BKT asymptotics for the quenched free energy of hierarchical pinning

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Abstract

We study the disordered pinning model of Derrida, Hakim and Vannimenus on the diamond hierarchical lattice. Its partition functions satisfy $R_{n+1} \overset{d}{=} (R_n^{(1)} R_n^{(2)} + B - 1)/B$, where $B > 2$, the two variables on the right are independent copies of R_n, and $R_0 = e^h W$ with $W > 0$ and $\mathbb{E}W = 1$. Let F be the quenched free energy and h_c its critical point. If $h_c > \log(B-1)$, we prove $F(h_c + u) = \exp[-(K + o(1))/\sqrt{u}]$ as $u \downarrow 0$, for a finite positive constant K characterized by the response at criticality to a change of field. We identify a limiting profile of the rescaled tilted measures and extend their convergence in exponential weights to every time before its first pole. This yields the free-energy upper bound; a comparison with a max-type recursion and a second-moment estimate for pairs of leaves give the matching lower bound. A Lean 4 formalization of the main results accompanies the paper.

1 Introduction

Pinning models describe a path interacting with a defect on which the contact rewards may be random; see Giacomin [15, 16] for background. We study the diamond hierarchical model of Derrida, Hakim and Vannimenus [11, Section 4.2, (4.10)], whose partition function satisfies an exact distributional recursion. Our aim is to determine how its quenched free energy vanishes as the contact reward h decreases to the critical point h_c.

The predicted Berezinskii–Kosterlitz–Thouless (BKT) singularity is an exponential decay in the inverse square root of the distance to criticality [12, 29]. We prove that $\log F(h_c + u) \sim -K/\sqrt{u}$ as $u \downarrow 0$, with $K \in (0, \infty)$, whenever the quenched and annealed critical points differ. We also characterize K through the response of the critical recursion to a change of field.

1.1 The model and main result

Let $B > 2$, let $W > 0$ satisfy $\mathbb{E}W = 1$, and set

$$R_0(h) = e^h W, \qquad R_{n+1}(h) \overset{d}{=} \frac{R_n^{(1)}(h) R_n^{(2)}(h) + c}{B}, \qquad c = B - 1, \tag{1.1}$$

where the two variables on the right are independent copies of $R_n(h)$. For integer B, this is the normalized partition function on a diamond lattice of longitudinal size 2^n, with random rewards on a distinguished path; the recursion also makes sense for every real $B > 2$ [17, Section 1.2]. The field h controls the contact reward, while the law of W is fixed.

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The quenched free energy and its critical point are

$$F(h) = \lim_{n\to\infty} 2^{-n}\mathbb{E}\log R_n(h), \qquad h_c = \sup\{h : F(h) = 0\}.$$

The annealed model follows the deterministic recursion for $\mathbb{E}R_n$ and has critical point $h_a = \log c$ [17, Theorem 1.2]. The standard existence and finite-volume bounds for the quenched free energy [17, Theorem 1.1 and Appendix A.1] are recalled in Lemma 2.1, with the elementary adaptation needed for the present finite-mean assumption. In particular, $F \geq 0$ and $h_a \leq h_c < \infty$. Our assumption $h_c > h_a$ means that disorder shifts the critical point.

To state the constant, write $a = \log B$ and $\theta = \log 2/a \in (0,1)$. Unless a field is specified, all moments below are evaluated at h_c. Set $A_n = \mathbb{E}R_n^\theta$, $\Pi_n = \prod_{j<n} A_j$, and

$$H_n(h) = \mathbb{E}[R_n(h)^\theta(\log R_n(h) - a)], \qquad J_n = \mathbb{E}[R_n^\theta(\log R_n - a)^3].$$

Theorem 1.1. *Suppose $h_c > h_a$. The limits*

$$\ell_3 = \lim_{n\to\infty} \frac{J_n}{\Pi_n}, \qquad \kappa = \lim_{n\to\infty} \frac{\partial_h H_n(h_c)}{\Pi_n} \tag{1.2}$$

exist and lie in $(0,\infty)$. Moreover, $\Pi_n \sim 3a^3n^2/(2\ell_3)$. With

$$\gamma = \frac{3a^2\kappa}{2\ell_3} = \lim_{n\to\infty} \frac{\partial_h H_n(h_c)}{an^2}, \tag{1.3}$$

one has

$$\lim_{u\downarrow 0} \sqrt{u}\,\log F(h_c + u) = -\frac{\pi\log 2}{\sqrt{\gamma}}. \tag{1.4}$$

Thus the constant in the BKT law is $K = \pi\log 2/\sqrt{\gamma}$. The assumption $\mathbb{E}W = 1$ normalizes a finite positive mean. It supplies every fixed tilted logarithmic moment $\mathbb{E}[R_n^\theta|\log R_n|^q]$ with $q \geq 0$, including at $n = 0$, as shown in Lemma 2.1; no additional logarithmic moment assumption is imposed. For $W = e^{\beta\omega-\beta^2/2}$ with ω standard normal and $\beta > 0$, the strict inequality $h_c > h_a$ holds whenever $B > 2 + \sqrt{2}$ [17, Theorem 1.4(2)]. The case $h_c = h_a$ is outside the theorem.

1.2 Relation to earlier work

The rigorous framework for (1.1) was developed by Giacomin, Lacoin and Toninelli [17]. Related results include the hierarchical smoothing bound for Gaussian disorder of Lacoin and Toninelli [23, Theorem 3.1 of the author version] and Gaussian marginal relevance at $B = 2 + \sqrt{2}$, in the present normalization, proved by Giacomin, Lacoin and Toninelli [18, Remark 2.1 and Theorem 2.3]. Tang and Chaté [29] predicted a BKT transition through rare-contact renormalization, and Derrida and Retaux [12, Section 4] supplied numerical evidence for (1.1). Monthus and Garel [26, Sections II–III] investigated a competing scenario with a broad critical law and a finite correlation-length exponent.

The link with max-type recursions is seen in logarithmic coordinates. Writing $X_n = \log R_n/a$, equation (1.1) becomes

$$X_{n+1} \overset{d}{=} \frac{1}{a}\log\left(B^{X_n^{(1)}+X_n^{(2)}-1} + 1 - B^{-1}\right). \tag{1.5}$$

This is a smooth analogue of the *hard recursion* $Y_{n+1} = (Y_n^{(1)} + Y_n^{(2)} - 1)^+$. We refer to (1.1) as the *soft recursion*. The hard recursion was studied in the spin-glass setting by Collet, Eckmann, Glaser and Martin [7, 8] and subsequently by Derrida and Retaux [12]; see also Derrida and Shi [13, Section 1]. Unlike Y_n, the logarithmic variable X_n need not be nonnegative; the rigorous comparison in Section 8 uses a deterministic lower bound for R_n.

The identity $R_n^\theta = 2^{X_n}$ gives $H_n = a\mathbb{E}[2^{X_n}(X_n-1)]$ and $J_n = a^3\mathbb{E}[2^{X_n}(X_n-1)^3]$. For nonnegative integer-valued hard laws with $\mathbb{P}(Y_0 \geq 2) > 0$ and $\mathbb{E}[Y_0 2^{Y_0}] < \infty$, the condition $\mathbb{E}[(Y_0-1)2^{Y_0}] = 0$ describes the critical manifold [8]. This explains the tilted moments used here; the soft recursion introduces correction terms in their evolution.

For the integer-valued hard recursion, Chen, Dagard, Derrida, Hu, Lifshits and Shi [2, Theorem 1.1] prove the exponent $1/2$ as $p \downarrow p_c$ along the family $(1-p)\delta_0 + p\mathcal{L}(Y_0^\star)$, where $Y_0^\star$ is positive and integer-valued, $\mathbb{P}(Y_0^\star \geq 2) > 0$ and $\mathbb{E}[(Y_0^\star)^3 2^{Y_0^\star}] < \infty$, without identifying a limiting constant. Other integrability regimes can lead to different exponents; see Chen, Dagard, Derrida, Hu, Lifshits and Shi [2, Theorem 1.2] for positive critical parameter, Hu and Shi [22] for regimes with zero critical parameter, and the survey of Derrida and Shi [13, Sections 6.2 and 8]. In the present logarithmic coordinates, the finite-mean assumption is $\mathbb{E}B^{X_0} < \infty$, with $B > 2$, so it already supplies an exponential moment beyond the tilt 2^{X_0}.

Precise BKT constants were obtained by Hu, Mallein and Pain [21, Theorem 1.10(i)–(ii)] for an exactly solvable continuous-time family. For discrete-time models with geometric offspring, Li and Zhang [24, Theorems 1.1 and 1.3] study geometric-type invariant laws, and Alsmeyer, Hu and Mallein [1, Theorems 1.3, 2.6(b) and 2.12(b)] establish precise constants for solvable families. These results concern recursions different from (1.1). Theorem 1.1 treats the diamond recursion itself, for arbitrary positive finite-mean disorder with a strict critical-point shift.

1.3 Main ideas and organization

The proof relates the response at criticality to the number of generations over which a small field perturbation changes the recursion. For the binary hard recursion, critical product bounds and the cubic-moment method go back to Collet, Eckmann, Glaser and Martin [8, Appendix III of the 1983 archival version]; see Chen, Derrida, Hu, Lifshits and Shi [4, Section 2] for the general-branching treatment. Here estimates on the Mellin transforms $G_n(q) = \mathbb{E}R_n^q$, including the soft correction terms, give the critical product scale $\Pi_n \asymp n^2$. Moments of the Gibbs contact count, the number of rewarded leaves in the partition-function expansion, control changes of field in the window $h - h_c = \lambda/N^2$. At rescaled time $t = n/N$, these estimates give compactness of the rescaled tilted measures, which weight the law of $\log R_n/(aN)$ by $N(R_n^\theta - 1)^+$; see (5.1). They also determine the measures' behavior as $t \downarrow 0$, which we call the *entrance data*. The field response fixes the normalization γ.

The limiting coalescence equation and its exponential profiles were identified by Derrida and Retaux [12] and Chen, Dagard, Derrida and Shi [3]. We select the profile from the entrance data by an entire-function argument. For $\lambda > 0$, put $v = \gamma\lambda$; its first pole occurs at rescaled time $T_v = \pi/\sqrt{v}$. Two points require additional control for the soft recursion: the additive correction in (1.1) must be estimated uniformly along the near-critical orbit, and the unweighted mass of the limiting profile becomes infinite before the pole. We therefore extend the convergence in exponential weights.

The weighted convergence gives the free-energy upper bound. For the lower bound, a comparison with the hard recursion shows that sufficient weighted mass in one bounded interval forces growth within a controlled number of generations. The pair-counting argument follows the two-leaf amplification method of Hu and Shi [22, Section 4.1]. For $u = \lambda/N^2$, the pole corresponds to the energy scale

$$NT_v = \frac{\pi}{\sqrt{\gamma u}}, \qquad 2^{-NT_v} = \exp\left(-\frac{\pi \log 2}{\sqrt{\gamma u}}\right).$$

The matching bounds turn this scale into the limit in (1.4).

Sections 2–4 establish the critical and field estimates. The proof of the scaling limit proceeds through compactness (Section 5), identification of the profile (Section 6), and continuation before the

pole (Section 7). Section 8 proves both free-energy bounds and concludes the proof of Theorem 1.1.

1.4 AI use

This work was developed through extensive, iterative interaction with OpenAI's ChatGPT, including the use of multiple AI agents. ChatGPT proposed proof strategies and generated and revised substantial portions of the mathematical arguments and manuscript text. This contribution included the central argument for the Berezinskii–Kosterlitz–Thouless asymptotics of the free energy in the disordered hierarchical pinning model on a diamond lattice. The author selected the research problem and directed the investigation and successive revisions, including the simplification of arguments, organization of the manuscript and attribution of prior work. ChatGPT also assisted with literature searches and reference checks. Claude was used for editorial and bibliographic audits.

All numbered results of the paper (the Gaussian example of the introduction excepted) were then formalized in Lean 4 with the Mathlib library, entirely by AI agents, in about 43 000 lines of Lean. The formalization is available at `https://github.com/igor-kortchemski/BKT-hierarchical-pinning`. The formalization was coordinated by Anthropic's Claude Code running Claude Opus 5.5, which delegated statements and proofs to agents running Claude Opus 5.5 and Claude Sonnet 5.5, OpenAI's GPT-6.2 Sol, GPT-6 Luna and GPT-6.2 Astra (through Codex), and Moonshot AI's Kimi K3.1 Pro (through Open Math OS). The author fixed the scope and the conventions of the formal statements, and delegated the approval of the Lean statements to GPT-6.2 Astra. Each proof was re-audited by a second model, usually of another family, for fidelity to the paper. The main theorem was checked with Comparator against a self-contained restatement depending only on Mathlib, by Lean's kernel and the independent checkers NanoDa, Lean4Lean, con-leche and con-ron, with no axioms beyond `propext`, `Classical.choice` and `Quot.sound`. No human has reviewed the Lean statements or proofs line by line. The formalization took about twelve hours of wall-clock time on 18–19 November 2026, largely unattended. It used about 1.5 billion tokens: Claude Opus 5.5 0.73 billion, Kimi K3.1 Pro 0.36 billion, GPT-6.2 Sol 0.21 billion, GPT-6 Luna 0.16 billion, Claude Sonnet 5.5 15 million and GPT-6.2 Astra 14 million. Almost all of these are input tokens, the agents' context re-read at each step and mostly served from cache. Output tokens, reasoning included, number about 6 million.

2 Basic bounds and critical moments

2.1 Floor and finite-volume bounds

Here $h \in \mathbb{R}$ is arbitrary and R_n denotes $R_n(h)$.

We recall the finite-volume argument of Giacomin, Lacoin and Toninelli [17, Theorem 1.1 and Appendix A.1], keeping the deterministic floor explicit under the present assumption $\mathbb{E}W = 1$.

Let $\phi_B(x) = (x^2 + c)/B$, $b_0 = 0$ and $b_{n+1} = \phi_B(b_n)$. We write $\rho = 2/B \in (0,1)$.

