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Can moving the mouths of a self-supporting wormhole make a shortcut? A no-shortcut theorem in a two-dimensional loop model

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Abstract

The Morris–Thorne–Yurtsever route to a time machine moves one wormhole mouth so that a twin-paradox lag becomes a time shift between the mouths. For wormholes supported by their own Casimir-like energy (Maldacena–Milekhin–Popov) the static throat is longer than the exterior path; that no motion can turn it into a shortcut is the standard expectation, since a shortcut would require matter violating the achronal averaged null energy condition, as those authors state; we know of no explicit dynamical calculation for moving mouths. We study a toy model in which the question can be posed exactly: a JT-gravity throat whose two boundaries are transparent to conformal matter that also propagates through a rigid flat exterior. We prove that in this model no timelike motion of the mouths produces a shortcut through the throat in either direction, as long as both mouth clocks keep running. The proof combines (i) a lap identity, by which there is no shortcut iff a Dirichlet operator built from one mouth's Schwarzian energy is non-negative on each lap of a matter stream; (ii) a sum rule equating the averaged null energy along each complete null geodesic of the throat to $-\frac{\beta}{2}$ times the sum of the clock rates at its ends, which is the dynamical form of a relation of Maldacena and Qi and is not claimed as new; and (iii) a variational step. With the quantum inequality for the streams the same steps exclude a shortcut for any agent acting locally on the streams in the exterior between the mouths, for any timelike motion of the mouths. Two partial lower bounds on the size of the margin are given; no positive floor is proven. Simulations from one code base illustrate the results. The model has a non-gravitating exterior and a classical Schwarzian mode, and nothing is claimed about four dimensions. The derivations, code and text were produced with AI assistance under the author's direction; Section 7 states what was checked and by whom. No physicist has checked this work.

1 Model

The throat is the global AdS_2 strip $\mathrm{d}s^2 = (-\mathrm{d}t^2 + \mathrm{d}\sigma^2)/\sin^2\sigma$, $0 < \sigma < \pi$, in JT gravity [1, 2]; every null ray crosses it in bulk time π. The mouths L, R are the two boundaries, with clocks $t_L(T)$, $t_R(T)$ as functions of exterior time T. Conformal matter of central charge c passes freely through the boundaries and through a flat exterior segment of light time d, forming two chiral streams around the loop, as in the two-dimensional reduction of [3]. We use units $c/24\pi = 1$. Each mouth carries the Schwarzian energy

$$E_i = -\beta\, S_i, \qquad S_i \equiv \{\tan(t_i/2), T\}, \qquad \frac{\mathrm{d}E_i}{\mathrm{d}T} = T_{\mathrm{in}} - T_{\mathrm{out}}, \tag{1}$$

where β is the renormalized dilaton in these units and the fluxes are measured in exterior time. This is the balance for a mouth at rest; for a moving mouth with rapidity η the balance in its

proper time u is $\mathrm{d}E/\mathrm{d}u = T_{\rm in} - T_{\rm out} + 2\eta''$. Each stream is the vacuum of a uniformizing phase w with period $P = 2\pi/\kappa$ per lap, so its flux in a coordinate y is $-\{\tan(\kappa w/2), y\}$. The static solution has $t_i = T$, $\beta = \frac{\pi}{2}(\kappa^2 - 1)$ and $P = \pi + d$; it exists for $\kappa > 1$, which is the statement that the throat is longer than the exterior [3]. The static margin is $m_0 = \pi - d = 2\pi - P$. A mouth is moved by prescribing its exterior worldline, which reparametrizes the streams by the Doppler map; a "kick" does work on the system.

A throat ray entering L at U exits R at bulk time $t_L(U) + \pi$. It is a *shortcut* if it exits before the exterior ray that left L at U arrives at R.

2 Exact criterion

Let $b(T)$ be the phase of the right-moving stream element emitted from L at T, and $F(U)$ the time at which the element emitted at U returns to L after one lap (exterior $L \to R$, then throat $R \to L$), so $b(F(U)) = b(U) + P$.

Lemma 1 (Lap identity)**.** *The bulk-time margin of the $L \to R$ throat ray entering at U is $\Phi_{LR}(U) = 2\pi - [t_L(F(U)) - t_L(U)]$. There is no shortcut iff the clock of mouth L advances by at most 2π during every lap of each stream past L.*

Proof. The exterior ray from L at U reaches R at bulk time $\tau_R(U)$; the stream element travelling with it crosses the throat and emerges at L at $F(U)$ with $t_L(F(U)) = \tau_R(U) + \pi$. Hence $\Phi_{LR} = t_L(U) + \pi - \tau_R(U) = 2\pi - [t_L(F(U)) - t_L(U)]$. The other stream gives the $R \to L$ rays in the same way. □

Proposition 1 (Eigenvalue criterion)**.** *Assume $t_L' > 0$ and $t_R' > 0$ on the lap (a rate reaching zero is closure of the throat, outside this statement), and let T in this proposition denote the proper time of mouth L, which is exterior time when L is at rest; for a moving mouth E_L is defined with its proper time, and the lap interval is to be read in that parameter. $\Phi_{LR}(U) \geq 0$ iff the operator $-\mathrm{d}^2/\mathrm{d}T^2 + E_L(T)/2\beta$ with Dirichlet conditions on $[U, F(U)]$ is non-negative, i.e. iff*

$$\int_U^{F(U)} E_L\, \psi^2 \,\mathrm{d}T \;\geq\; -2\beta \int_U^{F(U)} \psi'^2 \,\mathrm{d}T \qquad \text{for all } \psi \text{ with } \psi(U) = \psi(F(U)) = 0. \tag{2}$$

Proof. The functions $t_L'^{-1/2} \sin \frac{t_L - t_0}{2}$ solve $\psi'' + \frac{1}{2} S_L \psi = 0$. The solution vanishing at U vanishes next exactly when t_L has advanced by 2π. By Sturm oscillation theory the Dirichlet operator $-\mathrm{d}^2/\mathrm{d}T^2 - \frac{1}{2}S_L$ on $[U, F]$ is non-negative iff that solution has no zero in (U, F), i.e. iff $t_L(F) - t_L(U) \leq 2\pi$. Use Lemma 1 and $-\frac{1}{2}S_L = E_L/2\beta$. □

With $w = \psi^2$, (2) reads $\int E_L\, w \geq -\frac{\beta}{2} \int w'^2/w$, which has the form of the Flanagan / Fewster–Hollands inequality [6, 7] with $c'/24\pi = \beta$. Since β is dimensionful, this comparison is formal. The precise statement is: $E_L/\beta = -\{\tan(t_L/2), T\}$ is the flux of a unit-normalized $\kappa = 1$ cylinder vacuum uniformized by t_L, and (2) says that a lap is no longer than one period of that vacuum. The stream is the κ vacuum whose period is exactly one lap, which is why it saturates its own version (lowest Dirichlet eigenvalue exactly zero).

Lemma 2 (Pointwise sufficient condition)**.** *Let the stream be on the κ-vacuum orbit, with flux $-\{\tan(\kappa w/2), \cdot\}$. If $E_L(T)/\beta \geq T_{UU}^{\rm out}(T)$ throughout a lap, then $\Phi_{LR} \geq 0$ for that lap (and likewise with the incoming flux for Φ_{RL}).*

Proof. $T^{\rm out}_{UU} = -S_b$ with $S_b = \{\tan(\kappa b/2), T\}$, and the Dirichlet solution of $\psi'' + \frac{1}{2}S_b\psi = 0$ from U vanishes next at $F(U)$. If $S_L \le S_b$, Sturm comparison puts the next zero of the t_L-solution at or after $F(U)$. □

Statically the condition is $-\frac{1}{2} \ge -\frac{\kappa^2}{2}$, i.e. $\kappa > 1$.

Remark 1. Lemma 1 and Proposition 1 are exact for arbitrary mouth motion and arbitrary matter state; Lemma 2 needs the stream on the vacuum orbit. They are kinematic: they restate the question as a bound on E_L over a lap and do not show that the bound holds. In the equations it can be violated by prescribing a negative-energy-first flux into a mouth; Proposition 4 shows that the profiles that do so are not stress profiles of any state. The question addressed below is whether *moving a mouth*, which only Doppler-reparametrizes the streams and does positive work, can violate (2). Proposition 2 answers this in the negative, as long as the clocks run and the streams are unsourced.

3 The theorem

Use bulk time t as the variable at each mouth and let $q_i(t) = \mathrm{d}t/\mathrm{d}u > 0$ be the rate of mouth clock i per unit of its proper time. Let $T^+(t)$ be the bulk stress of the stream that enters the throat at L at bulk time t, $T^-(t)$ that of the stream arriving at L at t, and $\tau^\pm = T^\pm + \frac{1}{2}$. Define

$$J^\pm(a) = \tfrac{1}{2}\int_a^{a+2\pi} \tau^\pm(s)\,[1-\cos(s-a)]\,\mathrm{d}s. \tag{3}$$

$J^-(a)$ refers to the right-moving null geodesic from (L,a) to $(R,a+\pi)$ and $J^+(a)$ to the left-moving one from $(R,a+\pi)$ to $(L,a+2\pi)$. The latter meets the stream element that entered at s at $\sigma = (a+2\pi-s)/2$. The affine measure is $\mathrm{d}\sigma/\sin^2\sigma$, and the integrand $T_{\lambda\lambda}\,\mathrm{d}\lambda = T\sin^2\sigma\,\mathrm{d}s$ carries the weight $\sin^2\sigma = \frac{1}{2}[1-\cos(s-a)]$. So $J^\pm$ is the averaged null energy, with the $\frac{1}{2}$ subtracted, along a complete boundary-to-boundary null geodesic.

