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EHRHART POSITIVITY FOR POLYPOSITROIDS

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Abstract. We prove that all coefficients of the Ehrhart polynomial of an integral polypositroid are strictly positive. In particular, this proves the positroid conjecture of Ferroni, Jochemko and Schröter. More generally, inclusion of integral polypositroids in a common cyclic order implies coefficientwise comparison of their Ehrhart polynomials. Hence the hypersimplex dominates every positroid of the same rank and size. We express each coefficient as a sum of relative face volumes multiplied by explicit nonnegative rational weights. The weights count linear extensions and specialize, for hypersimplices, to Ferroni's formula involving weighted Lah and Eulerian numbers. In the appendix we exhibit two representable matroids whose quartic Ehrhart coefficients exceed those of the hypersimplices of the same rank and size.

Disclaimer: This text, including the mathematical proofs, was produced almost entirely by generative AI tools. The author's contributions were limited to minor changes in the exposition, the order of presentation, and the choice of what to include or omit. The author claims no mathematical credit beyond his earlier work, written with several coauthors, on which these models were trained.

1. Introduction

For a nonempty lattice polytope $P \subseteq \mathbb{R}^n$, its Ehrhart polynomial $E_P(t)$ counts the points of $tP \cap \mathbb{Z}^n$ at nonnegative integers t. Write $e_j(P) = [t^j]E_P(t)$ and $E_A \succeq E_B$ when $E_A - E_B$ has nonnegative coefficients in the monomial basis. We call the line spanned by $e_i - e_j$ a root direction. We use lattice-relative volume, normalized so that a fundamental parallelepiped has volume one.

Give $[n] = \{1, \dots, n\}$ a cyclic order and put $x(I) = \sum_{i \in I} x_i$. A *polypositroid* is a polytope in a hyperplane $x([n]) = r$ satisfying the two conditions below [LP24].

(i) Each edge is parallel to a difference $e_i - e_j$ of coordinate vectors.
(ii) Within its sum hyperplane the polytope is cut out by upper bounds on $x(I)$ for cyclic intervals I.

A lower bound is also permitted, by taking the complement interval. A polypositroid is *integral* if all its vertices are lattice points. Its dimension throughout the paper is its affine dimension.

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Theorem 1.1 *Let P be an integral polypositroid. Then $e_j(P) > 0$ for every $0 \leq j \leq \dim P$. Moreover, if $P \subseteq Q$ are integral polypositroids in the same cyclic order, then $E_P \preceq E_Q$.*

A positroid is a matroid representable by a real matrix whose maximal minors in the given column order are nonnegative. Its base polytope has root edge directions and a cyclic interval description [ARW16, Propositions 5.5–5.7]. This yields the following consequence.

Corollary 1.2 (Ferroni–Jochemko–Schröter conjecture) *For every positroid M, the Ehrhart polynomial of its base polytope has strictly positive coefficients through its degree.*

Ferroni, Jochemko and Schröter obtained the rank-two case and proposed the general statement in [FJS22, Conjecture 6.3]. For lattice path matroids, Ferroni, Morales and Panova used the order polynomials of fence posets [FMP25, FMP26]. Li and Shao proved positivity for series–parallel matroids and constructed a transversal matroid with a negative coefficient [LS26]. Liu and Zhang proved the paving positroid case [LZ26].

The proof works with sums along intervals in a fixed order. A small perturbation makes the intervals tight at any vertex nested or disjoint. We can then count integer vectors in each tangent cone by imposing one bound at a time. The issue is how the boundary terms from different cones fit together. Theorem 4.1 groups the terms by their chosen equalities. The remaining signed sum is the indicator of a single face. Opening a prefix facet gives coefficientwise domination over that facet. We pass from facets to arbitrary inclusions by interpolating the two systems of cyclic bounds and relaxing one bound at a time. Strict positivity then follows by comparing P with one of its faces in every dimension.

Proposition 7.1 makes the face weights explicit. A weight is a product of factors $a!b!/(a+b+1)!$, obtained by placing one label between a preceding labels and b following labels. We use this description in §7.1 to recover the weighted Lah numbers for hypersimplices. Appendix A disproves the upper bound in Ferroni's two-sided conjecture for matroids [Fer22, Conjecture 1.5]. The question for positroids, posed explicitly in §7.2, has an affirmative answer by inclusion in the hypersimplex.

2. Two elementary polyhedral identities

Lemma 2.1 (Closed tangent cones) *Let R be a nonempty polytope, and work in its affine hull. For a nonempty face F let $T_F R$ be the intersection of the closed facet halfspaces whose bounding facets contain F. The empty intersection, corresponding*

to $F = R$, is the whole affine hull. Then

$$\sum_{\varnothing \neq F \leq R} (-1)^{\dim F} \mathbf{1}_{T_F R} = \mathbf{1}_R.$$

Proof. Fix a point z in the affine hull. Let U be the union of the facets whose halfspaces z strictly violates, together with all their faces. The terms missing from the complete face sum at z are exactly the nonempty faces of U. The complete face sum is 1, by Euler's formula. If $z \in R$, then U is empty.

Suppose $z \notin R$. Strictly separate z from R by a hyperplane H. Each ray from z meeting R has a unique first point p of R. Some facet through p is strictly violated by z: otherwise a short step from p towards z would satisfy every facet inequality. Conversely, each point of a strictly violated facet is a first point. Central projection from z therefore gives a continuous bijection from U onto the convex section $H \cap \{z + \lambda(y - z) : y \in R, \lambda \geq 0\}$. The domain is compact, so this is a homeomorphism. In particular U is contractible, and its alternating face sum is 1. Here we use the elementary Euler formula for a finite polytopal complex: the alternating sum of its nonempty cells is its Euler characteristic, as follows by triangulating its cells and taking the alternating sum of the simplicial chain ranks. Subtraction gives zero at z. The use of strictly violated facets includes points z lying on any of the other facet hyperplanes. □

Lemma 2.2 (Two lattice summands) *If P and R are lattice polytopes in the same lattice, then*

$$B(a, b) = \#\big((aP + bR) \cap \mathbb{Z}^n\big)$$

agrees with a single polynomial for all nonnegative integers a, b, including the axes.

Proof. Take the cone spanned by $(v, 1, 0)$ for vertices v of P and $(w, 0, 1)$ for vertices w of R. Its lattice points of bidegree (a, b) are precisely the points being counted. Triangulate the cone using its generating rays; for instance, triangulate the section where the last two coordinates sum to one by successively pulling vertices. Pulling a vertex means coning it to the triangulations of the facets missing it, with the same vertex order used recursively on the facets.

