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A COUNTEREXAMPLE TO A STABILITY QUESTION FOR UNRAMIFIED MULTIPLICITY ONE SYMMETRIC PAIRS

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Abstract. We construct a split reductive symmetric pair over $\mathbb{Q}_{17}$ with multiplicity at most one for every irreducible unramified representation and which is not a stable symmetric pair.

1. Introduction

Let k be a p-adic local field of characteristic zero with ring of integers $\mathfrak{o}$. For a symmetric pair (G, H, θ), fix a hyperspecial compact subgroup $K \subset G(k)$. An irreducible smooth complex representation π is unramified with respect to K if $\pi^K \neq 0$. We say that the pair has *unramified multiplicity one* if

$$\dim \mathrm{Hom}_{G(k)}\big(\pi, C^\infty(H(k)\backslash G(k))\big) \leq 1$$

for every such π. Here $G(k)$ acts by right translation.

Aizenbud and Gourevitch call a symmetric pair satisfying Definition 3.2 a good pair [1, Definition 7.1.6]. We use Carmeli's term *stable* for the same property [2]. We consider the following question.

$$\text{Does unramified multiplicity one imply stability of a symmetric pair?} \tag{1.1}$$

We take

$$k = \mathbb{Q}_{17}, \qquad (n_1, \dots, n_6) = (4, 4, 8, 8, 4, 8), \qquad h_j = n_j/2.$$

Let $T = \mathbb{G}_m^6$, with coordinates $t = (s_1, t_1, s_2, t_2, u, v)$, and put

$$\tau(t) = (t_1, s_1, t_2, s_2, u, v).$$

Write $p_i : T \to \mathbb{G}_m$ for the i-th coordinate projection. Define $\chi_j = c_j^{h_j}$ using the characters

$$c_1 = p_1p_2p_3p_6, \qquad c_2 = p_3, \tag{1.2}$$

$$c_3 = p_1p_6, \qquad c_4 = p_1, \tag{1.3}$$

$$c_5 = p_1p_3p_5p_6, \qquad c_6 = p_1p_3p_5. \tag{1.4}$$

We introduce the group G as

$$G = \left\{ ((A_j, B_j)_j, t) \in \prod_{j=1}^{6} \mathrm{GL}_{n_j}^2 \times T \;\middle|\; \begin{matrix} \det A_j = \chi_j(t), \\ \det B_j = \chi_j(\tau t) \quad (1 \leq j \leq 6) \end{matrix} \right\}. \tag{1.5}$$

The involution

$$\theta((A_j, B_j)_j, t) = ((B_j, A_j)_j, \tau t) \tag{1.6}$$

has fixed subgroup $H = G^\theta$.

Theorem 1.1. *The symmetric pair defined by* (1.5)–(1.6) *has unramified multiplicity one (Proposition 2.3) and fails stability (Proposition 3.6).*

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2. Unramified multiplicity one

Write $S = T^{\tau} = \{(a, a, b, b, u, v)\}$ and let $p : G \to T$ be the torus projection. For a unit x in a coefficient ring set $\delta_n(x) = \mathrm{diag}(x, 1, \dots, 1)$. The homomorphism

$$\iota(t) = \big((\delta_{n_j}(\chi_j(t)), \delta_{n_j}(\chi_j(\tau t)))_j, t\big)$$

splits p over $\mathbb{Z}_{17}$ and restricts to a section $S \to H$. The kernels of $G \to T$ and $H \to S$ are respectively $\prod_j \mathrm{SL}_{n_j}^2$ and its blockwise diagonal copy $\prod_j \mathrm{SL}_{n_j}$. These descriptions give smooth group schemes with connected reductive geometric fibers. Indeed, the semidirect product descriptions give smoothness and connectedness, and in each fiber a connected normal unipotent subgroup maps trivially to the torus and hence lies in the semisimple kernel. The diagonal matrices give a split maximal torus in each group, since in each matrix factor the diagonal torus of SL_{n_j} already has diagonal centralizer. In particular G is split reductive and $K = G(\mathbb{Z}_{17})$ is hyperspecial.