Lemma 2.1. *We have $R_n \geq b_n \uparrow 1$ and $1 - b_n = O(\rho^n)$. For $n \geq 0$, $\mathbb{E}R_n < \infty$ and $\mathbb{E}[R_n^\theta |\log R_n|^q] < \infty$ for every fixed $q \geq 0$. The free energy F exists, is nonnegative and satisfies*

$$2^{-n}(\mathbb{E}\log R_n - a) \leq F(h) \leq 2^{-n}(\mathbb{E}\log R_n - \log b_n), \qquad n \geq 1. \tag{2.1}$$

Moreover, $0 \leq F(h+u) - F(h) \leq u$ for $u \geq 0$, $h_a \leq h_c < \infty$ and $F(h_c) = 0$.

Proof. Monotonicity of (1.1) gives $R_n \geq b_n \uparrow 1$. Since $1 - b_{n+1} = B^{-1}(1 - b_n)(1 + b_n) \leq \rho(1 - b_n)$, we obtain the stated rate. The recursion preserves a finite first moment. The tilted logarithmic moments are finite because $r^\theta |\log r|^q$ is bounded near zero and is $O(r)$ as $r \to \infty$; this also covers $n = 0$.

The two inequalities $\log R_{n+1} \geq \log R_n^{(1)} + \log R_n^{(2)} - a$ and $R_{n+1}/b_{n+1} \leq (R_n^{(1)}/b_n)(R_n^{(2)}/b_n)$ show that the left-hand side of (2.1) increases with n and the right-hand side decreases. Their difference tends to zero, proving existence of F and (2.1). The floor gives $F \geq 0$.

The partition function is multiaffine in its 2^n leaf weights, with nonnegative coefficients. Hence $R_n(h) \leq R_n(h+u) \leq e^{2^n u} R_n(h)$ for $u \geq 0$, which gives the asserted Lipschitz bound. At $h = h_a$, $\mathbb{E}R_n = c$ for every n, so Jensen's inequality gives $F(h_a) = 0$. As $h \to \infty$, $\log R_1(h)$ increases to infinity and is bounded below by $\log b_1$. Thus $\mathbb{E} \log R_1(h) \to \infty$, and (2.1) yields $h_c < \infty$. Continuity now gives $F(h_c) = 0$. □

For $S_n = R_n^{(1)} R_n^{(2)}$ and $n \geq 1$, set

$$C_n(z) = \mathbb{E}[(S_n + c)^z - S_n^z], \qquad \Re z < 1.$$

This correction term records the contribution of the additive constant in (1.1).

Lemma 2.2. *The functions C_n, $n \geq 1$, are holomorphic and uniformly bounded on compact subsets of $\{\Re z < 1\}$. They satisfy*

$$C_n(z) = z \int_0^c \mathbb{E}(S_n + t)^{z-1} \, \mathrm{d}t. \tag{2.2}$$

Proof. Integrate the derivative of $(S_n + t)^z$ over $t \in [0, c]$ to obtain (2.2). Since $S_n + t \geq b_1^2 > 0$, the integrand and its z-derivatives are uniformly bounded on each compact subset of $\{\Re z < 1\}$. Differentiation under the integral and expectation proves the assertion. □

2.2 Critical moment bounds

Write $G_n(q) = \mathbb{E}R_n^q$. We first control the fractional moment at θ and its logarithmic derivative when $F(h) = 0$.

Proposition 2.3. *If $F(h) = 0$, then, for every $n \geq 1$,*

$$G_n(\theta) \leq c^\theta, \qquad H_n := \mathbb{E}[R_n^\theta (\log R_n - a)] \leq C_B \rho^n. \tag{2.3}$$

Moreover, $\sup_n \mathbb{E}[R_n^\theta |\log R_n - a|] < \infty$.

Proof. Fractional moments. We first obtain a tail bound below θ and then use the moment recursion to reach θ. Fix n, $t \geq b_n$ and $p = \mathbb{P}(R_n \geq t) > 0$. In a tree of 2^k independent height-n blocks, retain the favorable blocks and their ancestors. In units of $\log(R/b_n)$, each retained block contributes at least $\log(t/b_n)$ and each ancestor costs at most $\log(B/b_n)$. The expected number of retained ancestors at height j above the blocks is at most $2^{k-j} \min(1, 2^j p)$. Dividing by 2^k and letting $k \to \infty$, we obtain, since $F = 0$,

$$p \log(t/b_n) \leq \log(B/b_n) \sum_{j \geq 1} \min(2^{-j}, p) \leq p \log(B/b_n) \big(\log_2(1/p) + 2\big).$$

Thus, with $\theta_n = \log 2 / \log(B/b_n) \uparrow \theta$,

$$\mathbb{P}(R_n \geq t) \leq \min\{1, 4(b_n/t)^{\theta_n}\}. \tag{2.4}$$

This is a floor-normalized adaptation of the favorable-block argument for the hierarchical model in Lacoin and Toninelli [23, proof of Theorem 3.1, (3.7)–(3.10)]. The earlier rare-stretch argument appears in Giacomin and Toninelli [19, Section 3.1], and the rare-contact mechanism in Tang and Chaté [29]. In particular, $\sup_n G_n(q) < \infty$ for each $0 < q < \theta$.

For $0 < q < 1$, convexity of $(v^{1/q} + c)^q$ gives, with $s_n = G_n(q)^{1/q}$,

$$s_{n+1} \geq (s_n^2 + c)/B. \tag{2.5}$$

If $s_n > c$, this recursion would force divergence, contrary to the tail bound. Thus $G_n(q) \leq c^q$ for $q < \theta$. Letting $q \uparrow \theta$ proves the first assertion.

The endpoint logarithmic moment. Set $H_n(q) = \mathbb{E}[R_n^q(\log R_n - a)]$ and $\psi_q(r) = (r+c)^q(\log(r+c) - 2a) - r^q(\log r - 2a)$. The exact identity is

$$H_{n+1}(q) = 2B^{-q}G_n(q)H_n(q) + B^{-q}\mathbb{E}\psi_q(R_n^{(1)}R_n^{(2)}). \tag{2.6}$$

For $0 < q \leq \theta$, $\psi_q(r) \geq 0$ on $r \geq 1$. Indeed, $r^q(\log r - 2a)$ decreases up to $r_0 = B^2 e^{-1/q} < B$ and then increases. If $r_0 > 1$, the difference ψ_q increases on $[1, r_0]$ from $\psi_q(1) = a(2 - B^q) \geq 0$; beyond r_0 the claim is immediate.

Using $R_n \geq b_n$ and $1 - b_n = O(\rho^n)$, we obtain, uniformly for $q \in [\theta/2, \theta]$, $H_{n+1}(q) \geq d_q G_n(q)H_n(q) - C_B\rho^n$, where $d_q = 2B^{-q}$. For fixed $q < \theta$, $H_j(q)$ is bounded by (2.4), whereas

$$\prod_{i=n}^{j-1} d_q G_i(q) \geq c_B d_q^{j-n}, \qquad d_q > 1.$$

We divide by these products and sum to infinity. The error is at most $C_B\rho^n \sum_{k\geq 0}(\rho/d_q)^k$, uniformly as $q \uparrow \theta$. Hence $H_n(q) \leq C'_B\rho^n$. Monotone convergence on $R_n \geq B$ and bounded convergence for the negative part give the endpoint bound. The bounded negative part also yields the asserted absolute moment. □

2.3 The critical barrier and returns to one

From now on, we assume $h = h_c > h_a$. The following barrier is a floor-centered version of the fractional-moment argument in Giacomin, Lacoin and Toninelli [17, Proposition 4.1, (4.1)–(4.3)].

Lemma 2.4 (Critical barrier). *For $\theta < q < 1$ and $n \geq 1$,*

$$G_n(q) \geq b_n^q + B^q - 2. \tag{2.7}$$

Proof. Since $(r+c)^q - r^q$ decreases, $Y_n(q) = G_n(q) - b_n^q \geq 0$ satisfies

$$Y_{n+1}(q) \leq B^{-q}\big(2b_n^q Y_n(q) + Y_n(q)^2\big) \leq B^{-q}\big(2Y_n(q) + Y_n(q)^2\big). \tag{2.8}$$

If $Y_n(q) < B^q - 2$, subsequent values decay geometrically and $F = 0$. At finite height, $R_n(h+u) \leq e^{2^n u}R_n(h)$, so this strict inequality persists for some $h_c + u > h_c$, a contradiction. □

Theorem 2.5. *If $h_c > h_a$, then*

$$\liminf_{n\to\infty} G_n(\theta) = 1. \tag{2.9}$$

Proof. We first rule out uniform moments above θ, then apply the same continuation argument at θ.

A uniform continuation criterion. For a related continuation argument for fixed laws of the hard recursion, see Derrida and Retaux [12, Appendix B]. The family $C(z) = (C_n(z))_{n\geq 1}$ defined in (2.2) is holomorphic with values in ℓ^∞ on $\Re z < 1$ and bounded on compact subsets. Each G_n is holomorphic there as well, because $\mathbb{E}R_n < \infty$, and

$$G_n(z)^2 = B^z G_{n+1}(z) - C_n(z). \tag{2.10}$$

Suppose $g = (G_n(q_0))_{n\ge N}$ is bounded, bounded away from zero, and

$$\sup_{n\ge N} \frac{B^{q_0}}{2G_n(q_0)} < 1. \tag{2.11}$$

The map $(\mathcal{K}_z v)_n = \sqrt{B^z v_{n+1} - C_n(z)}$, with the positive branch at (q_0, g), is a uniform contraction in a small ℓ^∞ ball: its radicands stay away from zero. Its fixed point is holomorphic in z and locally unique. If $G(q) \to g$ in ℓ^∞ as $q \uparrow q_0$, it agrees with this fixed point on the left. The coordinatewise identity theorem then gives a uniform moment bound to the right of q_0.

No uniform moment above θ. Suppose such a moment exists and set

$$q_* = \sup\{q \in (0,1) : \sup_{n\ge1} G_n(q) < \infty\} > \theta.$$

For $q < q_*$, (2.5) implies $G_n(q) \le c^q$. If $q_* < 1$, we discard finitely many indices and use (2.7) to obtain $G_n(q) \ge B^q - 1 - \varepsilon$ near q_*. Since $B^{q_*} > 2$, the neighborhood and ε can be chosen so that

$$\frac{B^{q_2}}{G_n(q_1) + G_n(q_2)} \le \eta < 1 \qquad (q_1 < q_2 < q_*).$$

Subtracting (2.10) at q_1, q_2 therefore gives

$$|G_n(q_2) - G_n(q_1)| \le \eta |G_{n+1}(q_2) - G_{n+1}(q_1)| + C|q_2 - q_1|.$$

Taking the supremum shows that $G(q)$ is Lipschitz in ℓ^∞ up to q_*. Its limit is $G(q_*)$ by coordinatewise continuity. The continuation criterion crosses q_*, a contradiction. Thus $q_* = 1$, and Fatou's lemma gives $\mathbb{E}R_n \le c$. This is impossible: the means follow $x \mapsto (x^2 + c)/B$ from $\mathbb{E}R_0 = e^{h_c} > c$.

Returns to one. If $G_n(\theta) \ge 1 + \delta$ for all large n, (2.11) holds at $q_0 = \theta$, since $B^\theta = 2$. Proposition 2.3 gives $\|G(q) - G(\theta)\|_\infty \le C(\theta - q)$ near θ. The criterion would produce a uniform moment above θ. Together with $G_n(\theta) \ge b_n^\theta \to 1$, this proves (2.9). □

3 The critical product

For the binary hard recursion, the critical product bounds and cubic-moment method already appear in Collet, Eckmann, Glaser and Martin [8, Appendix III of the 1983 archival version]. The general-branching treatment and further critical estimates are developed in Chen, Derrida, Hu, Lifshits and Shi [4, Section 2, Theorem 3 and Propositions 1–2]. Here the moment recurrences retain the soft correction kernels. For the diamond model, Giacomin, Lacoin and Toninelli [17, Theorem 1.6] prove tightness at criticality and convergence to one strictly below it; the argument below treats convergence at $h_c > h_a$.

At $h = h_c > h_a$, set

$$A_n = \mathbb{E}R_n^\theta, \quad \Pi_n = \prod_{j<n} A_j, \quad Z_n = \log R_n - a, \quad (K_n, J_n, L_n) = \mathbb{E}[R_n^\theta (Z_n^2, Z_n^3, Z_n^4)].$$

All constants below are independent of n.

Theorem 3.1. *For $n \ge 1$,*

$$C^{-1}(n+1)^2 \le \Pi_n \le C(n+1)^2, \qquad \sum_{n\ge1} \Pi_n^{-1} < \infty, \qquad K_n^2 \le C\Pi_n. \tag{3.1}$$

Moreover,

$$|H_n| \le C_B\big((A_n-1)^+ + \rho^n\big), \tag{3.2}$$

$$\ell_3 := \lim_{n\to\infty} J_n/\Pi_n \in (0,\infty), \tag{3.3}$$

$$L_n \le C(n+1)^3. \tag{3.4}$$

As $n \to \infty$,

$$A_n - 1 \asymp n^{-1}, \qquad K_n \asymp n, \qquad H_n = O(n^{-1}), \qquad R_n \longrightarrow 1 \quad \textit{in probability.} \tag{3.5}$$

Proof. We first show that the product diverges and that its reciprocal is summable. A positive limiting cubic residue will then give the matching upper bound on Π_n.

Exact identities. The tilted logarithmic moments are finite by Lemma 2.1. Let $U_n = R_n^{(1)} R_n^{(2)}$ and

$$\omega_r(x) = (x+c)^\theta(\log(x+c) - 2a)^r - x^\theta(\log x - 2a)^r.$$

For $r = 1,\dots,4$, these kernels are bounded on $[b_1^2,\infty)$ and vanish at infinity. Independence gives

$$H_{n+1} = A_n H_n + \tfrac{1}{2}\mathbb{E}\omega_1(U_n), \tag{3.6}$$

$$K_{n+1} = A_n K_n + H_n^2 + \tfrac{1}{2}\mathbb{E}\omega_2(U_n), \tag{3.7}$$

$$J_{n+1} = A_n J_n + 3H_n K_n + \tfrac{1}{2}\mathbb{E}\omega_3(U_n), \tag{3.8}$$

$$L_{n+1} = A_n L_n + 4H_n J_n + 3K_n^2 + \tfrac{1}{2}\mathbb{E}\omega_4(U_n). \tag{3.9}$$

Here $\omega_2(1) = -2a^2$ and $\omega_3(1) = 6a^3$. Proposition 2.3 gives $A_n \le c^\theta$, $|H_n| \le C_B$, $H_n^+ \le C_B\rho^n$ and $\mathbb{E}[R_n^\theta |Z_n|] \le C_B$. Also $J_n^- \le C_B$. Since $A_n \ge b_n^\theta$ and $\prod_{n\ge1} b_n > 0$, Π_n tends either to a positive finite limit or to infinity.

