hexagon:2611.00649v1

A NOTE ON TROPICAL CONVEXITY

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Abstract. We prove that the tropicalization of the regular function $h = (x_1^2 + x_2^2 + x_3^2)/(x_1x_2x_3)$ on the coefficient-free Markov cluster threefold is not min-convex. This disproves Conjecture 8.11 of Gross, Hacking, Keel, and Kontsevich.

1. Introduction

Tropical convexity relates regular functions on a log Calabi–Yau variety to the piecewise linear geometry of its tropicalization. On an algebraic torus, the value of a Laurent polynomial at a monomial valuation is the minimum of the linear functions defined by its exponents. For a cluster variety, every seed gives such a torus chart, while mutation changes the linear structure. Scattering diagrams provide compatible paths, called broken lines, on which one can test an intrinsic convexity condition.

Fomin and Zelevinsky introduced cluster algebras as an algebraic approach to dual canonical bases and total positivity [FZ02]. Their mutation rules generate overlapping Laurent coordinate systems. Fock and Goncharov constructed subtraction-free coordinate atlases on moduli spaces of local systems and used points over tropical semifields to describe lamination spaces [FG06]. Mutation identifies these tropical spaces by piecewise linear maps; their addition laws depend on the seed.

Gross, Hacking and Keel interpreted mutations through elementary transformations of toric blowups and described the associated $\mathcal{X}$ and principal-coefficient cluster varieties up to codimension two [GHK15a]. Their mirror construction for log Calabi–Yau surfaces uses scattering diagrams to correct toric gluings and broken lines to construct theta functions [GHK15b]. Gross, Hacking, Keel and Kontsevich subsequently developed this construction for cluster varieties in arbitrary dimension [GHKK18]. Their theta functions extend cluster monomials and give canonical bases under geometric hypotheses. The multiplication of theta functions also connects tropical convex sets with graded algebras and compactifications.

For the minimum convention, a piecewise linear function is *min-convex* along broken lines if its derivative on their directed velocity vectors is nonincreasing. Equivalently, the differential evaluated on the successive monomial exponents is nondecreasing, since those exponents are the negatives of the velocity vectors [GHKK18, Definition 8.2]. For a regular function f, write $f^{\mathrm{trop}}(v) = v(f)$, where v is a point of the geometric tropicalization viewed as a valuation. Gross, Hacking, Keel and Kontsevich conjectured that every nonzero regular function on a log Calabi–Yau manifold with maximal boundary has a min-convex tropicalization [GHKK18, Conjecture 8.11].[1] They proved the assertion for global monomials [GHKK18, Proposition 8.13]. Ordinary min-convexity in each seed follows from the Laurent polynomial expression of a regular function. The conjecture requires the corresponding inequality also at the bends prescribed by the scattering diagram.

The surface case was studied by Mandel while the cluster theta construction was being developed. In the positive Looijenga setting, where an affine surface occurs in the deformation class, he identified convexity along broken lines with convexity in every seed. Among integral piecewise linear functions, this condition characterizes tropicalizations of regular functions [Man16, Theorem 4.20]. His Lemma 4.17 compares the differential formulations of convexity,

Date: November 19, 2026.

[1]The conjecture is numbered 8.12 in arXiv:1411.1394v2. References to [GHKK18] in this note use the published numbering.

and Remark 4.21 discusses extensions beyond the positive case. In Conjectures 4.23 and 4.25 of the same paper, he proposed the corresponding seedwise criterion in higher dimension and restated the regular-function conjecture of Gross, Hacking, Keel and Kontsevich.

Davison and Mandel proved strong positivity for quantum theta bases and constructed the $\mathcal{X}$ scattering diagram directly in its charge lattice [DM21]. Their comparison with the principal-coefficient construction identifies its classical limit with the theta functions of [GHKK18]. Cheung, Magee and Nájera Chávez proved that the positive sets used in the compactification construction are precisely the closed sets convex along broken lines [CMN22, Theorem 1.2]. This provides a geometric criterion for the closure of the corresponding graded spaces of theta functions under multiplication.

Keel and Yu constructed mirror algebras for smooth affine log Calabi–Yau varieties containing an algebraic torus by counting non-archimedean analytic disks [KY23]. Their convexity lemma gives the regular-function convexity statement in that setting, and their cluster comparison identifies broken lines with the spines of the relevant analytic curves [KY23, Lemma 15.6 and §21]. The comparison theorems carry geometric hypotheses; for example, Theorem 1.19 assumes that the affinization of the $\mathcal{X}$ variety is smooth and that the canonical map to it is an open immersion.

In recent preprints, Frías-Medina and Magee prove the equivalence of the min-convex and decreasing conditions of [GHKK18], together with analogues of classical results in convex geometry [FM26, §4.2]. Cheung, Magee, Mandel and Muller prove valuative independence and theta reciprocity under finiteness hypotheses; their convexity result for tropical theta functions applies along broken line segments contained in the locus where the tropical value is finite [CMMM26, Proposition 5.20]. Keel, White and Yu extend the non-archimedean disk construction and establish convexity results under their standing geometric hypotheses, including a proper exhaustion [KWY26, Assumption 1.1 and §14].

We use the coefficient-free Markov cluster threefold. Over $\mathbb{C}$, set

$$\begin{aligned} &R = \mathbb{C}[x_1, x_2, x_3, h]/(x_1^2 + x_2^2 + x_3^2 - h x_1 x_2 x_3), \\ &\Sigma = \{x_1 = x_2 = x_3 = 0\} \subset \operatorname{Spec} R, \qquad U_0 = \operatorname{Spec} R \setminus \Sigma, \\ &V = U_0 \setminus \bigcup_{i=1}^{3} \{h = x_i = 0\}. \end{aligned} \tag{1.1}$$

On the initial torus the coordinate h is

$$h = \frac{x_1^2 + x_2^2 + x_3^2}{x_1 x_2 x_3}. \tag{1.2}$$

Zhou studies this mutation-invariant function in [Zho20, §5.3].

