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Finite model spaces and coherent sheaf models

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Abstract

Let T be a finitary coherent theory with finitely many sorts. We describe coherent sheaf-valued models over a Stone space as continuous families in a topological category of finite T-models and all their homomorphisms. We prove that every such family over every Stone base is constant up to isomorphism on a finite clopen partition if and only if each fixed finite carrier admits only finitely many T-structures. It suffices to test the partition property over $\beta\mathbb{N}$. Equivalently, at each bound on the sort cardinalities, a finite subsignature detects all homomorphisms between the bounded models. This condition holds for finite signatures and is invariant under coherent Morita equivalence. It characterizes when the canonical functor from the discrete condensed category of finite models to the coherent condensed category of points is an equivalence. For finite locally constant abelian-group sheaves, we compute first Yoneda extensions as sections of a finite locally constant sheaf and prove that all higher Yoneda extensions in their category vanish.

Keywords. coherent theory, finite model, Stone space, pretopos, condensed category, Yoneda extension.

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1 Introduction

Let T be a coherent first-order theory and let K be a Stone space. A $\mathbf{Sh}(K)$-valued model is coherent if the inverse-image functor of its classifying geometric morphism preserves coherent objects. Every coherent formula then defines a finite locally constant sheaf. We ask when a single finite clopen partition of K makes the entire model constant on each part.

The distinction already appears with one finite underlying sort. On $\beta\mathbb{N}$, take the constant set $\{0,1\}$, name its two elements, and interpret additional constants c_n by the characteristic functions of the clopen singletons $\{n\}$. Each formula involves only finitely many named elements and has a finite clopen description. The stalks at distinct natural numbers disagree on a named element, and an isomorphism must fix both 0 and 1. These stalks are pairwise nonisomorphic, so no finite partition trivializes the model.

Suppose now that T has finitely many sorts, while its sets of symbols and axioms may be arbitrary. For a finite carrier $A = (A_s)_s$, give the set $\mathrm{Str}_T(A)$ of labelled T-structures the topology induced by the product of all finite operation and relation tables. Each coherent axiom defines a clopen condition, so $\mathrm{Str}_T(A)$ is a Stone space. If the sort sheaves are fixed as the constant sheaves with fibres A_s, their coherent T-structures correspond to continuous maps $K \to \mathrm{Str}_T(A)$.

Assembling the structure spaces and the spaces of homomorphisms produces a topological category $\mathfrak{M}_T$. Its continuous families over K form a category $\mathrm{Fam}_T(K)$. The first main result is the natural equivalence

$$\mathrm{Fam}_T(K) \simeq \mathrm{CMod}_T(\mathbf{Sh}(K)) \tag{1}$$

of Theorem 3.5. For each fixed carrier, permutations of the labels act on the structures and account for all isomorphisms between structures on that carrier. Each coherent formula defines a clopen family of solutions whose pullback gives the formula's interpretation.

The classification of finite clopen model bundles depends on the condition

$$\mathrm{Str}_T(A) \text{ is finite for every finite carrier } A. \tag{F}$$

By Theorem 4.3, condition (F) is equivalent to each of the following assertions.

(i) Every coherent model on every Stone base is a finite clopen family of constant finite models.

(ii) The same assertion holds on the single Stonean base $\beta\mathbb{N}$.

(iii) There are only finitely many finite-model isomorphism types at each prescribed vector of sort cardinalities.

If a structure space is infinite, each orbit under relabelling is finite, so one can choose a sequence of pairwise nonisomorphic structures. Its extension to $\beta\mathbb{N}$ gives a coherent family with infinitely many stalk isomorphism types.

The condition has two further formulations. First, at every bound on the sort cardinalities, finitely many symbols determine the labelled structures and detect all homomorphisms between actual T-models (Theorem 4.5). Second, let $D(\mathcal{A})$ denote the discrete condensed category obtained from a small category $\mathcal{A}$ by finite clopen presentations. The natural comparison gives an equivalence

$$D(\mathrm{FinMod}(T)) \simeq \big[K \longmapsto \mathrm{CMod}_T(\mathbf{Sh}(K))\big] \tag{2}$$

precisely when (F) holds. This formulation proves invariance under coherent Morita equivalence (Corollary 5.2). For the associated 1-localic ∞-topos, (2) identifies Haine's coherent condensed category of points.

For a finite signature there are only finitely many tables on each finite carrier, regardless of the number of axioms. Infinitely many new operations may be defined by terms in the finite signature without changing the number of structures. Independent operations can behave differently. Over $\{0,1\}^{\mathbb{N}}$ the coordinate functions give a family of pairwise nonisomorphic $\mathbb{F}_2[t_1, t_2, \ldots]$-modules on one fixed two-element set.

For abelian groups, the finite-bundle description also gives an explicit calculation of extensions. Write $\mathrm{CAb}(K)$ for the category of finite locally constant abelian-group sheaves. For $M, N \in \mathrm{CAb}(K)$, there is a finite locally constant sheaf $\mathscr{E}^1(M,N)$ with stalks $\mathrm{Ext}^1_{\mathbb{Z}}(M_x, N_x)$, and

$$\mathrm{Ext}^1_{\mathrm{CAb}(K)}(M,N) \cong \Gamma(K, \mathscr{E}^1(M,N)). \tag{3}$$

Every two-extension admits a comparison over a finite clopen partition with one whose last short exact factor splits. Gluing these comparisons proves that $\mathrm{Ext}^n_{\mathrm{CAb}(K)}(M,N) = 0$ for $n \geq 2$, although $\mathrm{CAb}(K)$ has no nonzero projective or injective objects. The calculation respects primary decomposition and pullback in K.

1.1 Relation to earlier work

The coherent syntactic category and its pretopos completion give the classical description of models in a topos [MR77; MM92]. The comparison with geometric morphisms includes all natural transformations by [Lur18, Corollary C.3.6]. For the correspondence between coherent topoi and pretopoi, see [Hai19, Theorem 1.5 and Remark 1.6]. Over a Stone base, this description gives a criterion in terms of the sort sheaves and primitive relations (Theorem 2.7).

Boolean powers and Boolean products describe clopen variation and patching of algebraic data [BS81, Chapter IV, Sections 5 and 8]. Knoebel develops sheaf representations and their categorical formulations in [Kno12]. In our setting the carriers are finite locally constant sheaves. The space of tables records how the operations and relations can still vary on a part where every carrier is constant.

Awodey and Forssell use topological groupoids of models and isomorphisms in first-order logical duality [AF13]. Forssell also uses a topological category of models and homomorphisms to represent the classifying topos of a regular theory by equivariant sheaves [For13, §3]. Our object and arrow spaces use finite carriers and discrete table coordinates. Their continuous families

describe the coherent models over each Stone base. Kanalas studies coherent interpretations, Boolean-valued models and positive closure in [Kan22; Kan25; Kan26]. The finiteness question here concerns the inverse images of coherent objects.

Haine defines the coherent condensed category of points and records that the global coherent points of the classifying topos of abelian groups form **FinAb** [Hai26, Warning 0.4]. The equivalence (2) identifies the entire coherent condensed category under condition (F). For the theory of abelian groups, Section 6 computes extensions in these model categories over every Stone base.

2 Coherent syntax and finite sheaves

We work in fixed universes and take all theories and signatures to be small. We write $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. A coherent formula is built from atomic formulas by finite conjunctions, finite disjunctions, and existential quantification. A coherent theory is a set of sequents between such formulas. The signature is finitary and may have infinitely many sorts and symbols. Empty sorts are allowed. Nonemptiness, when required, is imposed by the coherent sequent $\top \vdash \exists x_s \top$. A model homomorphism commutes with each operation, including constants, and carries relation tuples to relation tuples. We write $\mathbf{Sh}(K)$ for the ordinary topos of sheaves of sets on K.

A Stone space is a compact Hausdorff space with a basis of clopen subsets. Equivalently, it is a profinite space. A Stonean space is a Stone space in which the closure of every open subset is open. The empty space is allowed, and its finite clopen partition is the empty partition.