For context we recall the general criterion of Sakellaridis. If X is a homogeneous affine spherical variety for a connected split reductive group L, write A^* for the unramified character torus of a maximal split torus of L, and use A_X^*, $\delta_{(X)}$, W, and W_X for the corresponding objects of [4, §8]. The restriction map in that criterion is

$$\mathbb{C}[A^*]^W \longrightarrow \mathbb{C}[\delta_{(X)}^{1/2} A_X^*]^{W_X}.$$

Theorem 2.1 ([4, Corollary 8.0.4])**.** *Assume the standing hypotheses of* [4, §§1.1, 1.7, 8]*, including its integral orbit axioms and the single rational open Borel orbit hypothesis, and assume Statements 6.3.1 and 7.1.5 of loc. cit. for the spherical roots and virtual weighted colors. If the restriction map is surjective, then every irreducible unramified representation π of $L(k)$ satisfies*

$$\dim \mathrm{Hom}_{L(k)}\big(\pi, C^{\infty}(X(k))\big) \le 1.$$

Proof. Let K_L be the hyperspecial subgroup and $\mathcal{H}$ its spherical Hecke algebra. By [4, Corollary 8.0.4(1)], $C_c^{\infty}(X(k))^{K_L}$ is cyclic over $\mathcal{H}$. Pair it with $C^{\infty}(X(k))^{K_L}$ by integration against an invariant measure. This pairing separates functions, since the compact open K_L-orbits have positive volume. If f is a simultaneous eigenfunction with character ω, its pairing with h applied to a cyclic generator is its pairing with the generator multiplied by $\omega(h^{\vee})$, where $h^{\vee}(g) = h(g^{-1})$. Thus each eigenspace has dimension at most one. An $L(k)$-equivariant map out of π is determined by the image of a nonzero spherical vector, which spans π^{K_L} and must map into the eigenspace for its Hecke character. □

For the pair (1.5)–(1.6) we give a direct proof. It rests on a choice of double-coset representatives for which averaging a spherical vector determines every $H(k)$-invariant functional from its value on that vector.

Lemma 2.2. *Every double coset in $H(k)\backslash G(k)/K$ has a representative g_0 such that $Kg_0K \subset H(k)g_0K$.*

Proof. Put $\varpi = 17$ and let ν be the valuation with $\nu(\varpi) = 1$. For $t = (s_1, t_1, s_2, t_2, u, v)$ the class of t modulo $S(k)T(\mathfrak{o})$ is determined by the two integers $x = \nu(t_1/s_1)$ and $y = \nu(t_2/s_2)$. Indeed, division by an element of $S(k)$ allows us to set s_1, s_2, u, v equal to 1, and $T(\mathfrak{o})$ removes the remaining unit factors. Thus, using $\iota(S(k))$ on the left and $\iota(T(\mathfrak{o}))$ on the right, we may take the torus component to be

$$t = (1, \varpi^x, 1, \varpi^y, 1, \varpi^e), \qquad e = \begin{cases} 1 & x \text{ odd and } y \text{ even}, \\ 0 & \text{otherwise}. \end{cases}$$

The last coordinate may be chosen freely because S contains the last coordinate subtorus. The valuations of the six characters in (1.2)–(1.4) are

$$(\nu c_j(t))_j = (x+e, 0, e, 0, e, 0),$$
$$(\nu c_j(\tau t))_j = (x+y+e, y, x+e, x, x+y+e, x+y).$$

In each column at least one entry is even. For x even this holds throughout the first row. For x, y both odd only its first entry is odd, and $x+y$ is even. In the remaining case $e=1$, the only odd entries of the first row are in columns 3 and 5, where the second row has even entries.

Write the matrix components as (A_j, B_j). Since t has pure uniformizer powers as coordinates and $\chi_j = c_j^{n_j/2}$, at least one matrix in each pair has determinant $\lambda_j^{n_j}$ for some $\lambda_j \in k^\times$. If that matrix is M_j, multiplication on the left by $\lambda_j M_j^{-1}$ on both blocks is allowed in the diagonal $\mathrm{SL}_{n_j}(k)$ and makes M_j scalar. We can also make the other block diagonal. Indeed, elementary divisors over $\mathfrak{o}$ give $P, Q \in \mathrm{GL}_{n_j}(\mathfrak{o})$ with PMQ diagonal for the other block M. Multiplying P on the left and Q on the right by diagonal unit matrices makes both determinants one and preserves diagonality. Apply P on the left of both blocks, then P^{-1} on the right of the scalar block and Q on the right of the other. These operations belong to $H(k)$ and K, respectively. Denote the resulting representative by g_0.

The subgroup $\iota(T(\mathfrak{o}))$ commutes with g_0, since all the matrices are diagonal. As $K = (\prod_j \mathrm{SL}_{n_j}(\mathfrak{o})^2)\iota(T(\mathfrak{o}))$, it remains to move a pair $(U, V) \in \mathrm{SL}_{n_j}(\mathfrak{o})^2$ past the jth block (A, B). If A is scalar, then

$$(UA, VB) = (V, V)(A, B)(V^{-1}U, I).$$

If B is scalar, then

$$(UA, VB) = (U, U)(A, B)(I, U^{-1}V).$$

In each identity the left factor belongs to the diagonal subgroup and the right factor is integral of determinant one. Taking the product over j proves the inclusion. □