Choose a vector h in the relative interior of the cone outside all spans of facets of the triangulation. Assign x to the unique maximal simplicial cone containing $x + \epsilon h$ in its relative interior for all sufficiently small $\epsilon > 0$. This partitions the closed cone. In the ray coordinates of a maximal cone, the coordinates of x must be nonnegative, and strictly positive wherever the coordinate of h is negative. Suppose this cone has p rays of the first bidegree and q of the second. Both are positive. In each group at least one coordinate of h is positive, since their group sums are the two positive degree coordinates of h.

Every lattice point in the assigned part of this cone has a unique expression as a nonnegative integer combination of its ray generators plus a lattice point of their fundamental parallelepiped. Use $[0,1)$ for a closed ray coordinate and $(0,1]$ for an open one. In each group at least one interval is $[0,1)$, so every such parallelepiped point has integral bidegree (i,j) with $0 \le i < p$ and $0 \le j < q$. Its contribution to $B(a,b)$ is therefore

$$\binom{a-i+p-1}{p-1}\binom{b-j+q-1}{q-1}.$$

The upper arguments are nonnegative even on the axes; either factor automatically vanishes when its required number of extra ray steps is negative. This is a polynomial for all $a, b \ge 0$. Sum over the finitely many cones and parallelepiped points. □

In particular, taking the second summand to be a point proves polynomiality of E_P. Its constant term is 1. For completeness, if P has dimension d, translate a vertex to the origin and choose lattice coordinates in its linear span. The grid count $t^{-d}|tP \cap \mathbb{Z}^n|$ is a Riemann sum for the indicator of P in those coordinates. The boundary has measure zero. Its limit is the positive relative volume, so E_P has degree d and leading coefficient $\mathrm{vol}(P)$.

We will also use a direct consequence of Lemma 2.2. If $R_L = D(L-L_0)P + R_{L_0}$ for large integers L and fixed $D \in \mathbb{Z}_{>0}$, then, for $j \ge 1$,

$$e_j(R_L) = (DL)^j e_j(P) + O(L^{j-1}). \tag{1}$$

Indeed, substitute $a = D(L-L_0)t$ and $b = t$ in $B(a,b)$. Only terms $a^u b^{j-u}$ contribute to $[t^j]$, and the coefficient of $a^j b^0$ is $e_j(P)$ by evaluation at $b = 0$.

3. A counting rule for a bounded sum

Fix an integer $t \ge 1$. We record the sum of integer coordinates by the power of z in their generating polynomial. Multiplying two such polynomials concatenates the two lists of coordinates. We need a rule for retaining totals in a closed interval that also records what happens at either endpoint.

Use two symbols S, D. Their generating functions are z^t and $(1-z)^{-1}$, respectively. Thus S adds t to a total and D allows any nonnegative integer contribution. We use finite linear combinations of $S^j D^p$, $j, p \ge 0$, with coefficients in $\mathbb{Q}[t]$. These combinations also have a continuous interpretation, in which each step either stays at its current position or moves to the right. Choose a one-periodic function $m\colon \mathbb{R} \to [0,1]$ that is piecewise affine on each period and takes uniformly distributed values there. Point values at the finitely many breakpoints can be assigned arbitrarily, subject to periodicity. Set

$$(Sg)(u) = g(u+1), \qquad (Dg)(u) = m(u)g(u) + t\int_0^\infty g(u+s)\,ds$$

on piecewise continuous functions vanishing sufficiently far to the right. Periodicity makes S and D commute. A product acts by taking its steps in order, starting at u. Its expansion consists of atoms at integer displacements and a density $K_O(u, w; m)$ at final positions $w > u$. For example the box step

$$A_c = D + S^c(1 - D), \qquad c \in \mathbb{Z}_{>0}, \tag{2}$$

has generating polynomial $1 + z + \cdots + z^{ct}$. Its continuous rule selects displacement 0 with weight $m(u)$, displacement c with weight $1 - m(u)$, or $s \in (0, c)$ with measure $t\,ds$.

Lemma 3.1 (Closed truncation) *For any such combination O and integer $k \geq 0$ there are combinations U_kO, V_kO which retain exactly the powers of z of degrees at most kt and at least kt, respectively. Their continuous rules retain the old atoms and density on the corresponding closed displacement range. Each also adds one atom at displacement k. Putting $x = m(u)$, its additional weight is respectively*

$$(1 - x) \lim_{w \downarrow k} \frac{K_O(0, w; m_x^-)}{t}, \qquad x \lim_{w \uparrow k} \frac{K_O(0, w; m_x^-)}{t}, \tag{3}$$

where $m_x^-(h) = x$ at integers h and $m_x^-(s) = 1 - \{s\}$ elsewhere. The density at negative displacements is zero.

Proof. It suffices to treat $S^j D^p$ for all p at once. Introduce a formal variable λ and write

$$R = \sum_{p \geq 0} \lambda^p D^p, \qquad b = (1 - \lambda)^{-1}, \qquad a(u) = (1 - \lambda m(u))^{-1}.$$

An identity for these series gives a finite identity in D upon taking $[\lambda^p]$. On the discrete side

$$R(z) = \frac{1 - z}{1 - \lambda - z}, \qquad [z^0]R(z) = b, \qquad [z^h]R(z) = \lambda b^{h+1} \quad (h \geq 1).$$

In particular the tail is geometric. For $j < k$ and $C = (k - j)t$ this gives

$$U_k(S^j R) = S^j R + S^k b^C (b - R), \qquad V_k(S^j R) = S^k b^C (R - 1). \tag{4}$$

For $j = k$ the upper and lower expressions are $S^k b$ and $S^k R$. For $j > k$ they are 0 and $S^j R$. The coefficients $[\lambda^h]b^C$ are polynomials in t.

To read the same identities continuously, expand a word of p steps by its sequence of strictly increasing positions. At a visited position v, an arbitrary string of stays has generating series $a(v)$. If there are $q \geq 1$ moves from u to w, the $q - 1$ intermediate positions are ordered. Integrating the product of their a-weights gives $(\int_u^w a(s)\,ds)^{q-1}/(q-1)!$. Summing over q gives the atom $a(u)$ and the density

$$K_R(u, w; m) = \lambda t\, a(u) a(w) \exp\left(\lambda t \int_u^w a(s)\,ds\right), \qquad w > u. \tag{5}$$

This computation is coefficientwise and retains the specified point values at the two ends of the ordered sequence.