Theorem 1.1. *The variety V in* (1.1) *is the coefficient-free Markov $\mathcal{A}$ cluster variety. It is a smooth nonaffine log Calabi–Yau threefold with maximal boundary. The nonzero regular function $h \in \Gamma(V, \mathcal{O}_V)$ has tropicalization which is not min-convex. Consequently, Conjecture 8.11 of* [GHKK18] *is false.*

The proof combines an explicit projective log Calabi–Yau model with a single broken line. Gross, Hacking, Keel and Siebert computed the relevant cubic-surface wall function in [GHKS22, Proposition 2.4 and (2.2)]. We transport that formula through the toric comparison of [GHK15b], keeping six independent incoming variables before specializing to the three Markov variables. The resulting wall has a term that changes the directional derivative of h^{trop} from 1 to 2 along a directed broken line. Section 2 proves the geometric assertions; Sections 3 and 4 establish the scattering formula; Section 5 constructs the broken line and proves the theorem.

2. The Markov threefold

The coefficient-free Markov seed has three mutable vertices and exchange matrix

$$B = \begin{pmatrix} 0 & 2 & -2 \\ -2 & 0 & 2 \\ 2 & -2 & 0 \end{pmatrix}. \tag{2.1}$$

Write

$$G = x_1^2 + x_2^2 + x_3^2 - hx_1x_2x_3.$$

Proposition 2.1. *The ring R is a normal domain of dimension three, with singular locus Σ. The variety V is the smooth, nonaffine Markov $\mathcal{A}$-cluster variety, and*

$$\Gamma(V, \mathcal{O}_V) = R.$$

The function h is nonzero and regular on V and has the same Laurent expression in every seed.

Proof. The coefficients of G, viewed as a polynomial in h over $\mathbb{C}[x_1, x_2, x_3]$, are coprime: none of the variables x_i divides $x_1^2+x_2^2+x_3^2$. Thus G is primitive. It is linear and nonconstant over $\mathbb{C}(x_1, x_2, x_3)$, hence irreducible there. Gauss's lemma makes G irreducible, and therefore prime, in the unique factorization domain $\mathbb{C}[x_1, x_2, x_3, h]$. Consequently R is a domain of dimension three.

If all x_i are nonzero, then $\partial G/\partial h = -x_1x_2x_3 \neq 0$. If exactly one coordinate, say x_i, vanishes, then for either remaining index j the derivative $\partial G/\partial x_j$ equals $2x_j \neq 0$. If two coordinates vanish, $G = 0$ forces the third to vanish. All four partial derivatives vanish on Σ. This proves that the singular locus is precisely $\Sigma \simeq \mathbb{A}^1$ and that U_0 and V are smooth. The hypersurface ring R is Cohen–Macaulay, so it satisfies Serre's condition S_2. Its singular locus has codimension two, giving R_1. Serre's criterion therefore proves normality.

Mutation at vertex 1 reverses the incident entries of B and changes the remaining upper triangular entry from 2 to $2 - 2 \cdot 2 = -2$. Thus $\mu_1(B) = -B$. Cyclic symmetry gives the same result at vertices 2 and 3. The matrix mutation rule also gives $\mu_i(-B) = -\mu_i(B)$. Every seed consequently has matrix B or $-B$, and every exchange relation is

$$x_ix_i' = x_j^2 + x_k^2, \qquad \{i, j, k\} = \{1, 2, 3\}. \tag{2.2}$$

Define polynomial maps σ_i on $\operatorname{Spec} R$ by

$$\begin{aligned} \sigma_i^*(x_i) &= hx_jx_k - x_i, & \sigma_i^*(x_j) &= x_j, \\ \sigma_i^*(x_k) &= x_k, & \sigma_i^*(h) &= h. \end{aligned} \tag{2.3}$$

The identity

$$(hx_jx_k - x_i)^2 - h(hx_jx_k - x_i)x_jx_k = x_i^2 - hx_ix_jx_k$$

shows that σ_i preserves G. Applying its coordinate formula twice gives the identity, so σ_i is an automorphism. In R,

$$x_i\sigma_i^*(x_i) = hx_1x_2x_3 - x_i^2 = x_j^2 + x_k^2.$$

It therefore realizes the exchange (2.2). The maps preserve Σ. On $h = 0$ each map changes only one coordinate's sign, so they also preserve the union of the deleted loci $\{h = x_i = 0\}$ and hence preserve V.

The initial torus

$$T_0 = \{x_1x_2x_3 \neq 0\} \subset \operatorname{Spec} R$$

is isomorphic to $(\mathbb{C}^*)^3$, because its equation uniquely gives $h = (x_1^2 + x_2^2 + x_3^2)/(x_1x_2x_3)$. Every seed torus is an open image of T_0 under a word in the σ_i. A point of U_0 has at most one zero coordinate. If $x_i = 0$ at a point of V, then $h \neq 0$ and $x_jx_k \neq 0$, so $\sigma_i^*(x_i) = hx_jx_k \neq 0$ there. Such a point belongs to $\sigma_i(T_0)$. Hence

$$V = T_0 \cup \sigma_1(T_0) \cup \sigma_2(T_0) \cup \sigma_3(T_0), \tag{2.4}$$

and this is also the union of all seed tori. Equivalently, a point with $h = x_i = 0$ retains a zero coordinate under every mutation and belongs to none of these tori.

Their open embeddings realize the maximal-open gluing used for cluster varieties in [GHK15a, Proposition 2.4]. Indeed, on the intersection of two torus images the rational cluster transition is an isomorphism. Conversely, suppose that transition extends to an isomorphism on an open subset of the source torus. Its source embedding and its composition with the target embedding are morphisms to $\operatorname{Spec} R$ which agree on a dense open subset. Since $\operatorname{Spec} R$ is separated, they agree throughout that open subset. Its image therefore belongs to the intersection of the two torus images. Thus these intersections are exactly the maximal isomorphism loci, proving the assertion about the full cluster variety.

