositive power of m; hence checking $m = m_0$ proves uniformity.

Since $k \in 4\mathbb{Z}$ and $\omega(\omega + \lambda)$ is even, the integer $k - \omega(\omega + \lambda)$ is even. By (6) and (7) its absolute value is less than two, so it vanishes. Thus $k = \omega(\omega + \lambda)$, and $k \geq 4$ gives $\omega \geq 2$. As $u > 4\omega$, this contradicts item (iv) of Lemma 7.13. Therefore $u < \rho m^{3/5}$, and $a < u^2$ gives the bound on a. □

Lemma 7.15. *Let e, a, u be positive integers with $e < a$ and $u^2 = ae + 1$. Let integers $X_0 > 0, Y_0$ satisfy $eY_0^2 - aX_0^2 = e - a$. Let $\lambda \in \{-1, 1\}$, $\epsilon \in \{0, 1\}$, let g be a positive divisor of a, and suppose $Y_0 \equiv -\lambda u^\epsilon \pmod{a/g}$. Suppose positive integers c, s, X satisfy $s^2 = ac + 1$, $X^2 = ec + 1$, and*

$$s\sqrt{e} + X\sqrt{a} = (Y_0\sqrt{e} + X_0\sqrt{a})(u + \sqrt{ae})^{N_s}$$

for an integer $N_s \geq 0$ with $N_s \equiv \epsilon \pmod 2$. If nonzero $j \in \mathbb{Z}$ and integers m, δ satisfy

$$g \mid 2j, \qquad j(2\lambda s - am + 2) = 4a^2 m\delta, \qquad 4|j|(Y_0 + \lambda u^\epsilon)^2 < a^2,$$

then $\epsilon = 0$, $(Y_0, X_0) = (-\lambda, 1)$, and $N_s \geq 2$ is even.

The hypothesis $4|j|(Y_0 + \lambda u^\epsilon)^2 < a^2$ holds in each of the cases

(a) $|Y_0|^2 \leq a(a - e)/(2(u + 1))$, $e \leq \sqrt{u}/2$ and $0 < |j| < u/8$;

(b) $\epsilon = 0$, $u \geq 16$, $|Y_0|^2 \leq a(a - e)/(2(u + 1))$ and $0 < |j| < u/4$.

Proof. Write $Y_0 = -\lambda u^\epsilon + (a/g)t$ with $t \in \mathbb{Z}$. The integral Chebyshev formula for the orbit gives $s = u^\epsilon Y_0 + aw$ for some integer w: $T_{N_s}(u) - u^\epsilon$ is divisible by $u^2 - 1 = ae$. Since $u^{2\epsilon} - 1$ is divisible by a,

$$2\lambda s - am + 2 = \frac{a}{g}(2\lambda u^\epsilon t + gC)$$

for an integer C. Write $2j = gj_0$. Substituting in the exact identity and cancelling $a/2$ gives

$$j_0(2\lambda u^\epsilon t + gC) = 8am\delta.$$

Because $g \mid a$ and $\gcd(u, g) = 1$, reduction modulo g yields

$$g^2 \mid 4jt.$$

If $t \neq 0$, this implies $g^2 \leq 4|j||t|$. But $(a/g)t = Y_0 + \lambda u^\epsilon$ gives

$$a^2|t| \leq 4|j|(Y_0 + \lambda u^\epsilon)^2,$$

contradicting the strict hypothesis. Therefore $t = 0$. The norm equation gives $X_0 = 1$ if $\epsilon = 0$, while $\epsilon = 1$ would give $X_0^2 = e^2 + 1$, impossible. Thus $\epsilon = 0$ and the principal seed is as stated. The case $N_{\mathrm{s}} = 0$ would give $X = 1$ and $c = 0$, so $N_{\mathrm{s}} \geq 2$.

In case (a), $e \leq \sqrt{u}/2$ implies $u \geq 4$ and $a \geq (15/8)u^{3/2}$. Thus

$$\frac{|Y_0| + u^\epsilon}{a} < \frac{1}{\sqrt{2u}} + \frac{8}{15\sqrt{u}} < \frac{5}{4\sqrt{u}}.$$

Consequently $0 < |j| < u/8$ gives $4|j|(Y_0 + \lambda u^\epsilon)^2/a^2 < 25/32 < 1$. In case (b), $a > u$ and $u \geq 16$ give

$$\frac{|Y_0| + 1}{a} < \frac{1}{\sqrt{2u}} + \frac{1}{u}, \qquad 4|j|\frac{(Y_0 + \lambda)^2}{a^2} < \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{u}}\right)^2 \leq \left(\frac{1}{\sqrt{2}} + \frac{1}{4}\right)^2 < 1.$$

Both cases therefore imply the strict hypothesis. □

Lemma 7.16. *Let $0 < a < b < c < d$ be an irregular Diophantine quadruple with $\deg\{a, b, c\} \geq 2$ and $e = d_-(a, b, c) < b$, let z have a type I representation (λ, m, n), and put $u = \sqrt{ae + 1}$, $\eta = u + \sqrt{ae}$ and $\delta = m^2 - n^2$. Suppose that $e < a$ and*

$$s\sqrt{e} + \sqrt{ec + 1}\,\sqrt{a} = (-\lambda\sqrt{e} + \sqrt{a})\,\eta^8.$$

Put $T_4 = 8u^4 - 8u^2 + 1$, $U_3 = 8u^3 - 4u$ and $P_1 = a\delta - (e + 2\lambda u)n^2 - n + m$. Then the following hold.

- *(i) $A_1 = P_1/T_4$ and $B_1 = (P_1 - 2m)/U_3$ are integers with $T_4A_1 - U_3B_1 = 2m$, and $R = (A_1 + 2m)/(8ae + 4)$ is an integer.*
- *(ii) $\sqrt{ec + 1} = \frac{1}{2}\big[(1 - \lambda\sqrt{e/a})\eta^8 + (1 + \lambda\sqrt{e/a})\eta^{-8}\big]$.*
- *(iii) If $u \geq 3$, $|P_1| < 2am^2$ and $a > 10m/81$, then $|A_1| < 2m^2/(7ae^2)$ and $R \geq 0$; and $R = 0$ implies $7ae^2 < m$.*

Proof. Write $T_i = T_i(u)$ and $U_{i-1} = U_{i-1}(u)$ for the integers with $T_i + U_{i-1}\sqrt{u^2 - 1} = (u + \sqrt{u^2 - 1})^i$; thus $\eta^i = T_i + U_{i-1}\sqrt{ae}$, $T_4 = 8u^4 - 8u^2 + 1$ and $U_3 = 8u^3 - 4u$. Expanding the hypothesis gives

$$s = aU_7 - \lambda T_8, \qquad \sqrt{ec + 1} = T_8 - \lambda eU_7.$$

Item (ii) follows from $\eta^{\pm 8} = T_8 \pm U_7\sqrt{ae}$.

By canonical descent (Theorem 2.7), $\{a, e, b, c\}$ is a regular quadruple with $b < c$, so $b = d_-(a, e, c)$. The regular-descent identities (Lemma 2.6) give $ab + 1 = (us - a\sqrt{ec + 1})^2$ and $eb + 1 = (u\sqrt{ec + 1} - es)^2$. Both inner integers are positive, since

$$(us)^2 - a^2(ec + 1) = a(e + c - a) + 1 > 0, \qquad u^2(ec + 1) - e^2s^2 = e(a + c - e) + 1 > 0.$$

They are the coordinates of the inverse step $(s\sqrt{e} + \sqrt{ec + 1}\sqrt{a})\eta^{-1}$, so

$$r = aU_6 - \lambda T_7, \qquad \sqrt{be + 1} = T_7 - \lambda eU_6.$$

Hence the denominator-free expressions

$$b = \frac{r^2-1}{a} = (a+e)U_6^2 - 2\lambda T_7 U_6, \qquad t = (a+e)U_6U_7 - \lambda(T_7U_7 + T_8U_6),$$

and the duplication identities $U_7 = 2T_4U_3$, $T_8 = 2T_4^2 - 1$ give

$$c = \frac{s^2-1}{a} = 4T_4U_3(aU_3 - \lambda T_4)(T_4 - \lambda e U_3).$$

Reducing with $U_6 = uU_7 - T_8$ and $T_7 = uT_8 - aeU_7$, one finds $b \equiv a+e+2\lambda u$ and $t \equiv \lambda$ modulo both T_4 and U_3, while $s \equiv \lambda \pmod{T_4}$ and $s \equiv -\lambda \pmod{U_3}$. Since $c \equiv 0$ modulo T_4 and U_3, the defining relation $kc = bn^2 + \lambda tn - am^2 - \lambda sm$ of the second-order defect (Definition 2.20) gives $T_4 \mid P_1$ and $U_3 \mid P_1 - 2m$, and $T_4A_1 - U_3B_1 = 2m$ by definition. Put $P = 2u^2 - 1$, which is odd. Then $T_4 = 2P^2 - 1 \equiv -1 \pmod{P}$ and $U_3 = 4uP$, so $P \mid A_1 + 2m$; and $T_4 \equiv 1$, $U_3 \equiv 0 \pmod 4$ give $A_1 \equiv 2m \pmod 4$, so $4 \mid A_1 + 2m$. As $\gcd(P,4) = 1$ and $4P = 8u^2 - 4 = 8ae + 4$, item (i) follows.

For (iii), $u \geq 3$ gives $T_4 > 7u^4$, and $u^4 > (ae)^2$, so

$$|A_1| < \frac{2am^2}{7u^4} < \frac{2m^2}{7ae^2}.$$

Put $\hat{x} = a/m > 10/81$. Then

$$\frac{A_1 + 2m + 8ae + 4}{m} > 2 + 8e\hat{x} - \frac{2}{7\hat{x}e^2},$$

and the right side increases with $\hat{x}, e > 0$ and equals $2 + 80/81 - 81/35 > 0$ at $(\hat{x}, e) = (10/81, 1)$. Hence $R(8ae+4) = A_1 + 2m > -(8ae+4)$, so $R > -1$ and $R \geq 0$. If $R = 0$, then $A_1 = -2m$, and $2m = |A_1| < 2m^2/(7ae^2)$ gives $7ae^2 < m$. □

Lemma 7.17. *Let $0 < a < b < c < d$ be an irregular Diophantine quadruple with $\deg\{a,b,c\} \geq 2$ and $e = d_-(a,b,c) < b$, and let z have a type I representation (λ, m, n) with second-order defect k. Let j, Y and T_δ be the integers of Lemma 2.22, and put $u = \sqrt{ae+1}$, $\delta = m^2 - n^2$, $q = T_\delta - 2\lambda Y$ and $\mu_j = (4j^2 - q^2)/a$.*

(i) If $j \neq 0$, then $q(am + 2\lambda s) = 4a\delta(n^2-1) + 12akn^2 + j(a^2m^2 - 4)$.

(ii) Suppose that $c \geq c_0$ and $c \geq Cm^8$ for one of the rows

	c_0	C	B
row 1	10^{90}	4000	9
row 2	10^{99}	3/10	3/4

Then

$$0 < |j| < \frac{201}{100}\,\frac{a^{3/2}m\delta}{\sqrt{c}}, \qquad |q| < B\,\frac{\sqrt{a}\,m^4}{\sqrt{c}}, \qquad \mu_j \in \mathbb{Z}_{\geq 1},$$

and consequently $c < (201/50)^2 a^2 m^6$.

Proof. The target is $\mu_j \geq 1$ in both rows; the bound on c then follows from $a \leq a\mu_j \leq 4j^2$. Put $A_0 = 201/100$, $Q_* = \sqrt{a}\,m^4/\sqrt{c}$ and $m_0 = 2^{156/5}$.

Item (i). The integers of Lemma 2.22 satisfy $T_\delta + 2\lambda Y = j(am + 2\lambda s)$, and its discriminant identity reads

$$4Y^2 - T_\delta^2 = 4ja\delta(1-n^2) - 12ajkn^2 + 4j^2 - j^2a^2m^2.$$

As $q(T_\delta + 2\lambda Y) = T_\delta^2 - 4Y^2$, division by the nonzero integer j gives (i). Since $T_\delta = 2a^2m\delta + jam$ and $2\lambda Y = T_\delta - q$, one has $2\lambda js = 4a^2m\delta + jam - q$; substituting this and $q^2 = 4j^2 - a\mu_j$ in j times (i) gives

$$am(4\delta q - j^2 m) = 4j\delta(n^2-1) + 12jkn^2 - 2jmq - \mu_j. \tag{1}$$

Common bounds. Suppose that $c \ge c_0$ and $c \ge Cm^8$ for one of the rows. By Lemma 7.13, $c < 2^{18}m^9$, and $c \ge 10^{90} > 2^{1494/5}$ gives $m > m_0$. Hence $c \ge \frac{3}{10}m^8 > 100m^4$ and $c \ge m^8/20$, so items (i) and (ii) of Lemma 7.13 and Lemma 7.14 give

$$\frac{m}{2} \le n < \frac{5}{7}m, \quad m > 4u, \quad a < \frac{m^2}{9}, \quad m \le \delta \le \frac{3}{4}m^2, \quad 4 \le k < \frac{m^2}{u^2}, \quad k < \frac{m^2}{a},$$

$$\lambda j > 0, \quad |jc - 2\lambda asm\delta| < 5a^{3/2}m^3\delta, \quad 0 < |j| < \frac{6a^{3/2}m\delta}{\sqrt{c}}, \quad \lambda Y > 0, \quad u < \frac{71}{100}m^{3/5}. \tag{2}$$

Since $s^2 = ac + 1$,

$$\frac{|j|\sqrt{c}}{a^{3/2}m\delta} < 2\sqrt{1 + \frac{1}{ac}} + \frac{5m^2}{\sqrt{c}} < 2 + \frac{1}{ac} + \frac{5}{\sqrt{C}\,m^2} < A_0, \qquad \text{so} \qquad 0 < |j| < \frac{A_0a^{3/2}m\delta}{\sqrt{c}}. \tag{3}$$

Since $a \mid T_\delta$, $Y \equiv js \pmod a$ and $s^2 \equiv 1 \pmod a$, one has $q^2 \equiv 4j^2 \pmod a$, so μ_j is an integer.

Row 1: $\mu_j \ge 0$. Here $C = 4000$; put $\rho = 113/100$, so that $u < \frac{71}{100}m^{3/5} < \rho m^{3/5}$ and $a < u^2 < \rho^2 m^{6/5}$. By (3),

$$\frac{|j|m}{\delta} < \frac{A_0\rho^3}{\sqrt{C}\,m^{1/5}} < \frac{1}{512}. \tag{4}$$

In particular $|j| < a\delta$, so $T_\delta > a^2m\delta > 0$; with $\lambda Y > 0$ this gives $q = T_\delta - 2|Y|$ and $|q| = |4Y^2 - T_\delta^2|/(T_\delta + 2|Y|) < |4Y^2 - T_\delta^2|/T_\delta$. The discriminant identity, $|1-n^2| < n^2$, $n < m$ and $k < m^2/a$ give the four-term bound

$$|q| < \frac{4|j|m}{a} + \frac{12|j|m^3}{a^2\delta} + \frac{4j^2}{a^2m\delta} + \frac{j^2m}{\delta}. \tag{5}$$

Dividing by Q_* and using (3), $a \ge 1$, $a < \rho^2m^{6/5}$ and $\delta < m^2$ gives

$$\frac{|q|}{Q_*} < 4A_0 + \frac{12A_0}{a} + \frac{4A_0^2\sqrt{a}}{m\sqrt{c}} + \frac{A_0^2a^{5/2}m}{\sqrt{c}} < 16A_0 + \frac{4A_0^2\rho}{\sqrt{C}\,m^{22/5}} + \frac{A_0^2\rho^5}{\sqrt{C}} < 32.277695 < 33. \tag{6}$$

Consequently $q^2/a < 1089m^8/c \le 1089/C < 1$; and $\mu_j \ge 0$, since $\mu_j < 0$ would give $q^2 \ge 4j^2 + a \ge a$.

Row 2: $\mu_j \ge 0$. Here $C = 3/10$; put $\rho = 71/100$. By (3), $|j| \ge 1$ and $\delta < m^2$,

$$a > \Big(\frac{\sqrt{C}\,m}{A_0}\Big)^{2/3} > 767362. \tag{7}$$

Put $\theta_0 = \rho/(2\sqrt{C}\,m^{12/5})$ and $\tau = \rho^3/(4\sqrt{C}\,m^{1/5})$. Since $2s - am > 2\sqrt{ac}\,(1-\theta_0)$, item (i), (3), $k < m^2/a$ and $\delta n^2 \le m^4/4$ give, with (7) for $6/a$,

$$\frac{|q|}{Q_*} < \frac{\frac{1}{2} + 6/a + 3A_0\rho^5/(8\sqrt{C})}{1-\theta_0} < 0.748298 < \frac{3}{4}. \tag{8}$$

Suppose $\mu_j < 0$. Then $|j| < |q|/2$. Absorbing the last term of item (i), using $a^2m^2 - 4 < a^2m^2$, gives

$$|q|\Big(2\sqrt{ac} - am - \frac{a^2m^2}{2}\Big) < 4a\delta n^2 + 12akn^2,$$

and the bracket, divided by $2\sqrt{ac}$, exceeds $1 - \theta_0 - \tau$. Hence, by (7),

$$\frac{|q|}{Q_*} < \frac{\frac{1}{2} + 6/a}{1 - \theta_0 - \tau} < 0.501091, \qquad \frac{q^2}{a} < \frac{0.501091^2}{C} < 0.836975 < 1,$$

so $\mu_j = (4j^2 - q^2)/a > -1$, against $\mu_j < 0$. Hence $\mu_j \geq 0$.

Exclusion of $\mu_j = 0$. In row 2 with $m < m_1 := 2^{79/2}$ one has $c/m^8 > 10^{99}/2^{316} > 7490 > 4000$ and $c \geq 10^{90}$, so row 1 applies and gives $\mu_j \geq 1$. Suppose now, in row 1, or in row 2 with $m \geq m_1$, that $\mu_j = 0$. By (3) and $u < \rho m^{3/5}$,

$$\frac{|j|m}{\delta} < \frac{A_0\rho^3}{\sqrt{C}\, m^{1/5}} < \frac{1}{128}; \tag{9}$$

in row 1 this is (4), and in row 2 the bound is less than 0.005499 at $m = m_1$. Then $q = 2\omega j$ with $\omega = \pm 1$, and (1) divided by j reads $am(8\omega\delta - jm) = 4\delta(n^2 - 1) + 12kn^2 - 4\omega jm$. Its right side is positive by (9), whereas $8\omega\delta - jm$ has the sign ω. Hence $\omega = 1$, and

$$q = 2j, \qquad am(8\delta - jm) = 4\delta(n^2 - 1) + 12kn^2 - 4jm. \tag{10}$$

Dividing (10) by $m^2\delta$ and using $n \geq m/2$ and (9) gives

$$\hat{x} := \frac{a}{m} > \frac{1 - (4 + 1/32)/m^2}{8 + 1/128} > \frac{1}{8.1}; \tag{11}$$

and if $a \geq 2m$, then $k/\delta < m/a \leq 1/2$, so the left side of (10) divided by $m^2\delta$ exceeds $2(8 - 1/128)$ while its right side is less than 11. Hence $m/8.1 < a < 2m$. Now $|j| \geq 1$, (2) and $a < 2m$ give $\delta > \sqrt{c}/(6a^{3/2}m) > \sqrt{C}\, m^{3/2}/(12\sqrt{2})$, and with $k < m^2/a < 9m$,

$$\frac{k}{\delta} < \frac{108\sqrt{2}}{\sqrt{C}\,\sqrt{m}}. \tag{12}$$

In row 1, dividing (10) by $m^2\delta$ and using (4) gives

$$\hat{x} < \frac{4 + 12k/\delta + 1/(128m^2)}{8 - 1/512} < 0.501, \tag{13}$$

so $a < 0.51m$. In row 2, put $\hat{t} = n^2/m^2$, so that $1/4 \leq \hat{t} < 25/49$. The correction $-4\delta - 4jm$ in (10) is negative by (9), so

$$\hat{x} < \hat{t}\,\frac{4 + 12k/\delta}{8 - 1/128} < \xi_0\hat{t}, \qquad \xi_0 = \frac{501}{1000}, \tag{14}$$

where the unrounded coefficient is less than 0.500963603; in particular $a < 0.29m$.

In both rows $e < u^2/a < \rho^2 m^{6/5}/a < 8.1\rho^2 m^{1/5}$ and $e/a < u^2/a^2 < 81\rho^2 m^{-4/5} < 1/1024$, so $e < a$. By the definitions of q, Y and T_δ, the equality $q = 2j$ is the square-gap relation

$$j(2\lambda s - am + 2) = 4a^2m\delta. \tag{15}$$

Apply Lemma 2.12 with $(P,Q,S)=(e,a,u)$ in the notation of that lemma to the solution $(\sqrt{ec+1},s)$ of $eY^2-aX^2=e-a$. It gives integers $X_0>0$, Y_0 and $N_{\rm s}\ge 0$ with

$$s\sqrt{e}+\sqrt{ec+1}\,\sqrt{a}=(Y_0\sqrt{e}+X_0\sqrt{a})\,\eta^{N_{\rm s}},\qquad \eta=u+\sqrt{ae},\qquad eY_0^2-aX_0^2=e-a,$$

and $Y_0^2\le (u-1)(a-e)/(2e)=a(a-e)/(2(u+1))<a^2/(2(u+1))$, as $u^2-1=ae$. Put $g=\gcd(a,2j)$ and let $\epsilon\in\{0,1\}$ with $\epsilon\equiv N_{\rm s}\pmod 2$. By (15), $s\equiv-\lambda\pmod{a/g}$; a unit step sends the first coordinate Y to $uY+aX$, and $u^2\equiv 1\pmod{a/g}$, so $Y_0\equiv-\lambda u^{\epsilon}\pmod{a/g}$. We show

$$\frac{4|j|(Y_0+\lambda u^{\epsilon})^2}{a^2}<\frac{4|j|}{u}\Big(\frac{1}{\sqrt{2}}+\frac{u^{3/2}}{a}\Big)^2<1. \tag{16}$$

In row 1, (3), $a<2m$ and $u>\sqrt{a}$ give $4|j|/u<8A_0/\sqrt{C}$, and $u^{3/2}/a<9\rho^{3/2}m^{-1/10}$, so (16) is less than $(8A_0/\sqrt{C})L(m)^2<0.968$, where $L(m)=1/\sqrt{2}+9\rho^{3/2}m^{-1/10}$ decreases and the value at m_0 is less than 0.967379. In row 2 the factor e must be retained. From (3), $|j|/u<A_0\hat{x}(1-\hat{t})/(\sqrt{C}\sqrt{e})$, and $u^{3/2}/a<K_1e^{3/4}/(\hat{x}^{1/4}m^{1/4})$ with $K_1=1.000001$, as $ae>m/8.1$. Using $e<\rho^2m^{1/5}/\hat{x}$, $e\ge 1$, (11), (14), $\hat{t}(1-\hat{t})\le 1/4$ and $\sqrt{\hat{t}}(1-\hat{t})\le 2/(3\sqrt{3})$, the middle expression of (16) is less than

$$\frac{A_0\xi_0}{2\sqrt{C}}+\frac{4\sqrt{2}\,A_0K_1\sqrt{\rho}}{\sqrt{C}\,m^{1/5}}\,\frac{2\sqrt{\xi_0}}{3\sqrt{3}}+\frac{4A_0K_1^2\rho^2}{\sqrt{C}\,m^{3/10}}\,\frac{3}{4}\sqrt{8.1}<0.919271+0.019952+0.004279=0.943502$$

for $m\ge m_1$. By (15), (16) and Lemma 7.15, $\epsilon=0$, $(Y_0,X_0)=(-\lambda,1)$, and $N_{\rm s}=2H_{\rm s}$ with $H_{\rm s}\ge 1$.

Squaring the second coordinate of $(-\lambda\sqrt{e}+\sqrt{a})\eta^{2H_{\rm s}}$ gives

$$c=\frac{(\sqrt{a}-\lambda\sqrt{e})^2\eta^{4H_{\rm s}}+(\sqrt{a}+\lambda\sqrt{e})^2\eta^{-4H_{\rm s}}-2(a+e)}{4ae}.$$

Since $e/a<1/1024$, the leading coefficient exceeds $(9/10)a$, both coefficients are less than $2a$, and $2(a+e)<4a$; with $\eta^2>4ae$, $\eta^2<6ae$ and $\eta^{4H_{\rm s}}>10$ this gives

$$\frac{(4ae)^{2H_{\rm s}}}{8e}<c<\frac{(6ae)^{2H_{\rm s}}}{e}. \tag{17}$$

If $H_{\rm s}\ge 5$, the lower bound, $a>m/8.1$ and $c<2^{18}m^9$ give $m<2(8.1)^{10}<m_0$, a contradiction. If $H_{\rm s}\le 3$, then $c<6^6a^6e^5$; in row 1, $a<0.51m$ and $e<8.1\rho^2m^{1/5}<11m^{1/5}$ give $c<6^6(0.51)^6\,11^5m^7<2^{39}m^7$, so $m<2^{39}/4000<m_0$; in row 2, $a<0.29m$ and $e<6.5m^{1/5}$ give $c<6^6(0.29)^6(6.5)^5m^7<2^{19}m^7<Cm^8$. Both are impossible, so $H_{\rm s}=4$, and Lemma 7.16 applies. Its hypotheses in (iii) hold: $u^2>a>m/8.1$ gives $u\ge 3$; $e<u$ and $3u/a+1/(am)<27\rho m^{-2/5}+9/m^2<0.0054$ give $|P_1|<(a+3u)m^2+m<2am^2$; and $a>10m/81$ is (11). Put $\hat{y}=ae/m$. By item (iii) there, $R\ge 0$, and $R=0$ gives $\hat{y}<1/(7e)$. If $R\ge 1$, then $A_1=R(8ae+4)-2m>(8\hat{y}-2)m$.

Row 1. If $R\ge 1$ and $\hat{y}\ge 3/8$, then $A_1>m$, whereas $|A_1|<2m^2/(7ae^2)=2m/(7e\hat{y})\le 16m/21$. Hence $\hat{y}<3/8$. As $ae>200$, $\eta^2<4ae+2<4.01ae$, and item (ii) there with $e/a<1/1024$ gives

$$c<\frac{0.27}{e}(4.01ae)^8\le 0.27(4.01\hat{y})^8m^8<0.27(4.01\cdot 3/8)^8m^8<8m^8<Cm^8,$$

a contradiction.

Row 2. For $R\ge 1$ we retain the index ratio. As $\delta=(1-\hat{t})m^2$ and $e<u$,

$$|P_1|<am^2\Big(1-\hat{t}+\frac{3u}{a}+\frac{2}{am}\Big)<am^2(1-\hat{t}+0.001),$$

since $3u/a + 2/(am) < 24.3\rho m^{-2/5} + 16.2/m^2 < 0.000303$. Also $T_4 > 7.99u^4$, because $u^2 > a > m/8.1$, and $u^4 > a^2e^2$, so

$$\frac{|A_1|}{m} < \frac{1 - \hat{t} + 0.001}{7.99e\hat{y}}. \tag{18}$$

For $e = 1$, (14) gives $\hat{y} = \hat{x} < \xi_0 \cdot 25/49$. For $e = 2$, $\hat{y} \geq 0.272$ would contradict (18), $A_1 > (8\hat{y} - 2)m$ and $\hat{t} > \hat{y}/(2\xi_0)$, since $(8 \cdot 0.272 - 2) \cdot 7.99 \cdot 2 \cdot 0.272 > 1.001 - 0.272/(2\xi_0)$ and the left side increases while the right side decreases with $\hat{y}$. For $e \geq 3$, $\hat{y} \geq 0.27$ would give $(8\hat{y} - 2) \cdot 7.99e\hat{y} \geq (8 \cdot 0.27 - 2) \cdot 7.99 \cdot 3 \cdot 0.27 > 0.751$, contrary to $\hat{t} \geq 1/4$ and (18). These bounds also hold when $R = 0$. Now $ae > m/8.1$ gives $\eta^2 < 4ae + 2 < 4.000001ae$, and $e/a < 52.65m^{-4/5}$ and $\eta^{-16} < 10^{-80}$ give, by item (ii) there,

$$c < \frac{0.25007}{e}(4.000001ae)^8 = \frac{0.25007}{e}(4.000001\hat{y})^8 m^8.$$

For $e = 1$, $e = 2$ and $e \geq 3$ the resulting bounds for c/m^8 are less than 0.298673, 0.245507 and 0.154288. All are less than C, a contradiction.

Conclusion. Thus $\mu_j \geq 1$ in both rows. By (3) and $\delta < m^2$,

$$a \leq a\mu_j = 4j^2 - q^2 \leq 4j^2 < \frac{4A_0^2 a^3 m^6}{c},$$

so $c < 4A_0^2 a^2 m^6 = (201/50)^2 a^2 m^6$. In row 2, (8) gives $|q| < \frac{3}{4}Q_*$. In row 1, this bound on c and $c \geq Cm^8$ give $a > \sqrt{C}\, m/(2A_0)$, and (6) improves to

$$\frac{|q|}{Q_*} < 4A_0 + \frac{24A_0^2}{\sqrt{C}\, m} + \frac{4A_0^2\rho}{\sqrt{C}\, m^{22/5}} + \frac{A_0^2\rho^5}{\sqrt{C}} < 9.$$

Every variable term in the numerical comparisons above decreases with m, so each holds at its stated endpoint m_0 or m_1 and thereafter. □

Lemma 7.18. *Let $0 < a < b < c < d$ be an irregular Diophantine quadruple with $\deg\{a, b, c\} \geq 2$, let $e = d_-(a, b, c)$, and suppose z has a type I representation. If $e < b$, then*

$$c \geq 2^{540} \quad \Longrightarrow \quad c < 2^{13} m^{42/5}.$$

Consequently every type I representation, including the case $e > b$, satisfies

$$c \geq 2^{540} \quad \Longrightarrow \quad m' = 2m > 2^{-23/42} c^{5/42}.$$

Proof. Suppose that $c \geq 2^{540}$ and, to the contrary, $c \geq 2^{13} m^{42/5}$. Then $c \geq 10^{90}$ and, as $m \geq 1$, $c \geq 2^{13} m^8 > 4000m^8$; in particular $c \geq m^8/20$. Row 1 of item (ii) of Lemma 7.17 gives $c < (201/50)^2 a^2 m^6$, and Lemma 7.14 gives $a < (5041/10000)\, m^{6/5}$. Hence

$$c < \Big(\frac{201}{50}\Big)^2 \Big(\frac{5041}{10000}\Big)^2 m^{42/5} < 4.11\, m^{42/5} < 2^{13} m^{42/5},$$

a contradiction. This proves the first assertion.

The resulting bound gives $m' = 2m > 2^{1-65/42} c^{5/42} = 2^{-23/42} c^{5/42}$. The equality case $e = b$ is impossible because $b^2 + 1$ is not a square. If $e > b$, Lemma 7.1 gives the stronger $m' > 6.24c^{1/4}$. □

7.3 Type II: the bound $c < 10^{79.71}$

The type II bound plays caps against branch bounds of the form $c < Cm^k$, one for each position. For a middle or low triple ($c \le b^2$) the branch bound is $c < 0.0275\,m^7$ for $c \ge 10^{75}$ (Lemma 7.30), and $c < 10m^6$ on $10^{75} \le c < 10^{90}$ (Lemma 7.32). For a high triple ($c > b^2$) a nonzero first-order defect is impossible above 10^{90}, by its floor (Lemma 7.4) against the caps, and a vanishing one splits by the descent of $\{a, b, f\}$, by whether L vanishes, and by the size of a against $4f$ (Lemma 7.27); on $10^{75} \le c < 10^{90}$ an iteration of the caps finishes the high case (Proposition 7.28).

Proposition 7.19. *Let $0 < a < b < c < d$ be an irregular Diophantine quadruple with $\deg\{a, b, c\} \ge 2$, and let z have a type II representation. Then*

$$c < 10^{79.71},$$

and $c < 10^{77.28}$ if $c \le b^2$.

Proof. The bound comes from two applications of the index caps of Proposition 5.14 against the bounds of c in terms of m for the two positions of e, followed by the finer caps below 10^{90}. By Lemma 3.5, $c > b^2$ if and only if $e > b$, and $c \le b^2$ if and only if $e < b$.

Branch bounds. The following hold whenever $c \ge 10^{90}$ and $m' < Q$, where $Q \le 4.3 \cdot 10^{13}$:

1. if $c \le b^2$, then $c < 0.0275\,m^7$;
2. if $c > b^2$, then $c < \max\{Q^6,\ 4.71 \cdot 10^4\, m^{23/4},\ 1979\, m^{11/2},\ 36.02\,\ell_c^2 m^6\}$, where $\ell_c = \log c$.

Item 1 is Lemma 7.30, since $c \ge 10^{90} > 10^{75}$. For item 2, $e > b$. If $H \ne 0$, Lemma 7.4 gives $m' > 1.06\sqrt{a}\,c^{1/4} - 0.5 \ge 1.06 \cdot 10^{22.5} - 0.5 > 4.3 \cdot 10^{13}$, which is impossible; so $H = 0$. Then Lemma 7.27, which applies since $c \ge 10^{90} > 10^{75}$, gives $c < \max\{(m')^6,\ 4.71 \cdot 10^4\, m^{23/4},\ 1979\, m^{11/2},\ 36.02\,\ell_c^2 m^6\}$, and $(m')^6 < Q^6$.

The bound $c < 10^{92}$. By item 2 of Proposition 7.7, $c < 10^{154}$. Suppose $c \ge 10^{92}$. By Proposition 5.14, type II, on $[10^{65}, 10^{154})$, $m' < 4.3 \cdot 10^{13}$, so $m < 2.15 \cdot 10^{13}$, and $\ell_c < 154 \log 10$. If $c \le b^2$, the branch bound gives $c < 0.0275\,(2.15 \cdot 10^{13})^7 < 10^{91.77}$. If $c > b^2$, it gives

$$\begin{aligned} c < \max\big\{&(4.3 \cdot 10^{13})^6,\ 4.71 \cdot 10^4\,(2.15 \cdot 10^{13})^{23/4},\\ &1979\,(2.15 \cdot 10^{13})^{11/2},\ 36.02\,(154 \log 10)^2 (2.15 \cdot 10^{13})^6\big\} < 10^{86.66}. \end{aligned}$$

Both contradict $c \ge 10^{92}$.

The bound $c < 10^{90}$. Suppose $10^{90} \le c < 10^{92}$. By Proposition 5.14, type II, on $[10^{65}, 10^{92})$, $m' < 1.5 \cdot 10^{13}$, so $m < 7.5 \cdot 10^{12}$, and $\ell_c < 92 \log 10$. If $c \le b^2$, the branch bound gives $c < 0.0275\,(7.5 \cdot 10^{12})^7 < 10^{88.57}$. If $c > b^2$, it gives

$$\begin{aligned} c < \max\big\{&(1.5 \cdot 10^{13})^6,\ 4.71 \cdot 10^4\,(7.5 \cdot 10^{12})^{23/4},\\ &1979\,(7.5 \cdot 10^{12})^{11/2},\ 36.02\,(92 \log 10)^2 (7.5 \cdot 10^{12})^6\big\} < 10^{83.46}. \end{aligned}$$

Both contradict $c \ge 10^{90}$. Hence $c < 10^{90}$.

High triples. Let $c > b^2$ and $c \ge 10^{75}$. Then $e > b$ and $10^{75} \le c < 10^{90}$, so Proposition 7.28 applies: its item 1 gives $c < 10^{80.59} < 10^{81}$, then its item 2 gives $c < 10^{79.84} < 10^{80}$, and its item 3 gives $c < 10^{79.71}$.

Middle and low triples. Let $c \le b^2$ and $c \ge 10^{75}$. Then $10^{75} \le c < 10^{90}$, and Lemma 7.32 gives $c < 10\,m^6$. By Proposition 5.14, type II, on $[10^{65}, 10^{90})$, $m' < 1.44 \cdot 10^{13}$, so $m < 7.2 \cdot 10^{12}$

and $c < 10\,(7.2 \cdot 10^{12})^6 < 10^{78.15} < 10^{80}$. Then the same proposition on $[10^{65}, 10^{80})$ gives $m' < 1.03 \cdot 10^{13}$, so $m < 5.15 \cdot 10^{12}$ and

$$c < 10\,(5.15 \cdot 10^{12})^6 < 10^{77.28}.$$

In both positions $c < 10^{79.71}$. □

High triples. Lemma 7.20 bounds the correction of the type II linear form, records the inequalities for $\log\beta$ and $\log\gamma$ used below, and makes the index difference Δ grow with m. For $H = 0$ the branches are as follows. If $f > b$ or $g = 0$, Lemma 7.10 gives $c < (m')^6$. Otherwise $b > f$, $g \geq 1$, and either $L = 0$ (Lemma 7.6, Lemma 7.21, Lemma 7.22) or $L \geq 1$, where Lemma 7.23 gives the Vieta bounds and the cases $a < 4f$ (Lemma 7.24, Lemma 7.25) and $a \geq 4f$ (Lemma 7.25, Lemma 7.26) follow. Lemma 7.27 collects them into one bound for $c \geq 10^{75}$.

Lemma 7.20. *Let $0 < a < b < c < d$ be an irregular Diophantine quadruple with $\deg\{a,b,c\} \geq 2$, and let z have a type II representation (λ, m, n). Put $\ell_c = \log c$, $B = \log\beta$ and $G = \log\gamma$, and let δ_λ be the seed quotient and $\Lambda = (2m+1)B - (2n+1)G + \log\delta_\lambda$ the linear form of Lemma 5.7. Then*

$$0 < \log(\delta_\lambda\beta/\gamma) < 0.288, \qquad \Delta > m/\ell_c.$$

Moreover

$$\Lambda = 2\Delta G - 2m(G-B) + \log(\delta_\lambda\beta/\gamma), \qquad G - B > \frac{1}{2}\log\frac{b}{a} - \frac{1}{4ac}, \qquad G < \frac{1}{2}\log(4bc) + \frac{1}{4bc},$$

and

$$\Delta G > m(G-B) - 0.144.$$

Proof. By Lemma 2.9, Lemma 2.10 and Lemma 2.14, $b > 4000$, $b > 8a$ and $n \geq m/2$; also $c > 4ab$ and $G < \ell_c$. By Theorem 2.7, the descent partner satisfies $e \geq 1$ and $c = d_+(a,b,e) > 4abe + a + b + e > 4ab + a + b$; hence $m \geq 4$ by Lemma 2.15. Define

$$\hat r = \frac{r}{\sqrt{ab}} = \sqrt{1 + 1/(ab)}, \quad \hat s = \frac{s}{\sqrt{ac}} = \sqrt{1 + 1/(ac)}, \quad \hat t = \frac{t}{\sqrt{bc}} = \sqrt{1 + 1/(bc)},$$

so $\hat r > \hat s > \hat t > 1$, and put $H_* = G - B$. Direct rewriting and rationalization give

$$H_* = \tfrac{1}{2}\log(b/a) + \log\frac{1+\hat t}{1+\hat s},$$

$$\delta_+ = \sqrt{b/a}\,\frac{\hat r + \hat t}{\hat r + \hat s}, \qquad \delta_- = \sqrt{b/a}\,\frac{c-a}{c-b}\,\frac{\hat r + \hat s}{\hat r + \hat t}.$$

In the rationalization, the respective product differences are $a(c-a)$ and $b(c-b)$. Put $\varepsilon_\lambda = \log\delta_\lambda - H_*$. Then

$$\varepsilon_+ = \log\frac{(\hat r + \hat t)(1+\hat s)}{(\hat r + \hat s)(1+\hat t)} > 0,$$

because the numerator minus the denominator is $(\hat r - 1)(\hat s - \hat t) > 0$. Moreover

$$\varepsilon_+ < \log\frac{1+\hat s}{1+\hat t} < (\hat s - \hat t)/2 < 1/(4ac).$$

For the other sign,

$$\varepsilon_- = \log\frac{c-a}{c-b} + \log\frac{\hat r+\hat s}{\hat r+\hat t} + \log\frac{1+\hat s}{1+\hat t} > 0.$$

Since $b/c < 1/4$, each of the last two logarithms is smaller than $(\hat s - \hat t)/2$, and $\hat s - \hat t < 1/(2ac)$. Also $ac > 4a^2 b \ge 16004$. Consequently

$$\varepsilon_- < \log(4/3) + 1/32008 < 0.288.$$

The elementary logarithmic error

$$0 < \log(\sqrt{X+1} + \sqrt{X}) - \log(2\sqrt{X}) < 1/(4X),$$

applied with $X = bc$ to G and with $X = ac$ to B, gives $G < \frac12\log(4bc) + \frac{1}{4bc}$ and $H_* = G - B > \frac12\log\frac{b}{a} - \frac{1}{4ac}$, the two bounds of the statement. As $b > 8a$,

$$H_* > \tfrac32\log 2 - 1/(4ac) > \tfrac32\log 2 - 1/64016 > 1.036.$$

The positive linear form of Lemma 5.7 is exactly

$$0 < \Lambda = (2m+1)B - (2n+1)G + \log\delta_\lambda = 2mB - 2nG + \varepsilon_\lambda = 2\Delta G - 2mH_* + \varepsilon_\lambda,$$

and $\varepsilon_\lambda = \log\delta_\lambda - (G-B) = \log(\delta_\lambda\beta/\gamma)$; this is the identity of the statement. Thus

$$\Delta G = mH_* - \varepsilon_\lambda/2 + \Lambda/2 > mH_* - 0.144,$$

which is the last inequality of the statement, and with $H_* > 1.036$ it gives $\Delta G > 1.036m - 0.144 \ge m$ for $m \ge 4$. As $G < \ell_c$, this proves $\Delta > m/\ell_c$. □

Lemma 7.21. *Let $0 < a < b < c < d$ be an irregular Diophantine quadruple with $\deg\{a,b,c\} \ge 2$ and $e > b$, and let z have a type II representation (λ, m, n) with first-order defect $H = 0$. Let f be the descent partner of the descent, and put $v = 2af + 1$ and $L = 2n(n+1) - (4af+1)\Delta$. If $L = 0$, then*

$$4av(v+S) < b < 4av(v+S) + 2(2v+1)(a+f),$$

$$\Delta > \frac{6}{25}\,m, \qquad v < \frac{361m+625}{150}, \qquad v+S < \frac{25(m+1)}{6},$$

and

$$c < 128\Big(a^3 + a + \frac{1}{4a}\Big)v^3(v+S)^2\Big(1+\frac{1.03}{v}\Big)^2\Big(1+\frac1b\Big),$$

$$c < 8v^4(v+S)^2\Big(1+\frac2v\Big)\Big(1+\frac{1.03}{v}\Big)^2\Big(1+\frac1b\Big).$$

Proof. By Lemma 7.5, $H = 0$ forces $\lambda = -1$ and $\Delta \ge 1$, so $N = 2n(n+1)+\Delta$, $M = 2m(m+1)-\Delta$ and $L = N - 2v\Delta$.

By Lemma 2.9, $b > 4000$. By Lemma 7.6, $1 \le \Delta \le n \le v-1$, $S < 3v$,

$$g := d_-(a,b,f) = 2a(v+S) + 2(a+f),$$

and $h = d_-(a,f,g)$ satisfies $a+1 \le h \le 4a$, while $f > 8a(a+1)$. Hence $v > 16a^2(a+1)+1$; in particular $v \ge 34$, $v > 16a^3$ and $16(a^3+a^2) \le v-2$. Since $M - N = 2(m^2-n^2) = 2S\Delta$, the relation $N = 2v\Delta$ gives

$$v\Delta = n^2 + \frac{S}{2}, \qquad (v+S)\Delta = m^2 + \frac{S}{2}, \qquad M = 2(v+S)\Delta. \tag{1}$$

The equation $H = 0$ reads $e\Delta = bN - aM$, so

$$e = 2vb - 2a(v + S). \tag{2}$$

By Lemma 7.6, $b = d_+(a, f, g)$. The two values $d_\pm(a, f, g)$ are b and h, so Definition 2.2 gives $b + h = 2(a + f) + 2(2af + 1)g$, that is

$$b = 2vg + 2(a + f) - h. \tag{3}$$

Since $1 \le h \le 4a < 2(a + f)$ (as $f > a$), (3) yields $2vg < b < 2vg + 2(a + f)$, and the value of g gives

$$4av(v + S) + 4v(a + f) < b < 4av(v + S) + 2(2v + 1)(a + f).$$

This contains the first displayed claim.

Since $c = d_+(a, b, e)$ (Theorem 2.7) and $f = d_-(a, b, e)$, so that c and f are the two values $d_\pm(a, b, e)$, Definition 2.2 gives $c = (4ab + 2)e + 2(a + b) - f$. By (2), $e < 2vb$, hence

$$c < 8avb^2 + 4vb + 2a + 2b \le 8avb^2 + 8vb \le 8avb^2\Big(1 + \frac{1}{b}\Big). \tag{4}$$

Write the upper bound for b as $b < 4av(v + S)(1 + \epsilon)$ with $\epsilon = \frac{(2v+1)(a+f)}{2av(v+S)}$. Since $2af = v - 1$,

$$\epsilon = \frac{2v + 1}{2v(v + S)} + \frac{(2v + 1)(v - 1)}{4a^2v(v + S)} < \frac{1.03}{v} + \frac{1}{2a^2},$$

using $v + S > v$, $2v + 1 \le 2.06v$ (as $v \ge 34$) and $(2v + 1)(v - 1) < 2v^2$. Hence $1 + \epsilon < (1 + \frac{1}{2a^2})(1 + \frac{1.03}{v})$ and

$$a^3(1 + \epsilon)^2 < \Big(a^3 + a + \frac{1}{4a}\Big)\Big(1 + \frac{1.03}{v}\Big)^2 \le \Big(a^3 + a^2 + \frac{1}{4}\Big)\Big(1 + \frac{1.03}{v}\Big)^2 \le \frac{v + 2}{16}\Big(1 + \frac{1.03}{v}\Big)^2.$$

Inserting the first of these bounds in $c < 128a^3(1+\epsilon)^2v^3(v+S)^2(1+\frac{1}{b})$, which follows from (4), gives the first displayed bound for c, and the last one gives the second.

For the bound on Δ, put $x = b/a$, $B = \log\beta$ and $G = \log\gamma$. By Lemma 7.20,

$$\Delta G > m(G - B) - 0.144, \tag{5}$$

$$G - B > \frac{1}{2}\log x - \frac{1}{4ac}, \qquad G < \frac{1}{2}\log(4bc) + \frac{1}{4bc}. \tag{6}$$

By (4), $4bc < 32avb^3(1 + \frac{1}{b}) = 32a^4vx^3(1 + \frac{1}{b})$. From $v > 16a^3$ we get $a^4v < 2^{-16/3}v^{7/3}$, and from $x > 4v(v + S) > 4v^2$ we get $v < \frac{1}{2}x^{1/2}$. Hence

$$4bc < 2^{5-16/3-7/3}x^{25/6}\Big(1+\frac{1}{b}\Big) = 2^{-8/3}x^{25/6}\Big(1+\frac{1}{b}\Big), \qquad \frac{6}{25}G < \frac{1}{2}\log x - \frac{8}{25}\log 2 + \frac{3}{25b} + \frac{3}{50bc}.$$

Together with (6), $b > 4000$, $c > b$ and $a \ge 1$,

$$(G - B) - \frac{6}{25}G > \frac{8}{25}\log 2 - \frac{1}{4ac} - \frac{3}{25b} - \frac{3}{50bc} > 0.2214.$$

By (5), $\Delta G > \frac{6}{25}mG + 0.2214m - 0.144 > \frac{6}{25}mG$, and $G > 0$ gives $\Delta > \frac{6}{25}m$.

The bounds for v and $v + S$ follow from (1). Since $n = m - \Delta < \frac{19}{25}m$ and $\frac{S}{2} < m$,

$$v = \frac{n^2 + S/2}{\Delta} < \frac{(19/25)^2m^2 + m}{(6/25)m} = \frac{361m + 625}{150},$$

$$v + S = \frac{m^2 + S/2}{\Delta} < \frac{m^2 + m}{(6/25)m} = \frac{25(m + 1)}{6}.$$

□

Lemma 7.22. *Assume the hypotheses of Lemma 7.21 with $L = 0$, and suppose $c \geq 10^{75}$. Let $g = d_-(a, b, f)$, $h = d_-(a, f, g)$ and $\ell = d_-(a, f, h)$ be as in Lemma 7.6. Then $v > 1.38 \cdot 10^{12}$, $m > 5.7 \cdot 10^{11}$, $h > 1.7313\, a$, and either $\ell > a$, or $\ell < a$ and $\{\ell, a, h\}$ is an Euler triple, $h = a + \ell + 2\sqrt{a\ell + 1}$. If $a \geq 100$, then $\ell > 0.0995\, a$ and $v > 1.378\, a^4$. In all cases*

$$c < 4.71 \cdot 10^4\, m^{23/4}.$$

Proof. By Lemma 7.21, $\Delta > \frac{6}{25}m$, $v < \frac{361m+625}{150}$, $v + S < \frac{25(m+1)}{6}$, $4av(v + S) < b < 4av(v + S) + 2(2v + 1)(a + f)$ and $c < 8v^4(v + S)^2(1 + \frac{2}{v})(1 + \frac{1.03}{v})^2(1 + \frac{1}{b})$. By Lemma 7.6, $S < 3v$, $g = 2a(v + S) + 2(a + f) > d_+(a, f, 0)$, where $d_+(a, f, 0) = a + f + 2\sqrt{af + 1}$ is the Euler extension of the pair $\{a, f\}$, $a + 1 \leq h \leq 4a$, $\ell \geq 2$, $f = d_+(a, h, \ell) > 8a(a + 1)$ (so $v \geq 34$) and $2v(h - a) = 2aS + \ell$. By Lemma 2.9, $b > 4000$. Since $L = 0$, $2v\Delta = N$, so $v\Delta = n^2 + \frac{S}{2}$; with $S\Delta = m^2 - n^2$ this gives $(v + S)\Delta = m^2 + \frac{S}{2}$.

In the bound for c, $S < 3v$, $v \geq 34$ and $b > 4000$ bound the correction factor by $(1 + \frac{2}{34})(1 + \frac{1.03}{34})^2(1 + \frac{1}{4000}) < 1.1243$, so $c < 8 \cdot 16 \cdot 1.1243\, v^6 < 144v^6$. Since $144\,(1.38 \cdot 10^{12})^6 < 10^{75} \leq c$, we get $v > 1.38 \cdot 10^{12}$. As $v < \frac{361m+625}{150}$, we get $m > \frac{150v-625}{361} > 5.7 \cdot 10^{11}$.

From $2v(h - a) = 2aS + \ell > 2aS$,

$$h > a\,\frac{v + S}{v} = a\,\frac{m^2 + S/2}{n^2 + S/2} > \frac{am^2}{n^2 + m}.$$

Since $\Delta > \frac{6}{25}m$, $n < \frac{19}{25}m$, hence $h > a/(0.5776 + 1/m) > 1.7313\, a$, as $m > 5.7 \cdot 10^{11}$.

By Lemma 7.6, $\{a, f, h\}$ is a Diophantine triple. Its largest element is f, since $f > 8a(a + 1) > 4a \geq h$, and it is not Euler, since $d_-(a, f, h) = \ell \geq 2$. Applying Theorem 2.7 again, $\{a, h, \ell\}$ is a Diophantine triple and $\ell \neq a$. Suppose $\ell < a$. Then h is the largest element of $\{\ell, a, h\}$. Were this triple not Euler, Theorem 2.7 would give $h = d_+(\ell, a, w)$ with $w \geq 1$, hence $h > 4\ell aw \geq 8a$ (as $d_+(A, B, X) > 4ABX$ and $\ell \geq 2$), contradicting $h \leq 4a$. So $\{\ell, a, h\}$ is Euler, $h = a + \ell + 2\sqrt{a\ell + 1}$. Otherwise $\ell > a$.

Let $a \geq 100$. If $\ell > a$, then $f = d_+(a, h, \ell) > 4ah\ell > 4 \cdot 1.7313\, a^3$. If $\ell < a$, then $\sqrt{a\ell + 1} \leq \sqrt{a\ell} + \frac{1}{2\sqrt{a\ell}}$ and $\ell \geq 2$ give

$$0.7313\, a < h - a = \ell + 2\sqrt{a\ell + 1} \leq \ell + 2\sqrt{a\ell} + \frac{1}{\sqrt{2a}}.$$

With $\omega = \sqrt{\ell/a}$ this reads $(1 + \omega)^2 > 1.7313 - \frac{1}{\sqrt{2}\,a^{3/2}} \geq 1.7313 - \frac{1}{1000\sqrt{2}}$, so $\omega > 0.3155$ and $\ell = \omega^2 a > 0.3155^2 a > 0.0995\, a$. In both cases $f > 4ah\ell > 4 \cdot 1.7313 \cdot 0.0995\, a^3 > 0.689\, a^3$, and $v = 2af + 1 > 1.378\, a^4$.

By Lemma 7.21, $c < 128\big(a^3 + a + \frac{1}{4a}\big)v^3(v + S)^2(1 + \frac{1.03}{v})^2(1 + \frac{1}{b})$. If $a \geq 100$, then $a^3 + a + \frac{1}{4a} \leq 1.0002\, a^3 < 1.0002\,(v/1.378)^{3/4}$. If $a \leq 99$, then $a^3 + a + \frac{1}{4a} \leq 99^3 + 99 + \frac{1}{4} < 10^6 < (v/1.378)^{3/4}$, as $v > 1.38 \cdot 10^{12}$. Therefore, with $v > 1.38 \cdot 10^{12}$ and $b > 4av(v + S) > 4v^2$,

$$c < 128 \cdot 1.0002 \cdot 1.378^{-3/4}\, v^{15/4}(v + S)^2\Big(1 + \frac{1.03}{v}\Big)^2\Big(1 + \frac{1}{b}\Big) < 100.7\, v^{15/4}(v + S)^2,$$

and the bounds for v and $v + S$, with $m > 5.7 \cdot 10^{11}$, give

$$c < 100.7\Big(\frac{361}{150}\Big)^{15/4}\Big(\frac{25}{6}\Big)^2\Big(1 + \frac{625}{361m}\Big)^{15/4}\Big(1 + \frac{1}{m}\Big)^2 m^{23/4} < 4.71 \cdot 10^4\, m^{23/4},$$

the leading constant being $100.7\,(\frac{361}{150})^{15/4}(\frac{25}{6})^2 = 47088.70\ldots$.

□

Lemma 7.23. *Let $0 < a < b < c < d$ be an irregular Diophantine quadruple with $\deg\{a,b,c\} \geq 2$ and $e > b$, and let z have a type II representation (λ, m, n) with first-order defect $H = 0$, so that $\lambda = -1$ and $\Delta \geq 1$. Let f be the descent partner of the descent, and put $v = 2af + 1$, $L = 2n(n+1) - (2v-1)\Delta$ and $g = d_-(a,b,f)$. If $b > f$, $g \geq 1$ and $L \geq 1$, then*

$$v\Delta < m^2, \qquad b < \frac{4am^2}{L}, \qquad 0 < g < \frac{b}{2(v-1)}, \qquad 7.6\,avb^2 < c < 8.003\,avb^2,$$

and

$$c < \frac{128.048\,a^3vm^4}{L^2}.$$

If $a < 4f$, then $c < 362.18\,v^{5/2}m^4$.

Proof. By Lemma 7.5, $\lambda = -1$, $\Delta \geq 1$ and $b < 4am^2/L$, and Lemma 2.9 gives $b > 4000$. Put $N = 2n(n+1)+\Delta$. As $n = m-\Delta \leq m-1$, $N = 2n^2+n+m \leq 2(m-1)^2+(m-1)+m < 2m^2$, so $2v\Delta = N - L < 2m^2$. By Theorem 2.7, $f \geq 1$, $c = d_+(a,b,e)$ and $e = d_+(a,b,f) > 4abf$. By Definition 2.2, $e+g = (4af+2)b+2(a+f)$ and $eg = (b-a-f)^2 - 4(af+1)$. As b is the largest element of $\{a,f,b\}$ and $g = d_-(a,f,b) \geq 1$, Theorem 2.7 also gives $b = d_+(a,f,g) > a+f$. Hence $eg < b^2$ and $0 < g < b/(4af) = b/[2(v-1)]$. Since c and f are the two values $d_\pm(a,b,e)$, $c = (4ab+2)e + 2(a+b) - f$, and substituting $e = 2vb + 2(a+f) - g$ gives exactly

$$c = 8avb^2 + (8a^2 + 16af - 4ag + 6)b + 6a + 3f - 2g. \tag{1}$$

Discarding the g terms and using $8a^2 + 16af + 6 \leq 10av$ and $6a + 3f \leq 3av$ gives $c < av(8b^2 + 10b + 3) < 8.003avb^2$; here

$$10av - (8a^2 + 16af + 6) = 6(2a+1)(a-1) + 4a(5a-4)(f-1) \geq 0,$$
$$3av - 6a - 3f = 6a^2f - 3a - 3f \geq 0.$$

Using $g < b/(4af)$ and $avf \geq 3$ in (1) gives $c > (8av - 1/f)b^2 \geq \frac{23}{3}avb^2 > 7.6avb^2$. With $b < 4am^2/L$ we obtain $c < 128.048a^3vm^4/L^2$.

If $a < 4f$, then $a^2 < 4af = 2(v-1) < 2v$, hence $a^3 < 2^{3/2}v^{3/2}$ and $c < 128.048 \cdot 2^{3/2}v^{5/2}m^4 < 362.18v^{5/2}m^4$. □

Lemma 7.24. *Assume the hypotheses of Lemma 7.23, and suppose moreover $c \geq 10^{75}$ and $a < 4f$. Put $x = b/a$. Then*

$$x > \sqrt{2(v-1)}, \qquad 4bc < 16.006\,x^9\Big(1 + \frac{2}{x^2}\Big), \qquad c < 4.0015\,x^7\Big(1 + \frac{2}{x^2}\Big),$$

and

$$\Delta > 0.1097\,m, \qquad v < 7.2255\,m.$$

Proof. Since b is the largest element of the Diophantine triple $\{a,f,b\}$ and $g = d_-(a,f,b) \geq 1$, Theorem 2.7 gives $b = d_+(a,f,g)$, and $d_+(A,B,X) > 2ABX + 2\sqrt{AB \cdot AX \cdot BX} = 4ABX$. Hence $b > 4afg \geq 4af = 2(v-1)$. Since $a < 4f$, $a^2 < 4af = 2(v-1)$. Therefore

$$x > \frac{2(v-1)}{a} > \sqrt{2(v-1)}, \qquad \text{so} \qquad v - 1 < \frac{x^2}{2}, \quad a < x.$$

By Lemma 7.23, $c < 8.003avb^2 = 8.003a^3vx^2$. With $a^2 < 2(v-1) < x^2$ and $v < \frac{x^2}{2} + 1$,

$$c < 8.003x^3\Big(\frac{x^2}{2} + 1\Big)x^2 = 4.0015x^7\Big(1 + \frac{2}{x^2}\Big),$$

$$\begin{aligned} 4bc &< 32.012\, avb^3 = 32.012\, a^4vx^3 < 32.012 \cdot 4(v-1)^2 v\, x^3 \\ &< 32.012\, x^4\Big(\frac{x^2}{2} + 1\Big)x^3 = 16.006\, x^9\Big(1 + \frac{2}{x^2}\Big). \end{aligned}$$

By Lemma 2.10, $x > 8$, so $c < 4.0015 \cdot \frac{33}{32}x^7 < 4.1266x^7$; since $4.1266 \cdot 10^{74.34} < 10^{75} \le c$, we get $x > 10^{10.62}$. Then $2/x^2 < 10^{-20}$, and $X = \log x$ satisfies

$$X > \frac{75\log 10 - \log\big(4.0015(1 + 10^{-20})\big)}{7} > 24.4724.$$

Also $v\Delta < m^2$ gives $v < m^2$, so Lemma 7.23 gives $c < 362.18v^{5/2}m^4 < 362.18m^9$; since $362.18\,(1.11 \cdot 10^8)^9 < 10^{75}$, we get $m > 1.11 \cdot 10^8$.

Put $B = \log\beta$ and $G = \log\gamma$. By Lemma 7.20, $\Delta G > m(G-B) - 0.144$, $G - B > \frac{1}{2}X - \frac{1}{4ac}$ and $G < \frac{1}{2}\log(4bc) + \frac{1}{4bc}$. By the bound for $4bc$ and $\frac{1}{2}\log 16.006 < 1.386482$,

$$G < 4.5X + 1.386482 + \frac{1}{x^2} + \frac{1}{4bc} < 4.5X + 1.38649,$$

using $1/x^2 < 10^{-20}$ and $1/(4bc) < 10^{-70}$. Since $\frac{1}{2} - \frac{9}{2} \cdot 0.1097 = \frac{127}{20000}$ and $X > 24.4724$,

$$(G-B) - 0.1097\, G > \frac{127}{20000}X - 0.1097 \cdot 1.38649 - \frac{1}{4ac} > 0.0033.$$

Therefore $\Delta G > 0.1097\, mG + 0.0033m - 0.144 > 0.1097\, mG$, as $m > 1.11 \cdot 10^8$, and $G > 0$ gives $\Delta > 0.1097m$.

As $L \ge 1$, $2v\Delta = N - L \le 2n(n+1) + \Delta - 1$, so $v < \frac{n(n+1)}{\Delta} + \frac{1}{2}$. With $n = m - \Delta$ the right side decreases in Δ, so $\Delta > \kappa_0 m$ with $\kappa_0 = 0.1097$ yields

$$v < \frac{(1-\kappa_0)^2}{\kappa_0}\, m + \frac{1-\kappa_0}{\kappa_0} + \frac{1}{2} < 7.2255\, m,$$

the last step because $\frac{(1-\kappa_0)^2}{\kappa_0} + \big(\frac{1-\kappa_0}{\kappa_0} + \frac{1}{2}\big)/m < 7.2255$ for $m > 1.11 \cdot 10^8$.

□

Lemma 7.25. *Assume the hypotheses of Lemma 7.23. Then exactly one of the following holds:*

1. *$av \le 2\Delta$; in this case $a \le 3$ and $c < 1153\, m^5$;*
2. *$av > 2\Delta$; in this case $b \le 3m^2$ and $c < 72.03\, avm^4$.*

In particular, if $a < 4f$, then $c < \max\{1153\, m^5,\ 101.87\, v^{3/2}m^4\}$, and if moreover $c \ge 10^{75}$, then $c < 1979\, m^{11/2}$. If $a \ge 4f$, then case 2 holds and $c < 36.02\, v^2m^4/f$; in particular $c < 36.02\, \ell_c^2 m^6$, where $\ell_c = \log c$.

Proof. By Lemma 7.23, $v\Delta < m^2$, $0 < g < b/[2(v-1)]$, $7.6avb^2 < c < 8.003avb^2$ and $c < 128.048a^3vm^4/L^2$. By Lemma 2.9, Lemma 2.10, Lemma 2.15 and Lemma 2.14 (applicable as $c = d_+(a,b,e) > 4abe > 4ab + a + b$), $b > 4000$, $b > 8a$, $m, n \ge 4$ and $m/2 \le n < m$; hence

$\Delta \le n$, $\Delta \le m/2$, and with $S = m+n$, $S\Delta = m^2 - n^2 < m^2$. Put $N = 2n(n+1) + \Delta$ and $M_0 = 2m(m+1)$. Since $2v\Delta = N - L$,

$$(v+S)\Delta = m^2 + \frac{S-L}{2} < m^2 + m.$$

Finally Lemma 7.5 gives the identity

$$Lb + g\Delta = aM_0 + (a+2f)\Delta. \tag{1}$$

Claim: if $L \ge n^2+2n$ and $2f\Delta < n^2/(2a)$, then $b \le 8a$. Indeed $g\Delta > 0$, $M_0 \le 2\cdot 2n(2n+1)$ (as $m \le 2n$) and $a\Delta \le an$ turn (1) into $Lb < 8an^2 + 5an + n^2/(2a)$, and

$$\Big(8a + \frac{1}{2a}\Big)(n^2+2n) - \Big(8an^2 + 5an + \frac{n^2}{2a}\Big) = 11an + \frac{n}{a} > 0,$$

so $b < 8a + \frac{1}{2a} \le 8a + \frac{1}{2}$, and $b \le 8a$ as b is an integer. By Lemma 2.10 the conclusion $b \le 8a$ is impossible.

Case 1: $av \le 2\Delta$. Suppose $a \ge 4$. Then $2v\Delta \le 4\Delta^2/a \le \Delta^2 \le n^2$, so $L = 2n(n+1) + \Delta - 2v\Delta > n^2 + 2n$; and $2f = (v-1)/a < v/a \le 2\Delta/a^2$ gives $2f\Delta < 2\Delta^2/a^2 \le 2n^2/a^2 \le n^2/(2a)$ (the last step uses $a \ge 4$). The claim yields a contradiction. Hence $a \le 3$, and

$$c < 128.048\, a^3 v m^4 = 128.048\, a^2 (av) m^4 \le 128.048 \cdot 9 \cdot 2\Delta\, m^4 \le 1152.5\, m^5 < 1153\, m^5.$$

Case 2: $av > 2\Delta$. By Lemma 7.2 with $H = 0$ and $\lambda = -1$, $c \mid \Delta F$, where $F = (4ab-4)(S+1) - 6(2ab+1) - 2E_1(a+b)$ and $E_1 = e - a - b = (2v-1)b + a + 2f - g$. Thus $\omega = -\Delta F/c$ is an integer. Put $P = (2v-1)F + 2fc$, so $c \mid \Delta P$. Here c and f are the two values $d_\pm(a,b,e)$, while $e = d_+(a,b,f)$ and $g = d_-(a,b,f)$ are the two values $d_\pm(a,b,f)$ (Theorem 2.7); so Definition 2.2 gives $c = (4ab+2)e + 2(a+b) - f$ and $e = 2vb + 2(a+f) - g$, hence exactly

$$c = 8avb^2 + (8a^2 + 16af - 4ag + 6)b + 6a + 3f - 2g.$$

Substituting this and $v = 2af + 1$, exact expansion gives $P = -2b^2 + C_1 b + C_0$ with

$$C_1 = 4a(2v-1)S - 4av(2v+1) + 8fv + 2g, \qquad C_0 = (2v-1)(2ag - 2a^2 - 4S) + 6f^2 - 4fg - 4v^2 - 8v + 2.$$

Since $8fv = 4v(v-1)/a < 4v^2/a$, we have $|C_1| < 8av(S+v) + 4av + 4v^2/a + 2g$. Using $g < b/[2(v-1)]$, $v \ge 3$, $2af = v-1$ and $a < b/8$, we get $(2v-1)2ag < \frac{5}{2}ab$, $6f^2 < \frac{3}{2}v^2$, $4fg < b/a \le b$ and $4v^2 + 8v + 2 \le 6.9v^2$, hence $|C_0| < 3.5ab + 4a^2 v + 8vS + 8.4v^2$. Dividing $|\Delta P| \le \Delta(2b^2 + |C_1|b + |C_0|)$ by $c > 7.6avb^2$,

$$\frac{|\Delta P|}{c} < \frac{2\Delta}{7.6av} + \frac{8\Delta(S+v)}{7.6b} + \frac{4\Delta}{7.6b} + \frac{4v\Delta}{7.6a^2 b} + \frac{\Delta}{7.6av(v-1)} + \frac{3.5\Delta}{7.6vb} + \frac{4a\Delta}{7.6b^2} + \frac{8S\Delta}{7.6ab^2} + \frac{8.4v\Delta}{7.6ab^2}.$$

Suppose $b > 3m^2$. With $av > 2\Delta$, $(v+S)\Delta < m^2 + m$, $v\Delta < m^2$, $S\Delta < m^2$, $\Delta \le m/2$, $m \ge 4$, $v \ge 3$, $a \ge 1$ and $a < b/8$, the nine terms are less than

$$\frac{1}{7.6},\quad \frac{8}{22.8}\cdot\frac{5}{4},\quad \frac{2}{91.2},\quad \frac{4}{22.8},\quad \frac{1}{30.4},\quad \frac{1.75}{273.6},\quad \frac{1}{364.8},\quad \frac{8}{1094.4},\quad \frac{8.4}{1094.4},$$

whose sum is less than 0.83. Hence $|\Delta P| < c$, and $c \mid \Delta P$ forces $P = 0$, i.e. $(2v-1)F = -2fc$. Multiplying by Δ gives $(2v-1)\omega = 2f\Delta > 0$. Since $\gcd(2v-1, 2f) = \gcd(4af+1, 2f) = 1$, $\Delta = (2v-1)w$ with an integer $w \ge 1$. Then $L = 2n(n+1) - \Delta^2/w \ge 2n(n+1) - n^2 = n^2 + 2n$, and

$4af+1 = 2v-1 \le \Delta \le n$ gives $2f \le (n-1)/(2a)$, so $2f\Delta \le n(n-1)/(2a) < n^2/(2a)$. The claim yields a contradiction. Therefore $b \le 3m^2$, and $c < 8.003avb^2 \le 72.027\,avm^4 < 72.03\,avm^4$.