A complete period multiplies the exponential by b^t, since $\lambda \int_0^1 a(s)\,ds = \lambda \int_0^1 (1-\lambda y)^{-1} dy = \log b$. Thus, above displacement k, the density of $S^j R$ agrees exactly with that of $S^k b^C R$. Equation (4) removes it on the upper truncation and retains it on the lower truncation. Below k the terms shifted by k contribute nothing. For $j < k$ the extra atoms at k have weights, respectively,

$$b^C(b - a(u)) = \lambda(1-x) b^{C+1} a(u), \qquad b^C(a(u) - 1) = \lambda x b^C a(u).$$

For the profile m_x^- the terminal value $m_x^-(w)$ approaches 1 from the right of k and 0 from the left. In (5) the two limits of $K_R(j, w; m_x^-)/t$ are therefore $\lambda b^{C+1} a(u)$ and $\lambda b^C a(u)$. This proves (3) for $j < k$. When $j = k$, the upper truncation adds $b - a(u)$ to the old atom, and the lower truncation adds zero, as the same one-sided rules give. When $j > k$ both limits vanish. Taking coefficients completes the proof. □

Lemma 3.2 (A final slack coordinate) *Fix $x \in [0,1]$ and evaluate the step rule from the origin with the profile m_x^-. If the discrete specialization $F_t(z)$ of O is a polynomial and the step rule has bounded displacement support, then $(O\mathbf{1})(0) = F_t(1)$.*

Proof. Append one unrestricted step, $H = OD$. Choose k larger than every displacement in the support of O, every power of S occurring in H, and $\deg(F_t)/t$. Then $[z^{kt}]H(z) = F_t(1)$, since the last coordinate supplies the slack. The formulas in the proof of Lemma 3.1 give the symbol identity

$$(U_k + V_k - \mathrm{Id})H = S^k [z^{kt}] H(z)$$

by coefficient extraction from $S^j R$. In the step rule there is no old atom at k. Near k the density of H is $t(O\mathbf{1})(0)$: the last step must move, and it can reach the final position from every endpoint of O. The two extra atom weights in (3) therefore sum to $(1 - x + x)(O\mathbf{1})(0)$, since the reset at the origin uses the same profile m_x^-. Thus the step rule of the left-hand side is a single atom at k of weight $(O\mathbf{1})(0)$, while that of the right-hand side is the atom $F_t(1)$ at k. Comparing these atoms proves the identity. □

4. A face formula for interval inequalities

In this section intervals belong to one fixed linear order on $[d]$. A family of intervals is *laminar* if each pair is disjoint or nested. A facet carries its interval I, the direction of its inequality and its integer threshold k_I.

Theorem 4.1 *Let $Q \subseteq \mathbb{R}^d$ be a full-dimensional simple lattice polytope given by interval bounds. Assume the facet intervals tight at each vertex form a laminar*

family. There are weights $w_G \geq 0$ on its nonempty faces such that

$$E_Q(t) = \sum_{\varnothing \neq G \leq Q} w_G t^{\dim G}.$$

Moreover $E_Q \succeq E_H$ whenever H is a facet defined by a bound on a prefix $[1, b]$.

Proof. Take an integral box strictly containing Q and translate its endpoints to $0, c_i$ in each coordinate. All facet thresholds of the translated polytope are now positive. Fix $x \in [0, 1]$ and use the decreasing profile m_x^- of Lemma 3.1, starting at zero. The box step in coordinate i is A_{c_i}.

For a nonempty face F, let $B(F)$ be its containing facets. The tangent polyhedron $T_F Q$ imposes just these bounds. They are laminar, being tight at any vertex of F, and distinct members of $B(F)$ have distinct intervals since Q is full-dimensional. So the coordinate enumerator in the clipped tangent polyhedron has a symbol O_F. Read the coordinates from left to right, multiplying the factors for each interval and applying its closed truncation, proceeding from smaller intervals to larger ones. This records each lattice vector once by the power of z equal to its coordinate sum. By Lemmas 2.1 and 3.2,

$$E_Q(t) = \sum_{\varnothing \neq F \leq Q} (-1)^{\dim F} (O_F \mathbf{1})(0). \tag{6}$$

Expanding a summand. At each bound Lemma 3.1 gives the restricted old measure plus its extra boundary atom. Fix the set $S \subseteq B(F)$ where this atom is chosen; we call its intervals *pinned*. The construction below also fixes whether each coordinate step uses an endpoint or its continuous part. Put $\sigma_I = 1$ for an upper pin and -1 for a lower pin. In the density for I evaluate at $k_I + \sigma_I \epsilon_I$, with $0 < \epsilon_I < 1/2$. Its displacement in its parent is still the exact integer k_I.

Order the pins by containment and put a formal root above them, even if $[d]$ is pinned. The children of a node are its maximal proper pins. The coordinates outside those children are its *direct leaves*. A pinned singleton therefore has one direct leaf. Writing z_i for the actual coordinate displacements, at each pin I we have

$$\sum_{i \text{ direct in } I} z_i = k_I + \sigma_I \epsilon_I - \sum_{J \text{ child of } I} k_J. \tag{7}$$

The endpoint leaves have integer displacements. A nonzero term therefore requires a continuous direct leaf at each pin. Use the last such leaf to solve (7), with Jacobian one. The pivot leaves at distinct pins are distinct. If p leaves were continuous, the resulting integral is over $p - |S|$ variables and has the single factor $t^{p-|S|}$. Indeed every continuous step supplies t and every boundary density in (3) divides by t.

An unpinned bound just tests its interval sum. At a pin whose parent profile has value v at its starting position, the rule gives weight v for a lower pin and $1 - v$ for an upper pin. Inside it, start at zero with profile m_v^-; outside it resume after its integer

displacement k_I. Thus all profiles, starting positions and weights are determined by S and the leaf choices. All of these weights are nonnegative.
One parameter suffices. These integration rules follow by convolution of successive steps. In a nested boundary density the inner limit is initially taken first. To evaluate all limits, in each node replace its free continuous direct leaf variables by their successive partial sums $q_1, q_2, \ldots$, starting with $q_0 = 0$. This is triangular with determinant one. At a pin the last partial sum has already been fixed at an integer plus $\sigma_I \epsilon_I$ by (7); at the root all nonempty partial sums are free. Child pins and endpoint leaves contribute only integers to apparent displacements. Consequently a profile argument is an integer plus one partial sum, and each unpinned bound in its nearest enclosing node compares a difference $q_b - q_a$ with an integer. The range indicators of pivot leaves have this same form.