The equations $h = x_i = 0$ in U_0 describe the two disjoint curves

$$h = x_i = 0, \qquad x_j = \pm\sqrt{-1}\, x_k, \qquad x_k \neq 0.$$

Each is isomorphic to $\mathbb{C}^*$. Curves belonging to distinct indices cannot intersect, since that would give two zero coordinates. Thus $U_0 \setminus V$ consists of six curves of codimension two.

Every σ_i fixes h. If $(x_{1;s}, x_{2;s}, x_{3;s})$ denotes the coordinates in any seed s, applying the corresponding mutation word gives

$$h = \frac{x_{1;s}^2 + x_{2;s}^2 + x_{3;s}^2}{x_{1;s}x_{2;s}x_{3;s}}. \tag{2.5}$$

The coordinate h restricts to a global regular function on V and is nonzero, as its value at $(x_1, x_2, x_3) = (1, 1, 1)$ is 3. The complement of V in the normal affine scheme $\operatorname{Spec} R$ has codimension two. Extension of regular functions across such a complement gives $\Gamma(V, \mathcal{O}_V) = R$. The canonical morphism from V to its affinization is consequently the open inclusion $V \subset \operatorname{Spec} R$, which omits the nonempty line Σ. An affine scheme is isomorphic to its affinization, so V is nonaffine. □

Proposition 2.2. *The variety V is a log Calabi–Yau threefold with maximal boundary. More precisely, there is a smooth projective threefold Y and a reduced simple normal crossings divisor D such that*

$$K_Y + D \sim 0, \qquad Y \setminus D \simeq U_0,$$

and D has a zero-dimensional stratum.

Proof. Use homogeneous coordinates $[X_1 : X_2 : X_3 : W]$ on $\mathbb{P}^3$. On $W \neq 0$ write $x_i = X_i/W$. Blow up $o = [0 : 0 : 0 : 1]$, obtaining the smooth toric threefold Y_0. Let D_0 be its reduced toric boundary: the strict transforms of the four coordinate hyperplanes together with the exceptional divisor E_0. Then $K_{Y_0} + D_0 \sim 0$.

For each cyclic permutation (i, j, k) of $(1, 2, 3)$, consider the lines

$$C_i^{\pm} = \{X_i = 0,\ X_j = \pm\sqrt{-1}\, X_k\} \subset \mathbb{P}^3.$$

Two distinct lines in this list intersect only at o, and their tangent directions there are distinct. Their strict transforms in Y_0 are therefore disjoint smooth projective curves. Each lies in the strict transform of $X_i = 0$ and meets E_0 and the strict transform of $W = 0$ transversely, at one point on each. These points lie on exactly two components of D_0. For example, in the blowup chart

$$x_1 = a, \qquad x_2 = ab, \qquad x_3 = ac,$$

the strict transforms of the two lines with $x_2 = 0$ are $b = 0$, $c = \pm\sqrt{-1}$, while E_0 is $a = 0$. This verifies transversality at E_0. At infinity the same two lines meet $[1 : 0 : \pm\sqrt{-1} : 0]$. In coordinates

$$b = X_2/X_1, \qquad c = X_3/X_1, \qquad d = W/X_1,$$

they are $b = 0$, $c = \pm\sqrt{-1}$, and the boundary at infinity is $d = 0$. This proves transversality there as well. Cyclic permutation gives these descriptions for the other four lines.

Blow up these six strict transforms, writing $\rho : Y \to Y_0$ for the result and $E_i^{\pm}$ for the new exceptional divisors. Let D be the strict transform of D_0. Every center has codimension two and lies generically on one boundary component. The blowup formulas give

$$K_Y = \rho^* K_{Y_0} + \sum_{i,\pm} E_i^{\pm}, \qquad D = \rho^* D_0 - \sum_{i,\pm} E_i^{\pm},$$

and hence $K_Y + D \sim 0$.

At a point where a center meets E_0 or $W = 0$, take local coordinates (a, b, c) in which the boundary is $ab = 0$ and the center is $a = c = 0$. The two blowup charts have equations

$$c = au : \quad D = \{b = 0\}, \qquad a = cv : \quad D = \{vb = 0\}.$$

Thus D remains simple normal crossings. Away from these points the same calculation applies with the factor b omitted. Moreover, the entire exceptional fiber over such a point satisfies $b = 0$ and belongs to D. The divisor $b = 0$ does not contain the center, so its total transform equals its strict transform, including this fiber. It follows that all points lying over the original origin or over $W = 0$ are removed in $Y \setminus D$. Away from the origin, the coordinate axes meet none of the six centers and remain in the boundary. Consequently every point of $Y \setminus D$ lies over one of the three opens $x_j x_k \neq 0$ in the affine chart $W \neq 0$.

On such an open, the blowup of the origin is an isomorphism and the four centers on $X_j = 0$ or $X_k = 0$ are absent. The two remaining centers have ideals

$$I_{\pm} = (x_i, x_j \mp \sqrt{-1}\, x_k).$$

They are comaximal because $2\sqrt{-1}\, x_k$ is a unit. Their product contains x_i, by subtracting the two generators $x_i(x_j \pm \sqrt{-1}\, x_k)$, and contains $x_j^2 + x_k^2$. All its generators belong to $(x_i, x_j^2 + x_k^2)$, so

$$I_+ \cap I_- = I_+ I_- = (x_i, x_j^2 + x_k^2).$$

Blowing up this ideal blows up the disjoint union of the two centers. In the x_i-chart its equation is

$$x_i t = x_j^2 + x_k^2. \tag{2.6}$$

The other chart has equation $(x_j^2 + x_k^2)s = x_i$, and the strict transform of $x_i = 0$ is $s = 0$. Its complement has s invertible and is the overlap with (2.6), where $t = s^{-1}$. Removing the strict boundary therefore leaves precisely the chart (2.6).