2.1 The syntactic category and its pretopos completion

Let $\mathcal{C}_T$ be the coherent syntactic category of T. Its objects are coherent formulas in context, and its arrows are equivalence classes of T-provably functional coherent formulas. Write $\mathcal{P}_T$ for its pretopos completion. A pretopos has finite limits, disjoint pullback-stable finite coproducts, and pullback-stable effective quotients of equivalence relations. A pretopos functor preserves finite limits, finite coproducts, and effective epimorphisms. We write $\mathrm{Pretop}(\mathcal{P}, \mathcal{Q})$ for the category of these functors and all natural transformations.

The classifying topos $\mathcal{X}_T = \mathbf{Set}[T]$ is coherent, and there is an equivalence

$$\mathcal{P}_T \simeq \mathcal{X}_T^{\mathrm{coh}}. \tag{4}$$

In an ordinary coherent topos, the coherent objects are the quasicompact, quasiseparated objects. They form an essentially small pretopos and generate the topos. Every jointly epimorphic family with coherent target admits a finite subcover. Formula objects generate $\mathcal{P}_T$ under finite limits, finite coproducts, and effective quotients. These facts are part of the coherent syntactic construction and pretopos completion [MR77; MM92].

Proposition 2.1. *For every Grothendieck topos $\mathcal{E}$, restriction to the syntactic category and extension to its pretopos completion give natural equivalences*

$$\mathrm{Mod}_T(\mathcal{E}) \simeq \mathrm{Coh}(\mathcal{C}_T, \mathcal{E}) \simeq \mathrm{Pretop}(\mathcal{P}_T, \mathcal{E}). \tag{5}$$

If $\mathcal{E}$ is coherent, the full subcategory $\mathrm{CMod}_T(\mathcal{E})$ of models whose classifying geometric morphisms are coherent satisfies

$$\mathrm{CMod}_T(\mathcal{E}) \simeq \mathrm{Pretop}(\mathcal{P}_T, \mathcal{E}^{\mathrm{coh}}). \tag{6}$$

Both statements include all model homomorphisms.

Proof. A model interprets contexts, formula objects, and functional formulas in $\mathcal{E}$. The resulting coherent functor preserves finite limits, finite unions, and images. Conversely, its values on sorts, operation arrows, and relation subobjects recover the model. The universal property of pretopos completion gives the second equivalence in (5), including natural transformations. The coherent topology on $\mathcal{P}_T$ presents $\mathcal{X}_T$. Its covering families are generated by finite families $(P_i \to P)_i$ for which $\coprod_i P_i \to P$ is an effective epimorphism. Restriction along Yoneda identifies geometric inverse-image functors $\mathcal{X}_T \to \mathcal{E}$ and their natural transformations with pretopos functors and their natural transformations [Lur18, Corollary C.3.6]. A geometric inverse-image functor is coherent exactly when its restriction to $\mathcal{X}_T^{\mathrm{coh}}$ lands in $\mathcal{E}^{\mathrm{coh}}$. This proves (6). The ordinary coherent-topos/pretopos equivalence is recalled in [Hai19, Theorem 1.5 and Remark 1.6]. □

Remark 2.2. *Objects of $\mathcal{P}_T$ need not be single formula objects in the original language. For the one-sorted pure equality theory, $1 \amalg 1$ belongs to $\mathcal{P}_T$. Its interpretation in a singleton model has two elements, whereas any original-language formula in any finite context has at most one solution in that model. Effective quotients similarly require imaginary sorts in general. The generation statement preceding Proposition 2.1 is the property used below.*

2.2 Coherent sheaves on a Stone space

For a Stone space K, put $\mathrm{Cons}(K) = \mathbf{Sh}(K)^{\mathrm{coh}}$. If $U \subseteq K$ is open, 1_U denotes its subterminal sheaf. For a set A, $\underline{A}_K$ denotes the associated constant sheaf. The clopen subterminal objects form a coherent generating family of $\mathbf{Sh}(K)$. For a discrete set D, write $\mathrm{LC}(K, D)$ for the set of locally constant maps from K to D.

Lemma 2.3. *For $F \in \mathbf{Sh}(K)$, the following conditions are equivalent.*

(i) *F is coherent.*

(ii) *F is locally constant with finite stalks.*

(iii) *There is a finite clopen partition $K = \coprod_{i=1}^{m} U_i$ and finite sets A_i such that $F|_{U_i} \cong \underline{A_i}_{U_i}$.*

These are also equivalent to constructibility in the spectral-space sense.

Proof. Suppose first that F is coherent. Sections over clopen subsets form a jointly epimorphic family into F. Quasicompactness gives an epimorphism $E = \coprod_{i=1}^{n} 1_{V_i} \twoheadrightarrow F$ with each V_i clopen. Its kernel pair $R = E \times_F E$ is coherent. For each pair (i, j), the component

$$R_{ij} = R \times_{E \times E} (1_{V_i} \times 1_{V_j})$$

is a coherent subterminal object, corresponding to a quasicompact open subset $W_{ij} \subseteq V_i \cap V_j$. Such an open is closed because K is Hausdorff. A finite clopen partition generated by all V_i and W_{ij} makes both E and its equivalence relation R constant. The effective quotient $F = E/R$ is therefore constant finite on each part.

Conversely, an object as in (iii) is a finite coproduct of clopen subterminal objects, hence coherent. Condition (iii) implies (ii). Under (ii), choose clopen trivializing neighbourhoods and a finite subcover using compactness. The nonempty atoms of their finite Boolean algebra give (iii). Finally, the quasicompact opens of a Stone space are exactly its clopens. Thus spectral constructible subsets are clopen, and a finite constructible stratification with finite locally constant restrictions is equivalent to (iii). □

Lemma 2.4. *For finite sets A, B there is a natural bijection*

$$\mathrm{Hom}_{\mathbf{Sh}(K)}(\underline{A}_K, \underline{B}_K) \cong \mathrm{LC}(K, \mathrm{Map}(A, B)). \tag{7}$$

Every finite diagram in $\mathrm{Cons}(K)$ *is represented on one finite clopen partition by diagrams of finite sets. Finite limits, finite coproducts, images, and effective quotients are computed on that partition.*

Proof. A sheaf map between the displayed constant sheaves is a continuous map $K \times A \to B$, or equivalently a continuous map from K to the finite discrete set $\mathrm{Map}(A, B)$. Its fibres are clopen. First trivialize the finitely many objects of the diagram using Lemma 2.3, and then refine by the fibres of its finitely many arrow maps. All diagram identities hold pointwise. The listed constructions agree with their sheaf-theoretic counterparts because stalk functors preserve finite limits and colimits and are jointly conservative. □

Let $\mathrm{Part}_{\mathrm{cl}}(K)$ be the filtered poset of finite clopen partitions, ordered by refinement. A finer partition repeats an object assigned to a coarser part. The preceding lemmas give a natural equivalence

$$\mathrm{Cons}(K) \simeq \operatorname*{2\text{-}colim}_{\mathcal{P} \in \mathrm{Part}_{\mathrm{cl}}(K)} \mathbf{Fin}^{\mathcal{P}}. \tag{8}$$

An arrow on the right is a family of finite-set maps on a common refinement, with two families identified when they agree on a further refinement. Formula (7) proves fullness. Equality of the induced sheaf maps implies equality on a common refinement, proving faithfulness. Pullback along $f : L \to K$ replaces the parts U by the nonempty inverse images $f^{-1}U$.