Outside the null hyperplanes on which a free partial sum, or a difference of distinct free partial sums, is integral, every nonconstant comparison is eventually fixed. A constant comparison either has no error or tests the sign of its own node's $\sigma_I \epsilon_I$. At an exactly integral start the profile inherits the value from its parent; at a pinned last partial sum it approaches its integer from the indicated side. These rules retain the contribution of a purely atomic subproduct. If an enclosing density is evaluated at a noninteger total, a subproduct containing all its direct continuous steps likewise has noninteger total; a proper nonempty subset of these steps has integral total only on a null hyperplane. Thus no choice of the point value of a continuous density at an integer is required. The free variables can be confined to a fixed bounded box and the integrands, after their power of t is removed, are bounded by one. Each weight and indicator depends on at most one ϵ_I and converges, outside a null set, as that $\epsilon_I \downarrow 0$. Bounded convergence therefore shows that the integrals converge along every joint approach of the errors to zero. The joint limit thus exists and agrees with the iterated limit in the prescribed nested order; in particular we may take all $\epsilon_I = \epsilon \downarrow 0$.
Summing over faces. For the fixed choices above the remaining faces in (6) are precisely the nonempty faces of $G = Q \cap \{z(I) = k_I : I \in S\}$, with $G = Q$ for $S = \varnothing$. Put

$$\delta_A = \epsilon \sum_{I \in S,\ I \subseteq A} \sigma_I$$

for every facet interval A. Summing (7) gives $z(I) = k_I + \delta_I$ for a pin I. An unpinned active interval A sees each contained pin at its external displacement, so it tests its bound at $k_A + \delta_A$ in the actual coordinates z. These are the same shifted thresholds for every $F \leq G$.

Shift all the facet thresholds in this way, taking equality at S, and denote the resulting face by G_ϵ. For sufficiently small ϵ it has the labeled face lattice of G. Indeed each vertex of the simple polytope Q is the solution of a nonsingular active system with the other inequalities strict. Its feasible systems persist, and any other

candidate independent system remains infeasible for sufficiently small changes. This preserves the full labeled face lattice and its face at S. In the affine space of the shifted pinned equalities the remaining tests for F are exactly the tangent inequalities of G_ϵ at the corresponding face. The signed sum of those tests over $F \le G$ is therefore $\mathbf{1}_{G_\epsilon}(z)$ by Lemma 2.1, with the common weights unchanged.

Each grouped integral is now nonnegative, and so is its limit. Moreover, every G_ϵ remains strictly inside the auxiliary box. A leaf at a box endpoint then contributes zero after grouping. Only the all-continuous choice remains and gives degree $d-|S| = \dim G$. Every face of a simple polytope has exactly one such set S, its containing facets. The surviving integral is w_G.

For the last assertion, make the bound defining H strict precisely in the tangent cones at $F \le H$. The signed indicator sum becomes $\mathbf{1}_Q - \mathbf{1}_H$: the pieces removed on the supporting hyperplane are exactly the tangent cones of H, with the same face signs. For a strict upper cutoff use $O - V_k O$. By Lemma 3.1 this restricts the old measure to displacements $< k$ and subtracts one atom with factor v, the incoming profile value. Choose $x = 0$. Every selected interval containing the specified prefix is itself a prefix, so the cutoff is read at zero throughout all its enclosing intervals and $v = x = 0$. For a strict lower cutoff use $O - U_k O$ and $x = 1$; the subtracted factor $1 - v$ again vanishes.

Thus the open facet never belongs to S. In the shifted face G_ϵ it either misses the face or cuts out a proper facet by simplicity. In the first case the signed residual tests give the whole face indicator. In the second case the same tangent-cone identity with one facet opened gives the indicator of G_ϵ minus the indicator of that facet, including at its boundary. Both are nonnegative pointwise. The preceding common-integral proof applies with the strict comparison at this one bound and gives $E_Q - E_H \succeq 0$. □

5. Perturbing the cyclic intervals

Let P be an integral polypositroid on $[n]$, with $n \ge 2$. Denote its tight bounds by $a_A = \max_{x \in P} x(A)$ for all $A \subseteq [n]$. In the defining cyclic inequalities we may replace each right-hand side by its tight bound. Since P is integral, these numbers and the coordinate sum r are integers.

We record why a is submodular. All linear functionals whose coordinates have a given strict order orient each edge of P in the same direction, since its edges are root directions. Their maximizing vertex is the same: a vertex is a maximizer precisely when none of its incident edges increases the functional. By taking limits, this vertex also maximizes the sums on every prefix of the order. Choose the elements of $A \cap B$ first and the remaining elements of $A \cup B$ next. At the resulting vertex v,

$$a_A + a_B \ge v(A) + v(B) = a_{A \cap B} + a_{A \cup B}.$$

Delete the last coordinate. The chart identifies the integer lattice on each integral sum hyperplane with $\mathbb{Z}^{n-1}$. A cyclic inequality on I avoiding n is an upper bound on a linear interval; if $n \in I$ it is a lower bound on the linear interval $[n] \setminus I$. For each nonempty proper cyclic interval put

$$q_I = |I|(n - |I|), \qquad c_I = q_I + \epsilon_I, \qquad |\epsilon_I| < 1/10,$$

where the ϵ_I are generic rational numbers. At total Lr, replace the bound a_I by $La_I + c_I$. Call the resulting polytope $\widetilde{Q}_L$. It is bounded, by the one-coordinate bounds and their complements. Since $c_I > 0$, every point of LP is in its relative interior in the sum hyperplane.

Suppose two linear intervals overlap without either containing the other. Then the corresponding cyclic bipartitions cross, regardless of which side contains n. For the original intervals A, B, all four regions of their Venn diagram are nonempty. Cutting the circle outside their union shows that their intersection and union are both nonempty proper cyclic intervals. Moreover,

$$q_A + q_B - q_{A\cap B} - q_{A\cup B} = 2|A \setminus B|\,|B \setminus A| > 0.$$

Adding La preserves this inequality by submodularity, and the four small errors cannot exhaust its slack. Simultaneous equality in the bounds on A, B would now give $x(A) + x(B) > x(A \cap B) + x(A \cup B)$, an impossibility. Thus the intervals of coactive bounds in the chart form a laminar family.

Any $n - 1$ independent bound normals determine a candidate vertex affine in L. For any other bound, its slack at this candidate is affine in L. The independent perturbations can avoid that slack vanishing identically: the offset of the other bound has nonzero coefficient and does not enter the candidate. There are finitely many candidates and bounds. For all sufficiently large L the feasible bases thus stabilize and every vertex is simple. Take a common denominator D for the c_I and put $Q_L = D\widetilde{Q}_L$. In the cumulative coordinates $y_i = x_1 + \cdots + x_i$, $y_0 = 0$, a row is a signed difference $y_b - y_a$. Independent difference rows form a forest on the index set; a cycle gives a dependence, and a forest is independent by leaf removal. In a full system their graph is a tree rooted at 0, so integer right-hand sides give integer coordinates by summing along paths. Consequently Q_L is integral.