Write $\alpha_n = A_n - 1$, $d(x) = 1 - (x+c)^\theta + x^\theta$ and $D_n = \mathbb{E}d(U_n)$. On $x \ge 1$, $d(x) \ge 0$ and $0 \le \omega_1(x) \le C_B d(x)$. The sign follows from the proof of Proposition 2.3. For the comparison, we differentiate near one and use $d(x) \to 1$, $\omega_1(x) \to 0$ at infinity. The floor yields

$$D_n \ge -C_B\rho^n, \qquad |\mathbb{E}\omega_1(U_n)| \le C_B D_n + C_B'\rho^n. \tag{3.10}$$

Since $\alpha_{n+1} = \alpha_n + \alpha_n^2/2 - D_n/2$, telescoping gives

$$\sum_{j=n}^{m} \frac{\Pi_n D_j}{\Pi_{j+1}} = 2\alpha_n - \sum_{j=n}^{m} \frac{\Pi_n \alpha_j^2}{\Pi_{j+1}} - \frac{2\Pi_n \alpha_{m+1}}{\Pi_{m+1}}. \tag{3.11}$$

The product diverges. Suppose $\Pi_n \to \Pi_\infty < \infty$. Its negative logarithmic increments are summable, so $\sum \alpha_n^+ < \infty$ and $R_n \to 1$ in probability. By (3.10), the bounded sequence H_n/Π_n has summable negative increments. Thus $H_n \to h \in [-a, 0]$; the lower bound follows from $r^\theta(\log r - a) \ge -a$ on $r \ge 1$ and the floor. If $h > -a$, (3.7) would make the increments of K_n/Π_n tend to $(h^2 - a^2)/\Pi_\infty < 0$, contradicting $K_n \ge 0$. Consequently $M_n := \mathbb{E}[R_n^\theta \log R_n] = H_n + aA_n \to 0$.

For small $t \ge 0$, the nonnegative Taylor remainder $E_n(t) = G_n(\theta + t) - A_n - tM_n$ satisfies

$$E_{n+1}(t) \le (A_n + Ct)E_n(t) + CE_n(t)^2 + Ct^2.$$

Indeed, we expand (2.10); the first two derivatives of C_n are uniformly bounded near θ. After division by the bounded product, $e_{n+1} \le (1+Ct)e_n + Ce_n^2 + Ct^2$ and $e_1 = O(t^2)$. Choosing a large constant and then a small τ, we obtain by induction

$$E_n(t) \le C(n+1)t^2 \qquad ((n+1)t \le \tau). \tag{3.12}$$

The induction costs $O(\tau t^2)$ from te_n and $O(\tau^2 t^2)$ from e_n^2 at each step.

The summable sequence $d_n = \alpha_n^+ + \rho^{n/2}$ has a subsequence along which $nd_n \to 0$. Fix $k > 1/(2a)$ and put $t_n = kd_n$. Along this subsequence,

$$G_n(\theta + t_n) \le 1 + d_n + t_n M_n + C(n+1)t_n^2 = 1 + d_n + o(d_n),$$

whereas (2.7) gives $G_n(\theta + t_n) \ge 1 + 2akd_n - C_B\rho^n$. This contradiction proves $\Pi_n \to \infty$.

We can now let $m \to \infty$ in (3.11). All ratios Π_{j+1}/Π_n are uniformly bounded below, so (3.10) gives

$$\sum_{j \ge n} \frac{\Pi_n |\mathbb{E}\omega_1(U_j)|}{\Pi_{j+1}} \le C_B(\alpha_n^+ + \rho^n).$$

Summing (3.6) backwards, with $H_n/\Pi_n \to 0$, proves (3.2).

A summable inverse product. To use an increasing product, we remove the floor by setting

$$1 + z_n = A_n/b_n^\theta, \qquad P_n = \prod_{j=1}^{n-1}(1+z_j) \asymp \Pi_n, \qquad T_n = 1 + \sum_{j=1}^{n-1} P_j^{-1}.$$

The sequence P_n is increasing, $P_n \ge 1$, and its ratios are bounded. Equations (3.7)–(3.8) imply $K_n/\Pi_n \le CT_n$ and $J_n/\Pi_n \le CT_n$: the only additional positive term is $O(\rho^n T_n)$, summable since $T_n \le n$. Cauchy–Schwarz and the bounded negative part of Z_n give

$$K_n^2 \le C(J_n + C_B) \le CP_nT_n. \tag{3.13}$$

For small z_n and large n, (3.2) gives $\mathbb{E}(V \log(R_n/b_n)) \ge a/2$, where $V = (R_n/b_n)^\theta$. Since $\mathbb{E}(V-1) = z_n$ and $\mathbb{E}\log(R_n/b_n) \le z_n/\theta$, Cauchy–Schwarz under the positive measure $(V-1)\,d\mathbb{P}$ yields

$$(a/4)^2 \le z_n \mathbb{E}[(V-1)\log^2(R_n/b_n)] \le Cz_n(K_n+1).$$

For larger z_n the same lower bound is immediate. Hence, from a fixed index onwards,

$$z_n \ge \frac{c_0}{K_n+1} \ge \frac{c_1}{\sqrt{P_nT_n}}. \tag{3.14}$$

For $x_n = \sqrt{P_n}$ this gives

$$x_{n+1} - x_n \ge cT_n^{-1/2}, \qquad T_{n+1} - T_n = x_n^{-2}, \qquad x_n^{-1} - x_{n+1}^{-1} \ge c\frac{T_{n+1}-T_n}{\sqrt{T_n}}.$$

Summing the last inequality bounds $\sqrt{T_n}$, hence T_n. It follows that $x_n \ge cn$, $\sum \Pi_n^{-1} < \infty$ and $K_n^2 \le C\Pi_n$. Also $J_n \le C\Pi_n$.

The cubic residue is positive. The positive increments of J_n/Π_n are summable by (3.8), since $H_n^+ \le C\rho^n$ and $\sum \Pi_n^{-1} < \infty$. As $J_n \ge -C_B$, this sequence converges to $\ell_3 \ge 0$. Suppose $\ell_3 = 0$. Summing backwards gives

$$J_n \le CP_n \sum_{j \ge n} \frac{(z_j + \rho^j)K_j + 1}{P_{j+1}}. \tag{3.15}$$

Assume $K_n \le MP_n^g$ for $n \ge N$, where $M \ge 1$ and $0 < g \le 1/2$. The index N is fixed independently of M, g; the estimate already holds for $g = 1/2$. Bounded ratios and (3.14) imply $P_{n+1}^g - P_n^g \ge cg/M$, so

$$\sum_{j \ge n} P_j^{-1} \le P_n^{-1} + \frac{M}{c(1-g)} P_n^{g-1}.$$

Moreover, $1-r^{-(1-g)} \geq (1-g)(1-r^{-1})$ for $r \geq 1$ gives

$$\sum_{j \geq n} \frac{z_j P_j^g}{P_{j+1}} \leq \frac{P_n^{g-1}}{1-g}.$$

The term with ρ^j has the same bound. Thus (3.15) and (3.13) improve the estimate to

$$J_n \leq CMP_n^g, \qquad K_n \leq \sqrt{C_0 M}\, P_n^{g/2},$$

with C_0 independent of g and M. We choose $M \geq C_0$ large enough for the initial estimate and iterate. The same M, N work at every step; letting $g \downarrow 0$ at fixed n gives $K_n \leq M$. Then (3.14) contradicts Theorem 2.5. Hence $\ell_3 > 0$.

The matching upper bound. Since $J_n^+ \leq C\Pi_n$, $J_n^- \leq C_B$ and $H_n^+ \leq C\rho^n$, we have $H_n J_n \leq C\rho^n \Pi_n + C_B$. Dividing (3.9) by Π_{n+1} and using $K_n^2 \leq C\Pi_n$, we obtain $L_n \leq C(n+1)\Pi_n$. The positive cubic residue and Cauchy–Schwarz now give

$$c\Pi_n^2 \leq J_n^2 \leq K_n L_n \leq C(n+1)\Pi_n^{3/2}.$$

Thus $\Pi_n \leq C(n+1)^2$, proving (3.1) and (3.4).

Rates. Subtracting the floor recurrence, as in (2.8), gives

$$z_{n+1} \leq \frac{b_n^{2\theta}}{b_{n+1}^{\theta}}(z_n + z_n^2/2) \leq z_n + z_n^2/2.$$

Reading backwards gives $z_j \geq z_n/[1+(n-j)z_n/2]$. On the other hand, bounded z_j and $P_n \asymp n^2$ imply

$$\sum_{j=\lfloor n/2 \rfloor}^{n} z_j \leq C \log \frac{P_{n+1}}{P_{\lfloor n/2 \rfloor}} \leq C'.$$

Summing the lower bound yields $2\log(1+nz_n/4) \leq C''$. Hence $z_n = O(n^{-1})$, $A_n - 1 = O(n^{-1})$ and, by (3.2), $H_n = O(n^{-1})$. The floor and $\mathbb{E}(R_n^\theta - b_n^\theta) \to 0$ imply $R_n \to 1$.

Finally, $K_n^2 \leq C\Pi_n \leq Cn^2$ gives $K_n \leq Cn$, while (3.14) and $z_n \leq C/n$ give $K_n + 1 \geq c/z_n \geq cn$. Thus $K_n \asymp n$, and (3.14) also gives $z_n \geq c/n$. Since $1 - b_n^\theta = O(\rho^n)$, this proves (3.5). □

4 Field response and contact moments

Unless a field is specified, all quantities are evaluated at $h_c > h_a$. Set

$$Q_n = \partial_h \log R_n(h)|_{h_c}, \qquad \mathcal{D}_n = \partial_h \mathbb{E} R_n(h)^\theta\Big|_{h_c}, \qquad \mathcal{E}_n = \partial_h H_n(h)|_{h_c}.$$

We will use the critical estimates of Theorem 3.1:

$$\begin{aligned} &\Pi_n \asymp (n+1)^2, \qquad K_n \asymp n+1, \qquad |H_n| \leq C(n+1)^{-1}, \qquad H_n^+ \leq C\rho^n, \\ &\mathbb{E}[R_n^\theta |\log R_n - a|^4] \leq C(n+1)^3. \end{aligned} \tag{4.1}$$

4.1 The spine and a passage estimate

The distinguished-leaf change of measure uses the classical spine viewpoint [25, Section 2], already applied to the hard recursion by Hu and Shi [22, Section 4, (4.1)–(4.3)]. It represents the field derivatives in terms of a path with independent increments. The following passage estimate will control its minimum.

Lemma 4.1. *Let Y_j be independent integrable random variables and $S_n = \sum_{j<n} Y_j$. Suppose that $Y_j \geq -b$ for $j \geq 1$, $\mathbb{E}Y_0^- < \infty$, $\sum_j (\mathbb{E}Y_j)^+ < \infty$, and $\mathbb{P}(S_n \geq \eta n) \geq \delta > 0$ for all sufficiently large n. Then*

$$\mathbb{P}\left(\min_{0\leq k\leq n} S_k \geq -z\right) \leq \frac{C(z+1)}{n+1}, \qquad z \geq 0.$$

Proof. For $\tau = \inf\{k \geq 1 : S_k < -z\}$, independence gives $\mathbb{E}S_{\tau\wedge n} = \sum_{j<n} \mathbb{E}Y_j\, \mathbb{P}(\tau > j) \leq C$. The overshoot below $-z$ is at most b, except at the first step, whose expected cost is bounded by $\mathbb{E}Y_0^-$. Hence

$$\mathbb{E}[(S_n + z + b)\mathbf{1}_{\{\tau>n\}}] \leq C(z+1).$$

Both $\{\tau > n\}$ and $\{S_n \geq \eta n\}$ are increasing events in the independent increments. Harris's inequality [20, Lemma 4.1], in its standard product-measure form, bounds the left-hand side below by $\eta n \delta\, \mathbb{P}(\tau > n)$. Adjusting C covers the remaining n. □

Lemma 4.2 (Spine identities). *Under $\widehat{\mathbb{P}}$, let $R_0^*, U_0, U_1, \ldots$ be independent, with respective laws $r^\theta \mathbb{P}(R_0 \in \mathrm{d}r)/A_0$ and $r^\theta \mathbb{P}(R_j \in \mathrm{d}r)/A_j$. Set $S_k = \sum_{j<k}(\log U_j - a)$, $p = 1-\theta$ and $d_0 = 1/\theta - a > 0$. Then*

$$\frac{\mathcal{D}_n}{\Pi_n} = \theta A_0 \widehat{\mathbb{E}} w_n, \qquad w_n = \left(1 + \frac{c}{BR_0^*}\sum_{k=1}^{n} e^{-S_k}\right)^{-p}, \tag{4.2}$$

$$\frac{\mathcal{E}_n}{\Pi_n} = \theta A_0 \widehat{\mathbb{E}}[w_n(\log T_n + d_0)], \qquad T_n = R_0^* e^{S_n} w_n^{-1/p}. \tag{4.3}$$

Proof. We first justify the field differentiations used in this section. At fixed height n, on the event that all leaves are positive, $R_n(h) = \sum_{k=0}^{2^n} c_{n,k} e^{kh}$ with nonnegative coefficients. For $x(h) = \log R_n(h)$, the probability weights proportional to $c_{n,k}e^{kh}$ give $0 \leq x'(h) \leq 2^n$ and $0 \leq x''(h) \leq 4^n$. Thus $|x(h) - x(h_0)| \leq 2^n$ on $|h - h_0| \leq 1$. For each integer $j \geq 0$, the first two field derivatives of $e^{\theta x(h)}(x(h)-a)^j$ are bounded on this interval by

$$C_{n,j} R_n(h_0)^\theta (1 + |\log R_n(h_0) - a|)^j.$$

The same bound with exponent $7/2$ applies to $e^{\theta x(h)}(1 + (x(h)-a)^2)^{7/4}$. These envelopes are integrable by Lemma 2.1, including when $n = 0$. Dominated convergence therefore permits the first two field derivatives to pass under expectation at every h_0.

