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# OBSERVATION COST OF NONEQUILIBRIUM IN NONNORMAL NETWORKS: A MATCHED EXPONENT FOR AN EXPLICIT FOUR-STATE GENERATOR

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Abstract. We consider detection of nonequilibrium from passive, causally scheduled snapshots of a stationary continuous-time Markov chain, with gaps between snapshots of at least $\tau$. False alarms must be controlled uniformly over every finite stationary detailed-balance chain observed through a deterministic labelling, with no restriction on hidden dimension, rates, stationary label law or mixing time. Aznagulov showed that for normal generators the optimal snapshot cost grows like $e^{2\gamma_{\mathrm{irr}}\tau}$, and that for nonnormal generators its exponent lies in a spectral window. For an explicit nonnormal four-state generator $Q_4$, with a real slow mode and residual damping $\beta$, we prove that $\tau^{-1}\log N^* \to 2\gamma$, where $\gamma = (701 - \sqrt{158761})/126 \in (2, \beta)$ is a relaxation rate of an explicit reversible birth–death comparator; for $Q_4$ this exponent lies strictly inside the window $[2, 2\beta]$. The comparator comes from an ordered-eigenfunction completion lemma: among $\pi$-reversible generators with a prescribed strictly monotone eigenfunction and eigenvalue, a birth–death generator maximizes the entire Dirichlet form. The upper bound uses symmetric two-lag scores whose stationary mean is nonpositive under every hidden reversible model, together with a positive-operator tilt lemma that gives the moment-generating bound $e^{32n\theta^2}$ for a deterministic number $n$ of dependent transitions, uniformly over the null class. We also show that the two-lag pair-law approximation exponent of $Q_4$ by hidden reversible models is $\gamma$, that the cost exponent $2\gamma$ holds uniformly over a four-parameter box of nonnormal deformations, and that it extends to interacting path families on $\{1,2,3,4\}^d$ built from tensor lifts. More generally, if $u$ has distinct entries and is an eigenvector of $Q$ and of its $L^2(\pi)$-adjoint with eigenvalue $-\lambda$, the exponent is $2\gamma^*$ whenever $\gamma^* < \beta$, where $\gamma^*$ is the restricted gap of the completion.

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## 1. Introduction

What controls the optimal cost of detecting nonequilibrium when relaxation and circulation do not commute? We consider a finite irreducible row generator $Q$ started in its stationary law $\pi$, with all states even under time reversal. An observer records passive snapshots of the state at causally chosen times whose gaps are at least $\tau$, and must decide whether the record could have come from an equilibrium system. The null hypothesis contains all finite stationary detailed-balance chains with deterministic labels, without bounds on dimension, rates or mixing time. The question is the exponential rate in $\tau$ of the minimal expected number of snapshots, and the target is a matching lower and upper bound.

Aznagulov [1] introduced this observation contract and null class. For an irreversible normal generator, the minimal expected record length is of order $e^{2\gamma_{\mathrm{irr}}\tau}\log(1/\alpha)$ [1, Eq. (3)], where $\gamma_{\mathrm{irr}}$ is the smallest damping among non-real eigenvalues. For general $Q$, the exponent is bracketed between twice the spectral gap of the additive reversible part and twice the slowest damping among irreversibly visible eigenvalues, the upper bound being stated when the slowest visible level is a single real eigenvalue or a single complex pair [1, Eq. (12)]. The passive causal observation contract, the unrestricted stationary hidden-equilibrium null, the causal row-relative-entropy lower-bound method and the concentration of antisymmetric Poissonized currents [1, Eq. (8)] are taken from that work.

This note determines the exponent for explicit nonnormal targets in which that window does not close. The main results are the following.

(1) An ordered-eigenfunction completion lemma (Lemma 3.1): given a stationary law and a strictly increasing mean-zero function $u$, an explicit birth–death generator $B$ with $Bu = -\lambda u$ maximizes the Dirichlet form among all reversible generators with the same eigenpair.

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(2) For an explicit nonnormal generator $Q_4$ on four states, the two-lag pair-law approximation exponent by hidden reversible models equals $\gamma$, a relaxation rate of the completion $B$ (Theorem 5.1).
(3) A positive-operator tilt lemma (Lemma 6.2): if two bounded symmetric two-lag scores have nonpositive stationary mean under *every* hidden reversible model, then their sum along one stationary dependent trajectory with a deterministic number of transitions has moment-generating function at most $e^{32n\theta^2}$, uniformly over the null class.
(4) The matched cost exponent $2\gamma$ for $Q_4$ (Theorem 7.1). For $Q_4$ the window of [1, Eq. (12)] is $[2, 2\beta]$, and $2 < 2\gamma < 2\beta$.
(5) The same exponent uniformly over a four-parameter box $|\theta| \le 10^{-3}$ of nonnormal deformations of $Q_4$ (Theorem 8.1), and for interacting path families on $\{1,2,3,4\}^d$ obtained by tensor lifts (Theorem 9.1).
(6) For a class of generators with a two-sided slow eigenvector $Qu = Q^\sharp u = -\lambda u$, the exponent is $2\gamma^*$ whenever $\gamma^* < \beta$, where $\gamma^*$ is the restricted gap of the completion; the lower bound $2\min\{\beta, \gamma^*\}$ holds throughout the class, and this expression is not the exponent in general when $\gamma^* \ge \beta$ (Theorem 10.1).

The scores in (3) are symmetric under endpoint exchange and generally nonzero on diagonal pairs, so they are outside the antisymmetric-score hypothesis of [1, Eq. (8)]; the lemma uses the quantifier over all hidden stationary laws, which permits reversible changes of stationary law in its proof.

**Scope.** The exponents are proved for the displayed targets only: the four-state generator $Q_4$, the box around it, the path families built from it, and the class of Theorem 10.1 in the regime $\gamma^* < \beta$. No general nonnormal spectral formula, sharp prefactors, or optimal dependence on $\alpha$ is asserted. For the path families, no simultaneous limit in $d$ and $\tau$ is claimed, the history-comparison constant depends on $d$, and general noncommuting interactions and unbounded degree are not covered. Our comparison with earlier work is limited to the works cited in Section 11.

## 2. Setting

All rates use one fixed physical time unit, and logarithms are natural. For a probability vector $\pi$ with positive entries, $\langle f, g\rangle_\pi = \sum_i \pi_i f_i g_i$ and $\|f\|_\pi^2 = \langle f, f\rangle_\pi$; $D_\pi = \mathrm{diag}(\pi)$, and $\mathbf{1}$ is the constant vector. For a matrix, $\|\cdot\|_{\max}$ is the largest absolute entry.

**Definition 2.1** (Observation contract and null class)**.** Let $Q$ be a finite irreducible reciprocal row generator with stationary law $\pi$, started from $\pi$. A *passive causal procedure with minimum gap* $\tau$ records the state at an initial time and at subsequent times whose gaps are at least $\tau$; each gap, the decision to stop, and the final decision may depend on the past record and on private randomness. Let $N \in \{0, 1, 2, \ldots\} \cup \{\infty\}$ be the number of sampled transitions before stopping, with $N = \infty$ if the procedure never stops. The *null class* $\mathcal{H}$ consists of all observations $\ell(Z_t)$ of finite stationary continuous-time Markov chains $Z$ satisfying detailed balance, through deterministic label maps $\ell$ into the observed state space; the hidden dimension, rates, stationary label marginal and mixing time are unrestricted, and reducible stationary models are permitted. For $0 < \alpha < 1/2$, $N^*(Q, \tau, \alpha)$ is the infimum of $\mathbb{E}_Q N$ over passive causal procedures with minimum gap $\tau$ whose probability of stopping and rejecting equilibrium is at most $\alpha$ under every member of $\mathcal{H}$ and at least $1 - \alpha$ under $Q$, with $\inf \emptyset = +\infty$.

Counting the initial snapshot as well gives $N + 1$ observations; this additive change does not affect any logarithmic exponent below.

*Remark* 2.2 (Comparison with the contract of [1]). In [1], $N^*(Q, \tau_0, \alpha)$ is the least expected number $\mathbb{E}_Q N$ of observed transitions over tests with false-positive probability at most $\alpha$ for every hidden equilibrium model and false-negative probability at most $\alpha$ under $Q$; gaps are at least $\tau_0$, are chosen from the past record and private randomness, and the process starts in stationarity. Definition 2.1 counts the same quantity under the same two error constraints. The requirement "rejects with probability at least $1 - \alpha$ under $Q$" coincides with "false-negative probability at most $\alpha$" for every procedure with $\mathbb{E}_Q N < \infty$, which are the only procedures that affect the infimum when it is finite. Allowing reducible stationary hidden models can only enlarge $\mathcal{H}$, hence only increase $N^*$. Every lower bound below uses an irreducible hidden

model, and every upper bound controls the enlarged class, so all exponents below hold for either reading of the null class. Our results are stated for $0 < \alpha < 1/2$; [1, Eq. (3)] is stated for $0 < \alpha \le 1/4$.