For L in the stable range, write each feasible vertex as $DLv + w$. On dividing its equations and inequalities by DL and letting L tend to infinity, one sees that $v \in P$ and the same basis inequalities are tight at v. The cone of maximizing functionals at $DLv + w$ consists of the nonnegative combinations of the outward normals of these rows. This follows by duality from the feasible infinitesimal directions at the vertex. Each such functional is also maximized on P at v. Take L_0 in the stable range. Equality of support functions gives

$$Q_L = D(L - L_0)P + Q_{L_0} \qquad (L \geq L_0) \tag{8}$$

in the projected chart. In particular (1) and the face formula give $e_j(P) \geq 0$. Below we compare nested polytopes by using compatible perturbations of both.

6. Monotonicity in a fixed cyclic order

The following is the only face comparison needed before taking limits. Write $d = n - 1$ for the dimension of the sum hyperplane.

Lemma 6.1 *If an integral polypositroid R has dimension d and H is a facet of R, then $E_R \succeq E_H$.*

Proof. Choose a supporting cyclic bound $x(A) \leq a_A$ of H. Rotate the ground set so that A starts at 1, and omit its predecessor as the last coordinate. The interval is now a prefix. For the deformation of R in (8), let H_L be the maximizing face in direction $\mathbf{1}_A$. Faces exposed by a fixed functional add under Minkowski addition, so $H_L = D(L - L_0)H + H_{L_0}$. For $L > L_0$ this face has dimension at least $d - 1$. The polytope Q_L is full-dimensional, so H_L is a facet. Its supporting bound is the prefix bound on A: in the full chart no other raw upper bound has the same outward normal. By Theorem 4.1, $E_{Q_L} \succeq E_{H_L}$. Since H and H_{L_0} are lattice polytopes, (1) applies to both sides; divide in degree $j \geq 1$ by $(DL)^j$ and let $L \to \infty$. Their constant coefficients agree. □

Lemma 6.2 *Suppose $K' \subsetneq K$ are full-dimensional integral polypositroids in the same sum hyperplane and cyclic order, and that $K' = K \cap \{x(A) \leq a\}$ for one integral cyclic interval bound. Then $E_K \succeq E_{K'}$.*

Proof. The cutting plane meets the interior of K. Let R be the other closed half and H their common facet. Every edge of R in H is an edge of H, hence an edge of the polypositroid K'. At a relative interior point of any other edge the cut is inactive, so its direction is that of an edge of K. Thus R has root edge directions. Its description is still by integral cyclic bounds, since the new lower bound can be written on the complementary interval. The cumulative-coordinate tree argument of §5 gives integrality. Lemma 6.1 and inclusion–exclusion give $E_K - E_{K'} = E_R - E_H \succeq 0$. □

Proof of Theorem 1.1. Suppose $P \subseteq Q$. Their coordinate sums agree and their tight bounds satisfy $a_I(P) \leq a_I(Q)$ for every cyclic interval I. Use the same perturbation $c_I = q_I + \epsilon_I$ for both endpoints, avoiding the finite exceptional systems in §5 for either polytope. Enlarge the common denominator D to at least 10. For fixed L the two systems have integral right-hand sides

$$b_I^- = D(La_I(P) + c_I), \qquad b_I^+ = D(La_I(Q) + c_I).$$

Along the segment between b^- and b^+, each crossing slack $b_A + b_B - b_{A\cap B} - b_{A\cup B}$ is at least $(2 - 4/10)D$. Indeed this holds at both ends by submodularity and the calculation in §5.

Follow the componentwise floors of the segment from b^- to b^+. Resolve simultaneous jumps by increasing one entry at a time. Every intermediate integer vector differs by at most one in each entry from a point on the segment. Rounding can decrease any crossing slack by at most four, so it remains strictly positive. The intermediate polytopes contain the lower endpoint polytope and lie in the upper one, so they are bounded and full-dimensional. We check that they all have root edge directions.

At a relative interior point of an edge take $n-2$ independent chart rows tight there. Positive crossing slack ensures that, after complementing the raw intervals containing n, their supports are laminar. In the homogeneous equations for edge directions, subtract from each selected interval its maximal proper selected subintervals, and from $[n]$ all maximal selected intervals. The resulting $n-1$ equations say that the coordinate displacement sums to zero on each of $n-1$ disjoint blocks partitioning $[n]$. None is empty, by independence; the last one contains n. Exactly one block has size two, proving that the edge has a root direction. Integrality follows from the difference rows as before. This proves that the intermediate polytopes are integral polypositroids.

Successive distinct polytopes now satisfy Lemma 6.2. Chaining its inequalities compares the Ehrhart polynomials at the two deformed endpoints. Dividing the degree-j inequality by $(DL)^j$ and applying (1) proves $E_P \preceq E_Q$, also when one or both original polytopes have smaller dimension. For $n = 1$ both are the same point.

Finally, every nonempty face F of an integral polypositroid is an integral polypositroid for the induced cyclic interval description: turn the defining tight bounds into equalities, and observe that its edges are edges of the original polytope. Applying the inclusion inequality to $F \subseteq P$ yields, in each degree,

$$e_j(P) \;\geq\; \max_{\substack{F\leq P\\ \dim F=j}} \operatorname{vol}_j(F) > 0 \qquad (0 \leq j \leq \dim P). \tag{9}$$

Here a point has volume one. □

7. Face volumes and linear extensions

For the closed polytope we take $m(u) = 1 - \{u\}$ off the integers and $m(h) = 1$ at integer h. For simple Q in Theorem 4.1, its own dilates LQ already have a constant normal fan. The same computation covers the possibly nonsimple P using the deformations in (8). Put $d = n - 1$ and let G_L be a stable j-face of Q_L. Its $d - j$ incident facets give a laminar set S of signed intervals. Choose a functional in the relative interior of their normal cone and write G for its maximizing face on P.

From (8),

$$G_L = D(L - L_0)G + G_{L_0}, \qquad \dim G \le j.$$

Make the containment forest of S, with an additional formal root. In each node H the *direct leaves* are the individual coordinates outside all its selected children. Each real node has at least one direct leaf: otherwise its interval normal would be a sum of child normals. For its h_H direct leaves, in order, let $q_{H,a}$ be the sum of the first a coordinate values. The free prefixes are $q_{H,1}, \dots, q_{H,h_H-1}$ at a real node and $q_{H,1}, \dots, q_{H,h_H}$ at the formal root. There are j such variables. They give lattice coordinates in the pinned affine space: differences recover the direct leaves, and each last leaf at a real node then follows from the pinned sum minus its child sums. Thus arbitrarily assigned integer free prefixes give integral leaf coordinates, and conversely. This chart has the same volume form as in the face formula. Write $u_{H,a} = \{q_{H,a}\}$ for each free prefix.