The mutually inverse regular formulas

$$h = \frac{x_i + t}{x_j x_k}, \qquad t = h x_j x_k - x_i$$

identify this chart with the open subset $x_j x_k \neq 0$ of U_0. Indeed, substituting either formula transforms (2.6) into $G = 0$. The three opens cover U_0, since outside Σ at least two coordinates are nonzero. Their pairwise overlaps lie in T_0, where the displayed formula for h is its initial Laurent expression. The identifications consequently glue to $Y \setminus D \simeq U_0$.

The vertex $[1 : 0 : 0 : 0]$ lies on none of the centers. It remains a point where the three boundary components arising from $X_2 = 0$, $X_3 = 0$ and $W = 0$ meet. Thus D has a zero-dimensional stratum. Proposition 2.1 shows that $U_0 \setminus V$ has codimension two. The criterion in [GHK15a, Lemma 1.2 and Remark 1.3] therefore makes (Y, D) a minimal model of the log Calabi–Yau variety V; the zero-dimensional stratum gives maximal boundary by [GHK15a, Definition 1.5]. □

3. The induced scattering diagram

Let $N = \mathbb{Z}^3$, with standard basis e_1, e_2, e_3, and put $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$. The matrix (2.1) defines the skew form $B(a, b) = a^{\mathsf{T}} B b$. Set

$$\delta = (1, 1, 1), \qquad \pi(a_1, a_2, a_3) = (a_1 - a_3, a_2 - a_3). \tag{3.1}$$

Then $\pi : N \to \mathbb{Z}^2$ is surjective, its kernel is $\mathbb{Z}\delta$. Expanding the pairing gives

$$\begin{aligned} B(a,b) &= 2(a_1b_2 - a_1b_3 - a_2b_1 + a_2b_3 + a_3b_1 - a_3b_2) \\ &= 2((a_1-a_3)(b_2-b_3) - (a_2-a_3)(b_1-b_3)) \\ &= 2\det(\pi(a),\pi(b)). \end{aligned} \tag{3.2}$$

We use the same notation for the real linear extension of π. Its surjectivity and (3.2) also give $\ker B = \mathbb{Z}\delta$.

Write

$$N_{\geq 0} = \sum_{i=1}^{3} \mathbb{Z}_{\geq 0} e_i, \qquad N^+ = N_{\geq 0} \setminus \{0\}, \qquad |a| = a_1 + a_2 + a_3 \quad (a \in N_{\geq 0}).$$

The commuting variables Y_1, Y_2, Y_3 are formal scattering variables; for $m \in N$, write $Y^m = \prod_i Y_i^{m_i}$. Coefficient series are completed in $\widehat{A} = \mathbb{C}[[Y_1, Y_2, Y_3]]$ by total degree. An action on Y^m takes values in the formal module $Y^m\widehat{A}$, with the filtration given by the degree of the increment from m. For $a \in N^+$ define

$$D_a(Y^m) = B(a,m)Y^{a+m}. \tag{3.3}$$

Their commutator is given by

$$\begin{aligned} [D_a, D_b](Y^m) &= (B(b,m)B(a,b+m) - B(a,m)B(b,a+m))Y^{a+b+m} \\ &= B(a,b)B(a+b,m)Y^{a+b+m} = B(a,b)D_{a+b}(Y^m). \end{aligned}$$

Each D_a raises the increment degree by $|a|$. The positive-degree completion therefore admits formal exponentials.

We use the classical direct $\mathcal{X}$ scattering diagram $\mathfrak{D}_{\mathcal{X}}$ of [DM21, §4.3]. Its initial walls are the hyperplanes

$$e_i^{B\perp} = \{\eta \in N_{\mathbb{R}} : B(e_i,\eta) = 0\}, \qquad i = 1,2,3,$$

with positive crossing automorphisms

$$G_i(Y^m) = Y^m(1+Y_i)^{B(e_i,m)}. \tag{3.4}$$

Equivalently,

$$G_i = \exp\left(\sum_{k\geq 1} \frac{(-1)^{k+1}}{k^2} D_{ke_i}\right).$$

Indeed, the derivation inside the exponential sends Y^m to $B(e_i,m)Y^m\log(1+Y_i)$ and annihilates Y_i. At a transverse crossing with velocity v, the positive operator is used when $B(e_i,v) < 0$; the reverse crossing uses its inverse. This is the classical limit of the crossing convention in [DM21, §§2.2.4, 2.3.2].

For $a \in N$, write $a^{B\perp} = \{\eta \in N_{\mathbb{R}} : B(a,\eta) = 0\}$. A wall of primitive charge $a \in N^+ \setminus \mathbb{Z}\delta$ has a rational polyhedral support $\mathfrak{d} \subset a^{B\perp}$ of codimension one in $N_{\mathbb{R}}$ and an automorphism $\exp(\sum_{k\geq 1} c_k D_{ka})$, with $c_k \in \mathbb{C}$. It is incoming if $\mathfrak{d} + \mathbb{R}_{\geq 0}a \subseteq \mathfrak{d}$, and outgoing otherwise. Joints are wall boundaries and intersections of walls of codimension two in $N_{\mathbb{R}}$. There are finitely many nontrivial walls modulo every fixed increment degree. Consistency means that the ordered crossing product around every transverse closed path avoiding the wall boundaries and joints is the identity. By [DM21, Theorem 2.13], the initial diagram has a unique consistent completion, up to equality of these products, obtained by adding outgoing walls. We will construct a conical representative in which an outgoing charge a has support

$$\pi^{-1}(-\mathbb{R}_{\geq 0}\pi(a)) = \mathbb{R}\delta + \mathbb{R}_{\geq 0}(-a). \tag{3.5}$$

The quotient π describes the supports. Monomials retain their indices in N, and

$$T = Y^\delta = Y_1Y_2Y_3 \tag{3.6}$$

has degree three.