Lemma 3 (Linear mouth equations)**.** $\{\tan(t/2), u\} = \frac{1}{2}q^2 + qq'' - \frac{1}{2}q'^2$ *(primes are* $\mathrm{d}/\mathrm{d}t$*), and the mouth equations are*

$$\beta\,(q_L''' + q_L') = \tau^-(t) - \tau^+(t), \qquad \beta\,(q_R''' + q_R') = \tau^+(t-\pi) - \tau^-(t+\pi). \tag{4}$$

Proof. Writing a dot for $\mathrm{d}/\mathrm{d}u$ and a prime for $\mathrm{d}/\mathrm{d}t$, $\dot t = q$, $\ddot t = qq'$, $\dddot t = q(q'^2 + qq'')$, and one finds $\{t,u\} = qq'' - \frac{1}{2}q'^2$, and $\mathrm{d}S/\mathrm{d}u = q^2(q''' + q')$. The Schwarzian balance is intrinsic to the boundary: in terms of the *bulk* stresses on the throat side, $\mathrm{d}E_L/\mathrm{d}u = q_L^2(T^+ - T^-)$, with no further term, for a mouth at rest or moving. The term $2\eta''$ of Section 1 arises only when the same balance is rewritten with the fluxes measured in the exterior frame (if the fluxes are evaluated directly in the proper time u there is no extra term, and adding $2\eta''$ would count it twice): the two streams are related to the mouth's proper time by $\mathrm{d}U/\mathrm{d}u = e^{-\eta}$, $\mathrm{d}V/\mathrm{d}u = e^{\eta}$, whose Schwarzians differ by $2\eta''$. It is therefore already contained in the equations below and is not dropped. Divide by q^2. At R the roles of the streams are exchanged and their arguments shifted by the crossing time π. □

Lemma 4 (Sum rule)**.** *Assume (i) the two linear mouth equations of Lemma 3 hold, with* q_i *of class* C^3*, throughout the history up to the anchors, (ii) both clocks exist and are smooth at* a*,* $a+\pi$*,* $a+2\pi$*, and (iii) the* $SL(2)$ *gauge constraint* $G^1 = G^{\cos} = G^{\sin} = 0$ *defined in the proof. Then*

$$J^+(a) = -\frac{\beta}{2}\big[q_R(a+\pi) + q_L(a+2\pi)\big], \qquad J^-(a) = -\frac{\beta}{2}\big[q_L(a) + q_R(a+\pi)\big]. \tag{5}$$

In particular $J^{\pm} < 0$ when the clock rates at the anchors are positive. A throat that was static in the past with $2\beta = \pi(\kappa^2 - 1)$ satisfies (iii), for any subsequent history of the exterior, as long as the mouth equations hold in between (no closure).

Proof. For ξ with $\xi''' + \xi' = 0$ define the boundary charge $Q_i^{\xi} = \beta[\xi(q_i + q_i'') - \xi' q_i' + \xi'' q_i]$; then $\mathrm{d}Q_i^{\xi}/\mathrm{d}t = \xi\,\beta(q_i''' + q_i')$. Take $f \in \mathrm{span}\{1, \cos t, \sin t\}$ at L and $f_R(t) = f(t+\pi)$ at R, and for a slice joining (L, t_1) to (R, t_2) let

$$G^f = Q_L^f(t_1) + Q_R^{f_R}(t_2) + \int_{t_2-\pi}^{t_1} \tau^+ f \,\mathrm{d}s + \int_{t_1}^{t_2+\pi} \tau^- f \,\mathrm{d}s.$$

By Lemma 3, $\partial G^f/\partial t_1 = f(\tau^- - \tau^+) + f(\tau^+ - \tau^-) = 0$ and $\partial G^f/\partial t_2 = f(t_2+\pi)[\tau^+(t_2-\pi) - \tau^-(t_2+\pi)] - \tau^+(t_2-\pi)f(t_2-\pi) + \tau^-(t_2+\pi)f(t_2+\pi) = 0$, since f is 2π-periodic. So $G^1, G^{\cos}, G^{\sin}$ are three constants, the total $SL(2)$ charges of throat plus streams. For $f_a = 1 - \cos(t-a)$ one has $f_a = f_a' = 0$, $f_a'' = 1$ at the anchors, so $Q = \beta q$ there. The slice $(L, a+2\pi)$–$(R, a+\pi)$ gives $2J^+(a) + \beta[q_R(a+\pi) + q_L(a+2\pi)] = G^{f_a}$ and the slice (L, a)–$(R, a+\pi)$ gives $2J^-(a) + \beta[q_L(a) + q_R(a+\pi)] = G^{f_a}$, with $G^{f_a} = G^1 - \cos a\, G^{\cos} - \sin a\, G^{\sin}$, which vanishes under (iii). In the static solution $q = 1$, $\tau^{\pm} = \frac{1}{2}(1-\kappa^2)$: $G^1 = 2\beta - \pi(\kappa^2-1) = 0$ and $G^{\cos} = G^{\sin} = 0$. □

The mouth equations alone give only the conservation of G; that $G = 0$ is the gauge constraint of JT gravity and must be imposed on the data. Off the constraint the right-hand sides of (5) acquire $\frac{1}{2}G^{f_a}$, e.g. the constant $\beta - \frac{\pi}{2}(\kappa^2-1)$ for a stiffened static throat. Sources acting on the streams in the exterior do not change G; an agent term added to a mouth equation does. For the static two-sided solution, (5) reduces to eq. (2.10) of Maldacena and Qi [1], $-2\phi_r = \int T$ along a null line. The sum rule is that relation with dynamical boundary rates; we do not claim it as new.

Remark 2 (Positioning). With the Poincaré-type coordinate $X = -\cot\frac{s-a}{2}$ one has $\mathrm{d}X = \mathrm{d}s/[1 - \cos(s-a)]$ and $\{X, s\} = \frac{1}{2}$, so $2J = \int T_{XX}\,\mathrm{d}X$: the ANEC integral along a complete null geodesic, and (5) says that it equals $-\beta(q+q')$ with q, q' the clock rates at its two ends. The logic of this note is therefore the familiar one: negative averaged null energy is allowed on a complete null geodesic only if that geodesic is not achronal (self-consistent achronal ANEC [5]), and a wormhole supported this way is longer than the outside [4, 3]. What is added here is that in this model the achronality step is proved (Rayleigh quotient plus lap identity) instead of assumed, for arbitrary mouth motion, in a form that can be checked exactly.

The sum rule itself does not need conformal matter. The constraint $G^f = Q_L^f + Q_R^{f_R} + M^f = 0$ uses only that M^f is the Killing charge of the matter and $\mathrm{d}Q_i^f/\mathrm{d}t$ the Killing flux through the mouth, and on the null ray $t = a + \sigma$ from (L, a) to $(R, a+\pi)$ the weight of f_a is the affine measure, $\lambda = -\cot\sigma$. Hence, for any matter in the throat whose stress tensor is conserved on the fixed AdS_2 background,

$$\beta\,[q_L(a) + q_R(a+\pi)] = -\int T_{kk}\,\mathrm{d}\lambda$$

along that ray: a positive sum of the two anchor rates is equivalent to a negative affine ANEC integral across the throat (one clock alone is not controlled by one ray). Conformal free transport enters only afterwards, in the Rayleigh step and the lap identity. For massive or interacting matter of that kind the no-shortcut question in this model is therefore exactly the achronal ANEC question on the complete ray [5, 8], which we do not answer. The crossing segment is affinely complete in the throat geometry, so it is already a complete null geodesic; achronality is a statement about the glued spacetime (throat against exterior), and exterior matter enters the argument only there. In operator language the left side is the parabolic $SL(2)$ generator $Q^{f_a} = Q^1 - \cos a\, Q^{\cos} - \sin a\, Q^{\sin}$ of a mouth

evaluated as $\beta q(a)$; a parabolic generator is positive in a positive discrete-series representation, so if the mouth states lie in such representations the positivity of the clock rate is automatic there. The classical identity was checked symbolically; the operator positivity is inferred from the representation content, is not proven for the coupled system, and we draw no conclusion from it. (The general form has two internal derivations in the sense of Section 7. A numerical check exists for conformal matter only: 60 anchors per run at rest after an excursion, residual below 5×10^{-5}, 5×10^{-4}, 10^{-4} for $\kappa = 1.2$, 1.5, 1.05 with the integral ranging over 0.57–3.24, 2.37–6.24, 0.08–1.32.) It does not extend to matter coupled to the dilaton, such as the s-wave reduction of four-dimensional fields: there the background is no longer AdS_2 and the charge acquires a history-dependent term of no definite sign, of relative order ϕ/ϕ_0, the order of the other corrections to JT gravity (an estimate, not derived here).