Proposition 2.3. *The symmetric pair (G, H) defined by (1.5)–(1.6) has unramified multiplicity one with respect to $K = G(\mathbb{Z}_{17})$.*

Proof. Let π be irreducible and K-spherical. By the spherical representation theorem, $\pi^K = \mathbb{C}w$ for any nonzero spherical vector w [3, §6]. Fix $\ell \in \mathrm{Hom}_{H(k)}(\pi, \mathbb{C})$. For g_0 as in Lemma 2.2, the function $g \mapsto \ell(\pi(g)w)$ is constant on Kg_0K, since it is left $H(k)$-invariant and right K-invariant. Give K volume one. Then

$$\ell(\pi(g_0)w) = \ell\left(\int_K \pi(k'g_0)w\, dk'\right) = a_{g_0}\ell(w),$$

where the averaged vector lies in π^K and equals $a_{g_0}w$. The integral is a finite sum after passage to a sufficiently small open subgroup of K. If $\ell(w) = 0$, the matrix coefficient therefore vanishes on every $H(k)g_0K$. The $G(k)$-translates of w span π, so $\ell = 0$. Evaluation at w thus embeds $\mathrm{Hom}_{H(k)}(\pi, \mathbb{C})$ in $\mathbb{C}$. Finally, evaluation of a $G(k)$-map at the identity coset identifies the Hom space in the introduction with $\mathrm{Hom}_{H(k)}(\pi, \mathbb{C})$. The inverse sends ℓ to $v \mapsto [H(k)g \mapsto \ell(\pi(g)v)]$. □

2.1. **Verification of Sakellaridis's criterion.**

Proposition 2.4. *For $k = \mathbb{Q}_{17}$ and the integral model of $X = H\backslash G$ over $\mathfrak{o} = \mathbb{Z}_{17}$, all the hypotheses of Theorem 2.1 hold. In particular $\delta_{(X)} = 1$, and the restriction*

$$\mathbb{C}[A^*]^W \longrightarrow \mathbb{C}[A_X^*]^{W_X}$$

is surjective. Consequently Theorem 2.1 also gives Proposition 2.3.

Proof. The open orbit and its integral model. Take the Borel B lower triangular in the first matrix of each pair and upper triangular in the second. Write A for its diagonal torus. Identify $Q = T/S$ with $\mathbb{G}_m^2$ by $z(t) = (t_1/s_1, t_2/s_2)$. Define six characters of Q by

$$(\eta_j(z))_j = (z_2, z_2, z_1, z_1, z_1z_2, z_1z_2).$$

Indeed, $c_j(\tau t)/c_j(t) = \eta_j(z(t))$. Over $\mathfrak{o}$ the map $g \mapsto ((A_j^{-1}B_j)_j, z(t))$ identifies X with

$$\left\{ ((Y_j)_j, z) \in \prod_j \mathrm{GL}_{n_j} \times Q \;\middle|\; \det Y_j = \eta_j(z)^{h_j} \right\}. \tag{2.1}$$

To see this over any $\mathfrak{o}$-algebra, lift z to $t = (1, z_1, 1, z_2, 1, 1)$, set $A_j = \delta_{n_j}(\chi_j(t))$, and take $B_j = A_jY_j$. This gives a section, and two lifts differ by a unique left multiplier in H. In particular $X(k) = H(k)\backslash G(k)$. The model is smooth and affine, since multiplying Y_j on the left by $\delta_{n_j}(\eta_j(z)^{-h_j})$ identifies it as a scheme with $Q \times \prod_j \mathrm{SL}_{n_j}$. The derived group of G is $\prod_j \mathrm{SL}_{n_j}^2$ and is simply connected.

The open orbit through $x_0 = H$ is given by nonvanishing of all proper leading principal minors of the Y_j. Gaussian elimination factors each matrix there as lower unitriangular, diagonal, and upper unitriangular. The diagonal locus is $A_X = A/(A \cap H)$: the section just constructed takes diagonal tuples to A. Thus the open orbit is a product of A_X with the two families of unitriangular groups over $\mathfrak{o}$. It is smooth, has homogeneous special fiber, and its k-points are one $B(k)$-orbit. The group G and the stabilizer H are unimodular, so $X(k)$ has an invariant measure. The open stabilizer $B \cap H = A \cap H$ is a split torus, and δ_B is trivial on it because the two triangular factors have opposite roots. This gives a $B(k)$-invariant measure on the open orbit. Every simple minimal parabolic enlarges that orbit, so $P(X) = B$, its Levi is A, and $\delta_{(X)} = 1$.