The tropicalization of the Markov $\mathcal{A}$ variety V is the space on which the broken lines for its dual $\mathcal{X}$ variety are defined; here all seed multipliers are one [GHKK18, Appendix A.4]. The diagram above is the induced $\mathcal{X}$ diagram specified in [GHKK18, Construction 7.11, footnote 2]. For the coordinate identification, let $N^* = \mathrm{Hom}(N, \mathbb{Z})$, write $N^*_{\mathbb{R}} = N^* \otimes_{\mathbb{Z}} \mathbb{R}$, and define

$$p^* : N \longrightarrow N^*, \qquad p^*(a) = B(a, \cdot).$$

Extend p^* real-linearly and set $\xi(\eta) = (p^*(\eta), \eta)$. Its image in $N^*_{\mathbb{R}} \oplus N_{\mathbb{R}}$ is the zero set of the weight map $(\mu, \eta) \mapsto \mu - p^*(\eta)$. On the lattices, the weight map is split by $\mu \mapsto (\mu, 0)$, so its kernel $\xi(N)$ is saturated. The cone of principal charges is generated by the $\xi(e_i)$ and projects to the zero cone in the quotient by $\xi(N_{\mathbb{R}})$. This cone is strongly convex, as required in [DM21, Lemma 2.12]. That lemma supplies an equivalent consistent representative in which the boundaries of walls restricting to codimension one meet $\xi(N_{\mathbb{R}})$ in codimension at least two, and constructs the induced diagram. Its classical theta functions are identified with the GHKK construction by [DM21, §4.4, Lemma 4.6 and Proposition 4.8]. In GHKK's max-plus coordinate n, the graph map is $\xi^T(n) = (-p^*(n), -n)$, so $n = -\eta$ under this identification.

In the initial seed, the monomial valuation with weights $\eta = (\eta_1, \eta_2, \eta_3)$ assigns to a Laurent polynomial the minimum of the weights of its nonzero monomials. Hence

$$F(\eta) = h^{\mathrm{trop}}(\eta) = 2\min(\eta_1, \eta_2, \eta_3) - \eta_1 - \eta_2 - \eta_3. \tag{3.7}$$

Write h^T for GHKK's tropicalization in the max-plus coordinates. The conventions of [GHKK18, §2] give

$$h^T(n) = F(-n), \qquad d(F \circ (-\mathrm{id}))_{-\eta}(-m) = dF_\eta(m) \tag{3.8}$$

where the differential exists. Thus the differential evaluations on decoration vectors agree in the two coordinate conventions. Also,

$$F(\eta + t\delta) = F(\eta) - t \qquad (t \in \mathbb{R}). \tag{3.9}$$

The tropical function is consequently considered on the full space $N_{\mathbb{R}}$.

4. A wall of the Markov diagram

Proposition 4.1. *The diagram $\mathfrak{D}_{\mathcal{X}}$ has a conical representative whose total crossing automorphism on the half-wall $\eta_1 < \eta_2 = \eta_3$, in a direction v with $B(e_1, v) < 0$, is*

$$Y^m \longmapsto Y^m \Phi_1^{B(e_1,m)}, \qquad \Phi_1 = \frac{(1+Y_1)(1+Y_1T^2)}{(1-Y_1T)^2}. \tag{4.1}$$

Proof. The coefficient ring. Let S be the smooth cubic surface obtained by blowing up two general points on each toric boundary line of $\mathbb{P}^2$. Write L for the pullback of the line class and E_{ia} for the six exceptional classes, where $1 \le i \le 3$ and $1 \le a \le 2$. These classes form a basis of $\mathrm{Pic}(S)$, and the strict boundary components have classes $D_i = L - E_{i1} - E_{i2}$. Use the toric rays

$$v_1 = (1, 0), \qquad v_2 = (0, 1), \qquad v_3 = (-1, -1)$$

and the piecewise linear lift

$$\varphi : \mathbb{R}^2 \longrightarrow \mathrm{Pic}(S) \otimes_{\mathbb{Z}} \mathbb{R}, \qquad \varphi(a, b) = \max(0, -a, -b)L.$$

Thus $\varphi(v_1) = \varphi(v_2) = 0$ and $\varphi(v_3) = L$. Let $E = \bigoplus_{i,a} \mathbb{Z}E_{ia}$. Every effective curve class on S lies in $\mathbb{Z}_{\ge 0}L \oplus E$: its coefficient of L is the degree of its pushforward to $\mathbb{P}^2$. Conversely, L is effective, so adjoining the exceptional classes and their negatives gives exactly $\mathbb{Z}_{\ge 0}L \oplus E$. This is the exceptional localization used in the toric comparison. The monoid used in the toric comparison of [GHK15b, Theorem 3.33] is, in this case,

$$Q = \{((a, b), kL + e) : a, b, k \in \mathbb{Z},\ e \in E,\ k \ge \max(0, -a, -b)\}.$$

Its units are the elements $(0,e)$, $e \in E$. For an exponent $q \in \mathbb{Z}^2 \oplus \mathrm{Pic}(S)$, write z^q for the corresponding formal monomial; for a divisor class C, abbreviate $z^{(0,C)}$ to z^C. Set $\bar{X}_i = z^{(v_i,\varphi(v_i))}$. The correspondence

$$(a,b,k) \longmapsto (a+k, b+k, k)$$

identifies the quotient Q/E with $\mathbb{Z}^3_{\geq 0}$. Its inverse sends (n_1,n_2,n_3) to (n_1-n_3, n_2-n_3, n_3). Therefore

$$\mathbb{C}[Q] = \mathbb{C}[z^{\pm E_{ia}}][\bar{X}_1, \bar{X}_2, \bar{X}_3], \qquad \mathfrak{m}_Q = (\bar{X}_1, \bar{X}_2, \bar{X}_3). \tag{4.2}$$

We complete this ring in $\mathfrak{m}_Q$. In particular,

$$\bar{X}_1\bar{X}_2\bar{X}_3 = z^L. \tag{4.3}$$

Introduce six independent variables A_{ia} and define

$$\alpha(A_{ia}) = z^{-E_{ia}}\bar{X}_i. \tag{4.4}$$

For a monomial $\prod_{i,a} A_{ia}^{n_{ia}}$, the coefficient of E_{ia} in the exponent of its image is $-n_{ia}$. Thus α is injective on monomials and on their polynomial algebra. The $\mathfrak{m}_Q$-order of its image is $\sum_{i,a} n_{ia}$, exactly its total degree. Distinct source monomials have distinct images, so a nonzero polynomial and its image have the same lowest degree. Consequently α extends to a strict injective homomorphism

$$\mathbb{C}[[A_{11}, A_{12}, A_{21}, A_{22}, A_{31}, A_{32}]] \hookrightarrow \widehat{\mathbb{C}[Q]}.$$

Here strictness means that the inverse image of $\mathfrak{m}_Q^d\widehat{\mathbb{C}[Q]}$ is the ideal of series of total degree at least d.