Under the original law, distinguish one leaf and let U_j be the mass of its sibling subtree at height j. Along the distinguished path, $T_0 = R_0$ and $T_{j+1} = (T_j U_j + c)/B$. Iterating gives

$$T_n = e^{S_n}\left(R_0 + \frac{c}{B}\sum_{k=1}^{n} e^{-S_k}\right), \qquad R_0 \partial_{R_0} T_n = R_0 e^{S_n}.$$

By multiaffinity and symmetry of the 2^n leaves, $\mathcal{D}_n = \theta 2^n \mathbb{E}[T_n^{\theta-1} R_0 e^{S_n}]$. Differentiating H_n inserts the factor $\log T_n + d_0$ in this expectation, with differentiation justified by the preceding envelopes. Changing measure and using $B^\theta = 2$ gives both identities. The leaf factor A_0 is separate from the U_0 factor in Π_n: R_0^* and U_0 are independent tilted copies. □

Lemma 4.3. *For all n, $\mathcal{D}_n \leq C(n+1)$.*

Proof. Under the tilted law, $Y_j = \log U_j - a$ satisfies

$$\widehat{\mathbb{E}}Y_j = H_j/A_j, \qquad \widehat{\mathbb{E}}Y_j^2 = K_j/A_j, \qquad \widehat{\mathbb{E}}Y_j^4 \leq C(j+1)^3.$$

The floor bounds Y_j below for $j \geq 1$, and (4.1) yields

$$-C\log(n+1) \leq \widehat{\mathbb{E}} S_n \leq C, \qquad \operatorname{Var}(S_n) \asymp n^2, \qquad \widehat{\mathbb{E}}(S_n - \widehat{\mathbb{E}} S_n)^4 \leq Cn^4.$$

For the centered variable $Z = S_n - \widehat{\mathbb{E}} S_n$, $\widehat{\mathbb{E}}|Z| \geq (\widehat{\mathbb{E}} Z^2)^{3/2}/(\widehat{\mathbb{E}} Z^4)^{1/2}$. Thus $\widehat{\mathbb{E}} Z^+ = \widehat{\mathbb{E}}|Z|/2 \geq cn$, and Paley–Zygmund applied to Z^+ gives $\widehat{\mathbb{P}}(S_n \geq \eta n) \geq \delta > 0$. Lemma 4.1 therefore applies.

With $M_n = -\min_{0 \leq k \leq n} S_k \geq 0$,

$$\widehat{\mathbb{E}} e^{-pM_n} = p \int_0^\infty e^{-pz} \widehat{\mathbb{P}}(M_n \leq z)\, \mathrm{d}z \leq \frac{C}{n+1}, \qquad w_n \leq \max\{1, (BR_0^*/c)^p\}\, e^{-pM_n}.$$

The maximum also covers a minimum attained at time zero. Its initial factor is independent of M_n and integrable: $\widehat{\mathbb{E}}(R_0^*)^p = \mathbb{E} R_0 / A_0 < \infty$. Now use (4.2) and $\Pi_n \leq C(n+1)^2$. □

4.2 Strict transversality

The relation between field response, escape time and the leading constant also appears in the solvable systems of Alsmeyer, Hu and Mallein [1, Section 5]. We establish the critical response directly for (1.1).

Theorem 4.4. *There is a constant $\kappa \in (0, \infty)$ such that*

$$\frac{\mathcal{E}_n}{\Pi_n} \longrightarrow \kappa. \tag{4.4}$$

In particular, $\mathcal{E}_n \asymp n^2$.

Proof. Positive lower bound. Fix $\ell > 1$ and consider $V_k = z + S_k - \ell \log(k+1)$, stopped at $\tau = \inf\{k \geq 1 : V_k < 0\}$. The logarithmic correction is $o(n)$, so the modified increments still satisfy Lemma 4.1. Their means are bounded below by $-C/(k+1)$. Hence

$$\widehat{\mathbb{P}}(\tau > k) \leq \min\{1, C(z+1)/(k+1)\}.$$

Since $V_\tau < 0$, stopping in the other direction gives

$$\widehat{\mathbb{E}}[V_n \mathbf{1}_{\{\tau > n\}}] \geq z - C \sum_{j \geq 0} \frac{\min\{1, C(z+1)/(j+1)\}}{j+1} \geq z - C' \log(z+2) - C'.$$

Fix z large enough that this is positive. On $\{\tau > n\}$, $\sum_{k \leq n} e^{-S_k} \leq e^z \sum_{k \geq 1} (k+1)^{-\ell}$. Choose $r_0 > 0$ with $\widehat{\mathbb{P}}(R_0^* \geq r_0) > 0$. On $\{\tau > n, R_0^* \geq r_0\}$, w_n is bounded below, and $\log T_n + d_0 \geq V_n$ for large n. The integrand in (4.3) is nonnegative everywhere for such n, since $T_n \geq b_n \uparrow 1$. Independence therefore gives

$$\liminf_n \mathcal{E}_n / \Pi_n > 0. \tag{4.5}$$

Convergence. For $n \geq 1$, let $S = R_n^{(1)} R_n^{(2)}$. Differentiation gives

$$\mathcal{D}_{n+1} = A_n \mathcal{D}_n - \mathcal{L}_n, \qquad \mathcal{L}_n = \theta \mathbb{E}\left[S^\theta Q_n^{(1)} \left(1 - \left(\frac{S}{S+c} \right)^p \right) \right] \geq 0.$$

Thus $\sum_{n \geq 1} \mathcal{L}_n / \Pi_{n+1} < \infty$. For $\psi(s) = (s+c)^\theta \log((s+c)/B^2) - s^\theta \log(s/B^2)$, differentiation with $z = s/(s+c)$ and $z^p \log(1/z) \leq (1 - z^p)/p$ give

$$s\psi'(s) \leq C s^\theta \left(1 - \left(\frac{s}{s+c} \right)^p \right), \qquad s \geq b_1^2.$$

The functions $\psi(s)$ and $s\psi'(s)$ are bounded on $[b_1^2, \infty)$, while $0 \le \partial_h \log S \le 2^{n+1}$. The floor holds at every field, so dominated convergence also justifies differentiating $\mathbb{E}\psi(S)$ locally in h. Differentiating $H_{n+1} = A_n H_n + \frac{1}{2}\mathbb{E}\psi(S)$ consequently yields

$$\mathcal{E}_{n+1} \le A_n \mathcal{E}_n + H_n \mathcal{D}_n + C\mathcal{L}_n.$$

Since $\sum |H_n| \mathcal{D}_n / \Pi_{n+1} < \infty$, the positive variations of $\mathcal{E}_n / \Pi_n$ are summable. This sequence is eventually nonnegative, so it has a finite limit, which is positive by (4.5). Since $\Pi_n \asymp n^2$, the last assertion follows. □

4.3 Contact moments

We use the contact representation of Giacomin, Lacoin and Toninelli [17, Section 1.2 and Appendix A, (A.11)]. At a fixed environment, the Gibbs measure chooses the product term $S = R_n^{(1)} R_n^{(2)}$ with probability $S/(S+c)$, adding the contact counts of the two independent subtrees; otherwise its count is zero. The initial count is $\mathcal{C}_0 = 1$. We write $\langle \cdot \rangle_n$ for the critical Gibbs expectation.

Weighted open-path moments [5, Lemma 3.2] and uniform factorial inductions [6, Theorems 1.2 and 2.3] give hard-model precedents for the estimates below. Our observable is the Gibbs contact count.

Proposition 4.5. *The contact representation satisfies*

$$Q_n = \langle \mathcal{C}_n \rangle_n, \qquad R_n(h_c + u) = R_n(h_c) \langle e^{u\mathcal{C}_n} \rangle_n. \tag{4.6}$$

There are $M, L < \infty$ such that

$$M_{k,n} := \mathbb{E}[R_n^\theta \langle \mathcal{C}_n^k \rangle_n] \le k! M L^{k-1} (n+1)^{2k-1}, \qquad k \ge 1,\ n \ge 0. \tag{4.7}$$

Consequently, if $u \ge 0$ and $Lu(n+1)^2 < 1$,

$$\mathbb{E}[R_n^\theta \langle e^{u\mathcal{C}_n} - 1 \rangle_n] \le \frac{Mu(n+1)}{1 - Lu(n+1)^2}. \tag{4.8}$$

The same bound holds for $\mathbb{E}R_n(h_c+u)^\theta - A_n$.

Proof. Induction over the height, using the root choice, gives the second identity in (4.6). Differentiating it at $u = 0$ gives the first.

The case $k = 1$ is Lemma 4.3. The root choice and $B^{-\theta} S(S+c)^{\theta-1} \le S^\theta/2$ give

$$M_{k,n+1} \le A_n M_{k,n} + \frac{1}{2} \sum_{\ell=1}^{k-1} \binom{k}{\ell} M_{\ell,n} M_{k-\ell,n}.$$

Divide by $k!\Pi_{n+1}$ and sum. Using $\Pi_n / \Pi_{j+1} \le C_\Pi (n+1)^2/(j+1)^2$, induction in k yields

$$\frac{M_{k,n}}{k!} \le (n+1)^{2k-1} \left(\frac{C_0}{k!} + \frac{C_\Pi (k-1)}{2(2k-3)} M^2 L^{k-2} \right), \qquad k \ge 2,$$

because $\sum_{j<n} (j+1)^{2k-4} \le (n+1)^{2k-3}/(2k-3)$. Choose M to cover $k = 1$ and C_0, then $L \ge \max(2, C_\Pi M)$. This closes the induction uniformly in k. Summing the exponential series proves (4.8); (4.6) and $z^\theta \le z$ for $z \ge 1$ give its last assertion. □

4.4 Uniform field estimates

The contact bounds give uniform Taylor estimates in the window $u(n+1)^2 \le \eta$. We also need one logarithmic moment above order three to control the entrance-law remainder.

Proposition 4.6. *There are $\eta, C > 0$ such that, for $n \ge 1$ and $0 \le u(n+1)^2 \le \eta$,*

$$A_n(h_c+u) = A_n + O(u(n+1)), \tag{4.9}$$
$$H_n(h_c+u) = H_n + u\mathcal{E}_n + O(u^2(n+1)^4), \tag{4.10}$$
$$K_n(h_c+u) = K_n + O(u(n+1)^3), \tag{4.11}$$
$$J_n(h_c+u) = J_n + O(u(n+1)^4), \tag{4.12}$$
$$\mathbb{E}[R_n(h_c+u)^\theta |\log R_n(h_c+u) - a|^{7/2}] \le C(n+1)^{5/2}. \tag{4.13}$$

Also, $\Pi_n(h_c+u)/\Pi_n(h_c) = 1 + O(u(n+1)^2)$.

Proof. We first combine contact and logarithmic moments. Write $\mathrm{d}\mu_n = R_n(h_c)^\theta \,\mathrm{d}\mathbb{P}$ and $Z_n = \log R_n(h_c) - a$. For $s \in \{0, 1, 2, 3, 7/2\}$ and integers $r \ge 1$,

$$\int (1+|Z_n|)^s \langle \mathcal{C}_n^r \rangle_n \,\mathrm{d}\mu_n \le C_s^r r!(n+1)^{2r+s-1}. \tag{4.14}$$

For $s = 0$, use (4.7). For $s = 1, 2$, apply Cauchy under the joint environment–Gibbs measure, using logarithmic moments of order two and four. For $s = 3, 7/2$, use Hölder with exponents $(4, 4/3)$ and $(8, 8/7)$, respectively, and the fourth logarithmic moment. The factorials are controlled by $((pr)!)^{1/p} \le p^r r!$. Expanding the exponential therefore gives, for $d = 1, 2$,

$$\int (1+|Z_n|)^s \langle \mathcal{C}_n^d e^{v\mathcal{C}_n} \rangle_n \,\mathrm{d}\mu_n \le C(n+1)^{2d+s-1} \tag{4.15}$$

whenever $v(n+1)^2 \le \eta$; here $\sum_{k\ge 0}(k+d)!x^k/k! = d!/(1-x)^{d+1}$.

We now transfer these estimates to the perturbed field. Put $T(v) = \langle e^{v\mathcal{C}_n} \rangle_n \ge 1$. The change of Gibbs measure gives

$$R_n(h_c+v)^\theta \langle \mathcal{C}_n^d \rangle_{n,h_c+v} = R_n(h_c)^\theta T(v)^{\theta-1} \langle \mathcal{C}_n^d e^{v\mathcal{C}_n} \rangle_n.$$

Since $t^{\theta-1}(1+\log t)^s$ is bounded on $[1,\infty)$, (4.15) also bounds this expression with the additional factor $(1+|\log R_n(h_c+v) - a|)^s$.

Now $\partial_h \log R_n = \langle \mathcal{C}_n \rangle$ and $\partial_h^2 \log R_n = \mathrm{Var}(\mathcal{C}_n)$ under Gibbs. The fixed-height envelopes in the proof of Lemma 4.2 justify differentiating the expectations. Differentiating $e^{\theta x}(x-a)^j$ then gives, throughout the window,

$$|A_n'| \le C(n+1), \quad |H_n''| \le C(n+1)^4, \quad |K_n'| \le C(n+1)^3, \quad |J_n'| \le C(n+1)^4.$$

For the second derivative, use $(\partial_h \log R_n)^2 + \partial_h^2 \log R_n = \langle \mathcal{C}_n^2 \rangle$ and $(d,s) = (2,1)$. Integration proves (4.9)–(4.12).

The same calculation for $e^{\theta x}(1+(x-a)^2)^{7/4}$ gives a field derivative $O((n+1)^{9/2})$. Its critical expectation is $O((n+1)^{5/2})$ by interpolation between the absolute logarithmic moments of orders three and four. This proves (4.13). Finally, integrate $\partial_h \log \Pi_n(h) = \sum_{j<n} A_j'(h)/A_j(h)$; the floor bounds $A_j(h)$ below for $j \ge 1$, while $A_0(h)$ is bounded away from zero. The sum is therefore $O((n+1)^2)$. □

5 The near-critical scaling limit

We state the weighted convergence result first. Its proof starts with compactness near the entrance, identifies the limiting profile in Section 6, and continues the convergence up to the first pole in Section 7.

