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Grothendieck rings of complex fusion categories: rank 4

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Abstract

Every complex fusion category of rank four admits a spherical structure. Together with the pivotal classification of Dong, Palcoux and Plessis, this gives exactly fifteen Grothendieck rings. We prove sphericality by comparing the ordinary ring with the signed ring of the sphericalization. Formal codegrees and moments of central twists bound the dimension of a putative counterexample. An exact search leaves two signed Casimir polynomials for one ordinary ring. Both violate the same moment inequality. For the six rings with a non-self-dual simple, we also give a direct classification by central induction and twist moments.

GPT-6.2 Astra assisted with the research. Complete human verification is pending, and the results should be treated with caution.

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1 Classification and sphericality

A fusion category of rank four has four simple objects up to isomorphism. Its Grothendieck ring records the multiplicities N_{XY}^{Z} in their tensor products. This article is the rank-four part of a series on ranks at most four over $\mathbb{C}$. A based isomorphism preserves the unit and the basis of simple classes.

Theorem 1.1. *Every complex fusion category of rank four admits a spherical structure. Exactly fifteen Grothendieck rings occur up to based isomorphism, as listed in Table 1. Each has a unitary realization. Moreover,*

$$\mathrm{FPdim}(\mathcal{C}) \leq 24 + 12\sqrt{3}, \qquad \max_{X,Y,Z} N_{XY}^{Z} \leq 6.$$

Equality in either bound characterizes the near-group ring $\mathrm{NG}(C_3, 6)$.

The existence of a spherical structure concerns every category with the given fusion rules. Exhibiting a unitary model proves existence of one categorification for each ring. The step connecting the two assertions is the sphericality theorem proved here (Theorem 12.3). It allows us to apply the independently established pivotal classification [7, Theorem 1.1 and Table 1].

History and scope

Ostrik's rank-two and pivotal rank-three classifications established how much information about a fusion category its center retains [26, 24]. In rank four, Rowell, Stong and Wang treated unitary modular categories [27], and Bruillard classified premodular categories up to Grothendieck equivalence [3]. Hagge and Hong observed that a pivotal category of rank at most four admits a spherical structure [14, Lemma 4]. Obtaining pivotality for an arbitrary fusion category is the remaining hypothesis in that observation.

Larson reduced the pseudo-unitary case with a non-self-dual simple to seven rings and determined the near-group parameters for C_3 [17]. Edie-Michell, Izumi and Penneys excluded the remaining quadratic candidate under a unitarity assumption [9]. Dong, Chen and Wang proved sphericality for fusion categories with a subcategory of corank one and classified the rank-four cases [4]. Dong described the possible self-dual tables with a proper nontrivial fusion subcategory [6]. For prescribed rank and multiplicity, Vercleyen and Slingerland developed a general enumeration of fusion rings [31]. Liu, Palcoux and Ren classified the categorifiable multiplicity-one rings up to rank six [19]. Vercleyen and Slingerland subsequently computed explicit associativity, braiding and pivotal data in low-rank multiplicity-free examples [32].

Dong, Palcoux and Plessis proved the fifteen-ring theorem for pivotal complex fusion categories, combining an a priori dimension bound with a finite classification [7]. Our finite calculation asks a different question: can the signed table of a category without a spherical structure be consistent with its ordinary table? Ruling this out extends their theorem. The direct treatment of the non-self-dual case in Section 4 recovers its six rings without calling on the pivotal classification. Model constructions and their sources are collected in Table 1.

How the proof works

Pass to the sphericalization $\mathcal{S} \to \mathcal{C}$ of Etingof, Nikshych and Ostrik [10]. Each simple object has two lifts. Their tensor multiplicities add to give the ordinary structure constants and subtract to give the signed ones. If signs can be chosen to make the tables agree, the chosen lifts form a tensor subcategory equivalent to $\mathcal{C}$, which inherits a spherical structure. To detect such a choice, we use Ostrik's formal codegrees and central induction [25, 24] and the higher indicators of Ng and Schauenburg [21, 22].

Theorem 1.2 (A bound before the finite search)**.** *Suppose a rank-four complex fusion category has no spherical structure. Its simple objects are self-dual, its global Frobenius–Perron dimension D has degree two or four, and*

$$D < 1000, \qquad N_{XY}^{Z} \leq 31.$$

Three parts of the argument can be read separately.

1. Sections 2–6 establish the comparison of ordinary and signed tables. Squared central dimensions give a positive measure on roots of unity. Its first few moments bound the mass contributed by induction of the unit. These estimates give sphericality directly in several families and identify the two dimension fields in a putative counterexample.