Proposition 2 (No shortcut)**.** *In the model of Section 1, on the constraint surface $G = 0$ (in particular for a throat that was static in the past), with the streams unsourced (each stream phase is carried along the light rays around the loop) and in the vacuum of Section 1 (each phase κw gains exactly 2π per lap), no timelike motion of the mouths produces a shortcut in either direction while $q_L, q_R > 0$. Precisely: the throat ray with anchors a, $a+\pi$, $a+2\pi$ is not a shortcut if both clocks exist with positive rates through that window and until the stream element of its lap has returned. Windows reaching past a stopped clock are treated in the extension that follows the proof.*

Proof. For an unsourced stream κ is a conserved monodromy: the phase gains exactly $2\pi/\kappa$ per lap of a ray. The bulk stress of a stream with phase $w(t)$ is $T = -\{\tan(\kappa w/2), t\}$, so $w'^{-1/2}\sin\frac{\kappa(w-w_0)}{2}$ solves $\psi'' - \frac{1}{2}T\psi = 0$ and vanishes each time κw advances by 2π. The Rayleigh quotient of $-\mathrm{d}^2/\mathrm{d}t^2 + \frac{1}{2}T$ on $[a, a+2\pi]$ with the Dirichlet trial function $\sin\frac{t-a}{2}$ is

$$\frac{\int(\frac{1}{4}\cos^2\frac{t-a}{2} + \frac{1}{2}T\sin^2\frac{t-a}{2})\,\mathrm{d}t}{\int\sin^2\frac{t-a}{2}\,\mathrm{d}t} = \frac{J(a)}{2\pi} < 0$$

by Lemma 4. The lowest Dirichlet eigenvalue is therefore negative, so the solution vanishing at a vanishes again before $a+2\pi$: the stream phase κw advances by 2π in less than 2π of bulk time at L. By Lemma 1, applied to the stream leaving the throat at L (J^-) for $L\to R$ rays and to the stream entering at L (J^+) for $R\to L$ rays, $\Phi > 0$. □

Extension: rays that exit before a clock stops. Let mouth L be at rest, so that exterior time T is its proper time, let the throat be static in the past, and keep the hypotheses of Proposition 2 ($G = 0$, the mouth equations of Lemma 3, unsourced streams in the single-winding vacuum of Section 1). (a) The $L\to R$ throat ray entering L at exterior time T_e and leaving R at T_B has positive margin if the solution exists with $q_L, q_R > 0$ at all exterior times up to $\max(T_e, T_B)$. (b) The $R\to L$ throat ray entering R at T_r and leaving L at T_X has positive margin if the solution exists with $q_L, q_R > 0$ at all exterior times up to $\max(T_r, T_X)$. With a the bulk time of L at the first anchor of the window, in (a) neither clock need reach $a+2\pi$, and in (b) the clock of R need not. This is positivity only; no bound on the size of the margin is claimed, and nothing is claimed for sourced streams or exterior agents.

Proof. (a) Let $a = t_L(T_e)$. In the proof of Lemma 4 the slice (L,a)–$(R,a+\pi)$ uses the mouth equations only up to these two anchors, which are reached, and the stream integral in $J^-(a)$ is over the stream inside the throat on that slice, labelled by its null coordinate $s\in[a,a+2\pi]$; so $J^-(a) = -\frac{\beta}{2}[q_L(a) + q_R(a+\pi)] < 0$ whether or not either clock reaches $a+2\pi$. As in the proof of Proposition 2, the Rayleigh quotient is $J^-(a)/2\pi < 0$, so the solution of $\psi'' - \frac{1}{2}T^-\psi = 0$ vanishing at a has a first zero $s_1\in(a,a+2\pi)$, and there the Hill phase W of the stream ($T^- = -\{\tan(W/2), s\}$)

has gained exactly 2π. The stream element with label s_1 entered the throat at R at bulk time $s_1 - \pi < a + \pi$; the clock of R is strictly increasing up to T_B and, by the static past, passes every earlier value, so this happens at one event A' on the worldline of R before the exit of the throat ray. Before A' the element travelled through the exterior from L, and since the stream is unsourced and singly wound its phase at emission was one lap, 2π, behind $W(s_1)$, hence equal to $W(a)$. The phase of the emitted stream is strictly increasing on the clock of L while $q_L > 0$ and the stress is finite, and both the emission time and T_e are at most $\max(T_e, T_B)$; so the element was emitted at T_e, and A' is the arrival at R of the exterior ray that leaves L with the throat ray. That arrival is at bulk time $s_1 - \pi < a + \pi$ of R, i.e. before the throat ray exits: $\Phi = a + 2\pi - s_1 > 0$ and the exterior-time margin is positive. Nothing at bulk time $a + 2\pi$ of either clock was used. (b) Mirror, with $J^+(a) = -\frac{\beta}{2}[q_R(a+\pi) + q_L(a+2\pi)]$ from the slice $(L, a+2\pi)$–$(R, a+\pi)$, for the ray entering R at bulk time $a + \pi$ and leaving L at $a + 2\pi$. The Hill phase W^+ of the stream entering the throat at L gains 2π at a first $s_1 \in (a, a + 2\pi)$. The element entering L at s_1 came through the exterior from an event B on the worldline of R, where by single winding $W^+(t_R(B) - \pi) = W^+(a)$, so $t_R(B) = a + \pi$. Both B (which precedes T_X) and the entry event lie at exterior times up to $\max(T_r, T_X)$, where the clock of R is strictly increasing; so B is the entry event, and the exterior ray leaving R with the throat ray reaches L at bulk time $s_1 < a + 2\pi$, before the throat ray exits. □

An external automated check found a missing hypothesis and a gap in the mirror step of an earlier form of this argument; both are repaired here. Numerically, 375 rays in the $L \to R$ direction and 102 in the $R \to L$ direction whose window reaches past a stopped clock (four trajectories, $\kappa = 1.2003$, exact kinematics, step 10^{-3}, $|G| < 10^{-3}$ at the events of each ray) all have positive margin.

Remark 3. The single-winding condition is necessary. A static stream wound twice (stress $-c^2/2$ with $c = 4/3$ on a lap of bulk length 3π, $q = 7\pi/(18\beta)$, $qd = 2\pi$) satisfies $G = 0$ with unsourced streams and has margin $-\pi/q < 0$. It is not a state of the model of Section 1 and cannot be reached from the static vacuum past without sources. This example was found by an external automated check.

Proposition 3 (Several species)**.** *Let species $i = 1, \dots, n$ (or a continuum of field lines), with weights $f_i > 0$, $\sum f_i = 1$, circulate unsourced through the same throat and the same pair of clocks, each along its own exterior path (the streams are in the vacuum of Section 1: each phase $\kappa_i w_i$ gains exactly 2π per lap). On the constraint surface $G = 0$ and while both clocks run, every throat ray is beaten by at least one exterior route that carries a field (for a continuum, by a set of field lines of positive weight). Hence the throat is never faster than every exterior route.*

Proof. The mouth equations are those of Lemma 3 with $\tau^\pm = \sum_i \tau_i^\pm$, $\tau_i = T_i + \frac{1}{2} f_i$, so Lemma 4 holds for $\sum_i J_i$. The lap identity holds for each species with the same window $[a, a + 2\pi]$, and since $T_i = -f_i\{\tan(\kappa_i w_i/2), t\}$ the Rayleigh quotient of $-\mathrm{d}^2/\mathrm{d}t^2 + T_i/2f_i$ is $J_i/(2\pi f_i)$. As $\sum_i J_i < 0$, some $J_i < 0$, and for that species $\Phi_i > 0$. □

Remark 4. The statement does not hold species by species: a light species on a short path can have $J_i > 0$, and this occurs in the runs. Each lap identity is transport along that species' own path, so paths whose lengths change differently under the motion are covered by the argument, but that case was not simulated. Species confined to the throat (reflecting at the mouths) drop out of the mouth equations but enter G and the sum, and for them $J \geq 0$: a confined species completes one lap in exactly 2π of bulk time, so by the inequality used in Proposition 4 the Dirichlet operator $-\mathrm{d}^2/\mathrm{d}t^2 + T_i/2f_i$ is non-negative on $[a, a + 2\pi]$ and its Rayleigh quotient $J_i/(2\pi f_i)$ is non-negative. On the vacuum orbit, $T_i = -f_i\{\tan(\phi/2), t\}$ with $\phi(t + 2\pi) = \phi(t) + 2\pi$, this needs

no quantum inequality: $\phi'^{-1/2}\sin\frac{\phi(t)-\phi(a)}{2}$ is a zero mode without interior zeros, so the lowest Dirichlet eigenvalue is exactly zero (this short argument is due to one of the external automated checks of Section 7). Hence the circulating species alone have $\sum J_i < 0$ and the proposition holds with confined species present. This extension has had less internal checking than the rest of this section. The argument needs one common pair of clocks. It was tested on about 1800 rays with no violation; in the hardest multi-species run the sum rule holds to 1.2×10^{-3}.