Bruhat strata and Axiom 2.2.1. In (2.1) the B-strata are indexed by tuples $w = (w_j) \in \prod_j \mathfrak{S}_{n_j}$: the jth matrix lies in $B_{n_j}^- \dot{w}_j B_{n_j}^+$. Here $B_n^\pm$ are the triangular Borels of GL_n, and we take signed permutation representatives of determinant one. The torus Q is free to vary. Indeed, the diagonal factor used to trivialize (2.1) preserves these cells, the section ι moves between fibers, and the determinant-one triangular groups act transitively on each cell with fixed determinant. The integral cells are products of split tori and affine spaces by Bruhat elimination. Their frontiers are the usual unions of Bruhat cells, defined by the same northwest submatrix rank conditions over $\mathfrak{o}$ and in both fibers. In particular the boundaries, also for a union of two cells related by a minimal parabolic, acquire no extra special-fiber stratum: every boundary cell is flat over $\mathfrak{o}$ and its generic fiber is dense in it.

To identify the rank-one quotient, take the representative $((\dot{w}_j)_j, 1)$ of a stratum Y and enlarge one triangular factor to P_α. The stabilizer projected to $P_\alpha/\operatorname{rad}(P_\alpha) \simeq \mathrm{PGL}_2$ preserves one coordinate line: the other flag in the same block is a full coordinate flag in permutation order, and it singles out one of the two lines in the affected two-dimensional flag quotient. The image is its entire Borel. It contains the corresponding elementary unipotent matrices, acting on both flags with the permutation conjugacy, and the diagonal SL_2-torus maps onto the adjoint torus as a group scheme for the faithfully flat topology. These matrix descriptions work over $\mathfrak{o}$. Thus $YP_\alpha/\operatorname{rad}(P_\alpha)$ is a split Borel quotient of PGL_2, of type U in [4, §2.2]. Together with the boundary description this gives Axiom 2.2.1 and excludes all type-N edges.

The Cartan and Iwahori axioms. Let $\Lambda = X_*(A)$ and $\Lambda_X = X_*(A_X)$. Write $\ell_j(\xi) = \langle c_j, \xi \rangle$ for $\xi \in X_*(T)$. An element of Λ consists of an exponent vector ξ and two integer vectors a_j, b_j of length n_j in each block with

$$\sum_i a_{ji} = h_j\ell_j(\xi), \qquad \sum_i b_{ji} = h_j\ell_j(\tau\xi).$$

An element of Λ_X consists of $q \in X_*(Q)$ and integer vectors m_j with $\sum_i m_{ji} = h_j\langle \eta_j, q\rangle$. The projection of lattices is

$$\mathrm{pr}_X(a, b, \xi) = ((b_j - a_j)_j, z_*(\xi)), \tag{2.2}$$

where $z_*(\xi)$ denotes the image of ξ in $X_*(Q)$. The antidominant cone consists of the $m_{j1} \le \cdots \le m_{jn_j}$ in each block. Indeed for a symmetric variety it is the image of the ambient antidominant cone over $\mathbb{Q}$ [4, §2.1]. Here a_j is decreasing and b_j increasing in that cone. Conversely any increasing m_j lifts rationally by taking a_j constant with its prescribed sum and $b_j = a_j + m_j$.

The invariants of a K-orbit are the two valuations of z and the ordered elementary-divisor exponents of every Y_j. They determine the orbit. First normalize z to pure powers of 17 using $\iota(T(\mathfrak{o}))$. Smith normal form in block j gives integral invertible P, R with $PY_jR = D_j = \mathrm{diag}(17^{m_{ji}})_i$. At the normalized z the determinants of Y_j and D_j agree. Thus $\det P \det R = 1$, and replacing (P, R) by (EP, RE^{-1}) for a diagonal unit matrix E makes both determinants one while fixing D_j. These left and right operations on Y_j are induced by the two $\mathrm{SL}_{n_j}(\mathfrak{o})$ factors of K. Conversely the invariants are unchanged under K. Hence $X(k)/K$ is precisely the set of antidominant integral parameters just described (Axiom 2.4.1).

For Axiom 2.4.2 take any diagonal point in A_X^+, with block $D = \mathrm{diag}(d_1, \ldots, d_n)$, so that $\nu(d_i) \le \nu(d_{i+1})$. For an element of the Iwahori J write (C, E) for its blocks. They are respectively lower and upper triangular modulo 17. Gaussian elimination over $\mathfrak{o}$ gives $C^{-1} = LU$ with L integral lower triangular invertible and U upper unitriangular congruent to I modulo 17. Then $(D^{-1}UD)E$ is integral and upper triangular modulo 17, so it has a factorization $L'R$ with L' lower unitriangular and R upper triangular integral invertible. Now

$$C^{-1}DE = L(DL'D^{-1})\, D\, R.$$

The left factor is integral lower triangular invertible by the order of the valuations. Its determinant is $(\det C)^{-1}$, and $\det R = \det E$. Taking its inverse as the first block, R as the second block, and the original torus component in $T(\mathfrak{o})$ produces an element of $B(\mathfrak{o})$ with exactly the same action on the given point. Thus $xJ \subset xB(\mathfrak{o})$ for every $x \in A_X^+$.