Proposition 2.5. *If $F \in \mathrm{Cons}(K)$ and $A \hookrightarrow F$ is a subobject in $\mathbf{Sh}(K)$, then A is coherent if and only if this subobject has a complement. Consequently,* $\mathrm{Cons}(K)$ *is a Boolean pretopos.*

Proof. If A is coherent, represent its inclusion on a finite clopen partition by injections of finite sets, using Lemma 2.4. The complements glue to a complement in F. Conversely, a complemented subobject is classified by a map $F \to 1 \amalg 1$. It is the pullback of one summand and is coherent by finite-limit closure. Applied to subobjects internal to $\mathrm{Cons}(K)$, this proves the last assertion. □

Example 2.6. *For $K = \beta\mathbb{N}$, the open subset $V = \mathbb{N}$ is not closed. Thus $1_V \hookrightarrow 1_K$ is a subobject of a coherent object whose domain is not coherent. More generally, a subobject of $\underline{A}_K$ is specified by opens $U_a \subseteq K$, one for each $a \in A$. Its étalé space is $\coprod_{a \in A} U_a$. It is coherent precisely when each U_a is clopen and only finitely many are nonempty. Indeed, a coherent sheaf has compact étalé space by Lemma 2.3, so the open summands have finite support and each is compact. Conversely, a finite coproduct of clopen subterminals is coherent.*

2.3 Formula objects, relations, and parameters

For a model M and a coherent formula $\varphi(\bar{x})$, write $\|\varphi\|_M \hookrightarrow M_{\bar{x}}$ for its interpreted subobject. Here $M_{\bar{x}}$ is the finite product of the relevant sort sheaves, including the terminal object for an empty context.

Theorem 2.7 (Atomic criterion)**.** *Let T be any finitary coherent theory and let K be a Stone space. For a model $M \in \mathrm{Mod}_T(\mathbf{Sh}(K))$, the following are equivalent.*

(i) $M \in \mathrm{CMod}_T(\mathbf{Sh}(K))$.

(ii) $\|\varphi\|_M$ *is coherent for every coherent formula in context.*

(iii) *Every basic sort sheaf M_s and every primitive relation sheaf R^M is coherent.*

If T is algebraic, condition (iii) requires only the basic sorts.

Proof. By Proposition 2.1, (i) implies (ii). Conversely, the objects whose images under the model's pretopos functor are coherent form a full replete subcategory closed under finite limits, finite coproducts, and effective quotients. Formula objects generate $\mathcal{P}_T$, so (ii) implies (i). Taking the formulas $\top(x_s)$ and $R(\bar{x})$ shows that (ii) implies (iii).

Assume (iii). Context objects are coherent. An atomic equality is an equalizer of term maps between coherent objects. A relation applied to terms is a pullback of the corresponding primitive relation subobject. Finite conjunctions are finite intersections. A finite disjunction is the image of a finite coproduct in its common context, and existential quantification is the image of a projection. The coherent pretopos is closed under all these operations. Induction on formulas proves (ii). In an algebraic signature the only atomic formulas are equations. □

Corollary 2.8 (Local parameters)**.** *The conditions of Theorem 2.7 are also equivalent to requiring that, for every clopen $U \subseteq K$, every coherent formula $\varphi(\bar{x}, \bar{y})$, and every section $a \in M_{\bar{y}}(U)$, the parameter fibre $\|\varphi(\bar{x}, a)\|_{M|_U}$ is coherent.*

Proof. Work in $\mathbf{Sh}(U)$, with terminal object 1_U. Restriction to a clopen subset preserves finite locally constant sheaves. If M satisfies condition (ii) of Theorem 2.7, the maps in the pullback

$$\|\varphi(\bar{x}, a)\|_{M|_U} = 1_U \times_{M_{\bar{y}|_U}} \|\varphi(\bar{x}, \bar{y})\|_M|_U$$

have coherent domains and codomain, so the fibre is coherent. For the converse, take $U = K$ and the empty parameter tuple. □

Corollary 2.9. *For an arbitrary set of sorts, $\mathrm{CMod}_T(\mathbf{Set})$ consists precisely of the sortwise finite models. In particular, for finitely many sorts it is the category $\mathrm{FinMod}(T)$ of finite models and all homomorphisms.*

Proof. Coherent objects of **Set** are finite sets. A relation in a finite product of finite sorts is finite. Apply Theorem 2.7. □

Example 2.10. *For the theory with one sort and one unary relation and no axioms, put $M_s = 1_K$ and $R^M = 1_V$, where $V \subset K$ is open but not closed. The sort sheaf is coherent, while the model is not. The relation condition in Theorem 2.7 is therefore necessary.*

3 Continuous families of finite structures

From here through Section 5, the set S of basic sorts of T is finite unless stated otherwise. The finitary signature Σ and the axiom set may be infinite. A single finite clopen partition now trivializes all the sort sheaves of a coherent model. On each part the carriers are fixed, and Theorem 2.7 lets us describe continuity one operation or relation at a time. For an empty context the product of carriers is a singleton, so the same description includes nullary symbols.

3.1 The space of structures on a fixed finite carrier

Let $A = (A_s)_{s \in S}$ be a family of finite sets. For a finite context $\bar{s} = (s_1, \ldots, s_m)$, put $A_{\bar{s}} = \prod_{j=1}^{m} A_{s_j}$. The space of raw Σ-structures on A is

$$\mathscr{R}_\Sigma(A) = \prod_{\sigma : \bar{s} \to t} \mathrm{Map}(A_{\bar{s}}, A_t) \times \prod_{R : \bar{s}} \mathcal{P}(A_{\bar{s}}),$$

where the products range over symbols of the indicated sorts and every factor has the finite discrete topology. An empty factor makes the entire product empty, as it should when a function symbol has no possible interpretation. Write $\mathrm{Str}_T(A)$ for the subspace consisting of structures satisfying T.

Lemma 3.1. *For each coherent formula $\varphi(\bar{x})$ and each tuple $\bar{a} \in A_{\bar{s}}$, the set*

$$\{B \in \mathscr{R}_\Sigma(A) : B \models \varphi(\bar{a})\}$$

is clopen and depends on only finitely many coordinates of $\mathscr{R}_\Sigma(A)$. The structures satisfying one coherent sequent, or any finite set of coherent sequents, likewise form a clopen subset. Consequently, $\mathrm{Str}_T(A)$ is a Stone space, possibly empty.

Proof. For a term $t(\bar{x})$ of sort u and a fixed tuple $\bar{a}$, evaluation defines a continuous map $B \mapsto t^B(\bar{a})$ to the finite discrete set A_u. This follows by induction on terms. It is immediate for variables. For $t = \sigma(t_1, \ldots, t_m)$, partition the raw structure space according to the finitely many possible values of the terms t_j. On each resulting clopen piece, evaluation of t reads a single entry of the operation table for σ. The induction also shows that only the finitely many symbols occurring in t are relevant.

An atomic equation therefore has a clopen truth set. For an atomic relation, fixing the finitely many term values reduces its truth to membership of one tuple in the relation table. The truth sets of $\top$ and $\bot$ are the whole space and the empty set. Conjunction and finite disjunction give finite intersections and finite unions. Finally,

$$\{B : B \models \exists \bar{y}\, \psi(\bar{a}, \bar{y})\} = \bigcup_{\bar{b} \in A_{\bar{t}}} \{B : B \models \psi(\bar{a}, \bar{b})\},$$

a finite union, including the empty union when $A_{\bar{t}} = \varnothing$. This proves the assertion for formulas.

The truth set of a sequent $\varphi(\bar{x}) \vdash_{\bar{x}} \psi(\bar{x})$ is

$$\bigcap_{\bar{a} \in A_{\bar{s}}} \big(\{B : B \not\models \varphi(\bar{a})\} \cup \{B : B \models \psi(\bar{a})\}\big),$$

again a finite intersection of clopen sets. Thus the axioms of T define a closed subspace of the product of finite discrete spaces $\mathscr{R}_\Sigma(A)$, proving the last assertion. □

Proposition 3.2. *Let K be profinite. There is a natural bijection between continuous maps*

$$K \longrightarrow \mathrm{Str}_T(A)$$

and coherent T-model structures on the specified family of sheaves $(\underline{A_s}_K)_{s \in S}$. Here the identifications of the sort sheaves with these constant sheaves are part of the data.

Proof. A morphism between constant finite sheaves is specified by a continuous map from K to the finite set of maps between their carriers. Thus the function symbols give precisely continuous maps to the operation-table factors of $\mathscr{R}_\Sigma(A)$.