The proof of [GHK15b, Theorem 3.33] constructs the consistent diagram first over the free monoid generated by six symbols with spatial projections v_i, and then applies precisely the map (4.4). The incoming functions are

$$\prod_{a=1}^{2}(1+A_{ia}) \qquad \text{on } \mathbb{R}_{\geq 0}v_i.$$

The injection and strictness just proved identify every outgoing function with a unique series in these six variables. By [GHK15b, Definition 3.24, Theorem 3.25], the image is also the toric pushforward of the canonical scattering diagram of S.

Pairing the incoming variables. Define

$$\sigma : \mathbb{C}[[A_{11}, A_{12}, A_{21}, A_{22}, A_{31}, A_{32}]] \longrightarrow \widehat{A}, \qquad A_{ia} \longmapsto Y_i.$$

The exponent map is $(n_{ia}) \mapsto (n_{i1}+n_{i2})_{i=1}^3$. It preserves total degree, and the inverse image of (n_1,n_2,n_3) has $\prod_i(n_i+1)$ elements. Each target coefficient is therefore a finite sum, so σ is well defined and continuous. A planar wall with function f and primitive normal ν multiplies both A_{i1} and A_{i2} by the same power $f^{\langle \nu, v_i\rangle}$, since their spatial projections agree. Thus σ commutes with the wall actions and with every finite-degree ordered crossing product. Passing to the inverse limit preserves consistency. The specialized incoming functions are $(1+Y_i)^2$. The surface S remains fixed throughout this coefficient specialization.

The cubic-surface formula. The ray in the interior of the cone between the second and third boundary rays maps under the toric comparison to $\mathbb{R}_{\geq 0}(v_2+v_3) = \mathbb{R}_{\geq 0}(-v_1)$. On this cone the monomial transport is the isomorphism in [GHK15b, (3.7)]; the interior ray uses the first clause of [GHK15b, Definition 3.24]. The cyclic version of [GHKS22, (2.2)] gives its function

$$f_{\mathrm{cub}} = \frac{\displaystyle\prod_{j=1}^{8}(1+z^{D_1+L_{1j}}\bar{X}_2^{-1}\bar{X}_3^{-1})}{(1-z^{D_2+D_3+2D_1}\bar{X}_2^{-2}\bar{X}_3^{-2})^4}. \tag{4.5}$$

The eight line classes L_{1j} occurring here are listed in the proof of [GHKS22, Proposition 2.4]:

$$\begin{aligned}
&E_{11},\ E_{12},\\
&L - E_{2a} - E_{3b} \qquad (a,b \in \{1,2\}),\\
&2L - E_{1a} - E_{21} - E_{22} - E_{31} - E_{32} \qquad (a \in \{1,2\}).
\end{aligned}$$

By (4.3), the numerator monomial for a class C in this list equals

$$z^{D_1+C}\bar{X}_2^{-1}\bar{X}_3^{-1} = z^{C-E_{11}-E_{12}}\bar{X}_1.$$

The corresponding inverse images under α are as follows; each row is read for all indicated values of a, b.

Class C	Free monomial	Degree
E_{11}	A_{12}	1
E_{12}	A_{11}	1
$L - E_{2a} - E_{3b}$	$A_{11}A_{12}A_{2a}A_{3b}$	4
$2L - E_{1a} - E_{21} - E_{22} - E_{31} - E_{32}$	$A_{11}A_{12}A_{1a}A_{21}A_{22}A_{31}A_{32}$	7

For example, in the degree-four row the image of the free monomial is

$$z^{-E_{11}-E_{12}-E_{2a}-E_{3b}}\bar{X}_1^2\bar{X}_2\bar{X}_3 = z^{L-E_{11}-E_{12}-E_{2a}-E_{3b}}\bar{X}_1.$$

For the degree-seven row, its image has toric part $\bar{X}_1^3\bar{X}_2^2\bar{X}_3^2 = z^{2L}\bar{X}_1$ and exceptional exponent $-E_{11} - E_{12} - E_{1a} - E_{21} - E_{22} - E_{31} - E_{32}$, as required. The denominator monomial in (4.5) is

$$\begin{aligned}
&z^{D_2+D_3+2D_1}\bar{X}_2^{-2}\bar{X}_3^{-2}\\
&\quad = z^{2L-2E_{11}-2E_{12}-E_{21}-E_{22}-E_{31}-E_{32}}\bar{X}_1^2\\
&\quad = \alpha(A_{11}^2A_{12}^2A_{21}A_{22}A_{31}A_{32}).
\end{aligned}$$

It has degree eight, so one minus this monomial is invertible in the completed ring. These identities place every factor of (4.5) in the six-variable completion.

After applying σ, the eight numerator monomials are Y_1 twice, Y_1T four times, and Y_1T^2 twice; the denominator monomial is $Y_1^2T^2$. Write $f_{\mathrm{pl}} = \sigma(\alpha^{-1}(f_{\mathrm{cub}}))$. Then

$$\begin{aligned}
f_{\mathrm{pl}} &= \frac{(1+Y_1)^2(1+Y_1T)^4(1+Y_1T^2)^2}{(1-Y_1^2T^2)^4}\\
&= \left(\frac{(1+Y_1)(1+Y_1T^2)}{(1-Y_1T)^2}\right)^2. \qquad (4.6)
\end{aligned}$$

The second equality follows from $1 - Y_1^2T^2 = (1 - Y_1T)(1 + Y_1T)$; all factors being cancelled have constant term one.