Proposition 4 (Local exterior agents)**.** *Let agents act on the conformal streams inside a region of the exterior (absorbing, delaying, re-emitting, measuring and feeding forward), so that the streams are no longer pure transports of a vacuum phase. The mouths may move along arbitrary timelike worldlines. Assume: the agents are confined to a timelike worldtube in the exterior strictly between the two mouths and cannot signal faster than light there (they may use the throat); outside the agent region the streams are the chiral conformal fields of the model; the evolution is smooth; and the throat is on the constraint surface $G = 0$ with positive margins initially (e.g. a static past). Then no throat ray is a shortcut while both clocks run.*

Proof. The inequality. In units $c/24\pi = 1$ the quantum inequality of Flanagan and Fewster–Hollands [6, 7] reads $\int T\, g^2\, \mathrm{d}z \geq -2\int g_z^2\, \mathrm{d}z$, with z an inertial null coordinate. It is an operator inequality for the chiral stress on an interval, valid in every state in its domain (Hadamard-type states), so it does not matter how the agents prepared the stream, provided the state they prepare lies in that domain. The clocks are classical here, so the proposition is a semiclassical statement. Under a reparametrization $z(x)$ with $g = f\, z'^{1/2}$ one finds $\int g_z^2\, \mathrm{d}z = \int (f'^2 - \frac{1}{2}\{z, x\} f^2)\, \mathrm{d}x$, while $T_x = z'^2 T_z - \{z, x\}$, the same law by which the stream is glued across a mouth; the Schwarzian terms cancel and $\int T_x f^2\, \mathrm{d}x \geq -2\int f'^2\, \mathrm{d}x$ in any coordinate along the stream. So the Dirichlet operator $-\mathrm{d}^2/\mathrm{d}x^2 + \frac{1}{2}T_x$ is non-negative on any interval of the stream that lies on one spacelike curve in the agent-free region. Here $T = \tau - \frac{1}{2}$ is the bulk stress at the mouth, in bulk time.

Where it applies. Write E and E_{RL} for the exterior light maps from L to R and from R to L, and B, C for the throat maps (each throat ray takes bulk time π); all are increasing while the clocks run, and no constant light time is used below. The margins are $m_{LR}(l) = B(l) - E(l)$ and $m_{RL}(r) = C(r) - E_{RL}(r)$, read at departure on the clock of the emitting mouth, and Lemma 1 for the ray lap starting at l concerns the stream passing L during $[l, F(l)]$ with $F = C \circ E$. Take a point P in the agent-free collar between L and the worldtube, the stream element through P, the time l at which it passed L, $r = E(l)$, and the time l' at which the left-moving null ray through P reaches L. The curve Σ made of the null segment from P to l' and a throat segment from l' to r carries exactly the stream that passes L during $[l, F(l)]$, provided $C(r) \geq l'$, and it avoids the agents. It is achronal, so that the inequality applies on it, if and only if (i) $B(l') > r$ and (ii) $C(r) > l'$. Condition (i) follows from $m_{LR}(l) \geq 0$, since $B(l') > B(l) \geq E(l) = r$. Condition (ii) is the statement that the $R \to L$ throat ray leaving r is not too early; it is not implied by the $L \to R$ margin alone. (For mouths at rest it reads $m_{RL}(r) \geq 2a' - 2d$, with a' the light distance of P from L; version 2 of this note omitted it.)

Continuity. Let t_* be the first time on the clock of L at which either an $L \to R$ ray departing from L or an $R \to L$ ray arriving at L has zero margin. If an $L \to R$ ray departing at t_* has zero margin, take $l = t_*$; then $l' = t_* + \delta$ with $\delta \to 0$ as P approaches L. An $R \to L$ ray arriving before t_* has positive margin, and the $R \to L$ margin is continuous in the arrival time and non-negative at t_*, so for P close enough to L condition (ii) holds (a failure would require an $R \to L$ ray arriving within δ of t_* with margin below δ minus the round-trip exterior light time, contradicting positivity before t_* and continuity at t_*); (i) holds strictly. If instead an $R \to L$ ray arriving at t_* has zero margin, the mirror construction in the collar at R applies, with the roles of (i) and (ii) exchanged and the $L \to R$ margin at t_* non-negative by continuity. In either case the inequality holds on the

closed lap as a limit of sub-intervals. The mouth equations involve only bulk stresses, so Lemma 4 is unaffected by the agents, and the Rayleigh quotient on $[s, s+2\pi]$ is $J^{\mp}(s)/2\pi < 0$ for every s. If the clock concerned gained 2π or more during the lap, a full interval of length 2π would lie inside it, a contradiction; so it gains strictly less, i.e. the margin at t_* is positive by Lemma 1, whose proof uses only the rays. As the margins are continuous while both clocks run and positive initially, there is no first zero.

Status. The proof for moving mouths was reconstructed after version 2: the gap (a constant exterior light time and a missing condition (ii)) was found by external automated checks, and the main step was found independently by one of them (agy) and by two internal derivations. Inherited and not re-checked: the validity of the quantum inequality on a curve that passes from the exterior into the throat at a mouth, and the continuity of the margins while both clocks run. □

Remark 5. (a) For unsourced streams this is a second proof of Proposition 2 that does not use the vacuum-orbit form of the stress; the unsourced stream saturates the inequality on a full lap, $T = -\kappa^2/2$ on a lap of length $2\pi/\kappa$. (b) No closed causal curve is assumed; its absence follows. A direct argument by contradiction from $m(U) \le 0$ would not do for extended or communicating agents: once a shortcut exists, agents on either side of the free strip could measure early rays of a lap, signal through the throat and act on later rays of the same lap, and the inequality would not constrain that lap. (c) Defining a Hill phase W of any stress profile by $T = -\{\tan(W/2), x\}$, the inequality says that W gains at most 2π per achronal ray lap. In the equations a source that discards the incoming stream and emits a phase-advanced one does produce a shortcut, but such prescribed profiles are not stress profiles of any state: in 219 source runs (24 332 ray laps) all 1789 shortcut laps have $W > 2\pi$ and none has $W \le 2\pi$. For a delay implemented as a transformation of the incoming stream the Hill phase rides the rays, so the gain per lap is exactly 2π and the bound is saturated; this is a derivation and has not been simulated. What was run are monotone step delays with the discarding source (saturated, no shortcut) and delay bumps with that source, which do violate the bound on the rising part when a ray returns while the source is still on, an artefact of that source model. (d) With inequality-violating driving the classical equations of the model do reach a closed null curve with both clocks still running and only logarithmic back-reaction on the clocks. We read this as saying that in this model all the protection resides in the quantum inequality; it is not a protocol. (e) Covered as agent regions: a causal version of a coupling between the mouths (measure at one, signal through the exterior, act at the other) and a localized non-conformal object in the exterior, static or moving, provided light crosses it no faster than c. Excluded by assumption: an instantaneous coupling between the mouths, which is how the deformation of Gao, Jafferis and Wall [4] would appear in this setting (in their construction no exterior connects the two boundaries); agents inside the throat; agents that convert one species into another when the species travel different paths; an exterior with a faster side channel. (f) The checking of the proof is described in its last paragraph; the applicability of the inequality in the states the agents can prepare has no external check. This proposition is the part of the note with the least checking.

Remark 6. The statements hold while both clocks run; what happens when a rate reaches zero (closure) is outside them. The theorem gives the sign of the margin; Proposition 5 below gives partial information on its size. The theorem explains the decoupled-throat control below: for β far above its on-shell value, (5) has the positive constant $\beta - \frac{\pi}{2}(\kappa^2 - 1)$ added, and $J > 0$ needs only a slowdown of the clocks of order $\frac{\pi}{2}(\kappa^2 - 1)/\beta$.

Proposition 5 (Size of the margin)**.** *Under the hypotheses of Proposition 2, let* $[a, a + 2\pi]$ *be a window of bulk time at* L*,* $Q = q_L(a) + q_R(a + \pi)$ *the sum of the anchor rates, and* $W = \kappa w$ *the*

phase of the stream leaving the throat at L, as a function of bulk time, so that $T = -\{\tan(W/2), t\}$.

(i) (Two laps.) Let Φ_1 be the margin of the ray lap that starts at a and Φ_2 that of the ray lap that ends at $a + 2\pi$. Then

$$\Phi_1 + \Phi_2 \;\geq\; 2\pi - \frac{\pi^{3/2}}{\sqrt{\beta Q}}. \tag{6}$$

(ii) (One lap.) Let Φ be the margin of the ray lap that starts at a. Then

$$\Phi \;\geq\; \frac{\beta Q}{(\max W')^3}, \tag{7}$$

the maximum being taken over the window. More generally W may be replaced in (7) by $M \circ W$ for any lifted Möbius map M of the circle W mod 2π (i.e. $\tan(\tilde{W}/2)$ a real Möbius function of $\tan(W/2)$ with positive determinant), chosen separately for each window, the maximum again being over the window:

$$\Phi \;\geq\; \frac{\beta Q}{\inf_M (\max (M \circ W)')^3}. \tag{8}$$

The same holds for the other chirality with $Q = q_R(a + \pi) + q_L(a + 2\pi)$.

Proof. Let $H = -\mathrm{d}^2/\mathrm{d}t^2 + \frac{1}{2}T$ with Dirichlet conditions and λ_1 its lowest eigenvalue on the window. The Rayleigh step in the proof of Proposition 2 and Lemma 4 give $\lambda_1 \leq J^-(a)/2\pi = -\beta Q/4\pi$.