Spherical roots and virtual weighted colors. On the Bruhat indices the ambient Weyl group acts separately by left and right permutations, since each simple step exchanges adjacent positions or adjacent values. The open index is the identity tuple. Its stabilizer is the blockwise diagonal $W_X = \prod_j \mathfrak{S}_{n_j}$, using $P(X) = B$ and [4, §2.2]. The simple reflection for positions $i, i+1$ in block j is the product of the corresponding left and right simple reflections. Their length-two path passes through the same codimension-one cell reached from the open orbit on either side. Thus the normalized spherical root is the sum of the two corresponding simple roots, of type G in the notation of [4, §6.2]. Its supporting Levi has derived group $\mathrm{SL}_2 \times \mathrm{SL}_2$. Thus [4, Theorem 6.3.2] applies. In this case the rank-one computation is exactly the two-sided group computation in [4, Example 5.4.1]: the adjacent root subgroups act by left and right multiplication on the same two-dimensional matrix block. This verifies Statement 6.3.1.

For clarity, the color in this path is the divisor of the leading i-rowed principal minor Δ_{ji} in block j. Its valuation on A_X is the cocharacter $\beta_{ji} = e_{ji} - e_{j,i+1}$, with zero torus component, where e_{ji} denotes a standard coordinate vector in the jth block. Indeed the other leading minors and the Q-coordinates are units at its generic point, and Δ_{ji} vanishes there to order one. For the order of vanishing, take $z = (1, 1)$ and the identity matrices with one two-by-two block replaced by $\left(\begin{smallmatrix} s & -1 \\ 1 & 0 \end{smallmatrix}\right)$ at positions $i, i+1$, where $\Delta_{ji} = s$ and all the other leading minors equal one. On the diagonal locus successive quotients of the leading minors are the diagonal coordinates. These evaluations give β_{ji}, and the Bruhat cells of codimension one exhaust the colors. Each color is defined over k, belongs to its single type-G root and has weight $\langle \beta_{ji}, \rho_{P(X)}\rangle = 1$, as seen by lifting β_{ji} to the right simple coroot. The definition in [4, §7.1] therefore gives, with multiplicity one,

$$\Theta = \{(e_{ji} - e_{jl}, +, 1) : 1 \le j \le 6,\ 1 \le i, l \le n_j,\ i \ne l\}.$$

The cone of color valuations is generated by the adjacent differences β_{ji}, so the positive members have $i < l$. Each simple transposition sends exactly its own positive simple coroot to a negative one. This proves both assertions of Statement 7.1.5.

Surjectivity on invariants. The coordinate rings of A^* and A_X^* are the group algebras $\mathbb{C}[\Lambda]$ and $\mathbb{C}[\Lambda_X]$. Restriction sends e^λ to $e^{\mathrm{pr}_X(\lambda)}$ by (2.2). Fix $\mu = ((m_j)_j, (x, y)) \in \Lambda_X$. Lift (x, y) to $\xi = (0, x, 0, y, 0, e)$ with e chosen as in Lemma 2.2. The two valuation rows in its proof say, for each j, that either $\ell_j(\xi)$ or $\ell_j(\tau\xi)$ is even. In the first case choose the constant vector $a_j = (\ell_j(\xi)/2)(1, \dots, 1)$ and put $b_j = a_j + m_j$. In the other case choose $b_j = (\ell_j(\tau\xi)/2)(1, \dots, 1)$ and put $a_j = b_j - m_j$. In each case the constant vector has length n_j. Since $n_j = 2h_j$ and the sum of m_j is prescribed, both determinant equations hold. These choices give a lift $\lambda \in \Lambda$ of μ with a constant vector on at least one side of every block.

The two symmetric groups of each ambient block permute a_j, b_j independently, whereas W_X permutes the entries of m_j. With one vector constant, subtraction gives a bijection of the *distinct* ambient and relative orbit points in that block, including when entries repeat. The Weyl groups fix the torus coordinates. Taking products yields

$$\sum_{\lambda' \in W\lambda} e^{\lambda'}\Big|_{A_X^*} = \sum_{\mu' \in W_X \mu} e^{\mu'}.$$

The sums on the right form a basis of $\mathbb{C}[\Lambda_X]^{W_X}$. This proves surjectivity and completes the verification of the hypotheses, independently of the multiplicity conclusion. □

3. Stable symmetric pairs

3.1. **Symmetric pairs.**

Definition 3.1. A **symmetric pair** is a triple (G, H, θ), where G and $H \subset G$ are reductive groups, θ is an involution of G, and $H = G^\theta$. We call the pair **connected** if $H\backslash G$ is connected as an algebraic variety.