2. Sections 7–10 prove Theorem 1.2. The quadratic argument compares norms and pairs of conjugates. The cubic case gives sphericality. In degree four a trace bound reduces the remaining problem to a finite calculation with algebraic integers.

3. Sections 11 and 12 enumerate the bounded ordinary rings and test their signed lifts. The norm of a central dimension leaves only one ordinary tuple and two signed polynomials. The moment inequality excludes both by hand. Section 13 then identifies the rings and gives the sharp bounds.

Throughout, d_i are the simple Frobenius–Perron dimensions and x_i the signed spherical dimensions of chosen lifts. We write $D = \sum_i d_i^2$, $E = \sum_i x_i^2$. The formal codegrees of the ordinary and signed rings are f_i and g_i, respectively, counted with their character multiplicities. The two numbers that enter the final comparison are $u = \sum_i f_i^{-2}$ and $v = \sum_i g_i^{-2}$.

2 Sphericalization, central induction, and rigidity

Throughout, a fusion category is finite, semisimple, rigid and $\mathbb{C}$-linear, with simple tensor unit. Its rank is the number of isomorphism classes of simple objects. A fusion ring is a based ring with nonnegative integral structure constants and the usual coefficient-of-unit pairing. A *Grothendieck ring* is a fusion ring realized by such a category. The Frobenius–Perron dimension is the positive real character $X \mapsto d_X$ of its fusion ring, and $D = \sum_X d_X^2$. We call a category weakly integral if $D \in \mathbb{Z}$, and pseudo-unitary if its categorical global dimension agrees with D. A pivotal structure is a monoidal isomorphism from the identity to the double dual; it is spherical when its left and right traces agree. We impose no pivotal hypothesis; the general existence question is [10, Conjecture 2.8].

For a self-dual basis $1, X, Y, Z$ we use the convention

$$\begin{aligned} X^2 &= 1 + aX + bY + cZ, \quad XY = bX + dY + eZ, \quad XZ = cX + eY + fZ, \\ Y^2 &= 1 + dX + gY + hZ, \quad YZ = eX + hY + iZ, \quad Z^2 = 1 + fX + iY + jZ. \end{aligned} \tag{2.1}$$

Thus a table is denoted by $(a, b, c, d, e, f, g, h, i, j)$. The same convention applies to signed tables. All subsequent changes of labels or signs are explicitly local to the argument concerned.

Lemma 2.1 (Rank-four algebras and coefficient bounds)**.** *A semisimple complex algebra of dimension four admitting a one-dimensional representation is commutative. In particular every rank-four fusion ring, and every rank-four signed sphericalization quotient, has commutative complexification. For an ordinary fusion ring, write $d_i = \mathrm{FPdim}(b_i)$ and $D = \sum_j d_j^2$. Then*

$$N_{ij}^{k} \leq d_i < \sqrt{D}.$$

Proof. The only noncommutative semisimple complex algebra of dimension four is $M_2(\mathbb{C})$, which has no one-dimensional representation. The regular based $*$-algebras in question are semisimple; their Frobenius–Perron or signed dimension characters supply the required representation. In the ordinary ring, reciprocity gives $N_i^* = N_{i^*}$, so commutativity makes N_i normal. Its operator norm is its spectral radius d_i. Every matrix entry is bounded by that norm, and $d_i^2 < D = \sum_j d_j^2$. □

For a commutative self-dual table its characters have real values, and its formal codegrees are $f_\chi = \sum_i \chi(X_i)^2$. Orthogonality gives

$$\sum_\chi \frac{\chi(X_i)\chi(X_j)}{f_\chi} = \delta_{ij}, \qquad \sum_\chi f_\chi^{-1} = 1. \tag{2.2}$$

All codegrees are positive. The d-number and cyclotomicity conditions are collected in Proposition 2.8. In degree two, the d-number criterion says that the norm divides the square of the trace. A real character field of a categorification is totally real and abelian.

The sphericalization $\mathcal{S} \to \mathcal{C}$ of Etingof–Nikshych–Ostrik [10, Remark 3.1] is a spherical fusion category with two lifts of each simple. Tensoring with its central sign ε exchanges these lifts. The sign has dimension -1 and its canonical central lift has twist -1 (the sign representation in the equivariantization description of [2, Section 5.3]). Choose lifts X_i^+ of dimension x_i compatibly with duality. Such choices exist because the dimensions of duals agree and the two lifts have opposite nonzero dimensions. The group of lift signs therefore has to act simultaneously on a dual pair. Put

$$R = K_0(\mathcal{C}) = K_0(\mathcal{S})/(\varepsilon - 1), \qquad T = K_0(\mathcal{S})/(\varepsilon + 1), \qquad L = \mathbb{Q}(x_i).$$

The dimension character of $\mathcal{S}$ factors through T. All x_i are totally real [10, Corollary 2.10]. At any chosen embedding we may change the lifts so their embedded dimensions β_i are positive. Write $E = \sum_i \beta_i^2$, and use f_i, g_i for the formal codegrees of R, T, respectively. Set $u = \sum_i f_i^{-2}$ and $v = \sum_i g_i^{-2}$.