Fix $\lambda \geq 0$, let $h_N = h_c + \lambda/N^2$, and evaluate all variables below at h_N unless otherwise indicated. The scale $N > 0$ may be real; generation indices are always integers. For $n = \lfloor Nt \rfloor$, define

$$\nu_N(t,\,\mathrm{d}x) = N\mathbb{E}\Big[(R_n^\theta - 1)\mathbf{1}_{\{R_n \geq 1\}}\delta_{\log R_n/(aN)}(\,\mathrm{d}x)\Big], \tag{5.1}$$

$$q_N(s) = \theta - \frac{s}{aN}, \qquad L_{n,N}(s) = N\big(G_n(q_N(s)) - 1\big), \quad 0 \leq s \leq N\log 2. \tag{5.2}$$

We write $L_N(t,s) = L_{\lfloor Nt \rfloor,N}(s)$. Statements on a fixed compact set of nonnegative weights are understood for N large enough that the set lies in this range.

Put $v = \gamma\lambda$ and $T_v = \pi/\sqrt{v}$ for $v > 0$, with $T_0 = \infty$. The exponential family and its sine parametrization are due to Derrida and Retaux [12, Section 3.2]; see also Chen, Dagard, Derrida and Shi [3, Section 2, (24)–(29)]. Write

$$u_v(t) = \begin{cases} \sin(\sqrt{v}\,t)/\sqrt{v}, & v > 0, \\ t, & v = 0, \end{cases}$$

and, for $0 < t < T_v$, set

$$a_v(t) = \frac{4}{u_v(t)^2}, \qquad b_v(t) = 2\frac{u_v'(t)}{u_v(t)}, \qquad f_v(t,x) = a_v(t)e^{-b_v(t)x}. \tag{5.3}$$

For $s + b_v(t) > 0$, write

$$F_v(t,s) = \frac{a_v(t)}{s + b_v(t)}. \tag{5.4}$$

Exponential weights allow us to follow the profile after its total mass becomes infinite.

Theorem 5.1. *Suppose $\lambda > 0$. Let $I \subset (0, T_v)$ be compact and let $s > 0$ satisfy $s + \inf_{t\in I} b_v(t) > 0$. Then*

$$L_N(t,s) \longrightarrow F_v(t,s) \tag{5.5}$$

uniformly for $t \in I$. At each such time, $e^{-sx}\nu_N(t,\,\mathrm{d}x)$ converges weakly to $e^{-sx}f_v(t,x)\,\mathrm{d}x$.

5.1 Compactness and entrance data

For the initial compactness argument, take $T > 0$ with λT^2 sufficiently small for Proposition 4.6 to apply. For $0 < t \leq T$ and $n = \lfloor Nt \rfloor$, set

$$\begin{aligned} S_N(t) &= N(A_n - 1), \quad & d_N(t) &= H_n/a, \\ \mathcal{K}_N(t) &= K_n/(a^2N), \quad & \mathcal{P}_N(t) &= J_n/(a^3N^2), \end{aligned} \tag{5.6}$$

and define

$$\mathcal{H}_N(t,s) = s + L_{n,N}(s) - L_{n,N}(0). \tag{5.7}$$

Theorem 5.2. *Every sequence $N \to \infty$ has a subsequence for which $S_N \to S$ in $L^1_{\mathrm{loc}}((0,T])$, with $S \in BV_{\mathrm{loc}}$, while $d_N, \mathcal{K}_N, \mathcal{P}_N$ converge locally uniformly to locally Lipschitz functions d, K, P. Moreover, $\mathcal{H}_N \to \mathcal{H}$ uniformly on compact subsets of $(0,T] \times [0,\infty)$. The limits satisfy*

$$d' = Sd, \qquad K' = SK + d^2 - 1, \qquad P' = SP + 3dK, \tag{5.8}$$

$$\partial_t\mathcal{H} = S\mathcal{H} + \tfrac{1}{2}\mathcal{H}^2 - \tfrac{1}{2}s^2 \qquad (s \geq 0) \tag{5.9}$$

almost everywhere. For almost every t there is a positive measure ν_t on $[0,\infty)$ such that

$$\mathcal{H}(t,s) = s + \int (e^{-sx} - 1)\nu_t(\,\mathrm{d}x), \tag{5.10}$$

$$\int (1, x, x^2, x^3)\nu_t(\,\mathrm{d}x) = (S(t), 1 + d(t), K(t), P(t)), \tag{5.11}$$

$$\nu_t(\{0\}) = 0. \tag{5.12}$$

At the entrance,

$$S(t) \asymp t^{-1}, \qquad |d(t)| \le C\lambda t^2, \qquad K(t) \asymp t, \qquad P(t) \asymp t^2, \tag{5.13}$$

$$\mathcal{H}(t,s) = -sd(t) + \tfrac{1}{2}s^2K(t) - \tfrac{1}{6}s^3P(t) + O\big(|s|^{7/2}t^{5/2}\big), \tag{5.14}$$

$$\lim_{t\downarrow 0} \frac{d(t)}{P(t)} = \frac{\lambda\kappa a^2}{\ell_3}. \tag{5.15}$$

The remainder in (5.14) is uniform on compact subsets of $\{\Re s \ge 0\}$, after the analytic extension given by (5.10).

The proof uses the following two estimates. The Mellin estimate is stated under its own boundedness hypothesis and will also be used in Section 7, beyond the initial compactness interval.

Lemma 5.3 (Mellin defect estimates). *Put $r_n = R_n^{(1)}R_n^{(2)}$ and*

$$D_n(q) = \mathbb{E}d_q(r_n), \qquad d_q(r) = B^q - 1 - (r+c)^q + r^q.$$

Uniformly for q, q_0 in a fixed compact subinterval of $(0,1)$, $D_n(q) \ge -C\rho^n$. If $n \ge 1$ and F is Borel measurable, bounded on $[b_1^2,\infty)$ and differentiable in a neighbourhood of 1, then

$$|\mathbb{E}F(r_n) - F(1)| \le C_F\big(D_n(q_0) + C\rho^n\big). \tag{5.16}$$

In particular,

$$|D_n(q) - D_n(q_0)| \le C|q - q_0|\big(D_n(q_0) + C\rho^n\big). \tag{5.17}$$

The exact transform recurrence is

$$L_{n+1,N}(s) = e^{s/N}L_{n,N}(s) + \frac{e^{s/N}}{2N}L_{n,N}(s)^2 - \frac{N}{2}e^{s/N}D_n(q_N(s)). \tag{5.18}$$

Consequently, if $|L_{n,N}(s_0)| \le M$ on a fixed time interval $[r, r+\delta] \subset (0,\infty)$, then

$$\Delta L_{n,N}(s_0) \le N^{-1}\Psi_{s_0}(L_{n,N}(s_0)) + C_MN^{-2} + CN\rho^n, \qquad \Psi_s(z) = sz + \tfrac{1}{2}z^2, \tag{5.19}$$

$$\sum_{rN \le n < (r+\delta)N} |D_n(q_N(s_0))| = O(N^{-1}). \tag{5.20}$$

On this interval, $L_N(\cdot, s_0)$ is bounded in BV, and its positive increments satisfy a uniform one-sided Lipschitz bound up to an error tending to zero. For $s \ge s_0$ in a compact set,

$$\Delta\big(L_{n,N}(s) - L_{n,N}(s_0)\big) = \frac{\Psi_s(L_{n,N}(s)) - \Psi_{s_0}(L_{n,N}(s_0))}{N} + e_{n,N}(s), \tag{5.21}$$

where $\sum_n |e_{n,N}(s)| = O_s(N^{-1})$ on the same time interval.

Proof. For $r > 1$, $d_q(r) > 0$, whereas $d_q(1) = 0$ and $d_q(r) \to B^q - 1 > 0$. Its derivative in r at 1 is bounded away from zero on compact q-intervals. Hence $|F(r) - F(1)| \le C_F d_{q_0}(r)$ for $r \ge 1$. For $n \ge 1$, the floor $r_n \ge b_n^2 \ge b_1^2$ controls $r < 1$ within the domain on which F is bounded. Applying the same comparison to $\partial_q d_q$, which is bounded and vanishes at 1, gives (5.17).

Substituting $G_{n+1}(q) = B^{-q}(G_n(q)^2 + B^q - 1 - D_n(q))$ gives (5.18). Expanding yields (5.19); summing the recurrence, and using the exponentially small negative part of D_n, gives (5.20) and the BV bound. Fix $0 \le s_0 \le s_1 < \infty$ and take N large enough that $s_1 \le N \log 2$. Then, for $s \in [s_0, s_1]$, $0 \le q_N(s_1) \le q_N(s) \le q_N(s_0) \le \theta < 1$. On $b_n \le R_n < 1$, both $1 - R_n^{q_N(s)}$ and $R_n^{q_N(s)} - R_n^{q_N(s_0)}$ lie in $[0, 1 - b_n]$; on $R_n \ge 1$, the powers are monotone in their exponent. Hence

$$-N(1 - b_n) \le L_{n,N}(s) \le L_{n,N}(s_0) + N(1 - b_n), \qquad s \in [s_0, s_1].$$

Since $1 - b_n \le \rho^n$, the errors are $O(N\rho^n)$ uniformly on this compact weight interval. We now subtract the recurrences at s and s_0. By (5.17), the remaining pointwise error is $O_s(N^{-2} + |D_n(q_N(s_0))| + \rho^n)$, whose sum is $O_s(N^{-1})$. □

Lemma 5.4 (One-sided compactness)**.** *Let f_N be uniformly bounded on a compact interval and suppose*

$$f_N(t) - f_N(r) \le C(t - r) + \varepsilon_N \quad (r \le t), \qquad \varepsilon_N \to 0.$$

If $f_N \to f$ in L^1 and f is continuous, then convergence is uniform on every interior compact interval.

Proof. An upward excursion of fixed height persists on an interval of fixed length to its left; a downward excursion persists to its right. Choose that length using C and the modulus of continuity of f. Either excursion contradicts L^1 convergence. □

Proof of Theorem 5.2. Bounds and time compactness. Fix $r > 0$. The critical estimates and Proposition 4.6 give (5.13) for the approximations uniformly on $[r, T]$, with comparison constants independent of r and errors vanishing as $N \to \infty$ for each fixed r. All generations used then satisfy $n = \lfloor Nt \rfloor \ge 1$ for N large enough. We first pass to the limit on each such interval and only afterwards let $t \downarrow 0$. The lower bounds follow from monotonicity of A_n and, for large n, of $r^\theta(\log r - a)^3$. Increasing r can decrease $r^\theta(\log r - a)^2$ by only a bounded amount. The moment estimate (4.13) gives

$$\int x^{7/2} \nu_N(t, \mathrm{d}x) \le C(t + N^{-1})^{5/2}. \tag{5.22}$$

Apply Lemma 5.3 at $s_0 = 0$. Helly's theorem yields the asserted L^1 convergence of S_N; choose a Borel BV representative S and the subsequence so that pointwise convergence also holds at every continuity point of S. The exact recurrences (3.6)–(3.8) hold at every field. Their bounded kernels satisfy $\omega_1(1) = 0$ and $\omega_2(1) = -2a^2$. Equations (5.16) and (5.20) make their normalized cumulative errors vanish. After removing these errors, the linear interpolants are uniformly Lipschitz. This gives compactness of $d_N, \mathcal{K}_N, \mathcal{P}_N$ and (5.8).

Since $L_N(t, s) = S_N(t) + \mathcal{H}_N(t, s) - s$, (5.21) at $s_0 = 0$ gives

$$\Delta\mathcal{H}_N = N^{-1}\left(S_N\mathcal{H}_N + \tfrac{1}{2}\mathcal{H}_N^2 - \tfrac{1}{2}s^2\right) + e_{n,N}(s), \qquad \sum_n |e_{n,N}(s)| = O_s(N^{-1}). \tag{5.23}$$

The first tilted logarithmic moment bounds the derivative in s. Arzelà–Ascoli and a diagonal extraction therefore give joint compactness and (5.9). Products pass to the limit using L^1 convergence of S_N and uniform convergence of the other factors.

Positive measures and the boundary. The mass of ν_N differs from S_N by $O(N\rho^n)$. Together with (5.22), this gives tightness and uniform integrability through the third moment. Let E be the full-measure Borel set of continuity points of S. At each $t \in E$, any measure limit satisfies (5.10)–(5.11), which determine it uniquely. The transform discrepancy tends to zero because

$$\mathbb{E}(\log R_n)^+ \le C(A_n - 1 + \rho^n) = O(N^{-1}).$$

Denote this unique limit by ν_t for $t \in E$ and set $\nu_t = 0$ otherwise. For every integer $j \ge 0$,

$$t \longmapsto \int e^{-jx}\nu_t(\,\mathrm{d}x) = \mathbf{1}_E(t)\big(S(t) + \mathcal{H}(t,j) - j\big)$$

is Borel. The algebra spanned by $x \mapsto e^{-jx}$, $j \ge 0$, contains the constants and generates the Borel sigma-field of $[0,\infty)$. The functional monotone-class theorem therefore shows that $t \mapsto \nu_t(A)$ is Borel for every Borel set A. Thus (ν_t) is a measurable finite-measure kernel, with locally bounded mass.

For $t \in E$, the positive representation extends $\mathcal{H}(t,\cdot)$ holomorphically to $\Re s > 0$ and continuously to $\Re s \ge 0$. The kernel property makes the corresponding time integrals measurable. On compact time intervals, the bound $|\mathcal{H}(t,s)| \le |s| + 2S(t)$ gives domination on compact subsets of the closed right half-plane. The integral equation consequently extends by analytic continuation for endpoints in E. Fixing one such endpoint then defines the continuous extension in t at the remaining times.

For $t \in E$ and real $s \ge 0$, write $b(t,s) = s - \mathcal{H}(t,s) = \int(1 - e^{-sx})\nu_t(\,\mathrm{d}x)$. Then $0 \le b \le S$ and

$$\partial_t \mathcal{H} = s(S - b) - Sb + \tfrac{1}{2}b^2.$$

To exclude an atom at zero, we integrate between continuity points of S, divide by s, and let $s \to \infty$. The leading terms s cancel at the endpoints, while dominated convergence on the right gives $\int \nu_t(\{0\})\,\mathrm{d}t = 0$. This proves (5.12) without assuming S continuous.