For relations, a coherent subobject of $\underline{A_{\bar{s}}}_K$ is specified by a continuous map $K \to \mathcal{P}(A_{\bar{s}})$. After a finite clopen refinement the inclusion on each part is represented by a fixed injection of finite sets. Its image is a fixed subset of $A_{\bar{s}}$. Conversely, a continuous map to the finite power set gives a finite clopen partition and hence a coherent relation subobject.

The product topology says exactly that all symbol interpretations are continuous separately. Stalk functors preserve the interpretation of coherent formulas and jointly detect inclusions of subsheaves. It follows that a coherent sequent holds in the sheaf-valued structure if and only if it holds in every stalk. Thus the structure is a T-model precisely when the map to $\mathscr{R}_\Sigma(A)$ lands in $\mathrm{Str}_T(A)$. When this holds, the atomic criterion identifies its coherence exactly with the coherence of the relation subobjects already described. The construction commutes with pullback along continuous maps of profinite spaces. □

Proposition 3.3. *For a coherent formula $\varphi(\bar{x})$, the subset*

$$\mathscr{D}_\varphi(A) = \{(B, \bar{a}) \in \mathrm{Str}_T(A) \times A_{\bar{s}} : B \models \varphi(\bar{a})\}$$

is clopen. Its projection to $\mathrm{Str}_T(A)$ is the étalé space of a coherent sheaf. If a framed model on K corresponds to $a : K \to \mathrm{Str}_T(A)$, its interpretation of φ is the pullback of this sheaf along a.

Proof. There are finitely many tuples $\bar{a} \in A_{\bar{s}}$. For each of them, Lemma 3.1 gives a clopen subset of the structure space on which $\varphi(\bar{a})$ holds. Their products with the singleton $\{\bar{a}\}$ form the displayed subset, proving that it is clopen. Refine these finitely many truth sets to their Boolean partition. On each part, the set of solutions of φ is fixed, so the projection there is a product projection with finite fibre. It therefore represents a coherent sheaf. For the last assertion, both sheaves are subobjects of the same constant context sheaf. Their stalks agree by the stalkwise semantics of coherent formulas, and consequently the subobjects agree. □

The partition in Proposition 3.3 may depend on the formula. With infinitely many symbols a structure map can be continuous in every table coordinate and still have infinite image.

3.2 A topological category of finite models

We assemble the labelled finite structures and their homomorphisms into a topological category. For $\nu = (\nu_s)_{s \in S} \in \mathbb{N}_0^S$, let $A_s^\nu = \{0, \dots, \nu_s - 1\}$. Define the object space of $\mathfrak{M}_T$ by

$$\mathrm{Ob}(\mathfrak{M}_T) = \coprod_{\nu \in \mathbb{N}_0^S} \mathrm{Str}_T(A^\nu).$$

For two vectors ν, μ, put $H_{\nu\mu} = \prod_{s \in S} \mathrm{Map}(A_s^\nu, A_s^\mu)$, a finite discrete set. Let $\mathscr{H}_{\nu\mu}$ be the subspace of

$$\mathrm{Str}_T(A^\nu) \times \mathrm{Str}_T(A^\mu) \times H_{\nu\mu}$$

consisting of triples (B, C, h) for which h is a homomorphism of Σ-structures. Set

$$\mathrm{Ar}(\mathfrak{M}_T) = \coprod_{\nu,\mu} \mathscr{H}_{\nu\mu}.$$

Here preservation of a relation requires only the forward implication.

Lemma 3.4. *Each $\mathscr{H}_{\nu\mu}$ is a closed Stone subspace of the displayed product. Source, target, identity, and composition define a topological category $\mathfrak{M}_T$.*

Proof. For each function symbol $\sigma : \bar{s} \to t$ and tuple $\bar{a}$, impose

$$h_t\big(\sigma^B(\bar{a})\big) = \sigma^C\big(h_{\bar{s}}(\bar{a})\big).$$

For each relation symbol R and tuple $\bar{a}$, impose

$$\bar{a} \in R^B \quad \Longrightarrow \quad h_{\bar{s}}(\bar{a}) \in R^C.$$

Each condition is clopen, since it reads only finitely many entries in finite tables. Their possibly infinite intersection is closed. Source and target are projections, and the identity map is $B \mapsto (B, B, \mathrm{id})$. On a component of the space of composable pairs, composition is

$$((B, C, h), (C, D, k)) \longmapsto (B, D, k \circ h).$$

It is continuous because composition of the underlying finite sort maps is a map of finite discrete spaces. The category axioms hold pointwise. □

Let $\mathrm{Fam}_T(K)$ denote the category whose objects are continuous maps $a : K \to \mathrm{Ob}(\mathfrak{M}_T)$ and whose morphisms from a to b are continuous maps $h : K \to \mathrm{Ar}(\mathfrak{M}_T)$ satisfying $sh = a$ and $th = b$. Identities and composition are pointwise.

Theorem 3.5 (Continuous families)**.** *For every profinite space K, evaluation induces an equivalence*

$$\mathrm{Fam}_T(K) \simeq \mathrm{CMod}_T(\mathbf{Sh}(K)).$$

These equivalences are natural, up to the canonical pullback isomorphisms, in continuous maps of profinite spaces.

Proof. The cardinality-vector map from $\mathrm{Ob}(\mathfrak{M}_T)$ to the discrete set $\mathbb{N}_0^S$ is continuous. The compact image of K therefore meets only finitely many cardinality components. An object a determines a finite clopen partition $K = \coprod_\nu K_\nu$. On K_ν, apply Proposition 3.2 to the map $a|_{K_\nu} : K_\nu \to \mathrm{Str}_T(A^\nu)$, and glue the resulting models over the disjoint partition. This gives a coherent model. A continuous arrow family gives sortwise sheaf maps, and the defining conditions of $\mathscr{H}_{\nu\mu}$ make these maps a model homomorphism.

Conversely, let M be coherent. Each sort sheaf has a finite clopen trivialization. Since S is finite, a common refinement trivializes all sorts simultaneously. Choose bijections between their finite carriers and the corresponding A_s^ν. On each piece, Proposition 3.2 produces a continuous structure map. Their disjoint union is an object of $\mathrm{Fam}_T(K)$ whose evaluation is isomorphic to M.

To prove full faithfulness, take two continuous object families. Refine their cardinality partitions together. On a resulting piece with vectors ν, μ, a morphism between the evaluated models is a family of sortwise maps between constant finite sheaves, hence a continuous map to $H_{\nu\mu}$. It is a model homomorphism precisely when its stalk maps preserve every symbol, equivalently when the resulting triple lands in $\mathscr{H}_{\nu\mu}$. This is exactly a continuous arrow family, uniquely determined by the sheaf morphism. The argument includes noninvertible homomorphisms.

For $f : L \to K$, pullback on families is precomposition with f. Pullback of the evaluated sheaves restricts each carrier partition to its nonempty inverse images and pulls back the operation and relation tables. These constructions agree canonically, and the identifications respect identities and composition of base maps. □

Remark 3.6. *Changing labels on a fixed carrier A is governed by the finite group $G_A = \prod_{s\in S} \mathrm{Sym}(A_s)$. The subgroupoid of invertible arrows in the corresponding component is its action groupoid on $\mathrm{Str}_T(A)$. Passing only to the orbit set would discard automorphisms. The full topological category above also retains the noninvertible homomorphisms required by $\mathrm{CMod}_T(\mathbf{Sh}(K))$.*

More explicitly, fix a cardinality vector ν and consider coherent models whose sorts have those cardinalities at every point of K. Their maximal subgroupoid is equivalent to the action groupoid

$$\mathrm{Cont}(K, \mathrm{Str}_T(A^\nu)) /\!\!/ \mathrm{Cont}(K, G_{A^\nu}),$$

where a continuous permutation family acts by transport of structure. To see that every such model admits the required global labels, first trivialize its finitely many sort sheaves on a common finite clopen partition. On every part, the carrier of sort s has cardinality ν_s, so choose a bijection with A_s^ν. Since the parts are disjoint, the resulting local identifications glue to a global identification with $\underline{A_s^\nu}_K$. Two choices differ by a continuous family of permutations. Conversely, any isomorphism of the resulting models is exactly such a permutation family carrying one structure map to the other.