Lemma 2.2 (Sphericalization and induction)**.** *For every rank-four complex fusion category, with the choices above,*

1. *its ordinary and signed coefficients satisfy $|T_{ij}^k| \le N_{ij}^k$ and $T_{ij}^k \equiv N_{ij}^k \pmod 2$;*
2. *all ordinary and signed codegrees h belong to L, are d-numbers, and E/h is an algebraic integer;*
3. *$g_i \le D$ for every signed codegree. In particular $E \le D$. If $E = D$ the positive lifts form a spherical section of $\mathcal{S} \to \mathcal{C}$.*

All statements hold at each embedding of L, with lift signs chosen there. Thus a categorification admitting no spherical structure satisfies $E < D$ at every embedding.

Proof. The two lift multiplicities have sum N_{ij}^k and difference T_{ij}^k. Over $\mathbb{Q}$, the central idempotents $(1 \pm \varepsilon)/2$ split $K_0(\mathcal{S})$ into R and T. Both factors are commutative by Lemma 2.1. A character of either factor takes values of equal absolute value on the two lifts, so the codegrees of $\mathcal{S}$ are $2f_i, 2g_i$. Its spherical dimension is $2\sum_i x_i^2$. Central induction gives the integrality and field assertions at the original spherical embedding [24, Theorem 2.13 and Corollary 2.15]. They persist under every embedding since the sets of codegrees are Galois invariant. Ostrik's d-number theorem applies to $2f_i, 2g_i$ and multiplication by a rational scalar preserves Galois invariance of the principal fractional ideal.

Each signed multiplication matrix is dominated in absolute value by the corresponding nonnegative ordinary matrix. Spectral comparison bounds $|\chi(X_i^+)|$ by d_i for every character of T. Its codegree $\sum_i |\chi(X_i^+)|^2$ is thus at most D. If $E = D$, then $\beta_i = d_i$ for all i. Subtracting the dimension equations gives $\sum_k (N_{ij}^k - T_{ij}^k) d_k = 0$. Each summand is nonnegative, so the two tables agree. The chosen lifts form a tensor-closed subcategory on which the forgetful functor is an equivalence, transporting a spherical structure to $\mathcal{C}$. □

From here to the end of Section 3 the ordinary ring is self-dual. Its chosen lifts are then also self-dual.

The central quotients $E/f_i, E/g_i$ belong to the dimension image of the signed order, since the forgetful image is an integral signed combination of the chosen lifts. For a field algebra, the coefficient-of-unit functional satisfies

$$\tau_T(a) = \mathrm{Tr}_{L/\mathbb{Q}}(a/E). \tag{2.3}$$

Indeed $\mathrm{Tr}(a) = \sum_i \tau_T(x_i a x_i) = \tau_T(aE)$. In particular $\mathrm{Tr}(x_i/E) = \delta_{i0}$. When both ordinary and signed algebras are the same quartic field,

$$\mathrm{disc}(R) = N(D), \quad \mathrm{disc}(T) = N(E), \quad N(E/D) \in \mathbb{Z}_{>0}^2. \tag{2.4}$$

The first two identities follow by comparing the trace pairing with the orthonormal based pairing. The last follows because the ratio of two order discriminants is a rational square and E/D is integral.

2.1 Spectral and trace saturation

A self-dual based ring has a basis $b_0 = 1, b_1, b_2, b_3$ and a symmetric integral multiplication tensor, with coefficient-of-unit form $\langle b_i, b_j \rangle = \delta_{ij}$. A signed table below has the same unit and form and allows negative structure constants. A gauge change replaces each b_i by $s_i b_i$, where $s_0 = 1$ and $s_i \in \{1, -1\}$.