For a symmetric pair (G, H, θ), define the anti-involution $\sigma(g) = \theta(g^{-1})$ and the symmetrization map

$$s : G \longrightarrow G^\sigma, \qquad s(g) = g\sigma(g).$$

For $\mathfrak{g} = \mathrm{Lie}(G)$, we write $\mathfrak{g}^\sigma = \ker(d\theta + 1)$, since $d\sigma = -d\theta$. The adjoint action of H preserves this space.

Definition 3.2. The pair (G, H, θ) is *stable* if $\sigma(H(k)gH(k)) = H(k)gH(k)$ for every $g \in G(k)$ whose double coset is closed in the k-adic topology.

3.2. **An unstable symmetric pair.** In this section, we let

$$k = \mathbb{Q}_{17}, \qquad (n_1, \dots, n_6) = (4, 4, 8, 8, 4, 8), \qquad (d_1, \dots, d_6) = (3, 3, 17, 17, 51, 51).$$

Choose roots of unity $i, \zeta \in k$ characterized by

$$i^2 = -1, \qquad i \equiv 4 \pmod{17},$$

and

$$\zeta^4 = -1, \qquad \zeta \equiv 2 \pmod{17}.$$

These roots exist uniquely by Hensel's lemma, and $\zeta^2 = i$. Write

$$\zeta_4 = i, \qquad \zeta_8 = \zeta.$$

For $N \in \{4, 8\}$ and $d \in \{3, 17, 51\}$, let

$$R(N,d) = \begin{pmatrix} 0 & 0 & \cdots & 0 & d \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

In the basis $1, \rho, \dots, \rho^{N-1}$ of $k[X]/(X^N - d)$, with ρ the class of X, this is multiplication by ρ. Its characteristic polynomial is $X^N - d$, and $R(N,d)^N = dI_N$.

Define

$$Z(N,d) = R(N,d)^{N/2}\big(I_N + R(N,d)\big)\big(I_N + \zeta_N R(N,d)\big)^{-1}.$$

Writing $R = R(N,d)$, its characteristic polynomial gives $\det R = -d$ and $\det(I_N + aR) = 1 - a^N d$. In particular both $\det(I_N + R)$ and $\det(I_N + \zeta_N R)$ equal $1 - d \neq 0$. Thus $Z(N,d)$ is defined over k and

$$\det Z(N,d) = (-d)^{N/2} = d^{N/2}. \tag{3.1}$$

The six matrices used in the construction are

$$Z_1 = Z_2 = Z(4,3), \quad Z_3 = Z_4 = Z(8,17), \quad Z_5 = Z(4,51), \quad Z_6 = Z(8,51).$$

We set

$$A_1 = \operatorname{diag}(17^2, 1, 1, 1), \qquad A_j = I_{n_j}\ (j = 2, \dots, 6), \qquad B_j = A_j Z_j,$$

and put

$$t_* = (1, 17, 1, 3, 1, 1), \qquad g_* = \big((A_j, B_j)_{j=1}^6, t_*\big).$$

Writing $\chi(t) = (\chi_j(t))_{j=1}^6$ as defined in the introduction, we obtain

$$(\det A_j)_j = (17^2, 1, 1, 1, 1, 1) = \chi(t_*),$$
$$(\det B_j)_j = (51^2, 3^2, 17^4, 17^4, 51^2, 51^4) = \chi(\tau t_*),$$

by (3.1) and the characters (1.2)–(1.4). Thus $g_* \in G(\mathbb{Q}_{17})$.

For the remainder of this section $E_j = k(\rho_j)$, where $\rho_j^{n_j} = d_j$. We first determine the centralizers that enter the double-coset calculation.

Lemma 3.3. *Each E_j/k is cyclic of degree n_j. For $d_j = 3$ it is unramified, and for $d_j = 17, 51$ it is totally ramified. The matrix Z_j has irreducible characteristic polynomial of degree n_j. Its centralizer in $\mathrm{GL}_{n_j}(k)$ is $E_j^\times$ acting by multiplication, and the determinants of its elements form $\mathrm{N}_{E_j/k}(E_j^\times)$.*

Proof. Write $(N,d) = (n_j, d_j)$, $\rho = \rho_j$, and $E = E_j$. For $(N,d) = (4,3)$, a root α of $X^4 - 3$ over $\mathbb{F}_{17}$ satisfies $\alpha^{17} = 3^4\alpha = 13\alpha$. The element 13 has order four in $\mathbb{F}_{17}^\times$, so the four roots form one Frobenius orbit. The polynomial has irreducible separable reduction, giving an unramified extension of degree four. For $d = 17$ or 51 the polynomial $X^N - d$ is Eisenstein at 17, so E/k is totally ramified of degree N and ρ is a uniformizer. Since $\zeta_N \in k$ has order N, in all cases the extension is cyclic with generator $\rho \mapsto \zeta_N \rho$.