The particular cases follow. If $a < 4f$, then $a^2 < 4af = 2(v-1) < 2v$, so case 2 gives $c < 72.03\sqrt{2}\,v^{3/2}m^4 < 101.87\,v^{3/2}m^4$, and case 1 gives $c < 1153\,m^5$. If moreover $c \ge 10^{75}$, Lemma 7.24 gives $v < 7.2255\,m$, whence $101.87\,v^{3/2}m^4 < 101.87 \cdot 7.2255^{3/2}m^{11/2} < 1979\,m^{11/2}$; and $1153\,m^5 < 1979\,m^{11/2}$ as $m \ge 1$. So $c < 1979\,m^{11/2}$. If $a \ge 4f$, then $a \ge 4$, so case 1 is impossible, and $a = (v-1)/(2f) < v/(2f)$ gives $c < 36.02\,v^2m^4/f$. Finally Lemma 7.20 gives $\Delta > m/\ell_c$, so $v < m^2/\Delta < m\ell_c$, and $c < 36.02\,v^2m^4 \le 36.02\,\ell_c^2 m^6$.

□

Lemma 7.26. *Assume the hypotheses of Lemma 7.23, and suppose moreover $a \ge 4f$ and $10^{75} \le c < 10^{107}$. Put $\ell_c = \log c$ and, for $t \ge 4$,*

$$\varphi(t) = \frac{17.1 + 8\log t}{\log t - 0.5}, \qquad \Phi(t) = \max\bigl\{36.02\,\varphi(t)^2,\ 0.4447\,\varphi(t)^4\bigr\}\,t^6.$$

Then $m > 1.51 \cdot 10^9$ and

$$c < \Phi(m) < 2948\,m^6.$$

Moreover Φ is increasing on $[4,\infty)$; in particular $c < \Phi(t)$ for every real $t \ge m$.

Proof. Since $a \ge 4f$, Lemma 7.25 (case 2) gives $b \le 3m^2$, $av > 2\Delta$ and $c < 36.02v^2m^4/f$; Lemma 7.23 gives $v\Delta < m^2$ and $c < 8.003avb^2$; and $a = (v-1)/(2f) < v/2$. Put $x = b/a$. Then $c < 8.003a^3vx^2$ and $a^3 < v^3/8$ give

$$c < 1.0004\,v^4x^2, \qquad c < 36.02\,v^2m^4. \tag{1}$$

As $v < m^2$, the last bound gives $c < 36.02\,m^8$; since $36.02\,(1.51 \cdot 10^9)^8 < 10^{75} \le c$, we get $m > 1.51 \cdot 10^9$. By Lemma 7.20, $\Delta > m/\ell_c$, hence $v < m^2/\Delta < m\ell_c$, and $\ell_c < 107\log 10$ by the hypothesis $c < 10^{107}$. Therefore $4bc \le 12m^2c < 432.24\,v^2m^6 < 432.24\,\ell_c^2 m^8$, and

$$\log(4bc) < \log(432.24\,\ell_c^2) + 8\log m < \log\bigl(432.24\,(107\log 10)^2\bigr) + 8\log m < 17.09 + 8\log m. \tag{2}$$

Put $B = \log\beta$ and $G = \log\gamma$. By Lemma 7.20, $\Delta G > m(G-B) - 0.144$, $G - B > \frac{1}{2}\log x - \frac{1}{4ac}$ and $G < \frac{1}{2}\log(4bc) + \frac{1}{4bc}$. With (2) and $m \ge 4$ (so $0.288/m \le 0.072$), whenever $\log x > 0.0721$,

$$\frac{\Delta}{m} > \frac{\log x - 0.0721}{17.0901 + 8\log m}, \qquad v < \frac{m^2}{\Delta} < m\,\frac{17.0901 + 8\log m}{\log x - 0.0721}. \tag{3}$$

If $x \ge \frac{2}{3}m$, then $\log x - 0.0721 \ge \log m - \log\frac{3}{2} - 0.0721 > \log m - 0.4777$, so (3) gives $v < m\,\frac{17.0901+8\log m}{\log m - 0.4777} < \varphi(m)\,m$, and (1) gives $c < 36.02\,\varphi(m)^2m^6$.

If $\exp(3) \le x < \frac{2}{3}m$ (a nonempty range, as $m > 1.51 \cdot 10^9$), then (1) and (3) give

$$c < 1.0004\,m^4(17.0901 + 8\log m)^4\,\psi(x)$$

with $\psi(x) = x^2/(\log x - 0.0721)^4$. Since $\frac{d}{d\log x}\log\psi(x) = 2 - \frac{4}{\log x - 0.0721} > 0$ for $\log x \ge 3$,

$$\psi(x) < \psi\Bigl(\frac{2}{3}m\Bigr) < \frac{\frac{4}{9}m^2}{(\log m - 0.4777)^4},$$

and therefore $c < \frac{4}{9} \cdot 1.0004\,\varphi(m)^4m^6 < 0.4447\,\varphi(m)^4m^6$.

If $x < \exp(3)$, then (1) and $v < m\ell_c$ give $c < 1.0004\exp(6)\,\ell_c^4m^4 < 403.6\,\ell_c^4m^4$.

Since $\frac{d\varphi}{d\log t} = -21.1/(\log t - 0.5)^2$, φ is decreasing for $t > \exp(1/2)$, so $\varphi(m) < \varphi(1.51\cdot 10^9) < 9.0226$, whence $36.02\,\varphi(m)^2 < 2933$ and $0.4447\,\varphi(m)^4 < 2948$. Also $403.6\,\ell_c^4 < 403.6\,(107\log 10)^4 < 1.49\cdot 10^{12} < m^2$, so $403.6\,\ell_c^4 m^4 < m^6 < \Phi(m)$, the last step because $\varphi(m) > 8$. Hence $c < \Phi(m)$ in each of the three ranges, and $c < \Phi(m) < 2948\,m^6$.

Φ is increasing on $[4,\infty)$: for $k \in \{2,4\}$,

$$\frac{d}{d\log t}\log\bigl(\varphi(t)^k t^6\bigr) = 6 - \frac{21.1\,k}{(\log t - 0.5)(17.1 + 8\log t)},$$

and $(\log t - 0.5)(17.1 + 8\log t)$ is increasing for $t \geq 4$ and exceeds $4\cdot 21.1/6$ at $t = 4$. So both functions in the maximum increase on $[4,\infty)$, and so does Φ; with $c < \Phi(m)$ this gives $c < \Phi(t)$ for $t \geq m$.

□

Lemma 7.27. *Let $0 < a < b < c < d$ be an irregular Diophantine quadruple with $\deg\{a,b,c\} \geq 2$ and $e = d_-(a,b,c) > b$, let z have a type II representation with first-order defect $H = 0$, and suppose $c \geq 10^{75}$. Put $\ell_c = \log c$. Then*

$$c < \max\bigl\{(m')^6,\ 4.71\cdot 10^4\,m^{23/4},\ 1979\,m^{11/2},\ 36.02\,\ell_c^2 m^6\bigr\}.$$

If moreover $c < 10^{107}$, then

$$c < \max\bigl\{(m')^6,\ 4.71\cdot 10^4\,m^{23/4},\ 1979\,m^{11/2},\ \Phi(m)\bigr\},$$

with Φ as in Lemma 7.26.

Proof. By Lemma 7.5, $\lambda = -1$, $\Delta \geq 1$ and $L \geq 0$, where $f = d_-(a,b,e) \geq 1$ (Theorem 2.7), $g = d_-(a,b,f) \geq 0$, and $v = 2af + 1$ and L are as in Definition 2.20.

If $g = 0$ or $f > b$, Lemma 7.10 gives $c < (m')^6$. Equality $f = b$ is impossible, since $\{a,b,f\}$ is a Diophantine triple and $b^2 + 1$ is not a square.

In the remaining case $b > f$ and $g \geq 1$. If $L = 0$, the hypotheses of Lemma 7.21 with $L = 0$ hold, and Lemma 7.22 gives $c < 4.71\cdot 10^4 m^{23/4}$. If $L \geq 1$, the hypotheses of Lemma 7.23 hold. For $a < 4f$, Lemma 7.25 gives $c < 1979\,m^{11/2}$, since $c \geq 10^{75}$. For $a \geq 4f$, Lemma 7.25 gives $c < 36.02\,\ell_c^2 m^6$, and if moreover $c < 10^{107}$, Lemma 7.26 gives $c < \Phi(m)$. □

Proposition 7.28. *Let $0 < a < b < c < d$ be an irregular Diophantine quadruple with $\deg\{a,b,c\} \geq 2$ and $e > b$, let z have a type II representation (λ,m,n), and suppose $10^{75} \leq c < 10^{90}$. Then the first-order defect of the representation is $H = 0$, and with Φ as in Lemma 7.26,*

$$c < \Psi(m) := \max\bigl\{(2m+1)^6,\ 4.71\cdot 10^4\,m^{23/4},\ 1979\,m^{11/2},\ \Phi(m)\bigr\},$$

where Ψ is increasing on $[4,\infty)$. Consequently:

1. *$c < 10^{80.59}$;*
2. *if $c < 10^{81}$, then $c < 10^{79.84}$;*
3. *if $c < 10^{80}$, then $c < 10^{79.71}$.*

Proof. Since $10^{65} \le c < 10^{90}$, Proposition 5.14, type II, on $[10^{65}, 10^{90})$ gives $m' < 1.44 \cdot 10^{13}$. If $H \neq 0$, Lemma 7.4 gives $m' > 1.06\sqrt{a}\,c^{1/4} - 0.5 \ge 1.06 \cdot 10^{75/4} - 0.5 > 1.44 \cdot 10^{13}$, a contradiction. Thus $H = 0$.

Since $H = 0$, $e > b$ and $10^{75} \le c < 10^{90} < 10^{107}$, Lemma 7.27 gives $c < \max\{(m')^6,\ 4.71 \cdot 10^4 m^{23/4},\ 1979\, m^{11/2},\ \Phi(m)\} = \Psi(m)$, as $m' = 2m + 1$. The first three functions in Ψ are increasing in m, and Φ is increasing on $[4, \infty)$ by Lemma 7.26; hence Ψ is increasing there (and $m \ge 4$ by Lemma 2.15).

Let Q be a type II cap of Proposition 5.14 on a range containing c. Then $2m+1 < Q$, so $(2m+1)^6 < Q^6$, $m < Q/2$, and by monotonicity $c < \max\{Q^6,\ 4.71 \cdot 10^4 (Q/2)^{23/4},\ 1979 (Q/2)^{11/2},\ \Phi(Q/2)\}$. The certified values of $\log_{10}$ of the four entries are:

- the range $[10^{65}, 10^{90})$, $Q = 1.44 \cdot 10^{13}$: below 78.951, 78.603, 74.012, 80.583; hence $c < 10^{80.59}$;
- the range $[10^{65}, 10^{81})$, $Q = 1.08 \cdot 10^{13}$: below 78.201, 77.885, 73.325, 79.834; hence, if $c < 10^{81}$, then $c < 10^{79.84}$;
- the range $[10^{65}, 10^{80})$, $Q = 1.03 \cdot 10^{13}$: below 78.078, 77.766, 73.212, 79.710; hence, if $c < 10^{80}$, then $c < 10^{79.71}$.

In each line the binding entry is $\Phi(Q/2) = 36.02\, \varphi(Q/2)^2 (Q/2)^6$, with $\varphi(Q/2) < 8.7250$, 8.7322, 8.7335 respectively. □

Middle and low triples. Here $c \le b^2$ and the first-order quotient J is a positive integer. When $c \ge 10m^6$, the defect forces $K = 0$ and two further descent steps, $\{a, e, f\} \to \{a, e, g_1\} \to \{a, g_1, h_1\}$, with the alternatives (A) $h_1 = 0$, where $\{a, e, g_1\}$ is an Euler triple and the second step does not occur, and (B) $h_1 \ge 1$ (Lemma 7.29). Alternative (A) is excluded, and alternative (B) gives $c < 0.0274\, m^7/a^3 + 9.73\, m^4$ (Lemma 7.30). On $10^{75} \le c < 10^{90}$ the seed structure (Lemma 7.31) excludes $c \ge 10m^6$ altogether (Lemma 7.32).

Lemma 7.29. *Let $0 < a < b < c < d$ be an irregular Diophantine quadruple with $\deg\{a, b, c\} \ge 2$, $c \ge 10^{75}$ and $c \le b^2$, and let z have a type II representation (λ, m, n) with $c \ge 10\, m^6$; let H, J, K and u be as in Definition 2.20, which applies since $c \le b^2$ makes $\{a, b, c\}$ middle or low (Lemma 3.5). Put $p_0 = m(m+1)$, $q_0 = n(n+1)$, $\Pi = aM - \lambda e \Delta$ (so that $H = bN - \Pi$), $P = 4u^2 - 2 = 4ae + 2$ and $E_\pm = a + e \pm 2u$. Then all of the following hold.*

1. *$m > 1.74 \cdot 10^9$, $a \le 3m^{3/2}$, $\Delta > m/5$, $M/N > 1.29$.*
2. *J is a positive integer, $N = PJ$ (that is, $K = 0$), $\Pi > 0$, and $f = \Pi/J + 2(a + e) = d_+(a, e, g_1)$, where $g_1 = d_-(a, e, f)$ is a positive integer; moreover $h_1 = d_-(a, e, g_1)$ is a nonnegative integer with*

$$h_1 = P\Big(g_1 - \frac{\Pi}{N}\Big), \qquad f\, h_1 = (g_1 - E_+)(g_1 - E_-), \qquad \frac{\Pi}{N} \le g_1 < \frac{\Pi}{N}\Big(1 + \frac{1}{2ae}\Big) + \frac{1}{2a} + \frac{1}{2e}.$$

3. *Exactly one of the following holds.*
   - *(A) $h_1 = 0$, $g_1 = E_\mu$ for one $\mu \in \{-1, 1\}$, and $\Pi = E_\mu N$.*
   - *(B) $h_1 \ge 1$, $e > 4a$, $g_1 < E_-$, $g_1 = d_-(a, e, h_1)$, $\{a, g_1, h_1\}$ is a Diophantine triple and $e = d_+(a, g_1, h_1)$.*

4. With $S = m + n$: $n \le m \le 2n$, $\Delta \le m/2$, $ae \ge 3$, $J < 0.167m^2$, $ae < 0.501m^2/J$,

$$\tfrac{1}{2}m^2 < N < 2.0000001m^2, \qquad 2m^2 < M < 2.0000001m^2, \qquad M-N = 2\Delta(S+1+\lambda) \ge 0,$$

$M/N < 4.0000001$ and $\Delta/N < 1/m$; moreover $b = d_+(a,e,f)$, $c > 4abe$, $b > 4aef$ and $c < 5aeb$.

Proof. **Standing facts.** By Theorem 2.7 applied to $\{a,b,c\}$ (not Euler, as $\deg\{a,b,c\} \ge 2$), $e \ge 1$ is an integer, $\{a,b,e\}$ is a Diophantine triple and $c = d_+(a,b,e) > 4abe + a + b + e$; in particular $c > 4ab + a + b$, which is the hypothesis of Lemma 2.15. By Lemma 2.9, Lemma 2.10, Lemma 2.15, Lemma 2.14: $b > 4000$, $b > 8a$, $m, n \ge 4$, $n \le m \le 2n$, so $0 \le \Delta \le n$ and $\Delta \le m/2$. Since $c \le b^2$ and $c > 4abe$, $e < b/(4a) < b$, so b is the largest element of $\{a,b,e\}$, which is not Euler (as $\deg\{a,b,c\} \ge 2$); Theorem 2.7 applied to it gives an integer $f \ge 1$, the Diophantine triple $\{a,e,f\}$, and $b = d_+(a,e,f)$. The pair $\{a,e\}$ is a Diophantine pair of distinct integers, so $|a-e| \ge 2$, $ae \ge 3$, and $E_+ \le 2(a+e)$. The Vieta relations of the quadratic $x^2 - \big((4ae+2)b + 2(a+e)\big)x + (b-E_+)(b-E_-)$, whose roots are $d_\pm(a,b,e)$, give

$$c + f = Pb + 2(a+e), \qquad cf = (b-E_+)(b-E_-),$$

and $c > (4ae+1)b$, $c < 5aeb$ (the latter because $2b + 2(a+e) < aeb$).

By Lemma 7.8, $(2m+1)^8 > 22c \ge 22 \cdot 10^{75}$; since $(2 \cdot 1.74 \cdot 10^9 + 1)^8 < 22 \cdot 10^{75}$, we get $m > 1.74 \cdot 10^9$. As $\{a,b,c\}$ is middle or low, Lemma 7.3 gives that $J = H/c$, with $H = bN - \Pi$, is a positive integer and

$$J(4ae+1) < 2m(m+1).$$

Consequently, for $m > 1.74 \cdot 10^9$,

$$ae < \frac{m(m+1)}{2J} < \frac{0.501m^2}{J}, \qquad J < \frac{2m(m+1)}{13} < 0.167m^2.$$

From $n \ge m/2$, $n \le m$, $\Delta \le \min(n, m/2)$:

$$\tfrac{1}{2}m^2 < 2n^2 + n \le N \le 2n^2 + 3n < 2.0000001m^2, \qquad 2m^2 < M < 2.0000001m^2,$$

$M - N = 2\Delta(S+1+\lambda) \ge 0$, $M/N < 4.0000001$, $\Delta/N < 1/m$, and $|\Pi - aM| \le e\Delta \le em/2$. Every comparison of this proof that involves m states that a function decreasing in m is below a constant; it holds at $m = 1.74 \cdot 10^9$, hence for every larger m.

The bounds $\Delta > m/5$ and $M/N > 1.29$. By Lemma 7.20, $m(G-B) < \Delta G + 0.144$, $G - B > \frac{1}{2}\log\frac{b}{a} - \frac{1}{4ac}$ and $G < \frac{1}{2}\log(4bc) + \frac{1}{4bc}$. As $\Delta \ge 0$, this gives $m\log\frac{b}{a} < \Delta\log(4bc) + \frac{\Delta}{2bc} + \frac{m}{2ac} + 0.288$; since $\log(4bc) = \log\frac{b}{a} + \ell_c + \log a + 2\log 2$ and $m - \Delta = n$,

$$n\log\frac{b}{a} < \Delta\big(\ell_c + \log a + 2\log 2\big) + \frac{m}{2ac} + \frac{\Delta}{2bc} + 0.288. \tag{1}$$

Now $b/a > c/(5a^2e)$ and $5a^2e \le 5(ae)^2 < 1.25501m^4$, while $c \ge 10m^6$ gives $\log m \le (\ell_c - \log 10)/6$; thus $\log(b/a) > \ell_c - 0.22718 - \frac{2}{3}\ell_c + 1.53505 > \frac{1}{3}\ell_c + 1.307$, and $\log a \le \log(ae) < \log(0.501) + 2\log m < \frac{1}{3}\ell_c - 1.45866$. Also $\Delta/(2bc) < m/(2ac) \le 1/(20m^5) < 10^{-10}$, by $c \ge 10m^6$. Inserting these in (1),

$$n\Big(\frac{\ell_c}{3} + 1.307\Big) < \Delta\Big(\frac{4}{3}\ell_c - 0.07235\Big) + 0.2881,$$

and since $\frac{4}{3}\ell_c > 0.07235$ and $4(\frac{\ell_c}{3} + 1.307) - (\frac{4}{3}\ell_c - 0.07235) = 5.30035 > 4 \cdot 0.2881/n$, this forces $\Delta > n/4$, hence $m = n + \Delta < 5\Delta$, $n < 4m/5$. Then $M/N \ge 1 + 2\Delta S/N > 1 +$

$2(m/5)(3m/2)/(2.0000001m^2) > 1.29$. This proves item 1 except for $a \le 3m^{3/2}$, which is proved last.

Vanishing of $K = N - PJ$ (Definition 2.20). Substituting $c = Pb + 2(a+e) - f$ into $Jc = bN - \Pi$ gives the exact identity

$$bK = \Pi + 2(a+e)J - fJ. \tag{2}$$

Suppose $K \ge 1$. Then $b \le \Pi + 2(a+e)J \le aM + e\Delta + 2(a+e)J$ and $c < 5aeb$. Here $5a^2eM \le 5(ae)^2M/e < 2.51002m^6/(J^2e)$, $5ae^2\Delta \le 5(ae)^2m/(2a) < 0.6276m^5/(aJ^2)$, and, by $J < 2m(m+1)/(4ae)$ and $a+e \le ae+1 < 0.501m^2+1$, $10ae(a+e)J < 5(a+e)m(m+1) < 2.5051m^4$. Thus $c < 2.51002m^6 + 0.6276m^5 + 2.5051m^4 < 2.511m^6 < 10m^6$, a contradiction.

Suppose $K \le -1$ and $\Pi \ge 0$. By (2), $fJ \ge b + \Pi + 2(a+e)J > b$, so $J > b/f > 4ae$ (as $f < b/(4ae)$ by $b > 4aef$). Then $4ae(4ae+1) < 2m(m+1)$ gives $ae < 0.3537m$. From $bN \equiv \Pi \pmod c$ and $cf = (b-E_+)(b-E_-)$ we get $(b-a-e)N \equiv \Pi - (a+e)N$ and

$$(\Pi - (a+e)N)^2 \equiv 4u^2N^2 \pmod c.$$

Both sides lie in $[0,c)$: $4u^2N^2 = 4(ae+1)N^2 < 4(0.3537m+1)(2.0000001m^2)^2 < 5.66m^5 < c$; and $\Pi - (a+e)N = a(M-N) - 2eq_0$ is a difference of two nonnegative numbers each at most $\max(a,e)\cdot 2m(m+1) \le 2(0.3537m)m(m+1)$, so $(\Pi-(a+e)N)^2 < 0.5006m^6 < c$. Hence $\Pi = (a+e\pm 2u)N = E_\mu N$. If $E_\mu = 0$ then $\Pi = 0$, which forces $\lambda = 1$, $\Delta \ge 1$ and $e/a = M/\Delta \ge 2(m+1)+1$, while $E_\mu = 0$ forces $|a-e| = 2$, $e \le 3a$: impossible. So $E_\mu \ge 1$, $E_{-\mu} \ge 0$, and $Jc = (b-E_\mu)N$ with $b > E_\mu$. Multiplying $cf = (b-E_\mu)(b-E_{-\mu})$ by J and cancelling $b - E_\mu$, and multiplying $c + f = Pb + 2(a+e)$ by J, we get the exact linear system

$$Jb - Nf = JE_{-\mu}, \qquad Kb + Jf = E_\mu N + 2J(a+e),$$

whose determinant $D = NK + J^2$ satisfies $Db = (N+J)(NE_\mu + JE_{-\mu}) > 0$, so $D \ge 1$; with $K \le -1$ this gives $J^2 \ge N+1$. From $J(4ae+1)b < Jc < bN$ we get $J < N/13$; and $E_\pm \le E_+ \le \frac{8}{13}(4ae+1) < \frac{8}{13}N/J$. Therefore $Db < \frac{8}{13}N(N+J)^2/J < 0.72N^3/J$ and

$$c < \frac{bN}{J} < \frac{0.72N^4}{DJ^2} \le \frac{0.72N^4}{J^2} < 0.72N^3 < 5.7601m^6 < 10m^6,$$

a contradiction.

Suppose $K \le -1$ and $\Pi < 0$. Then $\lambda = 1$ and $e\Delta > aM \ge 2am^2$, so $e > 4am$. If $J > 2ae$, then $2ae(4ae+1) < 2m(m+1)$ gives $ae < 0.5001m$, contradicting $ae > 4a^2m$. So $J \le 2ae$ and $fJ < bJ/(4ae) \le b/2$. By (2), $b \le fJ - \Pi - 2(a+e)J < fJ + e\Delta \le b/2 + e\Delta$, so $b < 2e\Delta \le em$ and $c < 5aeb \le 5ae^2m = 5(ae)^2m/a < 1.2551m^5/a < 10m^6$, a contradiction. Hence $K = 0$, i.e. $N = PJ$, and (2) reads

$$f = \frac{\Pi}{J} + 2(a+e) = P\,\frac{\Pi}{N} + 2(a+e). \tag{3}$$

Positivity of Π. If $\Pi \le 0$, then by (3) $f \le 2(a+e)$, and the Vieta sums $b + d_-(a,e,f) = Pf + 2(a+e)$, $c + f = Pb + 2(a+e)$ with $d_- \ge 0$ (Theorem 2.7) give, with $X = ae < 0.501m^2$ and $a + e \le X + 1$,

$$c \le Pb + 2(a+e) \le 2(X+1)\big[(4X+2)(4X+3)+1\big] < 4.03m^6 < 10m^6,$$

a contradiction. Thus $\Pi > 0$, and by (3) $f > 2(a+e) \ge E_+ > \max(a,e)$.

Item 2. Since $f > E_+$, the triple $\{a,e,f\}$ has largest element f and is not Euler. By Theorem 2.7, $g_1 = d_-(a,e,f)$ is a positive integer, $\{a,e,g_1\}$ is a Diophantine triple and $d_+(a,e,g_1) =$

f. By the product formula, $g_1 b = (f - E_+)(f - E_-) < f^2$ and $b > 4aef$, so $g_1 < f/(4ae)$; by (3) and $N = PJ$,

$$\frac{f}{4ae} = \frac{P\Pi/N + 2(a+e)}{4ae} = \frac{\Pi}{N}\Big(1 + \frac{1}{2ae}\Big) + \frac{1}{2a} + \frac{1}{2e}.$$

Let $h_1 = d_-(a, e, g_1)$. If $\{a, e, g_1\}$ is Euler, $h_1 = 0$ (its two extensions are 0 and d_+); otherwise Theorem 2.7, applied with the largest element of $\{a, e, g_1\}$ in the last position (the maps $d_\pm$ are symmetric), shows that h_1 is a positive integer. In both cases $f = d_+(a, e, g_1)$ and $h_1 = d_-(a, e, g_1)$ are the two roots of the extension quadratic of $\{a, e, g_1\}$, so

$$f + h_1 = Pg_1 + 2(a+e), \qquad fh_1 = (g_1 - E_+)(g_1 - E_-).$$

With (3), $h_1 = Pg_1 + 2(a+e) - f = P(g_1 - \Pi/N)$, hence $g_1 \geq \Pi/N$. This proves item 2.

Item 3. Since $f > 0$ and $h_1 \geq 0$, the product formula excludes $E_- < g_1 < E_+$. If $g_1 \in \{E_-, E_+\}$ then $h_1 = 0$, so $g_1 = \Pi/N$ and $\Pi = E_\mu N$: this is (A). Suppose $g_1 > E_+$. Then $h_1 \geq 1$, g_1 is the largest element of $\{a, e, g_1\}$, and $h_1 = d_-(a, e, g_1) < g_1/(4ae)$. By item 2 and $\Pi/N = aM/N - \lambda e\Delta/N < 4.0000001a + e/m$,

$$g_1 < \Big(4.0000001a + \frac{e}{m}\Big)\Big(1 + \frac{1}{2ae}\Big) + \frac{1}{2a} + \frac{1}{2e}.$$

If $e \geq 2$, the right side is below $4ae$, because $4ae - 4.0000001a - e/m \geq 3.9999a(e-1) \geq 3.9999$ exceeds $2.5000001/e + 1/(2a) + 1/(2am)$; so $h_1 < 1$, impossible. If $e = 1$ (so $a \geq 3$), the right side is below $4.0000001a + 2.67$, so $h_1 < 1.23$, $h_1 = 1$; but then $\{a, e, h_1\}$ is a Diophantine triple by Theorem 2.7, so $eh_1 + 1 = 2$ would be a square, impossible. Hence $g_1 > E_+$ does not occur, and the only remaining case is $g_1 < E_-$, where $h_1 \geq 1$.

In that case $e > 4a$: indeed, if $e \leq 4a$ then $e^2 < 4ae+4$, so $E_- < a$; but $\Pi/N \geq a$ (for $\lambda = -1$, $\Pi/N \geq aM/N \geq a$; for $\lambda = 1$, $\Pi/N = a + \Delta(2a(S+2) - e)/N \geq a$ as $e \leq 4a < 2a(S+2)$), giving $g_1 \geq a > E_-$, a contradiction. Next, $g_1 < E_- < a + e \leq a + e + (2ae+1)h_1$, and g_1 is one of the two roots $a + e + (2ae+1)h_1 \pm 2u\sqrt{(ah_1+1)(eh_1+1)}$ of the extension quadratic of $\{a, e, h_1\}$ (the regular relation $(x + y - z - w)^2 = 4(xy+1)(zw+1)$ is symmetric in z, w), so $g_1 = d_-(a, e, h_1)$. Finally, the largest element of the triple $\{a, e, g_1\}$ is e (since $e > 4a$ and $e > E_- > g_1$), and the triple is not Euler because $h_1 = d_-(a, e, g_1) \geq 1$; so Theorem 2.7, applied to the sorted triple with largest element e, gives that $\{a, g_1, h_1\}$ is a Diophantine triple and $e = d_+(a, g_1, h_1)$. This is (B). The two cases are exclusive since $h_1 = 0$ in (A) and $h_1 \geq 1$ in (B).

The bound $a \leq 3m^{3/2}$. Suppose $a > 3m^{3/2}$. Then $e < ae/a < 0.167\sqrt{m}$, $e < a$, and $u < \sqrt{0.501m^2 + 1} < 0.708m + 1$, so $E_+ = a + e + 2u < a(1 + 0.0557/m + 1.416/\sqrt{m} + 2/m^{3/2}) < 1.0001a$. Case (B) is excluded by $e < a < 4a$. In case (A), $g_1 = E_\mu \leq E_+ < 1.0001a$, while $g_1 = \Pi/N = aM/N - \lambda e\Delta/N > 1.29a - e/m > 1.28a$, a contradiction. Item 4 was established in the standing facts and the paragraph after them. □

Lemma 7.30. *Let $0 < a < b < c < d$ be an irregular Diophantine quadruple with $\deg\{a, b, c\} \geq 2$, $c \geq 10^{75}$ and $c \leq b^2$, and let z have a type II representation (λ, m, n); use the notation of Lemma 7.29. If $c \geq 10\,m^6$, then $\lambda = -1$, alternative (B) of Lemma 7.29 (so $h_1 = d_-(a, e, g_1) \geq 1$, $g_1 < E_-$, $e > 4a$), $e > 33\sqrt{m}$, $a^3 < 0.00275\,m$, and*

$$c < \frac{0.0274\,m^7}{a^3} + 9.73\,m^4.$$

In every case $m > 1.74 \cdot 10^9$ and

$$c < \max\big\{10\,m^6,\ 0.0274\,m^7 + 9.73\,m^4\big\} < 0.0275\,m^7.$$

Proof. In the notation of Lemma 7.29, items 1 and 4 of that lemma give $m > 1.74 \cdot 10^9$, $n \le m \le 2n$, $\Delta \le m/2$, $S = m+n$, $ae < 0.501m^2/J$, $J < 0.167m^2$, $\frac{1}{2}m^2 < N < 2.0000001m^2$, $2m^2 < M < 2.0000001m^2$, $M/N < 4.0000001$, $\Delta/N < 1/m$, $c > 4abe$, $b > 4aef$, $c < 5aeb$, $E_\pm = a+e \pm 2u$, $P = 4ae+2$, $\Pi = aM - \lambda e\Delta$. All conclusions of Lemma 7.29 are available: $\Delta > m/5$, $n < 4m/5$, $M/N > 1.29$, $N = PJ$, $\Pi > 0$, $f = P\Pi/N + 2(a+e) = d_+(a,e,g_1)$, $h_1 = P(g_1 - \Pi/N)$, $\Pi/N \le g_1 < \frac{\Pi}{N}(1+\frac{1}{2ae}) + \frac{1}{2a} + \frac{1}{2e}$, and the dichotomy (A)/(B). We use throughout

$$\frac{\Pi}{N} = \frac{aM}{N} - \frac{\lambda e\Delta}{N} < 4.0000001a + \frac{e}{m}, \qquad c \le P^3\frac{\Pi}{N} + 2(a+e)(P^2+P+1), \tag{0}$$

the second because $c \le Pb + 2(a+e)$, $b \le Pf + 2(a+e)$ (Vieta sums with nonnegative d_-), and $f = P\Pi/N + 2(a+e)$. Also $c > 64a^3e^3g_1$, since $c > 4abe$, $b > 4aef$, $f > 4aeg_1$. Every comparison below that involves m states that a function decreasing in m is below a constant, and is checked at $m = 1.74 \cdot 10^9$.

The primitive second-order congruence. Put $C_0 = a+b-e$, $F_1 = 4a(S+1)+2\lambda(2a-e)$, $F_0 = -4(S+1) + 2\lambda(e^2 - ae - 1)$ and

$$\mathcal{T} = \Delta(F_1\Pi + F_0 N) - 6JN.$$

By Lemma 7.2, $c \mid \Delta[(8ab - C_0^2)(S+1) + 2\lambda(6ab + 3 - C_0(a+b))] - 6J$. Since $e = d_-(a,b,c)$ (Theorem 2.7), we have the exact relation $2rst = (2ab+1)c + C_0$, so $C_0 \equiv 2rst$ and $C_0^2 \equiv 4r^2s^2t^2 \equiv 4(ab+1) \pmod c$. Hence $c \mid \Delta F - 6J$ with $F = (4ab-4)(S+1)+2\lambda(6ab+3-C_0(a+b))$. The polynomial identity $F + 2\lambda\big[(b-a-e)^2 - 4(ae+1)\big] = F_1 b + F_0$ and $cf = (b-a-e)^2 - 4(ae+1)$ give $F \equiv F_1 b + F_0 \pmod c$; multiplying by N and using $bN \equiv \Pi \pmod c$ yields $c \mid \mathcal{T}$. Moreover

$$F_1\Pi + F_0 N = 4aM\big[a(S+1+\lambda) - \lambda e\big] + 4\lambda e^2 q_0 - 2(2S+2+\lambda)N. \tag{1}$$

In the present situation $6JN = 6N^2/P \le 6(2.0000001m^2)^2/14 < 1.72m^4$. Since $(S+2)\Delta = (m+n+2)(m-n) \le m^2 + 2m$, $\Delta \le m/2$, $q_0 \le m(m+1)$ and $2S+3 \le 4m+1$,

$$\begin{aligned} 4a^2M(S+2)\Delta &< 8.0001a^2m^4, \qquad 4aMe\Delta < 4.0000002aem^3, \\ 4e^2q_0\Delta &< 2.0000001e^2m^3, \qquad 2(2S+3)N\Delta < 8.0001m^4, \end{aligned} \tag{2}$$

and therefore, for either sign λ,

$$|\mathcal{T}| < 8.0001a^2m^4 + 4.0000002aem^3 + 2.0000001e^2m^3 + 9.73m^4. \tag{3}$$

Positivity of $\mathcal{T}$ for $\lambda = 1$. By (1), $F_1\Pi + F_0N$ is a quadratic in e with leading coefficient $4q_0 > 0$, and completing the square gives $F_1\Pi + F_0N \ge 4a^2M(S+2) - a^2M^2/q_0 - 2(2S+3)N$. Since $M/q_0 < 8$ and $a \ge 1$, $F_1\Pi + F_0N > 4a^2MS - (4S+6)N \ge 4MS - (4S+6)N = 4S(M-N) - 6N = 8\Delta S(S+2) - 6N$. Hence $\mathcal{T} > 8\Delta^2(S+1)^2 - 8\Delta^2 - 6\Delta N - 6N^2/P$. Here $\Delta(S+1) = p_0 - q_0 > m^2 - n^2 > 0.36m^2$, so $8\Delta^2(S+1)^2 > 1.0367m^4$; $8\Delta^2 \le 2m^2$; $6\Delta N < 6.0000003m^3$; and $N < 2n^2+3n < 1.28m^2+2.4m$ with $P \ge 14$ give $6N^2/P < 0.7022m^4$. Thus $\mathcal{T} > 0.33m^4 > 0$.

Alternative (A) is impossible. Suppose $\Pi = E_\mu N$. Then $E_\mu(2q_0 - \lambda\Delta) = a(2p_0 + \lambda\Delta) - \lambda e\Delta$, i.e.

$$E_\mu q_0 - ap_0 = \lambda\Delta(a + \mu u). \tag{4}$$

Now $p_0/q_0 \le 4(m+1)/(m+2) < 4$, and $p_0/q_0 > (m/n)^2(1-1/n) > \frac{25}{16}(1-2/m) > 1.5624$; also $\Delta/q_0 < 2/m$ and $u < \sqrt{0.501m^2+1} < 0.708m + 1$, so $|\Delta(a+\mu u)/q_0| < 2a/m + 1.4161$. Hence

$$1.5623a - 1.4161 < E_\mu < 4.0000001a + 1.4161. \tag{5}$$

If $\mu=1$, (5) gives $e<e+2u<3.0000001a+1.4161$. If $\mu=-1$, (5) gives $e-2u<3.0000001a+1.4161$; writing $e=(u^2-1)/a$, the quadratic inequality $u^2-2au-(3.0000001a^2+1.4161a+1)<0$ gives $u<a+\sqrt{4.0000001a^2+1.4161a+1}<3.0000001a+0.605$, hence $e<9.0001a+3.7$. So in both cases

$$e<9.0001a+3.7. \tag{6}$$

Next, $\mathcal{T}\neq 0$: for $\lambda=1$ this was proved above. For $\lambda=-1$, (1) reads $F_1\Pi+F_0N=4aM(aS+e)-4e^2q_0-2(2S+1)N$, and $\mathcal{T}=0$ means $\Delta(F_1\Pi+F_0N)=6N^2/P$, so

$$4a^2SM<4e^2q_0+2(2S+1)N+\frac{6N^2}{P\Delta},\qquad a^2<\frac{e^2q_0}{SM}+\frac{(2S+1)N}{2SM}+\frac{6N}{4P\Delta S}\cdot\frac{N}{M}.$$

Here $q_0/M<\frac{1}{2}(1+1/m)$ and $S\geq 1.5m$ give $e^2q_0/(SM)<e^2/(2.9m)$; $N\leq M$ gives $(2S+1)N/(2SM)<1.0000001$; and $N<2.0000001m^2$, $P\geq 14$, $\Delta>m/5$, $S\geq 1.5m$ give the last term below 0.7143. With (6), $e^2<161.3a^2$, and $161.3/(2.9m)<3.2\cdot 10^{-8}$, so $a^2<1.7144+3.2\cdot 10^{-8}a^2$ and $a^2<4$, forcing $a=1$, $e\leq 12$, $P\leq 50$, and by (0) and (5), $c<50^3\cdot 5.42+26\cdot 2551<10^6$, contradicting $c\geq 10^{75}$. Thus $\mathcal{T}\neq 0$ and $c\leq|\mathcal{T}|$. Inserting (6) into (3): $4.0000002aem^3+2.0000001e^2m^3<373.5a^2m^3$, and $8.0001+373.5/m<8.0002$, so

$$c<8.0002a^2m^4+9.73m^4.$$

With $c\geq 10m^6$ this gives $a^2>(10-9.73m^{-2})m^2/8.0002>1.1179^2m^2$, so $a>1.1179m$. On the other hand $c>64a^3e^3E_\mu$ with $E_\mu>1.5623a-1.4161>1.56a$. If $\mu=-1$, then $E_-=a+e-2u>1.56a$ by (5), so $e-2u>0.56a>0$, $e>2u>2\sqrt{ae}$, hence $e>4a$ and $c>64\cdot 64\cdot 1.56a^7>6389a^7$. If $\mu=1$, then $e+2u>0.56a$ excludes $e\leq 0.01a$ (which would give $u<0.1a+1$ and $e+2u<0.21a+2$), so $e>0.01a$ and $c>64(0.01)^3(1.56)a^7>9.9\cdot 10^{-5}a^7$. In both cases $c<8.0002a^2m^4+9.73m^4<18a^2m^4$ gives $a^5<1.82\cdot 10^5m^4$, so $a<11.3m^{4/5}$, which contradicts $a>1.1179m$ because $1.1179m^{1/5}>11.3$. Hence (A) does not occur, and we are in alternative (B): $h_1\geq 1$, $g_1<E_-$, $e>4a$.

The sign $\lambda=-1$. Suppose $\lambda=1$. Then $\mathcal{T}>0$, so $c\leq\mathcal{T}<\Delta(F_1\Pi+F_0N)<\Delta\big[4a^2M(S+2)+4e^2q_0\big]$ by (1). Since $\Pi>0$ means $e\Delta<aM$, we get $4e^2q_0\Delta<4aeMq_0$, and with $q_0<(4m/5)(4m/5+1)$, $ae<0.501m^2$:

$$c<8.0001a^2m^4+4aeMq_0<8.0001a^2m^4+2.5652m^6.$$

With $c\geq 10m^6$ this gives $a^2>(10-2.5652)m^2/8.0001>0.964^2m^2$, so $ae>4a^2>3.7m^2$, contradicting $ae<0.501m^2$. Hence $\lambda=-1$, $\Pi=aM+e\Delta$, $\Pi/N\geq aM/N>1.29a$ and $\Pi/N\geq e\Delta/N$.

Nonvanishing of $\mathcal{T}$, and $e>33\sqrt{m}$. In (B), $fh_1=(E_+-g_1)(E_--g_1)<E_+E_-=(a-e)^2-4<e^2$ and $f>4aeg_1$, so $h_1<e/(4ag_1)$; as $h_1\geq 1$ and $g_1>1.29a$, this gives

$$e>4ag_1>5.16a^2. \tag{7}$$

Suppose $\mathcal{T}=0$. Then, as in the exclusion of alternative (A), $4e^2q_0<4aM(aS+e)$, so $e^2<a^2SM/q_0+aeM/q_0<16a^2m+8ae$, whence $e<4a(1+\sqrt{m+1})<4.0001a\sqrt{m}$. With (7), $a<4.0001\sqrt{m}/5.16<0.7753\sqrt{m}$. Then $\Pi/N<4.0000001a+e/m<4.0001a$, $P<16.0005a^2\sqrt{m}$, $P+1<16.0006a^2\sqrt{m}$ and $a+e<4.0002a\sqrt{m}$. By (0) and $P^2+P+1<(P+1)^2$,

$$\begin{aligned}c&<\big(16.0005^3\cdot 4.0001+2\cdot 4.0002\cdot 16.0006^2\big)a^7m^{3/2}<18435\,a^7m^{3/2}\\&<18435\cdot 0.7753^7m^5<3105\,m^5<10\,m^6,\end{aligned}$$

a contradiction. So $\mathcal{T} \neq 0$ and $c \le |\mathcal{T}|$. Suppose next $e \le 33\sqrt{m}$. Then (7) gives $a^2 < \frac{33}{5.16}\sqrt{m} < 6.4\sqrt{m}$, $a^4 < 41m$; further $\Pi/N < (4.0000001 + 33m^{-1/2})a < 4.0009a$, $P \le 132a\sqrt{m} + 2 < 132.1a\sqrt{m}$, $P + 1 < 132.2a\sqrt{m}$, $a + e < 33.1a\sqrt{m}$, and (0) gives

$$c < \big(132.1^3 \cdot 4.0009 + 2 \cdot 33.1 \cdot 132.2^2\big)a^4 m^{3/2} < 1.04 \cdot 10^7 a^4 m^{3/2} < 4.3 \cdot 10^8 m^{5/2} < 10\, m^6,$$

a contradiction. Hence $e > 33\sqrt{m}$.

The bound on c. Write $F_1\Pi + F_0 N = A_+ - A_-$ with $A_+ = 4aM(aS + e)$ and $A_- = 4e^2 q_0 + 2(2S + 1)N$, both positive, so $c \le |\mathcal{T}| \le \Delta \max(A_+, A_-) + 1.72m^4$. If $A_+ \ge A_-$, then, since $S\Delta = m^2 - n^2 < m^2$, $c < 8.0000004a^2m^4 + 2.0041m^5 + 1.72m^4$, and $c \ge 10m^6$ gives $a^2 > (10 - 2.0041m^{-1} - 1.72m^{-2})m^2/8.0000004 > 1.1179^2 m^2$, contradicting $ae > 4a^2$ and $ae < 0.501m^2$. Hence $A_- > A_+$ and, by (2),

$$c < 4e^2 q_0 \Delta + 9.7201m^4. \tag{8}$$

On the other hand $c > 64a^3e^3g_1 \ge 64a^3e^4\Delta/N$. Dividing by e^2 and using $4q_0\Delta = 2(N - \Delta)\Delta < 2N\Delta$, $N\Delta > 0.1m^3$ and $e > 33\sqrt{m}$,

$$\frac{64a^3e^2\Delta}{N} < 2N\Delta + \frac{9.7201m^4}{e^2} < 2N\Delta + 0.008926m^3 < 2.08926N\Delta, \qquad e^2 < \frac{0.032645N^2}{a^3}.$$

Put $\omega(n) = (2n^2+3n)^2(n^2+n)(m-n)$, so that $N^2q_0\Delta \le \omega(n)$ (as $N = 2n^2+n+m \le 2n^2+3n$). For $4 \le n \le 4m/5$ its logarithmic derivative is

$$\frac{1}{n}\Big(6 - \frac{6}{2n+3} - \frac{1}{n+1}\Big) - \frac{1}{m-n} \ge \frac{5.25}{n} - \frac{4}{n} > 0,$$

because $m - n \ge n/4$; so ω increases on $m/2 \le n \le 4m/5$, and since $n < 4m/5$,

$$N^2 q_0 \Delta < \omega(\tfrac{4}{5}m) = m^7\big(1.28 + 2.4m^{-1}\big)^2\big(0.64 + 0.8m^{-1}\big)(0.2) < 0.2098\, m^7.$$

Inserting this in (8),

$$c < \frac{4 \cdot 0.032645 \cdot 0.2098\, m^7}{a^3} + 9.7201m^4 < \frac{0.0274\, m^7}{a^3} + 9.73m^4.$$

Finally $c \ge 10m^6$ gives $a^3 < 0.0274m/(10 - 9.73m^{-2}) < 0.00275m$.

Consequences. Let $c \ge 10^{75}$ and $c \le b^2$. By Lemma 7.8, $(2m + 1)^8 > 22c \ge 22 \cdot 10^{75}$, so $m > 1.74 \cdot 10^9$, because $(2 \cdot 1.74 \cdot 10^9 + 1)^8 < 22 \cdot 10^{75}$. If $c < 10m^6$, then $c < 10m^6 < 0.0275m^7$, as $10 < 0.0275m$. Otherwise the main assertion gives $c < 0.0274m^7/a^3 + 9.73m^4 \le 0.0274m^7 + 9.73m^4 < 0.0275m^7$, as $0.0274 + 9.73m^{-3} < 0.0275$. □

Lemma 7.31. *Let $0 < a < b < c < d$ be an irregular Diophantine quadruple with $\deg\{a, b, c\} \ge 2$, $10^{75} \le c < 10^{90}$ and $c \le b^2$, and let z have a type II representation (λ, m, n) with $c \ge 10\, m^6$; use the notation of Lemma 7.29. Then $\lambda = -1$, alternative (B) of Lemma 7.29 holds, $1.74 \cdot 10^9 < m < 7.2 \cdot 10^{12}$, and, with $\ell = d_-(a, g_1, h_1)$, the following hold.*

1. *$N = 2n^2 + 2n + \Delta = PJ$ and $\Pi = a(2m^2 + m + n) + e\Delta = J(Pg_1 - h_1)$.*
2. *ℓ is a positive integer, $\{a, \ell, g_1\}$ is a Diophantine triple, $g_1 > \max(a, \ell)$, and $h_1 = d_+(a, \ell, g_1)$.*
3. *$a \le 2705$, $J^4 < 0.010486\, m/a + 0.954226\, aJ$, and*

$$\frac{a\ell}{8.0001} < J < \frac{h_1}{1.877\, g_1}.$$

4. $f = Pg_1 - h_1 + 2(a+e)$, $b = Pf - g_1 + 2(a+e)$, $c = Pb - f + 2(a+e)$, *and hence*

$$c = (P^3 - 2P)g_1 - (P^2 - 1)h_1 + 2(P^2 + P)(a+e).$$

Proof. **Item 1.** By Lemma 7.30, $\lambda = -1$, alternative (B) holds, $e > 33\sqrt{m}$ and $a^3 < 0.00275m$. By Lemma 7.29, $m > 1.74 \cdot 10^9$, $\Delta > m/5$, $M/N > 1.29$, $J \geq 1$, $N = PJ$, $\Pi = aM + e\Delta > 0$, $\Pi/N \leq g_1$, $h_1 = P(g_1 - \Pi/N) \geq 1$, $f = d_+(a, e, g_1)$, $g_1 < E_-$, $\{a, g_1, h_1\}$ is a Diophantine triple and $e = d_+(a, g_1, h_1)$. By Proposition 5.14, type II, on $[10^{65}, 10^{90})$, $2m + 1 < 1.44 \cdot 10^{13}$, so $m < 7.2 \cdot 10^{12}$, and then $a^3 < 1.98 \cdot 10^{10}$ gives $a \leq 2705$. By Lemma 2.14, $n \leq m \leq 2n$, so $\Delta \leq n$, and $\Delta > m/5$ gives $n < 4\Delta$. As $M \leq 2p_0$ and $N \geq 2q_0$, $M/N \leq m(m+1)/(n(n+1)) \leq 2(2n+1)/(n+1) < 4$. At $\lambda = -1$ one has $N = 2n^2 + 2n + \Delta$ and $M = 2m^2 + m + n$, and $h_1 = Pg_1 - P\Pi/N = Pg_1 - \Pi/J$ gives $\Pi = J(Pg_1 - h_1)$.

Bounds for h_1 and g_1. Since $0 < g_1 < E_- < 2(a+e)$, $fh_1 = (E_+ - g_1)(E_- - g_1) < E_+E_- = (a-e)^2 - 4 < e^2$, and $f = d_+(a, e, g_1) > 4aeg_1$; hence $h_1 < e/(4ag_1)$, and $h_1 \geq 1$ gives $e > 4ag_1$. Also $g_1 \geq \Pi/N \geq aM/N > 1.29a$.

Four exact identities. Put $\delta_1 = 4aJg_1 - \Delta \in \mathbb{Z}$. From $N = (4ae + 2)J$,

$$e = \frac{N - 2J}{4aJ}. \tag{1}$$

From item 1, $e\Delta = \Pi - aM = (4ae + 2)Jg_1 - Jh_1 - aM$, that is

$$e\delta_1 = aM + J(h_1 - 2g_1). \tag{2}$$

Since $e = d_+(a, g_1, h_1)$ is an integer, $\ell = d_-(a, g_1, h_1) = 2(a + g_1 + h_1 + 2ag_1h_1) - e$ is an integer; by Theorem 2.7 it is positive if $\{a, g_1, h_1\}$ is not Euler, and it is 0 if the triple is Euler (then $d_+d_- = 0$). From $e + \ell = 4ag_1h_1 + 2(a + g_1 + h_1)$ we get $h_1(4ag_1 + 2) = e - 2a - 2g_1 + \ell$; inserting (1), $g_1 = (\Delta + \delta_1)/(4aJ)$ and $4ag_1 + 2 = (\Delta + \delta_1 + 2J)/J$,

$$\frac{h_1}{g_1} = \frac{J\big[N - 2J - 8a^2J - 2(\Delta + \delta_1) + 4aJ\ell\big]}{(\Delta + \delta_1)(\Delta + \delta_1 + 2J)}. \tag{3}$$

Finally, the Vieta sums $f + h_1 = Pg_1 + 2(a+e)$, $b + g_1 = Pf + 2(a+e)$, $c + f = Pb + 2(a+e)$ in the $\{a, e\}$-orbit give

$$c = (P^3 - 2P)g_1 - (P^2 - 1)h_1 + 2(P^2 + P)(a+e). \tag{4}$$

Small quantities and δ_1. By (1) and $e > 33\sqrt{m}$, $N - 2J = 4aeJ > 132J\sqrt{m}$, so $J/N < 1.82 \cdot 10^{-7}$ and $N/(N-2J) < 1.0000004$; also

$$4a^2J/N < a/e < (0.00275m)^{1/3}/(33\sqrt{m}) < 1.23 \cdot 10^{-4}.$$

By (1), $aM/e = 4a^2JM/(N - 2J)$. Hence (2), $h_1 < e/(4ag_1)$, $g_1 < e/(4a)$, $1.29 < M/N < 4$ and $g_1 > 1.29a$ give

$$5.16a^2J - \frac{J}{2a} < \delta_1 < 16.00001\,a^2J + \frac{J}{5.16a^2} \leq 16.00001\,a^2J + 0.1938J,$$

so $\delta_1 \geq 1$.

The bound for J^4. In (4) drop $-2Pg_1 - (P^2 - 1)h_1 < 0$ and write $P = N/J$, $g_1 = (\Delta + \delta_1)/(4aJ)$ and, by (1), $2(P^2 + P)(a+e) = N(N+J)(N + (4a^2 - 2)J)/(2aJ^3)$. Then

$$c < \frac{N^3}{4aJ^4}\Big[\Delta + \delta_1 + 2J\Big(1 + \frac{J}{N}\Big)\Big(1 + \frac{4a^2J}{N}\Big)\Big] < \frac{N^3}{4aJ^4}\big[\Delta + 18.2\,a^2J\big],$$

by the bounds just proved $(2(1+1.82\cdot10^{-7})(1+1.23\cdot10^{-4})<2.0003$ and $16.00001+0.1938+2.0003<18.2)$. For fixed m and real $\Delta\in[m/5,m/2]$, $n=m-\Delta$, the quantity $N=2(m-\Delta)^2+2m-\Delta$ decreases, and so does $N^3\Delta$: its logarithmic derivative is $1/\Delta-3(4n+1)/N<0$, because $N=2n^2+2n+\Delta<(12n+3)\Delta$ when $\Delta>n/4$ and $n\ge2$. Hence $N^3\Delta<(1.28m^2+1.8m)^3m/5$ and $N^3<(1.28m^2+1.8m)^3<2.0972m^6$. With $c\ge10m^6$ this gives $40aJ^4<2.0972(m/5+18.2a^2J)$, that is

$$J^4<0.010486\,\frac{m}{a}+0.954226\,aJ. \tag{5}$$

The ratio h_1/g_1. Put $Y=a^2J$. If $0.954226aJ\le0.010486m/a$, then $J^4<0.020972m/a$ and, with $a^{7/4}<(0.00275m)^{7/12}$, $Y<c_am^{5/6}$, $c_a=(0.00275)^{7/12}(0.020972)^{1/4}<0.012210$. Otherwise $J^3<1.908452a$ and $Y<c_bm^{7/9}$, $c_b=(1.908452)^{1/3}(0.00275)^{7/9}<0.012646$. For $m>1.74\cdot10^9$ either case gives

$$\frac{91Y}{m}<0.0322,\qquad\frac{40Y}{m}<0.0142,\qquad\frac{20Y}{m}<0.0071,\qquad J^2<\frac{m}{200},\qquad\frac{J^2}{\Delta^2}<\frac{25J^2}{m^2}<10^{-13}.$$

Put $\epsilon_1=(\delta_1+2J)/\Delta$; as $\delta_1+2J<18.2a^2J$ and $\Delta>m/5$, $\epsilon_1<91Y/m<0.0322$. In (3) drop $4aJ\ell\ge0$. Since $n\ge\Delta$, $N\ge2\Delta^2+3\Delta$, and $8a^2J<40Y\Delta/m<0.0142\Delta$; so the bracket in (3) exceeds $2\Delta^2+3\Delta-2(1+\epsilon_1)\Delta-0.0142\Delta>2\Delta^2$, while the denominator is at most $(1+\epsilon_1)^2\Delta^2$. Therefore

$$\frac{h_1}{g_1}>\frac{2J}{(1.0322)^2}>1.877\,J. \tag{6}$$

In particular $h_1>g_1>a$, so h_1 is the largest element of $\{a,g_1,h_1\}$.

Item 2. From $\delta_1>0$, $g_1>\Delta/(4aJ)>m/(20aJ)\ge m/(20Y)>91/(20\cdot0.0322)>141$ (from $91Y/m<0.0322$) and $a/g_1<20Y/m<0.0071$. If $\{a,g_1,h_1\}$ were Euler, then $h_1=a+g_1+2\sqrt{ag_1+1}$ and $h_1/g_1<1+0.0071+2\sqrt{0.0071+141^{-2}}<1.1763$, contradicting (6). Hence $\ell\ge1$, and Theorem 2.7, applied to the sorted triple $a<g_1<h_1$, shows that ℓ is distinct from a and g_1, that $\{a,g_1,\ell\}$ is a Diophantine triple and that $d_+(a,g_1,\ell)=h_1$, i.e. $h_1=d_+(a,\ell,g_1)$.

Item 3. Since $h_1\ge\max(a,g_1)$, $\ell<h_1/(4ag_1)$. In (3) the numerator is below $J(N+4aJ\ell)$ and the denominator exceeds Δ^2, so $4a\ell<JN/\Delta^2+4aJ^2\ell/\Delta^2$. As $n<4\Delta$, $N<32\Delta^2+9\Delta$, and $9/\Delta<45/m$; with $J^2/\Delta^2<10^{-13}$,

$$a\ell<\frac{J(32+45/m)}{4(1-10^{-13})}<8.0001\,J.$$

Together with (6) this gives $a\ell/8.0001<J<h_1/(1.877g_1)$, and with (5) and $a\le2705$ it is item 3. Finally $\ell<8.0001J/a<m/(20aJ)<g_1$ by $J^2<m/200$, and $g_1>a$; so $g_1>\max(a,\ell)$, completing item 2. Item 4 consists of the three Vieta sums before (4) and of (4) itself.

Every displayed bound is monotone in m in the direction used and is checked at $m=1.74\cdot10^9$. □

Lemma 7.32. *Let $0<a<b<c<d$ be an irregular Diophantine quadruple with $\deg\{a,b,c\}\ge2$,*

$$10^{75}\le c<10^{90}\qquad\text{and}\qquad c\le b^2,$$

and let z have a type II representation (λ,m,n). Then $c<10\,m^6$.

Proof. Suppose, for contradiction, that $c\ge10m^6$. Then Lemma 7.31 applies: $\lambda=-1$, $m<7.2\cdot10^{12}$, and with $\ell=d_-(a,g_1,h_1)$ items 1 to 3 hold. By Lemma 7.29 and Lemma 2.14, $m/5<\Delta\le n$, i.e. $n/4<\Delta\le n$.

The orbit of $\{a,\ell\}$ through g_1. Put $y_0 = g_1$ and, as long as $y_i > \max(a,\ell)$, $y_{i+1} = d_-(a,\ell,y_i)$. By item 2, $\{a,\ell,y_0\}$ is a Diophantine triple with largest element y_0. If $\{a,\ell,y_i\}$ (largest element y_i) is Euler, then $y_{i+1} = 0$ and $d_+(a,\ell,0) = a+\ell+2\sqrt{a\ell+1} = y_i$; otherwise Theorem 2.7 gives that y_{i+1} is a positive integer, $\{a,\ell,y_{i+1}\}$ is a Diophantine triple, $d_+(a,\ell,y_{i+1}) = y_i$ and $a\ell y_{i+1} < a\ell y_i/12$. The sequence decreases strictly, so there is a first $j \geq 1$ with $x^* := y_j \leq \max(a,\ell)$. Then $x^* \geq 0$, ax^*+1 and ℓx^*+1 are squares, g_1 is obtained from x^* by j applications of $x \mapsto d_+(a,\ell,x)$, and $h_1 = d_+(a,\ell,g_1)$ by item 2.

A finite family. Let $J_*(a)$ be the largest integer $J \geq 1$ with $10^6 aJ^4 < 10486 \cdot 7.2 \cdot 10^{12} + 954300\,a^2J$; the set of such J is an interval, since $10^6 aJ^4 - 954300a^2J$ is convex in J. By item 3 and $m < 7.2 \cdot 10^{12}$, $J \leq J_*(a)$, $a \leq a\ell < 8.0001 J_*(a)$ and $a \leq 2705$. Hence (a,ℓ) is one of the pairs with $a \leq 2705$, $\ell \geq 1$, $\ell \neq a$, $a\ell+1$ a square and $a\ell < 8.01 J_*(a)$; x^* is one of the integers in $[0,\max(a,\ell)]$ with ax^*+1, ℓx^*+1 squares; and g_1 is an iterate of $x \mapsto d_+(a,\ell,x)$ from x^*. Given (a,ℓ,g_1), the integers $h_1 = d_+(a,\ell,g_1)$, $e = d_+(a,g_1,h_1)$, $P = 4ae+2$, $f = Pg_1 - h_1 + 2(a+e)$, $b = Pf - g_1 + 2(a+e)$, $c = Pb - f + 2(a+e)$ are determined (the last three by item 4 of Lemma 7.31), and c increases along the iterates; so only the finitely many iterates with $10^{75} \leq c < 10^{90}$ occur. For each such seed, item 3 confines J to the integers with $a\ell/8.01 < J < h_1/(1.8g_1)$ and $J \leq J_*(a)$. For each such J, $N = PJ = 2n^2+2n+\Delta$ with $n/4 < \Delta \leq n$ lies in $(2n^2+2.25n,\, 2n^2+3n]$; these intervals are pairwise disjoint, so N determines n, then $\Delta = N - 2n^2 - 2n$ and $m = n+\Delta$; and item 1 requires

$$a(2m^2+m+n) + e\Delta = J(Pg_1 - h_1). \tag{1}$$

The finite family contains no solution of (1). An exhaustive enumeration in exact integer arithmetic runs over exactly the family just described: 786 values of a (every other $a \leq 2705$ has $a \geq 8.01 J_*(a)$), 521 pairs $\{a,\ell\}$, 996 orbit bottoms x^*, 632 seeds with $10^{75} \leq c < 10^{90}$, 113860 values of J, of which 20689 give an n in one of these intervals; for every seed it verifies $f = d_+(a,e,g_1)$, $b = d_+(a,e,f)$, $c = d_+(a,e,b)$, $4ag_1h_1 + 2(a+g_1+h_1) - e = \ell$ and that $xy+1$ is a square for the nine pairs from $\{a,e,g_1,h_1,\ell\}$ involved. No pair (J,n) satisfies (1). This contradicts item 1 of Lemma 7.31. Hence $c < 10m^6$. □

7.4 Type I: the bound $c < 10^{99}$

The type I bound comes in two steps below the a priori bound $c < 10^{163}$. The analytic bound $c < 10^{106}$ (Proposition 7.34) plays a gap of exponent 5/39 against the caps. On $10^{99} \leq c < 10^{106}$, caps on short windows of c (Proposition 7.39) meet floors from Pell covers of the seeds (Lemma 7.43, Lemma 7.44).

Proposition 7.33. *Let $0 < a < b < c < d$ be an irregular Diophantine quadruple with $\deg\{a,b,c\} \geq 2$, and let z have a type I representation. Then*

$$c < 10^{99}.$$

Proof. The proof plays the finer type I caps of Proposition 7.39 against the exact exclusions of Lemma 7.43 and Lemma 7.44, one range of c at a time. By Proposition 7.34, $c < 10^{106}$. Suppose, to the contrary, that $c \geq 10^{99}$. Since $10^{65} \leq c < 10^{111}$, Proposition 5.14, type I, on $[10^{65}, 10^{111})$, gives

$$m' < 8.5 \cdot 10^{12}.$$

We exclude the four ranges $[10^{103}, 10^{106})$, $[10^{102}, 10^{103})$, $[10^{100}, 10^{102})$ and $[10^{99}, 10^{100})$ in turn; in each, the cap and the exclusion meet at the same value of m'.

If $10^{103} \le c < 10^{106}$, the hypotheses of row E1 of Proposition 7.39 hold, as $c < 10^{107}$ and $m' < 8.5 \cdot 10^{12} < 1.5 \cdot 10^{13}$; so $m' < 7 \cdot 10^{12}$. But row Y1 of Lemma 7.44 ($10^{103} \le c < 10^{130}$) gives $m' \ge 7 \cdot 10^{12}$.

If $10^{102} \le c < 10^{103}$, the hypotheses of row E2 of Proposition 7.39 hold, as $10^{100} \le c < 10^{103}$ and $m' < 8.5 \cdot 10^{12} < 9.15 \cdot 10^{12}$; so $m' < 6.2 \cdot 10^{12}$. But row Y2 of Lemma 7.44 ($10^{102} \le c < 10^{130}$) gives $m' \ge 6.2 \cdot 10^{12}$.

If $10^{100} \le c < 10^{102}$, then row E2 of Proposition 7.39 gives $m' < 6.2 \cdot 10^{12}$ as before, and row E3 then gives $m' < 5.6 \cdot 10^{12}$; but row X2 of Lemma 7.43 gives $m' \ge 5.6 \cdot 10^{12}$.

If $10^{99} \le c < 10^{100}$, the hypotheses of row E4 of Proposition 7.39 hold, as $m' < 8.5 \cdot 10^{12} < 9.15 \cdot 10^{12}$; so $m' < 5.45 \cdot 10^{12}$. But row X1 of Lemma 7.43 gives $m' \ge 5.45 \cdot 10^{12}$.

Every range leads to a contradiction, so $c < 10^{99}$. □

The analytic bound. Two gaps, $c < 2^{12} m^{58/7}$ for $c \ge 10^{90}$ (Lemma 7.37) and $c < 2^{24} m^{39/5}$ for $10^{90} \le c < 10^{130}$ and $m < 2.45 \cdot 10^{13}$ (Lemma 7.38), played against the type I caps of Proposition 5.14, give $c < 10^{106}$. Lemma 7.36 puts a type I representation with $c \ge 2^{15} m^8$ into a normal form: a positive defect μ_j (Lemma 7.17), a seed of exponent $N_{\mathrm{s}} \ge 7$ for the pair $\{e, a\}$, with $N_{\mathrm{s}} = 7$ unless $a < m^{12/11}$, and the nonzero integer $W^2 - e\mu_j$ of Lemma 7.35. The two gaps bound this integer against the normal form, and the second needs one finite cover.

Proposition 7.34. *Let $0 < a < b < c < d$ be an irregular Diophantine quadruple with $\deg\{a, b, c\} \ge 2$, and let z have a type I representation. Then*

$$c < 10^{106}.$$

Proof. The bound comes from two applications of the type I index caps of Proposition 5.14 against the gaps $c < 2^{12} m^{58/7}$ and $c < 2^{24} m^{39/5}$. Suppose, to the contrary, that $c \ge 10^{106}$.

By item 1 of Proposition 7.7, $c < 10^{163}$, so Proposition 5.14, type I, on $[10^{65}, 10^{163})$, gives $m' < 1.75 \cdot 10^{13}$. If $e > b$, Lemma 7.1 gives $m' > 6.24\, c^{1/4} \ge 6.24 \cdot 10^{26.5} > 1.75 \cdot 10^{13}$, which is impossible. Equality $e = b$ is impossible, since $b^2 + 1$ is not a square. Hence $e < b$.

Suppose $c \ge 10^{111}$. Then $m < 8.75 \cdot 10^{12}$, and Lemma 7.37 (which applies since $c \ge 10^{90}$) gives

$$c < 2^{12} m^{58/7} < 2^{12} \, (8.75 \cdot 10^{12})^{58/7} < 10^{110.85},$$

a contradiction.

Hence $10^{106} \le c < 10^{111}$, and Proposition 5.14, type I, on $[10^{65}, 10^{111})$, gives $m' < 8.5 \cdot 10^{12}$, so $m < 4.25 \cdot 10^{12} < 2.45 \cdot 10^{13}$, and Lemma 7.38 (which applies since $10^{90} \le c < 10^{130}$) gives

$$c < 2^{24} m^{39/5} < 2^{24} \, (4.25 \cdot 10^{12})^{39/5} < 10^{105.73},$$

a contradiction. Hence $c < 10^{106}$.

□

Lemma 7.35. *Let a, e, u, X be positive integers and let $h \in \mathbb{Z}_{\ge 0}$, $Z \in \mathbb{Z}$, with*

$$u^2 = ae + 1, \qquad Z^2 = ah + 1, \qquad X^2 = eh + 1.$$

Let $j, q \in \mathbb{Z}$, $j \ne 0$, and $\lambda \in \{-1, 1\}$, with

$$q \equiv -2\lambda j u Z \pmod{a}.$$

Then

$$\mu_j = \frac{4j^2 - q^2}{a}, \quad W = \frac{2ju + \lambda Zq}{a}, \quad V = \frac{2j + \lambda uZq}{a}$$

are integers, and

$$E := W^2 - e\mu_j = V^2 - ehq^2,$$

$$a^2 E = 4j^2 + 4\lambda juZq + (Z^2 + u^2 - 1)q^2.$$

If $h > 0$, then $E \neq 0$. If $h = 0$, then $E = 0$ if and only if $uq = -2\lambda jZ$.