(i) The two laps are $I_1 = [a, a + 2\pi - \Phi_1]$ and $I_2 = [a + \Phi_2, a + 2\pi]$, and $H \geq 0$ on each, with a zero mode. If they do not overlap, $\Phi_1 + \Phi_2 \geq 2\pi$ and there is nothing to prove. Otherwise the overlap has length $\ell = 2\pi - \Phi_1 - \Phi_2$. Let θ be 0 to the left of the overlap, $\pi/2$ to its right and linear across it, and $\chi_1 = \cos\theta$, $\chi_2 = \sin\theta$. For ψ vanishing at the ends of the window, $\chi_i\psi$ is a Dirichlet function on I_i, and the localisation identity

$$\langle \psi, H\psi\rangle = \sum_i \langle \chi_i\psi, H\chi_i\psi\rangle - \int (\chi_1'^2 + \chi_2'^2)\psi^2 \,\mathrm{d}t \;\geq\; -\frac{\pi^2}{4\ell^2}\int \psi^2 \,\mathrm{d}t$$

gives $\lambda_1 \geq -\pi^2/4\ell^2$. Hence $\beta Q/4\pi \leq \pi^2/4\ell^2$, which is (6).

(ii) Write $\psi = W'^{-1/2}\chi(W)$. Since $\frac{1}{2}T = -\frac{1}{2}\{W, t\} - \frac{1}{4}W'^2$, one finds $\langle \psi, H\psi\rangle = \int(\chi_W^2 - \frac{1}{4}\chi^2)\,\mathrm{d}W$ and $\langle \psi, \psi\rangle = \int \chi^2 W'^{-2}\,\mathrm{d}W$. Over the lap $[a, a + 2\pi - \Phi]$ the phase W gains exactly 2π, so over the window it gains $2\pi + \Delta$ with $0 < \Delta \leq \Phi \max W'$. The Poincaré inequality on an interval of length $2\pi + \Delta$ gives

$$\langle \psi, H\psi\rangle \;\geq\; -\Big[\frac{1}{4} - \frac{\pi^2}{(2\pi + \Delta)^2}\Big]\int \chi^2 \,\mathrm{d}W \;\geq\; -\frac{\Delta}{4\pi}\,(\max W')^2\,\langle \psi, \psi\rangle,$$

using $1 - (1 + x)^{-2} \leq 2x$ for $x \geq 0$. So $\beta Q/4\pi \leq -\lambda_1 \leq \Delta(\max W')^2/4\pi \leq \Phi(\max W')^3/4\pi$. For (8): the argument used only that the potential is $-\frac{1}{2}\{\tan(W/2), t\}$, which is unchanged by M, and that W gains 2π on the lap, which is unchanged because $M(W + 2\pi) = M(W) + 2\pi$. □

Remark 7. (a) Static values. In (6): $2\pi - \pi/\sqrt{\kappa^2 - 1}$ against $2m_0 = 4\pi(\kappa - 1)/\kappa$; for $\kappa = 1.2$ this is 1.547 against 2.094. The bound (6) is non-trivial only for $\beta Q > \pi/4$ (static throats with $\kappa > 1.118$) and is empty near $\kappa = 1$. In (7): $W' = \kappa$ and the bound is $\pi(\kappa^2 - 1)/\kappa^3$ against $m_0 = 2\pi(\kappa - 1)/\kappa$; the ratio $(\kappa + 1)/2\kappa^2$ tends to 1 as $\kappa \to 1$, so (7) is sharp where (6) is empty. (b) Kinematics behind (7). Assume in this item that both mouths are at rest in the exterior, so that proper time is

exterior time and the exterior light time is the constant d. Then $\Phi_{LR}(U) = \pi + t_L(U) - t_R(U+d)$, so $\mathrm{d}\Phi/\mathrm{d}U = q_L(U) - q_R(U+d)$ and at an extremum of the margin the rate of L at departure equals the rate of R at the arrival of the exterior ray. The lap map in bulk time is $e(a) = a + 2\pi - \Phi(a)$ with $e'(a) = q_R(U+d)/q_L(U)$, and $W(e(a)) = W(a) + 2\pi$, so $W'(e(a)) = W'(a)\, q_L(U)/q_R(U+d)$: along a chain of laps W' is κ times the accumulated bunching factor of the rays since the static era. The bound (7) is therefore a statement about clock rates only. (c) Numerical test (one code base, one step size per run): (6) on about 520 000 windows ($\kappa = 1.1$ to 1.5, delay kicks and moving mouths), no violation, smallest slack +0.353; (7) on about 60 000 windows of nine moving-mouth runs ($\kappa = 1.05$ to 1.5), no violation, Φ above the bound by a factor 1.16 to 1.40 near static windows and by factors 56 to 512 at the smallest margins, where $\max W'$ is 3.9 to 9.0 (the largest value in any window of these runs is 18.9). Part (ii) was also tested as mathematics on 300 random profiles W', with no violation. So (7) is weak at small margins. (d) What is not shown. Neither bound gives a positive floor on the margin for clock rates bounded away from zero: (6) is empty when $\beta Q \le \pi/4$, and in (7) $\max W'$ is not bounded by the rates and depends on the choice of phase, which is why (8) is the form to use. A drive at the lap frequency shows the difference: in linear theory the clocks do not respond there, and in a run of 40 laps the margin and the rates stayed flat while $\max W'$ grew from 2 to 197, so (7) becomes empty (of order $10^{-7}\beta Q$) without the margin moving. In that run the product of the maximum and minimum of W' stays at 1.34 to 1.41 against $\kappa^2 = 1.44$, i.e. the drive acts on the phase as a boost, and (8) is estimated at about $0.62\,\beta Q$. The form (8) was also tested on 60 random profiles boosted by rapidity up to 3, with no violation. The frequencies at which the clocks respond resonantly are the real roots of the linear response (1.563 for $\kappa = 1.2$, symmetric mode); a drive there saturates, with smallest margin $0.79\, m_0$ and slowest clock rate 0.78. Drives at 2κ and 3κ lie close to further roots and end in closure by a stopped clock, with $\Phi \ge 0.957\, m_0$ up to that point. These driven runs are single runs (two step sizes for one of them). Even (8) does not bound the phase rate by the clock rates. A positive floor on the margin for bounded clock rates is not proven; this is open. The stronger statement $\Phi \ge c_0\, \beta Q$ with a universal constant is false (Q is unbounded while $\Phi < 2\pi$). In all our runs small margins occur only together with slow anchor clocks, $\Phi \ge 0.51\, m_0 \min(q_L(a), q_R(a+\pi))$; this is an observation on twelve undriven runs, and by item (g) it is a statement in the frame of the static past. In the driven runs, a separate and larger set (about 100 runs), the corresponding statement holds only with the smallest rate over the whole history. We do not conjecture either as a bound. (e) With local exterior agents (Proposition 4) the sum rule is unchanged and the Hill phase of the stress gains at most 2π on an achronal ray lap, so the gain over the window is still at most $2\pi + \Delta$ and (7) holds with W the Hill phase, and so does (8) by the same invariance argument; (6) holds as well, since it uses only $H \ge 0$ on the two laps. (f) The two bounds have three internal derivations in the sense of Section 7 and no external check. (g) Frame dependence. The gauged $SL(2)$ acts on the bulk time of both mouths and of the matter together, $t \to M(t)$ with $\tan(M(t)/2)$ a Möbius function of $\tan(t/2)$. Invariant under it are the sign of Φ (a causal relation), everything defined by null rays and proper times (the exterior-time margin, the round-trip time through the throat), the Schwarzians S_i and the energies. The clock rates q_i, hence Q, the phase rate W' and the *value* of Φ are not invariant. With a static past the solution is unique, and all values of Φ, q_i and Q in this note are in the frame fixed by that past ($t_i = T$ initially). After a disturbance the throat can be left ringing about the static solution in a boosted frame: in a run driven at frequency 1 at $\kappa = 1.2$ (one run) q_L swings between 0.60 and 1.67 and Φ/m_0 between 0.61 and 1.61, while the exterior-time margin stays within 10^{-4} of m_0, both Schwarzians within 10^{-3} of $\frac{1}{2}$, and the energy is unchanged. Proposition 5 is covariant and holds in every frame, but a small value of Φ is not by itself a physical statement. Two exact relations to invariant quantities, for mouths at rest (two internal derivations, no external check): $\Phi_{LR}(U) = \bar{q}_R\, m_{\mathrm{ext}}(U)$, with m_{ext} the exterior-time margin

and $\bar{q}_R$ the mean rate of R between the arrival of the exterior ray and the exit of the throat ray; and the round-trip time through the throat, $T_{\rm rt} = 2\pi/\bar{q}_L$ with the mean over the trip, is the distance between conjugate points of $\psi'' + \frac{1}{2}S_L\psi = 0$, so that $T_{\rm rt} \geq 2\pi/\sqrt{2\max S_L}$ by Sturm comparison.

4 Numerical illustration and checks

All simulations are of the full nonlinear closed loop (both mouths, both streams). The theorem and the propositions do not depend on any run; the runs illustrate the size of the effect and test the formulas. They come from one code base, with a second-order and a fourth-order integrator, and have not been independently re-implemented except where stated.