4 Finite partitions and finite detection

We call a model over K a *finite model bundle* if some finite clopen partition of K makes it isomorphic on each part to a constant finite T-model.

4.1 The finite-partition criterion

Definition 4.1. *We say that T satisfies the* finite structure condition *if $\mathrm{Str}_T(A)$ is finite for every family A of finite carriers.*

For any T under the standing finite-sort assumption, finite model bundles give a fully faithful functor

$$\mathrm{FinBund}_T(K) := \operatorname*{2\text{-}colim}_{\mathcal{P}\in \mathrm{Part}_{\mathrm{cl}}(K)} \mathrm{FinMod}(T)^{\mathcal{P}} \longrightarrow \mathrm{CMod}_T(\mathbf{Sh}(K)).$$

Indeed, a constant finite model is coherent by the atomic criterion, and a morphism between two such models is a continuous map into the finite set of sortwise maps satisfying the homomorphism conditions. This last set remains finite when the signature is infinite. Its maps become constant on finite clopen refinements, proving full faithfulness. The issue is therefore essential surjectivity.

Lemma 4.2. *A coherent T-model M on a profinite base K is a finite model bundle if and only if its stalks represent finitely many isomorphism classes. These conditions are also equivalent to finiteness of the image of a representing family $K \to \mathrm{Ob}(\mathfrak{M}_T)$ from Theorem 3.5.*

Proof. A finite model bundle plainly has only finitely many stalk isomorphism classes. Suppose the latter condition holds, and choose a representing object family. Within any fixed cardinality component, a model isomorphism class contains at most $|G_{A^\nu}|$ labelled structures. Finitely many isomorphism classes therefore allow only finitely many values of the object family. A continuous map with finite image in a Hausdorff space has discrete image and hence clopen fibres. These fibres give the required finite partition. The argument also shows that finite image is independent of the chosen representing family. □

We can now ask when the theory forces every continuous family to have only finitely many stalk isomorphism types.

Theorem 4.3 (Finite partitions)**.** *The following conditions are equivalent.*

1. *For every profinite K, every coherent T-model over $\mathbf{Sh}(K)$ is constant, with finite carriers, on a finite clopen partition.*
2. *The same statement holds for every extremally disconnected profinite K.*
3. *Every coherent T-model over $\mathbf{Sh}(\beta\mathbb{N})$ is constant, with finite carriers, on a finite clopen partition of $\beta\mathbb{N}$.*
4. *The theory T satisfies the finite structure condition.*
5. *For each finite cardinality vector, there are only finitely many isomorphism classes of T-models with that vector.*

Under these conditions, the displayed finite-bundle functor is an equivalence for every profinite K, naturally under base change.

Proof. The first condition implies the second, and the second implies the third, since $\beta\mathbb{N}$ is the Stone space of the complete Boolean algebra $\mathcal{P}(\mathbb{N})$ and hence is extremally disconnected. For a fixed carrier A, the orbits of the finite group G_A on $\mathrm{Str}_T(A)$ are precisely the isomorphism classes with that carrier vector. Each orbit has at most $|G_A|$ elements. Thus the fourth and fifth conditions are equivalent.

Assume the fourth condition. In Theorem 3.5, the image of a compact base meets finitely many carrier components, each of which is now finite discrete. The object map consequently has finite image and clopen fibres. Its evaluation is a finite model bundle, proving the first condition.

For the converse from the third condition, suppose that $X = \mathrm{Str}_T(A)$ is infinite for some A. Since every G_A-orbit is finite, choose a sequence $(B_n)_{n\geq 1}$ of pairwise nonisomorphic structures in X. The map $n \mapsto B_n$ from the discrete space $\mathbb{N}$ to the compact Hausdorff space X extends continuously to

$$q : \beta\mathbb{N} \longrightarrow X.$$

By Proposition 3.2, the pullback of the tautological structure along q is a coherent model with constant carriers A over $\beta\mathbb{N}$. Its principal stalks are the pairwise nonisomorphic models B_n. It therefore cannot be a finite model bundle, contrary to the third condition. This proves the fourth condition. The final assertion combines essential surjectivity with the full faithfulness above and the base-change compatibility of Theorem 3.5. □

Corollary 4.4. *If T has finitely many basic sorts, function symbols, and relation symbols, then*

$$\mathrm{FinBund}_T(K) \simeq \mathrm{CMod}_T(\mathbf{Sh}(K))$$

for every profinite K. No finiteness assumption on the axiom set is required.

Proof. For fixed finite carriers, the raw structure space is a finite product of finite sets. Its subset $\mathrm{Str}_T(A)$ is therefore finite, so Theorem 4.3 applies. □

An infinite signature can also satisfy the criterion. For example, adding infinitely many function symbols, each equated by T with a term in a fixed finite signature, introduces no additional choices of finite structure. The finite structure condition records the actual choices permitted by the theory.

4.2 Detection by finite reducts

The criterion admits a finite-language formulation at each bound on the carriers. A subsignature below retains all basic sorts and some of the function and relation symbols. For a vector $b \in \mathbb{N}_0^S$, let $\mathrm{FinMod}_{\leq b}(T)$ be the full category of finite models A satisfying $|A_s| \leq b_s$ for every sort s.

Theorem 4.5 (Finite reducts)**.** *For a theory with finitely many sorts, the finite structure condition is equivalent to the following property. For each $b \in \mathbb{N}_0^S$, there is a finite subsignature Σ_b such that the reduct functor*

$$\mathrm{FinMod}_{\leq b}(T) \longrightarrow \mathrm{Mod}_{\Sigma_b}(\mathbf{Set})$$

is fully faithful. For every such subsignature and every specified carrier A^ν with $\nu \leq b$, the map taking a T-structure to its Σ_b-reduct is injective.

Proof. Assume the finite structure condition. There are finitely many vectors $\nu \leq b$, and each set $\mathrm{Str}_T(A^\nu)$ is finite. Consider all triples (B, C, h) where B, C are structures on these standard carriers and h is a family of sortwise maps. There are only finitely many triples. Whenever h fails to be a model homomorphism, choose a function or relation symbol witnessing that failure. Such a witness exists by the definition of a homomorphism. Take Σ_b to consist of the sorts and these witnesses. A sortwise map between the chosen models preserves this finite subsignature if and only if it is a T-homomorphism. Relabelling the carriers gives the same conclusion for all objects of $\mathrm{FinMod}_{\leq b}(T)$. Forgetting structure is faithful, so the reduct functor is fully faithful.

To see both the injectivity assertion and the converse, let Σ_b be any such fully faithful reduct signature and fix carriers A with $|A_s| \leq b_s$ for every s. If two T-structures on A have the same reduct, the identity sort maps are homomorphisms in both directions between their reducts. Fullness implies that these same identity maps are T-homomorphisms in both directions. Their function tables coincide, and preservation of relations in both directions makes their relation tables equal. Thus the structures are identical. There are only finitely many reducts on A in the finite signature Σ_b, so $\mathrm{Str}_T(A)$ is finite. □

The theorem concerns reducts of actual T-models. Its conclusion is a fully faithful description of this category at a prescribed cardinality bound. For a coherent family on a compact base, the underlying sort sheaves provide such a bound. Under the finite structure condition, a single finite set of symbols consequently detects both the model structures and all homomorphisms occurring in that family. The set of symbols may depend on the cardinality bound.

4.3 Two failures of a common finite partition

Example 4.6 (Infinitely many named elements)**.** *Let T have one sort, constants $0, 1, c_1, c_2, \ldots$, and no axioms. On $K = \beta\mathbb{N}$, take the constant sort sheaf $\underline{\{0,1\}}_K$, interpret the first two constants by their indicated distinct values, and set*

$$c_n = \mathbf{1}_{\{n\}} : K \longrightarrow \{0, 1\}.$$

Each singleton $\{n\}$ is clopen, so these are valid sections. The model is coherent because its single sort sheaf is coherent. At a principal point n, precisely c_n among the c_j has value 1. At a nonprincipal point every c_j has value 0. An isomorphism must fix the named elements 0 and 1, and hence must be the identity on the carrier. The principal stalks are therefore pairwise nonisomorphic. This coherent model admits no finite clopen partition into constant models, even though the base is Stonean and the underlying sort is globally constant.