Proof. Modulo a, one has $u^2 \equiv Z^2 \equiv 1$. Squaring the given congruence proves $\mu_j \in \mathbb{Z}$. Multiplication by λZ, respectively λuZ, proves integrality of W, respectively V. Expanding their squares gives the displayed identities, using

$$(u^2 - 1)(Z^2 - 1) = a^2 eh.$$

If $h > 0$, then $X \geq 2$, so $eh = X^2 - 1$ is strictly between the consecutive squares $(X-1)^2$ and X^2. If $E = 0$ and $q \neq 0$, the identity $V^2 = ehq^2$ would make eh a rational square, hence an integer square, a contradiction. If $q = 0$, then $E = V^2$, and $V = 2j/a \neq 0$. Thus $E \neq 0$.

If $h = 0$, then $Z^2 = 1$ and

$$a^2 E = (uq + 2\lambda jZ)^2.$$

The final equivalence follows. □

Lemma 7.36. *Let $0 < a < b < c < d$ be an irregular Diophantine quadruple with $\deg\{a,b,c\} \geq 2$ and $e = d_-(a,b,c) < b$, let z have a type I representation (λ, m, n), and suppose*

$$c \geq 10^{90}, \qquad c \geq 2^{15} m^8.$$

Let k be the second-order defect of the representation, let j, Y and T_δ be the integers of Lemma 2.22, and put

$$u = \sqrt{ae+1}, \qquad \delta = m^2 - n^2, \qquad q = T_\delta - 2\lambda Y, \qquad \mu_j = \frac{4j^2 - q^2}{a}.$$

Then the following hold.

(i) $m > 2^{156/5}$, $m/2 \leq n < m$, $m > 2u$, $a < m^2/9$, $m \leq \delta < m^2$, $4 \leq k < m^2/u^2$, $k < m^2/a$, $0 < |j| < 6a^{3/2} m\delta/\sqrt{c}$, $u < \frac{71}{100} m^{3/5}$ *and* $a < \frac{5041}{10000} m^{6/5}$.

(ii) j *has the sign* λ, $|jc - 2\lambda asm\delta| < 5a^{3/2} m^3 \delta$, $|q| < 100\sqrt{a}\, m^4/\sqrt{c}$, *and* μ_j *is a positive integer. Consequently* $c < 144a^2 m^6$, $a > \sqrt{c}/(12m^3)$ *and* $e < a/1024$.

(iii) $am(4\delta q - j^2 m) = 4j\delta(n^2 - 1) + 12jkn^2 - 2jmq - \mu_j$, *and* $q/j > m/(8a)$.

(iv) Let $X_0 \geq 1$, Y_0 and $N_{\mathrm{s}} \geq 0$ be integers given by Lemma 2.12 for $(P,Q,S) = (e,a,u)$ and the solution $(X,Y) = (\sqrt{ec+1}, s)$, so that

$$s\sqrt{e} + \sqrt{ec+1}\,\sqrt{a} = (Y_0\sqrt{e} + X_0\sqrt{a})\,\eta^{N_{\mathrm{s}}}, \qquad \eta = u + \sqrt{ae},$$

$eY_0^2 - aX_0^2 = e - a$ and $Y_0^2 \leq a(a-e)/(2(u+1))$. Then $h = (X_0^2 - 1)/e$ is a nonnegative integer, $Y_0^2 = ah + 1$, $X_0^2 < (u+1)/2$, $c < 4X_0^2 \eta^{2N_{\mathrm{s}}}/e$ and $N_{\mathrm{s}} \geq 7$. If $N_{\mathrm{s}} \geq 8$, then $a < m^{12/11}$.

(v) If $N_{\mathrm{s}} = 7$, then $W = (2ju + \lambda Y_0 q)/a$ and $V = (2j + \lambda u Y_0 q)/a$ are integers, and $W^2 - e\mu_j = V^2 - ehq^2$ is a nonzero integer, equal to

$$\frac{4j^2 + 4\lambda j u Y_0 q + (Y_0^2 + u^2 - 1)q^2}{a^2};$$

moreover $c > 2^{11} X_0^2 a^7 e^6$ if $Y_0 > 0$, while $c < 2^{11} a^7 e^6 / X_0^2$ if $Y_0 < 0$ and $X_0 \geq 2$.

Proof. Items (i) to (iii) follow from the type I size bounds and the identity of Lemma 2.22. In (iv) the hypothesis $c \geq 2^{15} m^8$ forces the solution $(\sqrt{ec+1}, s)$ of $eY^2 - aX^2 = e - a$ to arise from its seed with exponent $N_{\mathrm{s}} \geq 7$, and in (v) $N_{\mathrm{s}} = 7$ yields the nonzero integer $W^2 - e\mu_j$.

Items (i) and (ii). By Lemma 7.13, $c < 2^{18} m^9$; with $c \geq 10^{90} > 2^{1494/5}$ this gives $m > 2^{156/5}$. Put $\rho = 11/10$. Since $c \geq 2^{15} m^8$, we have $c > 100m^4$, $c \geq m^8/20$ and $c \geq 4000 m^8$, so items (i) and (ii) of Lemma 7.13, Lemma 7.14 and row 1 of item (ii) of Lemma 7.17 apply. The first two give (i) and the first two claims of (ii); the comparisons below use only the weaker $u < \rho m^{3/5}$ and $a < u^2 < \rho^2 m^{6/5}$. Row 1 of Lemma 7.17 gives $|q| < 9\sqrt{a}\, m^4/\sqrt{c} < 100\sqrt{a}\, m^4/\sqrt{c}$, $\mu_j \in \mathbb{Z}_{\geq 1}$ and $c < (201/50)^2 a^2 m^6 < 144 a^2 m^6$, that is $a > \sqrt{c}/(12m^3)$. With $c \geq 2^{15} m^8$ this gives $a > \sqrt{2^{15}}\, m/12 > 15m$, and then $e < u^2/a$ and (i) give $e/a < u^2/a^2 < \rho^2 m^{6/5}/(225 m^2) < 1/1024$. We record

$$0 < |j| < \frac{6a^{3/2} m\delta}{\sqrt{c}}, \qquad u < \rho m^{3/5}, \qquad a < \rho^2 m^{6/5}. \tag{1}$$

Item (iii). The identity of Lemma 2.22 (4), with $2\lambda Y = T_\delta - q$, $T_\delta - ajm = 2a^2 m\delta$ and $a\mu_j = 4j^2 - q^2$, gives

$$am(4\delta q - j^2 m) = 4j\delta(n^2 - 1) + 12jkn^2 - 2jmq - \mu_j. \tag{2}$$

Dividing (2) by j gives

$$4am\delta\, \frac{q}{j} = ajm^2 + 4\delta(n^2 - 1) + 12kn^2 - 2mq - \frac{\mu_j}{j}.$$

We show that the right side exceeds $2\delta n^2$. The term $12kn^2$ is positive. By $m \leq 2n$, $|ajm^2| \leq 4a|j|n^2$, and by (1),

$$\frac{4a|j|}{\delta} < \frac{24a^{5/2} m}{\sqrt{c}} < \frac{24\rho^5 m^4}{\sqrt{c}} \leq \frac{24\rho^5}{\sqrt{2^{15}}} < 0.2136.$$

By $\delta \geq m$, $n \geq m/2$ and $0 < \mu_j \leq 4j^2/a$, the remaining terms 4δ, $2m|q|$ and $\mu_j/|j|$, divided by δn^2, are at most $4/n^2$, $8|q|/m^2$ and $16|j|/(am^3)$, and each of these is less than 0.001. Hence the right side exceeds $(4 - 0.2136 - 0.003)\delta n^2 > 2\delta n^2$, and

$$\frac{q}{j} > \frac{2\delta n^2}{4am\delta} = \frac{n^2}{2am} \geq \frac{m}{8a}.$$

Item (iv). By (ii), $e < a$, so Lemma 2.12 applies to $(P, Q, S) = (e, a, u)$ and the positive solution $(\sqrt{ec+1}, s)$ of $eY^2 - aX^2 = e - a$. It gives X_0, Y_0 and N_{s} as in the statement, with $X_0^2 \leq e(a-e)/(2(u-1))$ and $Y_0^2 \leq (u-1)(a-e)/(2e)$. Since $u^2 - 1 = ae$, we have $(u-1)/e = a/(u+1)$ and $e/(u-1) = (u+1)/a$; hence $Y_0^2 \leq a(a-e)/(2(u+1))$ and $X_0^2 \leq (a-e)(u+1)/(2a) < (u+1)/2 < u$. As $u^2 - ae = 1$, $\gcd(a, e) = 1$, and $e(Y_0^2 - 1) = a(X_0^2 - 1)$; so $h = (X_0^2 - 1)/e$ is a nonnegative integer and $Y_0^2 = ah + 1$.

Since $(Y\sqrt{e}+X\sqrt{a})(X\sqrt{a}-Y\sqrt{e})=a-e$ for both solutions and $\eta^{-1}=u-\sqrt{ae}$,

$$2\sqrt{ec+1}\,\sqrt{a}=(Y_0\sqrt{e}+X_0\sqrt{a})\eta^{N_{\rm s}}+(X_0\sqrt{a}-Y_0\sqrt{e})\eta^{-N_{\rm s}}, \tag{3}$$

and both terms are positive, as $eY_0^2<aX_0^2$. Both coefficients are less than $2X_0\sqrt{a}$, so $ec+1<X_0^2(\eta^{N_{\rm s}}+\eta^{-N_{\rm s}})^2\le 4X_0^2\eta^{2N_{\rm s}}$, and $c<4X_0^2\eta^{2N_{\rm s}}/e$.

The case $N_{\rm s}\le 6$ is impossible. As $\eta^2=2ae+1+2u\sqrt{ae}<4ae+2\le 6ae$ and $X_0^2<u$, it would give

$$c<4u(6ae)^6/e\le 4\cdot 6^6u^{13}<4\cdot 6^6\rho^{13}m^{39/5},$$

and this is less than $2^{15}m^8\le c$ whenever $m>2.94\cdot 10^6$, in particular for $m>2^{156/5}$. Hence $N_{\rm s}\ge 7$.

Suppose $N_{\rm s}\ge 8$. Since $0<X_0\sqrt{a}-Y_0\sqrt{e}<2X_0\sqrt{a}$ and $a-e>a/2$,

$$Y_0\sqrt{e}+X_0\sqrt{a}=\frac{a-e}{X_0\sqrt{a}-Y_0\sqrt{e}}>\frac{\sqrt{a}}{4X_0}.$$

By (3), $2\sqrt{ec+1}\sqrt{a}>(Y_0\sqrt{e}+X_0\sqrt{a})\eta^{N_{\rm s}}$, so $ec+1>\eta^{2N_{\rm s}}/(64X_0^2)$. As $\eta^2>4ae$, $\eta^{2N_{\rm s}}>(4ae)^8>128X_0^2$, and $X_0^2<u<\sqrt{2ae}$,

$$c>\frac{\eta^{2N_{\rm s}}}{128eX_0^2}>\frac{4^8a^8e^7}{128\sqrt{2ae}}=\frac{512}{\sqrt{2}}a^{15/2}e^{13/2}.$$

Combined with $c<144a^2m^6$ this gives $a^{11/2}<(144\sqrt{2}/512)m^6<0.398\,m^6$, hence $a<m^{12/11}$.

Item (v). Let $N_{\rm s}=7$. By (3), $s=Y_0T_7+aX_0U_6$ with $T_7=T_7(u)$, $U_6=U_6(u)$, and $T_7\equiv u \pmod a$ since $u^2\equiv 1 \pmod a$; hence $s\equiv uY_0 \pmod a$. As $a\mid T_\delta$ and $Y\equiv js \pmod a$, we get $q\equiv -2\lambda js\equiv -2\lambda juY_0 \pmod a$. Thus Lemma 7.35, applied with $Z=Y_0$ and $X=X_0$, shows that W and V are integers, and that the integer $E=W^2-e\mu_j$ equals V^2-ehq^2 and satisfies

$$a^2E=4j^2+4\lambda juY_0q+(Y_0^2+u^2-1)q^2.$$

If $h>0$, this is Lemma 7.35. If $h=0$, then $Y_0^2=1$, and by that lemma $E=0$ would give $uq=-2\lambda jY_0$, so $|q/j|=2/u$. With (iii) this gives $16a>mu$; since $a<u^2/e$, it follows that $me<16u<16\rho m^{3/5}$ by (1), so $m<(16\rho)^{5/2}<1300$, against (i). Hence $E\ne 0$.

Here $\eta^2=2ae+1+2u\sqrt{ae}>4ae$, and $ae>200$, since otherwise $c<4X_0^2\eta^{14}/e<4u(4u^2)^7\le 4^8\cdot 201^{15/2}<10^{90}$; so $\eta^2<4ae+2<4.01ae$.

If $Y_0>0$, then (3) gives $2\sqrt{ec+1}\sqrt{a}>X_0\sqrt{a}\,\eta^7$, so $4(ec+1)>X_0^2\eta^{14}$ and

$$c>\frac{X_0^2\eta^{14}}{8e}>\frac{X_0^2(4ae)^7}{8e}=2^{11}X_0^2a^7e^6.$$

If $Y_0<0$ and $X_0\ge 2$, put $W_+=|Y_0|\sqrt{e}+X_0\sqrt{a}$ and $W_-=X_0\sqrt{a}-|Y_0|\sqrt{e}=(a-e)/W_+$. Since $eY_0^2=aeh+e>a(X_0^2-1)$,

$$W_+^2>a\bigl(X_0+\sqrt{X_0^2-1}\bigr)^2>2.9aX_0^2,$$

so $W_-^2<a/(2.9X_0^2)$. By (3), $2\sqrt{ec+1}\sqrt{a}=W_-\eta^7(1+\vartheta)$ with $\vartheta=W_+^2/((a-e)\eta^{14})<8X_0^2\eta^{-14}<8/(3^7u^{13})$, as $W_+^2<4aX_0^2$, $a-e>a/2$ and $\eta^2>3u^2$; hence $(1+\vartheta)^2<1.001$. Consequently

$$ec+1<\frac{1.001\,\eta^{14}}{4\cdot 2.9X_0^2}<\frac{\eta^{14}}{11X_0^2},\qquad c<\frac{\eta^{14}}{11eX_0^2}<\frac{4.01^7}{11}\,\frac{a^7e^6}{X_0^2}<2^{11}\frac{a^7e^6}{X_0^2},$$

using $4.01^7/11 < 1516 < 2^{11}$.

Every variable term in the comparisons of (iii) to (v) decreases with m, and the remaining terms are constant; they are $24\rho^5/\sqrt{2^{15}} < 0.2136$, $(4\cdot 6^6\rho^{13}/2^{15})^5 < 2.94\cdot 10^6 < 2^{156/5}$, $144\sqrt{2}/512 < 0.398$, $(16\rho)^{5/2} < 1300$, $4.01^7/11 < 1516$ and $4^8\cdot 201^{15/2} < 10^{90}$. □

Lemma 7.37. *Let $0 < a < b < c < d$ be an irregular Diophantine quadruple with $\deg\{a,b,c\} \geq 2$, let $e = d_-(a,b,c) < b$, and suppose z has a type I representation. Then*

$$c \geq 10^{90} \quad\Longrightarrow\quad c < 2^{12} m^{58/7}.$$

Consequently every type I representation, including the branch $e > b$, satisfies

$$c \geq 10^{90} \quad\Longrightarrow\quad m' = 2m > 2^{-13/29} c^{7/58}.$$

Proof. Suppose that $c \geq 10^{90}$ and, to the contrary, $c \geq 2^{12} m^{58/7}$. By Lemma 7.13, $c < 2^{18} m^9$, so $m > 2^{156/5} > 1449 > 8^{7/2}$; hence $m^{2/7} > 8$ and $c \geq 2^{12} m^{58/7} > 2^{15} m^8$. Thus Lemma 7.36 applies, and we use its notation: $u = \sqrt{ae+1}$, $\delta = m^2 - n^2$, j, q, μ_j, the seed (Y_0, X_0) with exponent N_{s}, h and $\eta = u + \sqrt{ae}$; put $\rho = 11/10$ and $K = X_0^2$. By its items (i) and (ii),

$$0 < |j| < \frac{6a^{3/2} m\delta}{\sqrt{c}} < \frac{6a^{3/2} m^3}{\sqrt{c}}, \qquad |q| < \frac{100\sqrt{a}\, m^4}{\sqrt{c}}, \qquad u < \rho m^{3/5}, \tag{1}$$

$$c < 144 a^2 m^6, \qquad a > \frac{\sqrt{c}}{12 m^3} \geq \frac{2^6 m^{29/7}}{12 m^3} = \frac{16}{3} m^{8/7}. \tag{2}$$

Since $\frac{16}{3} m^{8/7} > m^{12/11}$, item (iv) gives $N_{\mathrm{s}} = 7$, and item (v) gives a nonzero integer E with

$$a^2 E = 4j^2 + 4\lambda j u Y_0 q + (Y_0^2 + u^2 - 1) q^2. \tag{3}$$

Each seed case gives $|E| < 1$, a contradiction.

If $Y_0 > 0$, item (v) gives $c > 2^{11} K a^7 e^6$. Together with (2), this yields

$$K e^6 < B_+ := \frac{9}{128} \frac{m^6}{a^5}, \qquad B_+ > 1.$$

Using $uY_0 < \sqrt{2}\, a\sqrt{K}$, $Y_0^2 + u^2 - 1 < a(e + K/e) \leq 2aB_+$, (1) and $\sqrt{K} < \sqrt{B_+}$, equation (3) gives

$$|E| < \frac{144 a m^6}{c} + \frac{900 m^{10}}{a^{3/2} c} + \frac{(5625/4) m^{14}}{a^5 c}.$$

Using $a < \rho^2 m^{6/5}$ in the first term and (2) in the others, this is at most

$$\frac{144\rho^2 m^{36/5}}{c} + 21600\sqrt{3}\, \frac{m^{29/2}}{c^{7/4}} + 349920000\, \frac{m^{29}}{c^{7/2}} < 0.018. \tag{4}$$

Suppose $Y_0 < 0$ and $K \geq 2$. Item (v) gives $c < 2^{11} a^7 e^6 / K$, so $K < 2048 a u^{12}/c$. From (3), (1) and (2) we obtain

$$|E| < \frac{144\rho^2 m^{36/5}}{c} + 153600\rho^9 \frac{m^{62/5}}{c^{3/2}} + 20480000\rho^{14} \frac{m^{82/5}}{c^2} + 240000\rho^2 \frac{m^{61/5}}{c^{3/2}} < 0.867. \tag{5}$$

Here the last term bounds the contribution of e, using $e < u^2/a < 12\rho^2 m^{21/5}/\sqrt{c}$.

Finally, if $Y_0 < 0$ and $K = 1$, then $h = 0$ and $Y_0 = -1$. Directly in (3),

$$|E| < \frac{144\rho^2 m^{36/5}}{c} + \frac{2400\rho m^{38/5}}{c} + 240000\rho^2 \frac{m^{61/5}}{c^{3/2}} < 0.008. \tag{6}$$

The bounds (4) to (6) hold at $m = 2^{156/5}$, $c = 2^{12} m^{58/7}$, and decrease in c and then in m: their endpoint values are below 0.017920, 0.866470 and 0.007902. Thus $|E| < 1$ in every case, so $E = 0$, contrary to item (v) of Lemma 7.36.

Hence $c < 2^{12} m^{58/7}$ when $c \geq 10^{90}$, and

$$m' = 2m > 2^{1-84/58} c^{7/58} = 2^{-13/29} c^{7/58}.$$

The equality case $e = b$ is impossible because $b^2 + 1$ is not a square. If $e > b$, Lemma 7.1 gives the stronger $m' > 6.24 c^{1/4}$. □

Lemma 7.38. *Let $0 < a < b < c < d$ be an irregular Diophantine quadruple with $\deg\{a, b, c\} \geq 2$ and $10^{90} \leq c < 10^{130}$, and suppose z has a type I representation with $m < 2.45 \cdot 10^{13}$. If $e = d_-(a, b, c) < b$, then*

$$c < 2^{24} m^{39/5}.$$

Consequently, under the same hypotheses and including the case $e > b$,

$$m' = 2m > 2^{-27/13} c^{5/39}.$$

Proof. Under $c \geq 2^{24} m^{39/5}$ the normal form of Lemma 7.36 leaves a nonzero integer E with $|E| < 3X_0$; an exact enumeration of the few seeds and orbits compatible with this bound then contradicts the divisibility $m \mid Q_0$ of (12).

First suppose $e < b$ and, to the contrary, $c \geq Cm^p$, with $C = 2^{24}$ and $p = 39/5$. The hypothesis $m < 2.45 \cdot 10^{13}$, Lemma 7.13 and the hypothesis $c < 10^{130}$ give

$$2^{156/5} < m < M := 24500000000000, \qquad c < 2^{18} m^9, \qquad c < 10^{130}. \tag{1}$$

Since $M < 2^{45}$, $Cm^p = 2^{15} m^8 \cdot 2^9 m^{-1/5} > 2^{15} m^8$, so Lemma 7.36 applies; we use its notation $u = \sqrt{ae + 1}$, $\delta = m^2 - n^2$, k, j, Y, T_δ, q, μ_j, the seed (Y_0, X_0) with exponent N_{s}, h and η; put $\rho = 11/10$ and $K = X_0^2$. By its items (i) and (ii),

$$m > 2u, \quad a < m^2/9, \quad m \leq \delta < m^2, \quad 4 \leq k < m^2/u^2, \quad k < m^2/a,$$

$$\operatorname{sign}(j) = \lambda, \quad u < \rho m^{3/5}, \quad a < \rho^2 m^{6/5}, \quad \mu_j \in \mathbb{Z}_{\geq 1}, \quad c < 144 a^2 m^6. \tag{2}$$

Put $A = 201/100$ and $B = 9$. Since $c \geq Cm^p > 2^{15} m^8 > 4000 m^8$ and $c \geq 10^{90}$, row 1 of item (ii) of Lemma 7.17 gives

$$|j| < A a^{3/2} m\delta/\sqrt{c}, \quad |q| < B\sqrt{a}\, m^4/\sqrt{c}, \quad c < 4A^2 a^2 m^6. \tag{3}$$

Consequently

$$a > \frac{\sqrt{C}}{2A} m^{p/2-3} > 46.6892m > 46m,$$

$$e < \frac{2A\rho^2 m^{21/5}}{\sqrt{c}} < 12.342448, \qquad \mu_j < \frac{4A^2\rho^4 m^{42/5}}{c} < 152.336019. \tag{4}$$

In particular $e \leq 12$, $\mu_j \leq 152$ and $e/a < 1/1024$.

By item (iv), $h = (X_0^2 - 1)/e$ is a nonnegative integer,

$$Y_0^2 = ah + 1, \qquad K = X_0^2 = eh + 1 < (u+1)/2, \qquad 2eY_0^2 \le (u-1)(a-e). \tag{5}$$

Since $M^{1/11} < 16.5 < 46$, (4) gives $a > m^{12/11}$, so $N_{\mathrm{s}} = 7$ by item (iv). Item (v) then gives integers $W = (2ju + \lambda Y_0 q)/a$, $V = (2j + \lambda u Y_0 q)/a$ and a nonzero integer $E = W^2 - e\mu_j = V^2 - ehq^2$ with

$$a^2 E = 4j^2 + 4\lambda j u Y_0 q + (Y_0^2 + u^2 - 1)q^2, \tag{6}$$

and item (iii) gives $q/j > m/(8a) > 0$.

Suppose $h = 0$. Then $Y_0^2 = 1$, and directly from (3), (4) and (6),

$$|E| < 4A^2 a m^6/c + 4ABum^7/c + B^2(e + 1/a)m^8/c < 0.027592 < 1,$$

so $E = 0$, contrary to item (v). Hence $h \ge 1$.

Put $X = X_0 \ge 2$ and $Z = |Y_0|$. The identity (6) gives

$$\frac{|E|}{X} < \frac{4A^2 a m^6}{cX} + 4AB\sqrt{2}\frac{am^7}{c} + B^2\left(\frac{e}{X} + \frac{X}{e}\right)\frac{m^8}{c}. \tag{7}$$

Here $e/X < \sqrt{e} < \sqrt{2A}\rho m^{21/10}/c^{1/4}$. The growth $c > 2048X^2a^7e^6$ of item (v) for $Y_0 > 0$ gives

$$F_+ = \frac{4A^2\rho^2 m^{36/5}}{c} + 4AB\sqrt{2}\rho^2\frac{m^{41/5}}{c} + B^2\sqrt{2A}\rho\frac{m^{101/10}}{c^{5/4}} + \frac{B^2(2A)^{7/2}}{\sqrt{2048}}\frac{m^{37/2}}{c^{9/4}}$$

as an upper bound for (7) when $Y_0 > 0$. For $Y_0 < 0$, item (v) gives $c < 2048a^7e^6/X^2$; replacing the last term by $B^2\sqrt{2048}\rho^7 m^{61/5}/c^{3/2}$ gives the bound F_-. At $c = Cm^p$, evaluate negative powers at $2^{156/5}$ and positive powers at M. The results are

$$F_+ < 1.749913, \quad F_- < 2.196560, \quad 0 < |E| < 3X. \tag{8}$$

Moreover

$$u < 2\cdot 10^8, \quad a < 4\cdot 10^{16}, \quad 2 \le X \le 10000, \quad Q := |q| < 2\cdot 10^7, \quad v := |V| < 3\cdot 10^{11}. \tag{9}$$

Indeed, (2) and $m < M$ give $u < \rho M^{3/5} < 1.19\cdot 10^8$ and $a < \rho^2 M^{6/5} < 1.42\cdot 10^{16}$; (5) gives $X^2 < (u+1)/2 < 6\cdot 10^7$; (2), (3) and $c \ge Cm^p$ give $Q < B\rho\, m^{7/10}/2^{12} < 6\cdot 10^6$; and $v^2 = (X^2-1)Q^2 + E < 2.2\cdot 10^{21}$, since $E = V^2 - ehq^2$ and $eh = X^2 - 1$.

Reduce $v + Q\sqrt{X^2-1}$ by a power θ^ℓ of $\theta = X + \sqrt{X^2-1}$ into $[\sqrt{|E|}\theta^{-1/2}, \sqrt{|E|}\theta^{1/2})$, obtaining $v_0 + q_0\sqrt{X^2-1}$. By the coordinate calculation in the proof of Lemma 2.18,

$$q_0^2 \le E/[2(X+1)]\ (E > 0), \qquad q_0^2 \le |E|/[2(X-1)]\ (E < 0). \tag{10}$$

Thus $|q_0| \le 1$ by (8). The original positive value exceeds $\sqrt{|E|}$, so the reduction exponent ℓ is nonnegative.

Put $\sigma = \mathrm{sign}(Y_0)$. Since q and j have the sign λ, $q/j > m/(8a)$ and $u^2Z^2 = (ae+1)(ah+1) > a^2eh$, we have $uZ|q|/(2|j|) > m\sqrt{eh}/16 > 1$; so $aV = 2\lambda|j| + \sigma uZ|q|$ has the sign σ. Hence $V = \sigma v$, $q = \lambda Q$, and $\mathcal{J} = av - uZQ$ satisfies $2j = \sigma\mathcal{J}$. Since $u^2 - ae = 1$, $uV - \lambda eY_0 q = W$, so $w := \sigma W = uv - eZQ$ is an integer with $w^2 = e\mu_j + E$. Eliminating Z from $w = uv - eZQ$ and $eZ^2 - hu^2 = e - h$ gives

$$Eu^2 - 2vwu + w^2 - e(e-h)Q^2 = 0,$$

whose discriminant is $4e^2Q^2(h\mu_j + E)$. It has the integer root u and $eQ \ne 0$, so $h\mu_j + E$ is the square of an integer z_μ, and for one sign of z_μ,

$$u = (vw + eQz_\mu)/E, \qquad Z = (hQw + vz_\mu)/E. \tag{11}$$

The bounded seed enumeration is the following.

1. For $q_0 = 0$, take $e = 1, \dots, 12$, $r = 1, \dots, 12$, $a_0 = 1, \dots, \lfloor 152/r^2 \rfloor$ with $ea_0 + 1$ square. There are 272 triples. Take all $X = 2, \dots, 10000$ with $h = (X^2 - 1)/e$ integral and $ha_0 + 1$ square. Set $\mu_j = r^2 a_0$, $E = r^2$, $(|v_0|, |q_0|) = (r, 0)$, $|w| = r\sqrt{ea_0 + 1}$ and $|z_\mu| = r\sqrt{ha_0 + 1}$. This covers the principal sector without cancelling any common factor: the unit formula makes w/r integral with square 1 modulo e, whence $r^2 \mid \mu_j$.

2. For $|q_0| = 1$, take the same ranges for e, X, h, put $D = X^2 - 1$, and take every $v_0 \geq 0$ with $\max(0, D - 3X + 1) \leq v_0^2 \leq D + 3X - 1$. Set $E = v_0^2 - D \neq 0$, retain the applicable condition (10), and enumerate nonnegative w with $\max(0, E+e) \leq w^2 \leq E+152e$, $w^2 \equiv E \pmod{e}$. Set $\mu_j = (w^2 - E)/e$ and retain only those with $h\mu_j + E$ a nonnegative square; set $|z_\mu|$ to its root.

This gives 2782 seed tuples. The second step considers 102451 norm pairs before the square tests. For each seed use all independent signs of both coordinate pairs, identifying zero-coordinate duplicates, and iterate

$$(v, Q) \longmapsto (Xv + DQ,\ v + XQ)$$

for $\ell = 0, \dots, 49$, retaining positive coordinates and (9). This exponent cover is conservative: the absolute-coordinate seed sum is less than 130000; its positive value is at least $1/130000$ by its nonzero integer norm. The original value is less than $6 \cdot 10^{11}$, so $\theta^\ell < 6 \cdot 10^{11} \cdot 130000 < 3^{50}$. Reconstruct (11), retaining integer u, Z with

$$0 < u < 2{\cdot}10^8, \quad Z > 0, \quad a = (u^2{-}1)/e \in \mathbb{Z}, \quad e < a < 4{\cdot}10^{16}, \quad h < a, \quad 2eZ^2 \leq (u{-}1)(a{-}e).$$

Identify duplicate $(e, \mu_j, h, u, Z, v, Q)$ and retain only even nonzero $\mathcal{J}$. For both $\sigma = \pm 1$, calculate

$$s = \sigma Z T_7(u) + aXU_6(u), \quad \sqrt{ec + 1} = XT_7(u) + \sigma eZU_6(u), \quad c = (s^2 - 1)/a,$$

retaining positive coordinates and $10^{90} \leq c < 10^{130}$. Set $j = \sigma\mathcal{J}/2$, $\lambda = \operatorname{sign}(j)$, $q = \lambda Q$.

The identities $\mathcal{J}^2 - Q^2 = a\mu_j$ and $aE + \mu_j = (e+h)Q^2 + 2\mathcal{J}v$ imply $\gcd(\mathcal{J}, a) = \gcd(Q, a) \mid \mu_j$. Hence $g = \gcd(j, 4a) \leq 4\mu_j \leq 608$. The definitions of Y and T_δ (Lemma 2.22) give the integer

$$Q_0 = (q + 2\lambda j s)/a = m(4a\delta + j). \tag{12}$$

If $g \nmid Q_0$, exclude the configuration; otherwise take $d = 4a/g$ and the least residue $m_0 \equiv (Q_0/g)(j/g)^{-1} \pmod{d}$. Since $m/d < 152/46 < 4$, every possible positive index is among $m_0, m_0 + d, m_0 + 2d, m_0 + 3d$. Retain only

$$0 < m < a/46, \quad m < M, \quad m^5 > 2^{156}, \quad c^5 \geq 2^{120} m^{39}, \quad c < 2^{18} m^9, \quad m > 2u, \quad 9a < m^2. \tag{13}$$

The condition $c^5 \geq 2^{120} m^{39}$ is the assumption $c \geq Cm^p$. The complete exact evaluation gives 22238 signed secondary orbits, 14458 integral reconstructions, 10128 reduced ones, 4103 with even nonzero $\mathcal{J}$, and 2806 primary configurations in the range $10^{90} \leq c < 10^{130}$. Of 1420 solvable congruences, only six residues have $0 < m < a/46$, and exactly two pass (13):

e	μ_j	m	$Q_0 \bmod m$
8	3	624620274835	186099341498
12	128	17561117320624	14291384337520

Both remainders are nonzero, contradicting $m \mid Q_0$ in (12). The computation is exact; it uses only the displayed recurrences, square tests, rational reconstruction and four modular residues.

Therefore $c < 2^{24} m^{39/5}$. Taking roots gives $m' = 2m > 2^{1-24(5/39)} c^{5/39} = 2^{-27/13} c^{5/39}$. Equality $e = b$ is impossible, and Lemma 7.1 is stronger when $e > b$. □

Caps near 10^{100}. Proposition 7.39 sharpens the caps of Proposition 5.14 on four short ranges of c, using the bounds $u < 0.71m^{3/5}$ and $a < 0.5041\, m^{6/5}$ of the Vieta cutoff (Lemma 7.14).

Proposition 7.39. *Let $0 < a < b < c < d$ be an irregular Diophantine quadruple with $\deg\{a,b,c\} \geq 2$, and let z have a type I representation. If $c \in [c_0, c_1)$ and $m' < p_*$ for one of the rows of the following table, then $m' < R$.*

	$[c_0, c_1)$	p_*	R
E1	$[10^{103}, 10^{107})$	$1.5 \cdot 10^{13}$	$7 \cdot 10^{12}$
E2	$[10^{100}, 10^{103})$	$9.15 \cdot 10^{12}$	$6.2 \cdot 10^{12}$
E3	$[10^{100}, 10^{102})$	$6.2 \cdot 10^{12}$	$5.6 \cdot 10^{12}$
E4	$[10^{99}, 10^{100})$	$9.15 \cdot 10^{12}$	$5.45 \cdot 10^{12}$

Proof. The proof is the type I argument of Proposition 5.14, whose paragraphs we cite by name; we state only what differs: the size bound $\log(2u) < h_0$ (from Lemma 7.14, and in case E4a from Lemma 7.13), the parameters $\mathfrak{a}_i$ with the joint bound (6), and the certified values of each case.

Put $p = m' = 2m$, $\ell_c = \log c$, $A = \log\alpha$, $B = \log\beta$ and $\eta = \beta/\alpha$. The proof treats five cases: the rows E1, E2, E3 and, for row E4, the cases E4a (with $p_* = 9.15 \cdot 10^{12}$ and target $R = 6.8 \cdot 10^{12}$) and E4b (with $p_* = 6.8 \cdot 10^{12}$ and $R = 5.45 \cdot 10^{12}$), applied in this order. Their parameters are the columns of the table at the end. In each case we assume its hypotheses, suppose $p \geq R$, and derive a contradiction. Throughout

$$\rho_{\mathrm{MV}} = 20, \qquad M_{\mathrm{MV}} = \tfrac{21}{8}, \qquad u_{\mathrm{MV}} = \rho_{\mathrm{MV}} + 3 = 23, \qquad v_{\mathrm{MV}} = \rho_{\mathrm{MV}} - 1 = 19.$$

If $e > b$, Lemma 7.1 gives $m' > 6.24c^{1/4} \geq 6.24 \cdot 10^{99/4} > p_*$; and $e = b$ is impossible, as $b^2 + 1$ is not a square. So $e < b$. The paragraph "Type I: sizes" of the proof of Proposition 5.14 gives

$$4.8 < A < \tfrac{\ell_c}{2}, \qquad 0 < B - A < \tfrac{1}{2}\log(c/b), \qquad \tfrac{\ell_c}{2} + \log 2 < B < \tfrac{3}{4}\ell_c. \tag{1}$$

Canonical descent (Theorem 2.7, Lemma 2.6) gives $f = d_-(a,e,b) \geq 1$, $b = d_+(a,e,f) > a + e$ and $c + f = (4u^2 - 2)b + 2(a + e)$, so $c < 4u^2 b$ and

$$0 < B - A < \log(2u). \tag{2}$$

In each case we bound $\log(2u)$ by h_0 and a by a^+, using $m < M_0 := p_*/2$. In E1, $99M_0^8 < 100 \cdot 10^{103} \leq 100c$, so $c \geq \frac{99}{100}m^8 > m^8/20$, and Lemma 7.14 gives $u < 0.71m^{3/5} < 1.21m^{3/5}$, and the E1 values of the table are certified with the weaker constant 1.21; hence $\log(2u) < \log 2.42 + \frac{3}{5}\log M_0 < h_0$ and $a < u^2 < a^+ := 1.21^2 M_0^{6/5}$. In E2 and E3, $M_0^8 < 20 \cdot 10^{100} \leq 20c$, so Lemma 7.14 gives $u < 0.71m^{3/5}$; hence $\log(2u) < \log 1.42 + \frac{3}{5}\log M_0 < h_0$ and $a < a^+ := 0.71^2 M_0^{6/5}$. In E4a, item (i) of Lemma 7.13 gives $2u < m$; hence $\log(2u) < \log M_0 < h_0$ and $a < u^2 < a^+ := M_0^2/4$. In E4b, applied after E4a has given $p < 6.8 \cdot 10^{12}$, we have $m < 3.4 \cdot 10^{12}$ and $c/m^8 > 10^{99}/(3.4 \cdot 10^{12})^8 > 0.0559975 > 1/20$, so Lemma 7.14 gives $u < 0.71m^{3/5}$, $\log(2u) < h_0$ and $a < a^+ := 0.71^2 M_0^{6/5}$. With (1), (2), $L_0 \leq \ell_c < U$ and

$$a_0 = \frac{1}{2} - \frac{h_0}{L_0}, \qquad b_0 = \frac{1}{2} + \frac{h_0 + 0.001}{L_0}, \qquad U_B = b_0 U,$$

we get

$$0 < B - A < h_0, \qquad \frac{\ell_c}{2} < B < \frac{\ell_c}{2} + h_0 + 0.001 \leq b_0\ell_c, \qquad A > \frac{\ell_c}{2} - h_0 \geq a_0\ell_c. \tag{3}$$

The paragraph "Type I: the exact form" of the proof of Proposition 5.14, with the even exponent $2l$ of Lemma 2.17 and the correction D_λ and $\psi = \log D_\lambda$ of Lemma 5.13, gives for $p \geq R$

$$0 < \Lambda := 2lA - pB + \psi, \quad \log\Lambda < -2pB < -p\ell_c, \quad 0 < \Lambda < \psi < 0.312, \quad p < 2l, \quad 2lA < pB, \tag{4}$$

with $\mathrm{N}(D_\lambda) < 1$ and $\mathrm{h}(D_\lambda) < \ell_c - \frac{1}{2}\log a - 0.39$.

We apply Theorem 5.4 to $\Lambda = (2l-p)\log\alpha + \log D_\lambda - p\log\eta$, with $(\eta_1,\eta_2,\eta_3) = (\alpha, D_\lambda, \eta)$ and $(b_1,b_2,b_3) = (2l-p, 1, p)$, as in the paragraph "Type I: the three-logarithm form" of the proof of Proposition 5.14 (positive coefficients with gcd one, multiplicative independence, $\mathcal{D} = 4$, ±1 the only roots of unity in $\mathcal{K}$, $0 < \Lambda < 1$), but with

$$\mathfrak{a}_1 = u_{\mathrm{MV}}A, \qquad \mathfrak{a}_2 = 8\ell_c - 4\log a + 2.809, \qquad \mathfrak{a}_3 = u_{\mathrm{MV}}B - v_{\mathrm{MV}}A.$$

The parameter $\mathfrak{a}_2$ is admissible, since $(\rho_{\mathrm{MV}}-1)\psi + 8\,\mathrm{h}(D_\lambda) < 8\ell_c - 4\log a + 2.808$ by (4), and $\mathfrak{a}_1$, $\mathfrak{a}_3$ are admissible as there. By $1 \leq a < a^+$ and $\ell_c \geq L_0$, the certificates give $d_-\ell_c < \mathfrak{a}_2 < d_+\ell_c$ with $d_\pm$ of the table; with (3), $\mathfrak{a}_1 > u_{\mathrm{MV}}a_0\ell_c > 2\ell_c$ and $\mathfrak{a}_3 > 4B > 2\ell_c$. So $\min\mathfrak{a}_i > \mathfrak{a}_- := 2L_0$, and

$$\Omega := \mathfrak{a}_1\mathfrak{a}_2\mathfrak{a}_3 > \Omega_- := 2u_{\mathrm{MV}}a_0d_-L_0^3, \qquad \mathfrak{a}_1+\mathfrak{a}_2+\mathfrak{a}_3 < \Sigma_+ := \big((\rho_{\mathrm{MV}}+7)b_0 + d_+\big)U,$$

the latter since $\mathfrak{a}_1 + \mathfrak{a}_3 = 4A + u_{\mathrm{MV}}B < (\rho_{\mathrm{MV}}+7)B$.

We bound Ω/B by the joint product. Put

$$\mathcal{F}(B,H) = u_{\mathrm{MV}}\big[4B + (v_{\mathrm{MV}}-4)H - v_{\mathrm{MV}}H^2/B\big], \qquad \mathcal{C}(\ell_c) = 12\ell_c + 8\log 2 + 2.810,$$

so that $\mathfrak{a}_1\mathfrak{a}_3/B = \mathcal{F}(B, B-A)$; and, as for (I.7) in the paragraph "Type I: the three-logarithm form", $\mathfrak{a}_2 < \mathcal{C}(\ell_c) - 8B$. On the box (3) the product $P(B,H) = \mathcal{F}(B,H)(\mathcal{C}(\ell_c)-8B)$ increases with H, because $v_{\mathrm{MV}} - 4 - 2v_{\mathrm{MV}}H/B > v_{\mathrm{MV}} - 4 - 4v_{\mathrm{MV}}h_0/L_0 > 0$; so it is largest at $H = h_0$. Its B-derivative divided by u_{MV} is

$$4\mathcal{C}(\ell_c) - 64B - 8(v_{\mathrm{MV}}-4)H + v_{\mathrm{MV}}H^2\mathcal{C}(\ell_c)/B^2. \tag{5}$$

At $H = h_0$, with $h = h_0/\ell_c \leq h_0/L_0 < 1/4$, the quantity (5) divided by ℓ_c exceeds

$$16 - 8u_{\mathrm{MV}}h + \frac{12v_{\mathrm{MV}}h^2}{(1/2+h+0.001/L_0)^2} - \frac{0.064}{L_0},$$

which decreases in h, since $12v_{\mathrm{MV}}h/(1/2+h)^3 < 8u_{\mathrm{MV}}$ for $h < 1/4$, and is positive at $h = h_0/L_0$ (in E1, E2, E3 and E4b already $16L_0 - 8u_{\mathrm{MV}}h_0 - 0.064 > 0$). So P is largest at $H = h_0$, $B = B_m := \ell_c/2 + h_0 + 0.001$. The factor $\mathcal{F}(B_m,h_0)/(u_{\mathrm{MV}}\ell_c)$ decreases with ℓ_c, and $(\mathcal{C}(\ell_c) - 8B_m)/\ell_c$ increases with ℓ_c; enlarging them separately to L_0 and U gives

$$\frac{\Omega}{B} < \Pi\ell_c^2 < C_\Pi\ell_c^2, \qquad \Pi = u_{\mathrm{MV}}\Big[4b_0 + (v_{\mathrm{MV}}-4)\frac{h_0}{L_0} - v_{\mathrm{MV}}\frac{(h_0/L_0)^2}{b_0}\Big]\Big[8 - \frac{8(h_0-\log 2)-2.810}{U}\Big], \tag{6}$$

and $\Omega < \Omega_+ := C_\Pi b_0 U^3$.

The paragraph "The parametrisation" of the proof of Proposition 5.14 applies with these $\mathfrak{a}_i$, $\mathfrak{a}_-$, $\Omega_\pm$, Σ_+ and the χ_{MV}, ℓ of the table. The certificates check (P) and $\theta_1+\theta_2+\theta_3 < C_*^+$. By (4), $2l - p < p(B-A)/A$, and $\mathfrak{a}_3 \geq u_{\mathrm{MV}}(B-A)$, so

$$\varpi_1 = \frac{p}{\mathfrak{a}_1} + \frac{2l-p}{\mathfrak{a}_3} < \frac{2p}{u_{\mathrm{MV}}A} < \frac{2p_*}{u_{\mathrm{MV}}a_0L_0} =: \varpi_1^+, \qquad \varpi_2 = \frac{p}{\mathfrak{a}_2} + \frac{1}{\mathfrak{a}_3} < \frac{p_*/d_- + 1/2}{L_0} =: \varpi_2^+;$$

so $\log \mathfrak{b} < \mathcal{B}$ with the value $\mathcal{B}$ of the table, and the certificates check (M) with the margin of the table.

Alternatives (O) and (C1). Let $X = U_B$ in E1, E2 and $X = U/2$ in E3, E4a, E4b; by (1) and (3), $\mathfrak{a}_1 < u_{\mathrm{MV}} X$ and $\mathfrak{a}_2 < d_+ U$ in every case. Put $T_0 = \ell\big(1 + C_*^+ u_{\mathrm{MV}} X d_+ U\big)$. Then $\ell T_{\mathrm{MV}} \le T_0$, and the certificates give $T_0 < R/2 \le b_3/2$. So (O) holds, and with (4) and (6), $2pB < M_{\mathrm{MV}} \ell^2 \Omega \log \rho_{\mathrm{MV}}$ gives

$$p < \frac{M_{\mathrm{MV}} \ell^2 \log 20}{2} \cdot \frac{\Omega}{B} < \frac{M_{\mathrm{MV}} \ell^2 C_\Pi U^2 \log 20}{2} < \mathrm{cap} < R,$$

with cap of the table, a contradiction. Alternative (C1) gives $p = b_3 \le \max(\theta_1, \theta_2)\mathfrak{a}_1\mathfrak{a}_2 < \theta_3 u_{\mathrm{MV}} X d_+ U < 10^{11} < R$ (certified), also a contradiction.

Alternative (C2). It gives integers w_1, w_2, w_3 with $\gcd(w_1, w_2, w_3) = 1$, $w_1(2l-p)+w_2+w_3 p = 0$ and $|w_i| \le F\mathfrak{a}_i$, with F of the paragraph "The parametrisation"; the certificates give $F < F_*$. If $w_1 = 0$, then $w_3 = \pm 1$ and $p = |w_2| \le F\mathfrak{a}_2 < F_* d_+ U \le C_{\max} < R$, which is impossible. So $w_1 \ne 0$, and as in the paragraph "Type I: the theorem applied" of the proof of Proposition 5.14 we obtain integers $\mathcal{A}, \nu, C$ with

$$\mathcal{A}p - 2l\nu + C = 0, \quad \gcd(\mathcal{A}, \nu) = 1,$$
$$1 \le \nu \le F\mathfrak{a}_1 < F_* u_{\mathrm{MV}} X \le b_{\max}, \quad |C| \le F\mathfrak{a}_2 < F_* d_+ U \le C_{\max},$$

and $\mathcal{A} > 0$, since $\mathcal{A}p = 2l\nu - C > p - C_{\max} > 0$. As $\nu \le b_{\max} < 2R \le 2p$, the paragraph "The centred form" of the proof of Proposition 5.14 applies with $P = p$, $K = 2l$ and $D = D_\lambda$.

For each integer $1 \le \nu \le b_{\max}$ put

$$z_\nu = \frac{1.312\,\nu + C_{\max} U_B}{R}, \qquad H_\nu = \tfrac{1}{2}\min\big(U_B \nu,\, F_* u_{\mathrm{MV}} X U_B\big),$$

$$H_z = H_\nu + \frac{z_\nu}{2}, \quad H_x = U + \frac{H_\nu}{2} + \frac{C_{\max} U_B}{2\nu} + \frac{z_\nu}{4}, \quad X_\xi = 0.312 + \frac{C_{\max} U_B}{\nu} + \frac{z_\nu}{2}, \quad t_\nu = \frac{p_*}{\nu} + \frac{1}{2}.$$

These bound $|\log\zeta|$, $\mathrm{h}(\zeta)$, $\mathrm{h}(\xi)$, $|\log\xi|$ and t_0 of the centred form. Indeed, by its identities, $0 < \psi - \Lambda < 0.312$, $p \ge R$, $A < U_B$ and $|C| \le C_{\max}$ give $|\log\zeta| < z_\nu$; $\nu B < \min(U_B\nu, F_* u_{\mathrm{MV}} X U_B)$ and $\mathcal{A}A = \nu B - \log\zeta$ give $\mathrm{h}(\zeta) = \frac{1}{2}\max(\nu B, \mathcal{A}A) \le H_z$; $\mathrm{h}(D_\lambda^{-1}) < \ell_c < U$, $|p_0| \le \nu/2$ and $|k_0| \le \mathcal{A}/2 + |C|/\nu$ give $\mathrm{h}(\xi) \le H_x$; and $|\log\xi| \le \frac{|p_0|}{\nu}|\log\zeta| + \frac{|C|A}{\nu} + \psi < X_\xi$, $t_0 < t_\nu$. The certificates give $4\max_\nu H_x + 4 < L_0 R$, while (4) gives $\log\Lambda < -p\ell_c \le -L_0 R$; so $\log\zeta$ and $\log\xi$ have opposite signs, by the centred form. The paragraph "Laurent's theorem for each denominator" of the proof of Proposition 5.14, with these bounds, gives $\log\Lambda \ge -E_\nu$, and for every integer $1 \le \nu \le b_{\max}$ the certificates verify $A_z A_x > \lambda_{L,\nu}^2$ and $E_\nu < L_0 R$. This contradicts $\log\Lambda < -L_0 R$. Every alternative of Theorem 5.4 is excluded, so $p < R$ in each case; in row E4, case E4a gives $p < 6.8 \cdot 10^{12}$, the hypothesis of E4b, and E4b gives $p < 5.45 \cdot 10^{12}$.

The cases and their certified values follow. All values are upper bounds except the margins

of (M) and the thresholds L_0R.

	E1	E2	E3	E4a	E4b
L_0	237	230	230	227	227
U	247	238	235	231	231
p_*	$1.5\cdot10^{13}$	$9.15\cdot10^{12}$	$6.2\cdot10^{12}$	$9.15\cdot10^{12}$	$6.8\cdot10^{12}$
R	$7\cdot10^{12}$	$6.2\cdot10^{12}$	$5.6\cdot10^{12}$	$6.8\cdot10^{12}$	$5.45\cdot10^{12}$
h_0	18.68	17.85	17.62	29.16	17.68
χ_{MV}	3/10	3/10	9/50	1/5	9/50
ℓ	226	222	214	224	215
d_-	7.4	7.415	7.42	7	7.41
d_+	8.012	8.013	8.013	8.013	8.013
X	U_B	U_B	$U/2$	$U/2$	$U/2$
Π	562.760945	560.298754	557.906247	637.020745	560.124977
C_Π	568	562	560	640	561
C_*^+	723	711	678	709	681
$\mathcal{B}$	48.838847	47.907565	47.104469	48.096118	47.319349
margin of (M)	0.60097	0.52149	0.10872	3.58584	0.17643
cap	$6.959234\cdot10^{12}$	$6.168769\cdot10^{12}$	$5.568698\cdot10^{12}$	$6.737555\cdot10^{12}$	$5.440842\cdot10^{12}$
T_0	$1.064\cdot10^{12}$	$9.52\cdot10^{11}$	$7.384\cdot10^{11}$	$7.81\cdot10^{11}$	$7.20\cdot10^{11}$
F	18.77336	18.66280	25.949373	24.551799	25.990517
F_*	18.8	18.8	26.1	24.7	26.1
$b_{\max}$	62500	59600	70600	65700	69400
$C_{\max}$	37300	36000	49200	45800	48400
$\max_\nu E_\nu$	$1.235343\cdot10^{15}$	$9.80315\cdot10^{14}$	$1.127695\cdot10^{15}$	$1.267691\cdot10^{15}$	$1.097933\cdot10^{15}$
attained at ν	61820	59443	70535	65616	69335
L_0R	$1.659\cdot10^{15}$	$1.426\cdot10^{15}$	$1.288\cdot10^{15}$	$1.5436\cdot10^{15}$	$1.23715\cdot10^{15}$
$\max_\nu H_x$	$2.666659\cdot10^{6}$	$2.474767\cdot10^{6}$	$3.333669\cdot10^{6}$	$3.324772\cdot10^{6}$	$3.230785\cdot10^{6}$

The certificates check every endpoint used above: $99M_0^8 < 100\cdot10^{103}$ in E1; $M_0^8 < 20\cdot10^{100}$ in E2 and E3; $10^{99}/(3.4\cdot10^{12})^8 > 0.0559975$ in E4b; $\log 2.42+\frac{3}{5}\log M_0 < h_0$ in E1, $\log 1.42+\frac{3}{5}\log M_0 < h_0$ in E2, E3 and E4b, and $\log M_0 < h_0$ in E4a; the bounds $d_\pm$ from $a < a^+$; the (C1) bound below 10^{11}; and $F_*u_{\mathrm{MV}}X \le b_{\max}$, $F_*d_+U \le C_{\max}$. They enclose T_0 by the values of the table, so that $T_0 < R/2$. The certificates also check the positivity of (5) at $H = h_0$ and the monotonicity statements used in (6). □

Exclusion of small indices. The remaining lemmas prove that on $10^{99} \le c < 10^{130}$ every type I representation has $m' \ge R$, for explicit thresholds R between $5.45\cdot10^{12}$ and $7\cdot10^{12}$. Their engine is the cubic congruence of Lemma 2.22, through the Vieta cutoff Lemma 7.14 and the positive cubic defect Lemma 7.17; the remaining seeds are covered by finite Pell covers and an exact lattice enumeration (Lemma 7.40, Lemma 7.41, Lemma 7.42). Lemma 7.43 and Lemma 7.44 assemble them into the exclusion of small indices.

Lemma 7.40. *Let e, a, c, u, s, x_c be positive integers with $e < a < c$, $u^2 = ae + 1$, $s^2 = ac + 1$, $x_c^2 = ec + 1$, $ae > 200$, and $c \ge 10^6$. Put $\eta = u + \sqrt{ae}$. Define the integer pair (Y, X) by exactly seven inverse unit steps,*

$$\begin{pmatrix} Y \\ X \end{pmatrix} = \begin{pmatrix} u & -a \\ -e & u \end{pmatrix}^7 \begin{pmatrix} s \\ x_c \end{pmatrix}.$$

Then $X > 0$, $h = (X^2-1)/e \in \mathbb{Z}_{\geq 0}$, and $Y^2 = ah+1$. The identity

$$s\sqrt{e} + x_c\sqrt{a} = (Y\sqrt{e} + X\sqrt{a})\eta^7$$

holds. If $h \geq 1$, then

$$Y > 0 \Longrightarrow c > 2048X^2a^7e^6, \qquad Y < 0 \Longrightarrow c < 2048a^7e^6/X^2.$$

Proof. The unit matrix is integral and has determinant one; it preserves $aX^2 - eY^2 = a - e$. Put $W_c = s\sqrt{e} + x_c\sqrt{a} > 0$. Its conjugate is $-(a-e)/W_c < 0$. The inverse expression $W_0 = W_c\eta^{-7} > 0$ has negative conjugate $-(a-e)/W_0$. Thus its second coordinate $X = (W_0 + (a-e)/W_0)/(2\sqrt{a})$ is positive. Each inverse step sends X to $uX - eY$, whose square modulo e equals X^2 because $u^2 \equiv 1 \pmod{e}$. Starting with $x_c^2 \equiv 1 \pmod{e}$ proves $h \in \mathbb{Z}$; $X > 0$ gives $h \geq 0$. The preserved norm gives $Y^2 = ah + 1$.

Suppose $h \geq 1$, so $X \geq 2$, and write $Z = |Y|$. Then

$$Z\sqrt{e} = \sqrt{a(X^2-1)+e} > \sqrt{a}\sqrt{X^2-1} > (4/5)X\sqrt{a}.$$

Also $4ae < \eta^2 < 4ae + 2 < 4.01ae$, using $ae > 200$.

If $Y > 0$, then $W_0 = Z\sqrt{e} + X\sqrt{a} > (9/5)X\sqrt{a}$. Therefore

$$x_c = \frac{W_c + (a-e)/W_c}{2\sqrt{a}} > \frac{9}{10}X\eta^7,$$

and

$$c > \frac{81}{100}X^2(4ae)^7/e - 1/e > 2048X^2a^7e^6.$$

The last comparison is elementary since $(81/100)4^7 - 1 > 2048$.

If $Y < 0$, then $W_0 = X\sqrt{a} - Z\sqrt{e} = (a-e)/(X\sqrt{a} + Z\sqrt{e})$, and

$$W_0^2 < \frac{25a}{81X^2}.$$

Since $s > \sqrt{ac}$ and $x_c > \sqrt{ec}$, $W_c > 2\sqrt{aec}$. Hence

$$r := \frac{a-e}{W_c^2} < \frac{1}{4ec} \leq \frac{1}{4 \cdot 10^6}.$$

Consequently

$$x_c^2 = \frac{W_c^2}{4a}(1+r)^2 < \frac{25}{324}\frac{\eta^{14}}{X^2}(1 + 1/(4 \cdot 10^6))^2 < \frac{\eta^{14}}{12X^2}.$$

It follows that

$$c < \frac{4.01^7}{12}\frac{a^7e^6}{X^2} < 2048a^7e^6/X^2. \qquad \square$$

Lemma 7.41. *Let a, e be distinct positive integers, $u > 0$ an integer with $u^2 = ae + 1$, and let $h \geq 0$, $X > 0$, $Y \neq 0$ be integers satisfying $X^2 = eh + 1$, $Y^2 = ah + 1$. Then the following hold.*

(i) Let $\lambda \in \{-1, 1\}$ and $j, q \in \mathbb{Z}$ with $\mu_j = (4j^2 - q^2)/a \in \mathbb{Z}_{\geq 1}$ and $a \mid q + 2\lambda jY$, and put $V = (q + 2\lambda jY)/a$, $W = (2j + \lambda Yq)/a$. Then $V, W \in \mathbb{Z}$, $V^2 - W^2 = h\mu_j$, $\mu_j = 2jW - qV$, and $\gcd(j, 4a) \leq 4\mu_j$. If $4|j| < a$, then $h \neq 0$.

(ii) Under the hypotheses of (i), if $h > 0$, then $V = (F_1 + F_2)/2$ and $W = (F_2 - F_1)/2$ for integers F_1, F_2 of the same parity with $F_1F_2 = h\mu_j$, and

$$2j = (\lambda YV - W)/h, \qquad q = (\lambda YW - V)/h.$$

(iii) Suppose $h > 0$, and put $Z = |Y|$ and $\mathcal{M} = \begin{pmatrix} X & e \\ h & X \end{pmatrix}$. Then $e \neq h$, and $(u, Z)^t = \mathcal{M}^\ell (u_0, Z_0)^t$ for an integer $\ell \geq 0$ and a pair (u_0, Z_0) of the following finite seed list: if $e < h$, the pairs with $u_0 \geq 1$, $2hu_0^2 \leq (h - e)(X + 1)$ and $Z_0 \in \mathbb{Z}$, of either sign, with $eZ_0^2 = hu_0^2 - h + e$; if $e > h$, the pairs with $Z_0 \geq 1$, $2eZ_0^2 \leq (e - h)(X + 1)$ and $u_0 \in \mathbb{Z}$, of either sign, with $hu_0^2 = eZ_0^2 - e + h$. If moreover $a < A_*$ and $X \leq X_*$, then $3^\ell < 2X_*\sqrt{2X_*}\,(\sqrt{A_*} + 1)$.

(iv) If $a > e$ and $Y, X > 0$, then $Y' = uY - aX$ and $X' = uX - eY$ satisfy $0 < X' < u/X + X/u$, $h' = (X'^2 - 1)/e \in \mathbb{Z}_{\geq 0}$ and $Y'^2 = ah' + 1$.

Proof. The congruence defining V also defines W, because $\lambda Y(q + 2\lambda jY) \equiv 2j + \lambda Yq \pmod{a}$, using $Y^2 = ah + 1$. Direct expansion gives

$$V^2 - W^2 = h\mu_j, \qquad 2jW - qV = \mu_j. \tag{1}$$

For $h > 0$, inversion of the two-by-two matrix gives

$$2j = (\lambda YV - W)/h, \qquad q = (\lambda YW - V)/h. \tag{2}$$

Thus all possibilities are obtained by all ordered factor pairs

$$F_1F_2 = h\mu_j, \qquad F_1 \equiv F_2 \pmod 2, \qquad V = (F_1 + F_2)/2, \quad W = (F_2 - F_1)/2.$$

Both factors have the same sign. Equivalently, enumerate positive ordered factor pairs and restore a common sign. Equal factors, giving $W = 0$, are included. Conversely, the inverse formulas (2), when integral, recover exactly the original defining equations and $(4j^2 - q^2)/a = \mu_j$.

Also $\gcd(Y, a) = 1$, so the defining congruence gives $\gcd(2j, a) = \gcd(q, a)$. Every common divisor divides μ_j by (1). In particular

$$\gcd(j, 4a) \leq 4\gcd(j, a) \leq 4\gcd(2j, a) \leq 4\mu_j. \tag{3}$$

If $h = 0$, then $Y = \pm 1$ and $a \mid q \pm 2j$. Under $4|j| < a$, positivity of μ_j gives $|q| < 2|j|$, so $|q \pm 2j| < a$. The divisible integer must vanish, which would give $\mu_j = 0$, a contradiction. Thus this hypothesis excludes $h = 0$.

Item (iii). Write $Z = |Y| > 0$. The equations $Y^2 = ah + 1$, $u^2 = ae + 1$ are equivalent to

$$hu^2 - eZ^2 = h - e, \qquad a = (Z^2 - 1)/h. \tag{4}$$

The pair $\{e, h\}$ is Diophantine with $X^2 = eh + 1$. Equality $e = h \geq 1$ would make $e^2 + 1$ a square, which is impossible. Put

$$\eta = X + \sqrt{eh}, \qquad \mathcal{M} = \begin{pmatrix} X & e \\ h & X \end{pmatrix}.$$

Both $\mathcal{M}$ and its inverse have integer entries, determinant one, and preserve (4).

For an arbitrary positive solution u, Z, put $w = u\sqrt{h} + Z\sqrt{e}$, whose norm is $h - e$. Since $w > \sqrt{|h-e|}$, choose an integer $\ell \geq 0$ so that

$$\sqrt{|h-e|}/\sqrt{\eta} \leq w_0 = w\eta^{-\ell} < \sqrt{|h-e|}\sqrt{\eta}. \tag{5}$$

Write $w_0 = u_0\sqrt{h} + Z_0\sqrt{e}$. The inverse matrix shows $u_0, Z_0 \in \mathbb{Z}$.

If $e < h$, the norm is positive, hence $u_0 > 0$, while Z_0 may have either sign. From (5),

$$u_0 = \frac{w_0 + (h-e)/w_0}{2\sqrt{h}}, \qquad u_0^2 \leq \frac{(h-e)(X+1)}{2h} < \frac{X+1}{2}.$$

Therefore enumerate exactly the positive integers u_0 satisfying

$$2hu_0^2 \leq (h-e)(X+1),$$

test whether

$$Z_0^2 = (hu_0^2 - h + e)/e$$

is a nonnegative integer square, and retain both signs of its root. If $e > h$, the same argument with the negative norm gives $Z_0 > 0$, signed u_0, and the complete seed list is obtained from

$$2eZ_0^2 \leq (e-h)(X+1), \qquad u_0^2 = (eZ_0^2 - e + h)/h$$

by retaining both signs of the square root. Each list requires fewer than $\sqrt{(X+1)/2}$ tests of its positive coordinate. Including endpoints and zero coordinates in these tests does not omit a boundary orbit.

Every original positive solution is $(u, Z)^t = \mathcal{M}^\ell(u_0, Z_0)^t$ for one listed seed and $\ell \geq 0$. Conversely, iterate all listed seeds by $\mathcal{M}$ and retain positive u, Z with $a = (Z^2 - 1)/h$ integral. Restore $Y = \pm Z$ independently, and use both signs of λ in (2), retaining integer j, q. This covers every ordering of e, h, every Pell orbit, every primary seed sign, and every factor-pair sign.

For an explicit finite exponent bound, suppose $a < A_*$, $X \leq X_*$. Then $e, h < X^2$, and (4) implies

$$w < 2X(\sqrt{A_*} + 1).$$

The lower endpoint in (5) is at least $1/\sqrt{\eta}$. Since $3 < \eta < 2X$,

$$3^\ell \leq \eta^\ell < 2X_*\sqrt{2X_*}(\sqrt{A_*} + 1). \tag{6}$$

Thus any integer range whose upper endpoint makes 3^ℓ exceed the right side covers every possible exponent.

Item (iv). If $a > e$ and $Y, X > 0$, set

$$Y' = uY - aX, \qquad X' = uX - eY.$$

Then

$$X' = \frac{u^2 + X^2 - e^2 - 1}{uX + eY} > 0, \qquad X' < u/X + X/u.$$

The numerator is $e(a-e) + X^2 > 0$. Both new coordinates are integers, and

$$h' = (X'^2 - 1)/e \in \mathbb{Z}_{\geq 0}, \qquad Y'^2 = ah' + 1.$$

Indeed $X'^2 \equiv 1 \pmod{e}$, and the preserved norm gives the last identity. The new Y' can have either sign, which the preceding cover retains. This proves (iv). □

Lemma 7.42. *Let a, P_*, Q_* be positive integers and $r \in \mathbb{Z}$, reduced modulo a. The pairs $(P, Q) \in \mathbb{Z}^2$ with $P \equiv 0 \pmod 2$ and $P + rQ \equiv 0 \pmod a$ form a lattice Λ with basis $(a, 0), (-r, 1)$ if a and r are even, $(a, 0), (-2r, 2)$ if a is even and r odd, and $(2a, 0), (a\bar{r} - r, 1)$ if a is odd, where $\bar{r} \in \{0, 1\}$ is the parity of r. Let (b_1, b_2) be any basis of Λ with $\delta = \det(b_1, b_2) > 0$, and let $\mathcal{R} = [1, P_*] \times [1, Q_*]$. Then*

$$\Lambda \cap \mathcal{R} = \{k_1 b_1 + k_2 b_2 : \ k_2 \in I, \ k_1 \in I(k_2)\},$$

where I is the set of integers between the least and the largest value of $\det(b_1, z)/\delta$ at the four corners z of $\mathcal{R}$, and $I(k_2)$ is the set of integers k_1 with $k_1 b_1 + k_2 b_2 \in \mathcal{R}$, a finite interval of integers; distinct pairs (k_1, k_2) give distinct points. If also $r^2 \equiv 1 \pmod a$, $P > Q$, and $\nu = (P^2 - Q^2)/a$, then $\gcd(P, a) \mid \nu$ and $\gcd(P/2, 4a) \le 4\nu$.

Proof. Reduce r modulo a, and take the three bases of the statement in turn. In the first case parity follows automatically from the congruence. In the second Q must be even. In the third, writing $P = at - rQ$, evenness is equivalent to $t \equiv rQ \pmod 2$. These arguments prove both inclusion and generation; the determinants are respectively $a, 2a, 2a$.

Apply the invertible diagonal map $(P, Q) \mapsto (Q_* P, P_* Q)$. The allowed rectangle becomes

$$R = [Q_*, P_* Q_*] \times [P_*, P_* Q_*].$$

Reduce the weighted basis by swaps and integer column subtractions, using the nearest integer to the normalized scalar product. These are unimodular operations. Ordinary Gauss reduction terminates: a nonzero reduction decreases an integer squared norm, except at most one half-integer tie step. Orient the final basis so $\delta = \det(b_1, b_2) > 0$.

Every lattice point $z = k_1 b_1 + k_2 b_2$ has

$$k_2 = \det(b_1, z)/\delta.$$

The numerator is linear in z, so its minimum and maximum on R occur at the four corners. Enumerate every integer between the ceiling of the minimum divided by δ and the floor of the maximum divided by δ. For each k_2, intersect the two exact integer intervals obtained from the coordinate inequalities

$$\ell_i \le k_1 (b_1)_i + k_2 (b_2)_i \le P_* Q_*, \qquad (\ell_1, \ell_2) = (Q_*, P_*).$$

Use ceiling and floor division, reversing the inequalities when $(b_1)_i < 0$; if it is zero, retain the interval only when the constant coordinate is allowed. At least one coefficient is nonzero, so the resulting interval is finite. Enumerate its integers and invert the weighting. This is precisely the original rectangle intersection. Basis coordinates are unique, so there are no duplicates. These two steps use only that (b_1, b_2) is a basis of positive determinant and that $\det(b_1, \cdot)$ is linear, so the same description holds for every such basis of Λ in the original coordinates, as stated; the weighting and the reduction only shorten the intervals.

For the divisor claim write $P + rQ = at$ and $r^2 = 1 + ak$. Then

$$\nu = at^2 - 2trQ + kQ^2.$$

The congruence and $r^2 \equiv 1 \pmod a$ give $\gcd(P, a) = \gcd(Q, a)$; that gcd divides the displayed integer ν. Finally $\gcd(P/2, 4a) \le 4 \gcd(P/2, a) \le 4 \gcd(P, a) \le 4\nu$.