We use the following elementary bound for relative entropy: for probability vectors $p, r$ with $r > 0$,

$$D_{\mathrm{KL}}(p\|r) \le \sum_j \frac{(p_j - r_j)^2}{r_j}. \tag{1}$$

## 3. The ordered-eigenfunction completion lemma

**Lemma 3.1** (Ordered completion)**.** *Let $n \ge 3$, $\pi_i > 0$, $\sum_i \pi_i = 1$, and $u_1 < \cdots < u_n$ with $\sum_i \pi_i u_i = 0$. Fix $\lambda > 0$. Define a reversible birth–death generator $B$ by its conductances*

$$c_k = \pi_k B_{k,k+1} = \pi_{k+1} B_{k+1,k} = -\lambda \frac{\sum_{i \le k} \pi_i u_i}{u_{k+1} - u_k},$$

*with no other off-diagonal edges and rows summing to zero. Then $c_k > 0$, $Bu = -\lambda u$, and every $\pi$-reversible generator $R$ satisfying $Ru = -\lambda u$ obeys*

$$\langle v, -Rv\rangle_\pi \le \langle v, -Bv\rangle_\pi \quad \text{for every real } v.$$

*Thus $B$ maximizes the entire Dirichlet form within this constrained class, and in particular maximizes the gap restricted to $\{\mathbf{1}, u\}^{\perp_\pi}$.*

*Proof.* Every proper prefix $S_k = \sum_{i \le k} \pi_i u_i$ is negative, since the weighted mean on a strict lower segment is below the full mean zero. Hence $c_k > 0$. Taking $S_0 = S_n = 0$, direct subtraction gives

$$(-Bu)_i = \frac{-\lambda S_{i-1} + \lambda S_i}{\pi_i} = \lambda u_i.$$

For any $v$, choose $z$ up to a constant by

$$z_{k+1} - z_k = \frac{(v_{k+1} - v_k)^2}{u_{k+1} - u_k}.$$

Weighted Cauchy–Schwarz on consecutive increments yields

$$(v_j - v_i)^2 \le (u_j - u_i)(z_j - z_i), \qquad i < j.$$

With $d_{ij} = \pi_i R_{ij} = \pi_j R_{ji} \ge 0$, reversibility and summation by parts imply

$$\begin{aligned}
\langle v, -Rv\rangle_\pi &= \sum_{i<j} d_{ij}(v_i - v_j)^2 \\
&\le \sum_{i<j} d_{ij}(u_j - u_i)(z_j - z_i) \\
&= \lambda \langle z, u\rangle_\pi = -\lambda \sum_{k<n} S_k (z_{k+1} - z_k) \\
&= \sum_{k<n} c_k (v_{k+1} - v_k)^2 = \langle v, -Bv\rangle_\pi.
\end{aligned}$$

Both operators preserve the stated orthogonal complement, so the restricted spectral conclusion follows from the Rayleigh variational formula. □

## 4. The four-state generator $Q_4$ and its comparator

For uniform $\pi$ on four states, let $u = (-10, -1, 2, 9)^\top$ and

$$Q_4 = \frac{1}{383160} \begin{pmatrix} -357728 & 295735 & 1126 & 60867 \\ 160327 & -767390 & 606361 & 702 \\ 194566 & 238825 & -759812 & 326421 \\ 2835 & 232830 & 152325 & -387990 \end{pmatrix}.$$

All off-diagonal rates are positive, $Q_4 u = Q_4^\top u = -u$, and its remaining eigenvalues are

$$-\beta \pm i\omega, \qquad \beta = \frac{254}{103}, \quad \omega^2 = \frac{3887}{424360}.$$

The comparator $B$ of Lemma 3.1 (with $\lambda = 1$) has consecutive rates $10/9, 11/3, 9/7$. These rational identities, and those below, can be checked by exact arithmetic.

Put

$$\Pi = \tfrac{1}{4}\mathbf{1}\mathbf{1}^\top, \qquad U = \frac{uu^\top}{186}, \qquad F = I - \Pi - U.$$

The identities $Q_4 u = Q_4^\top u = -u$, $Q_4\mathbf{1} = Q_4^\top \mathbf{1} = 0$, and

$$[(Q_4 + \beta I)^2 + \omega^2 I]F = 0$$

give the reducing decomposition

$$e^{tQ_4} = \Pi + e^{-t}U + e^{-\beta t}\left[\cos(\omega t)F + \frac{\sin(\omega t)}{\omega}(Q_4 + \beta I)F\right]. \tag{2}$$

For the symmetric path generator $B$, direct calculation gives

$$\det(xI + B) = x(x-1)\left(x^2 - \frac{701}{63}x + \frac{440}{21}\right).$$

Its relaxation rates are $0, 1, \gamma, \Gamma$, where

$$\gamma = \frac{701 - \sqrt{158761}}{126}, \qquad \Gamma = \frac{701 + \sqrt{158761}}{126}, \qquad 2 < \gamma < \beta.$$

**Proposition 4.1** (No same-state completion reaches $\beta$)**.** *For*

$$v = (-32, 37, 34, -39)^\top, \qquad z = (0, 529, 532, 9053/7)^\top,$$

*one has $v \perp \mathbf{1}, u$ and the exact certificate*

$$\frac{\langle z, u\rangle_\pi}{\langle v, v\rangle_\pi} = \frac{42611}{17745} < \beta, \qquad \beta - \frac{42611}{17745} = \frac{118297}{1827735} > 0.$$

*Consequently every same-state generator $R$ reversible with respect to the same uniform $\pi$ and retaining $Ru = -u$ has restricted gap at most $42611/17745$, below the residual damping of $Q_4$.*

The proposition concerns generators on the same four states. The hidden models in the remaining sections are unrestricted, and the proposition is not used in the statistical results.

## 5. Two-lag approximation by unrestricted hidden equilibrium models

For any finite stationary detailed-balance hidden CTMC $\mathcal{R}$ and deterministic label map into $\{1, 2, 3, 4\}$, let

$$J_{\mathcal{R}}(t)_{ij} = \Pr\{\ell(Z_0) = i, \ell(Z_t) = j\}.$$

The stationary label marginal need not equal $\pi$. Define

$$d_2(\tau) = \inf_{\mathcal{R}} \max_{k=1,2} \left\| J_{\mathcal{R}}(k\tau) - D_\pi e^{k\tau Q_4} \right\|_{\max}.$$

The same model supplies both lags, but its dimension, rates, stationary law and labeling may change with $\tau$. Reducible stationary models are permitted.

**Theorem 5.1** (Pair-approximation exponent)**.** *For this fixed target and null class,*

$$\lim_{\tau\to\infty} -\frac{1}{\tau}\log d_2(\tau) = \gamma.$$

This statement concerns joint pair expectations, not statistical detection on a dependent trajectory; the statistical results are in Sections 6–7.

*Proof. Upper approximation bound.* The identity-labeled generator $B$ is admissible. It has the same projectors $\Pi, U$ as $Q_4$ and residual decay at rates $\gamma, \Gamma$. Since $\beta > \gamma$, their finite spectral decompositions give

$$\|e^{tQ_4} - e^{tB}\|_{\max} \le C_0 e^{-\gamma t}, \qquad d_2(\tau) \le C_0' e^{-\gamma\tau}. \tag{3}$$

*A fixed pointwise certificate.* Choose a real vector $w$ with $Bw = -\gamma w$ and $\|w\|_\pi = 1$. The distinct eigenvalues and symmetry of $B$ imply $w \perp_\pi \mathbf{1}, u$ and $Fw = w$. Define $\psi_1 = 0$ and

$$\psi_{k+1} - \psi_k = \frac{(w_{k+1} - w_k)^2}{u_{k+1} - u_k}.$$

Weighted Cauchy–Schwarz gives

$$(w_i - w_j)^2 \le (u_i - u_j)(\psi_i - \psi_j) \quad \text{for every } i, j. \tag{4}$$

Using $-Bu = u$, consecutive-edge equality and $-Bw = \gamma w$ gives

$$\langle \psi, u\rangle_\pi = \langle \psi, -Bu\rangle_\pi = \sum_{k=1}^{3} c_k(\psi_{k+1} - \psi_k)(u_{k+1} - u_k) = \langle w, -Bw\rangle_\pi = \gamma.$$

*Uniform consequences of a putative faster approximation.* Fix $\gamma < b < \beta$ and suppose an unbounded sequence $\tau$ admits hidden comparators with pair error $\delta_\tau \le e^{-b\tau}$. On stationary support write their generators as $-H_\tau$ and their stationary laws as $\mu_\tau$. Zero-mass states can be discarded; detailed balance makes $H_\tau$ nonnegative and self-adjoint on $L^2(\mu_\tau)$. Let $V_\tau a = a \circ \ell_\tau$ and $P_\tau(s) = e^{-sH_\tau}$.

Row sums of the pair matrices imply that their stationary label marginals $\rho_\tau$ satisfy $|\rho_\tau(i) - \pi_i| \le 4\delta_\tau$. Therefore, for any fixed label functions $a, c$,

$$|\langle V_\tau a, V_\tau c\rangle - \langle a, c\rangle_\pi| \le 4\delta_\tau \sum_i |a_i c_i|,$$

$$|\langle V_\tau a, P_\tau(k\tau) V_\tau c\rangle - \langle a, e^{k\tau Q_4} c\rangle_\pi| \le \delta_\tau \|a\|_1 \|c\|_1, \quad k = 1, 2.$$

These constants depend only on fixed four-label functions, not hidden dimensions, rates or stationary masses.

*Slow-mode saturation.* Put $f_\tau = V_\tau u$ and $A = \|u\|_\pi^2 = 186/4$. Since $Q_4 u = -u$, the preceding bounds give

$$|\|f_\tau\|^2 - A| \le 744\delta_\tau,$$

$$|\langle f_\tau, P_\tau(k\tau) f_\tau\rangle - Ae^{-k\tau}| \le 484\delta_\tau, \quad k = 1, 2.$$

Self-adjointness and the semigroup identity yield

$$\begin{aligned}\|(e^\tau P_\tau(\tau) - I) f_\tau\|^2 &= e^{2\tau}\langle f_\tau, P_\tau(2\tau) f_\tau\rangle - 2e^\tau \langle f_\tau, P_\tau(\tau) f_\tau\rangle + \|f_\tau\|^2 \\ &\le 2196\delta_\tau e^{2\tau}.\end{aligned}$$

For $0 < s \le \tau$, the inequality $|x^{s/\tau} - 1| \le |x - 1|$ for $x \ge 0$, applied by spectral calculus to $e^\tau P_\tau(\tau)$, gives

$$\|(P_\tau(s) - e^{-s} I) V_\tau u\| \le \sqrt{2196}\, e^{\tau - s}\sqrt{\delta_\tau} \longrightarrow 0.$$

Here $b > 2$ is essential and follows from $b > \gamma > 2$. For each fixed $s$, the convergence is uniform over the proposed hidden comparators.