Example 4.7 (Infinitely many independent sorts)**.** *To see the separate role of finitely many sorts, let T have sorts $S_1, S_2, \dots$ and no other symbols or axioms. There is then exactly one structure on each specified family of carriers. Put $K = \{0,1\}^{\mathbb{N}}$ and $U_n = \{x : x_n = 1\}$. Interpret S_n by the finite locally constant sheaf whose étalé space is*

$$(K \times \{0\}) \amalg (U_n \times \{1\}) \longrightarrow K.$$

Every sort is coherent. Every coherent formula involves only finitely many sorts, so the atomic criterion, which does not require finitely many sorts, shows that the model is coherent. Its stalk at x has sort cardinalities $(1+x_n)_{n\geq 1}$. Distinct points thus give nonisomorphic models. No finite clopen partition can make the entire model constant, although each individual sort has a two-piece trivialization. To obtain a Stonean example, extend a dense sequence in K to a continuous map $\beta\mathbb{N} \to K$. Its image is compact and contains the dense sequence, so it is surjective. The pullback still has infinitely many stalk cardinality vectors.

5 Condensed categories and change of presentation

5.1 The locally constant condensed category

For a small category $\mathcal{A}$ and a Stone space K, define

$$D(\mathcal{A})(K) = \operatorname*{2\text{-}colim}_{\mathcal{P}\in \mathrm{Part}_{\mathrm{cl}}(K)} \mathcal{A}^{\mathcal{P}}. \tag{9}$$

Objects and arrows are given by finite clopen presentations and common refinements, as in (8). Inverse images of parts define pullback along continuous maps. Refining by the two summands gives

$$D(\mathcal{A})(K \amalg L) \simeq D(\mathcal{A})(K) \times D(\mathcal{A})(L), \qquad D(\mathcal{A})(\varnothing) \simeq *.$$

We regard an ordinary category as an ∞-category by its nerve whenever it occurs as the value of a condensed ∞-category. We can compute (9) by the ordinary filtered colimit, identifying presentations that agree after refinement. Any finite string of composable arrows is represented on one partition, and equality of such strings is detected on a further refinement. Thus the nerve of this model is degreewise the filtered colimit of the nerves and represents their colimit in Cat_∞. Restriction to extremally disconnected profinite spaces gives a condensed category in the finite-product formulation of [Hai26, Notation 2.4].

This construction is the discrete condensed category associated with $\mathcal{A}$. Let $j : \mathbf{Fin}^{\mathrm{op}} \hookrightarrow \mathrm{Extr}^{\mathrm{op}}$ be the inclusion, where Extr denotes the category of Stonean spaces. In the indexing category for Lan_j at K, finite surjective quotients $K \twoheadrightarrow I$ form a cofinal subcategory. Indeed, for $u : K \to I$, its image quotient $K \twoheadrightarrow u(K)$ is initial among the surjective refinements of u. In fact, any $q : K \twoheadrightarrow J$ through which u factors induces a unique map $J \to u(K)$ over I. Up to relabelling, these quotients are precisely finite clopen partitions, ordered by refinement. Thus (9) is the left Kan extension of $I \mapsto \mathcal{A}^I$. For a finite-product-preserving functor on extremally disconnected spaces, its value on a finite set I is the I-fold product of its value on a point. The left Kan extension property then shows that D is left adjoint to evaluation at a point, and $D(\mathcal{A})(*) \simeq \mathcal{A}$. This agrees with the usual discrete condensed construction [Ásg22].

Put $\mathscr{M}_T(K) = \mathrm{CMod}_T(\mathbf{Sh}(K))$. In applying D we use the equivalent small category of finite models on the standard carriers A^ν. Clopen restriction identifies models over $K \amalg L$ with pairs of models, so this also defines a condensed category on Stonean test spaces. The functor

assigning a constant sheaf model to a finite model and gluing over disjoint clopen parts gives a natural comparison

$$\eta_T : D(\mathrm{FinMod}(T)) \longrightarrow \mathscr{M}_T. \tag{10}$$

Under the finite-sort assumption this comparison is fully faithful at every Stone space, by the description of finite-model homomorphisms in Theorem 4.3.

Theorem 5.1 (The condensed comparison)**.** *For a coherent theory with finitely many sorts, η_T is an equivalence of condensed categories if and only if T satisfies the finite structure condition. It suffices that $\eta_T(\beta\mathbb{N})$ be an equivalence. When these conditions hold,*

$$\mathscr{M}_T \simeq D(\mathrm{FinMod}(T)). \tag{11}$$

The equivalence extends naturally to all Stone spaces.

Proof. For $f : L \to K$, both sides of (10) pull a constant model on U back to the same constant model on $f^{-1}U$, discarding empty parts. These identifications commute with refinement and with composition of base maps, proving the required naturality of η_T. By Theorem 4.3, the finite structure condition makes each component an equivalence. An equivalence of condensed categories is an equivalence on every test space, hence on $\beta\mathbb{N}$. Essential surjectivity at that single space implies the finite structure condition by the same theorem. □

Finite-model categories in this statement are idempotent complete. To split an idempotent homomorphism, take its image on every sort and restrict the relations. Inclusion and retraction are model homomorphisms. If a coherent antecedent holds in the image, it holds in the original model. Applying the retraction to the consequent shows that the image satisfies every axiom. Its carriers remain finite. Idempotents in $D(\mathrm{FinMod}(T))(K)$ are represented on a common finite clopen partition and split on its parts. Thus (11) also respects the idempotent-complete targets used for coherent condensed points.

5.2 Coherent Morita equivalence

The comparison (10) has an intrinsic formulation using $\mathrm{CMod}_T(\mathbf{Set})$ in place of $\mathrm{FinMod}(T)$. By Corollary 2.9, these categories agree when the sort set is finite.

Corollary 5.2. *Let T and T' be coherent theories with finitely many sorts, and suppose their classifying topoi are equivalent. Then T satisfies the finite structure condition if and only if T' does.*

Proof. An equivalence of classifying topoi preserves coherent objects and induces an equivalence $\mathcal{P}_T \simeq \mathcal{P}_{T'}$ of pretopoi. Precomposition therefore gives equivalences of coherent-model categories in every coherent target topos, naturally under inverse images of coherent geometric morphisms. The equivalences commute with the constant-sheaf inverse image $\mathbf{Set} \to \mathbf{Sh}(K)$ and with restriction to clopen components. They consequently identify the two natural comparisons

$$D(\mathrm{CMod}_T(\mathbf{Set})) \longrightarrow \mathscr{M}_T, \qquad D(\mathrm{CMod}_{T'}(\mathbf{Set})) \longrightarrow \mathscr{M}_{T'}.$$

One is an equivalence if and only if the other is. Apply Theorem 5.1. □

Thus the labelled spaces $\mathrm{Str}_T(A)$ depend on the chosen sorts, while their finiteness for every finite A can be read from the classifying topos.

5.3 Comparison with coherent categories of points

To relate the ordinary constructions to Haine's notation, let $\mathcal{X}_T^\infty$ be the 1-localic ∞-topos associated with the ordinary classifying topos $\mathcal{X}_T$. If a site with finite limits presents $\mathcal{X}_T$ by sheaves of sets, the same site with sheaves of spaces presents $\mathcal{X}_T^\infty$. Its 0-truncated subcategory is $\mathcal{X}_T$. Apply [Lur09, Lemma 6.4.5.6] with $n = 1$, the coherent site $\mathcal{P}_T$, and the ∞-topos $\mathbf{Sh}(K;\mathrm{Spc})$. Restriction to 0-truncated objects identifies geometric morphisms from $\mathbf{Sh}(K;\mathrm{Spc})$ to $\mathcal{X}_T^\infty$ with those from $\mathbf{Sh}(K)$ to $\mathcal{X}_T$, including transformations of direct images. Passing from direct images to inverse images reverses natural transformations on both sides. The resulting comparison therefore retains all transformations of inverse images. Coherence of the corresponding ordinary and higher morphisms agrees by [Hai19, Proposition 2.11].