Validity of runs. A run is accepted if the three constraint charges G of Lemma 4, which the integrator does not enforce, stay below 10^{-3}. This test was applied to one representative of each class of run quoted in this note, not to every run. They pass with G between 10^{-5} and 3×10^{-4}, with two qualifications: three outward runs at $\kappa = 1.0002$ carry G of 2×10^{-3} to 8×10^{-3} from mid-run and are marked as marginal where quoted, and three runs at $\kappa = 1.0002$, $v = 0.99$ fail the test and are not used (Section 5). The sum rule (5) holds to 10^{-6}–4×10^{-5} on kicked runs ($\kappa = 1.02$–1.2, v up to 0.9), on a run with an inequality-violating source and a shortcut, on a closing run, and in its off-shell form on stiffened throats. The identities used in the proofs were also checked symbolically.

Clock locking. A mouth is taken on an excursion and returned, accumulating a twin-paradox lag. For slow motion the residual shift between the mouth clocks is zero to the accuracy of the run ($\kappa = 1.2$, lag 0.612: residual 0.0000; lags up to 6.9, exceeding the loop period, were tested), and the margin never falls below the static value by more than the transient ringing. The effect is due to the matter stream: multiplying β by a factor s (stiffening the clock against the stream) gives residual shift 0 for $s = 1, 10$, a crossover for $s \sim 10^2$–10^4, and the full lag imprinted for $s = 10^9$ (invariant shift $\tau = +0.6123$ from circuit times), where the throat then has a genuine shortcut (minimum margin -0.27, stable under refinement). The decoupled case is the Morris–Thorne–Yurtsever behaviour and serves as a positive control.

Slow motion, first order in v. For $v \ll 1$ there is a closed description (item (i) and the simultaneity argument of (ii) have two internal derivations in the sense of Section 7, item (iii) one; all numbers are from the one code base). (i) *Adiabatic family.* For mouths at rest at separation x the static solution with β fixed has both clocks at a rate q determined by

$$2\beta q = \pi\Big[\Big(\frac{2\pi}{\pi + qx}\Big)^2 - 1\Big], \qquad m_{\rm ad}(x) = \frac{\pi}{q} - x > 0,$$

which follows from (5) with the lap of bulk length $\pi + qx$; slowly moving mouths follow this family. For $x \to \infty$, $qx \to \pi$ and $m_{\rm ad} \to 2\beta$: slow mouths far apart keep a margin 2β, above m_0 (1.382 against 1.047 for $\kappa = 1.2$). For $x \to 0$, $q \to 3\pi/2\beta$ and $m_{\rm ad} \to 2\beta/3$, so $m_{\rm ad} < m_0$ when the mouths are closer than in the reference configuration: part of the erosion seen for inward motion is this static effect. Runs: at $x = 4d$ margins 1.254 and 1.249 against 1.249; at $x = 11d$, 1.39 and 1.32 against 1.326; for $\kappa = 1.05$, 0.320 and 0.315 against 0.3155. (ii) *Clock offset.* At equal exterior time in the rest frame of L, with R receding at speed v, $t_L - t_R = qvx/2$ to first order: in the frame at half the rapidity the configuration is reflection symmetric; the adiabatic state is unique (the balance equation in (i) has one solution, its left side increasing and its right side decreasing

in q), hence it is the reflection-symmetric one, so the clocks agree at equal time of that frame, and the offset is the relativity of simultaneity. Corrections are of relative order v^2 (time dilation, the change of q during the time $vx/2$), plus a non-adiabatic memory of which mouth was accelerated. Runs: 0.0648 against 0.0647 ($v = 0.05$), 0.1303 against 0.1290 ($v = 0.1$), 0.2665 against 0.2557 ($v = 0.2$), with the sign reversed on the return leg. (iii) *Round trip.* For two rays entering the throat at the same exterior time, with both margins read on the clock of L, to first order in v the sum is twice the adiabatic margin at the separation half a transit later: $m_{LR} + m_{RL} = 2\, m_{\mathrm{ad}}(x_*)$, $x_* = x + v\pi/(2q) = x + v\,(x + m_{\mathrm{ad}}(x))/2$, q the adiabatic rate at separation x. The clock offset of (ii) cancels in the sum. In runs ($\kappa = 1.2$, $x = 0.3\,d$ and $2.69\,d$, $v = \pm 0.025$ and ± 0.05, two step sizes) the part of the sum odd in v agrees with this to about 0.1%; a second script with a different start-up ($\kappa = 1.2003$, $v = 0.025$, 0.05, 0.1) agrees to better than 1%. Both use the same integrator. The remainder is even in v and of second order, 5% of the first-order term at $v = 0.05$ and $x = 0.3\,d$. To the same order the two one-way margins differ by $m_{LR} - m_{RL} = -v\, m_{\mathrm{ad}}(x)$ when m_{LR} is read on the clock of L at emission ($v > 0$ for a receding mouth), and are equal when each is read at reception in exterior time; two scripts confirm the first to about 1% at $v = 0.05$. The asymmetry between the two one-way margins is thus a convention about where the interval is read, not a physical effect. Here q and the offset are in the frame of the static past; the margins are invariant. (iv) *Scope.* None of this applies at relativistic speeds, where the separation changes by a fraction of order $2v$ per lap and there is no adiabatic regime. There, at fixed peak speed the minimum margin rises with the path length ($\kappa = 1.2$: 0.81, 0.89, 0.95 of m_0 at $v = 0.6$ and 0.70, 0.77, 0.83 at $v = 0.8$, for paths $3d$, $10d$, $30d$), and high speed on short paths ends in closure. In those runs ($\kappa = 1.05$ to 1.5, $v \le 0.95$) the minimum exterior-time margin is 0.54 to 0.97 of m_0, and 0.70 to 0.79 of m_{ad} at closest approach for the inward ones; these are single-clock readings in the sense explained below. This suggests that the erosion is due to acceleration-driven ringing rather than to the Doppler factor as such.

Fast motion. Faster pumping excites ringing of the mouths rather than a mean shift ($\kappa = 1.2$, measured from invariants during a long hold after the pumping, at two step sizes: residual shift 0.0000 at pumping frequency 0.5 and +0.0007 at 0.75, with no drift over 115 loop periods); sufficiently abrupt kicks close the throat altogether. In about 400 runs with single and double excursions ($\kappa = 1.0002$ to 1.2, speeds up to 0.9997, each converged at two or more resolutions) the margin of every ray that gets through is positive.

How a margin is read. The theorem concerns the sign of the margin, which is a causal statement. Its size depends on what is measured, and three quantities must be kept apart. (a) The bulk-time margin Φ of Lemma 1, whose value depends on the $SL(2)$ frame (item (g) of the remark after Proposition 5). (b) A single-clock exterior margin: the interval, on the clock of one mouth, between two rays. In our code the $L \to R$ margin m is the interval on L's clock between the entry of the throat ray and the emission of the direct exterior ray that reaches R at the same event, and the $R \to L$ margin is the corresponding reception interval on L's clock. When the other mouth is moving at the event concerned, such a reading differs from the reading on that mouth's own clock by a Doppler factor, as large as 6.2 at $v = 0.95$; for the $L \to R$ ray, $m = \int e^{-\eta}\, \mathrm{d}\tau_R$ over the proper time of R between the arrival of the direct ray and the exit of the throat ray, with η the rapidity of R. (c) The invariant interval $\sigma = \sqrt{m(m + 2x)}$ between the entry and exit events of the throat ray, x being the separation at the exit event; it vanishes exactly at a shortcut. We quote it relative to its value σ_{ad} in the static solution at the same separation. A single-clock number is meaningful only together with the clock, the event and the rapidity; single-clock minima at large rapidity are

dominated by Doppler or time-dilation factors.

How far the margin can be reduced. In every run of every kind the margin of every ray that gets through is positive. To see how small it can be made we ran numerical searches over one family of smooth trajectories of mouth R (sums of up to eight smooth bumps, speed capped, closest approach bounded below by $0.02\,d$), minimising the invariant ratio $\sigma/\sigma_{\mathrm{ad}}$ of the previous paragraph. Every value quoted is a reduction relative to mouths at rest at the same separation, and is given with the event at which it is reached, since the moving mouth is often fast there (see the table). For the several-bump family the following comparison, prompted by an external automated check, shows why no kinematic share is quoted: if the throat delay on the receiving mouth's own clock had its static value, the invariant ratio on these trajectories would already be 0.50 to 0.55; with the static delay in exterior time instead it would be 0.24 to 0.43. For this $\kappa = 1.2003$ line we therefore also report the reduction on the receiving clock itself, 20 to 30% at the minimising event; for the single-bump rows the corresponding figures are not yet settled and are not given. For a single bump at a speed cap of 0.95 the smallest ratios found are given in Table 1, with the direction of the ray, the rapidity of the moving mouth and the separation at the minimising event; they lie between 0.52 and 0.72 and are lowest near $\kappa = 1.4$. With several bumps the ratio is lower, and in this bump family it is reached where the receiving mouth recedes at rapidity 1.6 to 2.3. The values with the throat left open and with the throat closing lie on one continuous family of trajectories ($\kappa = 1.2003$). In the bump family the ratio is about 0.50 where the final energy crosses zero; lower values, down to about 0.46, are reached in that family only by trajectories that leave positive energy, so that the throat closes afterwards; in this family the lowest values found on that side are 0.458 (cap 0.99) and 0.456 (cap 0.995). A second family, piecewise-constant velocity with smooth steps (three velocity steps), reaches 0.44 with the same cap and closest approach and 0.41 when the closest approach is lowered to $10^{-4}\,d$, with the speed cap inactive; there the receiving mouth recedes at rapidity about 2.8. The trajectories giving 0.41 close during the motion; for the second family we make no statement on which values are compatible with a throat that stays open. So neither the cap nor the first family's values are limits. All these values are upper bounds from local searches with one code base; within the bump family a population search, interpolation between the known trajectories and a basin-hopping search found nothing lower, which certifies no floor. We do not separate these reductions into a kinematic and a dynamical part: we know no unique kinematic baseline in this model. The naive kinematic picture, in which the clocks dilate freely, gives zero or negative margin in one direction; in the full model both directions stay positive in every run. None of the numbers is a bound on the model. No lower bound is proven: a margin tending to zero as the speed tends to that of light would be compatible with the theorem, and whether the infimum over smooth trajectories of bounded speed is positive is open.