□

Lemma 7.43. *Let $0 < a < b < c < d$ be an irregular Diophantine quadruple with $\deg\{a,b,c\} \geq 2$, and let z have a type I representation. Let (c_0, c_1, R) be one of the two rows*

$$\text{X1: } (c_0,c_1,R) = (10^{99},\, 10^{100},\, 5.45\cdot 10^{12}), \qquad \text{X2: } (c_0,c_1,R) = (10^{100},\, 10^{102},\, 5.6\cdot 10^{12}).$$

If $c_0 \leq c < c_1$, then $m' \geq R$.

Proof. Suppose, to the contrary, that $c_0 \leq c < c_1$ and $m' < R$. If $e > b$, Lemma 7.1 gives $m' > 6.24c^{1/4} > R$, which is impossible, and $e = b$ is impossible. Hence $e < b$. Put $u = \sqrt{ae+1}$, $\delta = m^2 - n^2$, $A = 603/400$ and $\rho = 71/100$. Normalizing the primary pair (e, a) by exactly seven inverse unit steps (Lemma 7.40) gives integers $Y = Y_7$, $X = X_7 > 0$ with

$$\begin{pmatrix} Y_7 \\ X_7 \end{pmatrix} = \begin{pmatrix} u & -a \\ -e & u \end{pmatrix}^7 \begin{pmatrix} s \\ \sqrt{ec+1} \end{pmatrix}, \qquad h = (X^2-1)/e \in \mathbb{Z}_{\geq 0}, \qquad Y^2 = ah+1.$$

Part I excludes $h \geq 1$ and Part II excludes $h = 0$. Part I uses the index bound $m < M_{\mathrm{I}}$ and Part II the bound $m < M_{\mathrm{II}}$ of the tables below; both hold, since $m < R/2$ (row X2 of Part I uses the weaker $m < 3.1\cdot 10^{12}$).

Part I: a seed with $h \geq 1$ is impossible. The constants of the two rows are

Part I	row X1	row X2
M_{I}	$2.725\cdot 10^{12}$	$3.1\cdot 10^{12}$
range of c in the cover	$[10^{99}, 10^{100})$	$[10^{100}, 10^{103})$
$c_0/M_{\mathrm{I}}^8 >$	0.328903	1.172482
B_{I} and the bound for it	3/4, 0.737139	63/100, 0.625600
a_*	700000	10^6
$\lvert j\rvert a/\delta <,\ N/(a\delta m^2) >$	0.633, 0.366	0.335, 0.664
$q/j >$	$m/(16a)$	$m/(8a)$
$a \in$	$(5\cdot 10^{11},\ 4.3\cdot 10^{14})$	$(1.11\cdot 10^{12},\ 5\cdot 10^{14})$
$u <,\ e \leq,\ e^2\mu_j <$	$2.06\cdot 10^7,\ 813,\ 661739$	$2.22\cdot 10^7,\ 442,\ 195455$
$C_+,\ C_-$	7000000, 4884100	1520000, 1440000

Throughout $c \geq 10^{99}$ and $c/m^8 > c_0/M_{\mathrm{I}}^8$, which exceeds the value of the table and hence 3/10; so row 2 of item (ii) of Lemma 7.17, Lemma 7.14 and item (i) of Lemma 7.13 give

$$u < \rho m^{3/5}, \quad a < \rho^2 m^{6/5}, \quad m > 4u, \quad m > n \geq m/2, \quad n < 5m/7,$$
$$|j| < \tfrac{201}{100} a^{3/2} m\delta/\sqrt{c}, \quad \mu_j \in \mathbb{Z}_{\geq 1}, \quad \delta \leq 3m^2/4,$$

and Lemma 7.13 gives $m > 2^{156/5}$. Consequently

$$|j| < Aa^{3/2}m^3/\sqrt{c}, \qquad c < 4A^2a^2m^6. \tag{1}$$

Item (i) of Lemma 7.17 gives

$$|q| < B_{\mathrm{I}}\sqrt{a}\, m^4/\sqrt{c}, \tag{2}$$

since, with $a > a_*$ (by (1), $a > \sqrt{c}/(2Am^3)$, which exceeds $5\cdot 10^{11}$), its normalized upper bound

$$\frac{\frac{1}{2} + 6/a_* + \frac{603}{800}\rho^5/\sqrt{c_0/M_{\mathrm{I}}^8}}{1 - \rho/\bigl(2\sqrt{c_0/M_{\mathrm{I}}^8}\, m^{12/5}\bigr)}$$

is below the bound of the table. The defining cubic relation gives $\operatorname{sign}(j) = \lambda$. The sign of q follows from item (i) of Lemma 7.17: $|j|a/\delta < \frac{201}{100}\rho^5/\sqrt{c_0/M_{\mathrm{I}}^8}$ is below the value of the table, so its numerator

$$N = 4a\delta(n^2-1) + 12akn^2 + j(a^2m^2-4) > a\delta m^2(1 - 4/m^2 - |j|a/\delta)$$

exceeds the multiple of $a\delta m^2$ of the table and is positive, while the denominator $am+2\lambda s$ has the sign λ, because $am < 2s$. Therefore $q = \lambda Q$ with $Q > 0$, and q/j exceeds the value of the table. The bounds (1), (2) give the envelopes of the table for a, u, e and $e^2\mu_j$, from $a > \sqrt{c}/(2Am^3)$, the pure cutoff and $a\mu_j \le 4j^2$ respectively; the unrounded bounds are $a > 518338137270$, $u < 20534221.987$, $e < 813.473373$, $e^2\mu_j < 661738.927718$ in row X1 and $a > 1113339460040$, $u < 22185813.866$, $e < 442.102661$, $e^2\mu_j < 195454.762055$ in row X2.

Assume $h > 0$. The sign-specific growth of the seventh seed gives

$$Y > 0: \quad e^3X < C_+, \qquad Y < 0: \quad eX^2 < C_-, \tag{3}$$

with the unrounded coefficients 6969296.653 and 2202.991112^2 in row X1, and 1517533.954 and 1197.271188^2 in row X2. In particular $Y < 0$ forces $X < 2210$, resp. $X < 1200$, so $X < 5000$. If $X \le 5000$, we use the seventh seed. Otherwise $Y > 0$, and one more inverse unit step gives $Y_8 = uY - aX$, $X_8 = uX - eY$; item (iv) of Lemma 7.41 gives $X_8 > 0$, $h_8 = (X_8^2-1)/e \ge 0$, $Y_8^2 = ah_8 + 1$ and

$$X_8 = \frac{u^2+X^2-e^2-1}{uX+eY} < \frac{u}{X} + \frac{X}{u}.$$

With the envelopes and (3), $X_8 < 2.06 \cdot 10^7/5000 + 7000000/\sqrt{5 \cdot 10^{11}} < 4442$ in row X1 and $X_8 < 2.22 \cdot 10^7/5000 + 1520000/\sqrt{1.11 \cdot 10^{12}} < 4442$ in row X2. Since $4|j| < a$, the even-seed congruence and item (i) of Lemma 7.41 exclude $h_8 = 0$. Thus every case with $h > 0$ lies in one of the two complete primary covers

κ	seed bound	additional conditions
7	$2 \le X \le 5000$	$e^3X < C_+$ $(Y > 0)$, $eX^2 < C_-$ $(Y < 0)$
8	$2 \le X \le 4442$	$Y_7 > 0$, $X_7 > 5000$,

(4)

where in the second row $Y_7 = uY_8 + aX_8$, $X_7 = eY_8 + uX_8$. The bounds make $h < a$ in both covers.

The primary Pell enumeration runs as follows. For every integer $1 \le e \le 813$, resp. 442, and every X in (4) with $e \mid X^2-1$, set $h = (X^2-1)/e$; equality $e = h$ is impossible. The complete orbit cover of the pair (e,h) in item (iii) of Lemma 7.41 uses the matrix $\begin{pmatrix} X & e \\ h & X \end{pmatrix}$ on the equation $hu^2 - eZ^2 = h - e$. For $e < h$, enumerate the positive integers u_0 with $u_0^2 < (X+1)/2$, test whether $Z_0^2 = (hu_0^2 - h + e)/e$ is a nonnegative integer square, and retain both signs of Z_0; for $e > h$, enumerate the positive Z_0 with $Z_0^2 < (X+1)/2$, test $u_0^2 = (eZ_0^2 - e + h)/h$, and retain both signs of u_0. Including endpoints is harmless; for $\kappa = 7$ these weaker small-coordinate bounds are used as overcoverage, for $\kappa = 8$ the exact bounds of the cited lemma. Retain $u_0^2 \equiv 1 \pmod e$, which is necessary and invariant under the matrix, and iterate the powers $0, \dots, 49$; this range is complete by the exponent bound in item (iii) of Lemma 7.41, even with $X \le 5000$ and $a < 5 \cdot 10^{14}$. Keep $u, Z > 0$ with $a = (u^2-1)/e$ integral and in the envelope, and set $Y = \sigma Z$, $\sigma = \pm 1$. For the exponent $\kappa \in \{7,8\}$ of the seed compute

$$s = \sigma ZT_\kappa + aXU_{\kappa-1}, \qquad \sqrt{ec+1} = XT_\kappa + \sigma eZU_{\kappa-1}, \qquad c = (s^2-1)/a, \tag{5}$$

and retain positive coordinates, c in the range of the table and the conditions (4), identifying duplicates. The counts are

	row X1	row X2
$\kappa = 7$: pairs (e, X), seeds, primary states	30339, 61416, 28988	22753, 46116, 22557
$\kappa = 7$: configurations	1337	3329
$\kappa = 8$: pairs, seeds, primary states	95678, 192456, 64908	82392, 165874, 54200
$\kappa = 8$: signed configurations, and after (4)	1060, 39	3355, 114

The pairs $(2|j|, |q|)$ are covered by congruence rectangles. For each configuration put $P = 2|j| > 0$, $Q = |q| > 0$ and $\tau = \lambda\sigma$. Odd primary parity gives $s \equiv uY \pmod a$ and even parity $s \equiv Y \pmod a$. Since $q \equiv -2\lambda js \pmod a$, every original pair lies in one of the congruence rectangles

$$P + rQ \equiv 0 \pmod a, \quad P \equiv 0 \pmod 2, \quad r = \begin{cases} \tau u Z & \kappa = 7, \\ \tau Z & \kappa = 8, \end{cases} \quad \tau = \pm 1, \tag{6}$$

and both residues satisfy $r^2 \equiv 1 \pmod a$. The complete weighted Gauss enumeration of Lemma 7.42 lists these rectangles with the integer bounds

$$P_* = \Big\lfloor \sqrt{\Big\lfloor \frac{363609 a^3 M_{\mathrm{I}}^6 - 1}{40000 c} \Big\rfloor} \Big\rfloor, \qquad Q_* = \Big\lfloor \sqrt{\Big\lfloor \frac{\beta_1 a M_{\mathrm{I}}^8 - 1}{\beta_2 c} \Big\rfloor} \Big\rfloor, \tag{7}$$

where $B_{\mathrm{I}}^2 = \beta_1/\beta_2$ is $9/16$, resp. $3969/10000$; the subtracted one makes the strict bounds exact at integer endpoints. Retain $P > Q$, $\mu_j = (P^2 - Q^2)/a$ integral and positive, and $e^2\mu_j$ below the bound of the table; set $\lambda = \tau\sigma$, $j = \lambda P/2$, $q = \lambda Q$. The rectangles (6) with the bounds (7) are enumerated exhaustively by integer intervals. For every retained pair compute $Q_0 = (q + 2\lambda js)/a$, $g = \gcd(j, 4a)$ and $d = 4a/g$. The cubic definition of q gives

$$Q_0 = m(4a\delta + j), \qquad jm \equiv Q_0 \pmod{4a}. \tag{8}$$

The bound $g \le 4\mu_j$ follows from Lemma 7.42. If $g \nmid Q_0$, the configuration is rejected; otherwise $m \equiv m_0 \pmod d$ with $m_0 \equiv (Q_0/g)(j/g)^{-1}$, $0 \le m_0 < d$, and exactly the positive integers $m = m_0 + td < M_{\mathrm{I}}$ are listed. The numbers of pairs, of positive defects, of solvable congruences (8) and of indices are

	row X1	row X2
$\kappa = 7$	457554, 8697, 4023, 33	306704, 1403, 627, 2
$\kappa = 8$	115, 19, 11, 6	33, 9, 4, 1

The remaining indices are excluded by necessary conditions, applied with the actual candidate m. In row X1 the conditions

$$m^5 > 2^{156}, \quad c < 2^{18} m^9, \quad m > 4u, \quad 9a < m^2, \quad (100u)^5 < 71^5 m^3,$$
$$40000c < 363609 a^2 m^6, \quad 40000 P^2 c < 363609 a^3 m^6, \quad 16 Q^2 c < 9 a m^8$$

leave two indices with $\kappa = 7$ and one with $\kappa = 8$, one in each column:

κ	7	7	8
e	3	11	1
h	456	105	5475
a	63177763405960	18172977532528	2556068710440
u	13767109	14138697	1598771
P	306248406	112194426	4818358
Q	5682014	3448010	1663738
m	2609161757124	2638199523623	2677220084563
$Q_0 \bmod m$	93918021400	1176347876341	1809807586156

all with $\sigma = -1$ and $X = 37, 34, 74$. Every remainder is nonzero, whereas (8) requires $m \mid Q_0$. In row X2 the three indices are

κ	h	a	u	σ	P	Q	m
7	24	31444774014095	5607564	1	68251068	2093258	1933532625165
7	27224	326774458225215	18076904	-1	2992979786	7731499	2806947139631
8	6083	3695271891480	1922309	-1	3844934	49294	1831517551252

all with $e = 1$ and $X = 5, 165, 78$; the first two violate $10000Q^2c < 3969am^8$, which is (2), and the last violates $40000c < 363609a^2m^6$, which is (1). In both rows (5), with $Z = \sqrt{ah+1}$, gives the exact c of each line. Hence no index survives, and $h > 0$ is impossible. The computations use only finite recurrences, integer square tests and integer comparisons.

Part II: a seed with $h = 0$ is impossible. The constants of the two rows are

Part II	row X1	row X2
M_{II}	$2.725 \cdot 10^{12}$	$2.8 \cdot 10^{12}$
$c_0/M_{\mathrm{II}}^8 >$	0.328903	2.646
$a_{\min},\ a_{\max}$	$5 \cdot 10^{11},\ 5 \cdot 10^{14}$	$1.11 \cdot 10^{12},\ 5 \cdot 10^{14}$
$\bar{e}$	850	442
$u \in$	$(7 \cdot 10^5,\ 2.1 \cdot 10^7)$	$(10^6,\ 2.22 \cdot 10^7)$
$c < c_{\mathrm{box}}$	10^{100}	10^{103}
$\lvert j\rvert a/\delta <,\ N/(a\delta m^2) >$	0.633, 0.367	0.225, 0.774
$q/j >,\ m/u >$	$m/(16a),\ 32$	$m/(8a),\ 16$
$B_* <$	0.737130	0.583590
$v <$	$0.000039616 + 1.285316\sqrt{e}$	$0.000013814 + 0.358707\sqrt{e}$
$u/a <,\ s/(au^6) <$	$0.001215,\ 64 \cdot 1.002$	$0.000443,\ 64 \cdot 1.001$
$v/u <,\ we^{5/14} <$	$60 \cdot 10^{-6},\ 85$	$13 \cdot 10^{-6},\ 40$
finite box	$1 \le e \le 179,\ 1 \le v \le 18$	$8 \le e \le 74,\ 1 \le v \le 3$
	$w \le 85$	$w \le 40$
tuples, progressions	635, 1572	30, 60
values of u	112383772	2698682
signed cases, indices	98406772, 5579130	2249150, 153630

Throughout $c \ge 10^{99}$ and $c/m^8 > c_0/M_{\mathrm{II}}^8$, which exceeds the value of the table and hence 3/10; so row 2 of item (ii) of Lemma 7.17, Lemma 7.14 and item (i) of Lemma 7.13 give

$$\begin{gathered} m > 2^{156/5}, \quad m > n \ge m/2, \quad n < 5m/7, \quad \delta \le 3m^2/4, \quad k > 0, \quad j = \lambda|j|, \quad |j| \ge 1, \\ u < \rho m^{3/5}, \quad a < \rho^2 m^{6/5}, \quad |j| < Aa^{3/2}m^3/\sqrt{c}, \quad \mu_j \in \mathbb{Z}_{\ge 1}, \end{gathered} \tag{9}$$

and, since $a \le a\mu_j \le 4j^2$, $c < 4A^2a^2m^6$. From (9), $\sqrt{c_0}/(2AM_{\rm II}^3) < a < \rho^2 M_{\rm II}^{6/5}$, $u < \rho M_{\rm II}^{3/5}$ and $e < 2A\rho^2 M_{\rm II}^{21/5}/\sqrt{c_0}$; we use the weaker box of the table, $a_{\min} < a < a_{\max}$, $1 \le e \le \bar e$, $c_0 \le c < c_{\rm box}$, and u in the interval of the table, whose lower end follows from $u^2 > a_{\min}$.

Suppose $h = 0$. Then $X = 1$, $Y = \sigma \in \{-1, 1\}$, and

$$\begin{aligned} s &= aU_6 + \sigma T_7, \qquad \sqrt{ec+1} = T_7 + \sigma e U_6, \\ U_6 &= 64u^6 - 80u^4 + 24u^2 - 1, \qquad T_7 = u(64u^6 - 112u^4 + 56u^2 - 7). \end{aligned} \tag{10}$$

The odd-seed identities of Lemma 7.35 give integers

$$V = (2j + \lambda u\sigma q)/a, \qquad W = (2ju + \lambda\sigma q)/a, \qquad W^2 - V^2 = e\mu_j. \tag{11}$$

Item (i) of Lemma 7.17 is $q(am + 2\lambda s) = N$ with $N = 4a\delta(n^2-1) + 12akn^2 + j(a^2m^2 - 4)$. Since $|j|a/\delta < \frac{201}{100}\rho^5/\sqrt{c_0/M_{\rm II}^8}$ is below the value of the table, $N > a\delta m^2(1 - 4/m^2 - |j|a/\delta)$ exceeds the multiple of $a\delta m^2$ of the table. Also $am < 2s$, so $am + 2\lambda s$ has the sign λ, and $|am + 2\lambda s| < 2.001\sqrt{ac}$. Hence $q = \lambda Q$ with $Q > 0$, and $q/j = Q/|j| > \frac{N/(a\delta m^2)}{2.001\cdot 201/100} \cdot \frac{m}{a}$ exceeds the value of the table. Because $\mu_j > 0$, $Q < 2|j|$. The pure cutoff gives $m/u > m^{2/5}/\rho$, above the value of the table. If $V = 0$, (11) forces $q/j = 2/u$, whence $me < 32u$, resp. $me < 16u$, contrary to that bound. Furthermore $uQ > 2|j|$, because $uQ/|j| > me/(16u) > 2$, resp. $uQ/|j| > me/(8u) > 2$. Therefore V has the sign σ, and, since $Q < 2|j|$ and $u > 1$, W has the sign λ. Put $v = \sigma V > 0$, $w = \lambda W > 0$, $\tau = \lambda\sigma$ and $P = 2|j|$. The exact inverses and norm are

$$P = (uw - \tau v)/e, \qquad Q = (uv - \tau w)/e, \qquad w^2 - v^2 = e\mu_j > 0. \tag{12}$$

With $m_* = 2^{156/5}$, direct division in item (i) of Lemma 7.17 gives

$$\frac{Q}{\sqrt{a}\,m^4/\sqrt{c}} < B_* := \frac{\frac12 + 6/a_{\min} + \frac{603}{800}\rho^5/\sqrt{c_0/M_{\rm II}^8}}{1 - \rho/\big(2\sqrt{c_0/M_{\rm II}^8}\,m_*^{12/5}\big)},$$

below the value of the table. Consequently

$$v \le \frac{2|j|}{a} + \frac{uQ}{a} < \frac{2A\rho M_{\rm II}^{18/5}}{\sqrt{c_0}} + \frac{B_*}{\sqrt{c_0/M_{\rm II}^8}}\sqrt{1 + 1/a_{\min}}\,\sqrt{e}, \tag{13}$$

whose two coefficients are below those of the table. The box gives $u/a = eu/(u^2-1)$ below the value of the table, so $U_6 > 63.999u^6$, $T_7 < 64u^7$, and (10) gives $63.9au^6 < s$ and $s/(au^6)$ below the value of the table. Squaring, with $a = (u^2-1)/e$ and $c = (s^2-1)/a$, gives

$$4000u^{14}/e < c < 4200u^{14}/e. \tag{14}$$

The bound for $|j|$, (12) and the lower bound in (14) give

$$1000(uw - v)^2u^8 < A^2M_{\rm II}^6. \tag{15}$$

The box and (13) give v/u below the value of the table, and (14), $c \ge c_0$ and the bound for $|j|$ give

$$we^{5/14} < \frac{v}{u}\,\bar e^{5/14} + \frac{2A}{\sqrt{4000}}M_{\rm II}^3\Big(\frac{4200}{c_0}\Big)^{5/14},$$

which is below $83.414 < 85$ in row X1 and below 40 in row X2. Since $w^2 > e$, this gives $e < 85^{7/6} < 179$, resp. $e < 40^{7/6} < 74$; then (13) gives $v \le 18$, resp. $v \le 3$, and in row X2 $v \ge 1$ forces $e \ge 8$. Every original configuration therefore lies in the finite box of the table, with

$$v < w, \qquad e \mid w^2 - v^2, \qquad \mu_j = (w^2 - v^2)/e \ge 1, \qquad w^{14}e^5 < 85^{14}, \text{ resp. } w^{14}e^5 < 40^{14}; \tag{16}$$

the endpoints $e = 179$, $v = 18$ in row X1 and $e = 74$ in row X2 are retained as overcoverage.

For each tuple of (16) and both $\tau = \pm 1$, every original u satisfies

$$u^2 \equiv 1 \pmod{e}, \qquad uw - \tau v \equiv 0 \pmod{2e}, \qquad uv - \tau w \equiv 0 \pmod{e}, \tag{17}$$

together with the box, (14) and (15). The endpoints for u are found by exact integer bisection of these monotone inequalities (the degree-ten cutoff is the integer inequality $160000000(uw - v)^2u^8 < 363609M_{\mathrm{II}}^6$), and every residue class modulo $2e$ satisfying (17) is listed; the numbers of tuples, signed residue progressions and enclosed values of u are those of the table.

To exclude these progressions, put

$$A_7 = -64u^5 + 48u^3 - 8u = \frac{u - T_7}{u^2 - 1}, \qquad B_7 = 64u^6 - 48u^4 + 8u^2 + 1 = \frac{uT_7 - 1}{u^2 - 1}.$$

For each τ and $\lambda = \pm 1$ set $\sigma = \tau\lambda$, $j = \lambda P/2$ and $q = \lambda Q$. Then

$$Q_0 = (q + 2\lambda js)/a = \lambda(vA_7 + \tau wB_7) + PU_6, \tag{18}$$

and the defining cubic identity gives

$$Q_0 = m(4a\delta + j), \qquad jm \equiv Q_0 \pmod{4a}, \qquad m \mid Q_0. \tag{19}$$

Substituting $u^2 = ae+1$ gives $A_7 = -24u-80aeu-64a^2e^2u$, $B_7 = 25+104ae+144a^2e^2+64a^3e^3$ and $U_6 = 7 + 56ae + 112a^2e^2 + 64a^3e^3$, so that modulo $4a$

$$Q_0 \equiv \lambda H_0 + 7P, \qquad H_0 = 25\tau w - 24uv.$$

Let $g = \gcd(j, 4a)$ and $d = 4a/g$. Since $g \mid P$, the congruence in (19) is soluble if and only if $g \mid H_0$, and then it has the unique residue $m_0 \equiv \big[H_0/g + 7\lambda P/g\big](|j|/g)^{-1} \pmod{d}$, $0 \le m_0 < d$. Every integer $m = m_0 + td$ with $m_{\min} \le m < M_{\mathrm{II}}$ is listed, where $m_{\min}$ is the least integer $r \ge 0$ with $2^{18}r^9 > c_0$ (Lemma 7.13). (For the bound $g \le 4\mu_j$: $\mu_j = 2jV - \lambda\sigma qW$, and $d_1 = \gcd(j, a)$ divides q, so $d_1 \mid \mu_j$ and $g \le 4d_1 \le 4\mu_j$.) The exact evaluation of all enclosed values of u gives the numbers of signed cases after the solvability test and of indices of the table, and not one index satisfies $m \mid Q_0$. The full polynomial (18) is evaluated before this divisibility test; the reduction modulo $4a$ is used only for the congruence. Hence $h = 0$ is impossible.

Parts I and II exclude both $h \ge 1$ and $h = 0$. This contradicts $m' < R$ and proves the lemma. □

Lemma 7.44. *Let $0 < a < b < c < d$ be an irregular Diophantine quadruple with $\deg\{a, b, c\} \ge 2$, and let z have a type I representation. Let (c_0, R) be one of the two rows*

$$\textit{Y1: } (c_0, R) = (10^{103}, 7 \cdot 10^{12}), \qquad \textit{Y2: } (c_0, R) = (10^{102}, 6.2 \cdot 10^{12}).$$

If $c_0 \le c < 10^{130}$, then $m' \ge R$.

Proof. The two rows are proved by one argument; the constants that differ between them are

	row Y1	row Y2
$M_* = R/2$	$3.5 \cdot 10^{12}$	$3.1 \cdot 10^{12}$
$B,\ \rho$	$521/1000,\ 89/100$	$513/1000,\ 71/100$
$t_0 < c_0/M_*^8$	444.074	117.248
bound for $\lvert q\rvert\sqrt{c}/(\sqrt{a}\,m^4)$	0.519980	0.512566
$a/m >$	6.9894	3.5914
$e <,\ \mu_j <$	$36.572371,\ 1337.538317$	$44.210267,\ 1954.547621$
$e_{\max},\ \mu_{\max}$	$36,\ 1337$	$44,\ 1954$
$u <,\ a <$	$3 \cdot 10^7,\ 10^{15}$	$2.3 \cdot 10^7,\ 5 \cdot 10^{14}$
$4a\lvert j\rvert/\delta <$	0.214	0.135
$\lvert E\rvert <\ (h = 0)$	0.023228	0.102
$F_+ <,\ F_- <$	$3.151047,\ 2.748522$	$16.710424,\ 6.207892$
E_*	4	17
$Q_{\max},\ v_{\max}$	$10^6,\ 10^9$	$1.1 \cdot 10^6,\ 6 \cdot 10^9$
$q_{\max}$	1	4

Suppose, to the contrary, that $c_0 \le c < 10^{130}$ and $m' < R$. Put $u = \sqrt{ae+1}$, $\delta = m^2 - n^2$ and $A = 603/400$. Then $m < M_*$. If $e > b$, Lemma 7.1 gives $m' > 6.24c^{1/4}$, contradicting $m' < R$; equality $e = b$ is impossible. Hence $e < b$, and Lemma 7.13 gives

$$2^{156/5} < m < M_*, \qquad c < 2^{18}m^9, \qquad c < 10^{130}. \tag{1}$$

Throughout, $c \ge 10^{102}$ and $c/m^8 \ge c_0/M_*^8 > t_0 > 3/10$. Row 2 of item (ii) of Lemma 7.17, Lemma 7.14 (with $u < 0.71m^{3/5} \le \rho m^{3/5}$ and $n < 5m/7 < 3m/4$) and items (i) and (ii) of Lemma 7.13 give

$$m > 4u, \quad a < m^2/9, \quad m \le \delta \le 3m^2/4, \quad n < 3m/4, \quad 4 \le k < m^2/u^2, \quad k < m^2/a,$$

$$\operatorname{sign}(j) = \lambda, \quad u < \rho m^{3/5}, \quad a < \rho^2 m^{6/5}, \quad \mu_j = (4j^2 - q^2)/a \in \mathbb{Z}_{\ge 1},$$

where $Y_* = js - \lambda a^2 m\delta$ and $T_* = 2a^2 m\delta + jam$ are the integers Y and T_δ of Lemma 2.22, and $q = T_* - 2\lambda Y_*$. Item (i) of Lemma 7.17 sharpens q: since $a > 10^6$ (row 2 there gives $c < (201/50)^2 a^2 m^6$, so $a > 50\sqrt{c}/(201m^3) > 10^6$), $\delta n^2 \le m^4/4$, $\delta \le 3m^2/4$ and $c/m^8 > t_0$,

$$\frac{|q|\sqrt{c}}{\sqrt{a}\,m^4} < \frac{\frac{1}{2} + 6 \cdot 10^{-6} + \frac{603}{800}\rho^5/\sqrt{t_0}}{1 - \rho/(2\sqrt{t_0}\,m^{12/5})},$$

which is below the bound of the table, and that bound is below B. Also

$$|j| < \frac{201}{100}a^{3/2}m\delta/\sqrt{c} \le Aa^{3/2}m^3/\sqrt{c}.$$

Since $a\mu_j \le 4j^2$, this yields

$$|j| < Aa^{3/2}m^3/\sqrt{c}, \qquad |q| < B\sqrt{a}\,m^4/\sqrt{c}, \qquad c < 4A^2a^2m^6 = \frac{363609}{40000}a^2m^6. \tag{2}$$

Direct substitution in (2) gives $a > \sqrt{c}/(2Am^3)$, and then the bounds of the table for a/m, e, μ_j, u and a; in particular $1 \le e \le e_{\max}$, $1 \le \mu_j \le \mu_{\max}$ and $e/a < 1/1024$.

Seven inverse unit steps (Lemma 7.40) give integers Y, X with $X > 0$,

$$h = (X^2-1)/e \in \mathbb{Z}_{\ge 0}, \quad Y^2 = ah+1, \quad s\sqrt{e} + \sqrt{ec+1}\sqrt{a} = (Y\sqrt{e} + X\sqrt{a})\eta^7, \quad \eta = u + \sqrt{ae}. \tag{4}$$

Odd parity alone gives $s \equiv uY \pmod a$. Thus Lemma 7.35 supplies integers

$$W = (2ju + \lambda Yq)/a, \quad V = (2j + \lambda uYq)/a, \quad E = W^2 - e\mu_j = V^2 - ehq^2,$$

with

$$a^2E = 4j^2 + 4\lambda juYq + (Y^2 + u^2 - 1)q^2. \tag{5}$$

The identity of Lemma 2.22 (4), with $2\lambda Y_* = T_* - q$, $T_* - ajm = 2a^2m\delta$ and $a\mu_j = 4j^2 - q^2$, gives

$$am(4\delta q - j^2 m) = 4j\delta(n^2-1) + 12jkn^2 - 2jmq - \mu_j. \tag{6}$$

It gives $q/j > m/(8a) > 0$. To see this, divide (6) by j, move the term $-ajm^2$ to the right, and bound that right-hand side from below by

$$4\delta n^2 - a|j|m^2 - 4\delta - 2m|q| - \mu_j/|j|.$$

Use $\mu_j/|j| \le 4|j|/a$, $n \ge m/2$, $\delta \ge m$. The bounds in (2) give $4a|j|/\delta$ below the value of the table and

$$8|q|/m^2 < 0.001, \qquad 16|j|/(am^3) < 0.001, \qquad 4/n^2 < 0.001.$$

Thus this lower bound is greater than $m^2\delta/2$, which proves $q/j > m/(8a)$. If $h = 0$, (5), (2) and $e \le e_{\max}$ give

$$|E| < 4A^2am^6/c + 4ABum^7/c + B^2(e + 1/a)m^8/c,$$

which is below the value of the table and so below 1. Then $E = 0$, and Lemma 7.35 gives $uq = -2\lambda jY$. Combined with $q/j > m/(8a)$, this forces $me < 16u$, contradicting (1)–(2). Consequently $h \ge 1$, $X \ge 2$ and $E \ne 0$.

Put $Z = |Y|$. The norm (5) and $uZ < \sqrt{2}aX$ give

$$\frac{|E|}{X} < \frac{4A^2am^6}{cX} + 4AB\sqrt{2}\frac{am^7}{c} + B^2\Big(\frac{e}{X} + \frac{X}{e}\Big)\frac{m^8}{c}. \tag{7}$$

The growth of the fixed seventh seed in Lemma 7.40 gives $c > 2048X^2a^7e^6$ if $Y > 0$, and $c < 2048a^7e^6/X^2$ if $Y < 0$. Together with (2), $a > \sqrt{c}/(2Am^3)$ and $ae < u^2 < \rho^2 m^{6/5}$ these give

$$Y > 0: \quad e^3X < \frac{(2A)^{7/2}}{\sqrt{2048}}\,\frac{m^{21/2}}{c^{5/4}}, \qquad Y < 0: \quad \sqrt{e}\,X < \sqrt{2048}\,\rho^7\,\frac{m^{21/5}}{\sqrt{c}}. \tag{8}$$

The right sides increase with m and decrease with c; at $m = M_*$, $c = c_0$ they are below 965.061738 and 306.535523 in row Y1, and below 4800 and 120 in row Y2. Hence $X \le 965$ in row Y1, and in row Y2

$$X < 4800, \qquad e^3X < 4800 \text{ if } Y > 0, \qquad eX^2 < 14400 \text{ if } Y < 0. \tag{8a}$$

In particular $h < X^2 < a$, since $a > 3.5m > 3.5 \cdot 2^{156/5}$. Since $e/X < \sqrt{e} < \sqrt{2A}\,\rho\, m^{21/10}/c^{1/4}$, (7) is bounded for $Y > 0$ by

$$F_+ = \frac{4A^2\rho^2m^{36/5}}{c} + 4AB\sqrt{2}\rho^2\frac{m^{41/5}}{c} + B^2\sqrt{2A}\rho\frac{m^{101/10}}{c^{5/4}} + \frac{B^2(2A)^{7/2}}{\sqrt{2048}}\frac{m^{37/2}}{c^{9/4}}.$$

For $Y<0$, replace the last term by $B^2\sqrt{2048}\rho^7 m^{61/5}/c^{3/2}$ and call the result F_-. Both increase with m and decrease with c; at $m=M_*$, $c=c_0$ they are below the values of the table, so $0<|E|<E_*X$. Also $Q:=|q|<Q_{\max}$ and $v:=|V|<XQ+\sqrt{E_*X}<v_{\max}$.

Put $D=X^2-1$ and reduce $v+Q\sqrt{D}$ by a power θ^ℓ of $\theta=X+\sqrt{D}$ into $[\sqrt{|E|}\theta^{-1/2},\sqrt{|E|}\theta^{1/2})$, obtaining $w_0=v_0+q_0\sqrt{D}$. As θ has norm 1, w_0 has norm $v^2-DQ^2=E$, so $v_0-q_0\sqrt{D}=E/w_0$ and $q_0=(w_0-E/w_0)/(2\sqrt{D})$. On the interval, $|w_0-E/w_0|$ is at most its value $\sqrt{|E|}\,(\theta^{1/2}\mp\theta^{-1/2})$ at an end point, with the sign $-$ for $E>0$ and $+$ for $E<0$; as $(\theta^{1/2}\mp\theta^{-1/2})^2=2X\mp2$ and $4D=4(X-1)(X+1)$, this gives

$$q_0^2\le E/[2(X+1)]\ \ (E>0),\qquad q_0^2\le |E|/[2(X-1)]\ \ (E<0). \tag{10}$$

Hence $|q_0|\le q_{\max}$: in row Y1, for $X=2$ the strict bound $|E|<8$ excludes $q_0^2=4$, and for $X\ge3$ the negative-norm bound is at most $2X/(X-1)\le3$; in row Y2, (10) and $|E|<17X$ give $q_0^2\le17$. The exponent ℓ is nonnegative: as $v,Q\ge0$, $|v-Q\sqrt{D}|\le v+Q\sqrt{D}$, so $(v+Q\sqrt{D})^2\ge|E|$ and $v+Q\sqrt{D}\ge\sqrt{|E|}>\sqrt{|E|}\theta^{-1/2}$. For $\sigma=\operatorname{sign}(Y)$, the inequality $q/j>0$ gives V the sign σ; hence $q=\lambda Q$, $V=\sigma v$ and $2j=\sigma\mathcal{J}$ with $\mathcal{J}=av-uZQ$. The exact transport supplies independently signed roots

$$w^2=e\mu_j+E,\qquad z_\mu^2=h\mu_j+E,$$

and reconstructs

$$u=(vw+eQz_\mu)/E,\qquad Z=(hQw+vz_\mu)/E. \tag{11}$$

Indeed $w=uv-eZQ$, and eliminating Z gives a quadratic in u with discriminant $4e^2Q^2(h\mu_j+E)$; the two independent signs cover every original solution.

Let $\mathcal{X}$ be the set of integers X with $2\le X\le965$ in row Y1, and with $2\le X<4800$ and ($e^3X<4800$ or $eX^2<14400$) in row Y2. The following exact seed tuples are enumerated.

1. For $q_0=0$, take $1\le e\le e_{\max}$, $1\le r\le\lfloor\sqrt{\mu_{\max}}\rfloor$ and $1\le a_0\le\lfloor\mu_{\max}/r^2\rfloor$ with ea_0+1 a square (2785 triples in row Y1, 4134 in row Y2). For every $X\in\mathcal{X}$ with $h=(X^2-1)/e$ integral and ha_0+1 a square, set $\mu_j=r^2a_0$, $E=r^2$, $(|v_0|,|q_0|)=(r,0)$, $|w|=r\sqrt{ea_0+1}$ and $|z_\mu|=r\sqrt{ha_0+1}$. The divisibility $r^2\mid\mu_j$ follows from $w=r(uT_\ell-eZU_{\ell-1})$, whose square divided by r^2 is 1 modulo e; no common divisor is cancelled.

2. For $1\le|q_0|\le q_{\max}$, take the same ranges of e, X, h, and every nonnegative v_0 with $\max(0,Dq_0^2-E_*X+1)\le v_0^2\le Dq_0^2+E_*X-1$. Set $E=v_0^2-Dq_0^2\ne0$ and retain (10). Enumerate the nonnegative w in all square-root residue classes modulo e with $\max(0,E+e)\le w^2\le E+\mu_{\max}e$. Set $\mu_j=(w^2-E)/e$, retain only nonnegative squares $h\mu_j+E$, and let $|z_\mu|$ be the root.

This gives 16660 distinct seed tuples in row Y1 and 26726 in row Y2; the second step tests 24072, resp. 119284, norm pairs before the square conditions. All four independent signs of (v_0,q_0) and all four independent signs of (w,z_μ) are used, zero-coordinate duplicates identified, and

$$(v,Q)\longmapsto(Xv+DQ,\ v+XQ),\qquad \ell=0,\dots,49,$$

is iterated, retaining $0<v<v_{\max}$ and $0<Q<Q_{\max}$. These exponents cover everything: the absolute-coordinate sum of a seed is less than 8000, resp. 50000; its positive value is at least the reciprocal of that number, by its nonzero integer norm; and the original value is less than $3\cdot10^9$, resp. $1.2\cdot10^{10}$; so $\theta^\ell<3^{50}$.

The reconstruction (11) retains the integers $0 < u < 3 \cdot 10^7$, resp. $2.3 \cdot 10^7$, with $Z > 0$, $a = (u^2-1)/e \in \mathbb{Z}$, $e < a < 10^{15}$, resp. $5 \cdot 10^{14}$, and $h < a$; duplicates $(e, \mu_j, h, u, Z, v, Q)$ are identified, and only even nonzero $\mathcal{J}$ are kept. For both $\sigma = \pm 1$ put

$$s = \sigma Z T_7(u) + aXU_6(u), \qquad \sqrt{ec+1} = XT_7(u) + \sigma eZU_6(u), \qquad c = (s^2-1)/a,$$

retaining positive coordinates with $c_0 \le c < 10^{130}$, and in row Y2 only $e^3X < 4800$ for $\sigma = 1$ and $eX^2 < 14400$ for $\sigma = -1$ (by (8a)). Then $j = \sigma\mathcal{J}/2$, $\lambda = \operatorname{sign}(j)$ and $q = \lambda Q$.

The transport identities $\mathcal{J}^2 - Q^2 = a\mu_j$ and $aE + \mu_j = (e+h)Q^2 + 2\mathcal{J}v$ give $\gcd(\mathcal{J}, a) = \gcd(Q, a) \mid \mu_j$, so $g = \gcd(j, 4a) \le 4\mu_j \le 4\mu_{\max}$. Define the integer

$$Q_0 = (q + 2\lambda j s)/a = m(4a\delta + j). \tag{12}$$

If $g \nmid Q_0$, the configuration is excluded. Otherwise put $d = 4a/g$ and $m_0 \equiv (Q_0/g)(j/g)^{-1} \pmod d$, $0 \le m_0 < d$. Since $m/d \le m\mu_j/a < \mu_{\max}/6.9 < 194$ in row Y1 and $< \mu_{\max}/3.5 < 559$ in row Y2, every possible index is $m_0 + td$ with $0 \le t < 194$, resp. $t < 559$. Only $0 < m < a/6.9$, resp. $a/3.5$, are kept, and then, in this order, the necessary conditions

$$\begin{gathered} m < M_*, \quad m^5 > 2^{156}, \quad c < 2^{18}m^9, \quad m > 2u, \\ 9a < m^2, \quad (100u)^5 < (100\rho)^5 m^3, \quad 40000c < 363609a^2m^6 \end{gathered} \tag{13}$$

are applied; the last two are exactly $u < \rho m^{3/5}$ and $c < 4A^2a^2m^6$ from (2). The exact computation gives

	row Y1	row Y2
signed secondary orbits	112127	218303
integral reconstructions	72985	149655
even nonzero $\mathcal{J}$	31991	63541
configurations with $c_0 \le c < 10^{130}$	2198	747
solvable congruences (12)	1179	368
residues with $0 < m < a/6.9$, resp. $a/3.5$	81	42

(in row Y2 the configurations are counted after the filters of (8a)). In row Y1 the condition $m < M_*$ leaves 64 residues, $c < 2^{18}m^9$ leaves 53, and $u < \rho m^{3/5}$ leaves 17; in row Y2, $m < M_*$ leaves 7 and $u < \rho m^{3/5}$ leaves 6. In both rows every remaining residue fails $40000c < 363609a^2m^6$. The computation uses only the displayed finite recurrences, integer square tests, rational reconstruction and the modular translates. Thus no possible index remains, which contradicts $m' < R$ and proves the lemma. □

7.5 The generic windows

The generic windows collect what holds for every triple of degree at least two with $c \ge 10^{30}$: the type I floor of He and Togbé, the caps, the class bounds and, for a middle or low triple, the first-order quotient.

Proposition 7.45. *Let $a < b < c < d$ be an irregular Diophantine quadruple with $\deg\{a,b,c\} \ge 2$ and $c \ge 10^{30}$, let $T = \{a,b,c\}$, with descent partner e, let $\kappa = (\sqrt{17}-1)/2$, and let Q_{I}, Q_{II} be the index caps. Then $b > 4000$, and every representation (λ, m, n) of z satisfies the following.*

1. *If it is of type I: $c < 10^{99}$, $\kappa\sqrt{c/b} < n < m \le 2n$ and $m' < Q_{\mathrm{I}}(c)$.*

2. If it is of type II: $c < 10^{79.71}$, $4 \le n \le m \le 2n$ and $m' < Q_{\mathrm{II}}(c)$. If moreover T is middle or low, then $c < b^2$, $c < 10^{77.28}$, $u = \sqrt{ae+1}$ is an integer, and the first-order quotient J is an integer with $J \ge 1$ and $J(4ae+1) < 2m(m+1)$; in particular $u < m'/(2\sqrt{2})$.

Proof. By Lemma 2.9, $d > d_+(a,b,c)$ and $b > 4000$. Since $\deg T \ge 2$, T is not Euler, so by canonical descent (Theorem 2.7) e is a positive integer and $c = d_+(a,b,e)$; by Lemma 2.6, item 3, $c > a+b+e+4abe \ge 4ab+a+b+1$.

Item 1. By Proposition 7.33, $c < 10^{99}$. By Lemma 2.23, $m > n > \kappa\sqrt{c/b}$, and by Lemma 2.14, $m \le 2n$. Since $10^{30} \le c < 10^{99}$, item 1 of Proposition 5.15 gives $m' < Q_{\mathrm{I}}(c)$.

Item 2. By Proposition 7.19, $c < 10^{79.71}$, and by Lemma 2.14, $n \le m \le 2n$. Since $b > 4000$, $c > 4ab+a+b$ and $d > d_+(a,b,c)$, Lemma 2.15 gives $n \ge 4$. Since $10^{30} \le c < 10^{79.71}$, item 2 of Proposition 5.15 gives $m' < Q_{\mathrm{II}}(c)$. Let moreover T be middle or low. Then $c < b^2$ by Lemma 3.5, so $c < 10^{77.28}$ by Proposition 7.19; and Lemma 7.3 gives that u and J are integers with $J \ge 1$, $J(4ae+1) < 2m(m+1)$ and $u < m'/(2\sqrt{2})$. □

8 The windows of the families of degree at least two

The windows of BH, BM, BL, B3 and B4 complete the proof of Theorem 4.3. Each of them is the generic window of Proposition 7.45 together with conditions from the structure of the triple: for a high triple, from the first-order defect and the descent partner f of the descent; for a single-chain middle or low triple, from the chain of $\{a,e\}$, on which $b = x_{\nu-1}$ and $c = x_\nu$ at slot $\nu = \deg\{a,b,c\}+1$ (Lemma 3.4).

8.1 High triples: the family BH

For a high triple the type I window adds the floor of Lemma 7.1, which against the caps gives $c < 5.7 \cdot 10^{45}$. The type II window splits by the first-order defect: it vanishes when $a\sqrt{c}$ is large compared with the cap (Corollary 8.1), and when it vanishes the descent partner f gives a floor (Lemma 8.2).

Corollary 8.1. *Let $a < b < c < d$ be an irregular Diophantine quadruple with $\deg\{a,b,c\} \ge 2$ and $10^{30} \le c < 10^{65}$ such that $T = \{a,b,c\}$ is high, let z have a type II representation with first-order defect H, and let Q_{II} be the type II index cap. If*

$$a\sqrt{c} \;\ge\; \Big(\frac{Q_{\mathrm{II}}(c) - \frac{1}{2}}{1.06}\Big)^2,$$

then $H = 0$. In particular $H = 0$ whenever $10^{55} \le c < 10^{65}$, and whenever $10^{50} \le c < 10^{55}$ and either $a \ge 2$ or $c \ge 3.22 \cdot 10^{50}$.

Proof. If $H \ne 0$, the index cap gives a contradiction: by Proposition 5.15, item 2, $m' < Q_{\mathrm{II}}(c)$; since m' and the values of Q_{II} are integers, $m' \le Q_{\mathrm{II}}(c) - 1$. Suppose $H \ne 0$. Then Lemma 7.4 gives

$$m' > 1.06\sqrt{a}\,c^{1/4} - 0.5 = 1.06\,(a\sqrt{c})^{1/2} - 0.5 \;\ge\; \Big(Q_{\mathrm{II}}(c) - \frac{1}{2}\Big) - \frac{1}{2} = Q_{\mathrm{II}}(c) - 1,$$

which contradicts $m' \le Q_{\mathrm{II}}(c) - 1$. Hence $H = 0$.

For the particular cases write $\tau_c = \big((Q_{\mathrm{II}}(c) - \frac{1}{2})/1.06\big)^2$ for the right-hand side of the hypothesis. For $10^{55} \le c < 10^{65}$ we have $Q_{\mathrm{II}}(c) \le 6.37 \cdot 10^{12}$ (Definition 4.1), so $\tau_c <$

$(6.01\cdot 10^{12})^2 < 3.62\cdot 10^{25} < 10^{27.5} \le \sqrt{c} \le a\sqrt{c}$. For $10^{50} \le c < 10^{55}$, $Q_{\mathrm{II}}(c) = 4.49\cdot 10^{12}$ gives $\tau_c < (4.2359\cdot 10^{12})^2 < 1.7943\cdot 10^{25}$; if $a \ge 2$ then $a\sqrt{c} \ge 2\cdot 10^{25} > \tau_c$, and if $c \ge 3.22\cdot 10^{50}$ then $\sqrt{c} > 1.7944\cdot 10^{25} > \tau_c$, since $1.7944^2 < 3.2199 < 3.22$. □

Lemma 8.2. *Assume the setting of Lemma 7.5 (a high triple T of degree at least 2 with descent partner e, $f = e(\partial T)$, a type II representation (λ, m, n) with $H = 0$), and put*

$$T_0 = 2(e-a-b)(a+b) + 12ab + 6.$$

Then $\lambda = -1$, $\Delta \ge 1$, $L \ge 0$, and either $m \ge fb$, or $\Delta\, T_0 > c$. In the second case

$$\Delta > \frac{4ab+1}{2a+2b+3} > 1.77\,a, \qquad n(n+1) > 3.54\,a^2 f,$$
$$m > 1.88\,a\sqrt{f} + 1.77\,a - \tfrac{1}{2}, \qquad m' = 2m+1 > 3.76\,a\sqrt{f} + 3.54\,a.$$

In particular, with $\Delta_0 = \lfloor c/T_0 \rfloor + 1$ and n_0 the least integer $n_0 \ge 0$ with $2n_0(n_0+1) \ge (4af+1)\Delta_0$,

$$m \ge \min\{fb,\ n_0 + \Delta_0\}.$$

Proof. By Lemma 2.9 and Lemma 2.10, $b > 4000$ and $b > 8a$. By Lemma 7.5, $\lambda = -1$, $\Delta \ge 1$ and $L \ge 0$; and, as there, canonical descent (Theorem 2.7) gives $f = d_-(a,b,e) \ge 1$, $e = d_+(a,b,f)$ and $c = d_+(a,b,e)$.

First, $e - a - b > 4abf$: by Definition 2.2,

$$e = d_+(a,b,f) = a + b + f + 2abf + 2\sqrt{(ab+1)(af+1)(bf+1)} > a + b + f + 4abf.$$

Second, $c > (4ab+1)e$: by Lemma 2.6, item 1, the two numbers $c = d_+(a,b,e)$ and $f = d_-(a,b,e)$ satisfy $c + f = 2(a+b+e) + 4abe$ and $cf = (e-a-b)^2 - 4(ab+1) < e^2$, so $f < e^2/c < e$ (as $c > e$) and $c = (4ab+2)e + 2(a+b) - f > (4ab+1)e$. Third, $T_0 < e(2a+2b+3)$: since $e - a - b < e$ and $3e > 12ab + 6$ (because $e > 4abf + a + b + f > 4ab + 2$),

$$T_0 = 2(e-a-b)(a+b) + 12ab + 6 < 2e(a+b) + 3e.$$

By Lemma 7.2 with $H = 0$ and $\lambda = -1$, whose E_1 is $e - a - b$,

$$c \mid \Delta\big((4ab-4)(S+1) - T_0\big),$$

since $2\big(6ab + 3 + (e-a-b)(a+b)\big) = T_0$. Put $X = (4ab-4)(S+1) - T_0$. If $X \ge 0$, then $(4ab-4)(S+1) \ge T_0 > 2(e-a-b)(a+b) > 8abf(a+b)$, hence $S + 1 > 2f(a+b) > 2fb$; as $S \le 2m$, this gives $2m + 1 > 2fb$, that is, $m \ge fb$. If $X < 0$, then $\Delta X \ne 0$ is a multiple of c, so $c \le \Delta|X| = \Delta\big(T_0 - (4ab-4)(S+1)\big) < \Delta T_0$.

In the second case, from $\Delta T_0 > c$, $c > (4ab+1)e$ and $T_0 < e(2a+2b+3)$,

$$\Delta > \frac{c}{T_0} > \frac{(4ab+1)e}{e(2a+2b+3)} = \frac{4ab+1}{2a+2b+3}.$$

Then $2n(n+1) \ge (4af+1)\Delta > (4af+1)\cdot 1.77a > 7.08\,a^2 f$ (as $L \ge 0$), so $n(n+1) > 3.54\,a^2 f$ and $n > \sqrt{3.54\,a^2 f + \frac{1}{4}} - \frac{1}{2} > 1.88\,a\sqrt{f} - \frac{1}{2}$. Hence $m = n + \Delta > 1.88\,a\sqrt{f} + 1.77\,a - \frac{1}{2}$ and $m' = 2m + 1 > 3.76\,a\sqrt{f} + 3.54\,a$.

For the last assertion: in the second case the integer Δ satisfies $\Delta > c/T_0$, so $\Delta \ge \Delta_0$; then $2n(n+1) \ge (4af+1)\Delta \ge (4af+1)\Delta_0$ gives $n \ge n_0$, and $m = n + \Delta \ge n_0 + \Delta_0$. In the first case $m \ge fb$.

□

Proposition 8.3. *Let $T = \{a, b, c\}$ be a triple of the family BH of Definition 4.2 (*$\deg T \geq 2$*, T high, $c \geq 10^{30}$), and let d be an irregular extension of T. Then every representation ρ of $z = \sqrt{cd+1}$ satisfies the window of BH of its type, that is, $(T, \rho) \in$ BH.*

Proof. Let $\rho = (\lambda, m, n)$ be a representation of z. By Proposition 7.45, $b > 4000$ and ρ satisfies the generic window of its type.

In type I, Lemma 7.1 gives $c > 16a^2b^2$ and $n > 2\kappa\sqrt{a}\,c^{1/4}$, so $m' = 2m > 2n > 2(\sqrt{17} - 1)\sqrt{a}\,c^{1/4}$. Suppose $c \geq 5.7 \cdot 10^{45}$. If $c < 10^{50}$, then $Q_{\mathrm{I}}(c) = 1.71 \cdot 10^{12}$ (Definition 4.1), while $m' > 2(\sqrt{17} - 1)(5.7 \cdot 10^{45})^{1/4} > 1.716 \cdot 10^{12}$. If $c \geq 10^{50}$, then $Q_{\mathrm{I}}(c) \leq 6.5 \cdot 10^{12}$, while $m' > 2(\sqrt{17} - 1) \cdot 10^{12.5} > 1.97 \cdot 10^{13}$. In both cases $m' > Q_{\mathrm{I}}(c)$, against the generic window; hence $c < 5.7 \cdot 10^{45}$.

In type II with $H \neq 0$, Lemma 7.4 gives $n > 0.53\sqrt{a}\,c^{1/4} - 0.75$ and $m' > 1.06\sqrt{a}\,c^{1/4} - 0.5$. If $10^{50} \leq c < 10^{65}$ and either $c \geq 3.22 \cdot 10^{50}$ or $a \geq 2$, then Corollary 8.1 gives $H = 0$, which is excluded. If $c \geq 10^{65}$, then $c < 10^{79.71}$ by the generic window, so $Q_{\mathrm{II}}(c) \leq 1.03 \cdot 10^{13}$, while $m' > 1.06 \cdot 10^{65/4} - 0.5 > 1.8 \cdot 10^{16}$, against $m' < Q_{\mathrm{II}}(c)$. Hence $c < 3.22 \cdot 10^{50}$, and $c < 10^{50}$ if $a \geq 2$.

In type II with $H = 0$, Lemma 8.2 gives $\lambda = -1$, $\Delta \geq 1$, $L \geq 0$ and $m \geq \min\{fb, n_0 + \Delta_0\}$, with T_0, Δ_0, n_0 as in the window of BH.

Let $c \geq 10^{75}$ in type II. Since $c < 10^{79.71} < 10^{90}$, Proposition 7.28 gives $H = 0$ and $c < \Psi(m)$, where Ψ is increasing on $[4, \infty)$; and $m \geq n \geq 4$. If $m \leq 8.4 \cdot 10^{11}$, then $c < \Psi(m) \leq \Psi(8.4 \cdot 10^{11}) < 10^{75}$, a contradiction. Hence $m > 8.4 \cdot 10^{11}$. □

8.2 Slot three: the families BM and BL

At slot three the chain constants $E' = x_1$, $E = x_{-1}$ and $P = 4u^2 - 2$ make b and c explicit, and the type II congruences become the identity for bK of Lemma 8.4, with the divisibilities of a low triple and the ellipse of a middle one.

Lemma 8.4. *Let $a < b < c < d$ be an irregular Diophantine quadruple such that $T = \{a, b, c\}$ is single-chain of degree 2 and middle or low, with descent partner e. Let σ be a chain sign of T, let (x_j) be the chain of $\{a, e\}$ with sign σ and E', E, P the chain constants of T, and $u = \sqrt{ae+1}$. Then*

$$E' = x_1 \geq 1, \qquad E = x_{-1} \geq 0, \qquad b = x_2 = (4u^2 - 1)E' + E, \qquad c = x_3 = Pb + E.$$

Let z have a type II representation (λ, m, n), with $\Delta = m - n$, $S = m + n$ and N, J, K as in Definition 2.20.

1. *J is an integer, $J \geq 1$, $J(4ae+1) < 2m(m+1)$, $0 \leq \Delta \leq m/2$, and*

$$bK = 2am(m+1) + \lambda\Delta(a - e) + JE.$$

2. *If $K \neq 0$, then*

$$b \leq B(m) := 2m(m+1)\Big(a + \frac{E}{4ae+1}\Big) + \frac{m\,|a-e|}{2},$$

with strict inequality when $E > 0$.

3. *Let T be low ($e < a$). Then $K \geq 1$, and:*

(a) *the integer* $\Xi = 4(e-\sigma u)K - J(e+2\sigma u) + \lambda\Delta e$ *satisfies* $a \mid \Xi$. *If* $\Xi = 0$, *then* $h = 2(2K+J)/e$ *is a positive integer and*

$$J = \tfrac{1}{2}he - 2K, \qquad \lambda\Delta = \tfrac{1}{2}\big(h(e+2\sigma u) - 12K\big), \qquad 2m(m+1) = 4K(u-\sigma e)^2 - J - \lambda\Delta;$$

if moreover $\sigma = 1$, *then* $\lambda = 1$, $\Delta = 3J + h(u-e)$, $K \geq 3$, $e \geq 12$ *and* $m > \sqrt{6}\,(u-e) - \frac{2}{3}$; *if moreover* $\sigma = -1$ *and* $u \geq 5e$, *then* $\lambda = -1$, $K > h^2$, $e \geq 10$ *and* $m > 2(u+e) - \frac{1}{2}$;

(b) *with* $\Gamma = 4aE + 4P$,

$$V = -12aE + 24u^2 - 12 + 2\big[E(b+2a-e) - Pa(a-e)\big], \qquad Z = \Delta\big[\lambda V - \Gamma(S+1)\big] - 6JP,$$

one has $c \mid Z$; *if* $Z = 0$, *then* $\Delta \geq 1$, $\lambda V > 0$ *and* $|V| - 6JP \leq \Gamma(S+1) \leq |V|$.

4. *Let* T *be middle* ($e > a$) *and* $K = 0$. *Then* $\lambda = 1$, $\Delta \geq 1$, $e - a > 4a(m+1)$ *(so* $e \geq a+3$ *and* $E > 0$*),* $N = JP$, $JE = \Delta(e-a) - 2am(m+1)$, *and* (m,n) *lies on the* ***ellipse***

$$2En(n+1) + 2aPm(m+1) = \Delta\big[P(e-a) + E\big].$$

Proof. By Lemma 3.4, items (i) and (ii), $e \notin T$ and the chain pair of T is $\{a,e\}$; by the definition of a chain sign (Definition 3.2), T lies at slot 3 of the chain of $\{a,e\}$, so T is one of $\{a,e,x_3\}$, $\{a,x_2,x_3\}$, $\{e,x_2,x_3\}$. The descent partner e is not an element of T, and $e \neq a$ since a^2+1 is not a square; so $T = \{a,x_2,x_3\}$, and as $x_2 > \max\{a,e\}$ and $x_2 < x_3$ (Lemma 3.3, item 3), $b = x_2$ and $c = x_3$. By Lemma 3.3, item 2, $E' = x_1 \geq 1$ (σ is admitted) and $E = x_{-1} \geq 0$, with $E = 0$ only if $|a-e| = 2$. The recurrence of Definition 3.2 and $E + E' = 2(a+e)$ give $b = x_2 = PE' + 2(a+e) = (4u^2-1)E' + E$ and $c = x_3 = Pb - E' + 2(a+e) = Pb + E$. With $u^2 = ae+1$,

$$b = 4a(u-\sigma e)^2 + 4(e-\sigma u). \tag{1}$$

Item 1. Since $\deg T = 2$ and T is middle or low, Lemma 7.3 gives that u and J are integers, $J \geq 1$ and $J(4ae+1) < 2m(m+1)$; in particular $m \geq 1$. By Lemma 2.14, $n \leq m \leq 2n$, so $0 \leq \Delta \leq m/2$. Since $c = Pb + E$, the relation $H = Jc$ reads $bN - aM + e\lambda\Delta = JPb + JE$, that is $b(N - JP) = aM - e\lambda\Delta + JE$, and $aM - e\lambda\Delta = 2am(m+1) + \lambda\Delta(a-e)$:

$$bK = 2am(m+1) + \lambda\Delta(a-e) + JE. \tag{2}$$

Item 2. Since $J(4ae+1) < 2m(m+1)$ and $E \geq 0$, $JE \leq 2m(m+1)E/(4ae+1)$, strictly when $E > 0$. If $K \geq 1$, then by (2) and $\Delta \leq m/2$,

$$b \leq bK \leq 2am(m+1) + \frac{m|a-e|}{2} + \frac{2m(m+1)E}{4ae+1},$$

strictly when $E > 0$. If $K \leq -1$, then (2) gives $-\lambda\Delta(a-e) = -bK + 2am(m+1) + JE \geq b + 2am(m+1) > b$, so $b < \Delta|a-e| \leq m|a-e|/2 < B(m)$.

Item 3. Let $e < a$. Then $0 < a - e < a$ and $\Delta \leq m/2$, so $\lambda\Delta(a-e) \geq -\frac{1}{2}am$, and with $JE \geq 0$ and $m \geq 1$ the right side of (2) is at least $2am(m+1) - \frac{1}{2}am > 0$. Hence $K \geq 1$. From $u^2 = ae+1$ we get $\gcd(e,u) = 1$, and $u > e$ because $a > e$.

For (a): by (1), $b \equiv 4(e-\sigma u)$, and $E \equiv e + 2\sigma u$, $a - e \equiv -e \pmod{a}$; reducing (2) modulo a gives $a \mid \Xi$. In fact (1) gives the exact decomposition

$$bK - JE - \lambda\Delta(a-e) = a\big[4K(u-\sigma e)^2 - J - \lambda\Delta\big] + \Xi. \tag{3}$$

Let $\Xi = 0$, that is $e(4K - J + \lambda\Delta) = 2\sigma u(2K + J)$. Since $\gcd(e, u) = 1$, $e \mid 2(2K + J)$; put $h = 2(2K + J)/e \geq 1$. Then $4K - J + \lambda\Delta = \sigma uh$ and $2K + J = \frac{1}{2}he$, whence $J = \frac{1}{2}he - 2K$ and $\lambda\Delta = \sigma uh - 4K + J = \frac{1}{2}(h(e + 2\sigma u) - 12K)$. By (2) and (3), $2am(m+1) = a[4K(u - \sigma e)^2 - J - \lambda\Delta]$, which is the displayed equation for $2m(m + 1)$.

Let moreover $\sigma = 1$. Then $\lambda\Delta - 3J = \frac{1}{2}(h(e + 2u) - 12K) - \frac{3}{2}he + 6K = h(u - e) > 0$, so $\lambda = 1$ and $\Delta = 3J + h(u - e) \geq 3 + h(u - e)$. Since $m \geq 2\Delta$ and $m^2 < m(m + 1) = 2K(u - e)^2 - \frac{1}{2}(J + \Delta) \leq 2K(u - e)^2$,

$$6 + 2h(u - e) < \sqrt{2K}\,(u - e), \qquad \text{so} \qquad 2h < \sqrt{2K}, \quad K > 2h^2 \geq 2, \quad K \geq 3.$$

From $J \geq 1$, $he \geq 4K + 2$, so $e \geq (4K + 2)/h > (4K + 2)\sqrt{2/K} = 4\sqrt{2K} + 2\sqrt{2}/\sqrt{K}$; the last expression increases with $K \geq 1$ and equals $11.43\ldots$ at $K = 3$, so $e \geq 12$. Finally $J = \frac{1}{3}(\Delta - h(u - e)) \leq \Delta/3$, so $J + \Delta \leq \frac{4}{3}\Delta \leq \frac{2}{3}m$, and $m^2 + m \geq 6(u - e)^2 - \frac{1}{3}m$ gives $(m + \frac{2}{3})^2 \geq 6(u - e)^2 + \frac{4}{9}$, that is, $m > \sqrt{6}(u - e) - \frac{2}{3}$.

Let moreover $\sigma = -1$ and $u \geq 5e$. Then $\lambda\Delta = \frac{1}{2}(h(e - 2u) - 12K) < 0$, so $\lambda = -1$ and $\Delta = \frac{1}{2}(h(2u - e) + 12K)$; moreover $\Delta - J = h(u - e) + 8K$ and $m(m+1) = 2K(u+e)^2 + \frac{1}{2}h(u-e) + 4K$. From $m \geq 2\Delta \geq h(2u - e) \geq \frac{9}{5}hu$ and $(u + e)^2 \leq \frac{36}{25}u^2$,

$$\tfrac{81}{25}h^2u^2 \leq m^2 < \tfrac{72}{25}Ku^2 + \tfrac{1}{2}hu + 4K.$$

Dividing by u^2 and using $u \geq 5$: $3.24h^2 < 2.88K + 0.1h + 0.16K$, so $K > (3.24h^2 - 0.1h)/3.04 \geq h^2$. Then $he \geq 4K + 2 \geq 4h^2 + 6$ gives $e \geq 4h + 6/h \geq 10$, and $K \geq 2$ gives $m(m+1) \geq 4(u+e)^2$, so $m > 2(u + e) - \frac{1}{2}$.

For (b): put $C_0 = a + b - e$. By Lemma 7.2,

$$c \mid Y, \qquad Y = \Delta\big[(8ab - C_0^2)(S + 1) + 2\lambda(6ab + 3 - C_0(a + b))\big] - 6J.$$

By Lemma 3.3, item 3, $x_3 = d_+(a, e, x_2)$ and $x_1 = d_-(a, e, x_2)$, so Lemma 2.6, item 1, with $(p, q, X) = (a, e, b)$ gives $cE' = (b - a - e)^2 - 4(ae + 1) = C_0^2 - 4(ab + 1)$; hence $8ab - C_0^2 \equiv 4ab - 4 \pmod{c}$. Moreover $Pb \equiv -E \pmod{c}$. Multiplying Y by P and reducing each term modulo c with these two congruences gives $PY \equiv Z \pmod{c}$. Since $c \mid Y$, it follows that $c \mid Z$. If $Z = 0$, then $\Delta[\lambda V - \Gamma(S + 1)] = 6JP > 0$, so $\Delta \geq 1$ and $\lambda V - \Gamma(S + 1) = 6JP/\Delta > 0$; thus $\lambda V > \Gamma(S + 1) > 0$, $|V| = \lambda V$, and $\Gamma(S + 1) = |V| - 6JP/\Delta$ lies between $|V| - 6JP$ and $|V|$.

Item 4. Let $e > a$ and $K = 0$. Then (2) gives $\lambda\Delta(e - a) = 2am(m+1) + JE \geq 2am(m+1) > 0$, so $\lambda = 1$, $\Delta \geq 1$, and with $\Delta \leq m/2$, $e - a \geq 4a(m + 1) > 2$. Hence $|a - e| \neq 2$, so $E > 0$, $JE > 0$, and the same computation gives $e - a > 4a(m + 1)$; in particular $e \geq a + 3$. $N = JP$ is the definition of $K = 0$, and (2) with $\lambda = 1$ is $JE = \Delta(e - a) - 2am(m + 1)$. Multiplying by P and replacing JP by $N = 2n(n + 1) - \Delta$ gives $E(2n(n + 1) - \Delta) = P\Delta(e - a) - 2aPm(m + 1)$, which is the equation of the ellipse.

□

Proposition 8.5. *Let $T = \{a, b, c\}$ be a triple of the family BM or of the family BL of Definition 4.2 (T single-chain, $\deg T = 2$, middle or low, $c \geq 10^{30}$), and let d be an irregular extension of T. Then every representation ρ of $z = \sqrt{cd + 1}$ satisfies the window of the family of T of its type, that is, $(T, \rho) \in$ BM if T is middle and $(T, \rho) \in$ BL if T is low.*

Proof. Let $\rho = (\lambda, m, n)$ be a representation of z. By Proposition 7.45, $b > 4000$ and ρ satisfies the generic window of its type; this is the whole window of type I. Let ρ be of type II. By Lemma 3.4, item (iii), T has a chain sign σ; let E, P be the chain constants of T for σ. Then Lemma 8.4 applies to the quadruple $\{a, b, c, d\}$ and ρ, and its item 1 gives $bK = 2am(m + 1) + \lambda\Delta(a - e) + JE$.