*Fast covariance.* Write $a_\tau = \|V_\tau w\|^2$ and $F_\tau(t) = \langle V_\tau w, P_\tau(t) V_\tau w\rangle$. The marginal estimate gives $|a_\tau - 1| \le 16\delta_\tau$. Since $Fw = w$, the target spectral decomposition bounds its autocorrelation by $Me^{-\beta t}$ for a fixed finite $M$. The positive hidden spectral measure then gives

$$0 \le F_\tau(\tau) \le Me^{-\beta\tau} + 16\delta_\tau \le Ke^{-b\tau}, \qquad K = M + 16.$$

For large $\tau$, $a_\tau > 0$. Normalize the spectral measure of $V_\tau w$ to a probability measure $\nu_\tau$. Concavity, with $0 < s < \tau$, implies

$$\begin{aligned}\frac{F_\tau(s)}{a_\tau} &= \int (e^{-\tau r})^{s/\tau}\, d\nu_\tau(r) \\ &\le \left(\int e^{-\tau r}\, d\nu_\tau(r)\right)^{s/\tau},\end{aligned}$$

so $F_\tau(s) \le a_\tau^{1 - s/\tau} K^{s/\tau} e^{-bs}$ and $\limsup_{\tau\to\infty} F_\tau(s) \le e^{-bs}$ for every fixed $s > 0$. No bound on hidden spectral support or mixing enters.

*Finite-lag transfer and contradiction.* For any fixed hidden model, detailed balance and stationarity give

$$\langle Va, (I - e^{-sH})Vc\rangle_\mu = \frac{1}{2}\sum_{i,j} J_{\mathcal{R}}(s)_{ij}(a_i - a_j)(c_i - c_j). \tag{5}$$

The weights are nonnegative. Applying the pointwise certificate (4) therefore yields

$$a_\tau - F_\tau(s) \le \langle V_\tau\psi, (I - P_\tau(s))V_\tau u\rangle.$$

The left side has lower limit at least $1 - e^{-bs}$. Slow-mode saturation and the marginal estimate make the right side converge to $(1 - e^{-s})\langle\psi, u\rangle_\pi = \gamma(1 - e^{-s})$; indeed $\|V_\tau\psi\| \le \|\psi\|_\infty$. Hence

$$1 - e^{-bs} \le \gamma(1 - e^{-s}) \quad \text{for every fixed } s > 0.$$

Taking $s = (b - \gamma)/b^2$ contradicts this inequality, because

$$(1 - e^{-bs}) - \gamma(1 - e^{-s}) \ge (b - \gamma)s - \tfrac{1}{2}b^2s^2 = \frac{(b - \gamma)^2}{2b^2} > 0.$$

The auxiliary $s$ constrains a proposed reversible semigroup; it is not an observation at a forbidden short gap.

*Infimum and sequence argument.* For each fixed $b \in (\gamma, \beta)$, an unbounded sequence with $d_2(\tau) < e^{-b\tau}$ would supply actual comparators with errors below that threshold, contradicting the preceding uniform argument. Thus $d_2(\tau) \ge e^{-b\tau}$ eventually, without an attained infimum. Combining with the upper approximation bound gives

$$\gamma - \frac{\log C_0'}{\tau} \le -\frac{1}{\tau}\log d_2(\tau) \le b.$$

Letting $\tau \to \infty$ and then $b \downarrow \gamma$ proves the theorem. □

The pair theorem does not itself construct a test or establish concentration of symmetric covariance estimates from one dependent record. Section 6 supplies a separate positive-operator argument. It adds no null mixing bound, independent reset or i.i.d.-pair assumption.

## 6. Symmetric two-lag scores and the positive-operator lemma

The following argument concerns the same fixed $Q_4$, physical clock and unrestricted stationary hidden-equilibrium null. The alternative is fully observed on four states. At each sampled transition the observer privately chooses lag $\tau$ or $2\tau$, independently and fairly, before seeing its endpoint. A fixed number of transitions suffices.

### 6.1. **A universal stationary score.**

Retain the normalized $\gamma$-eigenvector $w$ and certificate $\psi$ of Section 5. Given $\varepsilon > 0$, choose

$$\gamma < b < \min\{\beta, \gamma + \varepsilon/2\}, \qquad s = \frac{b - \gamma}{b^2}, \qquad \Delta = 1 - e^{-bs} - \gamma(1 - e^{-s}) > 0.$$

Indeed $\Delta \ge (b - \gamma)^2/(2b^2)$. Set $Z = e^{-s}\|\psi\|_\infty$ and $\kappa = Z^2/\Delta$. These constants are positive. For $\tau \ge s$, put

$$p = \frac{s}{\tau}, \quad A_\tau = 1 - (1 - p)e^{-bs}, \quad L_\tau = pe^{b(\tau - s)}, \quad c = 1 - e^{-s},$$

$$\phi_\tau(i) = A_\tau w_i^2 - c\psi_i u_i - \kappa u_i^2 - \Delta/4.$$

Define the symmetric scores

$$\begin{aligned} \ell_1(i,j) &= \tfrac{1}{2}(\phi_\tau(i) + \phi_\tau(j)) - 2L_\tau w_i w_j + 4\kappa e^\tau u_i u_j, \\ \ell_2(i,j) &= \tfrac{1}{2}(\phi_\tau(i) + \phi_\tau(j)) - 2\kappa e^{2\tau} u_i u_j. \end{aligned} \tag{6}$$

**Lemma 6.1** (Score means)**.** *The fair-lag stationary mean of $\ell_1, \ell_2$ is nonpositive under every hidden null, even with a different stationary label marginal. Under $Q_4$ it is at least $\Delta/2$ for all sufficiently large $\tau$.*

*Proof.* For a hidden null, write its semigroup as $e^{-tH}$ and label lift as $V$. Put

$$a = \|Vw\|^2, \quad f = \langle Vw, e^{-\tau H} Vw\rangle, \quad c_{\psi u} = \langle V\psi, Vu\rangle, \quad D = \|(e^{\tau} e^{-\tau H} - I)Vu\|^2.$$

The pointwise certificate and finite-lag Dirichlet identity (5) give

$$a - \langle Vw, e^{-sH} Vw\rangle \le c c_{\psi u} + Z\sqrt{D}.$$

Here spectral calculus gives $\|(e^{-sH} - e^{-s} I)Vu\| \le e^{-s}\sqrt{D}$. Jensen's inequality and weighted arithmetic–geometric mean imply

$$\langle Vw, e^{-sH} Vw\rangle \le a^{1-p} f^p \le (1-p)e^{-bs} a + L_\tau f.$$

The zero cases follow by continuity. Since $Z\sqrt{D} \le \kappa D + \Delta/4$, it follows that

$$A_\tau a - L_\tau f - c c_{\psi u} - \kappa D - \Delta/4 \le 0.$$

Expanding $D$ shows that this expression is exactly the fair-lag mean of $\ell_1, \ell_2$.

Under $Q_4$, the analogous covariance combination defining $D$ is zero because $Q_4 u = -u$, while $a = 1$ and $c_{\psi u} = \gamma$. Thus the raw score mean is

$$\Delta + p e^{-bs} - \Delta/4 - L_\tau \langle w, e^{\tau Q_4} w\rangle_\pi \ge \Delta/2$$

eventually, using $b < \beta$. □

Define known finite arrays and a target margin by

$$M_\tau = \max\{1, \max_{k,i,j} |\ell_k(i,j)|\}, \quad r_k = \ell_k / M_\tau, \quad d_\tau = \Delta/(2M_\tau).$$

Then $|r_k| \le 1$, all null means are nonpositive and the target mean is at least $d_\tau$. Because $b > 2$, the coefficients $e^{2\tau}$ and $e^{\tau}$ are eventually bounded by a constant times $e^{b\tau}/\tau$. Hence

$$M_\tau \le C_b \tau^{-1} e^{b\tau}, \qquad d_\tau \ge c_b \tau e^{-b\tau}. \tag{7}$$

### 6.2. **The positive-operator lemma.**

**Lemma 6.2** (Positive-operator tilt lemma)**.** *Suppose two symmetric label scores $r_1, r_2$, bounded by one, have nonpositive fair-lag stationary mean for* every *finite stationary reversible hidden CTMC, allowing arbitrary label marginals. On one stationary null trajectory with independent fair successive gaps $\tau, 2\tau$, let*

$$S_n = \sum_{j=1}^{n} r_{K_j}(Y_{j-1}, Y_j).$$

*Then, uniformly over the hidden class,*

$$\mathbb{E} e^{\theta S_n} \le e^{32 n \theta^2}, \qquad 0 \le \theta \le 1.$$

*Proof.* For one null, use its stationary support and write its generator as $R = -H$, stationary law as $\mu$, and

$$T = e^{-\tau H}, \quad K = \tfrac{1}{2}(T + T^2), \quad A_{xy} = \tfrac{1}{2}\sum_{k=1}^{2} (T^k)_{xy} r_k(\ell(x), \ell(y)).$$

All three operators are self-adjoint in $L^2(\mu)$. Entrywise $|A| \le K$ and Markov contraction imply $\|A\| \le 1$. For every nonnegative unit $h$, set $d = \langle h, (I - K)h\rangle$. We claim

$$\langle h, Ah\rangle \le 10\sqrt{d}.$$

For $d \ge 1/4$ the norm bound suffices. For $d < 1/4$, initially take $h > 0$, put $q = \|Th\|$ and $g = Th/q$. Spectral calculus on $\tau H \ge 0$ gives

$$q^2 \ge 1 - 2d, \quad \|g - h\| \le 2\sqrt{d}, \quad \tau\langle g, Hg\rangle \le 2d, \quad \langle g, (I-K)g\rangle \le 2d.$$

For completeness, use $I - T \preceq I - K$, $I - T^2 \preceq 2(I - T)$ and $x e^{-2x} \le 1 - e^{-x}$. Also $(1-t)^2 \le 1 - (t + t^2)/2$ for $0 \le t \le 1$, so $\|Th - h\| \le \sqrt{d}$ and $1 - q \le \sqrt{d}$. These prove the stated normalization bound; division by $q^2 > 1/2$ gives the two remaining estimates.