Entrance data. Taylor's formula and (5.22) give (5.14); its remainder is $o(P(t))$ for fixed s. The uniform field expansions are

$$H_n(h_c + u) = H_n(h_c) + u\,\partial_h H_n(h_c) + O(u^2 n^4), \qquad J_n(h_c + u) = J_n(h_c) + O(un^4).$$

For each fixed $t > 0$, insert $u = \lambda/N^2$ and $n = \lfloor Nt \rfloor$ and first let $N \to \infty$ along the subsequence. Using $\partial_h H_n(h_c)/\Pi_n(h_c) \to \kappa$ and $J_n(h_c)/\Pi_n(h_c) \to \ell_3$, we obtain

$$d(t) - \frac{\lambda\kappa a^2}{\ell_3}P(t) = O(\lambda^2 t^4).$$

Now let $t \downarrow 0$. Since $P(t) \asymp t^2$, this gives (5.15). The common factor Π_n cancels; its precise asymptotic constant is not needed here. □

6 Identification of the limiting profile

We identify the profile (5.3) from the entrance data by showing that a regularized Laplace transform is a polynomial of degree at most one.

Theorem 6.1. *Every subsequential limit in Theorem 5.2 agrees, on a common initial interval, with $f_v(t,x)\,\mathrm{d}x$, where f_v is defined in (5.3). At criticality,*

$$n(A_n - 1) \to 2, \qquad \frac{K_n}{a^2 n} \to 1, \qquad \frac{J_n}{a^3 n^2} \to \frac{3}{2}, \qquad \frac{\Pi_n}{n^2} \to \frac{3a^3}{2\ell_3}. \tag{6.1}$$

The constants $2, 1, 3/2$ in (6.1) agree with the critical moment predictions for the binary hard recursion; see Appendix III of the 1983 archival version of Collet, Eckmann, Glaser and Martin [8] and Derrida and Shi [13, Section 5.1]. These predictions go beyond the earlier proved quadratic product bounds. The limits asserted here concern the soft diamond recursion.

The general inverse correspondence for canonical systems is due to de Branges [9, 10]; a modern statement is Eckhardt, Kostenko and Teschl [14, Theorem 2.2]. We prove the rigidity needed for the present entrance law directly. For the first assertion of the following lemma, compare the Krein–de Branges exponential-type formula in Romanov [28, Section 6, Theorem 11]. The second assertion is the classical Phragmén–Lindelöf principle at minimal exponential type [27, Section II, no. 5]. We include short proofs.

Lemma 6.2 (Growth bound and Phragmén–Lindelöf principle)**.** *Fix a measurable $w : (0, T) \to (0, \infty)$ such that w, w^{-1} are bounded. The fundamental matrix of $Y' = -z \left(\begin{smallmatrix} 0 & w^{-1} \\ w & 0 \end{smallmatrix}\right) Y$ normalized by $M(0, z) = I$ satisfies*

$$\|M(T, z)\| \le C_{\varepsilon, T, w} e^{T|\Re z| + \varepsilon |z|} \qquad (\varepsilon > 0).$$

A holomorphic function of minimal exponential type in a half-plane, continuous and bounded by one on its boundary, is bounded by one throughout that half-plane.

Proof. By density in $L^1(0, T)$, we choose a step function $w_0 \in [w_-, w_+]$, where $0 < w_- \le w \le w_+$, constant on each interval of a finite partition, with $\int_0^T |w - w_0| < w_- \varepsilon$. On each interval where $w_0 = b$, conjugation by $\operatorname{diag}(\sqrt{b}, 1/\sqrt{b})$ gives the symmetric matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ plus an error of norm at most $|w - b|/w_-$. Gronwall proves the first assertion; the finitely many changes of coordinates contribute a constant which may depend on ε, T, w.

For the second assertion, we use a right half-disc of radius R. The harmonic measure of its semicircle at $x \in (0, R)$ is $(4/\pi) \arctan(x/R)$. The maximum principle gives

$$\log |f(x)| \le \max(0, \log C_\varepsilon + \varepsilon R) \frac{4}{\pi} \arctan(x/R).$$

Let $R \to \infty$, then $\varepsilon \downarrow 0$. Vertical translation and reflection give the general case. □

Proof of Theorem 6.1. Let $S, d, K, P, \mathcal{H}$ be a subsequential limit from Theorem 5.2. It satisfies

$$d' = Sd, \quad K' = SK + d^2 - 1, \quad P' = SP + 3dK, \quad \partial_t \mathcal{H} = S\mathcal{H} + \tfrac{1}{2}\mathcal{H}^2 - \tfrac{1}{2}s^2,$$

with $S \le C/t$, $d = O(t^2)$, $K \asymp t$, $P \asymp t^2$ and $d/P \to \alpha := \lambda \kappa a^2 / \ell_3$. The entrance expansion is

$$\mathcal{H}(t, s) = -sd + \tfrac{1}{2}s^2 K - \tfrac{1}{6}s^3 P + o(P) \qquad (t \downarrow 0). \tag{6.2}$$

Regularization at the entrance. Set

$$Q = K - d \int_0^t d, \qquad R = P - 3d \int_0^t K, \qquad w = Q^2/R.$$

Then $Q' = SQ - 1$, $R' = SR$, $d/R = \alpha$. Since $K - Q = O(t^5)$ and $P - R = O(t^4)$, $Q \asymp t$, $R \asymp t^2$, and w, w^{-1} are bounded and locally absolutely continuous on a common initial interval. For $s > 0$, define

$$y = \exp\left(-\tfrac{1}{2} \int_0^t \mathcal{H}(v, s)\, \mathrm{d}v\right), \qquad U = \frac{2\mathcal{H} y}{s^2 Q}.$$

Direct differentiation gives

$$U' = (U - y)/Q, \qquad y' = -s^2 Q U/4, \qquad (wU')' = s^2 w U/4. \tag{6.3}$$

Moreover,

$$wU' = \frac{y}{R}(2\mathcal{H}/s^2 - Q), \qquad U(0,s) = 1, \qquad (wU')(0,s) = -s/3 - 2\alpha/s. \tag{6.4}$$

The traces follow from (6.2); no trace of w is assumed.

Entire-function rigidity. The value and flux of sU at zero are the polynomials s and $-s^2/3 - 2\alpha$. The integral form of (6.3) therefore makes $\mathcal{B}_t(s) = se^{st/2}U(t,s)$ entire. For $\Re s \geq 0$, positivity of the limiting measure gives

$$|\mathcal{H} + sd| \leq |s|^2 K/2, \qquad |\mathcal{H}| \leq |s|(2+d), \qquad 0 \leq \Re(s - \mathcal{H}) \leq C\min(t^{-1}, |s|).$$

For $s \neq 0$, split the time integral at $\min(t, |s|^{-1})$ to obtain

$$\exp\left(\tfrac{1}{2}\int_0^t \Re(s - \mathcal{H}(v,s))\,\mathrm{d}v\right) \leq C(1 + t|s|)^C.$$

Since $\mathcal{B}_t(s) = 2\mathcal{H}(t,s)\exp(\frac{1}{2}\int_0^t (s - \mathcal{H}(v,s))\,\mathrm{d}v)/(sQ(t))$, this gives polynomial growth in the right half-plane, with degree bound independent of t; the value at $s = 0$ follows by continuity. In the left half-plane with $|s| \geq 1$, apply Lemma 6.2 to $(U, -2wU'/s)$ with $z = s/2$. Its entrance value is $(1, 2/3 + 4\alpha/s^2)$, which is bounded in this range. For each fixed t, the factor $e^{st/2}$ cancels $e^{t|\Re s|/2}$ and gives $|\mathcal{B}_t(s)| \leq C_{\varepsilon,t,w,\alpha}|s|e^{\varepsilon|s|}$. For a fixed sufficiently large integer k, $\mathcal{B}_t(s)/(1-s)^k$ is bounded on the imaginary axis and has minimal exponential type in the left half-plane. The second part of the lemma and Cauchy's estimates show that $\mathcal{B}_t$ is a polynomial of degree at most k.

Write $\mathcal{B}_t(s) = \sum_{j=0}^k b_j(t)s^j$. For each fixed s, the integral system makes the value and flux of sU locally absolutely continuous. Since w is locally absolutely continuous, the same holds for $\mathcal{B}_t(s)$ and $w\partial_t\mathcal{B}_t(s)$. Interpolation at $k+1$ distinct positive weights therefore shows that both b_j and wb_j' are locally absolutely continuous. Direct differentiation gives

$$(w\partial_t\mathcal{B}_t)' = s\left(w\partial_t\mathcal{B}_t + \tfrac{1}{2}w'\mathcal{B}_t\right) \tag{6.5}$$

for almost every t at each fixed s. Choose $k+2$ distinct positive weights and a common full-measure set of times where the derivatives exist and these identities hold. The difference of the two sides of (6.5) has degree at most $k+1$ in s and vanishes at these weights, so all its coefficients vanish almost everywhere. The entrance values at any $k+1$ of these weights give $b_j(t) \to \mathbf{1}_{\{j=1\}}$ as $t \downarrow 0$. If $k \geq 2$, the coefficient of s^{k+1} gives $(\sqrt{w}\,b_k)' = 0$. Boundedness of w and the zero entrance trace force $b_k = 0$. Repeating this argument leaves degree at most one.

A scalar Riccati equation. Put $\ell = d/Q$ and $I = \int_0^t d$. Expanding at $s = 0$, and using $K = Q + dI$, yields

$$\mathcal{B}_t(s) = s(1 - t\ell) - 2\ell. \tag{6.6}$$

Equation (6.5) gives $w(1-t\ell)^2 = w_0 > 0$, while $\ell' = \alpha/w$. Since $\ell = O(t)$, $w \to w_0$, and

$$wU' = e^{-st/2}\left[-\tfrac{s}{2}w(1 - t\ell) - \alpha t - \frac{2\alpha}{s}\right].$$

The flux in (6.4) forces $w_0 = 2/3$. Consequently $r = \ell/(1 - t\ell)$ solves

$$r' = v + r^2, \qquad r(0) = 0, \qquad v = 3\alpha/2 = \gamma\lambda.$$

Hence $r = vu_v/u_v'$ initially. The identities $Q = w\int_0^t w^{-1}$, $S = w'/w + 2/Q$ and $R = Q^2/w$ recover the moments. Uniqueness in (6.3) recovers $\mathcal{H}$, and uniqueness of Laplace transforms gives (5.3).

Critical asymptotics. For $\lambda = 0$, the moments are $S = 2/t$, $K = t$, $P = 3t^2/2$. Lemma 5.4 upgrades the L^1 convergence of S_N to uniform convergence on interior compact intervals. Fix such a time t_0 and take $N = n/t_0$ to obtain the first three limits in (6.1). The last follows from $J_n/\Pi_n \to \ell_3$. Together with $\partial_h H_n(h_c)/\Pi_n \to \kappa$, it also proves the second identity in (1.3). □

7 Continuation up to the pole

Fix $\lambda > 0$ and use the notation of Section 5. We now complete the proof of the weighted convergence theorem by continuing the profile identified in Section 6.

Proof of Theorem 5.1. Integrating (5.3) gives $F_v(t,s) = \int_0^\infty e^{-sx} f_v(t,x)\,\mathrm{d}x$ whenever $s + b_v(t) > 0$; its unweighted mass is infinite when $b_v(t) \le 0$. We first extend the convergence from a time where the profile is identified. At the smallest weight used in this extension, possible loss of mass at infinity requires a separate argument.

Local bounds and compactness. On interior compact subsets of the initial interval of identification, Lemma 5.4 and the convergence of the transform differences give uniform convergence at every fixed nonnegative weight. Choose a time $r > 0$ in an interval where this convergence is known and a base weight $s_0 > 0$ with $s_0 + b_v(r) > 0$. We obtain a local bound by a first-exit argument. Choose $A > 1 + F_v(r, s_0)$ and put $M = A + 1$. For large N, $L_N(r, s_0) \le A$, while the floor gives $L_{n,N}(s_0) \ge -CN\rho^n > -1$ for $n \ge \lfloor Nr \rfloor$. The one-step estimate (5.19) uses only the bound at the current index. Before, and at, the first crossing of M, summing it therefore gives

$$L_{n,N}(s_0) \le A + C\delta + o_N(1), \qquad n \le \lfloor N(r+\delta) \rfloor,$$

where $C = 1 + \max_{-1 \le z \le M} \Psi_{s_0}(z)$; the accumulated N^{-2} and $N\rho^n$ errors tend to zero. Choose $C\delta < 1/2$. A first crossing is then impossible for large N. Thus $|L_N(t, s_0)| \le M$ on $[r, r+\delta]$, with δ depending only on s_0 and an upper bound for $F_v(r, s_0)$. Shrink δ if necessary so that the profile is finite at this weight throughout the interval. All further finite-N estimates in the weight variable are used on fixed compact sets before taking $N \to \infty$.

Lemma 5.3 gives BV compactness at s_0, the integrated defect estimate (5.20), and equicontinuity of the differences. After a diagonal extraction, $L_N(\cdot, s_0)$ converges in L^1 and $L_N(\cdot, s) - L_N(\cdot, s_0)$ converges uniformly for fixed $s \ge s_0$. Their limits satisfy

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(L(t,s) - L(t,s_0)\big) = sL(t,s) + \tfrac{1}{2}L(t,s)^2 - s_0 L(t,s_0) - \tfrac{1}{2}L(t,s_0)^2 \tag{7.1}$$

in integral form.