When a clock stops. The theorem assumes running clocks. In the runs, what ends that regime is closure: both clock rates fall to zero and no further ray gets through. With both mouths at rest the loop has a conserved energy $H = E_L + E_R + \int(\rho_a + \rho_b)\,\mathrm{d}x$, the sum of the two Schwarzian energies and of the energy of the two streams in the exterior, equal to $-(\beta + \kappa^2 d)$ in the static solution. If both clocks freeze exponentially with bounded bulk stress then $H \geq 0$, so a loop with $H < 0$ cannot close in that way; this is the bound/unbound dichotomy of Maldacena and Qi [1] (Section 4.1) carried over to two clocks with an exterior delay, and is not claimed as new. The converse, that $H > 0$ after the motion implies closure, is a numerical observation only: it held in all 342 sampled positive-energy states at $\kappa = 1.2003$ and in 20 at $\kappa = 1.4$ (run at one step size, with five extreme orbits checked at half the step); at $\kappa = 1.1$ and $\kappa = 1.05$ it was confirmed at two

κ	$\sigma/\sigma_{\mathrm{ad}}$	ray	rapidity	x/d
1.01	0.713	$R \to L$	-1.64	0.53
1.02	0.695	$R \to L$	-1.52	0.56
1.05	0.679	$L \to R$	1.65	0.55
1.2003	0.645	$L \to R$	1.06	0.69
1.3	0.552	$L \to R$	1.18	0.59
1.4	0.525	$L \to R$	0.81	0.68
1.5	0.561	$L \to R$	0.51	0.76
1.6	0.619	$L \to R$	0.32	0.79
1.7	0.689	$L \to R$	0.17	0.86

Table 1: Smallest invariant ratio found for a single smooth bump of mouth R at a speed cap of 0.95 (local searches, one code base, not independently reproduced), with the direction of the minimising ray, the rapidity of the moving mouth R at the minimising event (positive when R recedes from L; in the $R \to L$ rows R is the emitter and is approaching) and the separation there. The closest approach was bounded by $x \geq 0.02\,d$, and for $\kappa \geq 1.3$ the optimal trajectory sits on that bound (the minimum separation over the trajectory, not the separation at the minimising event listed in the table; at $\kappa = 1.4$ the ratio is 0.516 and 0.513 with the bound at $0.005\,d$ and $0.001\,d$). The final energy was not constrained: the $\kappa = 1.4$ trajectory ends with $H = +0.13$, so that throat closes afterwards (with $H \leq -0.5$ required the ratio is 0.535); the others end with the throat open.

step sizes on three states each, closure being much slower for thin throats (of order 60 to 80 units of $[H/(\beta + d)]^{-1/2}$ at $\kappa = 1.05$, with individual delays sensitive to the step). The approach to closure is not monotone and we know no monotone functional; three attempts at a proof failed. No state with $H < 0$ was seen to close, and in the free evolution after the drive, each ray that traversed the throat during the approach to closure did so with positive margin (lowest about $1.0\,m_0$ on the exterior clock, at $\kappa = 1.05$), and in most runs no ray completed a traversal after the drive; margins during the drive itself are covered by the motion studies above, not by this count. The derivation of the one direction and the details of the runs are not given in this note.

Single-clock readings for single excursions, each with its clock:

- The largest single-clock reduction among single excursions: $\kappa = 1.0002$, inward excursion at $v = 0.999$. The reception interval on L's clock for a ray entering R while R approaches at rapidity 2.8 is $0.428\,m_0$ (a reduction of 0.572, reproduced by both integrators to six digits with $G < 3 \times 10^{-5}$); the interval of the same pair of events on R's clock stays at or above m_0, and $\sigma/\sigma_{\mathrm{ad}} = 0.654$. Neighbouring inward runs give reductions of 0.548 to 0.568; three outward runs give 0.481 to 0.486, marginal in the validity test.

- One-way displacements at $\kappa = 1.01$ (two step sizes agreeing to four digits): the exterior-time margin stays between 0.91 and 1.06 of m_0.

- With both mouths back at rest (six runs) the round-trip margin stays above 0.974 of its static value; large reductions occur only while a mouth moves. This is an observation.

Bulk-time margin and an independent re-implementation. In the frame fixed by the static past the bulk-time margin Φ is reduced by up to 0.77 ($\kappa = 1.001$, outward excursion, $v = 0.997$); by item (g) this is not by itself a physical reduction. An external automated check (a different AI system, which wrote its own code from an earlier version of this text; Section 7) reported a reduction of Φ by 0.685 for an inward round trip of amplitude $0.65\,d$ at $\kappa = 1.2$, $v_{\max} = 0.75$,

converged at four step sizes, and 0.110/0.116 for a slow round trip at $v = 0.1$. Our code reproduces both ($\Phi_{\min} = +0.33001$ against its $+0.33000$; 0.110/0.116). Before the second re-implementation described in Section 7, this was the only quantity compared between two independently written codes. Its value at $\kappa = 1.1$ (0.75, one step size) compares with our 0.74.

Fits. For outward motion the single-clock reduction is fitted by $(1 - D)/2$ with D the Doppler factor; this is an empirical fit, not derived. For inward single excursions it is non-monotonic in v and peaks at 0.55–0.57. The light-clock limit $\beta \to 0$, where the reduction can be derived, gives $v/(1 + v)$ and $2v/(1 + v)$ for the two ray types; the external re-implementation finds 0.11–0.12 at $v = 0.1$, above $v/(1 + v) = 0.09$ and below $2v/(1 + v)$, so neither expression is established as an envelope. None of the numbers in this section is a bound.

Zero-stiffness equations. If the clock stiffness β is set to zero in the mouth equations, the phase margin equals m_0 exactly for every ray and any timelike mouth motion, so, in the deformed equations, the reduction of the phase margin is an effect of clock inertia; in a gentle test excursion a probe at one resolution finds the deviation linear in β, with both signs. This is a statement about deformed equations, not a limit of the physical family: on shell $2\beta = \pi(\kappa^2 - 1)$, so $\beta = 0$ at $\kappa > 1$ is off the $SL(2)$ constraint surface, and no perturbative bound follows at the physical ratio $\beta/m_0 \approx \frac{1}{2}$.

Test of the criterion. On 960 laps from five runs, the sign of the lowest Dirichlet eigenvalue computed from $E_L(T)$ alone equals the sign of the margin computed from the clocks in all 960, including 15 laps with a genuine shortcut in the decoupled control and laps within ± 0.03 of threshold. Independently, for 400 random monotonic clock functions (251 with $\Phi < 0$) the two signs agree in every case; this tests Proposition 1 as mathematics, independently of the dynamics.

Consistency. The $SL(2)$ charge Q_0 of the throat, which is not built into the integrator, is conserved to 1.3×10^{-4} against individual terms of order 1–5. $Q_\pm$ were not checked.

5 Changes from earlier versions

Version 4. One hypothesis is made explicit and two statements of earlier versions are corrected. (0) Clarification, not a correction: Propositions 2 and 3 now state in so many words that the streams are in the single-winding vacuum of Section 1, which earlier versions assumed through the definition of the model; a remark gives a doubly wound static configuration, found by an external automated check, showing that the condition is necessary. Extension, not a correction: Proposition 2 excluded rays whose window reaches past a stopped clock; the paragraph "Extension: rays that exit before a clock stops" now proves positivity of the margin for every ray that exits while both clocks still run, with a numerical check in both directions. Section 7 now reports a machine check in Lean 4 of the sum rule, the Sturm step and the lap argument from explicit hypotheses; this does not make the theorem formally verified, since the derivation of those hypotheses from the model is not machine-checked. (1) Versions 1 to 3 printed the round-trip formula of the slow-motion paragraph, item (iii), with q unspecified and an effective separation $x(2 - v)/(2(1 - v))$, which omits a term; it is wrong at first order and is replaced. The agreement figures quoted with it (0.1%, 0.4%, 1.7%) tested the total against the superseded form and are dropped. (2) Versions 2 and 3 said that on the closing side the search was limited by the imposed speed cap. This clause is withdrawn: a second family of trajectories reaches lower ratios, in one case with the cap inactive. The printed values for

the first family stand, and statements about them are now restricted to that family. The theorem, the propositions and the positivity of all margins are unaffected.