Construct another genuine reversible generator by

$$R^g_{xy} = R_{xy}\frac{g_y}{g_x} \quad (x \neq y), \qquad R^g_{xx} = -\sum_{y\neq x} R^g_{xy}.$$

Its stationary law is $\nu_x = \mu_x g_x^2$, since the stationary edge flows are $\mu_x R_{xy} g_x g_y$, symmetric in $x, y$. Compare its stationary path law on $[0, 2\tau]$ with the original generator started from this same $\nu$. The continuous-time path relative entropy is exactly

$$D_{\mathrm{KL}}(\mathbb{P}^{R^g,\nu} \| \mathbb{P}^{R,\nu}) = 2\tau\langle g, Hg\rangle \leq 4d.$$

Indeed, the logarithmic jump-rate contribution is the stationary mean of $\log g_y - \log g_x$, which vanishes. The remaining mean escape-rate difference is $\sum_x \mu_x g_x^2 \sum_{y\neq x} R_{xy}(1 - g_y/g_x) = \langle g, Hg\rangle$.

Add the same independent fair lag coin to both path laws. The score mean under $R^g$ is nonpositive by the universal stationary hypothesis. Pinsker's inequality and $|r_k| \leq 1$ therefore bound its expectation under the original generator started at $\nu$ by $2\sqrt{2}\sqrt{d}$. Replacing the endpoint weight $g_x^2$ by $g_x g_y$ changes that mean by at most

$$\frac{1}{2}\sum_{k=1}^{2}\sum_{x,y} \mu_x (T^k)_{xy} g_x |g_y - g_x| \leq \sqrt{2\langle g, (I-K)g\rangle} \leq 2\sqrt{d}.$$

This follows from Cauchy–Schwarz in the stationary marked pair measure. The resulting expectation is $\langle g, Ag\rangle$. Replacing $g$ by $h$ costs at most $2\|A\|\|g - h\| \leq 4\sqrt{d}$. Thus $\langle h, Ah\rangle \leq (6 + 2\sqrt{2})\sqrt{d} < 10\sqrt{d}$. For $h$ with zeros, apply this argument to normalized $h + \eta\mathbf{1}$ and let $\eta \downarrow 0$. All bounds are uniform, also for reducible nulls.

The tilted transfer matrix

$$(T_\theta)_{xy} = \tfrac{1}{2}\sum_{k=1}^{2} (T^k)_{xy} e^{\theta r_k(\ell(x),\ell(y))}$$

is self-adjoint and entrywise nonnegative. Its norm is its Perron eigenvalue, with a nonnegative unit eigenvector $h$. Taylor's entrywise remainder is bounded by $(e/2)\theta^2 K$ for $0 \leq \theta \leq 1$. Consequently,

$$\begin{aligned}\|T_\theta\| &\leq 1 - d + 10\theta\sqrt{d} + (e/2)\theta^2 \\ &\leq 1 + (25 + e/2)\theta^2 \leq e^{32\theta^2}.\end{aligned}$$

The exact dependent-history identity is

$$\mathbb{E}_\mu e^{\theta S_n} = \langle \mathbf{1}, T_\theta^n \mathbf{1}\rangle_\mu \leq \|T_\theta\|^n.$$

This proves the lemma without treating pairs as independent. □

*Remark* 6.3. The comparison generator $R^g$ changes rates and stationary marginals; the unrestricted null class permits both. No observable supermartingale or arbitrary-stopping validity is asserted.

## 7. The matched exponent for $Q_4$

**Theorem 7.1** (Matched exponent)**.** *For each fixed $0 < \alpha < 1/2$,*

$$\lim_{\tau\to\infty} \frac{\log N^*(Q_4, \tau, \alpha)}{\tau} = 2\gamma.$$

*More precisely, $N^*(Q_4, \tau, \alpha) \geq C^{-1} h_\alpha e^{2\gamma\tau}$ with $h_\alpha = (1 - 2\alpha)\log\frac{1-\alpha}{\alpha}$ for all large $\tau$, and for every $\varepsilon > 0$ a fixed-horizon test achieves $N \leq C_{\alpha,\varepsilon} e^{(2\gamma+\varepsilon)\tau}$ for all sufficiently large $\tau$. The null remains unrestricted in finite hidden dimension, rates, stationary label marginals and mixing time.*

7.1. **Upper bound: an admissible fixed-horizon test.** For the explicit scores (6) and $0 < \alpha < 1/2$, prespecify

$$n = \left\lceil d_\tau^{-2} \max\{512 \log(1/\alpha), 20/\alpha\} \right\rceil .$$

Collect $n$ transitions with the independent fair gap choices and reject equilibrium when $S_n \geq n d_\tau/2$. Since $0 < d_\tau < 1/2$, Lemma 6.2 applies with $\theta = d_\tau/128$ and gives

$$\sup_{\mathcal{R}} \Pr_{\mathcal{R}}\{S_n \geq n d_\tau/2\} \leq e^{-n d_\tau^2/512} \leq \alpha.$$

For power, use only the fixed known target's sampled-state kernel $P = \frac{1}{2}(e^{\tau Q_4} + e^{2\tau Q_4})$. Its finite spectral decomposition implies $\|P - \Pi\|_{L^2(\pi)\to L^2(\pi)} \leq 1/2$ eventually. Write $W_j = r_{K_j}(Y_{j-1}, Y_j)$ and $m = \mathbb{E}_Q W_j \geq d_\tau$. The conditional expectations of $W_j - m$ given its starting or ending state are mean-zero functions with $L^2(\pi)$ norm at most one, by conditional Jensen and $\mathrm{Var}(W_j) \leq 1$. Markov conditioning therefore gives

$$|\operatorname{Cov}_Q(W_1, W_{j+1})| \leq 2^{-(j-1)}, \quad j \geq 1, \qquad \operatorname{Var}_Q S_n \leq 5n.$$

Chebyshev yields

$$\Pr_Q\{S_n < n d_\tau/2\} \leq \frac{20}{n d_\tau^2} \leq \alpha.$$

No null mixing assumption enters this alternative-only variance calculation. The deterministic count terminates under every null and the target. Moreover, by (7),

$$N = n \leq C_{\alpha,b} \tau^{-2} e^{2b\tau} \leq C_{\alpha,\varepsilon} e^{(2\gamma+\varepsilon)\tau}$$

for sufficiently large $\tau$. No observation is taken at the auxiliary time $s$.

7.2. **Lower bound: stopped-history comparison.** The budget $N$ counts sampled transitions. The reversible comparator $B$ has the same stationary law as $Q_4$. Its strictly positive semigroup converges to $\Pi$. The spectral estimate (3) thus gives constants $C, \tau_* > 0$ such that, for all $t \geq \tau \geq \tau_*$,

$$\max_i D_{\mathrm{KL}}\big(e^{tQ_4}(i,\cdot) \| e^{tB}(i,\cdot)\big) \leq C e^{-2\gamma\tau}.$$

To see this, bound every comparator transition below by $\pi_j/2$ for $t \geq \tau_*$, use the entrywise discrepancy $O(e^{-\gamma t})$, and apply (1).

Consider any admissible passive causal sampling and stopping policy with minimum gap $\tau$, common stationary initialization, and integrable transition count $N$ under $Q_4$. Include times, private random choices, stopping flags and any terminal randomization in the transcript. Given an identical history, policy kernels are the same under $Q_4$ and $B$, so contribute zero conditional relative entropy. The initial distribution is also identical. Stop the record at $N \wedge M$ and pad it after stopping. The finite-prefix chain rule gives

$$D_{\mathrm{KL}}(\mathsf{P}_{Q_4}^{(M)} \| \mathsf{P}_B^{(M)}) \leq C e^{-2\gamma\tau} \mathbb{E}_{Q_4}(N \wedge M).$$

Relative entropy increases to that of the full transcript on its generated sigma-field as $M \to \infty$. Monotone convergence therefore proves

$$D_{\mathrm{KL}}(\mathsf{P}_{Q_4}^{\mathrm{stopped}} \| \mathsf{P}_B^{\mathrm{stopped}}) \leq C e^{-2\gamma\tau} \mathbb{E}_{Q_4} N.$$

This argument covers state-dependent timing and causal stopping; it does not require independent sampled pairs. Infinite expected count makes the resulting lower bound trivial. Since identity-labeled $B$ is an admissible hidden null, binary data processing for a test with both errors at most $\alpha$ gives

$$\mathbb{E}_{Q_4} N \geq C^{-1} h_\alpha e^{2\gamma\tau}, \qquad h_\alpha = (1-2\alpha)\log\frac{1-\alpha}{\alpha} > 0.$$

*Proof of Theorem 7.1.* Combine the lower bound with the fixed-horizon construction for every $\varepsilon > 0$. □

## 8. A uniform four-parameter family

This extension preserves the same physical clock, observation protocol and unrestricted hidden equilibrium null. It is a relatively open family on four observed states.