Here is a rule using only the marked intervals of S. Start with $v_{\text{root}} = 1$. If I is a selected child of H, let a be the number of direct leaves of H preceding I. Its incoming value is

$$v_I = \begin{cases} v_H, & a = 0, \\ 1, & a = h_H > 0,\ H \text{ a real upper pin}, \\ 0, & a = h_H > 0,\ H \text{ a real lower pin}, \\ 1 - u_{H,a}, & \text{otherwise}. \end{cases}$$

The factor for I is v_I for a lower bound, $1 - v_I$ for an upper bound. If a factor is identically zero, put $\beta(S) = 0$. Otherwise the product of all the factors has the form

$$W_S(u) = \prod_{k=1}^{j} u_k^{a_k}(1 - u_k)^{b_k}, \qquad \beta(S) = \prod_{k=1}^{j} \frac{a_k!\, b_k!}{(a_k + b_k + 1)!}.$$

Empty products equal one. In the zero case set $W_S = 0$.

Proposition 7.1 *For $0 \le j \le \dim P$, one has*

$$e_j(P) = \sum_{S:\, \dim G = j} \beta(S)\ \mathrm{vol}_j(G).$$

Proof. In the grouped integral for G_L the enclosing box is strict, so its only surviving leaf pattern integrates every leaf. The starting position of I inside its nearest selected parent H is $q_{H,a}$ plus the integer loads of preceding child pins. If $a = 0$ its profile has the value inherited from H. If the last direct leaf of a real parent has already been passed, the total prefix is an integer plus $\sigma_H \epsilon_H$ and its sawtooth limit is one on the right and zero on the left. All other nonzero prefixes are free. Outside a null set these give $1 - \{q_{H,a}\}$. This proves the rule for v_I. The indicator of the shifted face converges to the indicator of G_L almost everywhere in its free coordinates, since

for each fixed L the labeled faces are stable and their vertices converge. For $j = 0$ the unique point is retained. The contribution of S therefore is

$$\int_{\pi_S G_L} W_S(\{q\})\, dq,$$

where π_S is the projection to the free prefixes. An integral translation of the enclosing box shifts these prefixes by integers and leaves the integrand unchanged.

On every integer unit cube the integrand has integral $\beta(S)$, by $\int_0^1 u^a(1-u)^b\, du = a!b!/(a+b+1)!$. If $\dim G = j > 0$, then $\pi_S G_L$ has volume $(DL)^j \operatorname{vol}_j(G) + O(L^{j-1})$. Its bounded number of facet hyperplanes meet $O(L^{j-1})$ unit cubes: the entire face has diameter $O(L)$, and cubes meeting a hyperplane lie in a slab of bounded thickness. Thus its integral, divided by $(DL)^j$, tends to $\beta(S)\operatorname{vol}_j(G)$. If $\dim G < j$, the projected face stays in a bounded neighborhood of a dilated polytope of dimension less than j, so this limit is zero. The point case is exact. Summing over stable faces and using (1) proves the formula. □

Whenever $\beta(S) > 0$, the integer $d!\,\beta(S)$ counts linear extensions. Attach one label to every free prefix and one label to every pin. For a factor u_k require the pin label to precede prefix label k; for $1 - u_k$ require it to follow. A factor one gives an isolated label, and a zero factor makes the count zero. There are d labels in total. Conditional on the prefix positions in $(0, 1)$, independent uniform positions for the pin labels satisfy the comparisons with probability $W_S(u)$. All $d!$ relative orders are equally likely. This is also a direct order count: each nontrivial connected component consists of prefix label k, preceded by a_k pin labels and followed by b_k pin labels.

For a contributing face G the number $j!\operatorname{vol}_j(G)$ counts unit alcoves. Indeed in the original cumulative coordinates choose an independent set of tight difference equalities spanning the annihilator of the direction space of G, for instance those indexed by S. Their graph on the cumulative coordinate indices is a forest. Choose one representative for every connected component, fixing the component of the origin at zero. All other coordinates differ from their representative by integer constants. This also identifies the affine lattice with $\mathbb{Z}^j$; every remaining interval bound is a difference bound with an integral constant. Decomposing unit cubes according to the strict order of the j fractional parts partitions G into unit simplices, each of volume $1/j!$, up to their boundaries. Consequently $d!j!e_j(P)$ counts the triples formed by a contributing S, a unit alcove of G, and a linear extension of the indicated comparisons. For $j = 0$ the point is counted as one alcove.

Example 7.2 Take $0 \le z_1, z_2 \le 2$ and $1 \le z_1 + z_2 \le 3$, the hexagonal chart of the slice $x([3]) = 3$ of $[0, 2]^3$. Every edge has lattice length one. In the order lower and upper z_1, lower and upper z_2, lower and upper total, their weights are $1, 0, \frac{1}{2}, \frac{1}{2}, 1, 0$. For example, at a bound on z_2 its label is compared with the free prefix label from z_1,

and one of their two orders contributes. The vertex $(0,1)$ is the only one of weight one and the others have weight zero. The area is three, giving $E(t) = 1 + 3t + 3t^2$.

7.1. **Lattice paths and the weighted Lah numbers.** In the lattice path proof, Ferroni, Morales and Panova group the cells of a subdivision into snakes by the north-east path obtained by expanding the diagonal steps of a Delannoy path. For a ribbon plane partition of height at most t, summing over its marked high peaks cancels every labeling that attains t at one of those peaks. The cells still labeled t form an order filter. Removing it gives the fence order polynomials in [FMP26, Theorems 3.7, 3.11 and 3.12]. In our face formula the analogous grouping fixes the intervals supplying boundary terms; the remaining signed indicators sum to the indicator of one face. For nested lattice path diagrams the path formula gives coefficientwise monotonicity by inclusion of the admissible paths [FMP26, Theorem 4.1].

For $1 \le k < n$, the face weights of the hypersimplex $\Delta_{k,n} = \{x \in [0,1]^n : x([n]) = k\}$ recover the weighted Lah numbers of [Fer21]. Fix $1 \le j \le n-1$ and put $m = j+1$. Let S specify $n-m$ coordinates fixed at zero or one, and let $\ell(S)$ be the number fixed at one. Translate the bound on the omitted coordinate into the opposite bound on the sum of the remaining coordinates. With $\beta(S)$ as in Proposition 7.1, we claim that

$$(n-1)! \sum_{S:\,\ell(S)=\ell} \beta(S) = j!\, W(\ell, n, m). \tag{10}$$

Here $W(\ell, n, m)$ counts unordered partitions of $[n]$ into m linearly ordered blocks, with ℓ entries smaller than the first entry of their own block.