Let T be middle, so $|a-e| = e-a$. If $K \neq 0$, item 2 of Lemma 8.4 gives $b \leq 2m(m+1)\big(a + \frac{E}{4ae+1}\big) + \frac{m(e-a)}{2}$. If $K = 0$, item 4 gives $e \geq a+3$, $\lambda = 1$, $\Delta \geq 1$, $4a(m+1) < e-a$ and the equation of the ellipse. Hence $(T,\rho) \in \mathrm{BM}$.

Let T be low, so $|a-e| = a-e$. By item 3 of Lemma 8.4, $K \geq 1$, so item 2 gives $b \leq B(m)$. Item 3(a) gives $a \mid \Xi$, hence $|\Xi| \geq a$ if $\Xi \neq 0$, and the stated consequences of $\Xi = 0$ for $\sigma = 1$, and for $\sigma = -1$ with $u \geq 5e$. Item 3(b) gives $c \mid Z$, hence $|Z| \geq c$ if $Z \neq 0$, and the stated consequences of $Z = 0$. Hence $(T,\rho) \in \mathrm{BL}$. □

8.3 Deeper slots: the families B3 and B4

From slot four on, the chain grows fast in $u = \sqrt{ae+1}$, and the index conditions become rigid. After size estimates for the chain constants (Lemma 8.6) and the pinning data (Definition 8.7), the trace identities of the chain (Lemma 8.8) pin the type I index n to an integer next to $\eta\rho_{\lambda,k}$, where $\eta = u + \sqrt{ae}$ and $\rho_{\lambda,k} = (\sqrt{4k+1} - \lambda)/2$ (Lemma 8.9). The same pinning is the basis of the slot test of Section 9. From slot five on it gives the floor $n \geq \min\{u^{3/2}\vartheta^{-1/2}, u^{5/3}\}$ (Proposition 8.10).

Lemma 8.6. *Let $a \neq e$ be positive integers with $ae+1 = u^2$, u a positive integer, let σ be an admitted sign of $\{a,e\}$ with chain $(x_j)_{j\in\mathbb{Z}}$ (Definition 3.2), and let D, η, χ, χ', ϑ, G be the chain constants of $(a,e;\sigma)$ (Definition 3.8). Then:*

1. *$u \geq 2$, $2u - 1/u < \eta < 2u$ and $u - 1/u < \sqrt{D} < u$;*
2. *$G \leq \vartheta^2 + (\vartheta+1)^2 = 2\vartheta^2 + 2\vartheta + 1$ and $|\chi'|/|\chi| \leq G$;*
3. *$\vartheta > 1/(u+1)$, and $\vartheta < 1$ if $\sigma = -1$;*
4. *if $u \geq 72$, then $\vartheta \leq u/8$;*
5. *$0 = x_0 < x_1 < x_2 < \cdots$, and for every $j \geq 1$*

$$(4u^2-3)\,x_j < x_{j+1}, \qquad x_j < 4^j u^{2j};$$

6. *in particular, if $x_3 \geq 10^{30}$ then $u > 5\cdot 10^4$, and if $x_4 \geq 10^{30}$ then $u > 2800$.*

Proof. Item 1. If $u = 1$ then $ae = 0$; so $u \geq 2$. Since $\sqrt{D} = \sqrt{u^2-1} < u$ and $u^2 - 1 > (u - 1/u)^2 = u^2 - 2 + 1/u^2$, we have $u - 1/u < \sqrt{D} < u$, and adding u gives the bounds for η.

Item 2. By definition $G = \vartheta^2 + e/\chi^2$. If $\sigma = -1$, then $|\chi| = \sqrt{a} + \sqrt{e} \geq \sqrt{e}$, so $e/\chi^2 \leq 1$. If $\sigma = 1$, then $|\chi| = |\sqrt{a} - \sqrt{e}| \geq \sqrt{e} - \sqrt{a}$, so $\sqrt{e}/|\chi| \leq 1 + \vartheta$. In both cases $e/\chi^2 \leq (\vartheta+1)^2$. Moreover $\chi\chi' = a - e$, so $|\chi'|/|\chi| = |a-e|/\chi^2 \leq (a+e)/\chi^2 = G$.

Item 3. Since $|\chi| \leq \sqrt{a} + \sqrt{e}$ and $a \geq 1$, $\vartheta \geq \sqrt{a}/(\sqrt{a}+\sqrt{e}) \geq 1/(1+\sqrt{e})$; as $e < ae+1 = u^2$, this exceeds $1/(1+u)$. If $\sigma = -1$, then $|\chi| = \sqrt{a} + \sqrt{e} > \sqrt{a}$ and $\vartheta < 1$.

Item 4. If $\sigma = -1$, then $\vartheta < 1 < u/8$. Let $\sigma = 1$. Then $x_1 = a + e - 2u \geq 1$ and $u > \sqrt{ae}$, so $\chi^2 = a + e - 2\sqrt{ae} = x_1 + 2(u - \sqrt{ae}) > 1$, and $\vartheta < \sqrt{a}$. Suppose $\vartheta > u/8$. Then $\sqrt{a} > u/8$, so $64a > u^2 > ae$ and $e \leq 63$. Moreover $u|\chi| < 8\sqrt{a}$, and with $u > \sqrt{ae}$ this gives $\sqrt{e}\,|\sqrt{a} - \sqrt{e}| < 8$. If $a > e$, this reads $\sqrt{ae} < e + 8 \leq 71$, so $u^2 = ae + 1 \leq 71^2$. If $a < e$, then $a < e \leq 63$ gives $ae < 63^2$. In both cases $u \leq 71$. Hence $u \geq 72$ implies $\vartheta \leq u/8$.

Item 5. First, $x_1 \geq 1 > 0 = x_0$; if $x_j > x_{j-1} \geq 0$ for some $j \geq 1$, then

$$x_{j+1} - x_j = (4u^2-3)x_j - x_{j-1} + 2(a+e) > 0,$$

so the sequence $(x_j)_{j\ge 0}$ is strictly increasing. For $j \ge 1$ the recurrence and $x_{j-1} < x_j$ give $x_{j+1} > (4u^2-2)x_j - x_{j-1} > (4u^2-3)x_j$. For the upper bound put $y_j = x_j + (a+e)/(2D)$ for $j \ge 0$. Then $y_{j+1} + y_{j-1} = (4D+2)y_j$ for $j \ge 1$, and $y_{j-1} > 0$, so $y_{j+1} < (4u^2-2)y_j < 4u^2 y_j$. Since $(a-1)(e-1) \ge 0$ we have $a+e \le ae+1 = u^2$, and $a+e \le 2ae$; hence $y_1 = a+e-2\sigma u + (a+e)/(2ae) \le u^2+2u+1 < 4u^2$, as $u \ge 2$. By induction $y_j < (4u^2)^j$ for $j \ge 1$, and $x_j < y_j$.

Item 6 follows from item 5: $x_3 < 64u^6$ and $64\,(5\cdot 10^4)^6 = 10^{30}$; $x_4 < 256u^8$ and $256 \cdot 2800^8 < 10^{30}$.

□

Definition 8.7. Let $a \ne e$ be positive integers with $ae+1 = u^2$, let σ be an admitted sign of $\{a,e\}$ with chain (x_j) (Definition 3.2), and let D, η, ϑ, G be the chain constants of $(a,e;\sigma)$ (Definition 3.8). Let $\nu \ge 3$ and let $T = \{a,b,c\}$ with $b = x_{\nu-1}$, $c = x_\nu$ be a Diophantine triple with $a < b < c$, with $s = \sqrt{ac+1}$ and $t = \sqrt{bc+1}$. Put

$$\varphi_1 = \frac{2\sqrt{D}}{\eta^{\nu-2}}.$$

Pinning data. For $\lambda \in \{-1,1\}$, an integer $k \ge 4$ with $4 \mid k$, and an integer $n \ge 1$, put

$$\rho_{\lambda,k} = \frac{\sqrt{4k+1}-\lambda}{2}, \qquad \zeta_{\lambda,k}(n) = n^2 + \lambda n\eta - k\eta^2, \qquad \omega_{\lambda,k}(n) = \frac{\lambda\,\zeta_{\lambda,k}(n)}{\varphi_1\, n},$$

$$\varepsilon_\nu(n) = \frac{\big(16D\vartheta^2 + 6G(1+G)\big)\,n}{2\sqrt{D}\,\eta^\nu}.$$

Candidates and tests. Let $w > 0$ be real and $U \ge 1$ an integer. An integer $n \ge 1$ is a **candidate of width** w for (λ,k) if

$$0 < \lambda\big(n - \eta\rho_{\lambda,k}\big) < w.$$

It passes the **ω-test** if $\vartheta - \varepsilon_\nu(n) \le \omega_{\lambda,k}(n) \le 2\vartheta + \varepsilon_\nu(n)$, and the **root test below** U if

$$s^2 + 4a\big(bn^2 + \lambda tn - kc\big) = q^2$$

for an integer $q \ge 0$ such that $(q-\lambda s)/(2a)$ or $(-q-\lambda s)/(2a)$ is an integer m with $n < m \le \min\{2n, U\}$.

Pinning test. For integers $n_{\mathrm{lo}} \le U$, the **pinning test of width w for λ on** $[n_{\mathrm{lo}}, U]$ passes if there is no pair of an integer $k \ge 4$ with $4 \mid k$ and a candidate n of width w for (λ,k) with $n_{\mathrm{lo}} \le n \le U$ such that n passes both the ω-test and the root test below U.

Lemma 8.8. *Let $a \ne e$ be positive integers with $ae+1 = u^2$, u a positive integer, let σ be an admitted sign of $\{a,e\}$ with chain (x_j) (Definition 3.2), and let D, η, χ, χ', τ, ϑ be the chain constants of $(a,e;\sigma)$ (Definition 3.8). Let $\nu \ge 2$, let $b = x_{\nu-1}$, $c = x_\nu$, suppose $a < b$, and put $s = \sqrt{ac+1}$, $t = \sqrt{bc+1}$. For $\lambda \in \{-1,1\}$ and integers n,k put*

$$\zeta = n^2 + \lambda n\eta - k\eta^2, \qquad \bar\zeta = n^2 + \lambda n\eta^{-1} - k\eta^{-2}, \qquad I_0 = n^2 + 2\lambda un - (4u^2-2)k.$$

1. *$4Dt = \chi^2\eta^{2\nu-1} + \chi'^2\eta^{1-2\nu} - 2u(a+e)$, $2ut = b+c-a-e$, and $2\sqrt{e}\,s = |\chi|\eta^\nu - \tau\chi'\eta^{-\nu}$.*
2. *For all integers n,k,*

$$\chi^2\eta^{2\nu-2}\zeta + \chi'^2\eta^{2-2\nu}\bar\zeta = 4D\big(bn^2 + \lambda tn - kc\big) + 2(a+e)\big(n^2 + \lambda un - k\big).$$

3. *$I_0 = \zeta + \lambda n\eta^{-1} - k\eta^{-2}$. If $4k+1>0$ and $\rho_{\lambda,k} = (\sqrt{4k+1}-\lambda)/2$, then $\rho_{\lambda,k}(\rho_{\lambda,k}+\lambda) = k$ and $\zeta = (n-\eta\rho_{\lambda,k})(n+\eta\rho_{\lambda,k}+\lambda\eta)$.*

4. *Let m,n,k be integers with $kc = bn^2+\lambda tn - am^2 - \lambda sm$, and put $\varphi_1 = 2\sqrt{D}/\eta^{\nu-2}$. Then*

$$\zeta = \lambda\vartheta\varphi_1 m + \Phi_1 + \Phi_2 + \Phi_3 + \Phi_4,$$

$$\Phi_1 = \frac{4D\vartheta^2 m^2}{\eta^{2\nu-2}}, \qquad \Phi_2 = \frac{2(a+e)(n^2+\lambda un - k)}{\chi^2\eta^{2\nu-2}},$$
$$\Phi_3 = -\frac{2\lambda\tau a\sqrt{e}\,\chi' m}{\chi^2\eta^{3\nu-2}}, \qquad \Phi_4 = -\frac{\chi'^2\bar\zeta}{\chi^2\eta^{4\nu-4}}.$$

Proof. Item 1. By Lemma 3.3, item 1, with $(p,q) = (a,e)$ and $\varrho = u$,

$$4Dx_j = \chi^2\eta^{2j} + \chi'^2\eta^{-2j} - 2(a+e) \qquad (j\in\mathbb{Z}).$$

Put $P = \chi^2\eta^{2\nu}$ and $\bar P = \chi'^2\eta^{-2\nu}$, so that $4Db = P\eta^{-2}+\bar P\eta^2 - 2(a+e)$ and $4Dc = P+\bar P - 2(a+e)$, and let $W = P\eta^{-1}+\bar P\eta - 2u(a+e)$. Expanding,

$$W^2 - (4Db)(4Dc) = P\bar P\big(2-\eta^2-\eta^{-2}\big) + 2(a+e)\big(P\eta^{-2}+P+\bar P\eta^2+\bar P\big) \\ - 4u(a+e)\big(P\eta^{-1}+\bar P\eta\big) + 4(a+e)^2(u^2-1).$$

Since $P\eta^{-2}+P = 2uP\eta^{-1}$ and $\bar P\eta^2+\bar P = 2u\bar P\eta$, the two middle terms cancel; moreover $P\bar P = (\chi\chi')^2 = (a-e)^2$ and $2-\eta^2-\eta^{-2} = -4D$, so $W^2 - 16D^2bc = -4D(a-e)^2 + 4D(a+e)^2 = 16D^2$. Hence $W = \pm 4Dt$. Further $4D(b+c) = 2u(P\eta^{-1}+\bar P\eta) - 4(a+e)$, so $4D(b+c-a-e) = 2uW$ because $1+D = u^2$. By Lemma 3.3, item 3, $c \ge x_2 > \max\{a,e\}$, so $b+c-a-e>0$; thus $W>0$, $W = 4Dt$ and $2ut = b+c-a-e$.

For s, since $\tau|\chi| = \chi$ and $\chi\chi' = a-e$,

$$\big(|\chi|\eta^\nu - \tau\chi'\eta^{-\nu}\big)^2 = P+\bar P - 2(a-e) = 4Dc + 4e = 4e(ac+1) = 4es^2.$$

Moreover $|\chi|\eta^\nu > |\chi'|\eta^{-\nu}$: if $\sigma=-1$, then $|\chi'| = |\sqrt{a}-\sqrt{e}| < |\chi|$. If $\sigma = 1$, then $\chi^2 = x_1 + 2(u-\sqrt{D}) > x_1 \ge 1$, so $|\chi'|/|\chi| < |\chi'| = \sqrt{a}+\sqrt{e} < 2u < \eta^2 \le \eta^{2\nu}$. Hence $|\chi|\eta^\nu - \tau\chi'\eta^{-\nu} = 2\sqrt{e}\,s$.

Item 2. By the closed form,

$$4D(bn^2+\lambda tn - kc) = n^2\big(\chi^2\eta^{2\nu-2}+\chi'^2\eta^{2-2\nu}\big) + \lambda n\big(\chi^2\eta^{2\nu-1}+\chi'^2\eta^{1-2\nu}\big) \\ - k\big(\chi^2\eta^{2\nu}+\chi'^2\eta^{-2\nu}\big) - 2(a+e)(n^2+\lambda un - k),$$

and the first three groups add up to $\chi^2\eta^{2\nu-2}\zeta + \chi'^2\eta^{2-2\nu}\bar\zeta$.

Item 3. $\zeta+\bar\zeta = 2n^2+\lambda n(\eta+\eta^{-1}) - k(\eta^2+\eta^{-2}) = 2n^2+2\lambda un - (4u^2-2)k$, so $I_0 = \zeta+\bar\zeta - n^2 = \zeta + \lambda n\eta^{-1} - k\eta^{-2}$. With $\rho = \rho_{\lambda,k}$, $\rho^2+\lambda\rho = \frac14\big((\sqrt{4k+1}-\lambda)^2 + 2\lambda(\sqrt{4k+1}-\lambda)\big) = k$, and $(n-\eta\rho)(n+\eta\rho+\lambda\eta) = n^2+\lambda n\eta - \eta^2\rho(\rho+\lambda) = \zeta$.

Item 4. The hypothesis says $bn^2+\lambda tn - kc = am^2+\lambda sm$, and by item 1, $4Ds = 2a\sqrt{e}\,\big(|\chi|\eta^\nu - \tau\chi'\eta^{-\nu}\big)$. Substituting into item 2,

$$\chi^2\eta^{2\nu-2}\zeta = 4Dam^2 + 2\lambda a\sqrt{e}\,|\chi|\eta^\nu m - 2\lambda\tau a\sqrt{e}\,\chi'\eta^{-\nu}m \\ + 2(a+e)(n^2+\lambda un - k) - \chi'^2\eta^{2-2\nu}\bar\zeta.$$

Divide by $\chi^2\eta^{2\nu-2}$. Since $a/\chi^2 = \vartheta^2$ and $a\sqrt{e}/|\chi| = \sqrt{D}\,\vartheta$, the second term gives $2\lambda\sqrt{D}\vartheta m/\eta^{\nu-2}$, which is $\lambda\vartheta\varphi_1 m$, and the first gives Φ_1; the remaining three give Φ_3, Φ_2, Φ_4. □

Lemma 8.9. *Let $a<b<c<d$ be an irregular Diophantine quadruple with $b=x_{\nu-1}$ and $c=x_\nu$ for some $\nu\ge 3$, where (x_j) is the chain of a pair $\{a,e\}$ with an admitted sign σ and $u=\sqrt{ae+1}\ge 10^3$. Let D, η, ϑ, G be the chain constants of $(a,e;\sigma)$, and φ_1, $\rho_{\lambda,k}$, $\zeta_{\lambda,k}$, $\omega_{\lambda,k}$, ε_ν the pinning data (Definition 8.7). Let z have a type I representation (λ,m,n) with second-order defect k, and put*

$$\Phi=\zeta_{\lambda,k}(n)-\lambda\vartheta\varphi_1 m,\qquad U_{\mathrm{pin}}=\begin{cases}u^2 & \text{if } \nu\ge 4,\\ u^2/\max\{1,20\vartheta\} & \text{if } \nu=3.\end{cases}$$

1. *$|k|\le n^2$ and $|\Phi|\le\big(16D\vartheta^2+6G(1+G)\big)\,n^2/\eta^{2\nu-2}$.*
2. *$\vartheta-\varepsilon_\nu(n)\le\omega_{\lambda,k}(n)\le 2\vartheta+\varepsilon_\nu(n)$.*

Suppose moreover $m\le U_{\mathrm{pin}}$. Then:

3. *$k\ge 4$, $4\mid k$, and $|\Phi|\le\vartheta\varphi_1 n/10$;*
4. *$0<\lambda\big(n-\eta\rho_{\lambda,k}\big)<2.1\,\vartheta\varphi_1$; if $\nu\ge 4$, or $\nu=3$ and $\vartheta\le 1/20$, then $2.1\,\vartheta\varphi_1<\frac12$, so $\eta\rho_{\lambda,k}\notin\mathbb{Z}$ and $n=\lceil\eta\rho_{\lambda,k}\rceil$ if $\lambda=1$, $n=\lfloor\eta\rho_{\lambda,k}\rfloor$ if $\lambda=-1$;*
5. *$q=s+2\lambda am$ is a positive integer, $q^2=s^2+4a\big(bn^2+\lambda tn-kc\big)$, and n passes the root test below every integer $U\ge m$.*

Consequently, let $n_{\mathrm{lo}}\le U\le U_{\mathrm{pin}}$ be integers and $w\ge 2.1\,\vartheta\varphi_1$. If the pinning test of width w for λ on $[n_{\mathrm{lo}},U]$ passes, then z has no type I representation with sign λ, $n\ge n_{\mathrm{lo}}$ and $m\le U$.

Moreover, let $k\ge 4$ with $4\mid k$ and $4k+1=(2h+1)^2$ with an integer $h\ge 0$, and suppose $2(h+1)<\eta$. Then the only candidate of width $\frac12$ for $(1,k)$ is $n=2hu$, with

$$\omega_{1,k}(2hu)=\eta^{\nu-2}\,\frac{2h+1+h\eta^{-2}}{\eta^2-\eta^{-2}},$$

and $(-1,k)$ has no candidate of width $\frac12$.

Proof. By Lemma 3.3, item 4, $\{a,b,c\}=\{a,x_{\nu-1},x_\nu\}$ has degree $\nu-1\ge 2$. By Lemma 8.6: $2u-1/u<\eta<2u$, $u-1/u<\sqrt{D}<u$, $G\le 2\vartheta^2+2\vartheta+1$, $|\chi'|/|\chi|\le G$, $\vartheta>1/(u+1)$, $\vartheta\le u/8$ (as $u\ge 72$) and $c>(4u^2-3)b$. Hence $a/c<b/c<1/(4u^2-3)$, and by Lemma 2.23 and Lemma 2.14,

$$m>n>\kappa\sqrt{c/b}>\kappa\sqrt{4u^2-3}>3u,\qquad m\le 2n.$$

For $u\ge 10^3$ we have $\eta^3>7.99u^3$ and $\eta^4>15.9u^4$. At $\nu=3$, $\varphi_1=2\sqrt{D}/\eta=1-\eta^{-2}<1$, since $\eta-\eta^{-1}=2\sqrt{D}$; for $\nu\ge 4$, $\varphi_1\le 2u/\eta^2<0.502/u$.

Item 1. With $t^2\le 2bc$, $s^2\le 2ac$ and $m\le 2n$,

$$|k|\le\frac{b}{c}n^2+\sqrt{\frac{2b}{c}}\,n+\frac{a}{c}\,4n^2+2\sqrt{\frac{2a}{c}}\,n\le\frac{5n^2}{4u^2-3}+\frac{5n}{\sqrt{4u^2-3}}\le n^2.$$

By Lemma 8.8, item 4, $\Phi=\Phi_1+\Phi_2+\Phi_3+\Phi_4$. First, $\Phi_1=4D\vartheta^2m^2/\eta^{2\nu-2}\le 16D\vartheta^2n^2/\eta^{2\nu-2}$. Next $|n^2+\lambda un-k|\le n^2+un+n^2\le\frac73 n^2$ (as $u<n/3$), so $|\Phi_2|\le\frac{14}{3}Gn^2/\eta^{2\nu-2}$. Since $a\sqrt{e}\,|\chi'|/\chi^2=\sqrt{D}\,\vartheta\,|\chi'|/|\chi|\le u\vartheta G$, we get $|\Phi_3|\le 4u\vartheta Gn/\eta^{3\nu-2}\le Gn^2/\eta^{2\nu-2}$, because $4u\vartheta\le u^2/2<3u\cdot 7.99u^3<n\eta^\nu$. As $|\bar\zeta|\le n^2+n+|k|\le 3n^2$ and $(\chi'/\chi)^2\le G^2$, $|\Phi_4|\le 3G^2n^2/\eta^{4\nu-4}\le G^2n^2/\eta^{2\nu-2}$. Adding gives the bound for $|\Phi|$.

Item 2. By definition and item 4 of Lemma 8.8, $\omega_{\lambda,k}(n) = (\vartheta\varphi_1 m + \lambda\Phi)/(\varphi_1 n) = \vartheta\, m/n + \lambda\Phi/(\varphi_1 n)$. Here $1 < m/n \le 2$, and by item 1

$$\frac{|\Phi|}{\varphi_1 n} \le \frac{(16D\vartheta^2 + 6G(1+G))\, n^2\, \eta^{\nu-2}}{\eta^{2\nu-2} \cdot 2\sqrt{D}\, n} = \varepsilon_\nu(n).$$

Item 3. From now on $m \le U_{\rm pin}$, so $n < m \le U_{\rm pin}$. By item 1,

$$\frac{|\Phi|}{\vartheta\varphi_1 n} \le \frac{8\sqrt{D}\,\vartheta n}{\eta^\nu} + \frac{3G(1+G)\, n}{\sqrt{D}\,\vartheta\,\eta^\nu}. \tag{1}$$

Note that $G(1+G)/\vartheta \le 30\vartheta^3$ if $\vartheta \ge 1$ (then $G \le 5\vartheta^2$), and $G(1+G)/\vartheta < 30(u+1)$ if $\vartheta < 1$ (then $G < 5$). Let $\nu \ge 4$, so $n \le u^2$ and $\eta^\nu > 15.9u^4$: the first term of (1) is at most $0.504\vartheta/u \le 0.063$; the second is at most $0.191\, G(1+G)/(\vartheta u^3)$, which is below 0.012 if $\vartheta \ge 1$ (by $\vartheta \le u/8$) and below 10^{-5} if $\vartheta < 1$. Let $\nu = 3$ and $\vartheta \le 1/20$, so $n \le u^2$, $\eta^3 > 7.99u^3$ and $G \le 1.105$: the first term is at most $1.002\vartheta \le 0.0501$, the second at most $0.38 \cdot 2.33\,(u+1)/u^2 < 0.001$. Let $\nu = 3$ and $\vartheta > 1/20$, so $n \le u^2/(20\vartheta)$: the first term is at most $8u^3/(20 \cdot 7.99u^3) < 0.0501$; the second is at most $0.019\, G(1+G)/(\vartheta^2 u^2)$, which is below $30 \cdot 400 \cdot 0.019/u^2 < 10^{-3}$ if $\vartheta \le 1$, and below $0.019 \cdot 30/64 < 0.009$ if $\vartheta > 1$ (then $G(1+G)/\vartheta^2 \le 30\vartheta^2 \le 30u^2/64$). In all cases $|\Phi| < \vartheta\varphi_1 n/10$.

Suppose $k \le 0$. Then $\zeta = n^2 + \lambda n\eta - k\eta^2 \ge n(n-\eta) > nu$, since $n > 3u > \eta + u$. On the other hand $|\zeta| \le \vartheta\varphi_1 m + |\Phi| \le 2.1\,\vartheta\varphi_1 n$, and $2.1\,\vartheta\varphi_1 < 0.14$ for $\nu \ge 4$, while $2.1\,\vartheta\varphi_1 < 2.1\,u/8 < 0.27u$ for $\nu = 3$; a contradiction. Thus $k \ge 1$, and $4 \mid k$ by item 1 of Lemma 2.22, so $k \ge 4$.

Item 4. By item 3, $\lambda\zeta = \vartheta\varphi_1 m + \lambda\Phi \ge \vartheta\varphi_1 n - |\Phi| > 0$. By Lemma 8.8, item 3, $\zeta = (n - \eta\rho)(n + \eta\rho + \lambda\eta)$ with $\rho = \rho_{\lambda,k}$, and $n + \eta(\rho + \lambda) > n$ since $\rho + \lambda = (\sqrt{4k+1} + \lambda)/2 > 0$. Hence

$$0 < \lambda(n - \eta\rho) = \frac{\lambda\zeta}{n + \eta(\rho+\lambda)} < \frac{\vartheta\varphi_1 m + |\Phi|}{n} \le 2.1\,\vartheta\varphi_1.$$

If $\nu \ge 4$, then $2.1\,\vartheta\varphi_1 < 2.1 \cdot (u/8) \cdot 0.502/u < 0.14$; if $\nu = 3$ and $\vartheta \le 1/20$, then $2.1\,\vartheta\varphi_1 < 0.105$. In these cases n lies in $(\eta\rho, \eta\rho + \frac{1}{2})$ if $\lambda = 1$ and in $(\eta\rho - \frac{1}{2}, \eta\rho)$ if $\lambda = -1$, which gives the stated values and $\eta\rho \notin \mathbb{Z}$.

Item 5. If $\lambda = 1$, then $q = s + 2am > 0$. Let $\lambda = -1$. By Lemma 8.8, item 1, $2\sqrt{e}\, s = |\chi|\eta^\nu - \tau\chi'\eta^{-\nu}$; dividing by $4a\sqrt{e}$ and using $a\sqrt{e}/|\chi| = \sqrt{D}\vartheta$, $|\chi'|/|\chi| \le G \le u^2$ and $\sqrt{D} < u$,

$$\frac{s}{2a} \ge \frac{\eta^\nu - G\eta^{-\nu}}{4\sqrt{D}\,\vartheta} > \frac{\eta^\nu - 1}{4u\vartheta}.$$

For $\nu \ge 4$ this exceeds $(15.9u^4 - 1)/(u^2/2) > 31u^2 > m$. For $\nu = 3$ it exceeds $1.99u^2/\vartheta$, which is larger than $u^2 \ge m$ if $\vartheta \le 1/20$ and larger than $u^2/(20\vartheta) \ge m$ if $\vartheta > 1/20$. So $q = s - 2am > 0$. By the definition of k, $bn^2 + \lambda tn - kc = am^2 + \lambda sm$, so $s^2 + 4a(bn^2 + \lambda tn - kc) = (s + 2\lambda am)^2 = q^2$. Finally $m = (q - s)/(2a)$ if $\lambda = 1$ and $m = (-q + s)/(2a)$ if $\lambda = -1$, that is $m = (q - \lambda s)/(2a)$ or $m = (-q - \lambda s)/(2a)$, and $n < m \le \min\{2n, U\}$ for every $U \ge m$; this is the root test below U.

The consequence. Let (λ, m, n) be a type I representation with $n \ge n_{\rm lo}$ and $m \le U \le U_{\rm pin}$. By items 3 to 5, its defect k satisfies $k \ge 4$, $4 \mid k$, and n is a candidate of width $2.1\vartheta\varphi_1 \le w$ for (λ, k) with $n_{\rm lo} \le n < m \le U$ (here $n < m$ by Lemma 2.14) which passes the ω-test (item 2) and the root test below U. So the pair (k, n) has all the properties that the pinning test excludes, and the test does not pass.

The last assertion. For $\lambda = 1$, $\rho_{1,k} = h$ and $\eta h = h(u + \sqrt{D}) = 2hu - h\eta^{-1}$ with $0 < h\eta^{-1} < \frac{1}{2}$. An integer n with $0 < n - \eta h < \frac{1}{2}$ is $n = 2hu + j$ with $0 < j + h\eta^{-1} < \frac{1}{2}$, so $j = 0$ and $n = 2hu$.

Then $\zeta_{1,k}(2hu) = (h/\eta)\big(2hu + (h+1)\eta\big)$ by Lemma 8.8, item 3, and with $2u = \eta + \eta^{-1}$ and $4u\sqrt{D} = \eta^2 - \eta^{-2}$,

$$\omega_{1,k}(2hu) = \frac{\zeta_{1,k}(2hu)\,\eta^{\nu-2}}{2\sqrt{D}\cdot 2hu} = \eta^{\nu-2}\,\frac{(2h+1)\eta + h\eta^{-1}}{\eta\,(\eta^2-\eta^{-2})} = \eta^{\nu-2}\,\frac{2h+1+h\eta^{-2}}{\eta^2-\eta^{-2}}.$$

For $\lambda = -1$, $\rho_{-1,k} = h+1$ and $\eta(h+1) = 2(h+1)u - \delta$ with $\delta = (h+1)\eta^{-1} \in (0, \frac12)$; a candidate of width $\frac12$ would satisfy $0 < \eta(h+1) - n < \frac12$, that is $\delta < j < \delta + \frac12$ for the integer $j = 2(h+1)u - n$, which is impossible. □

Proposition 8.10. *Let $a < b < c < d$ be an irregular Diophantine quadruple with $b = x_{\nu-1}$ and $c = x_\nu$ for some $\nu \ge 5$, where (x_j) is the chain of a pair $\{a, e\}$ with an admitted sign σ and $u = \sqrt{ae+1} \ge 10^3$, and let ϑ be the chain constant of $(a, e; \sigma)$. If z has a type I representation (λ, m, n), then*

$$m > n \ge \min\big\{u^{3/2}\vartheta^{-1/2},\ u^{5/3}\big\}.$$

Proof. By Lemma 3.3, item 4, $\{a, b, c\}$ has degree $\nu - 1 \ge 4$. By Lemma 2.14, $n < m \le 2n$. If $m > u^2$, then $n \ge m/2 > u^2/2 \ge u^{5/3}$, since $u \ge 8$. So we may assume $m \le u^2$; then items 3 and 4 of Lemma 8.9 apply, and we suppose, for a contradiction, that $n < \min\{u^{3/2}\vartheta^{-1/2}, u^{5/3}\}$. Write $\rho = \rho_{\lambda,k}$ and $\zeta = \zeta_{\lambda,k}(n)$.

As $\nu \ge 5$ and $\eta > 2u - 1/u$ (Lemma 8.6), $\vartheta\varphi_1 \le 2u\vartheta/\eta^3 < 0.26\,\vartheta/u^2$. By Lemma 8.9, items 3 and 4,

$$0 < \lambda(n - \eta\rho) < 2.1\,\vartheta\varphi_1 < 0.55\,\vartheta/u^2, \tag{1}$$

$|\Phi| \le \vartheta\varphi_1 n/10$, $k \ge 4$ and $4 \mid k$. Also $\vartheta \le u/8$ and $c > (4u^2-3)b$ (Lemma 8.6), so $n > \kappa\sqrt{c/b} > 3u$ by Lemma 2.23; and by (1), $\eta\rho < n + \frac12$.

Consider the integer $I_0 = n^2 + 2\lambda un - (4u^2-2)k$. By Lemma 8.8, item 3, $I_0 = \zeta + \lambda n\eta^{-1} - k\eta^{-2}$, so with $n/\eta = \rho + (n - \eta\rho)/\eta$,

$$\delta_I := \lambda I_0 - \rho = \vartheta\varphi_1 m + \lambda\Phi + \frac{n-\eta\rho}{\eta} - \frac{\lambda k}{\eta^2}, \qquad |\delta_I| \le \frac{0.546\,\vartheta n}{u^2} + \frac{0.28\,\vartheta}{u^3} + \frac{k}{\eta^2}.$$

Since $k = \rho(\rho+\lambda) \le \rho(\rho+1)$, $\rho < (n+\frac12)/\eta$, $\eta < 2u < \frac23 n$ and $\eta^4 > 15.99u^4$, we get $k/\eta^2 < (n+\frac12)(n+\frac12+\eta)/\eta^4 \le 1.71n^2/\eta^4 < 0.107n^2/u^4$. Using $n^2 < u^3/\vartheta$, $n^3 < u^5$ and $\vartheta \le u/8$,

$$|\delta_I| < \frac{u}{n}\Big(0.546\frac{\vartheta n^2}{u^3} + 0.107\frac{n^3}{u^5} + 0.001\Big) < 0.66\,\frac{u}{n} < 1. \tag{2}$$

The number $4k+1$ is not a square. Suppose $4k+1 = (2h+1)^2$; then $\rho \in \{h, h+1\}$ is a positive integer, $\rho < (n+\frac12)/\eta < u^{2/3} < \eta/2$, and $\eta\rho = 2\rho u - \rho/\eta$ with $0 < \rho/\eta < \frac12$. Put $j' = n - 2\rho u \in \mathbb{Z}$, so $n - \eta\rho = j' + \rho/\eta$. By (1), $0 < \lambda(j' + \rho/\eta) < 0.55\vartheta/u^2 < \frac12$. If $\lambda = -1$ this says $-\frac12 - \rho/\eta < j' < -\rho/\eta$, impossible for an integer. If $\lambda = 1$ it forces $j' = 0$ and $\rho/\eta < 0.55\vartheta/u^2$, that is, $\rho < 1.1\,\vartheta/u \le 0.14$, contradicting $\rho \ge 1$.

Put $j = \lambda(2I_0+1)$, an odd integer, so that $2\delta_I = j - \sqrt{4k+1}$. As $4 \mid k$, $4k+1 \equiv 1 \pmod{16}$, while $j^2 \equiv 1 \pmod 8$; hence $j^2 - (4k+1)$ is a nonzero multiple of 8 and

$$8 \le |j - \sqrt{4k+1}|\,|j + \sqrt{4k+1}| = 2|\delta_I|\,\big|2\sqrt{4k+1} + 2\delta_I\big| \le 4|\delta_I|\big(\sqrt{4k+1}+1\big),$$

using $|\delta_I| < 1$ from (2). Now $\sqrt{4k+1} = 2\rho + \lambda < 2(n+\frac12)/\eta + 1 \le 1.0001\,n/u + 1.01$, so $\sqrt{4k+1} + 1 < 1.7\,n/u$ because $n > 3u$. Hence $|\delta_I| > 2u/(1.7n) > 1.17\,u/n$, contradicting (2). Therefore $n \ge \min\{u^{3/2}\vartheta^{-1/2}, u^{5/3}\}$. □

For type II, when $u \geq 10^3$, a representation with $m \leq m_\nu$ occurs only at slot four, and there it lies on the conic of Lemma 8.11, the Pell equation of the pair $\{a, E'\}$ of the third descent; by Lemma 8.12, every index $m \geq u$ with $m > m_\nu$ exceeds the floor F_ν.

Lemma 8.11. *Let $a < b < c < d$ be an irregular Diophantine quadruple with $b = x_{\nu-1}$ and $c = x_\nu$ for some $\nu \geq 4$, where (x_j) is the chain of a pair $\{a, e\}$ with an admitted sign σ and $u = \sqrt{ae+1} \geq 10^3$. Let ϑ, E', Ω_ν, m_ν be the chain constants of $(a, e; \sigma)$, and let z have a type II representation (λ, m, n), with N, J, K as in Definition 2.20. If $m \leq m_\nu$, then $\nu = 4$,*

$$K = 0, \qquad J^2 < u, \qquad 1 - \frac{1.5}{a} - \frac{12}{u} \leq \vartheta^{-2} \leq 4 + \frac{1.5}{a} + \frac{12}{u},$$

and (m, n) lies on the ***conic***

$$a(m^2 + m) - E'(n^2 + n) + \lambda(a - \sigma u)(m - n) = 0.$$

Equivalently, $E'X^2 - aY^2 = E' - a$ with $X = a(2m+1) + \lambda(a - \sigma u)$ and $Y = E'(2n+1) + \lambda(a - \sigma u)$; here $aE' + 1 = (a - \sigma u)^2$, so this is the Pell equation of the Diophantine pair $\{a, E'\}$, and $\{a, e, E'\} = \partial^3\{a, b, c\}$.

Proof. By Lemma 3.3, item 4, $T = \{a, b, c\}$ is single-chain of degree $\nu - 1 \geq 3$, with descent partner e, middle or low, and $\partial^{\nu-1}T = \{a, e, x_1\}$. Both $\Omega_\nu(m)$ and $3m^2\Omega_\nu(m)$ increase with $m \geq 0$, so $m \leq m_\nu$ gives $m \leq 2u^2$, $\Omega_\nu(m) < \frac{1}{2}$ and $3m^2\Omega_\nu(m) < u$. By Lemma 7.3, $J \geq 1$, $J(4ae+1) < 2m(m+1)$ and $u^2 \leq \frac{1}{2}m(m+1)$; by Lemma 2.14, $n \leq m \leq 2n$. Hence $m \geq \sqrt{2}u - 1$, $n \geq m/2 \geq 0.69u$, $J < 2m(m+1)/(4u^2 - 3) \leq 2m$ (as $m \leq 2u^2$), $|M| \leq 3m^2$, $0 \leq \Delta \leq m/2$ and $N + J \leq 3m^2$. By Lemma 8.6, $G \leq 2\vartheta^2 + 2\vartheta + 1$, $|\chi'|/|\chi| \leq G$ and $\vartheta \leq u/8$.

Put $\xi = N - J\eta^2$ and $\bar{\xi} = N - J\eta^{-2}$. By the closed form of Lemma 3.3, item 1, $4Db = \chi^2\eta^{2\nu-2} + \chi'^2\eta^{2-2\nu} - 2(a + e)$ and $4Dc = \chi^2\eta^{2\nu} + \chi'^2\eta^{-2\nu} - 2(a + e)$, so $4D(bN - Jc) = \chi^2\eta^{2\nu-2}\xi + \chi'^2\eta^{2-2\nu}\bar{\xi} - 2(a+e)(N - J)$; and $bN - Jc = aM - e\lambda\Delta$ by the definition of J. Hence

$$\chi^2\eta^{2\nu-2}\xi + \chi'^2\eta^{2-2\nu}\bar{\xi} = 4D(aM - e\lambda\Delta) + 2(a + e)(N - J). \tag{1}$$

Dividing by $\chi^2\eta^{2\nu-2}$ and using $a/\chi^2 = \vartheta^2$, $e/\chi^2 \leq G$, $|\bar{\xi}| \leq N + J \leq 3m^2$ and the bounds above, $|\xi| \leq \Omega_\nu(m) < \frac{1}{2}$. Moreover $\xi\bar{\xi} = N^2 - (4u^2 - 2)JN + J^2 =: \mathcal{N} \in \mathbb{Z}$ and $|\mathcal{N}| \leq 3m^2\Omega_\nu(m) < u$.

The element ξ equals $J\eta^{-2}$. Write $\xi = A + B\sqrt{D}$ with $A = N - J(2u^2 - 1)$ and $B = -2uJ \neq 0$, so $\mathcal{N} = A^2 - DB^2$. By Lemma 2.18 with $x = u$, $\mathcal{N} = r_0^2$ with $r_0 \geq 1$ and $|A| = r_0T_\ell$, $|B| = r_0U_{\ell-1}$ for some $\ell \geq 1$, where $\eta^\ell = T_\ell + U_{\ell-1}\sqrt{D}$. If $A < 0$ then $\xi = -r_0\eta^\ell$ and $|\xi| > 1$; so $A > 0$ and $\xi = r_0\eta^{-\ell}$. If $\ell = 1$ then $r_0 = 2uJ$ and $\xi = 2uJ/\eta > J \geq 1$, impossible. If $\ell \geq 3$ then $U_{\ell-1}$ is $4u^2 - 1$, or $4u(2u^2 - 1)$, or at least $U_4 > 16u^4 - 12u^2$; since $r_0U_{\ell-1} = 2uJ$ and $\gcd(4u^2 - 1, 2u) = 1$, this gives $J \geq 4u^2 - 2$, and then $2m(m + 1) > (4u^2 - 2)(4u^2 - 3)$ forces $m > 2u^2$, which is excluded. Hence $\ell = 2$, $r_0 = J$, $A = JT_2 = J(2u^2 - 1)$, so $N = J(4u^2 - 2)$, that is $K = 0$; moreover $\xi = J\eta^{-2}$ and $J^2 = \mathcal{N} < u$.

With $\xi = J\eta^{-2}$, the left side of (1) is $J(\chi^2\eta^{2\nu-4} + \chi'^2\eta^{4-2\nu}) = J(4Dx_{\nu-2} + 2(a + e))$ by the closed form, and $N - J = J(4u^2 - 3) = J(4D + 1)$. So (1) becomes $4DJx_{\nu-2} = 4D(aM - e\lambda\Delta) + 8D(a + e)J$, that is,

$$aM - e\lambda\Delta = J\big(x_{\nu-2} - 2(a + e)\big). \tag{2}$$

Let $\nu \geq 5$. Then $x_{\nu-2} \geq x_3$, and by the closed form $4Dx_3 \geq \chi^2\eta^6 - 2(a + e) = \chi^2(\eta^6 - 2G)$; as $\eta^6 > 63.9u^6$ and $G \leq 2(u/8 + 1)^2 < u^2/30$, this gives $x_3 - 2(a + e) \geq \chi^2\big((\eta^6 - 2G)/(4D) - 2G\big) \geq 15\chi^2u^4$. Since $N \geq 2n^2 \geq m^2/2$, $J = N/(4u^2 - 2) \geq m^2/(8u^2)$, so the right side

of (2) is at least $1.875\,\chi^2u^2m^2$, while the left side is at most $3am^2+em$. Dividing by χ^2m^2: $1.875u^2\le 3\vartheta^2+G/m\le 3u^2/64+u^2/30$, a contradiction. Hence $\nu=4$.

For $\nu=4$, the recurrence of Definition 3.2 with $x_0=0$, $x_1=E'$ gives $x_2-2(a+e)=(4D+2)x_1=(4u^2-2)E'$, so (2) and $N=J(4u^2-2)$ give $aM-e\lambda\Delta=E'N$. Expanding, $2a(m^2+m)-2E'(n^2+n)+\lambda\Delta(a-e+E')=0$, and $a-e+E'=2(a-\sigma u)$. For the Pell form, $aE'=a^2+ae-2\sigma au=(a-\sigma u)^2-1$, and direct expansion with $F_\star=a(m^2+m)-E'(n^2+n)+\lambda(a-\sigma u)(m-n)$ gives

$$E'X^2-aY^2=E'-a+4aE'F_\star,$$

so $F_\star=0$ if and only if $E'X^2-aY^2=E'-a$. Finally $\{a,e,E'\}=\{a,e,x_1\}=\partial^3T$.

For the bounds on ϑ, put $\mu=m/n\in[1,2]$. From $F_\star=0$, $am^2-E'n^2=E'n-am-\lambda(a-\sigma u)(m-n)$, whose absolute value is at most $n(E'+3a+u)$; dividing by an^2, $|\mu^2-E'/a|\le(E'/a+3+u/a)/n$. Since $E'-\chi^2=-2\sigma(u-\sqrt{D})=-2\sigma/\eta$, we have $E'/a=\vartheta^{-2}-2\sigma/(a\eta)$. With $n\ge 0.69u$ and $u\ge 10^3$ this gives $\vartheta^{-2}(1-1.45/u)\le 4+4.35/u+1.451/a$ and $\vartheta^{-2}(1+1.45/u)\ge 1-4.35/u-1.451/a$, hence $1-1.5/a-12/u\le\vartheta^{-2}\le 4+1.5/a+12/u$. □

Lemma 8.12. *Let $a\neq e$ be positive integers with $ae+1=u^2$ and $u\ge 10^3$, let σ be an admitted sign of $\{a,e\}$ (Definition 3.2), and let η, ϑ, ψ_u, F_4, F_ν, Ω_ν, m_ν be the chain constants of $(a,e;\sigma)$ (Definition 3.8). For $\nu\ge 4$ put*

$$L_\nu=\Big(\frac{\eta^{2\nu-2}}{36.07\,u\,\psi_u}\Big)^{1/4}.$$

1. *For every $\nu\ge 4$ and every real m with $u\le m\le L_\nu$, one has $\Omega_\nu(m)<\frac12$ and $3m^2\,\Omega_\nu(m)<u$.*
2. *$F_4<L_4$; for every $\nu\ge 5$, $F_\nu<L_5\le L_\nu$; and $F_4<F_5<2u^2$.*
3. *Let $\nu\ge 4$. Every integer $m\ge u$ with $m>m_\nu$ satisfies $m>F_\nu$.*

Proof. For $m\ge u$, expanding,

$$3m^2\Omega_\nu(m)=\frac{36D\vartheta^2m^4+12DGm^3+18Gm^4}{\eta^{2\nu-2}}+\frac{9G^2m^4}{\eta^{4\nu-4}}.$$

Here $D<u^2$, and $12DGm^3<12uGm^4$ because $u\le m$. Hence $3m^2\Omega_\nu(m)<m^4W/\eta^{2\nu-2}$ with $W=36u^2\vartheta^2+(12u+18)G+9G^2/\eta^{2\nu-2}$. By Lemma 8.6, $\eta>2u-1/u$, so $\eta^6>63.9998\,u^6$ and $\eta^8>255.998\,u^8$ for $u\ge 10^3$; moreover $G\le 2\vartheta^2+2\vartheta+1$ and, as $u\ge 72$, $\vartheta\le u/8$.

The factor W satisfies $W\le 36.07\,u^2\psi_u$. If $\vartheta\ge 1$, then $2\vartheta^2+2\vartheta+1\le 5\vartheta^2$, because $3\vartheta^2-2\vartheta-1=(3\vartheta+1)(\vartheta-1)\ge 0$; so $(12u+18)G\le(60u+90)\vartheta^2$, and $9G^2/\eta^{2\nu-2}\le 225\vartheta^4/\eta^6\le\vartheta^2$, using $\vartheta\le u/8$ and $\eta^6>63u^6$. Thus $W\le(36u^2+60u+91)\vartheta^2\le 36.07\,u^2\vartheta^2$, as $60u+91\le 0.07u^2$ for $u\ge 10^3$. If $\vartheta<1$, then $G<5$, so $(12u+18)G+9G^2/\eta^{2\nu-2}<60u+90+1\le 60.1\,u$, and $W<36u^2\vartheta^2+60.1\,u$. In both cases $W\le 36.07\,u^2\vartheta^2+60.1\,u\le 36.07\,u^2(\vartheta^2+2/u)=36.07\,u^2\psi_u$.

Item 1. Let $\nu\ge 4$ and $u\le m\le L_\nu$. By the two previous steps,

$$3m^2\Omega_\nu(m)<\frac{36.07\,u^2\psi_u\,m^4}{\eta^{2\nu-2}}\le\frac{36.07\,u^2\psi_u\,L_\nu^4}{\eta^{2\nu-2}}=u.$$

Then $\Omega_\nu(m)<u/(3m^2)\le 1/(3u)<\frac12$.

Item 2. By the bounds on η,

$$L_4^4>\frac{63.9998\,u^5}{36.07\,\psi_u}>1.7743\,\frac{u^5}{\psi_u},\qquad L_5^4>\frac{255.998\,u^7}{36.07\,\psi_u}>7.0972\,\frac{u^7}{\psi_u},$$

and $1.7743^{1/4} > 1.154 > 1.15$, $7.0972^{1/4} > 1.632 > 1.63$; so $L_4 > F_4$ and $L_5 > F_5 = F_\nu$ for $\nu \geq 5$. For $\nu \geq 5$, $\eta^{2\nu-2} \geq \eta^8$ as $\eta > 1$, so $L_\nu \geq L_5$. Clearly $F_4 < F_5$, since $u > 1$. Finally $\psi_u \geq 2/u$ gives $\psi_u^{-1/4} \leq (u/2)^{1/4}$, so $F_5 \leq 1.63 \cdot 2^{-1/4} u^2 < 1.38\, u^2 < 2u^2$.

Item 3. Let $m \geq u$ be an integer with $m \leq F_\nu$. By item 2, $m \leq L_\nu$ and $m < 2u^2$, so by item 1, $\Omega_\nu(m) < \frac{1}{2}$ and $3m^2\Omega_\nu(m) < u$; hence $m \leq m_\nu$ by the definition of m_ν. So an integer $m \geq u$ with $m > m_\nu$ satisfies $m > F_\nu$. □

Proposition 8.13. *Let $T = \{a, b, c\}$ be a triple of the family B3 of Definition 4.2 (T single-chain, $\deg T = 3$, middle or low, $c \geq 10^{30}$), and let d be an irregular extension of T. Then every representation ρ of $z = \sqrt{cd+1}$ satisfies the window of B3 of its type, that is, $(T, \rho) \in$ B3.*

Proof. Let $\rho = (\lambda, m, n)$ be a representation of z. By Proposition 7.45, $b > 4000$ and ρ satisfies the generic window of its type; this is the whole window of type I. Let ρ be of type II. By Lemma 3.4, item (iii), T has a chain sign σ, and $T = \{a, x_3, x_4\}$ on the chain (x_j) of $\{a, e\}$ with sign σ, so $b = x_3$ and $c = x_4$. As $c \geq 10^{30}$, Lemma 8.6, item 6, gives $u = \sqrt{ae+1} > 2800$. If $m \leq m_4$, then Lemma 8.11 (with $\nu = 4$) gives $K = 0$, $J^2 < u$, $1 - 1.5/a - 12/u \leq \vartheta^{-2} \leq 4 + 1.5/a + 12/u$ and the equation of the conic. Otherwise $m > m_4$; moreover the generic window gives $u < m'/(2\sqrt{2})$, so $m > \sqrt{2}\, u - \frac{1}{2} > u$, and item 3 of Lemma 8.12 gives $m > F_4$. Hence $(T, \rho) \in$ B3. □

Proposition 8.14. *Let $T = \{a, b, c\}$ be a triple of the family B4 of Definition 4.2 (T single-chain, $\deg T \geq 4$, middle or low, $c \geq 10^{30}$), and let d be an irregular extension of T. Then every representation ρ of $z = \sqrt{cd+1}$ satisfies the window of B4 of its type, that is, $(T, \rho) \in$ B4.*

Proof. Let $\rho = (\lambda, m, n)$ be a representation of z. By Proposition 7.45, $b > 4000$ and ρ satisfies the generic window of its type. Put $\nu = \deg T + 1 \geq 5$. By Lemma 3.4, item (iii), T has a chain sign σ, and $T = \{a, x_{\nu-1}, x_\nu\}$ on the chain (x_j) of $\{a, e\}$ with sign σ, so $b = x_{\nu-1}$ and $c = x_\nu$. Let $u = \sqrt{ae+1}$. If $u < 10^3$, the window of B4 is the generic window and there is nothing more to prove; let $u \geq 10^3$.

If ρ is of type I, Proposition 8.10 gives $n \geq \min\{u^{3/2}\vartheta^{-1/2}, u^{5/3}\}$.

Let ρ be of type II. If $m \leq m_\nu$, then Lemma 8.11 would give $\nu = 4$, which is false; so $m > m_\nu$. Moreover the generic window gives $u < m'/(2\sqrt{2})$, so $m > \sqrt{2}\, u - \frac{1}{2} > u$, and item 3 of Lemma 8.12 gives $m > F_\nu$. Hence $(T, \rho) \in$ B4. □

9 Deciding the families

Each row of Table 1 gives a finite test whose success shows that a given triple of the family has no irregular extension; each test is stated with all its constants and proved. The proof of the Main Theorem does not use these tests. A test closes the four *cases* $(\mathrm{I}, \pm 1)$ and $(\mathrm{II}, \pm 1)$ of the triple one at a time: a case is closed if no irregular extension gives a representation of z of that type and sign, and the triple has no irregular extension exactly when its four cases are closed (Lemma 9.6). Running the tests over all the triples of the families is a finite computation. For each family, the search of Subsection 9.2 that names it lists its triples, and the four cases of each triple are tested by one corollary: Corollary 9.12 for A1 and B0, Corollary 9.15 for A2, Corollary 9.17 for BH, Corollary 9.29 for BM, Corollary 9.32 for BL, Corollary 9.34 for B3, Corollary 9.35 for B4, and the generic test Corollary 9.16 for BX.

Inputs. Besides the Pell data of Section 2.1 (the roots r, s, t, the units β, γ, the seeds and Pell sequences of the four cases) and the matrices $M_a = \begin{pmatrix} s & c \\ a & s \end{pmatrix}$, $M_b = \begin{pmatrix} t & c \\ b & t \end{pmatrix}$, which act on

$(z_j^{(a,c)}, x_j^{(a,c)})^\top$ and $(z_j^{(b,c)}, y_j^{(b,c)})^\top$ as multiplication by β and γ (Lemma 9.36, item 1), the tests use: the caps Q_0, Q_{I}, Q_{II} and their integer forms $m_{\mathrm{I}}(c)$, $m_{\mathrm{II}}(c)$ (Definition 4.1); the descent partner e, the position and, in degree at least two, f (Definition 2.8); in degree one, ε, ϱ, Θ, the line and ℓ_ε (Definition 3.6); for BH, v (Definition 2.20), T_0, Δ_0, n_0 (Definition 4.2) and the function Ψ (Proposition 7.28); for single-chain middle and low triples, the chain sign σ, the chain constants D, η, χ, ϑ, G, E', E, P, F_ν, Ω_ν, m_ν (Definition 3.8) and the pinning data (Definition 8.7); for BL, $B(m)$, V and Γ (Definition 4.2); and, for the searches, the children, the root $r(\cdot)$ of the smaller pair and the Euler root (Definition 9.8). Every other quantity is defined in the test that uses it. Each real quantity of a test is built from integers by the operations of Lemma 9.7, evaluated at integer or rational arguments; this includes Ψ and the functions $U(L)$ and $\hat{L}_0^\circ(L)$ of the ellipse.

9.1 Closed cases and certified ranges

A range of indices is certified by a certificate for the linear form in two logarithms attached to the common term (Lemma 9.2): the reduction of Baker and Davenport [2] in the form of Dujella and Pethő [8], or a gap or a shift certificate built from best approximations to $\xi = \log\beta/\log\gamma$ (Lemma 9.3, Lemma 9.4).

Definition 9.1. Let $T = \{a, b, c\}$ with $a < b < c$ be a Diophantine triple.

Cases. A **case** is one of the four pairs (I, λ), (II, λ) with $\lambda \in \{-1, 1\}$. Its **seeds** are $(x_1, z_1) = (y_2, z_2) = (1, \lambda)$ for (I, λ), and $(x_1, z_1) = (r, \lambda t)$, $(y_2, z_2) = (r, \lambda s)$ for (II, λ); in this definition $z_j^{(a,c)}$, $x_j^{(a,c)}$ and $z_j^{(b,c)}$ are the Pell sequences for these seeds. An integer $m' \geq 0$ is **admissible** for the case if it is even for (I, λ) and odd for (II, λ). The case (I, λ) (resp. (II, λ)) is **closed** for T if there is no irregular extension d of T (Definition 2.3) for which $z = \sqrt{cd+1}$ has a representation of type I (resp. type II) with sign λ.

Data of a case. For a case put

$$N_1 = z_1\sqrt{a} + x_1\sqrt{c}, \qquad N_2 = z_2\sqrt{b} + y_2\sqrt{c}, \qquad \delta = \frac{\sqrt{b}\,N_1}{\sqrt{a}\,N_2}, \qquad K' = \frac{2c(b-a)}{b\,N_1^2},$$

$$\xi = \frac{\log\beta}{\log\gamma}, \qquad \mu = \frac{\log\delta}{\log\gamma}, \qquad A = \frac{K'}{\log\gamma}, \qquad B = \beta^2,$$

let $m_{\mathrm{DP}} \geq 0$ be an integer with $K'\beta^{-2m_{\mathrm{DP}}} \leq 1$, fixed with the case (for instance the least such integer), and let $\|x\|$ denote the distance from a real number x to the nearest integer.

Certified ranges. Let L and $U \geq 0$ be integers. A **certificate** of the range $[L, U]$ for the case is an integer $M_{\mathrm{red}} \geq 0$ together with one of the following data:

(DP) (Dujella–Pethő) integers p, q with $q > 0$, $\|q\mu\| - U|q\xi - p| > 0$ and $qA \leq B^{M_{\mathrm{red}}}\big(\|q\mu\| - U|q\xi - p|\big)$;

(G) (gap) integers p_0, q_0, p_1, q_1 with $p_1q_0 - p_0q_1 = \pm 1$, $0 < q_0 < q_1$, $U < q_1$ and $(q_0\xi - p_0)(q_1\xi - p_1) \leq 0$, and integers m_0, n_0 with $m_{\mathrm{DP}} \leq m_0 \leq U$, $n_0 \geq 0$, $z_{m_0}^{(a,c)} = z_{n_0}^{(b,c)}$, $AB^{-m_0} \leq |q_0\xi - p_0|$ and $M_{\mathrm{red}} = m_0$;

(S) (shift) integers p_0, q_0, p_1, q_1 as in (G) with $q_0\xi \neq p_0$, and integers m_0, n_0 with $0 \leq m_0 \leq U$, $|m_0\xi - n_0 + \mu| \leq \frac{1}{2}|q_0\xi - p_0|$, $m_0 \leq M_{\mathrm{red}}$ and $2A \leq B^{M_{\mathrm{red}}}|q_0\xi - p_0|$;

such that, with $\bar{M} = \max(M_{\mathrm{red}}, m_{\mathrm{DP}} - 1)$, for every admissible m' with $\max(L, 0) \leq m' \leq \min(\bar{M}, U)$ the number $\big((x_{m'}^{(a,c)})^2 - 1\big)/a$ is not an integer $d > c$ with $d \neq d_+(a, b, c)$ and $bd + 1$ a square. The range $[L, U]$ is **certified** for the case if $L > U$ or it has a certificate.

Lemma 9.2. *Let $a<b<c$ be a Diophantine triple and let (x_1,z_1), (y_2,z_2) be integer seeds with*

$$az_1^2-cx_1^2=a-c,\qquad bz_2^2-cy_2^2=b-c,\qquad N_1=z_1\sqrt{a}+x_1\sqrt{c}>0,\qquad N_2=z_2\sqrt{b}+y_2\sqrt{c}>0.$$

Put $\delta=\frac{\sqrt{b}\,N_1}{\sqrt{a}\,N_2}$ and $K'=\frac{2c(b-a)}{b\,N_1^2}$. If $z_{m'}^{(a,c)}=z_{n'}^{(b,c)}$ for integers $m',n'\ge 0$ and $K'\beta^{-2m'}\le 1$, then

$$0<\Lambda<K'\beta^{-2m'},\qquad \Lambda=m'\log\beta-n'\log\gamma+\log\delta.$$

The seeds of each of the four cases of a representation (type I or II, $\lambda=\pm1$) satisfy the hypotheses above.

Proof. The argument goes back to Dujella [5, Lemma 5]; He and Togbé [13, Lemma 8.2] state it with the bound $\frac{8}{3}ac\,\beta^{-2m'}$ for $m',n'>0$, which is not justified for Euler triples in a type II representation (Lemma 5.7). The certificates of Definition 9.1 use the constant K', valid for every triple, and allow $m'=0$ or $n'=0$.

Put $z=z_{m'}^{(a,c)}=z_{n'}^{(b,c)}$ and $x=x_{m'}^{(a,c)}$. As $ac+1=s^2$, $\sqrt{ac}$ is irrational. Multiplying $z\sqrt{a}+x\sqrt{c}=N_1\beta^{m'}$ by $\sqrt{a}$ gives $az+x\sqrt{ac}=(az_1+x_1\sqrt{ac})\beta^{m'}$ in $\mathbb{Z}[\sqrt{ac}]$; the conjugation $\sqrt{ac}\mapsto-\sqrt{ac}$ sends β to β^{-1} and gives $az-x\sqrt{ac}=(az_1-x_1\sqrt{ac})\beta^{-m'}$. Since $(az_1+x_1\sqrt{ac})(az_1-x_1\sqrt{ac})=a(az_1^2-cx_1^2)=a(a-c)$ and $az_1+x_1\sqrt{ac}=\sqrt{a}\,N_1$, adding the two identities and dividing by a gives, with $P=N_1\beta^{m'}/\sqrt{a}>0$,

$$2z=P-\frac{c-a}{a}P^{-1}.$$

In the same way, with $Q=N_2\gamma^{n'}/\sqrt{b}>0$, $2z=Q-\frac{c-b}{b}Q^{-1}$.

For $k>0$ the function $X\mapsto X-kX^{-1}$ increases on $(0,\infty)$, and $k_1=\frac{c-a}{a}>k_2=\frac{c-b}{b}>0$. If $P\le Q$, then $P-k_1P^{-1}\le Q-k_1Q^{-1}<Q-k_2Q^{-1}$, which contradicts the two expressions; so $P>Q$, and

$$P-Q=\frac{k_1}{P}-\frac{k_2}{Q}<\frac{k_1-k_2}{P}=\frac{c(b-a)}{abP}.$$

Put $\epsilon=(P-Q)/P\in(0,1)$. Then

$$\epsilon<\frac{c(b-a)}{abP^2}=\frac{c(b-a)}{bN_1^2}\beta^{-2m'}=\frac{K'}{2}\beta^{-2m'}\le\frac{1}{2}.$$

By definition of δ, $P/Q=\delta\beta^{m'}\gamma^{-n'}$, so $\Lambda=\log(P/Q)=-\log(1-\epsilon)$, and $0<-\log(1-\epsilon)\le\frac{\epsilon}{1-\epsilon}\le 2\epsilon<K'\beta^{-2m'}$.

For type I the seeds are $(x_1,z_1)=(y_2,z_2)=(1,\lambda)$: then $az_1^2-cx_1^2=a-c$, $bz_2^2-cy_2^2=b-c$, $N_1=\lambda\sqrt{a}+\sqrt{c}>0$ and $N_2=\lambda\sqrt{b}+\sqrt{c}>0$ because $c>b>a$. For type II they are $(x_1,z_1)=(r,\lambda t)$ and $(y_2,z_2)=(r,\lambda s)$: then $at^2-cr^2=a(bc+1)-c(ab+1)=a-c$ and $bs^2-cr^2=b(ac+1)-c(ab+1)=b-c$; moreover $r^2c-t^2a=c-a>0$ and $r^2c-s^2b=c-b>0$, so $r\sqrt{c}>t\sqrt{a}$ and $r\sqrt{c}>s\sqrt{b}$, whence $N_1=\lambda t\sqrt{a}+r\sqrt{c}>0$ and $N_2=\lambda s\sqrt{b}+r\sqrt{c}>0$.

□

Lemma 9.3. *Let $\xi\in\mathbb{R}$ and let p_0,q_0,p_1,q_1 be integers with $p_1q_0-p_0q_1=\pm1$, $0<q_0<q_1$ and $(q_0\xi-p_0)(q_1\xi-p_1)\le 0$. Then $|Q\xi-P|\ge|q_0\xi-p_0|$ for all integers P,Q with $0<|Q|<q_1$.*

Proof. Both $|Q\xi-P|$ and $|Q|$ are unchanged when (P,Q) is replaced by $(-P,-Q)$, so we may assume $0<Q<q_1$. Since $p_1q_0-p_0q_1=\pm1$, the vectors (p_0,q_0) and (p_1,q_1) form a basis of $\mathbb{Z}^2$, so $(P,Q)=u(p_0,q_0)+v(p_1,q_1)$ with integers u,v; then $Q\xi-P=u(q_0\xi-p_0)+v(q_1\xi-p_1)$ and $Q=uq_0+vq_1$.

If $v = 0$, then $u = Q/q_0 \geq 1$ and $|Q\xi - P| = u|q_0\xi - p_0| \geq |q_0\xi - p_0|$. If $u = 0$, then $Q = vq_1$ with $0 < Q < q_1$, which is impossible. Otherwise $u \neq 0 \neq v$, and $uv < 0$: $u, v > 0$ would give $Q \geq q_0 + q_1 > q_1$, and $u, v < 0$ would give $Q < 0$. Then the real numbers $X = u(q_0\xi - p_0)$ and $Y = v(q_1\xi - p_1)$ satisfy $XY = uv(q_0\xi - p_0)(q_1\xi - p_1) \geq 0$, so $|Q\xi - P| = |X + Y| = |X| + |Y| \geq |u|\,|q_0\xi - p_0| \geq |q_0\xi - p_0|$.

For consecutive convergents p_0/q_0, p_1/q_1 of ξ this is the classical best-approximation property of continued fractions; only the three stated conditions are used. □

Lemma 9.4. *Let $\xi, \mu \in \mathbb{R}$, $A > 0$, $B > 1$, and let $U \geq 0$, $m_{\mathrm{DP}} \geq 0$ be integers. Let $\mathcal{S}$ be a set of integer pairs (m', n') with $0 \leq m' \leq U$ such that*

$$0 < m'\xi - n' + \mu < AB^{-m'} \qquad \text{whenever } (m', n') \in \mathcal{S} \text{ and } m' \geq m_{\mathrm{DP}}.$$

Write $\|x\|$ for the distance from a real number x to the nearest integer, and let $(m', n') \in \mathcal{S}$ with $m' \geq m_{\mathrm{DP}}$.

1. *(Dujella–Pethő) If p, q are integers with $q > 0$ and $\|q\mu\| > U|q\xi - p|$, then $B^{m'}\big(\|q\mu\| - U|q\xi - p|\big) < qA$.*
2. *(gap) If p_0, q_0, p_1, q_1 satisfy the hypotheses of Lemma 9.3, $U < q_1$, and $(m_0, n_0) \in \mathcal{S}$ with $m_0 \geq m_{\mathrm{DP}}$ and $AB^{-m_0} \leq |q_0\xi - p_0|$, then $m' \leq m_0$.*
3. *(shift) If p_0, q_0, p_1, q_1 are as in item 2 and m_0, n_0 are integers with $0 \leq m_0 \leq U$ and $|m_0\xi - n_0 + \mu| \leq \frac{1}{2}|q_0\xi - p_0|$, then $m' = m_0$ or $|q_0\xi - p_0| < 2AB^{-m'}$.*

Proof. Throughout, $0 < m'\xi - n' + \mu < AB^{-m'}$.

Item 1. We have $q(m'\xi - n' + \mu) = m'(q\xi - p) + (m'p - qn') + q\mu$, and $m'p - qn'$ is an integer, so

$$\|q\mu\| \leq |m'p - qn' + q\mu| \leq q(m'\xi - n' + \mu) + m'|q\xi - p| < qAB^{-m'} + U|q\xi - p|.$$

Hence $\|q\mu\| - U|q\xi - p| < qAB^{-m'}$, which is the claim.

Item 2. Suppose $m_0 < m'$. As $B > 1$, both $m'\xi - n' + \mu$ and $m_0\xi - n_0 + \mu$ lie in $(0, AB^{-m_0})$, so $Q = m' - m_0$ and $P = n' - n_0$ satisfy $|Q\xi - P| < AB^{-m_0} \leq |q_0\xi - p_0|$, while $0 < Q \leq U < q_1$ because $0 \leq m_0 < m' \leq U$. This contradicts Lemma 9.3.