Retain $Q_0 = Q_4$, $\pi = (1/4, 1/4, 1/4, 1/4)$, $u$, $\Pi$, $U$, $F$, $\beta$ and the comparator $B$. Define

$$A = (Q_0 - Q_0^\top)/2, \qquad S = (Q_0 + Q_0^\top)/2 + U + \beta F, \qquad C = AS - SA.$$

For $\theta = (h, k, l, m)$ in the box $|h|, |k|, |l|, |m| \le \delta = 1/1000$, put

$$Q_\theta = -U - (\beta + h)F + (1 + k)S + lC + (1 + m)A.$$

**Theorem 8.1** (Uniform box)**.** *Every member is an irreducible reciprocal generator with stationary law $\pi$, satisfies $Q_\theta u = Q_\theta^\top u = -u$, and is nonnormal. For each fixed $0 < \alpha < 1/2$ and every $\varepsilon > 0$, there are positive constants independent of $\theta$ such that, eventually in $\tau$,*

$$c_\alpha e^{2\gamma\tau} \le N^*(Q_\theta, \tau, \alpha) \le C_{\alpha,\varepsilon} e^{(2\gamma+\varepsilon)\tau}.$$

*The upper test and its deterministic count can be chosen identically for the entire box. Consequently $\tau^{-1} \log N^*$ converges uniformly on the box to $2\gamma$.*

*Proof. All-parameter generator and spectral bounds.* Each of $F, S, C, A$ annihilates $\mathbf{1}, u$ on both sides. Their four-dimensional span is the full endomorphism space of $\operatorname{span}\{\mathbf{1}, u\}^\perp$. Direct rational identities give

$$a^2 = -\tfrac{1}{2}\operatorname{tr}(A^2) = \frac{125736}{265225}, \qquad \sigma^2 = \tfrac{1}{2}\operatorname{tr}(S^2) = \frac{986453}{2121800},$$

where $a, \sigma$ denote the positive square roots. Also

$$A^2 = -a^2 F, \qquad S^2 = \sigma^2 F, \qquad C^2 = 4a^2\sigma^2 F.$$

The three pairwise anticommutators vanish. These identities also establish independence: $A$ is nonzero and skew, while $S, C$ are nonzero independent traceless symmetric operators on the two-dimensional fast space, and $F$ is its identity.

For $i \ne j$, the affine entrywise estimate is

$$(Q_\theta)_{ij} \ge (Q_0)_{ij} - \delta\big(|F_{ij}| + |S_{ij}| + |C_{ij}| + |A_{ij}|\big) \ge \frac{1276299}{1644395000} > 0.$$

Thus positivity holds throughout the box, not just at its corners. The preserved row and column constraints then give the generator and stationary assertions.

Set $W_\theta = (1 + k)S + lC$ and

$$\omega_\theta^2 = (1 + m)^2 a^2 - (1 + k)^2\sigma^2 - 4l^2 a^2\sigma^2.$$

The exact lower bound is

$$\omega_\theta^2 \ge \Omega_*^2 := (1 - \delta)^2 a^2 - (1 + \delta)^2\sigma^2 - 4\delta^2 a^2\sigma^2 = \frac{4097319615580243}{562754405000000000} > 0.$$

Moreover $\beta + h \ge \beta_* := 253897/103000 > \gamma > 2$. The spectrum is $0, -1, -(\beta + h) \pm i\omega_\theta$. Since $W_\theta^2$ is a strictly positive scalar times $F$, both $W_\theta$ and $A$ are invertible on the fast space. They anticommute, so

$$Q_\theta Q_\theta^\top - Q_\theta^\top Q_\theta = 2(1 + m)[A, W_\theta] \ne 0.$$

This proves nonnormality even at interior parameters.

*Uniform semigroup bounds and proof of the statistical conclusion.* The residual square identity gives exactly

$$e^{tQ_\theta} = \Pi + e^{-t}U + e^{-(\beta+h)t}\left[\cos(\omega_\theta t)F + \frac{\sin(\omega_\theta t)}{\omega_\theta}\big(W_\theta + (1 + m)A\big)\right].$$

Let $\Omega_* = \sqrt{\Omega_*^2}$ and

$$L_* = (1 + \delta)(\sigma + a) + 2\delta a\sigma, \qquad M_* = 1 + L_*/\Omega_*.$$

The exact squares bound the operator norms of $S, A, C$ by $\sigma, a, 2a\sigma$. Hence, uniformly over the box,

$$\|e^{tQ_\theta}F\| \le M_* e^{-\beta_* t}, \qquad \|e^{tQ_\theta} - e^{tB}\| \le (M_* + 1)e^{-\gamma t}.$$

All norms are on $L^2(\pi)$. The fixed positive transition floor for $B$ now gives a uniform row-KL constant. The stopped-history proof of Section 7 supplies the claimed lower cost uniformly in $\theta$ and in every admissible causal policy.

Choose one $b$ with $\gamma < b < \min\{\beta_*, \gamma + \varepsilon/2\}$. The previously constructed scores use only the fixed slow eigenfunction, comparator eigenfunction and certificate, and this $b$. Their slow covariance cancellation and stationary moments are identical for every target. The sole fast remainder is bounded by the common envelope $M_* e^{-\beta_* t}$, so one common starting time gives the required positive score mean throughout the box. The unrestricted-null moment bound is unchanged. Finally,

$$\left\| \tfrac{1}{2}(e^{\tau Q_\theta} + e^{2\tau Q_\theta}) - \Pi \right\| \le e^{-\tau} + M_* e^{-\beta_* \tau}$$

is eventually at most $1/2$ at a uniform time. The same dependent-score variance bound and deterministic budget therefore prove the upper cost uniformly. □

*Remark* 8.2. Every target in the box has nonzero current with damping $\beta + h > \gamma$, whereas its statistical exponent is $2\gamma$. This local deformation keeps the stationary law and the slow eigenfunction fixed.

## 9. Interacting path families on increasing observed alphabets

Fix an integer $d \ge 2$. At site $i$ choose any member $Q_i = Q_{\theta_i}$ of the box of Section 8 independently, and choose $1/2000 \le \eta_i \le 1/1000$ for each path edge. On $\Omega_d = \{1,2,3,4\}^d$, with identity observation of every configuration, define tensor embeddings by a superscript $(i)$ and put

$$\widehat{Q} = \sum_{i=1}^{d} Q_i^{(i)} + \sum_{i=1}^{d-1} \eta_i Q_i^{(i)} Q_{i+1}^{(i+1)}, \qquad \widehat{B} = \sum_{i=1}^{d} B^{(i)} + \sum_{i=1}^{d-1} \eta_i B^{(i)} B^{(i+1)}.$$

Let $\mathcal{F}_d$ denote this $(5d-1)$-parameter target family. All rates use the original physical time unit.

**Theorem 9.1** (Path families)**.** *These are irreducible reciprocal generators with common stationary law $\mu_d(x) = 4^{-d}$; $\widehat{B}$ is reversible and every $\widehat{Q}$ is nonnormal. For each fixed $0 < \alpha < 1/2$, every $\varepsilon > 0$, and all sufficiently large $\tau$,*

$$\frac{h_\alpha}{C_d} e^{2\gamma\tau} \le N^*(\widehat{Q}, \tau, \alpha) \le C_{\alpha,\varepsilon} e^{(2\gamma+\varepsilon)\tau}, \qquad C_d = 2\,4^d (20^d + 1)^2.$$

*The lower bound holds for every admissible causal policy using full configuration histories. The upper bound uses one common deterministic-count test for all $d$ and all parameters, with its constant and starting time independent of $d$. Consequently the logarithmic exponent is $2\gamma$ uniformly over $\mathcal{F}_d$ for each fixed $d$. The same conclusion holds for the minimax budget of a common test over $\mathcal{F}_d$. No simultaneous growing-$d$ asymptotic is claimed.*

*Proof. Generator, interaction and invariant sectors.* The exact local identities of Section 8 imply

$$\|S\|, \|A\| < 7/10, \quad \|C\| < 98/100, \quad \|J_\theta\| \le 70119/50000,$$
$$7/1000 < \omega_\theta^2 < 1/50, \qquad \|J_\theta\|/\omega_\theta < 17,$$

where $J_\theta = (1+k)S + lC + (1+m)A$. In particular $\|Q_\theta\| < 4$. Writing $e_i(a) = -Q_i(a,a) < 4$, a jump changing just site $i$ has rate

$$Q_i(x_i, y_i) \left(1 - \sum_{j \sim i} \eta_{ij} e_j(x_j)\right),$$

and a jump changing precisely adjacent sites $i, j$ has rate $\eta_{ij} Q_i(x_i, y_i) Q_j(x_j, y_j)$. All other off-diagonal entries vanish. The single-jump bracket exceeds 0.99. For the comparator the same argument works since $\max_a[-B(a,a)] = 104/21 < 5$ and path degree is at most two. Thus both generators have nonnegative off-diagonals, positive connected single-jump graphs, and zero row and column sums. Reverse transitions exist for every allowed transition. Symmetry of $\widehat{B}$ proves reversibility.

Each coordinate-only subspace reduces both the target and its adjoint, with restrictions $Q_i, Q_i^\top$: an interaction touching another constant factor vanishes. Hence each coordinate is exactly the original-clock

CTMC $Q_i$, and its nonnormality implies global nonnormality. These interactions are nonzero: for adjacent sites,

$$\widehat{Q}[u(x_i)u(x_{i+1})] = -(2-\eta_i)u(x_i)u(x_{i+1}).$$

Every configuration is distinguished by $(u(x_1),\dots,u(x_d))$, since the four values of $u$ are distinct. Coordinate generators identify the local parameters; adjacent double-jump rates then identify $\eta_i$.