For the proof, fix the m free coordinates. They divide the cyclic word $1, \dots, n$ into m blocks, each consisting of one free coordinate followed by its consecutive fixed coordinates. Omit a coordinate q in the chart and call the block containing it B_0. In each other block of size b, the $b-1$ fixed steps all see the same free fractional part u. By the free-prefix rule each such u varies independently in $[0,1]$. At each position a lower bound contributes $1-u$ and an upper bound contributes zu, with z recording fixed ones. Integrating over the independent fraction for this block gives

$$H_b(z) = \int_0^1 (1 - u + zu)^{b-1}\, du = \frac{1 + z + \cdots + z^{b-1}}{b}.$$

Number the positions of B_0 from 0 at its free coordinate, and let r be the position of q. The contribution of B_0 is z^r. Indeed, the fixed coordinates just after q occur first in the chart, start at integers, and are forced to their lower bounds by the assigned profile value one. If $r > 0$, the bound on q itself must be its upper bound: a lower bound on x_q would be an upper bound on the chart total and has boundary factor zero at the origin. Its upper bound is a lower bound on the chart total, which is sampled from the left. All fixed steps after the last free step of the chart therefore

start just below integers and must also be upper. This accounts for precisely r fixed ones.

The sum over all free sets and choices of bounds is invariant under rotation of q. Average over q before summing over free sets. For fixed block sizes $b_1, \dots, b_m$, the average is $\prod_i H_{b_i}(z)$, since summing z^r over the positions of block i gives $b_i H_{b_i}(z)$ and $\sum_i b_i = n$. Marking one of the m free positions converts the cyclic blocks into a composition of n with a chosen first position. Hence

$$\sum_S \beta(S) z^{\ell(S)} = \frac{n}{m} \sum_{\substack{b_1+\cdots+b_m=n \\ b_i \geq 1}} \prod_{i=1}^{m} H_{b_i}(z).$$

For a prescribed ordered list of block sizes, distributing $[n]$ among the blocks gives $n!/\prod b_i!$ choices. In a block of size b the generating polynomial for its linear orders by the rank minus one of the first entry is $(b-1)!(1+\cdots+z^{b-1}) = b!H_b(z)$. Dividing by $m!$ to forget the order of the blocks proves (10).

To see the Eulerian factor as well, use the simple slices of $[0, L]^n$ at total $kL+1$ for integral $L \geq 2$. Maximizing a strictly ordered objective fills the first k positions with L and the next with 1. Hence these slices equal $(L-1)\Delta_{k,n} + \Delta_{k+1,n}$ by their support functions. Their projected facets are singletons and the total interval, so Proposition 7.1 applies to this simple deformation (its proof uses only simplicity, laminar tight intervals and a relation of the form (8), here with $D = L_0 = 1$). The limiting j-face for S is a hypersimplex on its $j+1$ free coordinates of rank $k - \ell(S)$ whenever $1 \leq k - \ell(S) \leq j$. Its relative volume is $A(j, k-\ell(S)-1)/j!$: across consecutive partial sums the integer part increases exactly at descents of their fractional parts. Here $A(j,h)$ counts permutations of $[j]$ with h descents and is zero when h is outside $0, \dots, j-1$. This recovers the coefficient formula

$$(n-1)!\, e_j(\Delta_{k,n}) = \sum_\ell W(\ell, n, j+1) A(j, k-1-\ell)$$

of [Fer21, Theorem 4.3].

7.2. **The hypersimplex upper bound for positroids.** For a matroid M, put $E_M = E_{P(M)}$. The bases of a rank-k matroid on $[n]$ give vertices of $\Delta_{k,n} = P(U_{k,n})$, so $E_M(t) \leq E_{U_{k,n}}(t)$ for every positive integer t. Appendix A shows how the coefficientwise inequality can fail. The counterexamples do not admit positroid orders.

Question 7.3 Does $E_M \preceq E_{U_{k,n}}$ hold for every rank-k positroid M on $[n]$?

Corollary 7.4 *The answer is yes. Every positroid is coefficientwise dominated by the hypersimplex of the same rank and ground-set size.*

Proof. Take $P = P(M)$ and $Q = \Delta_{k,n}$ in Theorem 1.1. Both are integral polypositroids in the positroid order, and $P \subseteq Q$. □

The bound was previously known for lattice path matroids and for elementary split matroids [FMP26, Theorem 4.1 and Corollary 4.2]. An exact finite check also verified the comparison for all 125,672 positroids on $[n]$ with $n \leq 8$ in the standard cyclic order, including disconnected positroids. The counts for $n = 1, \ldots, 8$ are

$$2, \quad 5, \quad 16, \quad 65, \quad 326, \quad 1957, \quad 13700, \quad 109601.$$

We enumerated decorated permutations and their Grassmann necklaces $(I_1, \ldots, I_n)$ as in [ARW16, Sections 4–5]. For an interval J starting at a, the greedy basis I_a gives $r(J) = |I_a \cap J|$. For $t = 1, \ldots, n-1$ we counted the points of $t\Delta_{k,n}$ satisfying $x(J) \leq t\, r(J)$ for every cyclic interval J. Both polynomials have degree at most $n-1$ and constant term one, so these counts suffice for exact interpolation in the monomial basis. Cyclic rotation and duality were used to reduce the enumeration.

Appendix A. The hypersimplex upper bound fails for matroids

Ferroni conjectured that a connected rank-k matroid on n elements satisfies $E_M \preceq E_{U_{k,n}}$ as part of a two-sided coefficient bound [Fer22, Conjecture 1.5]. The following examples disprove the upper bound.

Theorem A.1 *There is a connected, self-dual, real-representable matroid M of rank 23 on 46 elements such that*

$$[t^4](E_M - E_{U_{23,46}}) = \frac{824980663968746325186088725987688840 67}{244905774345198087517381265280000 0} > 0.$$

There is also a connected, self-dual, $\mathbb{Q}$-realizable example of rank 25 on 50 elements. In both cases the coefficients exceeding those of the hypersimplex occur precisely in degrees $4, 5, 6, 7$.

Proof. Let A have size m. On $A \cup \{p\}$ take the rank-r matroid F whose bases are all r-subsets except a specified family $\mathcal{H} \subseteq \binom{A}{r}$ of size h. We use the following two families:

(i) $A = \mathbb{Z}/23\mathbb{Z}$, $r = 11$, and $\mathcal{H}$ consists of the zero-sum 11-subsets in this cyclic group. Translation acts transitively on the possible sums, giving $h = \binom{23}{11}/23 = 58786$.