With N denoting the ordinary nerve, Haine's definition of coherent points and the classifying property in Proposition 2.1 now give, for every Stonean K, a natural equivalence [Hai26, Example 0.2 and Definition 2.10]

$$\mathbf{Pt}^{\mathrm{coh}}(\mathcal{X}_T^\infty)(K) \simeq N\big(\mathrm{CMod}_T(\mathbf{Sh}(K))\big).$$

In particular, under the finite structure condition,

$$\mathbf{Pt}^{\mathrm{coh}}(\mathcal{X}_T^\infty) \simeq D(\mathrm{FinMod}(T)). \tag{12}$$

All calculations on the right use ordinary pretopoi. In the higher comparison, the ordinary coherent objects correspond to the 0-truncated coherent objects. The truncated coherent ∞-pretopos can also contain objects in positive homotopy degrees. For example, the one associated with **Set** is the ∞-category of π-finite spaces [Hai19, Example 2.17].

6 Extensions of finite abelian-group bundles

Let K be a profinite space. Write $\mathrm{CAb}(K)$ for the full subcategory of $\mathbf{Ab}(\mathbf{Sh}(K))$ consisting of sheaves whose underlying sheaf of sets is coherent. Equivalently, these are the coherent models of the algebraic theory of abelian groups. Throughout this section, Ext^n in an abelian category denotes Yoneda extensions.

Proposition 6.1. *The category* $\mathrm{CAb}(K)$ *is abelian, and its inclusion into* $\mathbf{Ab}(\mathbf{Sh}(K))$ *is exact. Every finite diagram in* $\mathrm{CAb}(K)$ *becomes a diagram of constant finite abelian groups on a common finite clopen partition. Kernels, cokernels, and finite direct sums are computed on such a partition. Moreover,* $\mathrm{CAb}(K)$ *is closed under extensions in* $\mathbf{Ab}(\mathbf{Sh}(K))$.

Proof. The finite-model-bundle classification of Corollary 4.4 applies to the usual finite signature for abelian groups. After trivializing finitely many objects, a morphism between two constant finite group sheaves is a locally constant map into their finite set of group homomorphisms. Its fibres give a finite clopen refinement. This proves the assertion about finite diagrams. On each part, the kernel and cokernel are the constant sheaves associated with the corresponding finite-group kernel and cokernel. These are also the ambient sheaf kernel and cokernel, as can be checked on stalks. Finite direct sums have the same description. The canonical map from coimage to image is consequently an isomorphism, proving abelianness and exactness of the inclusion.

For extension closure, consider an ambient exact sequence $0 \to N \to E \to M \to 0$ with $M, N \in \mathrm{CAb}(K)$. Near any point, trivialize M and N with finite fibres A and B. The

epimorphism $E \to M$ locally lifts each of the finitely many constant sections indexed by A. Intersecting these finitely many neighbourhoods and taking a smaller clopen neighbourhood gives simultaneous lifts e_a, $a \in A$. Translation by these lifts identifies the underlying sheaf of E there with $\underline{B \times A}$. Compactness supplies a finite clopen partition of such neighbourhoods. Thus $E \in \mathrm{CAb}(K)$. □

6.1 The sheaf of first extension groups

For finite abelian groups A, B, every abelian-group extension of A by B has finite middle group. Hence the inclusion $\mathbf{FinAb} \hookrightarrow \mathbf{Ab}$ induces the isomorphism

$$\mathrm{Ext}^1_{\mathbf{FinAb}}(A, B) \cong \mathrm{Ext}^1_{\mathbb{Z}}(A, B).$$

These groups are finite. Indeed, a decomposition $A \cong \bigoplus_{j=1}^{r} \mathbb{Z}/n_j\mathbb{Z}$ gives

$$\mathrm{Ext}^1_{\mathbb{Z}}(A, B) \cong \bigoplus_{j=1}^{r} B/n_j B.$$

For a cyclic summand this is the cokernel of multiplication by n_j on B, as one sees from the resolution $0 \to \mathbb{Z} \xrightarrow{n_j} \mathbb{Z} \to \mathbb{Z}/n_j\mathbb{Z} \to 0$.

For $M, N \in \mathrm{CAb}(K)$, define a finite locally constant sheaf of abelian groups $\mathscr{E}^1(M, N)$ as follows. On any clopen piece where $M \cong \underline{A}$ and $N \cong \underline{B}$, use the constant sheaf with fibre $\mathrm{Ext}^1_{\mathbb{Z}}(A, B)$. If new labels are obtained from the old ones by isomorphisms $u : A \to A'$ and $v : B \to B'$, the new coordinate of a class e is $v_*(u^{-1})^* e$. Changes of labels are locally constant, so this rule defines isomorphisms of sheaves on overlaps. They satisfy the cocycle condition by functoriality of pullback and pushout of extensions, and they determine $\mathscr{E}^1(M, N)$ independently of the chosen presentation. In particular,

$$\mathscr{E}^1(M, N)_x \cong \mathrm{Ext}^1_{\mathbb{Z}}(M_x, N_x).$$

Theorem 6.2 (First extensions)**.** *There is a natural isomorphism of abelian groups*

$$\mathrm{Ext}^1_{\mathrm{CAb}(K)}(M, N) \xrightarrow{\sim} \Gamma\big(K, \mathscr{E}^1(M, N)\big).$$

For constant finite group sheaves, this specializes to

$$\mathrm{Ext}^1_{\mathrm{CAb}(K)}(\underline{A}, \underline{B}) \cong \mathrm{LC}\big(K, \mathrm{Ext}^1_{\mathbb{Z}}(A, B)\big).$$

The first extension group also agrees with that computed in $\mathbf{Ab}(\mathbf{Sh}(K))$.

Proof. Given an extension $0 \to N \to E \to M \to 0$ in $\mathrm{CAb}(K)$, choose a common finite clopen partition on which its entire diagram is constant. Its finite-group extension class on each part gives a section of $\mathscr{E}^1(M, N)$. Refinement and isomorphism of extensions preserve this section. The construction is additive because Baer sums are formed using finite direct sums, pullbacks, and pushouts, all computed on common refinements.

Conversely, a section of $\mathscr{E}^1(M, N)$ becomes constant on a finite clopen refinement of a partition trivializing M, N. Choose a finite-group extension representing its value on each part and glue these extensions over the disjoint clopen partition. This proves surjectivity. If two extensions determine the same section, trivialize both diagrams on a common partition. On

every nonempty part, their extension classes agree, so there is an isomorphism of finite-group extensions fixing both endpoints. These isomorphisms glue, proving injectivity. Pullbacks and pushouts establish naturality in the two variables.

The assertion about ambient first extensions follows from Proposition 6.1: every ambient extension has coherent middle object, and the subcategory is full. □

For a cyclic quotient the formula becomes

$$\mathrm{Ext}^1_{\mathrm{CAb}(K)}(\underline{\mathbb{Z}/n\mathbb{Z}}, \underline{B}) \cong \mathrm{LC}(K, B/nB).$$

In particular, take $K = \beta\mathbb{N}$, $n = p$ prime, and $B = \mathbb{F}_p$. Every function $\mathbb{N} \to \mathbb{F}_p$ has a unique continuous extension to $\beta\mathbb{N}$. Restriction therefore identifies the global extension group with the infinite product $\prod_{j\geq 1} \mathbb{F}_p$.

6.2 Vanishing of higher Yoneda extensions

For a point, higher vanishing also follows by primary decomposition from Brion's homological dimension formula for finite abelian p-groups with a continuous group action, applied to the trivial acting group [Bri19, Lemma 3.10]. We give a lifting construction in **FinAb** whose diagrams can be glued over a clopen partition of K.