The generators agree on the entire space $\operatorname{span}\{\mathbf{1},u\}^{\otimes d}$. Its product modes $u_I=\prod_{i\in I}u(x_i)$ have rates

$$r(I)=|I|-\sum_{\{i,i+1\}\subset I}\eta_i.$$

All modes with $|I|=1,2$ have rate below $\gamma$, while $|I|\ge 3$ gives $r(I)\ge |I|-(|I|-1)/1000\ge 2.998>\gamma$. Thus the common sector below $\gamma$ has dimension $1+d+\binom{d}{2}$.

*Global spectrum, semigroup and history comparison.* The tensor product of the local eigenbases diagonalizes each global generator. Its decay rates are

$$R=\sum_i r_i-\sum_i \eta_i r_i r_{i+1},$$

where the target local rates belong to $\{0,1,\beta+h_i+i\omega_i,\beta+h_i-i\omega_i\}$ and the comparator rates belong to $\{0,1,\gamma,\Gamma\}$. All active rates have real part at least one and modulus below nine. If $m$ sites are active their induced path has at most $m-1$ edges; if one is fast,

$$\operatorname{Re}R\ge r_{\mathrm{fast,min}}+(m-1)-81(m-1)/1000=r_{\mathrm{fast,min}}+919(m-1)/1000.$$

Here $r_{\mathrm{fast,min}}=\beta_*>\gamma$ for the target and $\gamma$ for the comparator. Tensor eigenbases remain bases at eigenvalue coincidences, so there are no resulting Jordan blocks. The comparator has gap exactly one, and its complementary sector has gap $\gamma$.

The local fast spectral projectors $E_\pm=(F\mp iJ_\theta/\omega_\theta)/2$, together with $\Pi,U$, have sum of norms less than twenty. Tensorizing and cancelling the entire common sector therefore gives, in the uniform stationary norm (equivalently the Euclidean matrix norm),

$$\|e^{t\widehat{Q}}-e^{t\widehat{B}}\|\le (20^d+1)e^{-\gamma t}.$$

Since $\widehat{B}$ is symmetric with gap one, $|e^{t\widehat{B}}(x,y)-4^{-d}|\le e^{-t}$, so every entry is at least $1/(2\,4^d)$ when $t\ge T_d=\log(2\,4^d)$. The row chi-square inequality (1) then yields

$$\sup_x D_{\mathrm{KL}}\big(e^{t\widehat{Q}}(x,\cdot)\|e^{t\widehat{B}}(x,\cdot)\big)\le 2\,4^d(20^d+1)^2e^{-2\gamma t}.$$

Applying the finite-prefix history argument of Section 7 to these full configuration kernels, with the same stationary start, gives $D_{\mathrm{KL}}(\mathsf{P}^{(M)}_{\widehat{Q}}\|\mathsf{P}^{(M)}_{\widehat{B}})\le C_d e^{-2\gamma\tau}\mathbb{E}_{\widehat{Q}}(N\wedge M)$. For completeness, one may apply binary data processing directly to $\{N\le M,\ \text{reject}\}$ and then let $M\to\infty$; finite target expectation implies target termination, and null rejection probability is at most $\alpha$. The limiting binary divergence is at least $h_\alpha$. Thus the displayed cost lower bound holds without imposing any coordinate-based restriction on the policy.

*One common test and dimension-independent target dependence.* Choose $\gamma<b<\min\{\beta_*,\gamma+\varepsilon/2\}$ and use exactly the earlier four-label arrays $r_1,r_2$ and margin $d_\tau$. The uniform local fast bound can here be taken as $\|e^{tQ_i}F\|\le 18e^{-\beta_* t}$, so their target means are at least $d_\tau\ge c_b\tau e^{-b\tau}$ at a common starting time. Define scores on full configurations by

$$R_k(x,y)=\frac{1}{d}\sum_{i=1}^{d}r_k(x_i,y_i),\qquad k=1,2.$$

They are symmetric and bounded by one. For every stationary hidden reversible null with configuration labels, projection to coordinate $i$ is an admissible four-label observation, with arbitrary marginal. The universal four-label mean inequality thus applies to every summand; the averaged scores have null mean at most zero. Under every target the coordinate restrictions give mean at least $d_\tau$. Lemma 6.2 applies on the full hidden space and gives $\mathbb{E}_{\mathrm{null}}e^{\theta S_n}\le e^{32n\theta^2}$.

To control dependence under the target, first note that the local symmetric fast block has damping at least $\beta_*-[1.001(0.7)+0.001(0.98)]>1$; the $u$-mode has damping one. Decompose the mean-zero global

space into orthogonal sectors specifying the $m$ mean-zero tensor factors. Each sector reduces $\widehat{Q}$. The local sum has numerical damping at least $m$. An interaction is zero unless both endpoints are active, has norm below $16/1000$, and there are at most $m-1$ active edges. Therefore

$$\operatorname{Re}\langle f, -\widehat{Q}f\rangle \geq [m - 0.016(m-1)]\|f\|^2 \geq \|f\|^2$$

on each such sector and hence on their orthogonal sum. Differentiating $\|e^{t\widehat{Q}}f\|^2$ proves $\|e^{t\widehat{Q}} - \Pi_d\| \leq e^{-t}$, uniformly in $d$. The fair-gap sampled kernel is thus eventually a contraction by $1/2$. The earlier endpoint-conditioning argument applies to the bounded scores $R_k$ and gives $\operatorname{Var}_{\widehat{Q}} S_n \leq 5n$. Use independent fair gaps $\tau, 2\tau$, the same threshold $nd_\tau/2$, and the same count

$$n = \left\lceil d_\tau^{-2} \max\{512 \log(1/\alpha), 20/\alpha\} \right\rceil .$$

The universal exponential bound and the target variance bound give the two errors at most $\alpha$, uniformly in all displayed parameters and $d$. The score-size estimate gives the claimed upper cost. All observations respect the minimum gap; the small auxiliary $s$ remains only a proof parameter. Counting the initial observation adds one as before. □

*Remark* 9.2. The interactions commute as polynomials in local tensor generators, preserve coordinate invariant spaces and require bounded neighbouring coupling. These are genuine simultaneously jumping path processes, and the common sector below $\gamma$ grows with $d$; the construction is a commuting polynomial lift preserving coordinate spaces, not a general theory of interacting nonnormal networks.

## 10. Generators with a two-sided slow eigenvector

The arguments of Sections 5–7 use only a few properties of $Q_4$. This section isolates them.

Let $n \geq 3$ and let $Q$ be an irreducible reciprocal generator on $\{1, \dots, n\}$ with stationary law $\pi$. Write $Q^\sharp = D_\pi^{-1} Q^\top D_\pi$ for its adjoint in $L^2(\pi)$. Assume that there are $\lambda > 0$ and a real vector $u$ with pairwise distinct entries such that

$$Qu = Q^\sharp u = -\lambda u. \tag{8}$$

Then $\lambda\langle \mathbf{1}, u\rangle_\pi = -\pi^\top Q u = 0$. Put $X = \{\mathbf{1}, u\}^{\perp_\pi}$ and let $F$ be the $\pi$-orthogonal projector onto $X$. For $f \in X$, $\langle \mathbf{1}, Qf\rangle_\pi = \langle Q^\sharp \mathbf{1}, f\rangle_\pi = 0$ because $Q^\sharp \mathbf{1} = D_\pi^{-1} Q^\top \pi = 0$, and $\langle u, Qf\rangle_\pi = -\lambda\langle u, f\rangle_\pi = 0$. Hence $X$ and $\operatorname{span}\{\mathbf{1}, u\}$ are both $Q$-invariant, and $F$ commutes with $Q$. Define

$$\beta(Q) = \min\{-\operatorname{Re} z : \ z \in \operatorname{spec}(Q|_X)\}.$$

Irreducibility gives $\beta(Q) > 0$, since 0 is a simple eigenvalue with eigenvector $\mathbf{1} \notin X$. Relabel the states so that $u_1 < \cdots < u_n$, and let $B = B(\pi, u, \lambda)$ be the birth–death generator of Lemma 3.1. It is $\pi$-self-adjoint, irreducible, and preserves $X$. Put

$$\gamma^*(\pi, u, \lambda) = \min \operatorname{spec}(-B|_X) > 0.$$

By Lemma 3.1 and the Rayleigh formula, $\gamma^*$ is the largest value of $\min \operatorname{spec}(-R|_X)$ over all $\pi$-reversible generators $R$ on the same $n$ states with $Ru = -\lambda u$, and $B$ attains it.

For $Q_4$, (8) holds with uniform $\pi$, $u = (-10, -1, 2, 9)^\top$ and $\lambda = 1$; Section 4 gives $\beta(Q_4) = 254/103$ and $\gamma^* = \gamma$.

**Theorem 10.1** (Two-sided slow mode)**.** *Let $Q, \pi, u, \lambda$ satisfy the assumptions above, write $\beta = \beta(Q)$, $\gamma^* = \gamma^*(\pi, u, \lambda)$, and fix $0 < \alpha < 1/2$.*

(i) $\liminf_{\tau\to\infty} \dfrac{\log N^*(Q, \tau, \alpha)}{\tau} \geq 2\min\{\beta, \gamma^*\}$.

(ii) *If $\gamma^* < \beta$, then* $\limsup_{\tau\to\infty} \dfrac{\log N^*(Q, \tau, \alpha)}{\tau} \leq 2\gamma^*$.