Identification with the direct diagram. Each specialized coefficient retains its full exponent in $N_{\geq 0}$. Every nonconstant monomial on an outgoing planar wall has nonzero spatial projection in the ray opposite its support [GHK15b, Definition 3.21]. Products preserve this property, so it also holds for the logarithm of the wall function. Suppose a logarithmic term on a planar wall is c_aY^a with $\pi(a) = \ell v$, where $v \in \mathbb{Z}^2$ is primitive and $\ell \in \mathbb{Z}_{>0}$. The normal $\det(v, \cdot)$ annihilates the spatial projections of the monomials in this wall function. For this orientation, the term's contribution to the logarithm of the crossing automorphism acts on Y^m by

$$c_a \det(v, \pi(m))Y^{a+m} = \frac{c_a}{2\ell}D_a(Y^m). \qquad (4.7)$$

Charges on one planar ray pair to zero under B, so their derivations commute. Group the logarithm by primitive rays of N^+ and exponentiate each group. This produces full-charge

walls, preserves the total crossing, and has only finitely many terms below every degree. Lift the supports by π^{-1}. Each outgoing support is (3.5), a rational cone of codimension one in $N_{\mathbb{R}}$ contained in $a^{B\perp}$, and its direction is $-a$. A closed transverse path avoiding $\mathbb{R}\delta$ projects to a closed planar path avoiding the origin, with the same ordered wall crossings. At each finite degree its product is the planar product, hence is the identity. Taking the inverse limit proves consistency of the lifted operators.

The action is faithful on the Lie algebra generated by the initial charges. Indeed, if $a+b \in \mathbb{Z}\delta$, then $B(a,b) = B(a,a+b) = 0$; hence a bracket with central total charge vanishes. The initial charges and all nonzero iterated brackets therefore have noncentral charge. For every such a there is an $m \in N$ with $B(a,m) \neq 0$, while distinct charges give distinct monomial increments in (3.3). Consequently an identity of the lifted operators is also an identity in the completed scattering group.

On an incoming ray the specialized planar action is

$$Y^m \longmapsto Y^m(1+Y_i)^{2\det(v_i,\pi(m))} = Y^m(1+Y_i)^{B(e_i,m)}.$$

These are the initial operators (3.4). Splitting each full initial hyperplane into its incoming and outgoing halves identifies the full-hyperplane presentation with the incoming-half-wall presentation: both have the same incoming walls and are completed by outgoing walls. The uniqueness theorem [DM21, Theorem 2.13], which allows incoming cones as initial walls, identifies the resulting diagram with $\mathfrak{D}_{\mathcal{X}}$.

Finally, $\eta_1 < \eta_2 = \eta_3$ projects to the outgoing ray $\mathbb{R}_{>0}(-v_1)$. Its primitive planar normal is $\det(v_1,\cdot)$. Equations (3.2) and (4.6) give

$$f_{\mathrm{pl}}^{\det(v_1,\pi(m))} = \Phi_1^{B(e_1,m)}.$$

For the crossing velocity $-e_2$ this normal has value -1, so the positive operator in (4.1) applies. The same orientation is characterized by $B(e_1,v) < 0$. □

Lemma 4.2. *Put $r = e_1 + 2\delta = (3,2,2)$. The positive crossing in Proposition 4.1 sends Y^{e_2} to a series in which the coefficient of $Y^{e_2+r} = Y^{(3,3,2)}$ is exactly 2.*

Proof. Since $B(e_1,e_2) = 2$, the multiplier is

$$\Phi_1^2 = (1+Y^{e_1})^2(1+Y^r)^2(1-Y^{e_1+\delta})^{-4}.$$

A term of charge r is determined by nonnegative integers a,b,c satisfying

$$ae_1 + b(e_1+\delta) + cr = r.$$

Its second and first coordinates give, respectively,

$$b + 2c = 2, \qquad a + 2b + 3c = 3,$$

or equivalently $b + 2c = 2$ and $a + b + c = 1$. The second equation allows only one nonzero entry, equal to one; the first then forces $(a,b,c) = (0,0,1)$. Thus the desired term comes from the linear term of $(1+Y^r)^2$ and the constant terms of the other two factors. Its coefficient is 2, which proves the assertion after multiplication by Y^{e_2}. □

5. A broken line and the failure of convexity

We recall the broken-line convention in the induced diagram [GHKK18, Construction 7.11 and footnote 2], [DM21, §3.1 and §4.3]. A broken line with endpoint b is a continuous proper piecewise affine map $\gamma : (-\infty, 0] \to N_{\mathbb{R}}$ with finitely many bends and $\gamma(0) = b$. Each linear piece carries a nonzero monomial cY^m, and its velocity is $-m$. The initial decoration is Y^{m_0} for a nonzero $m_0 \in N$. At a crossing, the next decoration is a nonzero term of the image of the preceding decoration under the crossing automorphism. When several walls have the same supporting hyperplane, their commuting automorphisms are combined at the crossing. The path avoids the singular locus of the diagram and crosses the walls it meets transversely; its endpoint lies off the walls. These conditions are interpreted in the positive-degree completion.

For a rational conical representative, the singular locus is contained in a countable union of rational cones of codimension at least two: the boundaries of walls and their intersections in distinct supporting hyperplanes. Only finitely many walls are effective at any fixed order. Selecting a constant term at a crossing leaves the decoration and the velocity unchanged.

By [GHKK18, Definition 8.2], a piecewise linear function is *min-convex* if, along every broken line directed toward its endpoint, its differential evaluated on the decoration exponents is nondecreasing. The differential is evaluated at points where the function is linear. Equivalently, its derivative on the path velocities is nonincreasing. The simultaneous change of signs of positions and exponents described in Section 3 preserves this condition.