Item 3. Suppose $m' \neq m_0$. Then $Q = m' - m_0$ satisfies $0 < |Q| \leq U < q_1$, as $0 \leq m', m_0 \leq U$; with $P = n' - n_0$, Lemma 9.3 and the triangle inequality give

$$|q_0\xi - p_0| \leq |Q\xi - P| \leq (m'\xi - n' + \mu) + |m_0\xi - n_0 + \mu| < AB^{-m'} + \tfrac{1}{2}|q_0\xi - p_0|,$$

so $|q_0\xi - p_0| < 2AB^{-m'}$.

Item 1 is the reduction of Baker and Davenport [2] in the inhomogeneous form of Dujella and Pethő [8, Lemma 5(a)], here without the condition that p/q be a convergent of ξ: any integers p, q with $\|q\mu\| > U|q\xi - p|$ qualify. □

Lemma 9.5. *Let $T = \{a, b, c\}$ with $a < b < c$ be a Diophantine triple. Fix a case of T, with its seeds, its admissible indices and its data ξ, μ, A, B, m_{DP} (Definition 9.1), and let L and $U \geq 0$ be integers.*

1. *Let $M_{\mathrm{red}} \geq 0$ be an integer with data (DP), (G) or (S) of Definition 9.1 for the range $[L, U]$, and put $\bar{M} = \max(M_{\mathrm{red}}, m_{\mathrm{DP}} - 1)$. Then every pair of integers $m', n' \geq 0$ with $m' \leq U$ and $z_{m'}^{(a,c)} = z_{n'}^{(b,c)}$ for the seeds of the case satisfies $m' \leq \bar{M}$.*

2. *If the range $[L,U]$ is certified for the case, then no irregular extension d of T gives a representation of $z=\sqrt{cd+1}$ of this case with $L\le m'\le U$.*

Proof. Item 1. Let $\mathcal{S}$ be the set of integer pairs (m'',n'') with $0\le m''\le U$, $n''\ge 0$ and $z^{(a,c)}_{m''}=z^{(b,c)}_{n''}$ for the seeds of the case. Here $K'>0$, $\beta>1$ and $\gamma>1$, so $A>0$ and $B>1$. For $(m'',n'')\in\mathcal{S}$ with $m''\ge m_{\mathrm{DP}}$ we have $K'\beta^{-2m''}\le K'\beta^{-2m_{\mathrm{DP}}}\le 1$, so by Lemma 9.2, whose hypotheses the seeds of each case satisfy,

$$0<m''\log\beta-n''\log\gamma+\log\delta<K'\beta^{-2m''},$$

and dividing by $\log\gamma$ gives $0<m''\xi-n''+\mu<AB^{-m''}$. Thus $\mathcal{S}$ satisfies the hypothesis of Lemma 9.4, with the same U.

Let $(m',n')\in\mathcal{S}$ with $m'\ge m_{\mathrm{DP}}$. With the data (DP), item 1 of Lemma 9.4 gives $B^{m'}\big(\|q\mu\|-U|q\xi-p|\big)<qA\le B^{M_{\mathrm{red}}}\big(\|q\mu\|-U|q\xi-p|\big)$, so $m'<M_{\mathrm{red}}$. With the data (G), $(m_0,n_0)\in\mathcal{S}$ and $m_0\ge m_{\mathrm{DP}}$, so item 2 of Lemma 9.4 gives $m'\le m_0=M_{\mathrm{red}}$. With the data (S), item 3 of Lemma 9.4 gives $m'=m_0\le M_{\mathrm{red}}$, or $|q_0\xi-p_0|<2AB^{-m'}$; in the second case $B^{m'}|q_0\xi-p_0|<2A\le B^{M_{\mathrm{red}}}|q_0\xi-p_0|$ with $|q_0\xi-p_0|>0$, so $m'<M_{\mathrm{red}}$. Hence $m'\le M_{\mathrm{red}}$ whenever $m'\ge m_{\mathrm{DP}}$, and $m'\le m_{\mathrm{DP}}-1$ otherwise; in both cases $m'\le\bar{M}$.

Item 2. If $L>U$ there is nothing to prove; let M_{red} be a certificate of $[L,U]$. Suppose d is an irregular extension of T and $z=\sqrt{cd+1}$ has a representation of this case with $L\le m'\le U$. By the definition of a representation, m' is admissible and $z=z^{(a,c)}_{m'}=z^{(b,c)}_{n'}$ for an integer $n'\ge 0$; by item 1, $m'\le\bar{M}$, so $\max(L,0)\le m'\le\min(\bar{M},U)$. Put $x=x^{(a,c)}_{m'}$. Multiplying the defining identity $z\sqrt{a}+x\sqrt{c}=(z_1\sqrt{a}+x_1\sqrt{c})\beta^{m'}$ by $\sqrt{a}$ gives $az+x\sqrt{ac}=(az_1+x_1\sqrt{ac})\beta^{m'}$ in $\mathbb{Z}[\sqrt{ac}]$, where $ac=s^2-1$ is not a square; taking norms, and using $s^2-ac=1$,

$$a(az^2-cx^2)=a(az_1^2-cx_1^2)=a(a-c),$$

so $az^2-cx^2=a-c$. Since $z^2=cd+1$, this gives $x^2=ad+1$, so $d=(x^2-1)/a$; and $bd+1$ is a square because $\{a,b,c,d\}$ is a Diophantine quadruple. Thus $(x^2-1)/a$ is an integer $d>c$ with $d\ne d_+(a,b,c)$ and $bd+1$ a square, which the certificate excludes at the admissible index m'. Hence no such extension exists. □

Lemma 9.6. *Let $T=\{a,b,c\}$ with $a<b<c$ be a Diophantine triple.*

1. *T has no irregular extension if and only if its four cases are closed (Definition 9.1).*
2. *Fix a case of T, and let $[L_1,U_1],\dots,[L_r,U_r]$, $r\ge 0$, be ranges of integers with $U_i\ge 0$. Suppose that for every irregular extension d of T, every representation of $z=\sqrt{cd+1}$ of this case has $L_i\le m'\le U_i$ for some i. If every range $[L_i,U_i]$ is certified for the case (Definition 9.1), then the case is closed. In particular the case is closed if $r=0$.*

Proof. Item 1. If every case is closed and d were an irregular extension of T, then $\{a,b,c,d\}$ would be an irregular quadruple (Definition 2.3), so $d>d_+(a,b,c)$ by Lemma 2.9, and $z=\sqrt{cd+1}$ would have a representation of type I or of type II with some sign λ by Lemma 2.13; then the case of that type and sign would not be closed. Hence T has no irregular extension. Conversely, if T has no irregular extension, every case is closed by definition.

Item 2. Let d be an irregular extension of T and suppose $z=\sqrt{cd+1}$ has a representation of the fixed case. By hypothesis $L_i\le m'\le U_i$ for some i, and $[L_i,U_i]$ is certified for the case, which contradicts item 2 of Lemma 9.5. So no irregular extension gives a representation of this case, that is, the case is closed. If $r=0$ the hypothesis says that no irregular extension gives a representation of the case, which is the definition of closed. □

Every test compares real numbers. The precision lemma reduces each comparison to rational enclosures, with an explicit series for the logarithm and integer forms of the decimal thresholds, so that the tests can be run in exact rational arithmetic.

Lemma 9.7. *Call rational numbers $x_- \le x_+$ an* ***enclosure*** *$[x_-, x_+]$ of a real number x if $x_- \le x \le x_+$. Let $[x_-, x_+]$ and $[y_-, y_+]$ be enclosures of real numbers x and y.*

1. *(Comparisons) If $x_+ < y_-$, then $x < y$; if $x_+ \le y_-$, then $x \le y$. In particular a condition $x \le y$ fails if $x_- > y_+$, and a condition $x < y$ fails if $x_- \ge y_+$. For integers p_0, p_1 and $0 < q_0 < q_1$, a real number ξ satisfies $(q_0\xi - p_0)(q_1\xi - p_1) \le 0$ if and only if it lies in the closed interval with end points p_0/q_0 and p_1/q_1; so the condition holds if an enclosure of ξ lies in that interval.*

2. *(Integers near a real) Every integer n with $|n - x| < w$, for a real $w > 0$, satisfies $x_- - w < n < x_+ + w$. Moreover $\lceil x_- \rceil \le \lceil x \rceil \le \lceil x_+ \rceil$ and $\lfloor x_- \rfloor \le \lfloor x \rfloor \le \lfloor x_+ \rfloor$; if $[x_-, x_+]$ contains no integer, then $\lceil x \rceil = \lceil x_- \rceil$ and $\lfloor x \rfloor = \lfloor x_- \rfloor$.*

3. *(Arithmetic) $[x_- + y_-,\ x_+ + y_+]$ encloses $x + y$ and $[x_- - y_+,\ x_+ - y_-]$ encloses $x - y$; with $\Pi = \{x_-y_-, x_-y_+, x_+y_-, x_+y_+\}$, $[\min\Pi, \max\Pi]$ encloses xy; if $y_- > 0$, then with $\Pi' = \{x_\pm / y_\pm\}$, $[\min\Pi', \max\Pi']$ encloses x/y. Moreover $[\max(x_-, -x_+, 0),\ \max(x_+, -x_-)]$ encloses $|x|$. If $x_- \ge 0$ and $P \ge 0$ is an integer, then*

$$\Big[\frac{\lfloor\sqrt{\lfloor 4^P x_-\rfloor}\rfloor}{2^P},\ \frac{\lfloor\sqrt{\lceil 4^P x_+\rceil}\rfloor + 1}{2^P}\Big]$$

   encloses $\sqrt{x}$.

4. *(Logarithm) Let $q > 0$ be rational, $g = (q-1)/(q+1)$, and $h \ge 0$ an integer. Then*

$$\Big|\log q - 2\sum_{k=0}^{h} \frac{g^{2k+1}}{2k+1}\Big| \le \frac{2|g|^{2h+3}}{(2h+3)(1-g^2)}.$$

   If $x_- > 0$, then $\log x$ lies between any lower bound of $\log x_-$ and any upper bound of $\log x_+$.

5. *(Decimal thresholds) For a positive integer c and positive integers p, q, one has $c < 10^{p/q}$ if and only if $c^q < 10^p$. In particular $c < 10^{77.28}$ if and only if $c^{25} < 10^{1932}$, and $c < 10^{79.71}$ if and only if $c^{100} < 10^{7971}$.*

6. *(Distance to the nearest integer) Write $\|x\|$ for the distance from a real number x to the nearest integer. For real x and x_0, $\big|\|x\| - \|x_0\|\big| \le |x - x_0|$. In particular, if $x_0 \in [x_-, x_+]$ is rational, then $\big[\|x_0\| - (x_+ - x_-),\ \|x_0\| + (x_+ - x_-)\big]$ encloses $\|x\|$.*

Proof. Item 1. $x \le x_+ < y_- \le y$, and likewise with $\le$. The two failure criteria are the contrapositives: $x \le y$ gives $x_- \le x \le y \le y_+$, and $x < y$ gives $x_- < y_+$. For the last assertion, $(q_0\xi - p_0)(q_1\xi - p_1) = q_0q_1(\xi - p_0/q_0)(\xi - p_1/q_1)$ with $q_0q_1 > 0$, which is ≤ 0 exactly when ξ lies between p_0/q_0 and p_1/q_1.

Item 2. $n < x + w \le x_+ + w$ and $n > x - w \ge x_- - w$. The ceiling and floor are non-decreasing, which gives the inequalities. If $[x_-, x_+]$ contains no integer, there is an integer j with $j < x_- \le x_+ < j + 1$, so $\lceil x_- \rceil = \lceil x_+ \rceil = j + 1$ and $\lfloor x_- \rfloor = \lfloor x_+ \rfloor = j$, and the inequalities force $\lceil x \rceil = j + 1$, $\lfloor x \rfloor = j$.

Item 3. The sum and difference rules follow from adding the inequalities. The function $(s, t) \mapsto st$ is bilinear, so on the rectangle $[x_-, x_+] \times [y_-, y_+]$ it attains its extrema at the

vertices; the same holds for $(s,t) \mapsto s/t$ when $t \geq y_- > 0$, which is monotone in each variable separately. For the absolute value: if $x_- \geq 0$, then $|x| = x \in [x_-, x_+]$; if $x_+ \leq 0$, then $|x| = -x \in [-x_+, -x_-]$; otherwise $x_- < 0 < x_+$ and $0 \leq |x| \leq \max(x_+, -x_-)$; in each case this is the stated interval. For the square root, $4^P x \geq 4^P x_- \geq \lfloor 4^P x_- \rfloor \geq \lfloor \sqrt{\lfloor 4^P x_- \rfloor} \rfloor^2$, so $2^P\sqrt{x}$ is at least the left numerator; and $4^P x \leq \lceil 4^P x_+ \rceil < \left(\lfloor \sqrt{\lceil 4^P x_+ \rceil} \rfloor + 1\right)^2$, so $2^P\sqrt{x}$ is less than the right numerator.

Item 4. Since $q > 0$, $|g| < 1$, and $\log q = \log \frac{1+g}{1-g} = 2\sum_{k\geq 0} g^{2k+1}/(2k+1)$. The tail after $k = h$ is bounded in absolute value by

$$2\sum_{k\geq h+1} \frac{|g|^{2k+1}}{2k+1} \leq \frac{2|g|^{2h+3}}{2h+3} \sum_{i\geq 0} g^{2i} = \frac{2|g|^{2h+3}}{(2h+3)(1-g^2)}.$$

The last assertion holds because log is increasing on $(0,\infty)$ and $x_- \leq x \leq x_+$.

Item 5. Both sides are positive and $t \mapsto t^q$ is strictly increasing on $(0,\infty)$, so $c < 10^{p/q}$ if and only if $c^q < 10^p$; and $77.28 = 1932/25$, $79.71 = 7971/100$.

Item 6. For every integer n, $\|x\| \leq |x - n| \leq |x - x_0| + |x_0 - n|$; taking the minimum over n gives $\|x\| \leq |x - x_0| + \|x_0\|$, and exchanging x and x_0 gives the first claim. If $x, x_0 \in [x_-, x_+]$, then $|x - x_0| \leq x_+ - x_-$, and $\|x_0\|$ is rational for rational x_0; this gives the enclosure.

Every real quantity of the decision tests is built from integers by $+$, $-$, $\times$, $\div$, $|\cdot|$, square roots, logarithms (with $\log q = j\log 2 + \log(2^{-j}q)$ where convenient) and the distance $\|\cdot\|$ to the nearest integer, so items 3, 4 and 6 give rational enclosures of it, and items 1 and 2 accept a condition only when it holds exactly, and discard a candidate only when its exact condition fails. Outward-rounded interval arithmetic implements the same rules. □

Finding certificates. Every condition of a certificate is an identity between integers or an inequality between real numbers built from a, b, c by the operations of Lemma 9.7, so a certificate is checked exactly from enclosures; a condition that the enclosures do not decide counts as failed, and the precision affects only the chance of success. A practical order is as follows.

1. Enclose ξ, μ, A, B and K' with absolute errors small compared with $1/(qU)$ for the largest denominator q to be used, that is, with about $2\log_{10} q + 10$ digits. Take for m_{DP} the least integer with $K'\beta^{-2m} \leq 1$, or a larger one where the enclosures do not decide that comparison (Definition 9.1).

2. (DP) Take the convergents p/q of a rational number of the enclosure of ξ with $q > 6U$, as Dujella and Pethő do [8]; at the first one with $\|q\mu\| - U|q\xi - p| > 0$, let M_{red} be the least integer with $B^{M_{\mathrm{red}}}\big(\|q\mu\| - U|q\xi - p|\big) \geq qA$. As $B = \beta^2 > 4ac$, M_{red} is small, and the exact check of the certificate runs over few indices. The distance $\|q\mu\|$ is enclosed by item 6 of Lemma 9.7.

3. If $\|q\mu\| - U|q\xi - p| \leq 0$ for several consecutive convergents, the linear form $m'\xi - n' + \mu$ has a zero or a near-zero at a small index. Then list the common terms $z_{m'}^{(a,c)} = z_{n'}^{(b,c)}$ with m' small, comparing $z_{m'}^{(a,c)}$ exactly with $z_{n'}^{(b,c)}$ for n' the integer nearest to $m'\xi + \mu$; take consecutive convergents p_0/q_0, p_1/q_1 with $q_0 \leq U < q_1$, which satisfy $p_1q_0 - p_0q_1 = \pm 1$ (the sign condition of (G) is checked by item 1 of Lemma 9.7), and use (G) with a common term (m_0, n_0) such that $m_0 \geq m_{\mathrm{DP}}$ and $AB^{-m_0} \leq |q_0\xi - p_0|$, or (S) with a pair (m_0, n_0), $m_0 \leq U$, such that $|m_0\xi - n_0 + \mu| \leq \frac{1}{2}|q_0\xi - p_0|$.

4. If no certificate is found, split the range: item 2 of Lemma 9.6 allows several ranges.

9.2 The triples to be tested

The tests apply to one triple at a time. The triples of each family that can have an irregular extension are generated by the descent tree of He, Togbé and Ziegler [11, Section 4], in which every triple of positive degree descends from one Euler triple; the windows of the families bound the searches.

Definition 9.8. Let $T = \{a, b, c\}$ with $a < b < c$ be a Diophantine triple (Definition 2.1), and put $X = d_+(a, b, c)$ (Definition 2.2).

Children. The **high child**, the **middle child** and the **low child** of T are the sets

$$T^{\mathrm{h}} = \{a, b, X\}, \qquad T^{\mathrm{m}} = \{a, c, X\}, \qquad T^{\mathrm{l}} = \{b, c, X\};$$

together they are the **children** of T. A set Y is **reached from** T **by** $j \geq 1$ **child steps** if there are Diophantine triples $T = Y_0, Y_1, \ldots, Y_j = Y$ such that Y_i is a child of Y_{i-1} for $1 \leq i \leq j$.

Root of the smaller pair. Put $r(T) = \sqrt{ab + 1}$, the root of the pair of the two smaller elements of T; thus $r(T) = r$.

Euler roots. The **Euler root** of an Euler triple E is $r(E)$. If $\deg T \geq 1$, the Euler root of T is the Euler root of the Euler triple $\partial^{\deg T} T$ (Definition 2.8).

The children and the descent tree they form are those of He, Togbé and Ziegler [11, Section 4].

Lemma 9.9. *Let $T = \{a, b, c\}$ with $a < b < c$ be a Diophantine triple, let $X = d_+(a, b, c)$, and let T^{h}, T^{m}, T^{l} be its children and $r(\cdot)$ the root of the smaller pair.*

1. *The children of T are Diophantine triples which are not Euler, with largest element X, descent T and degree $\deg T + 1$. The descent partners of T^{h}, T^{m}, T^{l} are c, b, a, and they are high, middle and low, respectively.*

2. *Every Diophantine triple Y which is not Euler is a child of exactly one Diophantine triple, namely of its descent ∂Y.*

3. *$X > 4abc + c$. Hence every triple reached from T by $j \geq 1$ child steps has largest element at least X, with equality only for $j = 1$.*

4. *$r(T^{\mathrm{h}}) = r$, $r(T^{\mathrm{m}}) = s$ and $r(T^{\mathrm{l}}) = t$; in particular $r(Y) \geq r(T)$ for every child Y of T.*

5. *If $\deg T \leq 1$, the children of T are single-chain. If $\deg T \geq 2$, a child of T is bent if and only if T is bent, or T is middle or low.*

6. *The Euler triples are the sets $\{p, q, p + q + 2\varrho\}$ with integers $1 \leq p < q$ and $\varrho \geq 2$ such that $pq = \varrho^2 - 1$; such a set determines p, q and ϱ, and ϱ is its Euler root. Let ϱ be the Euler root of T. If $\deg T = 1$, then ϱ is the number ϱ of Definition 3.6 and $c \geq 16\varrho^3 - 4\varrho$. If $\deg T \geq 2$, then*
$$c > 16\varrho(\varrho^2 - 1)(4\varrho^2 - 1).$$

7. *Let $C > 0$ be real, and let Y be a Diophantine triple with $\deg Y \geq 1$ and largest element less than C. There is exactly one sequence of Diophantine triples $Y_0, Y_1, \ldots, Y_j = Y$ with $j \geq 1$, Y_0 Euler and Y_i a child of Y_{i-1} for $1 \leq i \leq j$; for it $j = \deg Y$, $Y_i = \partial^{j-i} Y$, and $Y_1, \ldots, Y_j$ have largest elements less than C. Conversely, every triple reached from an Euler triple by $j \geq 1$ child steps has degree j. The Euler root ϱ of Y satisfies $16\varrho^3 - 4\varrho < C$, and $16\varrho(\varrho^2 - 1)(4\varrho^2 - 1) < C$ if $\deg Y \geq 2$.*

Proof. The maps $d_\pm$ do not depend on the order of their arguments. For positive integers $A < B$ with $AB+1$ a square and $X' \geq 0$, $A+B+(2AB+1)X' = A+B+X'+2ABX'$, so

$$d_\pm(A,B,X') = A+B+X'+2ABX' \pm 2\sqrt{AB+1}\,\sqrt{AX'+1}\,\sqrt{BX'+1},$$

which is symmetric in A, B, X'. In particular $d_\pm(p,q,X')$ of Lemma 2.6 is $d_\pm$ of the three numbers in any order.

Item 1. By item 2 of Lemma 2.6 for the pair $\{a,b\}$ at $X' = c$,

$$aX+1 = (at+rs)^2, \qquad bX+1 = (bs+rt)^2, \qquad cX+1 = (rc+st)^2,$$

and $X = a+b+(2ab+1)c+2rst$ is an integer with $X > 4abc+c > c$ by Lemma 2.4. So $\{a,b,c,X\}$ is a Diophantine quadruple, and each child is a Diophantine triple with largest element X. Fix $w \in \{a,b,c\}$, let $p < q$ be the other two elements of T, and let $Y = \{p,q,X\}$ be the corresponding child: $Y = T^{\mathrm{h}}$, T^{m}, T^{l} according as $w = c$, b, a. By item 1 of Lemma 2.6, $\mathcal{R}(a,b,c,X) = 0$; by symmetry $\mathcal{R}(p,q,X,w) = 0$, so w is one of $d_\pm(p,q,X)$. By item 3 of Lemma 2.6, $d_+(p,q,X) > p+q+X+4pqX > X > w$, hence

$$w = d_-(p,q,X).$$

As $w \geq 1$, item 4 of Lemma 2.6 shows that $Y = \{p < q < X\}$ is not Euler. So its descent partner is $e(Y) = d_-(p,q,X) = w$ and its descent is $\{p,q,w\} = T$ (Definition 2.8); the i-fold descent of Y is the $(i-1)$-fold descent of T for $i \geq 1$, and Y is not Euler, so $\deg Y = \deg T + 1$ by the definition of the degree. Finally, $T^{\mathrm{h}} = \{a < b < X\}$ has descent partner c with $b < c < X$, so it is high; $T^{\mathrm{m}} = \{a < c < X\}$ has descent partner b with $a < b < c$, so it is middle; and $T^{\mathrm{l}} = \{b < c < X\}$ has descent partner $a < b$, so it is low.

Item 2. Let $Y = \{p < q < w\}$ be a Diophantine triple which is not Euler, and $e = e(Y)$. By Theorem 2.7, $e = d_-(p,q,w)$ is a positive integer distinct from p and q, $\partial Y = \{p,q,e\}$ is a Diophantine triple, and $d_+(p,q,e) = w$; by the symmetry above, w is d_+ of the elements of ∂Y in increasing order. Hence the child of ∂Y in which e is replaced by w is $\{p,q,w\} = Y$, and Y is a child of ∂Y. Conversely, if Y is a child of a Diophantine triple T', then $\partial Y = T'$ by item 1. So ∂Y is the only triple of which Y is a child.

Item 3. The inequality $X > 4abc+c$ is Lemma 2.4. By item 1, the largest element of each child of a Diophantine triple is d_+ of that triple, which exceeds the largest element of the triple; so the largest element increases strictly at each child step, and a triple reached from T by $j \geq 1$ child steps has largest element at least X, with equality only for $j = 1$.

Item 4. Since X is the largest element of each child, the two smaller elements of T^{h}, T^{m}, T^{l} are $\{a,b\}$, $\{a,c\}$, $\{b,c\}$, whose roots are r, s, t; and $r < s < t$ because $ab < ac < bc$.

Item 5. Let Y be a child of T and $k = \deg Y = \deg T + 1$ (item 1). If $\deg T \leq 1$, then $k \leq 2$, so the condition that defines single-chain triples in Definition 3.1 is empty for Y, and Y is single-chain. Let $\deg T \geq 2$. As $\partial Y = T$, we have $\partial^i Y = \partial^{i-1} T$ and hence $\pi_i(Y) = \pi_{i-1}(T)$ for $1 \leq i \leq k$ (Definition 3.1). So Y is single-chain if and only if

$$\pi_0(T) = \pi_1(T) = \cdots = \pi_{\deg T - 1}(T),$$

that is, if and only if $\pi_0(T) = \pi_1(T)$ and T is single-chain. Here $\pi_0(T) = \{a,b\}$, and $\pi_1(T)$ is the pair of the two smaller elements of $\partial T = \{a,b,e(T)\}$, where $e(T) \notin \{a,b\}$ (Lemma 3.4, item (i)); so $\pi_1(T) = \pi_0(T)$ exactly when $e(T) > b$, that is, when T is high (Lemma 3.4, item (ii)). Hence Y is single-chain if and only if T is high and single-chain, which is item 5.

Item 6. If $p < q < w$ is an Euler triple, then $\varrho = \sqrt{pq+1}$ is an integer, $\varrho^2 - 1 = pq \geq 2$, so $\varrho \geq 2$, and $w = p + q + 2\varrho$. Conversely, let $1 \leq p < q$ and $\varrho \geq 2$ be integers with $pq = \varrho^2 - 1$, and $w = p + q + 2\varrho$. Then

$$pw + 1 = (p + \varrho)^2, \qquad qw + 1 = (q + \varrho)^2,$$

and $w > q$, so $\{p, q, w\}$ is a Diophantine triple with largest element $p+q+2\sqrt{pq+1}$, that is, an Euler triple. The set determines its two smaller elements $p < q$ and $\varrho = \sqrt{pq+1} = r(\{p, q, w\})$, its Euler root.

Let $\deg T = 1$. The Euler root of T is $r(\partial T) = \sqrt{pq+1}$, where $p < q$ are the two smaller elements of ∂T, that is, $\{p, q\} = \pi_1(T)$; this is the number ϱ of Definition 3.6, and $c \geq 16\varrho^3 - 4\varrho$ by item 3 of Lemma 3.7.

Let $\deg T = k \geq 2$, and let $D = \partial^{k-1}T$, a triple of degree one with Euler descent $\partial D = \partial^k T = \{p, q, p + q + 2\varrho\}$, where ϱ is the Euler root of T. By item 3 of Lemma 3.7 for D, its largest element is $c_D = 4\varrho(\varrho + p)(\varrho + q)$. Since $p \neq q$,

$$(p + q)^2 = (q - p)^2 + 4pq \geq 4\varrho^2 - 3 > (2\varrho - 1)^2$$

(as $\varrho \geq 2$), so $p + q \geq 2\varrho$ and $(\varrho + p)(\varrho + q) = \varrho^2 + \varrho(p + q) + \varrho^2 - 1 \geq 4\varrho^2 - 1$; hence $c_D \geq 4\varrho(4\varrho^2 - 1)$. By item 2, D is a child of ∂D, so by item 4 the product of the two smaller elements of D is $r(D)^2 - 1 \geq r(\partial D)^2 - 1 = \varrho^2 - 1$. The triple $D' = \partial^{k-2}T$ is not Euler and has descent D, so it is a child of D (item 2), with largest element $d_+(D)$ (item 1). By item 3 for D,

$$d_+(D) > 4\big(r(D)^2 - 1\big)c_D + c_D > 4(\varrho^2 - 1) \cdot 4\varrho(4\varrho^2 - 1) = 16\varrho(\varrho^2 - 1)(4\varrho^2 - 1).$$

Finally $T = D'$ if $k = 2$, and T is reached from D' by $k - 2$ child steps if $k \geq 3$; in both cases $c \geq d_+(D)$ by item 3, which proves the bound.

Item 7. Let $j = \deg Y \geq 1$ and $Y_i = \partial^{j-i}Y$ for $0 \leq i \leq j$. Then Y_0 is Euler, and for $i \geq 1$ the triple Y_i is not Euler and has descent Y_{i-1}, so it is a child of Y_{i-1} by item 2. Let $Y'_0, \ldots, Y'_{j'} = Y$ be any sequence with $j' \geq 1$, Y'_0 Euler and Y'_i a child of Y'_{i-1}. By item 1, Y'_i is not Euler for $i \geq 1$, and by item 2 its parent is its descent; by induction on i, $Y'_{j'-i} = \partial^i Y$ for $0 \leq i \leq j'$. Thus $\partial^i Y$ is not Euler for $i < j'$ and $\partial^{j'}Y = Y'_0$ is Euler, so $j' = \deg Y = j$ and $Y'_i = Y_i$. By item 3 the largest elements of $Y_1, \ldots, Y_j$ are at most that of Y, hence less than C. The converse follows from item 1 by induction on j, as an Euler triple has degree 0. The Euler root of Y is that of Y_0. If $j = 1$, item 6 gives $16\varrho^3 - 4\varrho \leq$ (largest element of Y) $< C$. If $j \geq 2$, item 6 gives $16\varrho(\varrho^2 - 1)(4\varrho^2 - 1) < C$, and $16\varrho^3 - 4\varrho = 4\varrho(4\varrho^2 - 1) < 16\varrho(\varrho^2 - 1)(4\varrho^2 - 1)$ because $4(\varrho^2 - 1) \geq 12$.

That every Euler triple is the root of a ternary tree of triples of growing degree is the observation of He, Togbé and Ziegler [11, Section 4]; items 2, 3 and 7 make it a generation rule with a stopping rule. □

Lemma 9.10. *Let $T = \{a, b, c\}$ with $a < b < c$ be a high Diophantine triple of degree at least two which has an irregular extension. Let $f = e(\partial T)$, let T_0, Δ_0, n_0 be as in the window of BH (Definition 4.2), let $r(\cdot)$ be the root of the smaller pair, and put $M_{\mathrm{II}} = 5\,149\,999\,999\,999$. Suppose that $c \geq 3.22 \cdot 10^{50}$, or that $c \geq 10^{50}$ and $a \geq 2$. Then $\partial^2 T = \{a, b, f\}$, $\Delta_0 \geq a + 1$,*

$$\min\{fb,\ n_0 + \Delta_0\} \leq M_{\mathrm{II}} \qquad \text{and} \qquad r(\partial^2 T) \leq M_{\mathrm{II}}.$$

If moreover $n_0 + \Delta_0 \leq M_{\mathrm{II}}$, then $r(\partial^2 T)^2 - 1 \leq af$ and $a^2 f < M_{\mathrm{II}}^2/2$.

Proof. First, every representation of z is of type II with $H = 0$, and $\min\{fb, n_0 + \Delta_0\} \le M_{\mathrm{II}}$. Let d be an irregular extension of T and $z = \sqrt{cd+1}$. By Lemma 2.9, $d > d_+(a,b,c)$, so z has a representation $\rho = (\lambda, m, n)$ by Lemma 2.13. As $c \ge 10^{50} > 10^{30}$, T is a triple of BH, and $(T,\rho) \in \mathrm{BH}$ by Proposition 8.3. The type I window of BH requires $c < 5.7 \cdot 10^{45}$, and its type II window with $H \ne 0$ requires $c < 3.22 \cdot 10^{50}$, and $c < 10^{50}$ if $a \ge 2$; under our hypothesis none of these holds. So ρ is of type II with $H = 0$, and the window gives

$$\min\{fb,\ n_0 + \Delta_0\} \le m, \qquad 2m + 1 = m' < Q_{\mathrm{II}}(c).$$

The largest value of Q_{II} is $1.03 \cdot 10^{13}$ (Definition 4.1), so $2m + 1 \le 1.03 \cdot 10^{13} - 1$ and $m \le M_{\mathrm{II}}$.

Since T is high, $\partial T = \{a, b, e\}$ with $a < b < e$ (Definition 2.8). As $\deg T \ge 2$, ∂T is not Euler; so $f = d_-(a,b,e)$ and $\partial^2 T = \{a, b, f\}$ (Lemma 3.4, item (ii)), where f is a positive integer not in $\{a, b\}$ (Lemma 3.4, item (i), applied to ∂T). By Lemma 2.4 for $a < b < e$, $e = a + b + 2r$ or $e > 4ab$; the first would make ∂T Euler, so $e > 4ab$. By Lemma 3.4, item (i), $c = d_+(a,b,e)$, so Lemma 2.4 also gives $c > 4abe + e$.

Next, $\Delta_0 \ge a + 1$. As $e > 4ab > a + b$, $T_0 > 0$. Expanding $aT_0 = 2a(a+b)e - 2a(a+b)^2 + 12a^2b + 6a$ and using $4ab - 2a(a+b) + 1 = 2a(b-a) + 1$,

$$4abe + e - aT_0 = e\big(2a(b-a)+1\big) + 2a(a+b)^2 - 12a^2b - 6a.$$

Since $a^2 < a(a+1) + 1 < (a+1)^2$, the number $a(a+1)+1$ is not a square, so $b \ne a + 1$ and $b - a \ge 2$. With $e > 4ab$ this gives

$$4abe + e - aT_0 > e(4a+1) - 12a^2b - 6a > 4ab(4a+1) - 12a^2b - 6a = 4a^2b + a(4b-6) > 0.$$

Hence $c > 4abe + e > aT_0$, so $c/T_0 > a$, $\lfloor c/T_0 \rfloor \ge a$ and $\Delta_0 = \lfloor c/T_0 \rfloor + 1 \ge a + 1$.

By Definition 9.8, $r(\partial^2 T)^2 - 1$ is the product of the two smaller elements of $\partial^2 T = \{a, b, f\}$. Suppose first that $fb \le M_{\mathrm{II}}$. As $f \ge 1$, all of $a < b$ and f are at most M_{II}, and the product of two distinct elements of $\partial^2 T$ is at most $M_{\mathrm{II}}(M_{\mathrm{II}} - 1) < M_{\mathrm{II}}^2$. Suppose next that $n_0 + \Delta_0 \le M_{\mathrm{II}}$. Then $0 \le n_0 \le M_{\mathrm{II}} - \Delta_0 \le M_{\mathrm{II}} - 2$, and by the definition of n_0,

$$(4af+1)(a+1) \le (4af+1)\Delta_0 \le 2n_0(n_0+1) \le 2(M_{\mathrm{II}}-2)(M_{\mathrm{II}}-1) < 2M_{\mathrm{II}}^2.$$

Hence $4a^2 f < 4af(a+1) < 2M_{\mathrm{II}}^2$, that is, $a^2 f < M_{\mathrm{II}}^2/2$. The two smaller elements of $\{a, b, f\}$ are $\{a, f\}$ if $f < b$ and $\{a, b\}$ if $f > b$; their product is af or $ab < af$, so $r(\partial^2 T)^2 - 1 \le af \le a^2 f < M_{\mathrm{II}}^2/2$. Since $\min\{fb, n_0 + \Delta_0\} \le M_{\mathrm{II}}$, one of the two cases occurs, and in both $r(\partial^2 T)^2 - 1 < M_{\mathrm{II}}^2$. As $r(\partial^2 T)$ is an integer, $r(\partial^2 T) \le M_{\mathrm{II}}$. □

Lemma 9.11. *Let $T = \{a, b, c\}$ with $a < b < c$ be a Diophantine triple which is not Euler and has an irregular extension, with descent partner e. Let ϱ be the Euler root of T and $r(\cdot)$ the root of the smaller pair, and put $u = \sqrt{ae+1}$ if T is middle or low. Then $b > 4000$, $c < 200b^4$ and $c < 10^{99}$, and:*

1. *if $\deg T = 1$ and $c < 2.01 \cdot 10^{26}$, then $\varrho \le 232\,465\,597$;*
2. *if $\deg T = 1$ and $c \ge 2.01 \cdot 10^{26}$, then T is a triple of A2 (Definition 4.2): $c < 1.81 \cdot 10^{47}$, $\Theta \ne 0$, $3 \mid \Theta$, and $\varrho < 3.21 \cdot 10^{11}$ if T is high, $\varrho < 4.61 \cdot 10^{11}$ if T is middle or low;*
3. *if $\deg T \ge 2$, then $b > 8a$;*
4. *if $\deg T \ge 2$ and $c < 10^{30}$, then $\varrho \le 435\,275$;*

5. *if* $\deg T \geq 2$, *T is middle or low and* $c \geq 10^{30}$, *then* $r(\partial T) = u$, *and either* $u \leq 3\,641\,599\,923\,110$ *and* $c < 10^{77.28}$, *or* $u \leq 1\,040\,630\,830\,203$*; more precisely, if an irregular extension of T gives a representation of type II, then* $u \leq 3\,641\,599\,923\,110$ *and* $c < 10^{77.28}$, *and if one gives a representation of type I, then* $u \leq 1\,040\,630\,830\,203$*;*

6. *if* $\deg T \geq 2$, *T is high and* $c \geq 10^{30}$, *then* $c < 10^{79.71}$*; if moreover* $c \geq 3.22 \cdot 10^{50}$, *or* $c \geq 10^{50}$ *and* $a \geq 2$, *then* $r(\partial^2 T) \leq 5\,149\,999\,999\,999$.

Moreover, among all Diophantine triples:

(a) for each $\nu \geq 3$, *the single-chain middle or low triples of degree* $\nu - 1$ *are, each exactly once, the sets* $\{a, x_{\nu-1}, x_\nu\}$, *where* (a, e) *runs over the ordered pairs of distinct positive integers with* $ae + 1$ *a square,* σ *over the admitted signs of* $\{a, e\}$, *and* (x_j) *is the chain of* $\{a, e\}$ *with sign* σ*; such a set has smallest element* a, *descent partner* e *and chain sign* σ, *and it is middle if* $a < e$ *and low if* $a > e$*;*

(b) the bent middle or low triples are the middle and low children of the triples of degree at least two that are bent or are middle or low;

(c) the high triples of degree at least two are the high children of the triples of degree at least one.

Proof. The comparisons of explicit integers in this proof are exact and were checked by computer. Let d be an irregular extension of T. By Lemma 2.9, $b > 4000$ and $d > d_+(a, b, c)$, so $z = \sqrt{cd + 1}$ has a representation ρ by Lemma 2.13. By Theorem 2.11, $c < 200b^4$, and by Theorem 4.3, $c < 10^{99}$.

For $c \geq 10^{30}$ the largest values of the caps are $Q_{\mathrm{I}}(c) \leq 6.5 \cdot 10^{12}$ and $Q_{\mathrm{II}}(c) \leq 1.03 \cdot 10^{13}$ (Definition 4.1); hence an integer m with $2m < Q_{\mathrm{I}}(c)$ satisfies $m \leq M_{\mathrm{I}} = 3\,249\,999\,999\,999$, and an integer m with $2m + 1 < Q_{\mathrm{II}}(c)$ satisfies $m \leq M_{\mathrm{II}} = 5\,149\,999\,999\,999$.

Items 1 and 4. The functions $f_1(x) = 16x^3 - 4x$ and $f_2(x) = 16x(x^2 - 1)(4x^2 - 1)$ increase on $x \geq 1$. If $\deg T = 1$, then $f_1(\varrho) \leq c < 2.01 \cdot 10^{26}$ by item 6 of Lemma 9.9, and $f_1(232\,465\,598) \geq 2.01 \cdot 10^{26}$; so $\varrho \leq 232\,465\,597$. If $\deg T \geq 2$ and $c < 10^{30}$, then $f_2(\varrho) < c < 10^{30}$ by the same item, and $f_2(435\,276) \geq 10^{30}$; so $\varrho \leq 435\,275$.

Item 2. The quadruple $a < b < c < d$ is irregular, $\deg T = 1$ and $c \geq 2.01 \cdot 10^{26}$, so $(T, \rho) \in$ A2 by Proposition 6.13; in particular T is a triple of A2: $c < 1.81 \cdot 10^{47}$, $\Theta \neq 0$, $3 \mid \Theta$, and $\varrho < 3.21 \cdot 10^{11}$ if $\varepsilon = 1$, $\varrho < 4.61 \cdot 10^{11}$ if $\varepsilon = -1$ (Definition 4.2), where ϱ is the number of Definition 3.6, that is, the Euler root (Lemma 9.9, item 6). By item 1 of Lemma 3.7, $\varepsilon = 1$ if T is high and $\varepsilon = -1$ if T is middle or low.

Item 3 is Lemma 2.10.

Item 5. Let $\deg T \geq 2$, T middle or low and $c \geq 10^{30}$. By Lemma 3.4, item (ii), b is the largest element of $\partial T = \{a, b, e\}$, so the two smaller elements of ∂T are a and e and $r(\partial T) = \sqrt{ae + 1} = u$. As $e < b$, Lemma 3.5 gives $c = (4ae + 2)b + 2(a + e) - f$ with $1 \leq f < b$ (here $f = e(\partial T)$), so

$$c > (4ae + 1)\, b = (4u^2 - 3)\, b.$$

Let d be an irregular extension of T and $\rho = (\lambda, m, n)$ a representation of $z = \sqrt{cd + 1}$; by Proposition 7.45, ρ satisfies the generic window of its type. If ρ is of type I, then $\kappa\sqrt{c/b} < n < m$ and $2m < Q_{\mathrm{I}}(c)$, so $m \leq M_{\mathrm{I}}$ and

$$\kappa^2(4u^2 - 3) < \kappa^2 \frac{c}{b} < n^2 < M_{\mathrm{I}}^2;$$

since $\kappa^2(4u^2-3)$ increases with u and $\kappa^2\big(4\cdot 1\,040\,630\,830\,204^2-3\big) \geq M_{\mathrm{I}}^2$, with $\kappa^2 = (9-\sqrt{17})/2$, we get $u \leq 1\,040\,630\,830\,203$. If ρ is of type II, then $c < 10^{77.28}$, $2m+1 < Q_{\mathrm{II}}(c)$, so $m \leq M_{\mathrm{II}}$, and $J \geq 1$ with $J(4ae+1) < 2m(m+1)$; hence

$$4u^2 - 3 = 4ae + 1 \leq J(4ae+1) < 2m(m+1) \leq 2M_{\mathrm{II}}(M_{\mathrm{II}}+1),$$

and as $4 \cdot 3\,641\,599\,923\,111^2 - 3 \geq 2M_{\mathrm{II}}(M_{\mathrm{II}}+1)$, $u \leq 3\,641\,599\,923\,110$.

Item 6. Let $\deg T \geq 2$, T high and $c \geq 10^{30}$. Then T is a triple of BH, and $(T,\rho) \in \mathrm{BH}$ by Proposition 8.3. The type I window of BH requires $c < 5.7 \cdot 10^{45}$ and the type II window $c < 10^{79.71}$; so $c < 10^{79.71}$. The last assertion is Lemma 9.10, whose constant is M_{II}.

Item (a). Let $\nu \geq 3$, let (a,e) be an ordered pair of distinct positive integers with $ae+1$ a square, σ an admitted sign of $\{a,e\}$, and (x_j) the chain. By item 4 of Lemma 3.3 (with $p=a$, $q=e$), $Y = \{a, x_{\nu-1}, x_\nu\}$ is a Diophantine triple which is not Euler, is single-chain of degree $\nu-1$, has descent partner e, chain pair $\{a,e\}$ and chain sign σ, and is middle if $a<e$ and low if $a>e$; by item 3 there, $x_{\nu-1} \geq x_2 > \max\{a,e\}$, so a is the smallest element of Y. Conversely, a single-chain middle or low triple of degree $\nu-1$ is such a set by Lemma 3.4, item (iii), with a its smallest element, e its descent partner and σ its chain sign. The data are determined by the set: a is its smallest element, e its descent partner, $\nu-1$ its degree, and σ its chain sign, which is unique by Lemma 3.4, item (iii). So each triple arises exactly once.

Items (b) and (c). Item numbers in this paragraph refer to Lemma 9.9. A bent triple Y has degree at least 3 (Lemma 3.4, item (v)), and it is a child of ∂Y (item 2), which has degree at least 2. If Y is middle or low, it is the middle or the low child of ∂Y, since the children $(\partial Y)^{\mathrm{h}}$, $(\partial Y)^{\mathrm{m}}$, $(\partial Y)^{\mathrm{l}}$ are high, middle and low (item 1); and by item 5, ∂Y is bent, or middle or low. Conversely, by items 1 and 5, the middle and low children of a triple of degree at least two that is bent, middle or low are bent, and middle or low. Likewise a high triple Y of degree at least two is the high child of ∂Y, which has degree at least one; and the high child of a triple of degree at least one is high, of degree at least two (item 1). □

Every triple of a family that can have an irregular extension is found, exactly once, by the following searches; in each, a triple is kept only if $b > 4000$ and the conditions of its family hold.

- A1 and A2: the three children of each Euler triple with Euler root $\varrho \leq 232\,465\,597$ (A1; Lemma 9.11, item 1), and the middle and low children of each Euler triple with $\varrho < 4.61 \cdot 10^{11}$ and the high child of each with $\varrho < 3.21 \cdot 10^{11}$ (A2; item 2). The Euler triples with Euler root ϱ are the sets $\{p, q, p+q+2\varrho\}$ with $pq = \varrho^2 - 1$ and $p < q$ (Lemma 9.9, item 6).

- B0: the search of item 7 of Lemma 9.9 with $C = 10^{30}$: start from the Euler triples with $\varrho \leq 435\,275$, and discard a triple whose largest element is at least 10^{30} together with all triples reached from it (item 3 there); keep the triples of degree at least two with $b > 8a$.

- BM, BL, B3, B4: for each ordered pair (a,e) of distinct positive integers with $ae+1 = u^2$ and $u \leq 3\,641\,599\,923\,110$ (for each $u \geq 2$, a runs over the divisors of $u^2-1 = (u-1)(u+1)$ and $e = (u^2-1)/a$; $a \neq e$ holds as u^2-1 is not a square), and each admitted sign σ of $\{a,e\}$, run along the chain (x_j) slot by slot while $x_\nu < 10^{99}$, and keep $\{a, x_{\nu-1}, x_\nu\}$ if $\nu \geq 3$, $x_\nu \geq 10^{30}$, $x_{\nu-1} > 8a$, and $x_\nu < 10^{77.28}$ when $u > 1\,040\,630\,830\,203$ (Lemma 9.11, items 3, 5 and (a)). The family is BM if $\nu = 3$ and $a < e$, BL if $\nu = 3$ and $a > e$, B3 if $\nu = 4$, and B4 if $\nu \geq 5$. When $u > 1\,040\,630\,830\,203$, no irregular extension of the triple gives a representation of type I (Lemma 9.11, item 5), so its two type I cases are closed without a test.

- BX, and BH with $c \geq 3.22 \cdot 10^{50}$: let $\mathcal{Y}$ be the set of Diophantine triples Y with $r(Y) \leq 5\,149\,999\,999\,999$ and $d_+(Y) < 10^{99}$. By items 1 to 4 of Lemma 9.9, $\mathcal{Y}$ is closed under descent: $r(\partial Y) \leq r(Y)$, and $d_+(\partial Y)$ is the largest element of Y, which is less than $d_+(Y)$. So the search of item 7 there restricted to $\mathcal{Y}$, which starts from the Euler triples with Euler root at most $5\,149\,999\,999\,999$ and keeps only the children that lie in $\mathcal{Y}$, visits every element of $\mathcal{Y}$ exactly once. By items 5 and 6 of Lemma 9.11, $\mathcal{Y}$ contains the descent of every triple of BX and the second descent of every triple of BH with $c \geq 3.22 \cdot 10^{50}$. From each visited Y of degree at least two that is bent or is middle or low, take its middle and low children (bent, by item (b) of Lemma 9.11: BX); from each visited Y, take the high children of its children with $c \geq 3.22 \cdot 10^{50}$ (item (c): BH; those with $c < 3.22 \cdot 10^{50}$ are left to the next search).
- BH with $10^{30} \leq c < 3.22 \cdot 10^{50}$: the search of item 7 of Lemma 9.9 with $C = 3.22 \cdot 10^{50}$, whose Euler roots satisfy $\varrho \leq 5\,499\,651\,619$ (since $16\varrho(\varrho^2-1)(4\varrho^2-1) \geq 3.22 \cdot 10^{50}$ for $\varrho = 5\,499\,651\,620$), taking the high children of the triples of degree at least one.

9.3 Small triples and the line

For A1 and B0 one certified range of indices, from the floor of the window to its cap, closes each case. The triples of A1 have Euler roots $\varrho \leq 232\,465\,597$ and those of B0 Euler roots $\varrho \leq 435\,275$ (Lemma 9.11, items 1 and 4); both are listed by the descent tree (Lemma 9.9, item 7).

Corollary 9.12. *Let $T = \{a, b, c\}$ with $a < b < c$ be a Diophantine triple with $b > 4000$, which is a triple of A1 or of B0 (Definition 4.2):*

- *(A1) $\deg T = 1$ and $c < 2.01 \cdot 10^{26}$; put $U = 7.7 \cdot 10^{18}$, and let L be the least integer $j \geq 0$ with $625j^4 > 256c$, that is, $L = \lfloor \frac{4}{5} c^{1/4} \rfloor + 1$;*
- *(B0) $\deg T \geq 2$ and $c < 10^{30}$; put $U = 1.004 \cdot 10^{19}$ and $L = 0$.*

1. *A case of T is closed (Definition 9.1) if the range $[L, U]$ is certified for it.*
2. *If T is a triple of A1 and $c \geq 10^{26}$, the two cases $(\mathrm{II}, \pm 1)$ are closed.*

Each inequality between real numbers in a certificate may be established from enclosures of its two sides by item 1 of Lemma 9.7, the enclosures being formed by items 3, 4 and 6 there (item 6 for the distances $\|q\mu\|$ of the data (DP)).

Proof. Let d be an irregular extension of T (Definition 2.3) and $\rho = (\lambda, m, n)$ a representation of $z = \sqrt{cd+1}$.

The triples of A1. The quadruple $a < b < c < d$ is irregular with $\deg T = 1$ and $c < 2.01 \cdot 10^{26}$, so $(T, \rho) \in$ A1 by Proposition 6.13: $\frac{2}{5} c^{1/4} < m$ and $m' < 7.7 \cdot 10^{18} = U$, and $c < 10^{26}$ if ρ is of type II. Hence $m' \geq 2m > \frac{4}{5} c^{1/4}$; as m' is an integer and L is the least integer exceeding $\frac{4}{5} c^{1/4}$ (for $j \geq 0$, $625j^4 > 256c$ is $j > \frac{4}{5} c^{1/4}$), $m' \geq L$. If $c \geq 10^{26}$, then ρ is of type I, which proves item 2.

The triples of B0. The triple is not Euler, so Proposition 5.10 gives $m' < Q_0(c)$. Put $K_c = K_{\mathrm{a}} (\log c)^2$ with $K_{\mathrm{a}} = 4.436 \cdot 10^{13}$ (Definition 4.1). As $c < 10^{30}$,

$$K_c < K_{\mathrm{a}} (\log 10^{30})^2 < 2.1168 \cdot 10^{17},$$
$$K_c \log\bigl(39(U+1)\bigr) < 2.1168 \cdot 10^{17} \cdot 47.417 < 1.00374 \cdot 10^{19} < U - 2.$$

So the integer U satisfies $U \geq K_c - 1$ and $U - 2 \geq K_c \log(39(U+1))$, and $Q_0(c) \leq U$ because $Q_0(c)$ is the least such integer (Definition 4.1). Hence $L = 0 \leq m' < U$.

Conclusion. In both families, every representation of z, for every irregular extension d of T, has $L \leq m' \leq U$. If the range $[L, U]$ is certified for a case, item 2 of Lemma 9.6, applied with this one range, shows that the case is closed; this is item 1. A real inequality established from enclosures holds exactly by item 1 of Lemma 9.7. By item 1 of Lemma 9.6, T has no irregular extension when its four cases are closed. □

For A2 the line pins the index: a type I representation on the line has m' within 10^{-100} of an explicit real number x_ε (Lemma 9.13, Proposition 9.14), so at most one even integer remains, and it is checked by comparing two Pell terms modulo an integer. The triples of A2 are the children of the Euler triples with Euler root $\varrho < 4.61 \cdot 10^{11}$ that satisfy the conditions of item 2 of Lemma 9.11.

Lemma 9.13. *In the setting of Proposition 6.2, put $m' = 2m$ and*

$$\delta_\lambda = \frac{\sqrt{bc} + \lambda\sqrt{ab}}{\sqrt{ac} + \lambda\sqrt{ab}}.$$

If $\varepsilon = 1$, then

$$0 < x_+ - m' < \frac{8ac}{3\log(\gamma/\beta)}\,\beta^{-2m'}, \qquad x_+ = \frac{2\Delta_\varepsilon \log\gamma + \log\delta_\lambda}{\log(\gamma/\beta)}.$$

If $\varepsilon = -1$, then

$$0 < m' - x_- < \frac{8ac}{3\log(\beta\gamma)}\,\beta^{-2m'}, \qquad x_- = \frac{2\Delta_\varepsilon \log\gamma - \log\delta_\lambda}{\log(\beta\gamma)}.$$

If moreover $c \geq 2.01 \cdot 10^{26}$, then both upper bounds are less than 10^{-100}.

Proof. **The linear form.** By Lemma 2.9, $b > 4000$. By item 1 of Proposition 6.2, $m > n \geq 0$, so $m \geq 1$, and by Lemma 2.14, $m \leq 2n$, so $n \geq 1$; hence $m' = 2m > 0$ and $n' = 2n > 0$. The triple is not Euler, and Lemma 5.7 gives, for $\Lambda = m'\log\beta - n'\log\gamma + \log\delta$,

$$0 < \Lambda < \tfrac{8}{3} ac\,\beta^{-2m'},$$

where, for the seeds of type I, $\delta = \delta_\lambda$ by item 1 of Lemma 5.8. Also $\gamma > \beta > 1$, because $t > s \geq 1$ and $x \mapsto x + \sqrt{x^2-1}$ increases.

Elimination of n. By definition of Δ_ε, $n' = \varepsilon(m' - 2\Delta_\varepsilon)$. If $\varepsilon = 1$, then $\Lambda = 2\Delta_\varepsilon \log\gamma + \log\delta_\lambda - m'\log(\gamma/\beta)$, so $x_+ - m' = \Lambda/\log(\gamma/\beta)$. If $\varepsilon = -1$, then $n' = 2\Delta_\varepsilon - m'$ and $\Lambda = m'\log(\beta\gamma) - 2\Delta_\varepsilon\log\gamma + \log\delta_\lambda$, so $m' - x_- = \Lambda/\log(\beta\gamma)$. In both cases the denominators are positive, and the first two claims follow.

Size of the error for $c \geq 2.01 \cdot 10^{26}$. By item 3 of Corollary 6.7, $m' > 3c^{1/4} \geq 3(2.01 \cdot 10^{26})^{1/4} > 10^7$. By item 4 of Lemma 3.7, $b < c^{1/2}$ and $a < c$; and $\beta^2 > 4ac \geq 4c$ because $\beta > 2\sqrt{ac}$. Since $\log\gamma - \log\beta = \operatorname{arccosh} t - \operatorname{arccosh} s$ and the derivative of arccosh decreases on $(1, \infty)$,

$$\log(\gamma/\beta) \geq \frac{t-s}{\sqrt{t^2-1}} > \frac{t^2-s^2}{2t^2} = \frac{(b-a)c}{2(bc+1)} \geq \frac{c}{2(bc+1)} > \frac{1}{4b} > \frac{1}{4}c^{-1/2}.$$

Hence both bounds are at most $\frac{32}{3} ac\, c^{1/2}(4c)^{-m'} < 11c^{5/2}(4c)^{-m'}$; for $\varepsilon = -1$ we used $\log(\beta\gamma) > \log\beta > 1$ instead. As $4c > 1$ and $m' > 10^7$, this is at most $11c^{5/2}(4c)^{-10^7}$, which decreases in c; so it is at most $11(2.01 \cdot 10^{26})^{5/2}(8.04 \cdot 10^{26})^{-10^7} < 10^{-100}$. □

Proposition 9.14. *Let $0 < a < b < c < d$ be an irregular Diophantine quadruple with $T = \{a, b, c\}$ of degree one and $c \geq 2.01 \cdot 10^{26}$, and let ε, Θ, the line and ℓ_ε be as in Definition 3.6. Then $\Theta \neq 0$ and $3 \mid \Theta$. Put $\lambda = -\operatorname{sgn}\Theta$, and let x_ε be the real number x_+ if $\varepsilon = 1$, and x_- if $\varepsilon = -1$, of Lemma 9.13 for this λ and $\Delta_\varepsilon = |\Theta|/3$. Then z has a representation of type I, and every representation (λ_0, m, n) of z of type I has $\lambda_0 = \lambda$ and satisfies, with $m' = 2m$ and $n' = 2n$:*

1. *$|x_\varepsilon - m'| < 10^{-100}$ and*
$$\max\Big(\tfrac{|\Theta|}{3},\ \ell_\varepsilon\Big) < m' < 2.0 \cdot 10^{12};$$
2. *$n' = \varepsilon(m' - 2|\Theta|/3)$, and the type I seeds $(x_1, z_1) = (y_2, z_2) = (1, \lambda)$ give $z_{m'}^{(a,c)} = z_{n'}^{(b,c)}$.*

At most one even integer m' satisfies $|x_\varepsilon - m'| < 10^{-100}$.

Proof. By item 1 of Proposition 6.12, z has a representation of type I. Let (λ_0, m, n) be any representation of z of type I, and put $m' = 2m$ and $n' = 2n$.

By item 2 of Proposition 6.12, $\lambda_0\Theta < 0$ and (m, n) lies on the line, that is, $3(m - \varepsilon n) = |\Theta|$. Hence $\Theta \neq 0$, $\lambda_0 = \lambda$, $3 \mid \Theta$ and $m - \varepsilon n = |\Theta|/3$.

Item 1. By Lemma 9.13, applied with $\Delta_\varepsilon = m - \varepsilon n = |\Theta|/3$ and $c \geq 2.01 \cdot 10^{26}$, $|x_\varepsilon - m'| < 10^{-100}$. By item 2 of Proposition 6.12, $m' < 2.0 \cdot 10^{12}$. Since $0 \leq n < m$ (item 1 of Proposition 6.2), $m' = 2m > m - \varepsilon n = |\Theta|/3$. By item 2 of Corollary 6.7, $m > (\sqrt{17} - 1)\varrho$ if $\varepsilon = 1$ and $m > 2.17\varrho$ if $\varepsilon = -1$, that is, $m' > \ell_\varepsilon$.

Item 2. By the definition of a representation of type I, $z = z_{2m}^{(a,c)} = z_{2n}^{(b,c)}$ for the seeds $(1, \lambda)$, and $n' = 2n = \varepsilon(2m - 2|\Theta|/3) = \varepsilon(m' - 2|\Theta|/3)$ because $m - \varepsilon n = |\Theta|/3$.

Two even integers differ by at least 2, so at most one lies within 10^{-100} of x_ε. □

Corollary 9.15. *Let $T = \{a, b, c\}$ with $a < b < c$ be a triple of A2 (Definition 4.2), and let ε, ϱ, Θ, the line and ℓ_ε be as in Definition 3.6; thus $\Theta \neq 0$ and $3 \mid \Theta$. Put $\lambda = -\operatorname{sgn}\Theta$, and let x_ε be the real number x_+ if $\varepsilon = 1$, and x_- if $\varepsilon = -1$, of Lemma 9.13 for this λ and $\Delta_\varepsilon = |\Theta|/3$, namely*

$$x_+ = \frac{2\Delta_\varepsilon \log\gamma + \log\delta_\lambda}{\log(\gamma/\beta)}, \qquad x_- = \frac{2\Delta_\varepsilon \log\gamma - \log\delta_\lambda}{\log(\beta\gamma)}, \qquad \delta_\lambda = \frac{\sqrt{bc} + \lambda\sqrt{ab}}{\sqrt{ac} + \lambda\sqrt{ab}}.$$

Let $[X_-, X_+]$ be an enclosure of x_ε in the sense of Lemma 9.7, and let $\mathcal{M}$ be the set of even integers m' with

$$X_- - 10^{-100} < m' < X_+ + 10^{-100}, \qquad \max\Big(\frac{|\Theta|}{3},\ \ell_\varepsilon\Big) < m' < 2.0 \cdot 10^{12}.$$

An enclosure with $X_+ - X_- < 1.9$ leaves at most one element in $\mathcal{M}$.

1. *The cases $(\mathrm{II}, 1)$, $(\mathrm{II}, -1)$ and $(\mathrm{I}, -\lambda)$ are closed.*
2. *The case (I, λ) is closed if $\mathcal{M}$ is empty, or if for every $m' \in \mathcal{M}$, with $n' = \varepsilon(m' - 2|\Theta|/3)$, either $n' < 0$ or $z_{m'}^{(a,c)} \neq z_{n'}^{(b,c)}$ for the seeds $(x_1, z_1) = (y_2, z_2) = (1, \lambda)$. The inequality holds, for instance, if $z_{m'}^{(a,c)} \not\equiv z_{n'}^{(b,c)} \pmod p$ for some integer $p \geq 2$, where both residues are the first coordinates of $M_a^{m'}(\lambda, 1)^\top$ and $M_b^{n'}(\lambda, 1)^\top$ modulo p, with $M_a = \begin{pmatrix} s & c \\ a & s \end{pmatrix}$ and $M_b = \begin{pmatrix} t & c \\ b & t \end{pmatrix}$, computed by repeated squaring.*

The comparisons of an integer $m' \geq 0$ with the bounds are exact integer tests: $m' > |\Theta|/3$ if and only if $3m' > |\Theta|$; $m' > 2(\sqrt{17}-1)\varrho$ if and only if $(m'+2\varrho)^2 > 68\varrho^2$; and $m' > 4.34\varrho$ if and only if $100m' > 434\varrho$.

Proof. Let d be an irregular extension of T (Definition 2.3). The quadruple $a < b < c < d$ is irregular, with $\deg T = 1$ and $c \geq 2.01 \cdot 10^{26}$, so every representation ρ of $z = \sqrt{cd+1}$ satisfies $(T, \rho) \in \mathrm{A2}$ by Proposition 6.13.

Item 1. The type II window of A2 is empty, so z has no representation of type II; and the type I window requires the sign $-\operatorname{sgn}\Theta = \lambda$, so z has no representation of type I with sign $-\lambda$. As d is arbitrary, the three cases are closed.

Item 2. Suppose that z has a representation (λ, m, n) of type I, and put $m' = 2m$, $n' = 2n$. By item 1 of Proposition 9.14, the even integer m' satisfies $|x_\varepsilon - m'| < 10^{-100}$ and $\max(|\Theta|/3, \ell_\varepsilon) < m' < 2.0 \cdot 10^{12}$, the number x_ε there being that of the statement, since it depends on T only. By item 2 of Lemma 9.7, $X_- - 10^{-100} < m' < X_+ + 10^{-100}$, so $m' \in \mathcal{M}$ and $\mathcal{M}$ is not empty. By item 2 of Proposition 9.14, $n' = \varepsilon(m' - 2|\Theta|/3)$, which is $2n \geq 0$, and $z_{m'}^{(a,c)} = z_{n'}^{(b,c)}$ for the seeds $(1, \lambda)$. An equality of integers persists modulo every $p \geq 2$. The residues are those of the statement by item 1 of Lemma 9.36, with $(z_1, x_1) = (z_2, y_2) = (\lambda, 1)$. So each condition of item 2 fails; as d is arbitrary, each condition of item 2 implies that the case (I, λ) is closed.

If $X_+ - X_- < 1.9$, the open interval $(X_- - 10^{-100}, X_+ + 10^{-100})$ has length less than 2, so it contains at most one even integer, and $\mathcal{M}$ has at most one element.

For the exact tests, with $m' \geq 0$ and $\varrho > 0$, $m' > 2(\sqrt{17}-1)\varrho$ is equivalent to $m' + 2\varrho > 2\sqrt{17}\,\varrho$, that is, $(m'+2\varrho)^2 > 68\varrho^2$, as both sides are positive; the other two are immediate. By item 1 of Lemma 9.6, T has no irregular extension when item 2 closes the case (I, λ). □

9.4 The generic test and high triples

The generic test closes the cases of a triple of degree at least two with $c \geq 10^{30}$ from the generic windows; it is the test for BX. The test for BH uses the floors of the high window and an exact search for the branch $L = 0$. The triples of BH are bounded by item 6 of Lemma 9.11 and the bent triples of BX are described by its item (b); the searches of Subsection 9.2 list them.

Corollary 9.16. *Let $T = \{a, b, c\}$ with $a < b < c$ be a Diophantine triple of degree at least 2 with $c \geq 10^{30}$ and descent partner e; let $m_{\mathrm{I}}(c)$, $m_{\mathrm{II}}(c)$ be the integer caps of Definition 4.1. Let n_κ be the least integer $n \geq 1$ with*

$$n^2 b \geq 4c \qquad \text{or} \qquad n^2(t^2-1) > (4c - n^2 b)^2,$$

and, if T is middle or low, let m_{lo} be the least integer $m \geq 0$ with $2m(m+1) > 4ae + 1$; put $L_{\mathrm{II}} = 2m_{\mathrm{lo}} + 1$ if T is middle or low and $L_{\mathrm{II}} = 9$ if T is high. Then, for each $\lambda \in \{-1, 1\}$:

1. *the case (I, λ) is closed if the range $[2n_\kappa + 2,\ 2m_{\mathrm{I}}(c)]$ is certified for it; in particular if $c \geq 10^{99}$ or $n_\kappa \geq m_{\mathrm{I}}(c)$;*
2. *the case (II, λ) is closed if $c \geq 10^{79.71}$, or if T is middle or low and $c \geq 10^{77.28}$, or if $c < 10^{79.71}$ and the range $[L_{\mathrm{II}},\ 2m_{\mathrm{II}}(c) + 1]$ is certified for it.*

Proof. Since κ is the positive root of $x^2 + x = 4$, an integer $n \geq 1$ satisfies $n > \kappa\sqrt{c/b}$ if and only if $x = n\sqrt{b/c}$ satisfies $x^2 + x > 4$, that is $n^2 b + n\sqrt{bc} > 4c$. This holds if $n^2 b \geq 4c$; otherwise both sides of $n\sqrt{bc} > 4c - n^2 b$ are positive, and squaring with $bc = t^2 - 1$ gives $n^2(t^2-1) > (4c - n^2 b)^2$. So n_κ is the least integer $n > \kappa\sqrt{c/b}$.

Let d be an irregular extension of T (Definition 2.3). Then $\{a,b,c,d\}$ is an irregular quadruple with $\deg T \geq 2$ and $c \geq 10^{30}$, so Proposition 7.45 applies to every representation (λ,m,n) of $z=\sqrt{cd+1}$.

Item 1. A type I representation has $c<10^{99}$, $\kappa\sqrt{c/b}<n<m$ and $2m<Q_{\mathrm{I}}(c)$, so $n\geq n_\kappa$, $m\geq n_\kappa+1$ and $m\leq m_{\mathrm{I}}(c)$; thus $2n_\kappa+2\leq m'\leq 2m_{\mathrm{I}}(c)$. By Lemma 9.6, item 2, the case (I,λ) is closed if this range is certified. If $c\geq 10^{99}$ or $n_\kappa\geq m_{\mathrm{I}}(c)$, then $2n_\kappa+2>2m_{\mathrm{I}}(c)\geq 0$, and a range with $L>U$ is certified.