The limiting measure and its trace. Uniformly on the local interval,

$$L_N(t,s) - \int e^{-sx} \nu_N(t,\,\mathrm{d}x) \longrightarrow 0, \qquad s \ge s_0.$$

Indeed, the discrepancy is bounded by $C_s \mathbb{E}(\log R_n)^+ + CN\rho^n$, and

$$\mathbb{E}(\log R_n)^+ \le q_N(s_0)^{-1}\big(G_n(q_N(s_0)) - 1 + C\rho^n\big) = O(N^{-1}).$$

A bound at s_0 need not give tightness at s_0. At every $\sigma > s_0$, however,

$$\int_{x>R} e^{-\sigma x} \nu_N(t,\,\mathrm{d}x) \le e^{-(\sigma - s_0)R} \int e^{-s_0 x} \nu_N(t,\,\mathrm{d}x).$$

Choose the Helly subsequence so that the base transforms converge at every continuity point of a Borel BV representative of their limit. On this common full-measure Borel set, weighted tightness and uniqueness of Laplace transforms give a positive measure μ_t with $\int e^{-s_0 x} \mu_t(\,\mathrm{d}x) \le M$ and

$$L(t,s) = \int e^{-sx} \mu_t(\,\mathrm{d}x), \qquad s > s_0.$$

Set $\mu_t = 0$ on the exceptional set. For fixed $\sigma > s_0$, the integrals of e^{-kx}, $k \in \mathbb{N}_0$, against $e^{-\sigma x}\mu_t(\,\mathrm{d}x)$ are Borel functions of t. The algebra generated by these functions of x contains the constants and

generates the Borel sets of $[0,\infty)$. The functional monotone-class argument used in the proof of Theorem 5.2 therefore shows that $e^{-\sigma x}\mu_t(\,\mathrm{d}x)$, and hence $\mu_t(\,\mathrm{d}x)$, are measurable measure kernels. Their convolutions are measurable kernels as well, by applying the same argument to rectangles and then integrating against the product kernels. Integrating (7.1), dividing by s, and letting $s \to \infty$ gives $\int \mu_t(\{0\})\,\mathrm{d}t = 0$ on every interior time interval: the endpoint terms divided by s vanish, and dominated convergence applies since $0 \le L(t,s) \le M$ almost everywhere. Thus there is no atom at zero almost everywhere; enlarge the exceptional Borel set accordingly and set the kernel to zero there.

To identify the trace at r, we use the continuity of the transform differences up to r. If $\Delta = L(r+,s_0) - F_v(r,s_0)$, their right traces are therefore $F_v(r,s) + \Delta$ for $s > s_0$. The one-sided bound gives $\Delta \le 0$; positivity and $F_v(r,s) \to 0$ as $s \to \infty$ give $\Delta \ge 0$. Hence $\Delta = 0$. The tail bound and uniqueness of Laplace transforms now show that, along the good times, $e^{-\sigma x}\mu_t(\,\mathrm{d}x)$ converges weakly as $t \downarrow r$ to $e^{-\sigma x} f_v(r,x)\,\mathrm{d}x$ for every $\sigma > s_0$. We denote this measure trace by $\mu_r = f_v(r,x)\,\mathrm{d}x$.

Transport and uniqueness. Linear combinations of (7.1) whose coefficients sum to zero yield

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \varphi, \mu_t\rangle = -\langle \varphi', \mu_t\rangle + \tfrac{1}{2}\langle \varphi, \mu_t * \mu_t\rangle, \qquad \varphi \in C_c^1((0,\infty)). \tag{7.2}$$

The identity is understood in integral form. To justify the passage from exponentials to test functions, write $\varphi(x) = e^{-\sigma x} g(e^{-x})$, with $\sigma > s_0$ and g zero near 0 and 1. Approximate g' uniformly by polynomials and integrate from 1. This approximates both g and g' uniformly and preserves $g(1) = 0$. The resulting exponential combinations vanish at zero, and the errors are controlled in the weighted norms of μ_t and $\mu_t * \mu_t$. In particular, each pairing $\langle \varphi, \mu_t\rangle$ has an absolutely continuous representative, whose value at r is $\langle \varphi, \mu_r\rangle$. Taking a countable family dense in C^1 on each spatial compact and using the local mass bounds gives a common full-measure set of times where all these representatives equal the measure pairings.

We also need time-dependent tests. Let $\psi(\tau,x)$ be C^1 with its spatial support in one compact subset of $(0,\infty)$. Its piecewise linear interpolants in τ are finite sums of fixed spatial tests with piecewise linear coefficients. The product rule for the preceding absolutely continuous pairings applies to these interpolants. As the mesh tends to zero, the interpolants and their spatial derivatives converge uniformly; their time derivatives converge uniformly away from the mesh points, in the essential-supremum sense. The local mass bounds for μ_τ and $\mu_\tau * \mu_\tau$ therefore give, at every endpoint t in the common full-measure set,

$$\langle \psi(t,\cdot), \mu_t\rangle - \langle \psi(r,\cdot), \mu_r\rangle = \int_r^t \left(\langle \partial_\tau \psi - \partial_x \psi, \mu_\tau\rangle + \tfrac{1}{2}\langle \psi, \mu_\tau * \mu_\tau\rangle\right)\mathrm{d}\tau.$$

Let T_τ be left translation with absorption at zero:

$$\langle \varphi, \mathsf{T}_\tau \mu\rangle = \int_{x>\tau} \varphi(x-\tau)\mu(\,\mathrm{d}x), \qquad \|\mu\|_\sigma = \int e^{-\sigma x}|\mu|(\,\mathrm{d}x).$$

Then

$$\|\mathsf{T}_\tau \mu\|_\sigma \le e^{\sigma\tau}\|\mu\|_\sigma, \qquad \|\mu * \eta\|_\sigma \le \|\mu\|_\sigma \|\eta\|_\sigma.$$

Apply the time-dependent identity to $\psi(\tau,x) = \varphi(x - t + \tau)$, where $\varphi \in C_c^1((0,\infty))$ is extended by zero to the real line. For $r \le \tau \le t$, these tests have a common compact support inside $(0,\infty)$ and satisfy $\partial_\tau \psi = \partial_x \psi$. This gives the mild equation

$$\mu_t = \mathsf{T}_{t-r}\mu_r + \tfrac{1}{2}\int_r^t \mathsf{T}_{t-\tau}(\mu_\tau * \mu_\tau)\,\mathrm{d}\tau. \tag{7.3}$$

The integral is defined as a measure integral of the positive kernels: on a Borel set $A \subset (0,\infty)$ its integrand is $\int_{x>t-\tau} \mathbf{1}_A(x-t+\tau)(\mu_\tau * \mu_\tau)(\,\mathrm{d}x)$. It has finite weighted mass, bounded by $\delta e^{\sigma\delta} \sup_\tau \|\mu_\tau\|_\sigma^2$. Equality of the compactly supported test integrals identifies the measures almost everywhere in time. The right-hand side of (7.3) then defines a measurable mild representative at every time, with bounded weighted norm; changing the source kernel on a null set does not change this formula.

For uniqueness, let two such solutions have the same trace and weighted norms bounded by M_σ on $[r, r+\delta]$. Put $C_\sigma = M_\sigma e^{\sigma\delta}$ and define deterministic functions by

$$E_0(t) = 2M_\sigma, \qquad E_{j+1}(t) = C_\sigma \int_r^t E_j(\tau)\,\mathrm{d}\tau.$$

If $\|\mu_\tau - \widetilde{\mu}_\tau\|_\sigma \le E_j(\tau)$ almost everywhere, the convolution bound gives $\|\mu_\tau * \mu_\tau - \widetilde{\mu}_\tau * \widetilde{\mu}_\tau\|_\sigma \le 2M_\sigma E_j(\tau)$ there. Test the difference of the mild equations with any Borel function bounded in absolute value by $e^{-\sigma x}$, integrate this deterministic bound, and then take the supremum over the test functions. It follows that

$$\|\mu_t - \widetilde{\mu}_t\|_\sigma \le E_{j+1}(t), \qquad E_j(t) = 2M_\sigma \frac{(C_\sigma(t-r))^j}{j!}.$$

Letting $j \to \infty$ proves uniqueness. This argument requires neither measurability of the total-variation distance nor strong continuity of translation in total variation. Since $a_v' = -b_v a_v$ and $b_v' = -a_v/2$, the density (5.3) solves $\partial_t f_v = \partial_x f_v + \frac{1}{2} f_v * f_v$. Its weighted norms are bounded on the local interval, and the same time-dependent testing argument shows that it satisfies (7.3) with trace μ_r. It is therefore the unique limit at every weight above s_0.

The base weight and continuation. It remains to exclude loss of mass at infinity at the base weight s_0. Put $e(t) = L(t,s_0) - F_v(t,s_0)$. Since the transform at $s > s_0$ has been identified with a smooth function, the integral form of (7.1) first shows that e is absolutely continuous. Subtracting the profile equation then gives

$$e' = (s_0 + F_v(t,s_0))e + \tfrac{1}{2}e^2, \qquad e(r) = 0. \tag{7.4}$$

Scalar uniqueness gives $e = 0$. All subsequential limits at the base weight are identified. Lemma 5.4 upgrades L^1 convergence to uniform convergence on interior compact intervals. The uniform convergence of the transform differences gives the same conclusion at every fixed larger weight.

Now fix I and s as in the theorem. Since $b_v' = -a_v/2 < 0$, we can choose $0 < s_0 < s$ and t_2 with $\max I < t_2 < T_v$ so that $s_0 + b_v(t_2) > 0$. Choose $t_0 < \min I$ in the initial interval of identification. The profile at weight s_0 is bounded on $[t_0, t_2]$, so the local comparison interval can be chosen with a uniform length δ, further reduced to be smaller than $t_2 - \max I$. Starting at t_0, take successive initial times separated by $\delta/2$, strictly inside the preceding intervals. Finitely many such intervals have interiors covering I, and the final interval ends strictly after $\max I$. The interior uniform upgrade therefore covers both endpoints of I and proves (5.5). At each time in I, the bound at $s_0 < s$ makes $e^{-sx}\nu_N(t,\,\mathrm{d}x)$ tight by the preceding tail estimate. The transform discrepancy and convergence at the fixed weights $s+u$, $u \ge 0$, identify every weak limit by its Laplace transform as $e^{-sx} f_v(t,x)\,\mathrm{d}x$. □

8 Free-energy asymptotics

We obtain both free-energy bounds from the profile near its pole. In the scaling arguments below, fix $\lambda > 0$, set $v = \gamma\lambda$ and $T_v = \pi/\sqrt{v}$, and use the rescaled quantities of Section 5, evaluated at $h_c + \lambda/N^2$.

8.1 Upper bound

Corollary 8.1. *The free energy satisfies*

$$\limsup_{u\downarrow 0} \sqrt{u}\, \log F(h_c + u) \le -\frac{\pi \log 2}{\sqrt{\gamma}}. \tag{8.1}$$

Proof. For $0 < t < T_v$, choose a positive admissible weight s. Theorem 5.1 gives $G_{\lfloor Nt \rfloor}(q_N(s)) = 1 + O(N^{-1})$. Since $q_N(s) \to \theta > 0$,

$$\mathbb{E} \log R_{\lfloor Nt \rfloor} \le q_N(s)^{-1} \log G_{\lfloor Nt \rfloor}(q_N(s)) = O(N^{-1}).$$

The finite-size bound yields $F(h_c + \lambda/N^2) \le C 2^{-\lfloor Nt \rfloor}$. Letting $N \to \infty$ and then $t \uparrow T_v$ gives

$$\limsup_{N\to\infty} N^{-1} \log F(h_c + \lambda/N^2) \le -\frac{\pi \log 2}{\sqrt{\gamma \lambda}}.$$

Take $N = \sqrt{\lambda/u}$ to obtain (8.1). □

8.2 Pair counting and the band criterion

Two-leaf witnesses, common-ancestor second moments and iterative tail amplification are used by Hu and Shi [22, Section 4.1, Lemma 4.1]. We give a finite-band version: its hypothesis concerns one bounded interval, so their global-tail lemma does not apply directly.

Let Y_0 be nonnegative and integer-valued. In this subsection, expectations of nonnegative random variables are taken in $[0, \infty]$. We use the recursion

$$Y_{j+1} = (Y_j^{(1)} + Y_j^{(2)} - 1)^+, \qquad F_{\mathrm{d}} = \lim_{j\to\infty} 2^{-j} \mathbb{E} Y_j \in [0, \infty],$$

where the two copies are independent.

Lemma 8.2 (Pair counting). *Let $L \ge 2$ be an integer, $I \subset \{\lceil 3L/4 \rceil, \dots, L\}$, and $W = \mathbb{E}[2^{Y_0} \mathbf{1}_{\{Y_0 \in I\}}]$. Then*

$$\mathbb{P}(Y_L \ge k) \ge 2^{-k} 2 \left(\frac{W}{W+2} \right)^2, \qquad 1 \le k \le \lfloor L/2 \rfloor. \tag{8.2}$$

Proof. Two leaves of masses $m, \ell \in I$ produce root mass k when their most recent common ancestor has height $d = m + \ell - L - k + 1$ above the leaves and all other leaves are zero. Indeed, $1 \le d \le \min(m, \ell)$ and the resulting root mass is $m + \ell - L - d + 1 = k$. Restoring the other leaves can only increase it.

Let Z count unordered pairs with this property. There are 2^{L+d-2} pairs with common-ancestor height d. Assigning m and ℓ to the left and right leaves of each pair, respectively, gives

$$\mu := \mathbb{E} Z = \sum_{m,\ell \in I} 2^{m+\ell-k-1} \mathbb{P}(Y_0 = m) \mathbb{P}(Y_0 = \ell) = 2^{-k-1} W^2. \tag{8.3}$$

Disjoint pairs have independent indicators. Given a leaf of mass $m \in I$, its expected number of partners is

$$g(m) = \sum_{\ell \in I} 2^{d(m,\ell)-1} \mathbb{P}(Y_0 = \ell) = 2^{m-L-k} W \le 2^{-k} W,$$

and we set $g(m) = 0$ outside I. Distinct partners are conditionally independent, so overlapping ordered pairs contribute at most $\sum_{\text{leaves } z} \mathbb{E} g(Y_z)^2 \le 2^{1-k} W \mu$. Consequently,

$$\mathbb{E} Z^2 \le \mu + \mu^2 + 2^{1-k} W \mu, \qquad \mathbb{P}(Y_L \ge k) \ge \frac{\mu}{1 + \mu + 2^{1-k} W}.$$

Substituting (8.3) and using $k \ge 1$ proves (8.2). □

Proposition 8.3 (Uniform band criterion). *There are absolute constants $D_*, L_* < \infty$ such that*

$$L \geq L_*, \qquad L\,\mathbb{E}[2^{Y_0}\mathbf{1}_{\{\lceil 3L/4\rceil \leq Y_0 \leq L\}}] \geq D_* \tag{8.4}$$

imply $\mathbb{E}Y_T \geq 3/2$ for some integer $T \leq 2L$. In particular,

$$F_{\mathrm{d}} \geq \tfrac{1}{2}\,2^{-2L}. \tag{8.5}$$

One may take $D_* = L_* = 10^4$.