Lemma 6.3. *Let $H \subseteq B$ be finite abelian groups. Every extension $0 \to N \to E \to H \to 0$ of finite abelian groups is the pullback of an extension $0 \to N \to G \to B \to 0$ with G finite.*

Proof. Choose a surjection $q : F \to B$ with F a free abelian group of finite rank, and set $R = \ker(q)$ and $L = q^{-1}(H)$. The subgroup $L \subseteq F$ is free. Lifting a basis of L to E gives a homomorphism $\sigma : L \to E$ lifting $q|_L : L \to H$. For $r \in R$, the element $\sigma(r)$ lies in N. Write $\alpha : R \to N$ for this homomorphism. Put

$$G = (N \oplus F)/\{(-\alpha(r), r) : r \in R\}.$$

The inclusion of N and projection to $F/R = B$ give an exact sequence $0 \to N \to G \to B \to 0$. In particular $|G| = |N||B|$, so G is finite. The inverse image of H is the quotient of $N \oplus L$ by the same relation subgroup. The map $(n, l) \mapsto n + \sigma(l)$ identifies this quotient with E, compatibly with N and H. □

Lemma 6.4. *For finite abelian groups A, N and every $n \geq 2$, $\mathrm{Ext}^n_{\mathbf{FinAb}}(A, N) = 0$.*

Proof. Consider a two-extension

$$0 \longrightarrow N \longrightarrow E \xrightarrow{d} B \xrightarrow{q} A \longrightarrow 0,$$

and set $H = \ker(q)$. By Lemma 6.3, the extension of H by N is the restriction of an extension $0 \to N \to G \xrightarrow{\pi} B \to 0$ with G finite. Identify E with $\pi^{-1}(H)$ and write $j : E \hookrightarrow G$. There is a morphism of two-extensions from the displayed sequence to

$$0 \longrightarrow N \longrightarrow G \xrightarrow{g \mapsto (\pi(g), 0)} B \oplus A \xrightarrow{\mathrm{pr}_2} A \longrightarrow 0,$$

whose middle maps are j and $b \mapsto (b, q(b))$, and whose endpoint maps are identities. The latter two-extension represents zero because its last short exact factor is split. This proves degree-two vanishing by a diagram entirely in **FinAb**. Every longer Yoneda extension factors as a product of short extensions through its finite intermediate images. Grouping two adjacent factors and applying degree-two vanishing proves the assertion for all $n \geq 2$. □

Theorem 6.5 (Higher extensions)**.** *For every profinite K, all $M, N \in \mathrm{CAb}(K)$, and all $n \geq 2$,*

$$\mathrm{Ext}^n_{\mathrm{CAb}(K)}(M, N) = 0.$$

Proof. An extension diagram has finitely many terms and arrows, and therefore becomes a diagram of finite abelian groups on a common finite clopen partition. For a two-extension, the construction in Lemma 6.4 gives, on each part, a morphism to a two-extension with split last factor. Gluing these finitely many diagrams gives the same equivalence in $\mathrm{CAb}(K)$, together with a global splitting of the last factor. Consequently every two-extension is zero. The factorization of a longer extension through its images takes place in the abelian category $\mathrm{CAb}(K)$, so grouping adjacent factors finishes the proof. □

6.3 Base change and primary components

Let $f : L \to K$ be a continuous map of profinite spaces. Inverse image restricts to an exact functor $f^* : \mathrm{CAb}(K) \to \mathrm{CAb}(L)$, as is immediate on finite clopen presentations. The local definition of the extension sheaf gives a canonical isomorphism

$$f^*\mathscr{E}^1(M, N) \cong \mathscr{E}^1(f^*M, f^*N).$$

Under Theorem 6.2, pullback of Yoneda extensions is precisely pullback of global sections along this isomorphism. For constant endpoints it sends a locally constant extension-class map c to $c \circ f$. This compatibility follows directly by pulling back the finite diagrams representing extensions.

Corollary 6.6. *An extension in $\mathrm{CAb}(K)$ splits if and only if its sequence on every stalk splits. The zero object is the only projective object of $\mathrm{CAb}(K)$ and the only injective object of $\mathrm{CAb}(K)$.*

Proof. For the first assertion, the extension class corresponds to a section of $\mathscr{E}^1(M, N)$. Its germ is zero at every point precisely when the stalk extension splits. A section whose every germ is zero is zero, so the original extension splits. Concretely, a common finite clopen presentation makes each extension constant, and finite-group splittings on the pieces glue to a global splitting.

Now let M be a nonzero object. Choose a nonempty clopen subset U on which M is constant with a nonzero finite group A. Choose a prime p dividing $|A|$, and let P be constant with value $\mathbb{F}_p$ on U and zero on $K \setminus U$. The finite-group calculations give

$$\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{F}_p, A) = A/pA \neq 0, \qquad \mathrm{Ext}^1_{\mathbb{Z}}(A, \mathbb{F}_p) \neq 0.$$

For the second inequality, a cyclic primary summand of A of order p^r contributes $\mathbb{F}_p$ to the displayed decomposition of first extension groups. A constant nonzero class on U, extended by zero on its clopen complement, gives nonzero elements of $\mathrm{Ext}^1_{\mathrm{CAb}(K)}(P, M)$ and $\mathrm{Ext}^1_{\mathrm{CAb}(K)}(M, P)$, respectively. Thus M is neither injective nor projective. □

On any nonempty clopen subset, the cyclic extension $0 \to \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p^2\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0$ gives a nonsplit extension of constant sheaves. Extending by zero to the complement gives a nonsplit extension over K.

Corollary 6.7. *Every $M \in \mathrm{CAb}(K)$ has a canonical finite primary decomposition*

$$M \cong \bigoplus_p M_{(p)}.$$

Morphisms preserve these summands. For $p \neq q$,

$$\mathrm{Hom}(M_{(p)}, N_{(q)}) = 0, \qquad \mathrm{Ext}^1_{\mathrm{CAb}(K)}(M_{(p)}, N_{(q)}) = 0.$$

Consequently first extension groups decompose as the direct sum of their same-prime contributions. These decompositions commute with base change.

Proof. Choose a finite clopen presentation of M and an integer $d > 0$ annihilating all its finite fibres. For each prime $p \mid d$, put $M_{(p)} = \ker(p^{v_p(d)} : M \to M)$. The ordinary primary decompositions on the pieces give the asserted isomorphism and its independence of d. The homomorphism assertion follows fibrewise. The formula for finite-group first extensions and Theorem 6.2 give the cross-prime vanishing. Additivity and exact inverse image give the remaining statements. □

6.4 Subobjects and infinitely many scalar operations

Even though its fibres have finite length, an object of $\mathrm{CAb}(K)$ need not be noetherian. Take $K = \beta\mathbb{N}$, fix a prime p, and put $M = \underline{\mathbb{F}_p}$. For $V_n = \{1, \ldots, n\} \subseteq K$, let H_n equal M on V_n and the zero group on its complement. Each V_n is clopen, so $H_n \in \mathrm{CAb}(K)$, while

$$H_1 \subsetneq H_2 \subsetneq \cdots \subseteq M.$$

The ambient union has fibre $\mathbb{F}_p$ on $\mathbb{N}$ and zero on $\beta\mathbb{N} \setminus \mathbb{N}$. Its underlying set sheaf is not locally constant, since $\mathbb{N}$ is not closed. Thus finite fibres neither give the ascending chain condition nor preserve coherence under ambient increasing unions.

An infinite family of scalar operations gives a different phenomenon. Let $R = \mathbb{F}_2[t_1, t_2, \ldots]$ and $K = \{0,1\}^{\mathbb{N}}$. On the constant one-dimensional $\mathbb{F}_2$-vector sheaf, define $t_i \cdot a = x_i a$ at $x \in K$. Every polynomial involves finitely many coordinates, so it acts continuously. The underlying sort is coherent, and Theorem 2.7 applies to this algebraic theory. The fibre at x is the module defined by evaluation $t_i \mapsto x_i$. Distinct points yield nonisomorphic modules because the only automorphism of the underlying one-dimensional $\mathbb{F}_2$-space is the identity. Hence the entire module structure is not constant on any nonempty open subset of K.

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