(iii) *Consequently, if $\gamma^* < \beta$, then*

$$\lim_{\tau\to\infty} \frac{\log N^*(Q, \tau, \alpha)}{\tau} = 2\gamma^* = 2\min\{\beta, \gamma^*\}.$$

(iv) *If $Q$ is $\pi$-reversible, then $\gamma^* \geq \beta$ and $N^*(Q, \tau, \alpha) = +\infty$ for every $\tau > 0$. If $Q$ is normal in $L^2(\pi)$, then $\gamma^* \geq \beta$.*

*Proof. Semigroup comparison.* Let $\Pi = \mathbf{1}\pi^\top$ and $U = uu^\top D_\pi/\|u\|_\pi^2$. Both $Q$ and $B$ act on $\operatorname{span}\{\mathbf{1}, u\}$ as 0 on $\mathbf{1}$ and $-\lambda$ on $u$, and both commute with $F$. Hence

$$e^{tQ} - e^{tB} = e^{tQ}F - e^{tB}F.$$

The Jordan form of $Q|_X$ gives a constant $C_Q$ with $\|e^{tQ}F\| \le C_Q(1+t)^n e^{-\beta t}$, and self-adjointness gives $\|e^{tB}F\| = e^{-\gamma^* t}$, all norms on $L^2(\pi)$. With $\mu = \min\{\beta, \gamma^*\}$,

$$\|e^{tQ} - e^{tB}\| \le (C_Q + 1)(1+t)^n e^{-\mu t}. \tag{9}$$

*(i) Lower bound.* The identity-labelled $B$ is an irreducible member of $\mathcal{H}$ with stationary law $\pi$. Since $e^{tB} \to \Pi$, every entry of $e^{tB}$ is at least $\pi_j/2$ for $t \ge t_0$. Entries are bounded by a constant multiple of the $L^2(\pi)$ operator norm, so (1) and (9) give

$$\max_i D_{\mathrm{KL}}\big(e^{tQ}(i,\cdot)\|e^{tB}(i,\cdot)\big) \le C(1+t)^{2n}e^{-2\mu t}, \qquad t \ge t_0.$$

The right side is decreasing for $t \ge n/\mu$. Thus for $\tau \ge \max\{t_0, n/\mu\}$ every sampled transition with gap $t \ge \tau$ contributes at most $C(1+\tau)^{2n}e^{-2\mu\tau}$. The stopped-history argument of Section 7 applies verbatim and gives

$$N^*(Q, \tau, \alpha) \ge \frac{h_\alpha}{C(1+\tau)^{2n}} e^{2\mu\tau},$$

which proves (i).

*(ii) Upper bound: certificate.* Fix $b$ with $\gamma^* < b < \beta$. Choose a real $w \in X$ with $-Bw = \gamma^* w$ and $\|w\|_\pi = 1$. Define $\psi_1 = 0$ and $\psi_{k+1} - \psi_k = (w_{k+1} - w_k)^2/(u_{k+1} - u_k)$. As in Lemma 3.1, Cauchy–Schwarz over consecutive increments gives

$$(w_i - w_j)^2 \le (u_i - u_j)(\psi_i - \psi_j) \qquad \text{for all } i, j.$$

With the conductances $c_k$ of $B$,

$$\lambda\langle\psi, u\rangle_\pi = \langle\psi, -Bu\rangle_\pi = \sum_{k<n} c_k(\psi_{k+1} - \psi_k)(u_{k+1} - u_k) = \sum_{k<n} c_k(w_{k+1} - w_k)^2 = \gamma^*,$$

so $\langle\psi, u\rangle_\pi = \gamma^*/\lambda$.

*(ii) Upper bound: lags and scores.* Set $s = (b - \gamma^*)/b^2$, $c = 1 - e^{-\lambda s}$ and

$$\Delta = 1 - e^{-bs} - \frac{\gamma^*}{\lambda}\, c.$$

Since $1 - e^{-bs} \ge bs - \frac{1}{2}b^2s^2$ and $c \le \lambda s$, $\Delta \ge (b - \gamma^*)^2/(2b^2) > 0$. Put $Z = \sqrt{2}\, e^{-\lambda s}\|\psi\|_\infty$, $\kappa = Z^2/\Delta$, and, for $\tau \ge s$, $p = s/\tau \in (0, 1]$, $A_\tau = 1 - (1-p)e^{-bs}$, $L_\tau = pe^{b(\tau - s)}$. The procedure uses the two gaps $t_1 = \tau$ and $t_2 = \tau + s$; both are at least $\tau$, and $t_2 \le 2\tau$. At each sampled transition the observer chooses $k \in \{1, 2\}$ independently and fairly before seeing its endpoint, waits $t_k$, and scores the transition with $\ell_k$, where $\phi_\tau(i) = A_\tau w_i^2 - c\psi_i u_i - \kappa u_i^2 - \Delta/4$ and

$$\ell_1(i,j) = \tfrac{1}{2}(\phi_\tau(i) + \phi_\tau(j)) - 2L_\tau w_i w_j + 2\kappa\Big(1 + \frac{\tau}{s}\Big)e^{\lambda\tau}u_i u_j,$$
$$\ell_2(i,j) = \tfrac{1}{2}(\phi_\tau(i) + \phi_\tau(j)) - 2\kappa\,\frac{\tau}{s}\,e^{\lambda(\tau+s)}u_i u_j.$$

For a hidden null with semigroup $e^{-tH}$ and lift $V$, let $a, f, c_{\psi u}$ be as in the proof of Lemma 6.1, and

$$E = \|Vu\|^2 - \Big(1 + \frac{\tau}{s}\Big)e^{\lambda\tau}\langle Vu, e^{-\tau H}Vu\rangle + \frac{\tau}{s}\,e^{\lambda(\tau+s)}\langle Vu, e^{-(\tau+s)H}Vu\rangle.$$

Both endpoint marginals of the stationary pair law are the label marginal, so expanding $E$ shows that the fair-lag mean of the scores equals

$$A_\tau a - L_\tau f - c\, c_{\psi u} - \kappa E - \Delta/4.$$

Put $Y = e^{\lambda\tau}e^{-\tau H}$, a nonnegative self-adjoint operator. Then $e^{-sH} = e^{-\lambda s}Y^p$, $e^{-(\tau+s)H} = e^{-\lambda(\tau+s)}Y^{1+p}$ and $E = \langle Vu, G_p(Y)Vu\rangle$, where

$$G_p(x) = 1 - \Big(1 + \frac{1}{p}\Big)x + \frac{x^{1+p}}{p}, \qquad x \ge 0.$$

We claim $(1-x^p)^2 \le 2G_p(x)$ for $x \ge 0$ and $0 < p \le 1$. Indeed $G_p(1) = 0$ and $G_p'(x) = \frac{1+p}{p}(x^p - 1)$. For $0 \le x < 1$, concavity gives $1 - z^p \ge p(1-z)$ on $[x, 1]$, hence $G_p(x) \ge \frac{1+p}{2}(1-x)^2$, while $0 \le 1 - x^p \le 1 - x$. For $x > 1$,

$$(x^p - 1)^2 = \int_1^x 2pz^{p-1}(z^p - 1)\,dz \le 2p \int_1^x (z^p - 1)\,dz = \frac{2p^2}{1+p} G_p(x).$$

In particular $G_p \ge 0$ on $[0, \infty)$, so $E \ge 0$. Spectral calculus for the positive self-adjoint operator $Y$ therefore gives $\|(e^{-sH} - e^{-\lambda s} I)Vu\|^2 = e^{-2\lambda s}\|(Y^p - I)Vu\|^2 \le 2e^{-2\lambda s}E$. The certificate and (5) give

$$a - \langle Vw, e^{-sH}Vw\rangle \le \langle V\psi, (I - e^{-sH})Vu\rangle \le c\, c_{\psi u} + Z\sqrt{E}.$$

The Jensen and weighted arithmetic–geometric mean step of Lemma 6.1 involves only $b, s, \tau$ and the lag-$\tau$ covariance $f$, and gives $\langle Vw, e^{-sH}Vw\rangle \le (1-p)e^{-bs}a + L_\tau f$. With $Z\sqrt{E} \le \kappa E + \Delta/4$, the null mean is nonpositive, for every hidden null and every stationary label marginal.

Under $Q$ the pair law at lag $t$ is $D_\pi e^{tQ}$. Then $a = 1$, $c_{\psi u} = \gamma^*/\lambda$, and the target analogue of $E$ is $\|u\|_\pi^2\big(1 - (1 + \tau/s) + \tau/s\big) = 0$ because $e^{tQ}u = e^{-\lambda t}u$. Since $w \in X$, $|\langle w, e^{\tau Q}w\rangle_\pi| = |\langle w, e^{\tau Q}Fw\rangle_\pi| \le C_Q(1+\tau)^n e^{-\beta\tau}$. The target mean is therefore

$$\Delta + pe^{-bs} - \Delta/4 - L_\tau\langle w, e^{\tau Q}w\rangle_\pi \ge \tfrac{3}{4}\Delta - C_Q(1+\tau)^n e^{-(\beta-b)\tau} \ge \Delta/2$$

for all large $\tau$, because $b < \beta$.

*(ii) Upper bound: test.* All quantities other than $L_\tau$ and the two $u_i u_j$ coefficients in $\ell_1, \ell_2$ are bounded independently of $\tau$, and those coefficients are at most $C\tau e^{\lambda\tau}$. By Lemma 10.2, $b > \gamma^* > \lambda$, so $M_\tau = \max\{1, \max_{k,i,j} |\ell_k(i,j)|\} \le C(e^{b\tau} + \tau e^{\lambda\tau}) \le C' e^{b\tau}$ for $\tau \ge s$. Normalize $r_k = \ell_k/M_\tau$ and $d_\tau = \Delta/(2M_\tau) < 1/2$. By Lemma 10.3, the moment bound of Lemma 6.2 holds for the gaps $\tau, \tau + s$; it does not use the size of the label alphabet, so it applies to $r_1, r_2$. Since $Q$ is irreducible, $\|\frac{1}{2}(e^{\tau Q} + e^{(\tau+s)Q}) - \Pi\|_{L^2(\pi)} \le 1/2$ for large $\tau$. The fixed-horizon test of Section 7, with the same $n$ and threshold and with the gaps $\tau, \tau + s$, therefore has both errors at most $\alpha$; the variance bound uses only this sampled kernel. Hence

$$N = n \le C_\alpha M_\tau^2 \le C'_\alpha e^{2b\tau}.$$

Letting $b \downarrow \gamma^*$ proves (ii). The constants and the starting time depend on $Q$, $\alpha$ and $b$; no uniformity in $Q$ is claimed. Part (iii) follows from (i) and (ii).