In the power basis of E, the matrix $Z(N,d)$ represents multiplication by

$$z = \rho^{N/2}\frac{1+\rho}{1+\zeta_N\rho}.$$

The unique involution of $\mathrm{Gal}(E/k)$ sends ρ to $-\rho$. It fixes $\rho^{N/2}$ because $N/2$ is even. If it fixed z, cross-multiplication would give $2(1 - \zeta_N)\rho = 0$, which is impossible. Every nontrivial subgroup of this cyclic 2-group contains the involution, so the stabilizer of z is trivial and $k[z] = E$.

Multiplication by z therefore has irreducible minimal and characteristic polynomials of degree N. A k-linear map commuting with it is E-linear on the one-dimensional E-space E, hence multiplication by an element of E. The determinant of multiplication by any such element is its field norm. □

Lemma 3.4. *The rational double coset $H(k)g_*H(k)$ is closed in $G(k)$.*

Proof. For $g = ((A_j, B_j)_j, t)$ put $Y_j(g) = A_j^{-1}B_j$. Under left multiplication by $H(k)$ the matrices Y_j are unchanged. A right multiplier with blocks (D_j, D_j) sends Y_j to $D_j^{-1}Y_jD_j$. Let F consist of the g with $t \in t_*S(k)$ such that each $Y_j(g)$ has the characteristic polynomial of Z_j. These conditions are closed in the k-adic topology and invariant under both copies of $H(k)$. Hence F is a closed union of double cosets containing g_*.

We show that each double coset in F is relatively open. Fix $g^0 = ((A_j^0, B_j^0)_j, t^0) \in F$ and put $Y_j^0 = Y_j(g^0)$. By Lemma 3.3 its characteristic polynomial is irreducible, so any nonzero vector w_j is cyclic. For Y near Y_j^0 the matrix

$$P_j(Y) = [\, w_j,\ Yw_j,\ \ldots,\ Y^{n_j-1}w_j \,]$$

is invertible. In these cyclic bases $Y = Y_j(g)$ and Y_j^0 have the same companion matrix whenever $g \in F$. Consequently $D_j = P_j(Y_j^0)P_j(Y)^{-1}$ satisfies $D_j^{-1}Y_j^0D_j = Y$ and tends to the identity as $g \to g^0$. For g sufficiently close to g^0, Hensel's lemma gives an n_jth root of $(\det D_j)^{-1}$ in k, since $17 \nmid n_j$. Multiplying D_j by this scalar preserves the conjugacy and gives $\det D_j = 1$ in every block.

Right multiplication of g^0 by $((D_j, D_j)_j, 1) \in H(k)$ now gives the required relative matrices $Y_j(g)$. For the left multiplier take torus component $t/t^0 \in S(k)$ and common blocks $C_j = A_j(A_j^0D_j)^{-1}$. Their determinants are $\chi_j(t/t^0)$, so the multiplier belongs to $H(k)$. This gives g itself, because $B_j = A_jY_j(g)$. All double cosets in F are thus relatively open. Their complements in F are unions of relatively open double cosets, so each is also relatively closed. Since F is closed in $G(k)$, the assertion follows. □

It remains to decide whether the anti-involution preserves this closed orbit. To record the necessary norm conditions, let $m(x) = \nu(x) \bmod 2$ and let $\epsilon(x) \in \mathbb{F}_2$ record the square class of the residue of the unit $17^{-\nu(x)}x$, with 0 for a square and 1 for a nonsquare. Both are homomorphisms from $k^\times$ to the additive group $\mathbb{F}_2$.

Lemma 3.5. *Suppose $x^{N/2}$ is a norm from $E = k(\sqrt[N]{d})$ for one of the pairs (N, d) occurring in the construction. Then*

d	*necessary condition on x*
3	$m(x) = 0$
17	$\epsilon(x) = 0$
51	$\epsilon(x) + m(x) = 0.$

Proof. If $d = 3$ then $N = 4$ and the extension is unramified. The valuation of a norm is divisible by four, since all four conjugates have the same integral valuation. Thus $4 \mid 2\nu(x)$.