(ii) $A = \{-12, -11, \ldots, 12\}$, $r = 12$, and $\mathcal{H}$ consists of the 12-subsets with integer sum zero. Here $h = [u^{12}z^0] \prod_{a=-12}^{12}(1 + uz^a) = 110780$.

In either case two forbidden sets have at most $r-2$ common elements, since $r-1$ elements determine the last one from the sum. Thus the allowed sets satisfy basis exchange. Among two or more candidates for one exchange at most one is forbidden,

and with only one candidate it is the other basis. Every $(r-1)$-subset extends to a basis and every $(r+1)$-subset contains one.

Form the 2-sum $M = F \oplus_2 F^*$ at p, on two disjoint copies of A. Its bases are obtained by taking a basis from each factor with exactly one containing p and removing p. Thus its rank is m, its ground set has size $2m$, and exchanging the copies identifies M with its dual. The element p is neither a loop nor a coloop, and $F/p = U_{r-1,m}$ is connected. It follows that F is connected: any component other than the one containing p would survive contraction as a separate component, as would the other elements of the component containing p. The dual is likewise connected. In the 2-sum, two elements on opposite sides belong to a common circuit: take a circuit through each of them and p in its factor, and remove p from their union. Thus the two sides lie in a single connected component.

Put $B_l(t,b) = [z^b](1+z+\cdots+z^t)^l$ and write $f_t(s)$ for the number of lattice points of $tP(F)$ with p-coordinate s. We claim

$$E_M(t) = \sum_{s=0}^{t} f_t(s)^2, \qquad E_{U_{m,2m}}(t) = B_{2m}(t, mt). \tag{11}$$

The fiber count is

$$f_t(s) = B_m(t, rt-s) - h \sum_{q=0}^{t-s-1} \binom{s+q+r-1}{r-1} \binom{q+m-r-1}{m-r-1}. \tag{12}$$

For the first identity, complement the basis from the second factor. The resulting pairs of bases of F have equal membership of p. Their convex hull consists of all pairs in $P(F)^2$ with equal p-coordinate. Indeed, in convex decompositions of two points at the same height, the total weight in each p-membership class agrees, so the two distributions can be coupled independently within each nonempty class. After dilation, projecting away p and complementing the second copy preserves the lattice, since $s = rt - x(A)$ recovers the omitted coordinate and the complement is $y_a \mapsto t - y_a$.

For the fiber count, the matroid rank inequalities on this slice are the box bounds, the equation $x(A) = rt - s$, and $x(H) \le (r-1)t$ for $H \in \mathcal{H}$. The rank assertions above give this description directly from the base-polytope inequalities. Two distinct forbidden inequalities cannot both be violated, since

$$x(H) + x(J) \le x(A) + t|H \cap J| \le 2(r-1)t.$$

For a violation at H let $q = x(A \setminus H)$. The r deficits $t - x_i$ on H sum to $s + q < t$. Both distributions, of the deficits and of the outside coordinates, consequently have no effective upper bound. Stars and bars gives (12).

Here are details certifying the coefficient comparisons. Set $d = 2m-1$, and use (11)–(12) to evaluate $v_t = E_M(t) - E_{U_{m,2m}}(t)$ for $0 \le t \le d$. Both polynomials have

degree at most d. If $s(j,k)$ is a signed Stirling number of the first kind, Newton interpolation gives

$$[t^k](E_M - E_{U_{m,2m}}) = \sum_{j=k}^{d} \frac{s(j,k)}{j!}\,\Delta^j v_0.$$

The following Python code evaluates the finite sums exactly. The returned pair consists of the quartic excess and the list of the degrees with positive excess.

```
from fractions import Fraction
from math import comb, factorial

def B(l,t,b):
    return sum((-1)**i*comb(l,i)*comb(b-i*(t+1)+l-1,l-1)
               for i in range(min(l,b//(t+1))+1))

def excess(m,r,h):
    d = 2*m-1
    v = []
    for t in range(d+1):
        f = []
        for s in range(t+1):
            bad = sum(comb(s+q+r-1,r-1)*comb(q+m-r-1,m-r-1)
                      for q in range(t-s))
            f.append(B(m,t,r*t-s)-h*bad)
        v.append(sum(y*y for y in f)-B(2*m,t,m*t))
    a, fall = [0]*(d+1), [1]
    for j in range(d+1):
        for k,c in enumerate(fall):
            a[k] += c*v[0]*(factorial(d)//factorial(j))
        fall = [x-j*y for x,y in zip([0]+fall,fall+[0])]
        v = [y-x for x,y in zip(v,v[1:])]
    return Fraction(a[4],factorial(d)), [k for k,c in
                                         enumerate(a) if c>0]
```

At $(m,r,h) = (23,11,58786)$ this gives the displayed fraction and $[4,5,6,7]$. At $(25,12,110780)$ it gives the same degree list and quartic excess

$$\frac{1272870008389390132327737065037854929280894 3}{503800566520993889793755348432793600 00}.$$

To verify representability in (i), take $z_a = \exp(2\pi i a/23)$ and use the columns $(z_a, z_a^2, \dots, z_a^{10}, z_a^{11}-1)^{\mathsf{T}}$ and $v_p = e_{11}$. On eleven finite columns the determinant is the Vandermonde times $(\prod_a z_a) - 1$, by expansion in the last row. Minors containing p are nonzero by Vandermonde. Rescaling each finite column by z_a^{-17} changes the first ten rows into the powers $7,\dots,16$ modulo 23, which occur in conjugate pairs.

Replace each pair by its real and imaginary parts. The last row is $z_a^{-6} - z_a^6$ on the finite columns, so multiplying it by i and rescaling v_p gives a real matrix. In (ii) use the integral columns $(1, a, \dots, a^{10}, a^{12})^\mathsf{T}$ and $v_p = e_{12}$. On twelve finite columns the determinant is the Vandermonde times their sum, by reducing X^{12} modulo $\prod_a (X - a)$. Again the minors containing p are nonzero. Representability over either field is preserved by duality and by 2-sums: for the latter, identify the two lines representing p in the direct sum of the representing spaces and delete p. □

Neither example is a positroid in any cyclic order. Indeed, $M|A = F \setminus p$, as follows from the basis description of the 2-sum. Each $H \in \mathcal{H}$ supports a distinct facet of this restriction. The bases $H - i + j$, for $i \in H$ and $j \in A \setminus H$, span the product of two simplices of total dimension $m - 2$ on the bounding hyperplane. There is also a basis disjoint from H, since $m - r \geq r + 1$. A positroid on m elements can have at most m such facets with a prescribed support cardinality r, since the support must be a cyclic interval [ARW16, Proposition 5.7]. In both cases $h > m$, and restrictions of positroids preserve positroidality [ARW16, Proposition 3.5].

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