Version 3. The proof of Proposition 4 in version 2 used a constant exterior light time, so it covered only mouths at rest while the agents act, although the statement did not say so; it also omitted a second achronality condition (condition (ii) in the proof). Both gaps were found by external automated checks. The proposition is now stated and proved for arbitrary timelike motion of the mouths, with the agents confined to a worldtube between the mouths; the conclusion is unchanged. The domain of the quantum inequality (Hadamard-type states) and the semiclassical character of the statement are now said explicitly. Section 7 adds a comparison with a second independently written code on three trajectories. For the $\kappa = 1.2003$ trajectory family the reduction is now also given on the receiving clock itself, and the wording on a kinematic baseline is "no unique" instead of "no well-defined". No other statement is changed.

Version 2. Version 1 (hexagon:2611.00646v1) gave no numerical values from the optimised-trajectory searches. This version adds the paragraph "How far the margin can be reduced" with Table 1, and the paragraph "When a clock stops". The abstract and the item on prior art now say explicitly that the conclusion is the one expected from the achronal averaged null energy condition and stated in [3]. In the paragraph on the zero-stiffness equations the clause on clock inertia is narrowed to the phase margin in the deformed equations. No statement of version 1 is withdrawn; the theorem, the propositions and their proofs are unchanged.

6 Limitations

1. **Scope of the theorem.** Not covered: an instantaneous coupling between the mouths; agents inside the throat; conversion between species on different paths; evolution after a clock stops (closure); throats or species that do not share one pair of clocks; $1/c$ corrections to the classical Schwarzian mode; a gravitating exterior. The checks that have and have not been made are listed in Section 7. No physicist has checked this work.

2. **No lower bound on the margin.** The infimum of the margin over smooth subluminal trajectories is not known to be positive, and Proposition 5 gives no floor in terms of the motion; only the theorem (margin > 0) protects that regime. Exterior-time margins for rays that cross while a mouth moves depend on the clock on which they are read (Section 4); this note does not give all clock readings for every run it quotes.

3. **Runs near the marginal throat.** Three runs at $\kappa = 1.0002$, $v = 0.99$ gave non-converging negative margins. Using (5) as a diagnostic shows these are integration failures: the formula holds up to a definite time and fails afterwards, and all negative margins lie in the invalid part; in the validated part the margins are about $0.51\, m_0$ or larger. Whether and when those throats close is numerically unresolved. These three runs also fail the constraint test of Section 4 with either integrator (G of 0.1 to 0.3), so no number from them is quoted as a result. There $\beta \approx 6 \times 10^{-4}$, outside the semiclassical regime of the model in any case.

4. **Quantum fluctuations of the clocks.** The statements are about the classical Schwarzian mode, i.e. expectation values at large c. Whether the margin is positive as an operator statement is not shown: for $\beta \gg 24\pi/c$ it would be equivalent to an achronal averaged null energy condition on clock-conditioned states, which is unproven, and the classical description first fails for $\kappa - 1 \lesssim 24/c$.

5. **Toy model.** Two dimensions, JT gravity, large c; the exterior is rigid and does not gravitate; mouth motion is prescribed. In four dimensions the relevant ray would be a complete achronal null geodesic and the generalized second law applies [8]; nothing here bears on that.

6. **Single code base** for the dynamical runs. This note does not specify the runs (excursion profiles, Doppler map, initial data, the invariant defining the residual shift) in enough detail to be re-implemented from the text alone; an external automated check confirmed this for the erosion runs. No reproducibility appendix is included, and the numerical values of Section 4 other than those of the paragraph on the independent re-implementation and the three trajectories of Section 7 should be regarded as unreproduced.

7. **Prior art.** The sum rule is the dynamical form of Maldacena–Qi eq. (2.10) [1], and integrated null energy is related to $SL(2)$ charges in the same way in [2] (which we did not study in detail). What may be new is its combination with the lap identity for a stream looping through an exterior. In Maldacena–Qi [1] (Section 4.1 and Appendix B) the potential is for the common rate of the two boundary clocks; the relative shift is an $SL(2)$ gauge mode without matter and is fixed by the matter charges with matter. The locking found here is that known structure (the relative shift is constrained, not a free modulus) in a new setting with a finite-light-time exterior and moving mouths. The conclusion of this note is not new as physics. The introduction of [3] states that short traversable wormholes, which would lead to causality violation, are excluded by the Einstein equations together with the achronal averaged null energy condition, that long ones are allowed, and that the null lines carrying the negative Casimir energy are not achronal; the construction is described there as a long wormhole that does not lead to causality violations (see also [9]). That a moved mouth cannot produce a shortcut is the direct consequence, and is the answer a specialist would give without calculation. What this note adds is narrower: an explicit model in which the statement is proved for arbitrary mouth motion without assuming the achronal condition for the state reached, the general theorems not being stated for a throat glued to a non-gravitating exterior. On a fixed classical background a potential difference between static mouths produces a growing time shift [10] (of which we read the abstract, one figure and the discussion only); our runs are the back-reacting counterpart. The use of Hill's equation and Sturm theory for Schwarzian functions is classical, and (2) has the form of known quantum inequalities [6, 7]. We did not find the lap criterion, the no-shortcut statement or the dynamical calculation in three keyword searches and in the citation lists we could check; this is not exhaustive.

7 Statement on the use of AI and on checking

Production. The derivations, the simulation code, the numerical work and nearly all of the text of this note were produced by an AI system (Claude, Anthropic) under the author's direction. The author is not a professional physicist, chose the question and the direction of the work, and takes responsibility for the note, but has not independently verified the mathematics.

Internal checks. The work was organised as several separately prompted instances of the same AI system with different roles: one deriving and running numerics, one refereeing, one writing. An "internal derivation" or "internal check" in this note means a derivation or check by one such instance; two internal derivations are by two instances. These are not independent in the sense that matters most: the instances share one underlying model and could share its errors. The lap identity, the eigenvalue criterion, the linear mouth equations, the sum rule and the no-shortcut proposition have each been derived or checked by three instances. Proposition 3 (in particular the

confined species), Proposition 4 and Proposition 5 have had fewer checks, as stated where they appear.

External automated checks. Two different AI systems from other providers were given an earlier version of this text, without the code, and asked to re-derive and criticise it. Both reproduced the kernel identity behind Lemma 4 and the Rayleigh step. One correctly objected that the sum rule was then stated for general histories but proved only for a static past; the present proof by conserved charges, with the constraint as an explicit hypothesis, is the response. The other wrote its own simulation and reproduced the zero residual clock shift, the accuracy of the sum rule, the conservation of the three $SL(2)$ charges and the shortcut of the decoupled throat, and gave the values compared in Section 4. A third AI system (Gemini) later wrote a further code from the equations of Sections 1 to 3; we do not know whether it had access to our code. We ran it against our code on three trajectories at $\kappa = 1.2$ (smooth outward excursions with peak speed 0.2 and 0.7, and an off-shell control with β multiplied by 10^4): the clock functions agree to 6×10^{-6}, 1×10^{-4} and 2×10^{-7}, the minimum exterior margins to 10^{-6}, the residual clock shift at speed 0.2 is 2×10^{-7} against a lag of 0.146 in both codes, and both give the same shortcut in the control. For these three trajectories the clock functions and minimum margins have therefore been computed by two independently written codes. This checks the implementation of the equations, not their derivation. The numerical table of the report that accompanied that code was not reproduced by the code supplied to us and is not used. An automated check by the second system (agy) independently reproduced the zero residual clock shift at slow speed with an invariant definition (residual about 10^{-8}, scaling like v^4); we have not rerun it. Apart from these comparisons, everything added after the first checks has had internal checks only: the conserved-charge proof of the sum rule, Propositions 4 and 5, the general-matter form of the sum rule, the slow-motion formulas and the invariant-interval results of Section 4.

Machine-checked steps. After version 3, the sum rule, the Sturm step and the lap argument of the no-shortcut proposition, including its extension to rays that exit before a clock stops, were machine-checked in Lean 4 from explicit hypotheses (linear mouth equations, a static past, existence of the Hill phase, single-winding transport, monotonicity and range of the clocks); the derivation of those hypotheses from the model is not machine-checked. In particular it is not checked that the model gives the mouth equations, that a Hill phase exists, or that an unsourced vacuum stream obeys the transport law; Propositions 3, 4 and 5 and all numerics are not covered. The formal statements and proofs were written by the AI system that produced this note; the statements were compared with the text by a separate session of the same AI system, and no person has read them. Details: Lean v4.37.0-rc1 with Mathlib (commit `03a891ad`); twelve statements, three of them auxiliary (the algebra of the linear mouth equations, a non-vacuity check, the doubly wound example), which use only the axioms `propext`, `Classical.choice` and `Quot.sound`; the kernel replay passes (run by the same AI system; not reproduced independently), and the exported proof terms are accepted by two further proof checkers, named `con-leche` and `con-ron` in the run log. The Lean files are not distributed with this note; for identification, the file containing the statements has SHA-256 `5ae345b1b0feb9d6...`.

Human checks. None. No physicist has examined the derivations. The prior-art search was done by the AI system with keyword and citation searches and is not exhaustive.

Numerics. All runs except the external re-implementations come from one code base written by the AI system. During the work several numerical artefacts were found in it and corrected (a tabulated trajectory differentiated three times; loss of the constraint at step-size changes), which is why a constraint-based validity test is used; it was applied to representatives of each class of run and not to every run.

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