In the ramified cases put $h = N/2$, $d = 17D$, $r = \nu(x)$, and $x = 17^r w$ with $w \in \mathfrak{o}^\times$. Every element of $E^\times$ is $\rho^l b$ for a unit b of E, where $\rho^N = d$ and $\mathrm{N}_{E/k}(\rho) = -17D$. If $\mathrm{N}_{E/k}(\rho^l b) = x^h$, valuation gives $l = hr$. Total ramification means all conjugates of b have the same residue in $\mathbb{F}_{17}^\times$, whence

$$\big(\overline{w}\,(-\overline{D})^{-r}\big)^h = \overline{\mathrm{N}_{E/k}(b)} = \overline{b}^{\,N}.$$

In the cyclic group $\mathbb{F}_{17}^\times$ of order 16, with $N = 2h$ dividing 16, an hth power a^h is an Nth power only if a is a square. Indeed, writing a as the qth power of a generator gives $hq \equiv 2hs \pmod{16}$ for some s, so q is even. Finally -1 is a square modulo 17 and 3 is a nonsquare ($3^8 \equiv -1$). For $D = 1$ this gives $\epsilon(x) = 0$, and for $D = 3$ it gives $\epsilon(x) + m(x) = 0$. □

Proposition 3.6. *The symmetric pair defined by* (1.5)–(1.6) *is not stable.*

Proof using Carmeli's second obstruction. Put $r = s(g_*)$ and $P_j = B_j A_j^{-1} = A_j Z_j A_j^{-1}$. In our coordinates,

$$r = \big((P_j^{-1}, P_j)_j,\ t_*/\tau(t_*)\big).$$

By Lemma 3.3, Z_j has separable characteristic polynomial, so the matrix components of r are semisimple. Its torus component is semisimple as well. Hence r is a semisimple element of $s(G(k))$, and the double coset of g_* is closed by Lemma 3.4.

Let $C = Z_G(r)(k)$. Since $\theta(r) = r^{-1}$, the involution θ preserves C and r represents a cocycle for its action on C. Carmeli identifies the second obstruction with the class represented by r and proves that r is stable exactly when this class is neutral in $H^1(\theta, C)$ [2, Propositions 4.18–4.19]. Here neutrality means that

$$r = c\,\theta(c)^{-1} \qquad \text{for some } c \in C. \tag{3.2}$$

Indeed, in the convention $y^{-1}\theta(y)$ one takes $y = c^{-1}$. In $G(k)$ the equation already has the solution g_*.

We determine when it has a solution in C. Set $F_j = A_j k[Z_j] A_j^{-1} \simeq E_j$ and $\mathcal{N}_j = \mathrm{N}_{E_j/k}(E_j^\times)$. The two blocks P_j^{-1} and P_j have the same centralizer $F_j^\times$ in $\mathrm{GL}_{n_j}(k)$, and the determinant on $F_j^\times$ is the field norm (Lemma 3.3). Thus an element $c = ((U_j, V_j)_j, w)$ of C has $U_j, V_j \in F_j^\times$, with $\det U_j = \chi_j(w)$ and $\det V_j = \chi_j(\tau w)$.

The torus component of (3.2) requires $w/\tau(w) = t_*/\tau(t_*)$, or equivalently $w = t_* s$ with $s \in S(k)$. The first matrix components require $U_j V_j^{-1} = P_j^{-1}$, hence $V_j = P_j U_j$. It follows that (3.2) is solvable exactly when there exists a *single* $s \in S(k)$ such that

$$\chi_j(t_* s) \in \mathcal{N}_j \qquad (1 \le j \le 6). \tag{3.3}$$

For sufficiency, choose $U_j \in F_j^\times$ with that determinant and set $V_j = P_j U_j$. Since $\det P_j = \chi_j(\tau t_*)/\chi_j(t_*)$ and $\tau(s) = s$, we get $\det V_j = \chi_j(\tau w)$. Also $V_j U_j^{-1} = P_j$, so these choices solve the full equation in C.

Write $s = (a, a, b, b, u, v)$. The six bases of the powers $\chi_j(t_* s) = c_j(t_* s)^{n_j/2}$ in (3.3) are

$$(17a^2bv,\ b,\ av,\ a,\ abuv,\ abu).$$

Apply Lemma 3.5 in each pair of blocks. For $j = 1, 2$ the quotient of the bases gives $m(17a^2 v) = 0$, so $m(v) = 1$. For $j = 3, 4$ the quotient gives $\epsilon(v) = 0$. For $j = 5, 6$ it gives $\epsilon(v) + m(v) = 0$. These three equalities contradict one another in $\mathbb{F}_2$. Hence (3.3) has no solution and $[r] \neq 1$ in $H^1(\theta, C)$. By Carmeli's Proposition 4.19 the semisimple symmetrization r is unstable. Thus the closed double coset of g_* is moved by σ, proving the proposition. □

Proof of Theorem 1.1. Proposition 2.3 gives the multiplicity bound for every irreducible K-unramified representation of $G(k)$. Proposition 3.6 exhibits a closed rational double coset for this same pair that is moved by σ. Together they give a negative answer to (1.1). □

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