Proof. Let $\mathcal{G}(\alpha, L)$, $0 \leq \alpha \leq 2$, be the distribution supported on $\{0, \dots, L\}$ with $\mathbb{P}(Y \geq k) = \alpha 2^{-k}$ for $1 \leq k \leq L$. Its upper-quarter weighted mass is $\alpha + \frac{\alpha}{2}(L - \lceil 3L/4\rceil) \geq \alpha L/8$. After L generations, Lemma 8.2 gives the stochastic lower law $\mathcal{G}(\alpha', L')$, where

$$L' = \lfloor L/2 \rfloor, \quad \alpha' = 2\left(\frac{\alpha L}{\alpha L + 16}\right)^2. \tag{8.6}$$

Set $M = 128$ and $A_* = 9(M+16)^2$. Suppose $\alpha L^2 \geq A_*$. As long as $w = \alpha L < M$,

$$A' := \alpha'(L')^2 \geq \frac{2(\alpha L^2)^2}{9(M+16)^2} \geq 2\alpha L^2.$$

The inequality $L' \geq L/3$ used here remains valid at every step: $\alpha \leq 2$ and $\alpha L^2 \geq A_*$ force $L > 300$. Since L is halved at each step, eventually $w \geq 128$. The next step has $\alpha' \geq 128/81$ and $L' \geq 150$, hence mean $\alpha'(1 - 2^{-L'}) > 3/2$. The elapsed time is at most $L + \lfloor L/2 \rfloor + \cdots \leq 2L$.

For the general initial law in (8.4), write W for its band moment. After L generations, Lemma 8.2 gives $\mathcal{G}(\alpha_1, L_1)$ with $\alpha_1 = 2(W/(W+2))^2$ and $L_1 = \lfloor L/2 \rfloor$. For $L \geq 3$,

$$\alpha_1 L_1^2 \geq \frac{2}{81}\min\{(WL)^2, L^2\} \geq A_*$$

with the stated constants. The preceding amplification takes at most $2L_1$ further generations, so the total is at most $L + 2L_1 \leq 2L$. Finally, $Y_{j+1} \leq Y_j^{(1)} + Y_j^{(2)}$ makes $2^{-j}\mathbb{E}Y_j$ nonincreasing in $[0, \infty]$, so its limit exists. If $\mathbb{E}Y_0 = \infty$, retaining one leaf gives $Y_j \geq (Y_0^{\mathrm{leaf}} - j)^+$; hence every mean and F_{d} are infinite. If $\mathbb{E}Y_0 < \infty$, iteration of $\mathbb{E}Y_{j+1} \geq 2\mathbb{E}Y_j - 1$ gives the standard bound [22, (1.6)]

$$F_{\mathrm{d}} \geq 2^{-j}(\mathbb{E}Y_j - 1). \tag{8.7}$$

This bound also holds in the infinite-mean case, with $\infty - 1 = \infty$. Apply this at the generation $T \leq 2L$ obtained above to deduce (8.5). □

8.3 Exact coupling at the deterministic floor

Fix a field h and a generation $n \geq 1$ of the diamond model, and set

$$a_n = \log(B/b_n), \qquad \theta_n = \frac{\log 2}{a_n}, \qquad Y_0 = \left\lfloor \frac{\log(R_n/b_n)}{a_n} \right\rfloor. \tag{8.8}$$

The next comparison transfers the band criterion to the diamond model.

Lemma 8.4. *For the initial law in (8.8),*

$$F(h) \geq 2^{-n} a_n F_{\mathrm{d}}. \tag{8.9}$$

If this law, or any stochastically smaller integer law, satisfies (8.4), then

$$F(h) \geq \frac{a_n}{2}\,2^{-n-2L}. \tag{8.10}$$

Proof. Here $n \geq 1$ and $0 \leq Y_0 \leq \log(R_n/b_n)/a_n$, so $\mathbb{E}Y_0 < \infty$ by Lemma 2.1. This also holds for any stochastically smaller nonnegative integer law. We keep the same floor b_n at all subsequent generations. Since $(b_n^2 + c)/B \geq b_n$,

$$\frac{R'}{b_n} \geq \max\left\{1, \frac{b_n}{B}\frac{R_1}{b_n}\frac{R_2}{b_n}\right\}.$$

We can therefore couple the integer recursion below the diamond so that $R_{n+j} \geq b_n e^{a_n Y_j}$. Taking free-energy limits proves (8.9). The same comparison holds for a stochastically smaller initial law, and Proposition 8.3 gives (8.10). □

8.4 The matching lower bound

Use the profile f_v and its pole time T_v from Theorem 5.1. The link between this pole and the BKT scale was identified in Derrida and Retaux [12, Section 3.2] and Chen, Dagard, Derrida and Shi [3, (27)–(29)]. The band criterion provides the lower bound for the present recursion.

Theorem 8.5. *For every fixed $\lambda > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N} \log F(h_c + \lambda/N^2) \geq -T_v \log 2. \tag{8.11}$$

Proof. Fix $\varepsilon > 0$, let $L = \lfloor \varepsilon N \rfloor$, and put $I_\varepsilon = [4\varepsilon/5, 9\varepsilon/10]$. At $n = \lfloor Nt \rfloor$, $0 < t < T_v$, use (8.8). On $\{\log R_n/(aN) \in I_\varepsilon\}$, the integer Y_0 lies in $[\lceil 3L/4 \rceil, L]$ for large N: the band has fixed interior margins and $a_n - a = O(\rho^n)$. Moreover,

$$2^{Y_0} \geq \tfrac{1}{2}(R_n/b_n)^{\theta_n}.$$

On this band, replacing $R_n^{\theta_n}$ by R_n^θ has relative error $O(N\rho^n)$. The weighted convergence of ν_N implies vague convergence on the band; its endpoints have zero limiting mass. Using $R_n^\theta \geq R_n^\theta - 1$, we obtain

$$\liminf_{N\to\infty} L\, \mathbb{E}[2^{Y_0} \mathbf{1}_{\{\lceil 3L/4 \rceil \leq Y_0 \leq L\}}] \geq \frac{\varepsilon}{2} \int_{I_\varepsilon} f_v(t,x)\, \mathrm{d}x. \tag{8.12}$$

As $t \uparrow T_v$, $u_v(t) \sim T_v - t$ and $2u_v'(t)/u_v(t) \sim -2/(T_v - t)$. Thus the right-hand side of (8.12) diverges. Choose $t_\varepsilon < T_v$ so that it exceeds D_* by a fixed margin. For large N, Proposition 8.3 and (8.10) yield

$$\liminf_{N\to\infty} N^{-1} \log F(h_c + \lambda/N^2) \geq -(t_\varepsilon + 2\varepsilon)\log 2 \geq -(T_v + 2\varepsilon)\log 2.$$

Let $\varepsilon \downarrow 0$. □

Proof of Theorem 1.1. The limits in (1.2) follow from Theorems 3.1 and 4.4, and Theorem 6.1 gives the second expression for γ in (1.3). Fix $\lambda > 0$. Since $N > 0$ is real, we may set $N = \sqrt{\lambda/u}$ in (8.11). Combining the resulting lower bound with Corollary 8.1, we obtain

$$\lim_{u\downarrow 0} \sqrt{u}\, \log F(h_c + u) = -\frac{\pi \log 2}{\sqrt{\gamma}}. \tag{8.13}$$

□

References

[1] G. Alsmeyer, Y. Hu and B. Mallein. The Derrida–Retaux model on a geometric Galton–Watson tree. arXiv:2502.02991v2 (2026).

[2] X. Chen, V. Dagard, B. Derrida, Y. Hu, M. Lifshits and Z. Shi. The Derrida–Retaux conjecture on recursive models. *Ann. Probab.* **49** (2021), 637–670. doi:10.1214/20-AOP1457.

[3] X. Chen, V. Dagard, B. Derrida and Z. Shi. The critical behaviors and the scaling functions of a coalescence equation. *J. Phys. A: Math. Theor.* **53** (2020), article 195202. doi:10.1088/1751-8121/ab8134.

[4] X. Chen, B. Derrida, Y. Hu, M. Lifshits and Z. Shi. A max-type recursive model: some properties and open questions. In V. Sidoravicius (ed.), *Sojourns in Probability Theory and Statistical Physics–III*, Springer Proceedings in Mathematics & Statistics **300**, Springer, Singapore, 2019, pp. 166–186. doi:10.1007/978-981-15-0302-3_6.

[5] X. Chen, Y. Hu and Z. Shi. The sustainability probability for the critical Derrida–Retaux model. *Probab. Theory Related Fields* **182** (2022), 641–684. doi:10.1007/s00440-021-01091-z.

[6] X. Chen and Z. Shi. The stable Derrida–Retaux system at criticality. In M. E. Vares, R. Fernández, L. R. Fontes and C. M. Newman (eds.), *In and Out of Equilibrium 3: Celebrating Vladas Sidoravicius*, Progress in Probability **77**, Birkhäuser, Cham, 2021, pp. 239–264. doi:10.1007/978-3-030-60754-8_12.

[7] P. Collet, J.-P. Eckmann, V. Glaser and A. Martin. A spin glass with random couplings. *J. Stat. Phys.* **36** (1984), 89–106. doi:10.1007/BF01015728.

[8] P. Collet, J.-P. Eckmann, V. Glaser and A. Martin. Study of the iterations of a mapping associated to a spin glass model. *Comm. Math. Phys.* **94** (1984), 353–370. doi:10.1007/BF01224830. Archival version: *Les rencontres physiciens-mathématiciens de Strasbourg–RCP25* **33** (1983), exposé 6, 117–142. Numdam.

[9] L. de Branges. Some Hilbert spaces of entire functions. II. *Trans. Amer. Math. Soc.* **99** (1961), 118–152. doi:10.1090/S0002-9947-1961-0133456-2.

[10] L. de Branges. *Hilbert Spaces of Entire Functions*. Prentice-Hall, Englewood Cliffs, 1968.

[11] B. Derrida, V. Hakim and J. Vannimenus. Effect of disorder on two-dimensional wetting. *J. Stat. Phys.* **66** (1992), 1189–1213. doi:10.1007/BF01054419.

[12] B. Derrida and M. Retaux. The depinning transition in presence of disorder: a toy model. *J. Stat. Phys.* **156** (2014), 268–290. doi:10.1007/s10955-014-1006-y. Author version: arXiv:1401.6919v1.

[13] B. Derrida and Z. Shi. Results and conjectures on a toy model of depinning. *Mosc. Math. J.* **20** (2020), 695–709. doi:10.17323/1609-4514-2020-20-4-695-709. Author version: arXiv:2005.10208v1.

[14] J. Eckhardt, A. Kostenko and G. Teschl. Spectral asymptotics for canonical systems. *J. Reine Angew. Math.* **736** (2018), 285–315. doi:10.1515/crelle-2015-0034.

[15] G. Giacomin. *Random Polymer Models*. Imperial College Press, London, 2007. doi:10.1142/p504.

[16] G. Giacomin. *Disorder and Critical Phenomena Through Basic Probability Models: École d'Été de Probabilités de Saint-Flour XL–2010*. Lecture Notes in Mathematics **2025**, Springer, Heidelberg, 2011. doi:10.1007/978-3-642-21156-0.

[17] G. Giacomin, H. Lacoin and F. L. Toninelli. Hierarchical pinning models, quadratic maps and quenched disorder. *Probab. Theory Related Fields* **147** (2010), 185–216. doi:10.1007/s00440-009-0205-y.

[18] G. Giacomin, H. Lacoin and F. L. Toninelli. Marginal relevance of disorder for pinning models. *Comm. Pure Appl. Math.* **63** (2010), 233–265. doi:10.1002/cpa.20301.

[19] G. Giacomin and F. L. Toninelli. Smoothing effect of quenched disorder on polymer depinning transitions. *Comm. Math. Phys.* **266** (2006), 1–16. doi:10.1007/s00220-006-0008-2.

[20] T. E. Harris. A lower bound for the critical probability in a certain percolation process. *Proc. Cambridge Philos. Soc.* **56** (1960), 13–20. doi:10.1017/S0305004100034241.

[21] Y. Hu, B. Mallein and M. Pain. An exactly solvable continuous-time Derrida–Retaux model. *Comm. Math. Phys.* **375** (2020), 605–651. doi:10.1007/s00220-019-03465-w.

[22] Y. Hu and Z. Shi. The free energy in the Derrida–Retaux recursive model. *J. Stat. Phys.* **172** (2018), 718–741. doi:10.1007/s10955-018-2066-1.

[23] H. Lacoin and F. L. Toninelli. A smoothing inequality for hierarchical pinning models. In A. Boutet de Monvel and A. Bovier (eds.), *Spin Glasses: Statics and Dynamics*, Progress in Probability **62**, Birkhäuser, Basel, 2009, pp. 271–278. doi:10.1007/978-3-7643-9891-0_12. Author version dated 16 May 2008.

[24] Z. Li and R. Zhang. Asymptotic behavior of the generalized Derrida–Retaux recursive model. *J. Stat. Phys.* **192** (2025), article 98. doi:10.1007/s10955-025-03480-3. Corrected author version: arXiv:2411.13068v3 (18 August 2025).

[25] R. Lyons, R. Pemantle and Y. Peres. Conceptual proofs of $L \log L$ criteria for mean behavior of branching processes. *Ann. Probab.* **23** (1995), 1125–1138. doi:10.1214/aop/1176988176.

[26] C. Monthus and T. Garel. Critical points of quadratic renormalizations of random variables and phase transitions of disordered polymer models on diamond lattices. *Phys. Rev. E* **77** (2008), article 021132. doi:10.1103/PhysRevE.77.021132.

[27] E. Phragmén and E. Lindelöf. Sur une extension d'un principe classique de l'analyse et sur quelques propriétés des fonctions monogènes dans le voisinage d'un point singulier. *Acta Math.* **31** (1908), 381–406. doi:10.1007/BF02415450.

[28] R. Romanov. Canonical systems and de Branges spaces. arXiv:1408.6022 (2014).

[29] L.-H. Tang and H. Chaté. Rare-event induced binding transition of heteropolymers. *Phys. Rev. Lett.* **86** (2001), 830–833. doi:10.1103/PhysRevLett.86.830.