*(iv)* If $Q$ is $\pi$-reversible, $Q$ itself is a $\pi$-reversible generator with $Qu = -\lambda u$. Lemma 3.1 gives $\langle v, -Qv\rangle_\pi \le \langle v, -Bv\rangle_\pi$ for all $v$, and both operators are self-adjoint and preserve $X$, so $\beta = \min\operatorname{spec}(-Q|_X) \le \gamma^*$. The identity-labelled $Q$ belongs to $\mathcal{H}$. An admissible procedure would reject with probability at most $\alpha$ and at least $1 - \alpha > \alpha$ under the same law, so none exists and $N^* = +\infty$. If $Q$ is normal in $L^2(\pi)$, then $R = \frac{1}{2}(Q + Q^\sharp)$ is a $\pi$-reversible generator: its off-diagonal entries $\frac{1}{2}(Q_{ij} + \pi_j Q_{ji}/\pi_i)$ are nonnegative and $R\mathbf{1} = 0$. It satisfies $Ru = -\lambda u$. Normality makes $Q|_X$ unitarily diagonalizable with $R|_X$ diagonal in the same basis, so $\min\operatorname{spec}(-R|_X) = \beta$. Lemma 3.1 then gives $\gamma^* \ge \beta$. □

The two lemmas used in part (ii) follow.

**Lemma 10.2** (The slow mode is the spectral gap of the completion)**.** *In the setting of Theorem 10.1, $\lambda$ is the smallest nonzero eigenvalue of $-B$, and* $\operatorname{spec}(-B|_X) \subset (\lambda, \infty)$. *In particular $\gamma^* > \lambda$.*

*Proof.* The eigenvalues of $-B$ are simple: $B$ is tridiagonal with positive off-diagonal entries, so the eigenvalue equation determines an eigenvector recursively from its first entry. Let $0 = \mu_0 < \mu_1 < \cdots < \mu_{n-1}$ be these eigenvalues, with real eigenvectors $g_0 = \mathbf{1}, g_1, \dots, g_{n-1}$, pairwise $\pi$-orthogonal. Since $-Bu = \lambda u$ and $u \ne 0$ is not constant, $\lambda = \mu_j$ for some $j \ge 1$, and $u$ is a multiple of $g_j$.

The semigroup $e^{tB}$ maps nondecreasing functions to nondecreasing functions. Indeed, for small $h > 0$ the matrix $P = I + hB$ is stochastic and tridiagonal with $P_{k,k+1} + P_{k+1,k} \le 1$; its tail sums $\sum_{l \ge m} P_{kl}$ are then nondecreasing in $k$ for every $m$ (they equal 0 for $k \le m-2$, $P_{m-1,m}$ for $k = m-1$, $1 - P_{m,m-1}$ for $k = m$, and 1 for $k \ge m+1$), so $P$ preserves nondecreasing functions, and $e^{tB} = \lim_{N\to\infty}(I + tB/N)^N$.

The indicators $\mathbf{1}_{\{\ge m\}}$, $m = 1, \dots, n$, span $\mathbb{R}^n$, so $\langle \mathbf{1}_{\{\ge m\}}, g_1\rangle_\pi \ne 0$ for some $m$. For $f = \mathbf{1}_{\{\ge m\}}$,

$$e^{\mu_1 t}\big(e^{tB}f - \langle f, \mathbf{1}\rangle_\pi \mathbf{1}\big) \longrightarrow \frac{\langle f, g_1\rangle_\pi}{\|g_1\|_\pi^2}\, g_1 \qquad (t \to \infty).$$

The left side is nondecreasing for every $t$, so $g_1$ or $-g_1$ is nondecreasing; call it $g$. It is nonconstant because $g \perp_\pi \mathbf{1}$ and $g \neq 0$. Since $u$ and $g$ have $\pi$-mean zero,

$$2\langle u, g\rangle_\pi = \sum_{i,k} \pi_i \pi_k (u_i - u_k)(g_i - g_k) > 0,$$

because every term is nonnegative and a pair with $g_i \neq g_k$ gives a positive term. Eigenvectors for distinct eigenvalues are $\pi$-orthogonal, so $j = 1$ and $\lambda = \mu_1$. The space $X$ is spanned by $g_2, \dots, g_{n-1}$, whose eigenvalues exceed $\mu_1$. □

**Lemma 10.3** (Second gap in $[\tau, 2\tau]$)**.** *Lemma 6.2 remains true when the fair gap choice $\tau, 2\tau$ is replaced by a fair choice between $\tau$ and any fixed $t_2 \in [\tau, 2\tau]$, with $r_2$ used after the gap $t_2$ and the hypothesis on the stationary mean read for these gaps.*

*Proof.* In the proof of Lemma 6.2, replace $T^2$ by $T_2 = e^{-t_2 H}$ in $K$, $A$ and $T_\theta$, and keep $T = e^{-\tau H}$. Since $T_2 \preceq T$, still $I - T \preceq I - K$. The normalization bounds hold as before: $\|Th - h\|^2 \le \langle h, (I-T)h\rangle \le d$ because $(1-t)^2 \le 1 - t$ on $[0,1]$; $1 - q^2 = \langle h, (I - T^2)h\rangle \le 2d$; $\tau\langle g, Hg\rangle \le d/q^2$ by $xe^{-2x} \le 1 - e^{-x}$; and $\langle g, (I-K)g\rangle \le d/q^2$ because $T$ commutes with $K$ and $\|T\| \le 1$. The path relative entropy is now computed on $[0, t_2]$ and equals $t_2\langle g, Hg\rangle \le 2\tau\langle g, Hg\rangle \le 4d$. The remaining steps do not use the value of the second gap. □

*Remark* 10.4. Theorem 7.1 is the case $Q = Q_4$ of part (iii); the proof in Section 7 gives sharper polynomial factors. It uses the gaps $\tau, 2\tau$, whose slow-mode penalty has coefficient $e^{2\lambda\tau}$; that suffices there because $\gamma > 2 = 2\lambda$, but in general the same argument with the gaps $\tau, 2\tau$ yields only the bound $2\max\{\gamma^*, 2\lambda\}$. The gaps $\tau, \tau + s$ reduce the coefficient to $O(\tau e^{\lambda\tau})$, which Lemma 10.2 makes negligible. Part (iv) shows that $2\min\{\beta, \gamma^*\}$ is not the exponent in general when $\gamma^* \ge \beta$: for reversible $Q$ it is finite while $N^* = +\infty$. The regime $\gamma^* < \beta$ forces irreversibility and, by (iv), nonnormality in $L^2(\pi)$.

**Outlook.** The regime $\gamma^* \ge \beta$ with irreversible $Q$ remains open. There part (i) gives the lower bound $2\beta$ up to polynomial factors, and part (iv) shows that the expression $2\min\{\beta, \gamma^*\}$ is not the exponent for reversible $Q$; we do not determine the exponent for irreversible $Q$ in this regime. The window of [1, Eq. (12)] applies under its own hypotheses. Lemma 3.1 also requires distinct entries of $u$; ties are not treated.

## 11. Relation to prior work

For Aznagulov [1], the normal theorem is Eq. (3), the nonnormal window is Eq. (12), and the universal concentration theorem is Eq. (8), with proofs in the ancillary supplement of that submission. That concentration statement requires antisymmetric bounded endpoint scores and an independently Poissonized count. The present symmetric scores, potentially nonzero on the diagonal, are outside that statement's hypotheses. The deterministic-count positive-operator Lemma 6.2 instead exploits nonpositive stationary mean under *every* hidden reversible model, including its reversible changes of stationary law; the earlier theorem does not directly supply that implication. For $Q_4$ the spectral window of [1, Eq. (12)] gives exponents $[2, 2\beta]$; the completion and symmetric-score arguments here give $2\gamma$, strictly between these endpoints.

Nagayama, Kolchinsky and Ito [3] decompose the stationary entropy production of Ornstein–Uhlenbeck processes into oscillatory and nonnormal contributions. This is relevant physical background, but those identities do not provide the snapshot-testing guarantees required here. Concentration bounds for time-integrated currents and activities of continuous-time Markov chains have earlier precedent in Bakewell-Smith, Girotti, Guţă and Garrahan [2].

## AI assistance disclosure

OpenAI GPT-5.6 Sol, accessed through ChatGPT and Codex, was used to develop and audit the proofs for the explicit generator and its extensions, and for drafting. Anthropic Claude Opus 5.5, used through Claude Code, was used to rewrite the note, verify the constants exactly, and derive Theorem 10.1. The author reviewed and takes responsibility for every statement and proof. AI systems are not authors of this work, and the work has not been peer reviewed.

## References

[1] Murad Aznagulov, *The spectral cost of detecting nonequilibrium at finite temporal resolution*, 2026, arXiv:2609.37205, version 1 (29 September 2026).
[2] George Bakewell-Smith, Federico Girotti, Mădălin Guţă, and Juan P. Garrahan, *General upper bounds on fluctuations of trajectory observables*, Physical Review Letters **131** (2023), 197101.
[3] Ryuna Nagayama, Artemy Kolchinsky, and Sosuke Ito, *Oscillatory-nonnormal decomposition of dissipation in Ornstein-Uhlenbeck processes*, 2026, arXiv:2606.07263, version 2 (1 July 2026).

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