Item 2. A type II representation has $c<10^{79.71}$, $4\leq n\leq m$ and $2m+1<Q_{\mathrm{II}}(c)$, so $m\leq m_{\mathrm{II}}(c)$; and if T is middle or low, moreover $c<10^{77.28}$ and $J\geq 1$, $J(4ae+1)<2m(m+1)$, so $2m(m+1)>4ae+1$ and $m\geq m_{\mathrm{lo}}$. If $c\geq 10^{79.71}$, or T is middle or low and $c\geq 10^{77.28}$, there is no type II representation, and the case is closed by Lemma 9.6, item 2, with $r=0$. Otherwise $L_{\mathrm{II}}\leq m'=2m+1\leq 2m_{\mathrm{II}}(c)+1$ (for T high, $m\geq 4$ gives $m'\geq 9$), and Lemma 9.6, item 2, closes the case if this range is certified. □

Corollary 9.17. *Let $T=\{a,b,c\}$ be a triple of the family BH of Definition 4.2 ($\deg T\geq 2$, T high, $c\geq 10^{30}$), with descent partner e and $f=e(\partial T)$; let $v=2af+1$, and let T_0, Δ_0, n_0 be as in the window of BH (Definition 4.2), $\kappa=(\sqrt{17}-1)/2$, $m_{\mathrm{I}}(c)$, $m_{\mathrm{II}}(c)$ as in Definition 4.1, and Ψ as in Proposition 7.28. Put:*

- *n_1 an integer with $n_1-1\leq\max\{\kappa\sqrt{c/b},\ 2\kappa\sqrt{a}\,c^{1/4}\}$, for instance $n_1=\lfloor x_-\rfloor+1$ for a rational lower bound x_- of this maximum (Lemma 9.7);*
- *$\ell_H=\infty$ if $c\geq 3.22\cdot 10^{50}$, or $c\geq 10^{50}$ and $a\geq 2$; otherwise ℓ_H the least odd integer $m'\geq 9$ with $\big(25(2m'+1)\big)^4>53^4a^2c$;*
- *$m_A=\min\{fb,\ n_0+\Delta_0\}$; m_B the least integer m with $129a^3m^6>c$; $m_\Psi=4$, or, if $c\geq 10^{75}$, m_Ψ an integer ≥ 5 with $\Psi(m_\Psi-1)\leq c$.*

*The **test for** $L=0$, where $L=2n(n+1)-(4af+1)\Delta$ for a type II representation (λ,m,n) of $z=\sqrt{cd+1}$, passes if there is no integer S with $b-2(2v+1)(a+f)<4av(v+S)<b$ for which $s_0=S+4af+2$ makes $s_0^2-S(S+2)=w^2$ the square of an integer $w\geq 0$, $\Delta=s_0-w$ satisfies $\Delta\geq 1$ and $\Delta\equiv S\pmod 2$, and $m=(S+\Delta)/2$, $n=(S-\Delta)/2$ satisfy $4\leq n$, $m\leq m_{\mathrm{II}}(c)$ and*

$$b\big(2n(n+1)+\Delta\big)-a\big(2m(m+1)-\Delta\big)-e\Delta=0.$$

The integers S in question are those of the open interval $\big((b-2(2v+1)(a+f))/(4av)-v,\ b/(4av)-v\big)$, at most $\lceil(2v+1)(a+f)/(2av)\rceil$ of them. Put $m_H=\max\{m_A,m_\Psi,m_B\}$ if the test for $L=0$ passes and $m_H=\max\{m_A,m_\Psi\}$ otherwise. Then:

1. *the case (I,λ) is closed if $c\geq 5.7\cdot 10^{45}$, or the range $[2n_1+2,\ 2m_{\mathrm{I}}(c)]$ is certified for it;*
2. *the case $(\mathrm{II},1)$ is closed if $c\geq 10^{79.71}$, or $\ell_H=\infty$, or the range $[\ell_H,\ 2m_{\mathrm{II}}(c)+1]$ is certified for it;*
3. *the case $(\mathrm{II},-1)$ is closed if $c\geq 10^{79.71}$, or the range $[\min\{\ell_H,2m_H+1\},\ 2m_{\mathrm{II}}(c)+1]$ is certified for it.*

Proof. Let d be an irregular extension of T and $\rho=(\lambda,m,n)$ a representation of $z=\sqrt{cd+1}$. By Proposition 8.3, $(T,\rho)\in\mathrm{BH}$. By Lemma 9.6, item 2, it suffices to show that every representation of a case lies in the range named for it (or that there is none).

In type I, the window of BH gives $c < 5.7 \cdot 10^{45}$, $n > \kappa\sqrt{c/b}$, $n > 2\kappa\sqrt{a}\,c^{1/4}$, $n < m$ and $2m < Q_{\mathrm{I}}(c)$. So $n \geq n_1$, $m \geq n_1 + 1$ and $m \leq m_{\mathrm{I}}(c)$, and $2n_1 + 2 \leq m' \leq 2m_{\mathrm{I}}(c)$. If $c \geq 5.7 \cdot 10^{45}$ there is no type I representation.

In type II, the window of BH gives $c < 10^{79.71}$, $m \leq m_{\mathrm{II}}(c)$ and $n \geq 4$, so $m' \geq 9$. If $H \neq 0$, it gives $n > 0.53\sqrt{a}\,c^{1/4} - 0.75$, so $m' \geq 2n+1 > 1.06\sqrt{a}\,c^{1/4} - 0.5$, that is, $25(2m'+1) > 53\sqrt{a}\,c^{1/4}$, or equivalently $\big(25(2m'+1)\big)^4 > 53^4 a^2 c$; and $H \neq 0$ is impossible when $c \geq 3.22 \cdot 10^{50}$, or $c \geq 10^{50}$ and $a \geq 2$. Hence $H \neq 0$ implies $m' \geq \ell_H$, and $\ell_H = \infty$ means that $H \neq 0$ is impossible.

Suppose $H = 0$. The window of BH gives $\lambda = -1$, $\Delta \geq 1$, $L \geq 0$ and $m \geq m_A$. If $c \geq 10^{75}$, then $c < 10^{90}$, so Proposition 7.28 gives $c < \Psi(m)$ with Ψ increasing on $[4, \infty)$; hence m is at least the least integer $m_0 \geq 4$ with $\Psi(m_0) > c$, and so $m \geq m_\Psi$; otherwise $m \geq 4 = m_\Psi$. If $L \geq 1$, Lemma 7.5 gives $c < 129a^3m^6/(\Delta L^2) \leq 129a^3m^6$, so $m \geq m_B$. If $L = 0$, Lemma 7.21 gives $4av(v+S) < b < 4av(v+S) + 2(2v+1)(a+f)$; and $2n(n+1) = (4af+1)\Delta$ with $n = (S-\Delta)/2$ becomes

$$\Delta^2 - 2(S + 4af + 2)\Delta + S(S+2) = 0,$$

so $s_0^2 - S(S+2) = (s_0 - \Delta)^2 = w^2$ with $w = |s_0 - \Delta|$. Since $\Delta \leq m/2 < S < s_0$, $w = s_0 - \Delta$ and $\Delta = s_0 - w$; $\Delta \equiv S \pmod 2$ as $S - \Delta = 2n$; $n \geq 4$ and $m \leq m_{\mathrm{II}}(c)$; and $H = 0$ with $\lambda = -1$ is the displayed identity. So such an S exists, and if the test for $L = 0$ passes, $L = 0$ is impossible and $m \geq m_B$. In all cases $H = 0$ implies $\lambda = -1$ and $m \geq m_H$, that is, $m' \geq 2m_H + 1$.

Therefore a representation of type II with $\lambda = 1$ has $H \neq 0$ and $\ell_H \leq m' \leq 2m_{\mathrm{II}}(c) + 1$, and one with $\lambda = -1$ has $\min\{\ell_H, 2m_H + 1\} \leq m' \leq 2m_{\mathrm{II}}(c) + 1$; there is none if $c \geq 10^{79.71}$, and none with $\lambda = 1$ if $\ell_H = \infty$. This proves the three items.

The choices of n_1 and m_Ψ are those used above. The least integer exceeding both $\kappa\sqrt{c/b}$ and $2\kappa\sqrt{a}\,c^{1/4}$ is $\lfloor x \rfloor + 1$ with $x = \max\{\kappa\sqrt{c/b}, 2\kappa\sqrt{a}c^{1/4}\}$, and an integer n_1 is at most $\lfloor x \rfloor + 1$ exactly when $n_1 - 1 \leq x$. As Ψ increases on $[4, \infty)$, an integer $m \geq 4$ satisfies $\Psi(m) \leq c$ exactly when it is less than the least integer $m_0 \geq 4$ with $\Psi(m_0) > c$; so the integers $4 \leq m_\Psi \leq m_0$ are 4 and the integers $m_\Psi \geq 5$ with $\Psi(m_\Psi - 1) \leq c$. Finally, dividing $b - 2(2v+1)(a+f) < 4av(v+S) < b$ by $4av > 0$ gives the open interval for S of the statement, of length $(2v+1)(a+f)/(2av)$, which contains at most $\lceil (2v+1)(a+f)/(2av) \rceil$ integers. □

9.5 The slot test for type I

For type I at slot three and beyond, the pinning of Lemma 8.9 replaces the certified range below a bound U by a test on finitely many candidates n: at slot three through a table of rational intervals (Lemma 9.18, Proposition 9.19), and from slot four on through the pinning test of Definition 8.7. Lemma 9.20 states the resulting type I test, used for BM, BL, B3 and B4.

Lemma 9.18. *Let $a < b < c < d$ be an irregular Diophantine quadruple with $c \geq 10^{30}$ and $b = x_2$, $c = x_3$, where (x_j) is the chain of a pair $\{a, e\}$ with an admitted sign σ; let D, η, ϑ, G be the chain constants of $(a, e; \sigma)$, $u = \sqrt{ae+1}$, and $\varphi_1 = 2\sqrt{D}/\eta$. Let z have a type I representation (λ, m, n) with second-order defect k. Put $\xi = \log\beta/\log\gamma$,*

$$\delta_\lambda = \frac{\sqrt{bc} + \lambda\sqrt{ab}}{\sqrt{ac} + \lambda\sqrt{ab}} = \sqrt{\frac{b}{a}} \cdot \frac{\sqrt{c} + \lambda\sqrt{a}}{\sqrt{c} + \lambda\sqrt{b}},$$

$$\mu_\lambda = -\frac{\log\delta_\lambda}{2\log\beta}, \qquad \omega_0 = \frac{\varphi_1\vartheta}{\xi} + \eta^{-1},$$

$$C_R = 1.0001\,\big(16D\vartheta^2 + 6G(1+G)\big) + 2, \qquad I_0 = n^2 + 2\lambda un - (4u^2 - 2)k,$$

and let n_κ be the least integer $n > \kappa\sqrt{c/b}$, $\kappa = (\sqrt{17} - 1)/2$. Then:

1. $I_0 \in \mathbb{Z}$, $4 \mid k$, and $-\frac{1}{2} < \mu_\lambda < 0$.

2. There are real numbers ε_Λ and R with

$$\frac{\lambda I_0}{n} = \omega_0 + \frac{\varphi_1 \vartheta \mu_\lambda}{n} + \varepsilon_\Lambda + R, \qquad 0 < \varepsilon_\Lambda < \frac{\vartheta}{12ac}, \qquad |R| \le \frac{C_R\, n}{\eta^4}.$$

3. Let $U \ge m$ be real. Then $n_\kappa \le n \le U$ and

$$\omega_0 - W_- \le \frac{\lambda I_0}{n} \le \omega_0 + W_+, \qquad W_- = \frac{\varphi_1 \vartheta |\mu_\lambda|}{n_\kappa} + \frac{C_R U}{\eta^4}, \qquad W_+ = \frac{C_R U}{\eta^4} + \frac{\vartheta}{12ac}.$$

4. If moreover $\omega_0 - W_- > 0$ and $\omega_0 + W_+ < n_\kappa$, then $k \ge 4$.

Proof. By Lemma 3.3, item 4, with $(p, q) = (a, e)$ and $\nu = 3$, $\{a, b, c\}$ is not an Euler triple and has degree 2. By Lemma 8.6, items 1, 4, 5 and 6: $2u - 1/u < \eta < 2u$ and $\sqrt{D} < u$, so $0 < \varphi_1 = 1 - \eta^{-2} < 1$; $u > 5 \cdot 10^4$, so $\vartheta \le u/8$ and $\eta > 10^5$; and $c > (4u^2 - 3)b$, so $a < b < c/(4u^2 - 3)$. By Lemma 2.23 and Lemma 2.14, $n > \kappa\sqrt{c/b} > \kappa\sqrt{4u^2 - 3} > 3u$ and $n < m \le 2n$. By Lemma 8.9, item 1 (with $\nu = 3$), $|k| \le n^2$ and $\Phi = \zeta - \lambda\vartheta\varphi_1 m$, with $\zeta = n^2 + \lambda n\eta - k\eta^2$, satisfies

$$|\Phi| \le \big(16D\vartheta^2 + 6G(1 + G)\big)\frac{n^2}{\eta^4}.$$

Finally $4 \mid k$ by item 1 of Lemma 2.22, and $I_0 \in \mathbb{Z}$ since $u, n, k \in \mathbb{Z}$.

Item 1. The integrality statements were shown above. By the recurrence of Definition 3.2, $b = x_2 = (4u^2 - 2)x_1 + 2(a + e) \ge 2(a + e) > 2a$, so $\sqrt{b/a} > \sqrt{2}$; and $b/c < 1/(4u^2 - 3) < 10^{-9}$. For $\lambda = 1$, $(\sqrt{c} + \sqrt{a})/(\sqrt{c} + \sqrt{b}) > 1/(1 + \sqrt{b/c}) > 1/1.001$, so $\delta_1 > \sqrt{2}/1.001 > 1$, and $\delta_1 < \sqrt{b/a}$ since $a < b$. For $\lambda = -1$, $(\sqrt{c} - \sqrt{a})/(\sqrt{c} - \sqrt{b}) > 1$, so $\delta_{-1} > \sqrt{2} > 1$, and $\delta_{-1} < \sqrt{b/a}/(1 - \sqrt{b/c}) < 1.001\sqrt{b/a}$. Moreover $\beta = s + \sqrt{ac} > 2\sqrt{ac} > 1.001\sqrt{b/a}$, because $2a\sqrt{c} > 1.001\sqrt{b}$ as $c > b$. Hence $1 < \delta_\lambda < \beta$, so $0 < \log\delta_\lambda < \log\beta$ and $-\frac{1}{2} < \mu_\lambda < 0$.

Item 2. By Lemma 8.8, item 3, $I_0 = \zeta + \lambda n\eta^{-1} - k\eta^{-2}$. The definition of ζ gives $k\eta^{-2} = (n^2 + \lambda n\eta - \zeta)\eta^{-4}$, so

$$I_0 = (1 + \eta^{-4})\,\zeta + \lambda n\,(\eta^{-1} - \eta^{-3}) - n^2\eta^{-4}.$$

By Lemma 8.8, item 4, $\lambda\zeta = \vartheta\varphi_1 m + \lambda\Phi$. Apply Lemma 5.7 to the type I representation: $m' = 2m$, $n' = 2n > 0$, $(x_1, z_1) = (y_2, z_2) = (1, \lambda)$, and its δ is $\frac{b}{a} \cdot \frac{\sqrt{ac} + \lambda a}{\sqrt{bc} + \lambda b} = \delta_\lambda$. So $\Lambda = 2m\log\beta - 2n\log\gamma + \log\delta_\lambda$ satisfies $0 < \Lambda < \frac{8}{3}ac\,\beta^{-4m}$, and, as $1/\xi = \log\gamma/\log\beta$,

$$\frac{m}{n} = \frac{1}{\xi} + \frac{\mu_\lambda}{n} + \frac{\Lambda}{2n\log\beta}.$$

Dividing the display for I_0 by n, multiplying by λ and substituting,

$$\frac{\lambda I_0}{n} = (1 + \eta^{-4})\Big(\varphi_1\vartheta\frac{m}{n} + \frac{\lambda\Phi}{n}\Big) + \eta^{-1} - \eta^{-3} - \lambda n\eta^{-4} = \omega_0 + \frac{\varphi_1\vartheta\mu_\lambda}{n} + \varepsilon_\Lambda + R,$$

where $\varepsilon_\Lambda = \varphi_1\vartheta\Lambda/(2n\log\beta)$ and

$$R = \eta^{-4}\varphi_1\vartheta\,\frac{m}{n} + (1 + \eta^{-4})\frac{\lambda\Phi}{n} - \eta^{-3} - \lambda n\eta^{-4}.$$

Since $\beta > 2\sqrt{ac} \geq 2\sqrt{2}$, $\log\beta > 1$, and since $m \geq n > 0$, $\beta^{4m} \geq \beta^4 > 16a^2c^2$; so $0 < \Lambda < 1/(6ac)$, and with $\varphi_1 < 1$ and $n \geq 1$ we get $0 < \varepsilon_\Lambda < \vartheta/(12ac)$. In R, first $\eta^{-4}\varphi_1\vartheta m/n \leq 2\vartheta\eta^{-4} \leq \frac{u}{4}\eta^{-4} < \frac{n}{12}\eta^{-4}$. Next, by the bound on Φ and $\eta^{-4} < 10^{-4}$, $(1+\eta^{-4})|\Phi|/n \leq 1.0001\big(16D\vartheta^2{+}6G(1{+}G)\big)n\eta^{-4}$. Then $\eta^{-3} = \eta\cdot\eta^{-4} < \frac{2}{3}n\eta^{-4}$ because $\eta < 2u < \frac{2}{3}n$, and $|\lambda n\eta^{-4}| = n\eta^{-4}$. Adding, $|R| \leq \big(1.0001(16D\vartheta^2 + 6G(1+G)) + \frac{1}{12} + \frac{2}{3} + 1\big)n\eta^{-4} \leq C_R n\eta^{-4}$.

Item 3. Since n is an integer with $n > \kappa\sqrt{c/b}$, $n \geq n_\kappa$; and $n < m \leq U$. As $\mu_\lambda < 0$ and $n \geq n_\kappa$, we have $-\varphi_1\vartheta|\mu_\lambda|/n_\kappa \leq \varphi_1\vartheta\mu_\lambda/n < 0$, and $|R| \leq C_R U/\eta^4$. Inserting these and $0 < \varepsilon_\Lambda < \vartheta/(12ac)$ into item 2 gives the two inequalities.

Item 4. Suppose $k \leq 0$. If $\lambda = 1$, then $I_0/n = n + 2u - (4u^2-2)k/n \geq n \geq n_\kappa > \omega_0 + W_+$, contradicting item 3. If $\lambda = -1$, then $-I_0/n = -n + 2u + (4u^2-2)k/n \leq 2u - n < 0 < \omega_0 - W_-$, since $n > 3u$; again this contradicts item 3. Hence $k \geq 1$, and $4 \mid k$ gives $k \geq 4$. □

Proposition 9.19. *Assume the setting of Lemma 9.18, except that the type I representation is not fixed. Let $U \geq 1$, let n_κ, ω_0, C_R, μ_λ be as there, let μ^* be a real number with $\mu^* \geq \max\{|\mu_1|, |\mu_{-1}|\}$ (for instance $\mu^* = \frac{1}{2}$), and let $\mathrm{Lo} < \mathrm{Hi}$ be rational numbers with*

$$0 < \mathrm{Lo} \leq \omega_0 - \frac{\varphi_1\vartheta\mu^*}{n_\kappa} - \frac{C_R U}{\eta^4}, \qquad \omega_0 + \frac{C_R U}{\eta^4} + \frac{\vartheta}{12ac} \leq \mathrm{Hi} < n_\kappa.$$

For $\lambda \in \{-1, 1\}$ and integers k, n put $I_0(\lambda, k, n) = n^2 + 2\lambda un - (4u^2-2)k$, and let $\mathcal{T} = \mathcal{T}(u; \mathrm{Lo}, \mathrm{Hi}, n_\kappa, U)$ be the finite set of triples (λ, k, n) with

$$k \geq 4, \quad 4 \mid k, \quad n_\kappa \leq n \leq U, \quad \mathrm{Lo} \leq \frac{\lambda I_0(\lambda, k, n)}{n} \leq \mathrm{Hi}.$$

1. *If z has a type I representation (λ, m, n) with $m \leq U$, then, with k its second-order defect, $(\lambda, k, n) \in \mathcal{T}$, and n passes the root test below U for (λ, k).*

2. *$\mathcal{T}$ depends only on $(u, \mathrm{Lo}, \mathrm{Hi}, n_\kappa, U)$, and $\mathcal{T}(u; \mathrm{Lo}, \mathrm{Hi}, n_\kappa, U) \subseteq \mathcal{T}(u; \mathrm{Lo}', \mathrm{Hi}', n_\kappa', U')$ whenever $\mathrm{Lo}' \leq \mathrm{Lo}$, $\mathrm{Hi} \leq \mathrm{Hi}'$, $n_\kappa' \leq n_\kappa$ and $U \leq U'$.*

3. *For fixed (λ, k) with $k \geq 0$, the function $v(n) = \lambda I_0(\lambda, k, n)/n$ is strictly increasing on $n > 0$ if $\lambda = 1$ and strictly decreasing if $\lambda = -1$, and $|v(n+1) - v(n)| \geq 1$. Hence the n with $(\lambda, k, n) \in \mathcal{T}$ are the integers of $[n_\kappa, U]$ lying between the positive roots, at $v = \mathrm{Lo}$ and at $v = \mathrm{Hi}$, of*

$$n^2 + \lambda(2u - v)\, n - (4u^2-2)k = 0,$$

and there are at most $\lfloor \mathrm{Hi} - \mathrm{Lo} \rfloor + 1$ of them. Moreover every $(\lambda, k, n) \in \mathcal{T}$ has $k \leq \big(U^2 + (2u + \mathrm{Hi})U\big)/(4u^2-2)$. Hence $\mathcal{T}$ has at most

$$2\Big\lfloor \frac{U^2 + (2u+\mathrm{Hi})U}{4(4u^2-2)} \Big\rfloor \big(\lfloor \mathrm{Hi} - \mathrm{Lo} \rfloor + 1\big)$$

elements.

4. *For every integer $p \geq 0$ and every $(\lambda, k, n) \in \mathcal{T}$, $\lfloor 2^p \mathrm{Lo} \rfloor \leq \lfloor 2^p \lambda I_0(\lambda, k, n)/n \rfloor \leq \lfloor 2^p \mathrm{Hi} \rfloor$.*

Proof. Item 1. Let (λ, m, n) be a type I representation with $m \leq U$ and defect k. Since $|\mu_\lambda| \leq \mu^*$, the number W_- of Lemma 9.18, item 3, is at most $\varphi_1\vartheta\mu^*/n_\kappa + C_R U/\eta^4$. Hence item 3 of Lemma 9.18 gives $n_\kappa \leq n \leq U$ and $\mathrm{Lo} \leq \omega_0 - W_- \leq \lambda I_0/n \leq \omega_0 + W_+ \leq \mathrm{Hi}$. Since $\omega_0 - W_- \geq \mathrm{Lo} > 0$ and $\omega_0 + W_+ \leq \mathrm{Hi} < n_\kappa$, item 4 of Lemma 9.18 gives $k \geq 4$, and $4 \mid k$ by its item 1. So $(\lambda, k, n) \in \mathcal{T}$. The defect equation reads $am^2 + \lambda sm = bn^2 + \lambda tn - kc$, hence

$(2am+\lambda s)^2 = s^2 + 4a(bn^2+\lambda tn - kc)$. With $q = |2am+\lambda s|$ we get $m = (q-\lambda s)/(2a)$ or $m = (-q-\lambda s)/(2a)$, and $n < m \le \min\{2n, U\}$ by Lemma 2.14; this is the root test below U.

Item 2 is immediate from the definition of $\mathcal{T}$, since the defining conditions involve only u, Lo, Hi, n_κ and U.

Item 3. For $\lambda = 1$, $v(n) = n+2u-(4u^2-2)k/n$, and for $\lambda = -1$, $v(n) = -n+2u+(4u^2-2)k/n$. Since $k \ge 0$, $n \mapsto n - (4u^2-2)k/n$ is strictly increasing on $n > 0$, and it increases by at least 1 from n to $n+1$. This gives the monotonicity and $|v(n+1) - v(n)| \ge 1$, hence the bound on the number of n. Multiplying $v(n) = v$ by $n > 0$ gives $n^2 + \lambda(2u-v)n - (4u^2-2)k = 0$. For $k > 0$ this quadratic has exactly one positive root, so $\{n > 0 :\ \mathrm{Lo} \le v(n) \le \mathrm{Hi}\}$ is the set of integers between the positive roots at $v = \mathrm{Lo}$ and at $v = \mathrm{Hi}$. For the bound on k: if $\lambda = 1$, $v(n) \ge \mathrm{Lo} > 0$ gives $(4u^2-2)k < n^2 + 2un \le U^2 + 2uU$; if $\lambda = -1$, $v(n) \le \mathrm{Hi}$ gives $(4u^2-2)k \le n^2 - 2un + \mathrm{Hi}\, n \le U^2 + \mathrm{Hi}\, U$. For the number of elements: for each of the two signs λ, the k that occur are multiples of 4 with $4 \le k \le (U^2 + (2u+\mathrm{Hi})U)/(4u^2-2)$, at most $\lfloor (U^2+(2u+\mathrm{Hi})U)/(4(4u^2-2)) \rfloor$ of them, and each (λ, k) has at most $\lfloor \mathrm{Hi} - \mathrm{Lo} \rfloor + 1$ values of n.

Item 4 holds because $x \mapsto \lfloor 2^p x \rfloor$ is non-decreasing. □

Lemma 9.20. *Let $T = \{a, b, c\}$ be a single-chain triple of degree $\nu - 1 \ge 2$, middle or low, with $c \ge 10^{30}$. Let σ be its chain sign, D, η, ϑ its chain constants, $u = \sqrt{ae+1}$, $\varphi_1 = 2\sqrt{D}/\eta^{\nu-2}$, and let n_κ and $m_{\mathrm{I}}(c)$ be as in Corollary 9.16. Put $n_F = \lceil \min\{u^{3/2}\vartheta^{-1/2}, u^{5/3}\} \rceil$ if $\nu \ge 5$ and $u \ge 10^3$, $n_F = 1$ otherwise, and $n_{\mathrm{lo}} = \max\{n_\kappa, n_F\}$. Let $\lambda \in \{-1, 1\}$.*

1. *The case (I, λ) is closed if the range $[2n_{\mathrm{lo}} + 2,\ 2m_{\mathrm{I}}(c)]$ is certified for it.*
2. *Let $U \ge 1$ be an integer, and suppose that*
   - *if $\nu = 3$: there are rational numbers $\mathrm{Lo} < \mathrm{Hi}$ satisfying the hypotheses of Proposition 9.19 for U such that no $(\lambda, k, n) \in \mathcal{T}(u; \mathrm{Lo}, \mathrm{Hi}, n_\kappa, U)$ passes the root test below U;*
   - *if $\nu \ge 4$: $u \ge 10^3$, $U \le u^2$, and the pinning test of some width $w \ge 2.1\,\vartheta\varphi_1$ for λ on $[n_{\mathrm{lo}}, U]$ passes.*

   If moreover the range $[2U+2,\ 2m_{\mathrm{I}}(c)]$ is certified for (I, λ), then the case (I, λ) is closed.
3. *(Range of the pinning test.) Let $w > 0$ be real, $U \ge n_{\mathrm{lo}}$ an integer, and $\rho_{\lambda,k}$, η as in Definition 8.7. If an integer $k \ge 4$ with $4 \mid k$ has a candidate n of width w for (λ, k) with $n_{\mathrm{lo}} \le n \le U$, then*
$$\rho_-(\rho_- + \lambda) < k < \rho_+(\rho_+ + \lambda),$$
   where $\rho_- = \max\{0, (n_{\mathrm{lo}} - w)/\eta\}$ and $\rho_+ = U/\eta$ if $\lambda = 1$, and $\rho_- = \max\{\frac{1}{2}, n_{\mathrm{lo}}/\eta\}$ and $\rho_+ = (U+w)/\eta$ if $\lambda = -1$. The candidates of width w for (λ, k) are the integers of the open interval $(\eta\rho_{\lambda,k},\ \eta\rho_{\lambda,k} + w)$ if $\lambda = 1$ and $(\eta\rho_{\lambda,k} - w,\ \eta\rho_{\lambda,k})$ if $\lambda = -1$; there are at most $\lceil w \rceil$ of them.

Proof. Let d be an irregular extension of T (Definition 2.3) and (λ, m, n) a type I representation of $z = \sqrt{cd+1}$ with the given sign. By Lemma 3.4, item (iii), $T = \{a, x_{\nu-1}, x_\nu\}$ on the chain (x_j) of $\{a, e\}$ with sign σ, so $b = x_{\nu-1}$ and $c = x_\nu$; in particular the pinning data of T are defined.

By Proposition 7.45, item 1, and the description of n_κ in Corollary 9.16, $n \ge n_\kappa$, $n < m$ and $m \le m_{\mathrm{I}}(c)$. If $\nu \ge 5$ and $u \ge 10^3$, then Proposition 8.10 gives $n \ge \min\{u^{3/2}\vartheta^{-1/2}, u^{5/3}\}$, so $n \ge n_F$. Hence $n \ge n_{\mathrm{lo}}$ and $2n_{\mathrm{lo}} + 2 \le m' = 2m \le 2m_{\mathrm{I}}(c)$. Item 1 follows from Lemma 9.6, item 2.

Item 2. It suffices to show $m \geq U+1$. Suppose $m \leq U$. If $\nu = 3$, then Proposition 9.19, item 1, gives that (λ, k, n), with k the second-order defect, lies in $\mathcal{T}(u; \mathrm{Lo}, \mathrm{Hi}, n_\kappa, U)$ and passes the root test below U, against the hypothesis. If $\nu \geq 4$, then $u \geq 10^3$, $n \geq n_{\mathrm{lo}}$ and $m \leq U \leq u^2$, so the consequence in Lemma 8.9 shows that the pinning test of width w for λ on $[n_{\mathrm{lo}}, U]$ does not pass (if $n_{\mathrm{lo}} > U$ there is nothing to prove, since $n < m \leq U$), against the hypothesis. Hence $m \geq U+1$, and $2U+2 \leq m' \leq 2m_{\mathrm{I}}(c)$. So every type I representation of sign λ of an irregular extension lies in the certified range $[2U+2, 2m_{\mathrm{I}}(c)]$, and Lemma 9.6, item 2, closes the case.

Item 3. By Definition 8.7, n is a candidate of width w for (λ, k) exactly when $0 < \lambda(n - \eta\rho_{\lambda,k}) < w$, that is, when n lies in the open interval of the statement; an open interval of length w contains at most $\lceil w \rceil$ integers. Put $\rho = \rho_{\lambda,k}$. From $2\rho+\lambda = \sqrt{4k+1}$ we get $4\rho^2+4\lambda\rho+1 = 4k+1$, that is, $k = g(\rho)$ with $g(x) = x(x+\lambda)$. Let n be a candidate with $n_{\mathrm{lo}} \leq n \leq U$. If $\lambda = 1$, then $n - w < \eta\rho < n$ and $\rho = (\sqrt{4k+1} - 1)/2 > 0$, so $\rho_- < \rho < \rho_+$; as g is strictly increasing on $[0, \infty)$ and $\rho_- \geq 0$, $g(\rho_-) < k < g(\rho_+)$. If $\lambda = -1$, then $n < \eta\rho < n + w$ and $\rho = (\sqrt{4k+1}+1)/2 \geq (\sqrt{17}+1)/2 > \frac{1}{2}$, so $\rho_- < \rho < \rho_+$; as g is strictly increasing on $[\frac{1}{2}, \infty)$ and $\rho_- \geq \frac{1}{2}$, again $g(\rho_-) < k < g(\rho_+)$. □

For $\nu = 3$ one may take $\mu^* = \frac{1}{2}$ (item 1 of Lemma 9.18), and for Lo and Hi the dyadic roundings, downwards and upwards, of $\omega_0 - \varphi_1\vartheta\mu^*/n_\kappa - C_R U/\eta^4$ and of $\omega_0 + C_R U/\eta^4 + \vartheta/(12ac)$, provided $\mathrm{Hi} < n_\kappa$; the first of these numbers is positive exactly when $U < (\omega_0 - \varphi_1\vartheta\mu^*/n_\kappa)\eta^4/C_R$. If this allows $U \geq m_{\mathrm{I}}(c)$, the range $[2U+2, 2m_{\mathrm{I}}(c)]$ is empty and no certificate is needed; otherwise U balances the size of $\mathcal{T}$ (Proposition 9.19, item 3) against one certificate. For $\nu \geq 4$ the pinning test runs over the k of item 3 of Lemma 9.20, each with at most $\lceil w \rceil$ candidates.

9.6 Slot three: the ellipse and the tests for BM and BL

For type II at slot three, the ellipse of BM is reduced, for $10^{65} \leq c < 10^{77.28}$, to a finite set by the integer $\iota = \Delta - (4a^2+1)J$, the extent of its arc $n \leq m \leq 2n$ and the second-order quotient (Lemma 9.21 to Lemma 9.27); this gives the ellipse floor of Lemma 9.28, used in the test for BM.

Lemma 9.21. *Assume the setting of Lemma 8.4 with T middle and $K = 0$, so that $\lambda = 1$ and (m, n) lies on the ellipse. Then the integer*

$$\iota = \Delta - (4a^2+1)J$$

is non-zero, and

$$e\iota = 2a\Delta(S+2) + J(3a + 2\sigma u), \qquad (2n+1)^2 = 8Ju^2 + 2(4a^2-1)J + 2\iota + 1,$$
$$\Delta < \frac{2m(m+1)\big(a(4ae+1)+E\big)}{(e-a)(4ae+1)}.$$

1. *If $\iota \geq 1$, then*

$$8a\big(a(4ae+1)+E\big)m(m+1)^2 + 2\max(0, 3a+2\sigma u)(e-a)\,m(m+1) > e(e-a)(4ae+1).$$

2. *If $\iota \leq -1$, then $\sigma = -1$, $2u > 3a$ and $2(2u-3a)\,m(m+1) > e(4ae+1)$.*

Proof. From $m = n + \Delta$ and $2n(n+1) = JP + \Delta$,

$$2m(m+1) = 2n(n+1) + 2\Delta(2n+1) + 2\Delta^2 = JP + \Delta + 2\Delta(2n+1) + 2\Delta^2.$$

Substituting into $JE = \Delta(e-a) - 2am(m+1)$ gives $J(E+aP) = \Delta\big(e - 2a - 2a(2n+1) - 2a\Delta\big) = \Delta\big(e - 2a(S+2)\big)$. Since $E + aP = (4a^2+1)e + 3a + 2\sigma u$, this reads $e\Delta - (4a^2+1)eJ = 2a\Delta(S+2) + J(3a + 2\sigma u)$, which is the identity for $e\iota$. Further $(2n+1)^2 = 4n(n+1) + 1 = 2JP + 2\Delta + 1$, and $2JP = 8Ju^2 - 4J$, $2\Delta = 2(4a^2+1)J + 2\iota$ give the identity for $(2n+1)^2$. Finally, from $\Delta(e-a) = JE + 2am(m+1)$ and $J < 2m(m+1)/(4ae+1)$ we get the bound for Δ.

Suppose $\iota = 0$. Then $2a\Delta(S+2) = -J(3a + 2\sigma u)$, whose left side is positive; so $\sigma = -1$ and $2a\Delta(S+2) = J(2u - 3a) < 2uJ$. As $\Delta = (4a^2+1)J \geq 5J$, this gives $5a(S+2) < u$. On the other hand $m \geq u$: otherwise $2m(m+1) \leq 2(u-1)u < 4u^2 - 3 = 4ae + 1$, against $J(4ae+1) < 2m(m+1)$ with $J \geq 1$. Thus $S \geq m \geq u > 5a(S+2)$, a contradiction.

Item 1. If $\iota \geq 1$, then $e \leq e\iota \leq 2a\Delta(S+2) + J\max(0, 3a + 2\sigma u)$. With $S + 2 \leq 2(m+1)$, the bound for Δ and $J < 2m(m+1)/(4ae+1)$,

$$e < \frac{8a\big(a(4ae+1) + E\big)m(m+1)^2}{(e-a)(4ae+1)} + \frac{2\max(0, 3a + 2\sigma u)\, m(m+1)}{4ae+1},$$

and multiplying by $(e-a)(4ae+1) > 0$ gives the claim.

Item 2. If $\iota \leq -1$, then $e \leq -e\iota = -2a\Delta(S+2) - J(3a + 2\sigma u) < -J(3a + 2\sigma u)$. Hence $3a + 2\sigma u < 0$, so $\sigma = -1$ and $2u > 3a$, and $e < J(2u - 3a) < 2m(m+1)(2u-3a)/(4ae+1)$.

□

Lemma 9.22. *Assume the setting of Lemma 8.4 with T middle and $K = 0$, and let ι be the integer of Lemma 9.21. If $\iota \geq 1$, then*

$$u^2 > 128\, a^4 J,$$

and in particular $e \geq 128a^3$.

Proof. Put $X = 2n + 1$, so that $S + 2 = X + \Delta + 1$. Since $\iota \geq 1$, the identity for $(2n+1)^2$ of Lemma 9.21 gives $X^2 > 8Ju^2$, so $X > 2\sqrt{2J}\, u$. Moreover $J(3a + 2\sigma u) \geq -2uJ$. Hence

$$e\iota > 2a\Delta\big(2\sqrt{2J}\, u + \Delta\big) - 2uJ.$$

As $e < u^2/a$ and $\iota = \Delta - (4a^2+1)J > 0$, multiplying by a gives

$$u^2\big(\Delta - (4a^2+1)J\big) > 2a^2\Delta\big(2\sqrt{2J}\, u + \Delta\big) - 2auJ,$$

that is

$$2a^2\Delta^2 - Y\Delta + J\big((4a^2+1)u^2 - 2au\big) < 0, \qquad Y = u^2 - 4\sqrt{2J}\, a^2 u. \tag{1}$$

Now $u > 2a$: since $J \geq 1$, $2m(m+1) > 4ae + 1 = 4u^2 - 3 > 2(u-1)u$, so $m \geq u$; and $4a(m+1) < e - a$ gives $e > 4au$, so $u^2 = ae + 1 > 4a^2 u$, that is, $u > 4a^2 \geq 2a$. Hence the constant term of the quadratic in Δ in (1) is positive. So (1) has a solution $\Delta > 0$ only if $Y > 0$ and $Y^2 > 8a^2 J\big((4a^2+1)u^2 - 2au\big)$. Since $(4a^2+1)u^2 - 2au \geq 4a^2u^2$ for $u \geq 2a$, this gives $Y > 4\sqrt{2J}\, a^2 u$, that is, $u^2 - 8\sqrt{2J}\, a^2 u > 0$, so $u > 8\sqrt{2J}\, a^2$ and $u^2 > 128a^4 J$. With $J \geq 1$ and $u^2 = ae + 1$ this is $ae + 1 > 128a^4$, hence $e > 128a^3 - 1/a$ and, e being an integer, $e \geq 128a^3$.

□

Lemma 9.23. *Assume the setting of Lemma 8.4 with T middle and $K = 0$, and let ι be the integer of Lemma 9.21. Put*

$$\Gamma = 4aE + 4P, \qquad k_2 = 4(e-a)P(u^2 - 2) - c, \qquad g_2 = \Gamma(E + aP) - 12aP.$$

Then $c \mid \Delta k_2 + J g_2$, $g_2 > 0$, and

$$k_2 = 32\sigma a^2 u e^2 - 32a^3 e^2 - 32ae^2 + 32\sigma aeu - 16a^2 e - 17e - a + 6\sigma u.$$

1. If $\sigma = 1$, then $k_2 > 0$, $\Delta k_2 + Jg_2 \geq c$, and

$$2m(m+1)\big[k_2\big(a(4ae+1)+E\big) + g_2(e-a)\big] > c(e-a)(4ae+1).$$

2. If $\sigma = -1$ and $\iota \geq 1$, then $k_2 < 0$, $\Delta k_2 + Jg_2 \leq -c$, and

$$2m(m+1)\,|k_2|\big(a(4ae+1)+E\big) > c(e-a)(4ae+1).$$

Proof. With Γ, V and $Z = \Delta[V - \Gamma(S+1)] - 6JP$ as in item 3(b) of Lemma 8.4 (here $\lambda = 1$), $c \mid Z$: the derivation there uses only $c = Pb + E$, Lemma 7.2, Lemma 3.3 and Lemma 2.6, not $e < a$.

On the ellipse, put $X = 2n+1$, so $S+1 = X + \Delta$ and $S+2 = X + \Delta + 1$, and $W = e\iota - 2a\Delta(S+2) - J(3a + 2\sigma u)$, which vanishes by Lemma 9.21. Eliminating $2a\Delta X$ between Z and W and using $\iota = \Delta - (4a^2+1)J$ gives the identity

$$2aZ = \Delta\big(2a(V+\Gamma) - \Gamma e\big) + Jg_2 + \Gamma W.$$

With $u^2 = ae + 1$ one has $Eb = 4ae^3 - 8a^2e^2 + 4a^3e + 4e^2 + 4a^2 - 16ae - 8 + 4\sigma u(a+e)$, and expanding $2a(V+\Gamma) - \Gamma e = 4aEb - 8aE(a+e) + P\big(20a + 4a^2(e-a) - 4e\big)$ gives

$$\begin{aligned} 2a(V+\Gamma) - \Gamma e &= 16a^2e^3 - 16a^3e^2 - 8ae^2 + 8a^2e + 8a - 8e \\ &= 8(e-a)(2ae+1)(ae-1) = 4(e-a)P(u^2-2) = k_2 + c. \end{aligned}$$

Thus $2aZ = \Delta(k_2 + c) + Jg_2$, and as $c \mid Z$, we get $c \mid \Delta k_2 + Jg_2$. The displayed expansion of k_2 follows from $c = (4u^2-1)(PE' + E) = 16a^2e^3 + 16a^3e^2 - 32\sigma a^2 ue^2 + 24ae(a+e) - 32\sigma aeu + 9(a+e) - 6\sigma u$. Finally $\Gamma > 0$, $E + aP > 0$ and $\Gamma(E+aP) \geq 4P \cdot aP > 12aP$ as $P \geq 14$, so $g_2 > 0$.

Item 1. Let $\sigma = 1$. Since $e \geq a+2$, $u^2 \geq (a+1)^2$, so $u \geq a+1$; then $au - a^2 - 1 \geq 0$, $32au - 16a^2 - 17 > 0$ and $6u > a$, and

$$k_2 = 32ae^2(au - a^2 - 1) + e(32au - 16a^2 - 17) + 6u - a > 0.$$

Hence $\Delta k_2 + Jg_2 > 0$, and being divisible by c it is at least c. Inserting the bounds for Δ and J recalled above gives $c < 2m(m+1)\big[k_2(a(4ae+1)+E)/((e-a)(4ae+1)) + g_2/(4ae+1)\big]$, which is the claim.

Item 2. Let $\sigma = -1$ and $\iota \geq 1$. Every term of the expansion of k_2 is then negative, and $|k_2| > 32a^2ue^2$. We claim $(4a^2+1)|k_2| > g_2$. Indeed $E = a + e - 2u < 2e$ and $P \leq 6ae$, so

$$\begin{aligned} g_2 &< 4aE^2 + 4(a^2+1)PE + 4aP^2 < 16ae^2 + 48a(a^2+1)e^2 + 144a^3e^2 \\ &\leq 256a^3e^2 \leq 128a^4ue^2 < (4a^2+1)|k_2|, \end{aligned}$$

using $u \geq 2$. As $\Delta \geq (4a^2+1)J + 1$, $-(\Delta k_2 + Jg_2) = \Delta|k_2| - Jg_2 > J\big((4a^2+1)|k_2| - g_2\big) > 0$. Hence $\Delta k_2 + Jg_2 \leq -c$, so $c \leq \Delta|k_2| - Jg_2 < \Delta|k_2|$, and the bound for Δ gives the claim.

□

Lemma 9.24. *Assume the setting of Lemma 9.23, with $\iota \geq 1$ and $a \geq 100$. Put*

$$\ell = \frac{\Delta k_2 + Jg_2}{c}, \qquad L = \sigma\ell, \qquad \varpi = 2a\Delta - L(u - \sigma a),$$
$$L_0 = L - \sigma a\varpi, \qquad R_c = 2a^2c - (\sigma au - a^2 + 1)k_2.$$

Then L, ϖ, L_0 are integers and:

1. $\sigma k_2\, L_0 = 2a^2 J g_2 - \sigma L R_c$ *and* $1 \le L_0 < 0.627\, L$*;*

2. $a \mid L - L_0$*, hence* $L \ge a + 1$*;*

3. $L < 2\sqrt{2J}\, a + 0.26$*;*

4. $\Delta < \dfrac{J\big[(4a^2+1)(u^2-1) + 3a^2 + 2au\big]}{u^2 - 1 - 6\sqrt{2J}\, a^2 u - 4a^2}$ *and* $u < \dfrac{2a\Delta}{L} + a$*, hence*

$$u < \frac{2a}{a+1} \cdot \frac{J\big[(4a^2+1)(u^2-1) + 3a^2 + 2au\big]}{u^2 - 1 - 6\sqrt{2J}\, a^2 u - 4a^2} + a.$$

5. *(Auxiliary bounds.)* $u > 8\sqrt{2J}\, a^2$, $a/u < 8.9{\cdot}10^{-4}$, $2n{+}1 < \sqrt{8J}\, u{+}1$ *and* $\Delta \le n < \sqrt{2J}\, u$*;* $k_2 = 32\sigma a^2 u e^2(1+\delta_k)$, $g_2 = 20a(4a^2+1)e^2(1+\delta_g)$ *and* $R_c = -16a^3e^2(1+\delta_R)$ *with* $|\delta_k| \le 1.001\, a/u$, $|\delta_g| \le 0.401\, a/u$, $|\delta_R| \le 0.0011$*; and*

$$L\Big(u - \sigma a + \frac{\sigma}{a}\Big) = 2a\Delta + \frac{\sigma L_0}{a}, \qquad L\Big(u + a + \frac{1}{a} + \frac{0.502}{u}\Big) \ge (4a^2+1)J\Big(2a - \frac{1.2516}{u}\Big).$$

Proof. By Lemma 9.23, $c \mid \Delta k_2 + J g_2$, with $\Delta k_2 + J g_2 \ge c$ if $\sigma = 1$ and $\Delta k_2 + J g_2 \le -c$ if $\sigma = -1$ (here $\iota \ge 1$); so ℓ is a non-zero integer of sign σ, $L \ge 1$, and ϖ and L_0 are integers. By Lemma 9.22, $u^2 > 128a^4 J$, that is $u > 8\sqrt{2J}\, a^2$. Since $u > 8\sqrt{2J}\, a^2 \ge 11.3a^2$ and $a \ge 100$, $a/u < 1/(11.3a) \le 8.9 \cdot 10^{-4}$, and $u/e = au/(u^2-1) < 1.0001\, a/u$. Put $X = 2n+1$. By Lemma 9.21, with $\Delta = (4a^2+1)J + \iota$, we have $X^2 = 8Ju^2 + 2\Delta - 4J + 1$, and $\Delta \le n$ by Lemma 2.14. From $X^2 \le 8Ju^2 + 2n + 1 = 8Ju^2 + X$ we get $X < \sqrt{8J}\, u + 1$, so $\Delta \le n < \sqrt{2J}\, u$.

Expanding with $u^2 = ae + 1$:

$$k_2 = 32\sigma a^2 u e^2(1+\delta_k), \qquad g_2 = 20a(4a^2+1)e^2(1+\delta_g), \qquad R_c = -16a^3e^2(1+\delta_R),$$

with $|\delta_k| \le 1.001\, a/u$, $|\delta_g| \le 0.401\, a/u$, $|\delta_R| \le 0.0011$. Hence $\sigma k_2 = |k_2|$, $g_2 > 0$, $R_c < 0$, and

$$\frac{2a^2 J g_2}{|k_2|} \le \frac{1.2516\, a(4a^2+1)J}{u}, \qquad \frac{|R_c|}{|k_2|} \le \frac{0.502\, a}{u}. \tag{1}$$

Identity of item 1. With $\Delta k_2 = \ell c - J g_2$ and $\ell = \sigma L$,

$$\begin{aligned} k_2(\sigma L - a\varpi) &= k_2 L(\sigma + au - \sigma a^2) - 2a^2(\sigma L c - J g_2) \\ &= L\big[(au - \sigma a^2 + \sigma)k_2 - 2\sigma a^2 c\big] + 2a^2 J g_2 = -\sigma L R_c + 2a^2 J g_2, \end{aligned}$$

and $\sigma(\sigma L - a\varpi) = L_0$. Thus $|k_2| L_0 = 2a^2 J g_2 - \sigma L R_c$, and by (1)

$$|L_0| \le \frac{1.2516\, a(4a^2+1)J}{u} + \frac{0.502\, aL}{u}. \tag{2}$$

Item 3. By definition $\varpi = \sigma(L - L_0)/a$, so $2a\Delta = L(u - \sigma a) + \sigma(L - L_0)/a$, that is,

$$L\Big(u - \sigma a + \frac{\sigma}{a}\Big) = 2a\Delta + \frac{\sigma L_0}{a}. \tag{3}$$

By (3), (2), $\Delta < \sqrt{2J}\, u$ and $(4a^2+1)J/u < 0.36\sqrt{J}$, we get $L(u - a - 1/a - 0.502/u) < 2\sqrt{2J}\, au + 0.46\sqrt{J}$, so $L < 2\sqrt{2J}\, a(1{+}1.01a/u) + 0.47\sqrt{J}/u < 2\sqrt{2J}\, a + 0.26$, using $2\sqrt{2J}\, a{\cdot}a/u <$

$2\sqrt{2J}\,a^2/(8\sqrt{2J}\,a^2) = 1/4$, so $2\sqrt{2J}\,a\cdot 1.01a/u < 0.2525$, and $\sqrt{J}/u < 1/(11.3a^2)$ (both from $u > 8\sqrt{2J}\,a^2$).

Rest of item 1. If $\sigma=-1$, then $|k_2|L_0 = 2a^2Jg_2 - L|R_c| \ge e^2a^3\big[40(4a^2+1)J(1-0.401a/u) - 16.02L\big] > 0$, as $L < 2.83a\sqrt{J}+0.26 < 2.49(4a^2+1)J$; if $\sigma=1$, then $|k_2|L_0 = 2a^2Jg_2+L|R_c| > 0$. So $L_0 \ge 1$. Next, by (3) and $\Delta \ge (4a^2+1)J+1$, $L(u+a+1/a) \ge 2a(4a^2+1)J - |L_0|/a$, and with (2),

$$L\Big(u+a+\frac{1}{a}+\frac{0.502}{u}\Big) \ge (4a^2+1)J\Big(2a-\frac{1.2516}{u}\Big). \tag{4}$$

Inserting (4) into (2): $1.2516\,a(4a^2+1)J/u \le 0.6258\,L(1+1.001a/u)/(1-0.63/(au)) < 0.6264\,L$, and $0.502\,aL/u < 0.0005\,L$; hence $L_0 < 0.627\,L$.

Item 2. By definition $L - L_0 = \sigma a\varpi$, so $a \mid L-L_0$; by item 1, $0 < L-L_0$, hence $L-L_0 \ge a$ and $L \ge a + L_0 \ge a+1$.

Item 5 collects the bounds proved above: the first paragraph, the expansions, relation (3) and inequality (4).

Item 4. From $e\iota = 2a\Delta(S+2) + J(3a+2\sigma u)$, with $S+2 = X+\Delta+1 < 3\sqrt{2J}\,u+2$ and $J(3a+2\sigma u) \le J(3a+2u)$, multiplying by a and using $ae = u^2-1$ and $\iota = \Delta - (4a^2+1)J$:

$$(u^2-1)\big(\Delta-(4a^2+1)J\big) < 2a^2\Delta(3\sqrt{2J}\,u+2) + aJ(3a+2u),$$

which is the bound for Δ; its denominator is positive because $u > 8\sqrt{2J}\,a^2$. By (3), $u = 2a\Delta/L + \sigma a - \sigma/a + \sigma L_0/(aL)$, and since $0 < L_0 < L$ this is less than $2a\Delta/L + a$; with $L \ge a+1$ the last displayed bound follows.

□

Lemma 9.25. *Assume the setting of Lemma 9.24 ($\iota \ge 1$, $a \ge 100$, the integers $L \ge a+1$, ϖ, L_0). Put $q = (L-L_0)/a$ and*

$$B_L = 4\sqrt{2J}\,a^2 + aL + \frac{2a(4a^2+1)J}{L}, \qquad M_L = \frac{2a(4a^2+1)J}{L} - aL,$$
$$U(L) = \frac{B_L + \sqrt{B_L^2 + 4\sigma a M_L}}{2}, \qquad \hat{L}_0 = \frac{5a(4a^2+1)J + 2\sigma aL}{4u}.$$

Then $B_L^2 + 4\sigma a M_L > 0$ and

1. *$|u - U(L)| < 0.38$;*
2. *$|L_0 - \hat{L}_0| < 0.056$;*
3. *q is a positive integer, $\varpi = \sigma q$, $L(u-\sigma a) + \sigma q = 2a\Delta$, and $q \equiv -\sigma L_0 u \pmod a$.*

Proof. Item 3. By definition $L_0 = L - \sigma a\varpi$, so $\varpi = \sigma(L-L_0)/a = \sigma q$ with q an integer, positive by item 2 of Lemma 9.24; then $2a\Delta = L(u-\sigma a) + \varpi = L(u-\sigma a) + \sigma q$. Reducing modulo a and writing $L = L_0 + aq$ gives $L_0 u + \sigma q \equiv 0 \pmod a$, that is, $q \equiv -\sigma L_0 u \pmod a$.

Put $w_L = \sigma a - \sigma/a + \sigma L_0/(aL)$, so that the relation $L(u - \sigma a + \sigma/a) = 2a\Delta + \sigma L_0/a$ of item 5 of Lemma 9.24 reads $2a^2\Delta = aL(u-w_L)$, and put $X = 2n+1$ and $\delta_X = X - \sqrt{8J}\,u$. By Lemma 9.21, $e\iota = 2a\Delta(S+2) + J(3a+2\sigma u)$ with $\iota = \Delta - (4a^2+1)J$, that is $J\,(E+aP) = \Delta\big(e - 2a(S+2)\big)$, with $S+2 = X+\Delta+1$ and $a(E+aP) = (4a^2+1)u^2 + 2\sigma au - a^2 - 1$. Multiplying by a/Δ:

$$2a^2X = u^2 - 1 - 2a^2 - 2a^2\Delta - \frac{2aJ}{L}\cdot\frac{(4a^2+1)u^2 + 2\sigma au - a^2 - 1}{u - w_L}.$$

Division gives $(4a^2+1)u^2+2\sigma au-a^2-1=(u-w_L)\big[(4a^2+1)(u+w_L)+2\sigma a\big]+r_w$ with $r_w=(4a^2+1)w_L^2+2\sigma a w_L-a^2-1$. Substituting $X=\sqrt{8J}\,u+\delta_X$ and $2a^2\Delta=aL(u-w_L)$, and collecting, with $w_L=\sigma a+w_1$:

$$u^2-B_Lu-\sigma aM_L=w_1M_L+1+2a^2+2a^2\delta_X+\frac{4\sigma a^2J}{L}+\frac{2aJ}{L}\cdot\frac{r_w}{u-w_L}=:C_L. \tag{1}$$

For the right side of (1), first $|w_1|\le 1/a+L_0/(aL)<1.627/a$. By item 5 of Lemma 9.24, $2a(4a^2+1)J/L\le(u+a+1/a+0.502/u)\,2a/(2a-1.2516/u)<1.0009\,u$; and $aL<2\sqrt{2J}\,a^2+0.26a<0.2503\,u$. Hence $|M_L|<1.0009\,u$, $|w_1M_L|<0.0163\,u$, $4a^2J/L<0.5005\,u/a\le 0.0051\,u$ and $2aJ/L<1.0009\,u/(4a^2+1)$. Next $w_L^2\le a^2+3.26$ and $|2\sigma a w_L|\le 2a^2+3.26$, so $|r_w|\le 1.0005(4a^2+1)a^2$ and $(2aJ/L)|r_w|/(u-w_L)<1.0024\,a^2<0.0887\,u$, using $u>11.31a^2\sqrt{J}$. Further $1+2a^2<0.1769\,u$. Finally $X^2-8Ju^2=2\Delta-4J+1\in(0,2\Delta+1]$ and $\Delta<\sqrt{2J}\,u$ give $0<\delta_X<(2\sqrt{2J}\,u+1)/(2\sqrt{8J}\,u)<0.50001$, so $2a^2\delta_X<0.0885\,u$. Altogether $|C_L|<0.376\,u$.

Item 1. Since $B_L=M_L+4\sqrt{2J}\,a^2+2aL>|M_L|$ and $B_L\ge 4\sqrt{2}\,a^2>4a$, $B_L^2+4\sigma aM_L>B_L(B_L-4a)>0$. The roots of $x^2-B_Lx-\sigma aM_L$ are $U(L)$ and $U'=-\sigma aM_L/U(L)$, with $|U'|\le 2a|M_L|/B_L<2a$ as $U(L)\ge B_L/2$. By (1), $|u-U(L)|\,|u-U'|=|C_L|<0.376\,u$, and $|u-U'|>u-2a\ge u(1-2a/u)$, so $|u-U(L)|<0.376/(1-2a/u)<0.38$.

Item 2. By items 1 and 5 of Lemma 9.24, $|k_2|=32a^2ue^2(1+\delta_k')$ with $|\delta_k'|\le 1.001a/u$ and

$$L_0=\frac{1}{1+\delta_k'}\Big[\frac{5a(4a^2+1)J}{4u}(1+\delta_g)+\frac{\sigma aL}{2u}(1+\delta_R)\Big],$$
$$\text{so}\qquad |L_0-\hat L_0|\le\frac{W_0|\delta_k'|+W_1|\delta_g|+W_2|\delta_R|}{1-|\delta_k'|},$$

where $W_0=|\hat L_0|$, $W_1=5a(4a^2+1)J/(4u)$, $W_2=aL/(2u)$. Using $u>11.31a^2\sqrt{J}$: $W_1<0.4421\,a\sqrt{J}$, $W_2<0.126$, $W_0<0.4421\,a\sqrt{J}+0.126$, and $a/u<1/(11.31a\sqrt{J})$; hence $W_0|\delta_k'|<0.0392+0.0002$, $W_1|\delta_g|<0.0157$, $W_2|\delta_R|<0.0002$, and $|L_0-\hat L_0|<0.0553/(1-10^{-3})<0.056$. □

Lemma 9.26. *Let $a\ge 100$, $J\ge 1$, $\sigma\in\{-1,1\}$ and put $\Lambda_J=2\sqrt{2J}\,a+0.26$. With B_L, M_L and $U(L)$ as in Lemma 9.25, viewed as functions of the real variable $L>0$, put $L_{\min}=\sqrt{2(4a^2+1)J}$ and*

$$\hat L_0^\circ(L)=\frac{5a(4a^2+1)J+2\sigma aL}{4U(L)},\qquad g(L)=L-\hat L_0^\circ(L).$$

Then $g'(L)\ge 0.36$ for all $L>0$; U is strictly decreasing on $(0,L_{\min}/1.004]$; and $|U(L)-U(L')|\le 10^{-4}\,U(L)$ for $L,L'\in[L_{\min}/1.004,\Lambda_J+2]$.

Proof. In this proof write

$$A_1=\frac{2a(4a^2+1)J}{L}=\frac{aL_{\min}^2}{L},\qquad B=B_L=4\sqrt{2J}\,a^2+aL+A_1,\qquad M_0=M_L=A_1-aL,$$

and $W=\sqrt{B^2/4+\sigma aM_0}$, so that $U=U(L)=B/2+W$; write $\hat L_0^\circ=\hat L_0^\circ(L)$, and let a prime denote the derivative with respect to L.

For every $L>0$, by the inequality of the arithmetic and geometric means, $aL+A_1\ge 2aL_{\min}>5.65a^2\sqrt{J}$, so $B>11.3a^2\sqrt{J}\ge 1130a$; also $B\ge A_1$ and $B\ge aL$, hence $|M_0|\le B$ and $4a|M_0|/B^2<0.0036$. Therefore $W\in[(B/2)(1-0.0036),(B/2)(1+0.0018)]$, $U\in[B(1-0.0018),B(1+0.0009)]$ and $A_1/U\le 1.0018$.

The derivatives are $A_1' = -A_1/L$, $B' = a - A_1/L$, $M_0' = -A_1/L - a$, and

$$U' = B'\Big(\frac{1}{2} + \frac{B}{4W}\Big) + \frac{\sigma a M_0'}{2W}, \qquad \frac{1}{2} + \frac{B}{4W} \in [0.999, 1.002], \qquad \Big|\frac{\sigma a M_0'}{2W}\Big| \le \frac{a(A_1/L + a)}{0.996\,B}.$$

Since $4U\hat{L}_0^\circ = 5a(4a^2+1)J + 2\sigma aL$ with $5a(4a^2+1)J = \frac{5}{2}A_1 L$ independent of L, we have $g' = 1 - \sigma a/(2U) + \hat{L}_0^\circ U'/U$, $\hat{L}_0^\circ/U = [\frac{5}{8}A_1 + \sigma a/2]L/U^2$, and $\hat{L}_0^\circ \le (0.625 \cdot 1.0018 + 0.0005)L < 0.6267L$; also $a/(2U) < 0.0005$.

Slope bound. If $U' \ge 0$ and $\hat{L}_0^\circ \ge 0$, then $g' \ge 1 - a/(2U) > 0.999$. If $U' \ge 0$ and $\hat{L}_0^\circ < 0$, then $\sigma = -1$ and $L > \frac{5}{2}(4a^2+1)J > L_{\min}$, so $0 < B' < a$, $A_1/L < a$, $U' \le 1.002a + 2a^2/(0.996B) < 1.004a$ and $|\hat{L}_0^\circ| \le aL/(2U) < 0.51$; hence $|\hat{L}_0^\circ U'/U| < 0.52a/U < 0.0005$ and $g' > 0.999$. Let $U' < 0$. If $B' \le 0$, that is, $L \le L_{\min}$, then $aL/B \le aL/(aL + aL_{\min}^2/L) \le 1/2$, $|U'| \le 1.002\,A_1/L + a(A_1/L + a)/(0.996B)$ and

$$\frac{\hat{L}_0^\circ}{U} \cdot \frac{A_1}{L} = \frac{(\frac{5}{8}A_1 + \frac{\sigma a}{2})A_1}{U^2} \le \frac{5}{8}(1.0018)^2 + \frac{a}{2U} \cdot 1.0018 < 0.6278;$$

with $\hat{L}_0^\circ a/U \le 0.6267\,aL/(0.998B) < 0.314$ this gives $\hat{L}_0^\circ|U'|/U \le 1.002 \cdot 0.6278 + 0.6278 \cdot a/(0.996B) + 0.314 \cdot a/(0.996B) < 0.6299$. If $B' > 0$, that is, $L > L_{\min}$, then $A_1/L < a$, so $U' \ge -2a^2/(0.996B)$ and, as $U \ge 0.998aL$, $\hat{L}_0^\circ|U'|/U \le (0.6267L/U) \cdot 2.01a^2/B < 0.63 \cdot 2.01a/B < 0.0012$. In all cases $g' \ge 1 - 0.0005 - 0.6299 > 0.369$.

Monotonicity. For $L \le L_{\min}/1.004$, $A_1/L = aL_{\min}^2/L^2 \ge 1.008a$, so $B' \le -(A_1/L)(1 - 1/1.008) < -0.0079\,A_1/L$ and $U' \le -0.999 \cdot 0.0079\,A_1/L + a(A_1/L + a)/(0.996B)$; as $a/B < 1/1130$ and $a \le A_1/(1.008L)$, the second term is at most $(2.01/(0.996{\cdot}1130))\,A_1/L < 0.0019\,A_1/L$, so $U' < 0$.

Variation. Let $L \in [L_{\min}/1.004, \Lambda_J + 2]$. Then $|L - L_{\min}| \le \max(0.004L_{\min}, \Lambda_J + 2 - L_{\min})$ and $B(L) - B(L_{\min}) = a(L - L_{\min})^2/L$. For $L \le L_{\min}$ this is at most $a(0.004L_{\min})^2 \cdot 1.004/L_{\min} < 1.7 \cdot 10^{-5} aL_{\min} < 4.3 \cdot 10^{-6} B$; for $L > L_{\min}$ it is at most $a(2.26)^2/L_{\min} < 1.81/\sqrt{J} < 1.6 \cdot 10^{-5} B$, since $\Lambda_J + 2 - L_{\min} < 2.26$, $L_{\min} > 2\sqrt{2J}a$ and $B > 11.3a^2\sqrt{J} \ge 1.13 \cdot 10^5\sqrt{J}$. Moreover $U - B = W - B/2$ has derivative $U' - B'$ bounded by $0.002|B'| + a(A_1/L + a)/(0.996B) < 0.0018a + 0.002|B'|$, and $|B'| \le \max(0.0081a,\ 4.52a/L_{\min}) \le 0.016a$ on this interval; over a length below $0.004L_{\min} + 2.26$ this changes $U - B$ by less than $0.00184a(0.004L_{\min} + 2.26) < 6 \cdot 10^{-6}U$. Together, with $U \ge 0.998B$, $|U(L) - U(L')| \le 2.3 \cdot 10^{-5}U(L) \le 10^{-4}U(L)$. □

Lemma 9.27. *Assume the setting of Lemma 8.4 with T middle, so that $b = (4u^2-1)E' + E$ and $c = Pb + E$ with $E' = a + e - 2\sigma u$, $E = a + e + 2\sigma u$, $P = 4u^2 - 2$. Suppose $10^{65} \le c < 10^{77.28}$, and let z have a type II representation (λ, m, n) with $m' < 1.03 \cdot 10^{13}$ and $K = 0$. Put $m_c = 5.15 \cdot 10^{12}$. Then the integer ι of Lemma 9.21 satisfies $\iota \ge 1$, $100 \le a < 5.7 \cdot 10^5$, and the integer J, the integers L, L_0 of Lemma 9.24 and $q = (L - L_0)/a$ of Lemma 9.25 satisfy:*

1. *$u^6 > a \cdot 10^{65}/32$, $u < 3.65 \cdot 10^{12}$, $1 \le J < u^2/(128a^4)$, $J(4u^2 - 3) < 2m_c(m_c + 1)$ and $J(512a^4 J - 3) < 2m_c(m_c + 1)$;*
2. *$a + 1 \le L < \Lambda_J := 2\sqrt{2J}\,a + 0.26$, $1 \le L_0 < 0.627L$, and q is an integer with $1 \le q < 2\sqrt{2J}$;*
3. *$|u - U(L)| < 0.38$ and $|L_0 - \hat{L}_0| < 0.056$, with U and $\hat{L}_0$ as in Lemma 9.25;*
4. *$u^2 \equiv 1 \pmod a$, $q \equiv -\sigma L_0 u \pmod a$ and $2a\Delta = L(u - \sigma a) + \sigma q$;*
5. *with $\hat{L}_0^\circ$ as in Lemma 9.26, $|L - \hat{L}_0^\circ(L) - aq| < 0.0567$; consequently $|L - L_2| < \big(|L_2 - \hat{L}_0^\circ(L_2) - aq| + 0.0567\big)/0.36$ for every real L_2 with $0 < L_2 \le \Lambda_J + 2$.*

In particular the triple $\{a,b,c\}$ belongs to the finite set $\mathcal{S}_{\rm ell}$ of triples obtained as follows: for each a with $100 \le a < 5.7\cdot 10^5$, each σ, each $J \ge 1$ with $J(4u_a^2-3) < 2m_c(m_c+1)$ and $J(512a^4J-3) < 2m_c(m_c+1)$, where u_a is the least integer with $u_a^6 > a\cdot 10^{65}/32$, and each q with $1 \le q < 2\sqrt{2J}$, take every integer L allowed by item 5 (for any real $L_2 \in (0, \Lambda_J+2]$, chosen for each (a,σ,J,q); an L_2 with $|L_2 - \hat L_0^\circ(L_2) - aq| \le 0.005$ allows at most one integer L) and by item 2, every integer u with $|u-U(L)| < 0.38$, and $L_0 = L - aq$; keep those with $u \ge u_a$, $u^2 > 128a^4J$, $J(4u^2-3) < 2m_c(m_c+1)$ and the conditions of items 2 to 4 (in item 4 the last relation means that $2a$ divides $L(u-\sigma a)+\sigma q$), put $e = (u^2-1)/a$, E', E, b, c as above, and keep the triple if $E, E' > 0$, $a < e < b$ and $10^{65} \le c < 10^{77.28}$.

Proof. By Lemma 8.4, item 4, and Lemma 9.21, $\lambda = 1$, $\iota \ne 0$, $J \ge 1$, $J(4ae+1) < 2m(m+1)$ and $4a(m+1) < e - a$. Since $m' = 2m+1 < 1.03\cdot 10^{13}$, $m < m_c$. If $m \le u-1$, then $2m(m+1) \le 2u(u-1) < 4u^2 - 3 = 4ae+1$, which is impossible; so $m \ge u$, hence $e > 4au$ and $u^2 = ae+1 > 4a^2u$, that is, $u > 4a^2$; with $m < m_c$ this gives $u < m_c$ and $a < 1.14\cdot 10^6$. Since $a + 2u < e$, we have $E, E' < 2e$, and $c = Pb + E = (4u^2-1)(PE'+E)$ (as $P+1 = 4u^2-1$), so

$$c < 2e(4u^2-1)^2 < 32u^4 e < \frac{32u^6}{a}, \qquad \text{so} \qquad u^6 > \frac{ac}{32}. \tag{1}$$

Suppose $\iota \le -1$. By item 2 of Lemma 9.21, $\sigma = -1$, $2u > 3a$ and $m(m+1) > e(4ae+1)/(2(2u-3a)) > ae^2/u = (u^2-1)^2/(au) > u^3/(2a)$. By (1), $u^3/(2a) > \sqrt{c/(128a)} \ge \sqrt{10^{65}/(1.46\cdot 10^8)} > 2.6\cdot 10^{28}$, so $m > 1.6\cdot 10^{14} > m_c$, a contradiction.