Lemma 5.1. *Let $r = e_1 + 2\delta = (3, 2, 2)$, and put*

$$q = (\sqrt{2}, 1 + \sqrt{2}, 1 + \sqrt{2}), \qquad \epsilon = \frac{\sqrt{3}}{100}, \qquad b = q - \epsilon(e_2 + r).$$

There is a broken line with endpoint b, initial decoration Y^{e_2}, and final decoration $2Y^{e_2+r}$, whose underlying path is

$$\gamma(t) = \begin{cases} q - (t + \epsilon)e_2, & t \le -\epsilon, \\ q - (t + \epsilon)(e_2 + r), & -\epsilon \le t \le 0. \end{cases} \tag{5.1}$$

Proof. The bend point belongs to the half-wall $\{\eta_1 < \eta_2 = \eta_3\}$. The incoming velocity is $-e_2$, so Proposition 4.1 and Lemma 4.2 permit the change of decoration from Y^{e_2} to $2Y^{e_2+r}$ at q. Both velocities in (5.1) are the negatives of the corresponding decoration exponents. Since $e_2 \neq 0$, the norm of $\gamma(t)$ tends to infinity as $t \to -\infty$; hence γ is proper.

We verify the path conditions. The only rational linear hyperplane containing q is

$$H = \{\eta_2 = \eta_3\}.$$

Indeed, a rational relation $\sum_i \lambda_i q_i = 0$ gives

$$\lambda_2 + \lambda_3 = 0, \qquad \lambda_1 + \lambda_2 + \lambda_3 = 0$$

by the independence of 1 and $\sqrt{2}$ over $\mathbb{Q}$. Thus $\lambda_1 = 0$ and $\lambda_2 = -\lambda_3$. Both e_2 and $e_2 + r = (3, 3, 2)$ are transverse to H.

Let J be a rational cone of codimension at least two, and let v be either of these two direction vectors. If the line $q + \mathbb{R}v$ met J, then

$$q \in \operatorname{span}_{\mathbb{R}}(J, v).$$

This span is a rational subspace of dimension at most two. A rational linear hyperplane containing it would contain q, hence would equal H. It would also contain v, contrary to transversality. The two supporting lines therefore avoid the singular locus. If one of them met a wall nontransversely, its direction would belong to the wall's rational linear supporting hyperplane. That hyperplane would then contain both q and v, giving the same contradiction. Every encountered wall is crossed transversely.

The endpoint lies in no rational linear hyperplane. A relation $\sum_i \lambda_i b_i = 0$, with $\lambda_i \in \mathbb{Q}$, expands as

$$(\lambda_2 + \lambda_3) + (\lambda_1 + \lambda_2 + \lambda_3)\sqrt{2} - \frac{3\lambda_1 + 3\lambda_2 + 2\lambda_3}{100}\sqrt{3} = 0.$$

The numbers $1, \sqrt{2}, \sqrt{3}$ are linearly independent over $\mathbb{Q}$. Indeed, squaring $c_0 + c_1\sqrt{2} = -c_2\sqrt{3}$ for rational c_0, c_1, c_2 gives $c_0 c_1 = 0$; if $c_2 \neq 0$, the alternatives force either $(c_0/c_2)^2 = 3$ or $(c_1/c_2)^2 = 3/2$, both impossible over $\mathbb{Q}$. Thus $c_2 = 0$, and then $c_0 = c_1 = 0$. All three coefficients in the displayed relation therefore vanish. These equations give $\lambda_1 = \lambda_2 = \lambda_3 = 0$. Hence b is off every wall.

At all crossings other than q, choose the constant term of the crossing factor. Each finite-order diagram has finitely many walls, and the two lines meet each supporting hyperplane at most once. The choices are therefore well defined at every degree bound at least seven. The coefficient of the charge-r term is exactly two in the complete crossing operator, so

these choices agree as the truncation order increases. They define the stated broken line in the completed diagram, with one bend. □

Proof of Theorem 1.1. Propositions 2.1 and 2.2 establish the asserted geometric properties of V and the regularity of h. Consider the broken line of Lemma 5.1. On its incoming ray, write $\eta = q + se_2$, with $s \geq 0$. Then

$$\eta_2 - \eta_1 = 1 + s, \qquad \eta_3 - \eta_1 = 1.$$

On the outgoing segment, write $\eta = q - s(e_2 + r)$, with $0 \leq s \leq \epsilon$. Then

$$\eta_2 - \eta_1 = 1, \qquad \eta_3 - \eta_1 = 1 + s.$$

Thus the first coordinate is strictly smallest along both pieces. The tropicalization (3.7) is consequently linear in a neighborhood of the whole path, with

$$F(\eta) = \eta_1 - \eta_2 - \eta_3, \qquad dF = (1, -1, -1).$$

Its evaluations on the two decoration exponents are

$$dF(e_2) = -1, \qquad dF(e_2 + r) = 3 - 3 - 2 = -2. \tag{5.2}$$

They strictly decrease at the bend, whereas min-convexity requires them to be nondecreasing. Equivalently, the derivatives on the path velocities increase from 1 to 2. Therefore h^{trop} is not min-convex. Since h is a nonzero regular function on the smooth log Calabi–Yau variety V with maximal boundary, this is a counterexample to [GHKK18, Conjecture 8.11]. □

Corollary 5.2. *The function h^{trop} is min-convex in the ordinary linear structure of every seed. Thus convexity in every seed and convexity along broken lines are distinct conditions.*

Proof. Fix a seed s, with coordinates $x_{1;s}, x_{2;s}, x_{3;s}$. For a monomial valuation v, put $\eta_i = v(x_{i;s})$. Equation (2.5) gives

$$h^{\mathrm{trop}}(\eta) = \min_{1 \leq i \leq 3} (2\eta_i - \eta_1 - \eta_2 - \eta_3).$$

It is a minimum of three linear functions, hence is min-convex in the ordinary sense of [GHKK18, Definition–Lemma 8.1]. Theorem 1.1 supplies the failure along broken lines. □

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