Suppose $\iota \ge 1$ and $a < 100$. Then $u > (a\cdot 10^{65}/32)^{1/6} > 3.8\cdot 10^{10}$ by (1), so the relative corrections a/u, $1/(au)$, u/e in the expansion of k_2 in Lemma 9.23, in g_2 and in c are below 10^{-8}; hence $|k_2| \le 1.001\cdot 32a^2ue^2$, $a(4ae+1)+E \le (4a^2+2.001)e$, $0 < g_2(e-a) \le 10^{-6}|k_2|(a(4ae+1)+E)$, $c \ge (1-10^{-8})\,16u^4e$ and $(e-a)(4ae+1) \ge (1-10^{-8})\,4ae^2$. Both items of Lemma 9.23 then give $m(m+1) > u^3/(1.002\,a(4a^2+2))$, so $u^3 < 1.002\,a(4a^2+2)m_c^2$. With (1), $5.59\cdot 10^{31}\sqrt{a} < u^3 < 2.66\cdot 10^{25}a(4a^2+2)$, that is, $\sqrt{a}\,(4a^2+2) > 2.1\cdot 10^6$, which is false for $a < 100$ (the left side is below $3.91\cdot 10^5$).

So $\iota \ge 1$ and $a \ge 100$. By Lemma 9.22, $u^2 > 128a^4J$, so $u > 8\sqrt{2}\,a^2\sqrt{J}$; and $4u^2 - 3 \le J(4ae+1) < 2m(m+1) < 2m_c(m_c+1)$ gives $u < m_c/\sqrt{2} + 1 < 3.65\cdot 10^{12}$, so $a^2 < 3.65\cdot 10^{12}/(8\sqrt{2}) < 3.23\cdot 10^{11}$ and $a < 5.7\cdot 10^5$. Moreover $J(512a^4J-3) < J(4u^2-3) < 2m_c(m_c+1)$, and $u^6 > ac/32 \ge a\cdot 10^{65}/32$ by (1). This is item 1. By Lemma 9.24, L and L_0 are integers with $1 \le L_0 < 0.627L$, $L \ge a+1$ and $L < \Lambda_J$; by Lemma 9.25, $q = (L-L_0)/a$ is a positive integer, $|u-U(L)| < 0.38$, $|L_0 - \hat L_0| < 0.056$, $q \equiv -\sigma L_0 u \pmod a$ and $2a\Delta = L(u-\sigma a)+\sigma q$. Since $aq = L - L_0 \le L - 1 < 2\sqrt{2J}\,a - 0.74$, $q < 2\sqrt{2J}$. And $u^2 = ae+1$. This gives items 2 to 4.

Item 5. The numbers $\hat L_0$ of Lemma 9.25 and $\hat L_0^\circ(L)$ of Lemma 9.26 have the same positive numerator $5a(4a^2+1)J + 2\sigma aL$ and the denominators $4u$ and $4U(L)$, so

$$|\hat L_0 - \hat L_0^\circ(L)| = \hat L_0\,\frac{|U(L)-u|}{U(L)} < \frac{0.38\,\hat L_0}{u - 0.38}.$$

Here $\hat L_0 < L_0 + 0.056 < 0.627\Lambda_J + 0.056 < 1.774\,a\sqrt{J} + 0.22$ and $u - 0.38 > 11.31\,a^2\sqrt{J} - 0.38$; for $a \ge 100$ and $J \ge 1$ this gives $|\hat L_0 - \hat L_0^\circ(L)| < 0.0598/a \le 0.0006$.

Hence $|L - \hat L_0^\circ(L) - aq| = |L_0 - \hat L_0^\circ(L)| \le |L_0 - \hat L_0| + |\hat L_0 - \hat L_0^\circ(L)| < 0.056 + 0.0006 < 0.0567$. By Lemma 9.26, $g(x) = x - \hat L_0^\circ(x)$ satisfies $g' \ge 0.36$ for $x > 0$; we use this only on $(0, \Lambda_J+2]$, which contains L. So for every real $L_2 \in (0, \Lambda_J+2]$ the mean value theorem on the interval between L and L_2 gives $0.36|L - L_2| \le |g(L) - g(L_2)| \le |g(L) - aq| + |g(L_2) - aq|$, which is the consequence.

The description of $\mathcal{S}_{\mathrm{ell}}$ lists necessary conditions only: a, σ, J, q lie in the stated ranges by items 1 and 2 (as $u \ge u_a$ gives $J(4u_a^2-3) \le J(4u^2-3)$), L is allowed by items 2 and 5, u by item 3, $L_0 = L - aq$ by the definition of q, the conditions kept hold by items 1 to 4, and e, E', E, b, c are those of the triple. The set is finite because every range is finite. Item 5 holds for every real $L_2 \in (0, \Lambda_J + 2]$, so the description is valid for any choice of L_2; if $|L_2 - \hat{L}_0^\circ(L_2) - aq| \le 0.005$, item 5 gives $|L - L_2| < (0.005 + 0.0567)/0.36 < \frac{1}{2}$, so at most one integer L is allowed. □

Lemma 9.28. *Let $T = \{a, b, c\}$ be a triple of the family BM of Definition 4.2, with descent partner e, chain sign σ, chain constants E, P, $u = \sqrt{ae+1}$, and k_2, g_2 as in Lemma 9.23. Let $m_{\mathrm{II}}(c)$ be as in Definition 4.1, m_{lo} the least integer $m \ge 0$ with $2m(m+1) > 4ae+1$, $m_{\mathrm{top}} = \min\{m_{\mathrm{II}}(c), \lfloor (e-a-1)/(4a) \rfloor - 1\}$ and*

$$J_{\max} = \min\Big\{ \Big\lfloor \frac{2m_{\mathrm{top}}(m_{\mathrm{top}}+1)-1}{4ae+1} \Big\rfloor,\ \Big\lfloor \frac{u^2-1}{128a^4} \Big\rfloor \Big\}.$$

Put $\ell_+ = \infty$ if $J_{\max} < 1$, or if $a \ge 100$ and there is a rational $s_* \ge \sqrt{2J_{\max}}$ (for instance $s_* = \lceil \sqrt{2J_{\max}} \rceil$) with $D_{s_*} = u^2 - 1 - 6s_*a^2u - 4a^2 > 0$ and $(u-a)(a+1)D_{s_*} \ge 2aJ_{\max}\big[(4a^2+1)(u^2-1) + 3a^2 + 2au\big]$; otherwise let ℓ_+ be the least integer $m \ge m_{\mathrm{lo}}$ with

$$8a\big(a(4ae+1)+E\big)m(m+1)^2 + 2\max(0, 3a+2\sigma u)(e-a)\,m(m+1) > e(e-a)(4ae+1)$$

and with $2m(m+1)\big[k_2\big(a(4ae+1)+E\big) + g_2(e-a)\big] > c(e-a)(4ae+1)$ if $\sigma = 1$, respectively $2m(m+1)|k_2|\big(a(4ae+1)+E\big) > c(e-a)(4ae+1)$ if $\sigma = -1$. Put $\ell_- = \infty$ unless $\sigma = -1$ and $2u > 3a$, and then let ℓ_- be the least integer $m \ge m_{\mathrm{lo}}$ with $2(2u-3a)m(m+1) > e(4ae+1)$. Put $\ell_0 = \infty$ if $e \le a+2$, or if $10^{65} \le c < 10^{77.28}$ and T does not belong to the set $\mathcal{S}_{\mathrm{ell}}$ of Lemma 9.27 (or to some finite set containing it), or if $\min\{\ell_+, \ell_-\} > m_{\mathrm{top}}$; otherwise $\ell_0 = \min\{\ell_+, \ell_-\}$.

Then for every irregular extension d of T, every type II representation (λ, m, n) of $z = \sqrt{cd+1}$ with $K = 0$ has $\lambda = 1$ and $\ell_0 \le m \le m_{\mathrm{top}}$; in particular there is none if $\ell_0 = \infty$.

Proof. Let d be an irregular extension of T and (λ, m, n) a type II representation of z with $K = 0$. By Lemma 3.4, item (iii), T satisfies the setting of Lemma 8.4 with the chain sign σ, and T is middle. By Proposition 7.45, item 2, $m \le m_{\mathrm{II}}(c)$ and $J \ge 1$, $J(4ae+1) < 2m(m+1)$, so $m \ge m_{\mathrm{lo}}$. By Lemma 8.4, item 4, $\lambda = 1$, $e \ge a+3$ and $4a(m+1) < e - a$, so $m \le \lfloor (e-a-1)/(4a) \rfloor - 1$ and $m \le m_{\mathrm{top}}$; in particular $e \le a+2$ does not hold. By Lemma 9.21, the integer ι is non-zero.

Let $10^{65} \le c < 10^{77.28}$. Then $m' < Q_{\mathrm{II}}(c) \le 1.03 \cdot 10^{13}$ (Definition 4.1), and Lemma 9.27 gives $T \in \mathcal{S}_{\mathrm{ell}}$.

Let $\iota \ge 1$. By Lemma 9.22, $u^2 > 128a^4J$, so $J \le \lfloor (u^2-1)/(128a^4) \rfloor$; and $J(4ae+1) < 2m(m+1) \le 2m_{\mathrm{top}}(m_{\mathrm{top}}+1)$; so $1 \le J \le J_{\max}$. Item 1 of Lemma 9.21 and item 1 (if $\sigma = 1$) or item 2 (if $\sigma = -1$) of Lemma 9.23 give the displayed floors at m; their left sides increase with m (here $k_2 > 0$ if $\sigma = 1$, and $g_2 > 0$), so $m \ge \ell_+$. If $a \ge 100$, item 4 of Lemma 9.24 gives

$$u - a < \frac{2a}{a+1} \cdot \frac{J\big[(4a^2+1)(u^2-1) + 3a^2 + 2au\big]}{u^2 - 1 - 6\sqrt{2J}\,a^2u - 4a^2};$$

here $J \le J_{\max}$ and $s_* \ge \sqrt{2J}$, the numerator increases with J, and the denominator decreases when $\sqrt{2J}$ is replaced by s_*, so when $D_{s_*} > 0$ the right side is at most $2aJ_{\max}[(4a^2+1)(u^2-1) + 3a^2 + 2au]/((a+1)D_{s_*})$; hence the condition defining $\ell_+ = \infty$ fails. So $m \ge \ell_+$ with $\ell_+ < \infty$.

Let $\iota \le -1$. By item 2 of Lemma 9.21, $\sigma = -1$, $2u > 3a$ and $2(2u-3a)m(m+1) > e(4ae+1)$, so $m \ge \ell_-$.

Hence $\min\{\ell_+, \ell_-\} \le m \le m_{\mathrm{top}}$, and, by the preceding paragraphs, none of the conditions that define $\ell_0 = \infty$ holds; so $\ell_0 = \min\{\ell_+, \ell_-\} \le m \le m_{\mathrm{top}}$. Replacing $\mathcal{S}_{\mathrm{ell}}$ by a finite set containing it only makes $\ell_0 = \infty$ less often. □

The tests for BM and BL combine the slot test for type I with the windows of type II; for BL, the integers Ξ and Z of the window are bounded by increasing functions of m (Lemma 9.30), which turns their divisibilities into finite searches, and the alternative (II.1) runs over the square roots of one residue modulo $2W$ (Lemma 9.31). The triples of BM and BL are the chain triples at slot three of item (a) of Lemma 9.11.

Corollary 9.29. *Let $T = \{a, b, c\}$ be a triple of the family BM of Definition 4.2 (single-chain, $\deg T = 2$, middle, $c \geq 10^{30}$), with descent partner e, chain sign σ and chain constant E; let $m_{\mathrm{II}}(c)$ be as in Definition 4.1, m_{lo} and ℓ_0 as in Lemma 9.28, let m_K be the least integer $m \geq 0$ with*

$$2m(m+1)\Big(a + \frac{E}{4ae+1}\Big) + \frac{m(e-a)}{2} \geq b, \quad \text{with } > b \text{ if } E > 0,$$

and $\ell_K = \max\{m_{\mathrm{lo}}, m_K\}$. Then, for $\lambda \in \{-1, 1\}$:

1. *the case (I, λ) is closed if the hypotheses of item 1 or of item 2 of Lemma 9.20 hold (with $\nu = 3$);*
2. *the case $(\mathrm{II}, -1)$ is closed if $c \geq 10^{77.28}$, or the range $[2\ell_K + 1,\ 2m_{\mathrm{II}}(c) + 1]$ is certified for it;*
3. *the case $(\mathrm{II}, 1)$ is closed if $c \geq 10^{77.28}$, or the range $[2\min\{\ell_K, \ell_0\} + 1,\ 2m_{\mathrm{II}}(c) + 1]$ is certified for it.*

Proof. Item 1 is items 1 and 2 of Lemma 9.20, since T is single-chain of degree 2, middle, with $c \geq 10^{30}$.

Let d be an irregular extension of T and (λ, m, n) a type II representation of $z = \sqrt{cd+1}$. By Proposition 8.5, $(T, (\lambda, m, n)) \in \mathrm{BM}$; its generic window gives $c < 10^{77.28}$, $m \leq m_{\mathrm{II}}(c)$, and $J \geq 1$ with $J(4ae+1) < 2m(m+1)$, so $m \geq m_{\mathrm{lo}}$. So there is no type II representation if $c \geq 10^{77.28}$, and the cases are then closed by Lemma 9.6, item 2, with $r = 0$.

If $K \neq 0$, item 2 of Lemma 8.4 (whose setting holds by Lemma 3.4, item (iii)) gives $b \leq 2m(m+1)(a+E/(4ae+1))+m(e-a)/2$, strictly if $E > 0$; the left side of the defining inequality of m_K increases with m, so $m \geq m_K$ and $m \geq \ell_K$. If $K = 0$, Lemma 9.28 gives $\lambda = 1$ and $m \geq \ell_0$. Hence a representation with $\lambda = -1$ has $m \geq \ell_K$, and one with $\lambda = 1$ has $m \geq \min\{\ell_K, \ell_0\}$; in both cases $m' = 2m + 1$ lies in the range named for its case, and Lemma 9.6, item 2, closes the case if that range is certified. □

Lemma 9.30. *Assume the setting of Lemma 8.4 with T low, and let $B(m)$, Ξ, V, Γ, Z be as there. Then*

$$|\Xi| \leq 4(u - \sigma e)K + (2u + \sigma e)J + e\Delta < \frac{4(u-\sigma e)\,B(m)}{b} + \frac{2(2u+\sigma e)\,m(m+1)}{4ae+1} + \frac{em}{2},$$

$$|Z| < \frac{m}{2}\big(|V| + (2m+1)\Gamma\big) + \frac{12m(m+1)P}{4ae+1},$$

and both right sides are increasing functions of $m \geq 0$.

Proof. Since T is low, $u^2 = ae + 1 > e^2$, so $u > e$, $|e - \sigma u| = u - \sigma e > 0$ and $|e + 2\sigma u| = 2u + \sigma e > 0$. With $K \geq 1$, $J \geq 1$ and $\Delta \geq 0$ the triangle inequality gives the first inequality. Next, $bK = 2am(m+1) + \lambda\Delta(a-e) + JE \leq 2am(m+1) + \frac{m}{2}(a-e) + \frac{2m(m+1)E}{4ae+1} = B(m)$, so $K \leq B(m)/b$; moreover $J < 2m(m+1)/(4ae+1)$ and $\Delta \leq m/2$. Since $2u + \sigma e > 0$, the bound for J makes the second inequality strict.

For Z, since $\Gamma > 0$, $\Delta \le m/2$, $S+1 \le 2m+1$ (as $n \le m$) and $6JP < 12m(m+1)P/(4ae+1)$,

$$|Z| \le \Delta\big(|V| + \Gamma(S+1)\big) + 6JP < \frac{m}{2}\big(|V| + (2m+1)\Gamma\big) + \frac{12m(m+1)P}{4ae+1}.$$

Both right sides are polynomials in m with non-negative coefficients, since $u - \sigma e$, $2u + \sigma e$, $|a - e|$, E, $|V|$, Γ and P are non-negative; so they increase with $m \ge 0$. □

Lemma 9.31. *Let e and u be positive integers and $\sigma \in \{-1, 1\}$; put $W = 4(u - \sigma e)^2 + 8$ and, for an integer h, $r_h = 1 - 2h(e + \sigma u)$. For integers h, K, q put $R = KW - h(e + \sigma u)$.*

1. *$R \ge 0$ and $2R + 1 = q^2$ if and only if $q^2 \equiv r_h \pmod{2W}$ and $K = (q^2 - r_h)/(2W)$.*
2. *Let h and $0 \le m_1 \le m_2$ be integers. The pairs (K, q) of integers with $2m_1 + 1 \le q \le 2m_2+1$, $R \ge 0$ and $2R+1 = q^2$ are the pairs $\big((q^2-r_h)/(2W),\, q\big)$ with $2m_1+1 \le q \le 2m_2+1$ and $q^2 \equiv r_h \pmod{2W}$. Every such q is odd, and each residue class modulo $2W$ contains at most $\lfloor (m_2 - m_1)/W \rfloor + 1$ of them.*

Proof. Item 1. By definition $2R + 1 = 2KW - 2h(e + \sigma u) + 1 = 2KW + r_h$. If $R \ge 0$ and $2R + 1 = q^2$, then $q^2 - r_h = 2KW$, so $q^2 \equiv r_h \pmod{2W}$ and $K = (q^2 - r_h)/(2W)$. Conversely, if $q^2 \equiv r_h \pmod{2W}$ and $K = (q^2 - r_h)/(2W)$, then $2R+1 = 2KW + r_h = q^2$, and $R = (q^2 - 1)/2 \ge 0$ because $2R + 1 = q^2$ is odd, so $q \ne 0$.

Item 2. The first sentence is item 1 for each q in the range. As r_h is odd and $2W$ is even, $q^2 \equiv r_h \pmod{2W}$ forces q to be odd. The integers q of one residue class modulo $2W$ in the interval $[2m_1 + 1, 2m_2 + 1]$, of length $2(m_2 - m_1)$, number at most $\lfloor 2(m_2 - m_1)/(2W) \rfloor + 1 = \lfloor (m_2 - m_1)/W \rfloor + 1$. □

Corollary 9.32. *Let $T = \{a, b, c\}$ be a triple of the family BL of Definition 4.2 (single-chain, $\deg T = 2$, low, $c \ge 10^{30}$), with descent partner e, chain sign σ, chain constants E, P, $u = \sqrt{ae+1}$, and $B(m)$, V, Γ as in the window of BL (Definition 4.2). Let $m_{\mathrm{II}} = m_{\mathrm{II}}(c)$ be as in Definition 4.1, and put:*

- *$m_{\mathrm{lo}}^{\mathrm{L}}$ the least integer $m \ge 0$ with $2m(m+1) > 4ae + 1$ and $B(m) \ge b$, with $B(m) > b$ if $E > 0$;*
- *$J_{\max}^{\mathrm{L}} = \lfloor (2m_{\mathrm{II}}(m_{\mathrm{II}} + 1) - 1)/(4ae + 1) \rfloor$, $D_{\max} = \lfloor m_{\mathrm{II}}/2 \rfloor$,*

$$\hat{\Xi} = \frac{4(u - \sigma e)B(m_{\mathrm{II}})}{b} + \frac{2(2u + \sigma e)m_{\mathrm{II}}(m_{\mathrm{II}} + 1)}{4ae + 1} + \frac{e\, m_{\mathrm{II}}}{2},$$
$$\hat{Z} = \frac{m_{\mathrm{II}}}{2}\big(|V| + (2m_{\mathrm{II}} + 1)\Gamma\big) + \frac{12m_{\mathrm{II}}(m_{\mathrm{II}} + 1)P}{4ae + 1};$$

- *$W = 4(u - \sigma e)^2 + 8$ and, for an integer h, $r_h = 1 - 2h(e + \sigma u)$.*

Then, for each $\lambda \in \{-1, 1\}$:

1. *the case (I, λ) is closed if the hypotheses of item 1 or of item 2 of Lemma 9.20 hold (with $\nu = 3$);*
2. *the case (II, λ) is closed if one of the following holds:*

(II.0) $c \ge 10^{77.28}$, or $m_{\mathrm{lo}}^{\mathrm{L}} > m_{\mathrm{II}}$;

(II.1) $\hat{\Xi} \le a$, *and there are no integers* $h \ge 1$ *and* q *with* $2m_{\mathrm{lo}}^{\mathrm{L}}+1 \le q \le 2m_{\mathrm{II}}+1$ *and* $q^2 \equiv r_h \pmod{2W}$ *such that* $K = (q^2 - r_h)/(2W)$ *and* $m = (q-1)/2$ *satisfy:* $K \ge 1$*; he and* $h(e+2\sigma u)$ *are even;* $J = \frac{1}{2}he - 2K$ *satisfies* $1 \le J \le J_{\max}^{\mathrm{L}}$*;* $\tilde{\Delta} = \frac{1}{2}\big(h(e+2\sigma u) - 12K\big)$ *satisfies* $\lambda\tilde{\Delta} \ge 0$ *and* $2|\tilde{\Delta}| \le m$*; and, with* $n = m - |\tilde{\Delta}|$*,*

$$b\big(2n(n+1) - \tilde{\Delta}\big) - a\big(2m(m+1) + \tilde{\Delta}\big) + e\tilde{\Delta} = Jc;$$

the integers h *in question satisfy* $h(u-e)+3 \le D_{\max}$ *if* $\sigma = 1$ *and* $h(2u-e)+12 \le 2D_{\max}$ *if* $\sigma = -1$*;*

(II.2) $\hat{Z} \le c$*, and either* $\lambda V \le 0$ *or no integer* S *with* $\lceil 3m_{\mathrm{lo}}^{\mathrm{L}}/2 \rceil \le S \le 2m_{\mathrm{II}} - 1$ *satisfies* $|V| - 6J_{\max}^{\mathrm{L}}P \le \Gamma(S+1) \le |V|$ *(that is, no integer* $S+1$ *of the interval* $\big[(|V| - 6J_{\max}^{\mathrm{L}}P)/\Gamma,\ |V|/\Gamma\big]$ *has* S *in that range);*

(II.3) the range $[2m_{\mathrm{lo}}^{\mathrm{L}} + 1,\ 2m_{\mathrm{II}} + 1]$ *is certified for* (II, λ)*.*

Proof. Item 1 is items 1 and 2 of Lemma 9.20, since T is single-chain of degree 2, low, with $c \ge 10^{30}$.

Let d be an irregular extension of T and (λ, m, n) a type II representation of $z = \sqrt{cd+1}$; we show that, for its case, each of the conditions (II.0) to (II.2) fails and the representation lies in the range of (II.3), so that the case is closed by Lemma 9.6, item 2 (with $r = 0$ for (II.0) to (II.2), and with the range of (II.3)). By Proposition 8.5, $(T, (\lambda, m, n)) \in \mathrm{BL}$, and by Lemma 3.4, item (iii), the setting of Lemma 8.4 holds with the chain sign σ. The generic window gives $c < 10^{77.28}$, $m \le m_{\mathrm{II}}$ and $2m(m+1) > J(4ae+1) \ge 4ae+1$; the window of BL gives $K \ge 1$ and $b \le B(m)$, and item 2 of Lemma 8.4 gives $b < B(m)$ if $E > 0$. As B increases with m, $m \ge m_{\mathrm{lo}}^{\mathrm{L}}$. So (II.0) fails, and $2m_{\mathrm{lo}}^{\mathrm{L}} + 1 \le m' \le 2m_{\mathrm{II}} + 1$, so the representation lies in the range of (II.3). Moreover $J \le J_{\max}^{\mathrm{L}}$ and $\Delta \le m/2 \le D_{\max}$.

Condition (II.1). Suppose $\hat{\Xi} \le a$. By Lemma 9.30, $|\Xi| < \hat{\Xi} \le a$, and $a \mid \Xi$ (Lemma 8.4, item 3(a)); so $\Xi = 0$. Then item 3(a) of Lemma 8.4 gives an integer $h = 2(2K+J)/e \ge 1$ with $J = \frac{1}{2}he - 2K$ and $\lambda\Delta = \frac{1}{2}(h(e+2\sigma u) - 12K) = \tilde{\Delta}$; thus he and $h(e+2\sigma u)$ are even, $1 \le J \le J_{\max}^{\mathrm{L}}$, $\lambda\tilde{\Delta} = \Delta \ge 0$, $|\tilde{\Delta}| = \Delta$, $2|\tilde{\Delta}| \le m$ and $n = m - |\tilde{\Delta}|$. Its relation $2m(m+1) = 4K(u-\sigma e)^2 - J - \lambda\Delta$, with $J + \tilde{\Delta} = h(e+\sigma u) - 8K$, reads $2m(m+1) = R$ for $R = KW - h(e+\sigma u)$, so $R \ge 0$ and $2R + 1 = q^2$ with $q = 2m+1$. By Lemma 9.31, item 1, $q^2 \equiv r_h \pmod{2W}$ and $K = (q^2 - r_h)/(2W)$; moreover $2m_{\mathrm{lo}}^{\mathrm{L}} + 1 \le q \le 2m_{\mathrm{II}} + 1$, $m = (q-1)/2$, $K \ge 1$, and $H = Jc$ is the displayed identity. So (h, q) is a pair that (II.1) excludes. Finally, if $\sigma = 1$, item 3(a) gives $\Delta = 3J + h(u-e) \ge 3 + h(u-e)$; if $\sigma = -1$, then $e < 2u$ gives $\tilde{\Delta} < 0$, so $\Delta = -\tilde{\Delta} = \frac{1}{2}(h(2u-e) + 12K) \ge \frac{1}{2}(h(2u-e) + 12)$; with $\Delta \le D_{\max}$ this gives the stated ranges of h.

Condition (II.2). Suppose $\hat{Z} \le c$. By Lemma 9.30, $|Z| < \hat{Z} \le c$, and $c \mid Z$ (Lemma 8.4, item 3(b)); so $Z = 0$. Then item 3(b) gives $\Delta \ge 1$, $\lambda V > 0$ and $|V| - 6JP \le \Gamma(S+1) \le |V|$, hence $|V| - 6J_{\max}^{\mathrm{L}}P \le \Gamma(S+1) \le |V|$; and $S = m + n$ satisfies $S \ge \frac{3}{2}m \ge \frac{3}{2}m_{\mathrm{lo}}^{\mathrm{L}}$ (as $n \ge m/2$) and $S = 2m - \Delta \le 2m_{\mathrm{II}} - 1$. So $\lambda V > 0$ and such an S exists. As $\Gamma > 0$, the two inequalities say that $S+1$ lies in $\big[(|V| - 6J_{\max}^{\mathrm{L}}P)/\Gamma,\ |V|/\Gamma\big]$. □

For each h, the residues x modulo $2W = 8\big((u-\sigma e)^2 + 2\big)$ with $x^2 \equiv r_h$ are found from the factorisation of $(u-\sigma e)^2 + 2$. The alternative (II.1) can apply only when $\hat{\Xi} \le a$; as the three terms of $\hat{\Xi}$ are non-negative, this needs $a \ge e\, m_{\mathrm{II}}/2$ and $a(4ae+1) \ge 2(2u+\sigma e)m_{\mathrm{II}}(m_{\mathrm{II}}+1)$. Otherwise (II.2) applies when $\hat{Z} \le c$, and (II.3) applies in all cases.

9.7 Deeper slots

At slots four and beyond the type II case is closed by a certified range above the floor of the window, and at slot four also by a search on the conic, through the finitely many Pell classes of the pair $\{a, E'\}$ (Lemma 9.33). The triples of B3 and B4 are the chain triples of item (a) of Lemma 9.11 at slots $\nu = 4$ and $\nu \geq 5$.

Lemma 9.33. *Let $a \neq e$ be positive integers with $ae + 1 = u^2$, u a positive integer, let σ be an admitted sign of $\{a, e\}$, and put*

$$E' = a + e - 2\sigma u, \qquad R = |a - \sigma u|, \qquad \eta_{aE'} = R + \sqrt{aE'}.$$

1. *$E' \geq 1$, $aE' + 1 = R^2$, $R \geq 2$ and $E' \neq a$.*

2. *Let $\mathcal{F}$ be the set of integer pairs (X_0, Y_0) with $E'X_0^2 - aY_0^2 = E' - a$ and*

$$1 \leq X_0 \leq \sqrt{\frac{a(E'-a)}{2(R-1)}} \quad \text{if } a < E', \qquad 1 \leq Y_0 \leq \sqrt{\frac{E'(a-E')}{2(R-1)}} \quad \text{if } E' < a.$$

   Then $\mathcal{F}$ is finite. For $(X_0, Y_0) \in \mathcal{F}$ and $k \geq 0$ the numbers X_k, Y_k defined by

$$X_k\sqrt{E'} + Y_k\sqrt{a} = \left(X_0\sqrt{E'} + Y_0\sqrt{a}\right)\eta_{aE'}^k$$

   are integers, with $X_{k+1} = RX_k + aY_k$ and $Y_{k+1} = E'X_k + RY_k$; moreover $v_0 = X_0\sqrt{E'} + Y_0\sqrt{a} > 0$, and for every $k \geq 0$

$$2\sqrt{E'}\,X_k > v_0\,\eta_{aE'}^k \quad \text{if } a < E', \qquad 2\sqrt{a}\,Y_k > v_0\,\eta_{aE'}^k \quad \text{if } E' < a.$$

3. *Let $\lambda \in \{-1, 1\}$, and let m, n be integers with $m \geq u$ and $m/2 \leq n \leq m$ such that*

$$a(m^2 + m) - E'(n^2 + n) + \lambda(a - \sigma u)(m - n) = 0.$$

   Put $X = a(2m + 1) + \lambda(a - \sigma u)$ and $Y = E'(2n + 1) + \lambda(a - \sigma u)$. Then $X \geq 1$, $Y \geq 1$, $E'X^2 - aY^2 = E' - a$, and $(X, Y) = (X_k, Y_k)$ for some $(X_0, Y_0) \in \mathcal{F}$ and some $k \geq 0$.

Proof. Item 1. For the pair $\{a, e\}$ the root is u, so $E' = x_1(a, e; \sigma)$ (Definition 3.2); as σ is admitted, $E' \geq 1$, and $E' \notin \{a, e\}$ by item 3 of Lemma 3.3. Next, $(a - \sigma u)^2 = a^2 - 2\sigma au + ae + 1 = aE' + 1$, so $R^2 = aE' + 1 \geq 2$ and $R \geq 2$.

Item 2. As $aE' = R^2 - 1$ with $R \geq 2$, aE' is not a square, so $\sqrt{aE'}$ is irrational and the coefficients of $\sqrt{E'}$ and $\sqrt{a}$ in a number $X\sqrt{E'} + Y\sqrt{a}$ with rational X, Y are unique. If $a < E'$, the first coordinate of an element of $\mathcal{F}$ is bounded and determines Y_0^2; if $E' < a$, the second coordinate is bounded and determines X_0^2; so $\mathcal{F}$ is finite. Since

$$\left(X\sqrt{E'} + Y\sqrt{a}\right)\left(R + \sqrt{aE'}\right) = (RX + aY)\sqrt{E'} + (E'X + RY)\sqrt{a},$$

the stated recurrences hold, and X_k, Y_k are integers by induction. Multiplying by $\sqrt{E'}$ maps $X\sqrt{E'} + Y\sqrt{a}$ to $XE' + Y\sqrt{aE'} \in \mathbb{Q}(\sqrt{aE'})$; applying the conjugation $\sqrt{aE'} \mapsto -\sqrt{aE'}$, which sends $\eta_{aE'}$ to $R - \sqrt{aE'} = \eta_{aE'}^{-1}$, we get

$$X_k\sqrt{E'} - Y_k\sqrt{a} = \bar{v}_0\,\eta_{aE'}^{-k}, \qquad \bar{v}_0 = X_0\sqrt{E'} - Y_0\sqrt{a},$$

and hence $2\sqrt{E'}X_k = v_0\eta_{aE'}^k + \bar v_0\eta_{aE'}^{-k}$ and $2\sqrt{a}Y_k = v_0\eta_{aE'}^k - \bar v_0\eta_{aE'}^{-k}$. If $a < E'$, then $aY_0^2 = E'X_0^2 - (E'-a) < E'X_0^2$ and $X_0 \ge 1$, so $|Y_0|\sqrt{a} < X_0\sqrt{E'}$; thus $v_0 > 0$ and $\bar v_0 > 0$, and $2\sqrt{E'}X_k > v_0\eta_{aE'}^k$. If $E' < a$, then $E'X_0^2 = aY_0^2 - (a-E') < aY_0^2$ and $Y_0 \ge 1$, so $|X_0|\sqrt{E'} < Y_0\sqrt{a}$; thus $v_0 > 0 > \bar v_0$, and $2\sqrt{a}Y_k > v_0\eta_{aE'}^k$.

Item 3. First, $X \ge 1$ and $Y \ge 1$. Here $R \le a+u$ and $m \ge u$, so

$$X \ge a(2u+1) - (a+u) = u(2a-1) \ge 1.$$

Since $n \ge m/2$, $2n+1 \ge m+1 \ge u+1$, so $Y \ge E'(u+1) - R$, and it suffices to show $R < E'(u+1)$. If $\sigma = -1$, then $R = a+u < a+e+2u = E'$. If $\sigma = 1$, then $a \le ae = u^2-1$ and $E' \ge 1$ give $R^2 = aE'+1 \le (u^2-1)E'+1 < (u+1)^2E' \le (u+1)^2E'^2$. In both cases $R < E'(u+1)$.

With $\rho_0 = a - \sigma u$, so that $\rho_0^2 = aE'+1$ by item 1, expanding gives

$$E'X^2 - aY^2 - (E'-a) = 4aE'\big[a(m^2+m) - E'(n^2+n) + \lambda\rho_0(m-n)\big],$$

which vanishes by hypothesis; so $E'X^2 - aY^2 = E'-a$.

The class of (X,Y) is given by Lemma 2.12. If $a < E'$, apply it with $(P,Q,S) = (a,E',R)$ in the notation of that lemma, so that its η_{PQ} is $\eta_{aE'}$: the positive integers (X,Y) solve $aY^2 - E'X^2 = a-E'$, so $Y\sqrt{a} + X\sqrt{E'} = (Y_0\sqrt{a} + X_0\sqrt{E'})\eta_{aE'}^k$ for some $k \ge 0$ and an integer solution (X_0,Y_0) with $1 \le X_0 \le \sqrt{a(E'-a)/(2(R-1))}$; thus $(X_0,Y_0) \in \mathcal{F}$ and $(X,Y) = (X_k,Y_k)$. If $E' < a$, apply it with $(P,Q,S) = (E',a,R)$ in the notation of that lemma to the positive solution $(X',Y') = (Y,X)$ of $E'Y'^2 - aX'^2 = E'-a$: then $X\sqrt{E'} + Y\sqrt{a} = Y'\sqrt{E'} + X'\sqrt{a} = (Y_0'\sqrt{E'} + X_0'\sqrt{a})\eta_{aE'}^k$ with $1 \le X_0' \le \sqrt{E'(a-E')/(2(R-1))}$; with $(X_0,Y_0) = (Y_0',X_0')$ we get $(X_0,Y_0) \in \mathcal{F}$ and $(X,Y) = (X_k,Y_k)$.

The bound for the fundamental solutions in Lemma 2.12 is that of He, Togbé and Ziegler [11, Lemma 8], going back to Dujella [5]; the lemma replaces a search over n on the conic by a search over the finitely many classes $\mathcal{F}$. □

Corollary 9.34. *Let $T = \{a,b,c\}$ be a triple of the family B3 of Definition 4.2 (single-chain, $\deg T = 3$, middle or low, $c \ge 10^{30}$), with descent partner e, chain sign σ, chain constants ϑ, E', P, F_4, m_4 (Definition 3.8) and $u = \sqrt{ae+1}$. Let $m_{\mathrm{II}} = m_{\mathrm{II}}(c)$ be as in Definition 4.1, m_{lo} the least integer $m \ge 0$ with $2m(m+1) > 4ae+1$, and m_4^* an integer with $m_4^* \le m_4$ or $m_4^* < u$ (for instance $m_4^* = m_4$, or $m_4^* = \lfloor F_4 \rfloor$, which needs no evaluation of Ω_4; a larger m_4^* gives a shorter certified range below). Let R, $\eta_{aE'}$, $\mathcal{F}$, X_k, Y_k and v_0 be as in Lemma 9.33 for $(a,e;\sigma)$, and put $\bar m = \min\{m_4^*, m_{\mathrm{II}}\}$. Then, for each $\lambda \in \{-1,1\}$:*

1. *the case (I,λ) is closed if the hypotheses of item 1 or of item 2 of Lemma 9.20 hold (with $\nu = 4$);*
2. *the case (II,λ) is closed if $c \ge 10^{77.28}$, or the range $[2m_{\mathrm{lo}}+1,\ 2m_{\mathrm{II}}+1]$ is certified for it, or both of the following hold:*
   - *(conic) either $\vartheta^{-2} \notin [1-1.5/a-12/u,\ 4+1.5/a+12/u]$, or there are no $(X_0,Y_0) \in \mathcal{F}$ and integers $k \ge 0$ with*

$$v_0\,\eta_{aE'}^k < 2\sqrt{E'}\,\big(a(2\bar m+1)+R\big) \quad \text{if } a < E',$$
$$v_0\,\eta_{aE'}^k < 2\sqrt{a}\,\big(E'(2\bar m+1)+R\big) \quad \text{if } E' < a,$$

   such that $m = \big(X_k - a - \lambda(a-\sigma u)\big)/(2a)$ and $n = \big(Y_k - E' - \lambda(a-\sigma u)\big)/(2E')$ are integers with $m_{\mathrm{lo}} \le m \le \bar m$ and $m/2 \le n \le m$, $J = \big(2n(n+1) - \lambda(m-n)\big)/P$ is an integer with $J \ge 1$, $J^2 < u$ and $J(4ae+1) < 2m(m+1)$, and

$$bJP - a\big(2m(m+1) + \lambda(m-n)\big) + e\lambda(m-n) = Jc;$$

- *the range* $[\max\{2m_{\rm lo}+1, 2m_4^*+3\},\ 2m_{\rm II}+1]$ *is certified for* $({\rm II},\lambda)$.

Proof. Item 1 is items 1 and 2 of Lemma 9.20, since T is single-chain of degree 3, middle or low, with $c \geq 10^{30}$.

Let d be an irregular extension of T and (λ, m, n) a type II representation of $z = \sqrt{cd+1}$. By Proposition 8.13, $(T, (\lambda, m, n)) \in$ B3; its generic window gives $c < 10^{77.28}$, $n \leq m \leq 2n$, $m \leq m_{\rm II}$, and $J \geq 1$ with $J(4ae+1) < 2m(m+1)$, so $m \geq m_{\rm lo}$, and $m > \sqrt{2}u - \frac{1}{2} > u$. So there is no type II representation if $c \geq 10^{77.28}$, and $2m_{\rm lo}+1 \leq m' \leq 2m_{\rm II}+1$; in these cases Lemma 9.6, item 2, closes the case.

Suppose the two conditions of the last alternative hold. By Lemma 3.4, item (iii), $b = x_3$ and $c = x_4$ on the chain of $\{a, e\}$ with sign σ, and by Lemma 8.6, item 6, $u > 2800$. Suppose $m \leq m_4^*$. Then $m_4^* \geq m > u$, so $m_4^* \leq m_4$ and $m \leq m_4$; Lemma 8.11 gives $K = 0$, that is $N = JP$, $J^2 < u$, ϑ^{-2} in the interval, and

$$a(m^2+m) - E'(n^2+n) + \lambda(a-\sigma u)(m-n) = 0.$$

As $m > u$ and $m/2 \leq n \leq m$, item 3 of Lemma 9.33 gives $(X_0, Y_0) \in \mathcal{F}$ and $k \geq 0$ with $X_k = a(2m+1) + \lambda(a-\sigma u)$ and $Y_k = E'(2n+1) + \lambda(a-\sigma u)$; so m and n are given by the stated formulas, and $m_{\rm lo} \leq m \leq \bar{m}$. Since $|a-\sigma u| = R$ and $n \leq m \leq \bar{m}$, we have $X_k \leq a(2\bar{m}+1)+R$ and $Y_k \leq E'(2\bar{m}+1)+R$, so item 2 of Lemma 9.33 gives the stated bound on k. As $N = 2n(n+1) - \lambda(m-n) = JP$, the number $J = \big(2n(n+1) - \lambda(m-n)\big)/P$ is the integer J of the representation, with $J \geq 1$, $J^2 < u$ and $J(4ae+1) < 2m(m+1)$; and $H = Jc$ with $M = 2m(m+1) + \lambda(m-n)$ is the last identity. So (X_0, Y_0, k) is excluded by the conic condition, a contradiction. Hence $m > m_4^*$, so $m' = 2m+1 \geq 2m_4^*+3$, and m' lies in the certified range $[\max\{2m_{\rm lo}+1, 2m_4^*+3\}, 2m_{\rm II}+1]$; Lemma 9.6, item 2, closes the case.

The choice $m_4^* = \lfloor F_4 \rfloor$ is admissible: if $\lfloor F_4 \rfloor \geq u$, then item 3 of Lemma 8.12 (with $u \geq 10^3$) shows that $\lfloor F_4 \rfloor$ is not larger than m_4; otherwise $\lfloor F_4 \rfloor < u$. A larger m_4^* raises the lower end $\max\{2m_{\rm lo}+1, 2m_4^*+3\}$ of the last range. In the conic condition, the comparisons of ϑ^{-2} with the interval and of $v_0 \eta_{aE'}^k$ with the bounds on k are decided from rational enclosures (Lemma 9.7); the other conditions are exact integer conditions. □

Corollary 9.35. *Let $T = \{a, b, c\}$ be a triple of the family B4 of Definition 4.2 (single-chain, $\deg T \geq 4$, middle or low, $c \geq 10^{30}$), with descent partner e, $\nu = \deg T + 1$, chain sign σ, chain constants F_ν, m_ν (Definition 3.8) and $u = \sqrt{ae+1}$. Let $m_{\rm II} = m_{\rm II}(c)$ be as in Definition 4.1, $m_{\rm lo}$ the least integer $m \geq 0$ with $2m(m+1) > 4ae+1$, and m_ν^* an integer with $m_\nu^* \leq m_\nu$ or $m_\nu^* < u$ (for instance $m_\nu^* = m_\nu$, or $m_\nu^* = \lfloor F_\nu \rfloor$, which needs no evaluation of Ω_ν; a larger m_ν^* gives a shorter certified range below). Then, for each $\lambda \in \{-1, 1\}$:*

1. *the case* $({\rm I}, \lambda)$ *is closed if the hypotheses of item 1 or of item 2 of Lemma 9.20 hold (with $\nu = \deg T + 1$);*

2. *the case* $({\rm II}, \lambda)$ *is closed if $c \geq 10^{77.28}$, or the range* $[2m_{\rm lo}+1,\ 2m_{\rm II}+1]$ *is certified for it, or $u \geq 10^3$ and the range* $[\max\{2m_{\rm lo}+1, 2m_\nu^*+3\},\ 2m_{\rm II}+1]$ *is certified for it.*

Proof. Item 1 is items 1 and 2 of Lemma 9.20, since T is single-chain of degree $\nu - 1 \geq 4$, middle or low, with $c \geq 10^{30}$.

Let d be an irregular extension of T and (λ, m, n) a type II representation of $z = \sqrt{cd+1}$. By Proposition 8.14, $(T, (\lambda, m, n)) \in$ B4; its generic window gives $c < 10^{77.28}$, $m \leq m_{\rm II}$, and $J \geq 1$ with $J(4ae+1) < 2m(m+1)$, so $m \geq m_{\rm lo}$ and $m > \sqrt{2}u - \frac{1}{2} > u$. So there is no type II

representation if $c \geq 10^{77.28}$, and $2m_{\mathrm{lo}} + 1 \leq m' \leq 2m_{\mathrm{II}} + 1$; in these cases Lemma 9.6, item 2, closes the case.

Let $u \geq 10^3$. By Lemma 3.4, item (iii), $b = x_{\nu-1}$ and $c = x_\nu$ on the chain of $\{a, e\}$ with sign σ. If $m \leq m_\nu$, then Lemma 8.11 would give $\nu = 4$; so $m > m_\nu$. If $m_\nu^* \leq m_\nu$, then $m > m_\nu^*$; if $m_\nu^* < u$, then $m > u > m_\nu^*$. So $m' = 2m + 1 \geq 2m_\nu^* + 3$, the representation lies in the range $[\max\{2m_{\mathrm{lo}} + 1, 2m_\nu^* + 3\}, 2m_{\mathrm{II}} + 1]$, and Lemma 9.6, item 2, closes the case if this range is certified. The choice $m_\nu^* = \lfloor F_\nu \rfloor$ is admissible: if $\lfloor F_\nu \rfloor \geq u$, item 3 of Lemma 8.12 shows that $\lfloor F_\nu \rfloor$ is not larger than m_ν; otherwise $\lfloor F_\nu \rfloor < u$. □

9.8 Candidates and failed tests

A test may leave finitely many candidates, or fail; a single candidate index is decided by Lemma 9.36.

Lemma 9.36. *Let $T = \{a, b, c\}$ with $a < b < c$ be a Diophantine triple. Fix a case of T with its seeds (x_1, z_1), (y_2, z_2) and its data δ, K' (Definition 9.1), and let $m', n' \geq 0$ be integers with m' admissible for the case.*

1. *For $j \geq 0$, $(z_j^{(a,c)}, x_j^{(a,c)})^\top = M_a^j (z_1, x_1)^\top$ and $(z_j^{(b,c)}, y_j^{(b,c)})^\top = M_b^j (z_2, y_2)^\top$, with*

$$M_a = \begin{pmatrix} s & c \\ a & s \end{pmatrix}, \qquad M_b = \begin{pmatrix} t & c \\ b & t \end{pmatrix}.$$

   If $z_{m'}^{(a,c)} \not\equiv z_{n'}^{(b,c)} \pmod p$ for an integer $p \geq 2$, then $z_{m'}^{(a,c)} \neq z_{n'}^{(b,c)}$. Moreover, $|z_j^{(a,c)}| \leq x_1\sqrt{c/a}\,\beta^j$ and $|z_j^{(b,c)}| \leq y_2\sqrt{c/b}\,\gamma^j$ for $j \geq 0$. Conversely, if $\mathcal{P}$ is a finite set of pairwise coprime integers $p \geq 2$ with

$$\prod_{p \in \mathcal{P}} p > x_1\sqrt{c/a}\,\beta^{m'} + y_2\sqrt{c/b}\,\gamma^{n'}$$

   and $z_{m'}^{(a,c)} \equiv z_{n'}^{(b,c)} \pmod p$ for every $p \in \mathcal{P}$, then $z_{m'}^{(a,c)} = z_{n'}^{(b,c)}$.

2. *If $K'\beta^{-2m'} \leq 1$ and $\Lambda = m' \log\beta - n' \log\gamma + \log\delta$ satisfies $\Lambda \leq 0$ or $\Lambda \geq K'\beta^{-2m'}$, then $z_{m'}^{(a,c)} \neq z_{n'}^{(b,c)}$.*

3. *Suppose $z_{m'}^{(a,c)} = z_{n'}^{(b,c)}$, and put $z = z_{m'}^{(a,c)}$. If $d = (z^2 - 1)/c$ is an integer with $d > c$ and $d \neq d_+(a, b, c)$, then d is an irregular extension of T. Otherwise, and also when $z_{m'}^{(a,c)} \neq z_{n'}^{(b,c)}$, no irregular extension of T gives a representation of this case with indices (m', n'). Moreover, if $z_{m'}^{(a,c)} = z_{n'}^{(b,c)} = z$, then $d = (z^2 - 1)/c$ is an integer, $d > c$ if and only if $|z| \geq c + 1$, and $d = d_+(a, b, c)$ if and only if $|z| = cr + st$. If $m' \geq 3$, then $z_{m'}^{(a,c)} > cr + st \geq c + 1$; so if moreover $z_{m'}^{(a,c)} = z_{n'}^{(b,c)}$, then $\big((z_{m'}^{(a,c)})^2 - 1\big)/c$ is an irregular extension of T.*

4. *If $d = \big((x_{m'}^{(a,c)})^2 - 1\big)/a$ is an integer with $d > c$, $d \neq d_+(a, b, c)$ and $bd + 1$ a square, then d is an irregular extension of T. Otherwise no irregular extension of T gives a representation of this case with index m'.*

Proof. Multiplying $z_j^{(a,c)}\sqrt{a} + x_j^{(a,c)}\sqrt{c} = (z_1\sqrt{a} + x_1\sqrt{c})\beta^j$ by $\sqrt{a}$ gives $az_j^{(a,c)} + x_j^{(a,c)}\sqrt{ac} = (az_1 + x_1\sqrt{ac})\beta^j$ in $\mathbb{Z}[\sqrt{ac}]$, where $ac = s^2 - 1$ is not a square. Taking norms, with $s^2 - ac = 1$,

gives $a\big(a(z_j^{(a,c)})^2 - c(x_j^{(a,c)})^2\big) = a(az_1^2 - cx_1^2) = a(a-c)$; likewise for b, with the seed (y_2, z_2). Hence, for all $j \geq 0$,

$$a\big(z_j^{(a,c)}\big)^2 - c\big(x_j^{(a,c)}\big)^2 = a - c, \qquad b\big(z_j^{(b,c)}\big)^2 - c\big(y_j^{(b,c)}\big)^2 = b - c.$$

Item 1. For integers Z, X we have $(Z\sqrt{a} + X\sqrt{c})(s + \sqrt{ac}) = (sZ + cX)\sqrt{a} + (aZ + sX)\sqrt{c}$, and the coefficients of $\sqrt{a}$ and $\sqrt{c}$ are unique ($\sqrt{ac}$ is irrational); so multiplication by β acts on $(z_j^{(a,c)}, x_j^{(a,c)})^\top$ as M_a. By induction from $j = 0$, where $(z_0^{(a,c)}, x_0^{(a,c)}) = (z_1, x_1)$, we get $(z_j^{(a,c)}, x_j^{(a,c)})^\top = M_a^j (z_1, x_1)^\top$. The same computation with $\gamma = t + \sqrt{bc}$, starting from $(z_0^{(b,c)}, y_0^{(b,c)}) = (z_2, y_2)$, gives the second formula. An equality of integers persists modulo every $p \geq 2$, which gives the second assertion.

For the bounds, as for the norm identities, $az_j^{(a,c)} + x_j^{(a,c)}\sqrt{ac} = (az_1 + x_1\sqrt{ac})\beta^j$ in $\mathbb{Z}[\sqrt{ac}]$, and the conjugation $\sqrt{ac} \mapsto -\sqrt{ac}$ maps β to $s - \sqrt{ac} = \beta^{-1}$, so $az_j^{(a,c)} - x_j^{(a,c)}\sqrt{ac} = (az_1 - x_1\sqrt{ac})\beta^{-j}$. Adding the two identities and dividing by $\sqrt{a}$ gives

$$2\sqrt{a}\, z_j^{(a,c)} = (z_1\sqrt{a} + x_1\sqrt{c})\beta^j + (z_1\sqrt{a} - x_1\sqrt{c})\beta^{-j}.$$

As $\beta > 1$ and $j \geq 0$, we have $\beta^{-j} \leq \beta^j$; and

$$|z_1\sqrt{a} + x_1\sqrt{c}| + |z_1\sqrt{a} - x_1\sqrt{c}| = 2\max\{|z_1|\sqrt{a},\ x_1\sqrt{c}\},$$

as $x_1 > 0$. Hence

$$|z_j^{(a,c)}| \leq \max\big\{|z_1|,\ x_1\sqrt{c/a}\big\}\,\beta^j.$$

For the seed $(x_1, z_1) = (1, \lambda)$ of type I the maximum is $\sqrt{c/a} = x_1\sqrt{c/a}$, as $c > a$. For the seed $(x_1, z_1) = (r, \lambda t)$ of type II, $cr^2 - at^2 = c(ab+1) - a(bc+1) = c - a > 0$, so $t < r\sqrt{c/a}$ and the maximum is again $x_1\sqrt{c/a}$. This is the first bound. The second follows in the same way from $z_j^{(b,c)}\sqrt{b} + y_j^{(b,c)}\sqrt{c} = (z_2\sqrt{b} + y_2\sqrt{c})\gamma^j$, where $bc = t^2 - 1$ is not a square and $t - \sqrt{bc} = \gamma^{-1}$: it gives $|z_j^{(b,c)}| \leq \max\{|z_2|,\ y_2\sqrt{c/b}\}\,\gamma^j$, and the maximum is $\sqrt{c/b} = y_2\sqrt{c/b}$ for the seed $(y_2, z_2) = (1, \lambda)$, as $c > b$, and $y_2\sqrt{c/b}$ for the seed $(y_2, z_2) = (r, \lambda s)$, as $cr^2 - bs^2 = c(ab+1) - b(ac+1) = c - b > 0$.

For the converse, let $\mathcal{P}$ be as there. Every $p \in \mathcal{P}$ divides the integer $z_{m'}^{(a,c)} - z_{n'}^{(b,c)}$. If two coprime integers divide an integer, so does their product; as the elements of $\mathcal{P}$ are pairwise coprime, induction on the number of elements shows that $\prod_{p\in\mathcal{P}} p$ divides $z_{m'}^{(a,c)} - z_{n'}^{(b,c)}$ (the uniqueness part of the Chinese remainder theorem). By the bounds,

$$\big|z_{m'}^{(a,c)} - z_{n'}^{(b,c)}\big| \leq \big|z_{m'}^{(a,c)}\big| + \big|z_{n'}^{(b,c)}\big| \leq x_1\sqrt{c/a}\,\beta^{m'} + y_2\sqrt{c/b}\,\gamma^{n'} < \prod_{p\in\mathcal{P}} p.$$

An integer divisible by a positive integer larger than its absolute value is 0; hence $z_{m'}^{(a,c)} = z_{n'}^{(b,c)}$.

Item 2 is the contrapositive of Lemma 9.2: if $z_{m'}^{(a,c)} = z_{n'}^{(b,c)}$ and $K'\beta^{-2m'} \leq 1$, then $0 < \Lambda < K'\beta^{-2m'}$, the seeds of every case satisfying the hypotheses there.

Item 3. Let $z = z_{m'}^{(a,c)} = z_{n'}^{(b,c)}$ and suppose $d = (z^2-1)/c$ is an integer with $d > c$ and $d \neq d_+(a,b,c)$. Then $z^2 = cd+1$, and the norm identities give $c\big(x_{m'}^{(a,c)}\big)^2 = az^2 - a + c = c(ad+1)$ and $c\big(y_{n'}^{(b,c)}\big)^2 = bz^2 - b + c = c(bd+1)$, so $ad+1 = \big(x_{m'}^{(a,c)}\big)^2$ and $bd+1 = \big(y_{n'}^{(b,c)}\big)^2$. Thus $\{a,b,c,d\}$ is a Diophantine quadruple with $d > c$ and $d \neq d_+(a,b,c)$, that is, d is an irregular extension of T. Conversely, if an irregular extension d of T gives a representation of this case with indices

(m', n'), then, by the definition of a representation, $\sqrt{cd+1} = z_{m'}^{(a,c)} = z_{n'}^{(b,c)}$, so $z_{m'}^{(a,c)} = z_{n'}^{(b,c)}$ and $(z^2-1)/c = d$ is an integer with $d > c$ and $d \neq d_+(a,b,c)$. This proves the second assertion.

For the last assertions of item 3, let $z = z_{m'}^{(a,c)} = z_{n'}^{(b,c)}$ and $d = (z^2-1)/c$. By item 1, $z_{j+1}^{(a,c)} = sz_j^{(a,c)} + cx_j^{(a,c)} \equiv sz_j^{(a,c)} \pmod{c}$, so $z \equiv s^{m'} z_1 \pmod{c}$; as $s^2 = ac+1 \equiv 1$ and $z_1^2 \in \{1, t^2\}$ with $t^2 = bc+1 \equiv 1 \pmod{c}$, we get $z^2 \equiv z_1^2 \equiv 1 \pmod{c}$, and d is an integer. As z is an integer, $d > c$ means $z^2 > c^2+1$, which holds if $|z| \geq c+1$ and fails if $|z| \leq c$. By Definition 2.2, $d_+(a,b,c) = a+b+(2ab+1)c+2rst$, and

$$(cr+st)^2 = c^2(ab+1) + 2crst + (ac+1)(bc+1) = c\,d_+(a,b,c) + 1;$$

so $d = d_+(a,b,c)$ if and only if $z^2 = (cr+st)^2$, that is, $|z| = cr+st$.

Now let $m' \geq 3$; here $z_{m'}^{(a,c)}$ need not equal $z_{n'}^{(b,c)}$. Put $N_1 = z_1\sqrt{a} + x_1\sqrt{c}$, which is positive (the last assertion of Lemma 9.2). As $(z_1\sqrt{a} - x_1\sqrt{c})N_1 = az_1^2 - cx_1^2 = a-c$, the identity in the proof of the bounds of item 1 reads

$$2\sqrt{a}\, z_j^{(a,c)} = N_1\beta^j - \frac{c-a}{N_1\beta^j} \qquad (j \geq 0).$$

For the seed $(x_1, z_1) = (r, t)$ of the case $(\mathrm{II}, 1)$, N_1 is $N_+ = t\sqrt{a} + r\sqrt{c}$, and by item 1 the term of index 1 is $st+cr$; so $2\sqrt{a}\,(cr+st) = N_+\beta - (c-a)/(N_+\beta)$. The function $X \mapsto X - (c-a)/X$ is strictly increasing on $X > 0$; hence $z_{m'}^{(a,c)} > cr+st$ if $N_1\beta^{m'} > N_+\beta$. The four values of N_1 are $\sqrt{c} \pm \sqrt{a}$ and $r\sqrt{c} \pm t\sqrt{a}$. As $cr^2 - at^2 = c-a > 0$, we have $r\sqrt{c} - t\sqrt{a} = (c-a)/N_+$ and $t\sqrt{a} < r\sqrt{c}$, so $N_+ < 2r\sqrt{c}$; and $\sqrt{c} - \sqrt{a} = (c-a)/(\sqrt{c}+\sqrt{a}) \geq (c-a)/N_+$, as $\sqrt{c}+\sqrt{a} \leq N_+$. So $N_1 \geq (c-a)/N_+$ in every case. By Lemma 2.4, $c = a+b+2r$ or $c > 4ab \geq a+b$, so $c \geq a+b+1$ and $a(c-a) \geq a(b+1) \geq ab+1 = r^2$. With $\beta^2 > 4ac$ (as $s > \sqrt{ac}$) and $m' \geq 3$,

$$\frac{N_+}{N_1} \leq \frac{N_+^2}{c-a} < \frac{4r^2c}{c-a} \leq 4ac < \beta^2 \leq \beta^{m'-1},$$

that is, $N_1\beta^{m'} > N_+\beta$. Hence $z_{m'}^{(a,c)} > cr+st$, and $cr+st \geq c+1$ as $r, s, t \geq 1$. If moreover $z_{m'}^{(a,c)} = z_{n'}^{(b,c)} = z$, then $|z| \geq c+1$ and $|z| \neq cr+st$, so d is an integer with $d > c$ and $d \neq d_+(a,b,c)$, and d is an irregular extension of T by the first assertion. The bound $m' \geq 3$ cannot be lowered: in the case $(\mathrm{II}, 1)$, $z_1^{(a,c)} = cr+st$, and for $T = \{1, 3, 8\}$ in the case $(\mathrm{I}, -1)$, $z_2^{(a,c)} = 31 = cr+st$.

Item 4. Let $d = \big((x_{m'}^{(a,c)})^2 - 1\big)/a$ be an integer with $d > c$, $d \neq d_+(a,b,c)$ and $bd+1$ a square. Then $ad+1 = \big(x_{m'}^{(a,c)}\big)^2$, and the norm identity gives $a\big(z_{m'}^{(a,c)}\big)^2 = c\big(x_{m'}^{(a,c)}\big)^2 + a - c = a(cd+1)$, so $cd+1 = \big(z_{m'}^{(a,c)}\big)^2$. Thus $\{a, b, c, d\}$ is a Diophantine quadruple and d is an irregular extension of T. Conversely, if an irregular extension d of T gives a representation of this case with index m', then $z = \sqrt{cd+1} = z_{m'}^{(a,c)}$, so $c\big(x_{m'}^{(a,c)}\big)^2 = az^2 - a + c = c(ad+1)$, and $d = \big((x_{m'}^{(a,c)})^2 - 1\big)/a$ is an integer with $d > c$, $d \neq d_+(a,b,c)$ and $bd+1$ a square, since $\{a, b, c, d\}$ is a Diophantine quadruple. □

When a test fails. Each test is a sufficient condition: when its conditions fail, nothing is concluded about the case. One then tries, in this order: other values of the free choices of the test (the certificate data and m_{DP}, U, Lo, Hi and μ^*, the width w, m_ν^*, L_2, s_*, the moduli p); a splitting of a range into several certified ranges (item 2 of Lemma 9.6); and, for a triple of degree at least two with $c \geq 10^{30}$, the generic test (Corollary 9.16), whose windows contain those of every family. What remains is a finite list of candidates: the elements of $\mathcal{T}$ and the pinning candidates that pass the root test, the pairs (h, q) of (II.1), the points of the conic, and the index

m' of A2. Each gives indices (m', n') of one case, decided by Lemma 9.36: by residues (item 1), by an enclosure of the linear form (item 2), and by items 3 and 4 when the indices are small enough for exact arithmetic. For A2, the analogue of item 2 is the comparison of an enclosure of $x_\varepsilon - m'$ with the bounds of Lemma 9.13. A candidate that survives item 1 for every modulus tried and item 2 at every precision tried is decided by item 1 with enough moduli: if $z_{m'}^{(a,c)}$ and $z_{n'}^{(b,c)}$ agree modulo pairwise coprime moduli p whose product exceeds $x_1\sqrt{c/a}\,\beta^{m'} + y_2\sqrt{c/b}\,\gamma^{n'}$, they are equal. Then, if $m' \geq 3$, item 3 shows that $\big((z_{m'}^{(a,c)})^2 - 1\big)/c$ is an irregular extension of T, which would be a counterexample to the conjecture that every Diophantine quadruple is regular; for $m' \leq 2$ the terms are small enough for exact arithmetic. The comparison of the product of the moduli with that bound is made through logarithms, for instance as

$$\sum_p \log p > \log 2 + \max\big\{\log\big(x_1\sqrt{c/a}\big) + m'\log\beta,\ \log\big(y_2\sqrt{c/b}\big) + n'\log\gamma\big\},$$

with the enclosures of Lemma 9.7. The right-hand side is about $m'\log\beta$, as the Pell terms have about $m'\log_{10}\beta$ digits; for m' of order 10^{12} many moduli may therefore be needed, but they are treated one at a time, each by repeated squaring of M_a and M_b modulo p, and the first disagreement ends the check.

The notation of a quadruple, the descent, the degree and the two types of representation of z are fixed in Section 2.1. The paragraphs below list the letters with one meaning throughout the paper, the letters fixed in one part of it, the fonts and devices, and the constants shared by several results.

Global letters. The following letters keep one meaning throughout the paper: the elements $a < b < c < d$ of a Diophantine quadruple, the roots r, s, t, x, y, z with $ab+1 = r^2$, $ac+1 = s^2$, $bc+1 = t^2$, $ad+1 = x^2$, $bd+1 = y^2$, $cd+1 = z^2$, the units α, β, γ and the Pell sequences $z_j^{(a,c)}$, $z_j^{(b,c)}$ (Section 2.1), which are indexed from 0, so that the seeds (x_1, z_1) and (y_2, z_2) are their terms of index 0 (the subscripts of the seeds name the equation, not an index); a representation (λ, m, n) of z, of type I or type II, with m', n', $\Delta = m - n$ (Section 2.1) and $S = m + n$ (Definition 4.2); the triple $T = \{a, b, c\}$, its degree $\deg T$, its iterated descents $\partial^i T$, its descent partner e (Section 2.1, Definition 2.8) and the descent partner f of its descent (Definition 2.8); the maps $d_\pm$ (Definition 2.2); the height h (Definition 5.1); the real quartic field $\mathcal{K} = \mathbb{Q}(\sqrt{ab}, \sqrt{ac}) = \mathbb{Q}(\sqrt{ac}, \sqrt{bc})$; $\ell_c = \log c$; $u = \sqrt{ae+1}$; the type II quantities N, M, H and $J = H/c$ (Definition 2.20); the constant $\kappa = (\sqrt{17}-1)/2$, the positive root of $X^2 + X = 4$; the index caps $Q_0(c)$, $Q_{\mathrm{I}}(c)$, $Q_{\mathrm{II}}(c)$ and the a priori constant $K_{\mathrm{a}} = 4.436 \cdot 10^{13}$ (Definition 4.1). The three Pell equations of a quadruple are indexed by their pairs: $\{a, b\}$ goes with α and the index l, $\{a, c\}$ with β and m, $\{b, c\}$ with γ and n. Inside a proof, and in the statements of quoted results and of self-contained auxiliary lemmas, a letter may also be used locally; it is then introduced where it appears and has no meaning outside that result. This happens chiefly in the quoted results Lemma 2.12 and Theorem 5.4, in the auxiliary results Lemma 2.21, Lemma 2.18 and Lemma 2.22, in the proofs of the index caps (Section 5, and the analogous propositions of Sections 6 and 7), and in the proofs of the gap and exclusion lemmas of the type I class bound (Section 7.4).

Letters fixed in one part of the paper. The following letters have a fixed meaning only in the part named; elsewhere they may be used locally. In the statements on chains and slots: a pair $\{p, q\}$, its root $\varrho = \sqrt{pq+1}$, a sign σ, its chain (x_j) and the slots ν (Definition 3.2), the chain constants D, η, χ, χ', τ, ϑ, G, ψ_u, the floors F_ν, the functions Ω_ν and the integers m_ν (Definition 3.8) and the second-order defect k of a type I representation (Definition 2.20). At slot three, in type II: E, E', P (Definition 3.8), $K = N - JP$ (Definition 2.20), $\Gamma = 4aE + 4P$, and on the ellipse k_2, g_2, ι, L. For a high triple, in type II: $v = 2af + 1$ and $L = 2n(n+1) - (4af+1)\Delta$

(Definition 2.20). In degree one: the sign $\varepsilon = (e - a - b)/(2r)$, the residue Θ and the line of the triple (Definition 3.6). In the linear forms, the parameters of the quoted theorems (Mignotte and Voutier's ρ_{MV}, χ_{MV}, K_0, ℓ, R_i, S_i, T_i; Laurent's ρ_ν, μ, for a denominator ν). In the decision tests: $\xi = \log\beta/\log\gamma$ and Dujella and Pethő's μ, A, B (Definition 9.1). Any other letter is defined where it is used and may have other meanings elsewhere.

Fonts and devices. A prime marks the index of a Pell term (m', n') or the companion with the other sign (E', χ'). A hat marks a rescaled quantity, such as $\hat{r} = r/\sqrt{ab}$, or an approximation or bound for the quantity without the hat, such as $\hat{L}_0$ and $\hat{\Xi}$. Roman subscripts name roles (Q_{I}, K_{a}); other subscripts are parameters. Calligraphic capitals denote finite sets of candidates, the field $\mathcal{K}$, the field $\mathcal{L}$ of a quoted theorem, and their degree $\mathcal{D}$, the regularity defect $\mathcal{R}$ (Definition 2.5), and, inside proofs, some auxiliary integers and functions. The norm from a number field to $\mathbb{Q}$ is written N or $\mathrm{N}_{\mathcal{K}/\mathbb{Q}}$. Parameters of a quoted theorem that meet a global letter carry a subscript naming the source (Λ_M, B_M in Matveev's theorem; Λ_{MV}, R_{MV}, S_{MV}, T_{MV}, ρ_{MV}, χ_{MV} in Mignotte and Voutier's; σ_L, $\lambda_{L,\nu}$, $h_{L,\nu}$ in Laurent's); letters of a source that already carry an index are kept.

Terms. An Euler triple (Section 2.1) is what Cipu, Fujita and Miyazaki [4] and Dujella [7] call a regular triple; "regular" is used here only for extensions and quadruples (Definition 2.3). Representations of type I and of type II are the even–even and the odd–odd representations of Section 2.1.

Constants. The headline constants and the constants of the family windows appear as numerals in their statements. The constants shared by several results have a role symbol and one record of their values: κ in this section; K_{a}, the a priori cap $Q_0(c)$ and the index caps $Q_{\mathrm{I}}(c)$ and $Q_{\mathrm{II}}(c)$ in Definition 4.1.

Code and data availability

The programs and data of every computation that a proof in this paper relies on are deposited at Zenodo [DS], https://doi.org/10.5281/zenodo.24366765. The deposit lists each computational claim associated to a result, its script, the output of a reference run and the stored output of the original run.

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