Lemma 2.3. *Let $N = (N_{ijk})$ be the structure tensor of a self-dual rank-four fusion ring, and let $T = (T_{ijk})$ be a symmetric integral structure tensor defining an associative unital algebra on the same orthonormal basis $1, X, Y, Z$, with $T_{0ij} = \delta_{ij}$. If*

$$|T_{ijk}| = N_{ijk} \qquad (0 \le i, j, k \le 3),$$

then there are signs $s_0 = 1$, $s_1, s_2, s_3 \in \{1, -1\}$ such that

$$T_{ijk} = s_i s_j s_k N_{ijk}.$$

Proof. Associativity and symmetry give, for every i, j,

$$\sum_k T_{iik} T_{jjk} = \sum_k T_{ijk}^2.$$

The same identity holds for N, and the right-hand sides agree. Each term $N_{iik}N_{jjk} - T_{iik}T_{jjk}$ is nonnegative; therefore each is zero. For a fixed k, all nonzero coefficients T_{iik} consequently have the same sign.

Use the usual ten-entry convention

$$(a, b, c, d, e, f, g, h, i, j) = (N_{111}, N_{112}, N_{113}, N_{122}, N_{123}, N_{133}, N_{222}, N_{223}, N_{233}, N_{333}).$$

The coefficients with a repeated index fall into three groups

$$(a, d, f), \qquad (b, g, i), \qquad (c, h, j).$$

Independent sign changes of X, Y, Z make every signed coefficient in these groups nonnegative. Hence the two tables now agree except possibly at e. If they agree there too, we are done. Otherwise $e > 0$ and the signed table has coefficient $-e$.

Compare the associativity equations for the two choices of sign of e. Their linear and constant parts give

$$a = d + f, \qquad g = b + i, \qquad j = c + h,$$
$$bh + ci = bc, \qquad dh = cd + fh, \qquad bf + di = fi,$$
$$e^2 = 1 + df + bi + ch.$$

Interchanging Y, Z if necessary, assume $d \geq f$. If $d = f = 0$ the first group is zero. If $d = f > 0$, the two middle relations involving d, f give $b = c = 0$, whence $e^2 = 1 + d^2$, impossible. If $d > f$, then $bf = i(f - d)$ gives $i = bf = 0$. When $f > 0$, the second group vanishes. When $f = 0$, the equation $dh = cd$ gives $h = c$ and $e^2 = 1 + c^2$, hence $c = h = j = 0$. Thus some diagonal group is zero.

Flip the basis sign belonging to a zero group. It preserves all nine diagonal coefficients and reverses the sign of e. The two tensors now agree, proving the lemma. □

Corollary 2.4. *Under the same hypotheses, suppose only that $|T_{ijk}| \leq N_{ijk}$. Let C_N and C_T be multiplication by the respective Casimir elements $\sum_i b_i^2$. If*

$$\mathrm{Tr}(C_T) = \mathrm{Tr}(C_N),$$

then T and N are gauge equivalent.

Proof. In the regular representation the multiplication matrices are symmetric, and

$$\mathrm{Tr}(C_N) = \sum_{i,j,k} N_{ijk}^2, \qquad \mathrm{Tr}(C_T) = \sum_{i,j,k} T_{ijk}^2.$$

Equality, together with termwise domination, forces equality of every absolute value. Apply Lemma 2.3. □

Corollary 2.5. *Equality of the ordinary and signed Casimir traces in a rank-four self-dual sphericalization implies that the original category is spherical.*

Proof. Corollary 2.4 supplies a basis sign change for which the signed and ordinary tensors agree. The chosen lifts are tensor closed, so the spherical structure restricts to a tensor subcategory equivalent to the original category. □

Lemma 2.6 (Galois transport)**.** *If a spherical dimension character belongs to the Galois orbit of the Frobenius–Perron character, the category has a pseudo-unitary Galois conjugate with the same fusion rules. Its central identities can therefore be evaluated with positive Frobenius–Perron dimensions.*

Proof. Extend the required embedding of the dimension field to an automorphism of $\mathbb{C}$. Apply it to skeletal tensor, duality and spherical structure constants. All their identities and the fusion rules persist, while the dimension character becomes the Frobenius–Perron character. This requires no algebraic field of definition for the associativity coefficients. Equivalently, apply the automorphism to the central identities; restriction multiplicities are fixed and twists remain roots of unity. □

Corollary 2.7 (Multiplicity-one sphericality)**.** *Every complex categorification of a self-dual rank-four fusion ring of multiplicity one admits a spherical structure.*

Proof. For its signed quotient, domination and parity give $|T_{ij}^k| \le N_{ij}^k$ and $T_{ij}^k \equiv N_{ij}^k \pmod 2$. Since $N_{ij}^k \in \{0,1\}$, $|T_{ij}^k| = N_{ij}^k$ for every coefficient. Lemma 2.3 makes the two tables gauge equivalent. The corresponding tensor-closed lifts form a spherical subcategory equivalent to the original category. □

2.2 Structural inputs and central traces

Proposition 2.8 (Structural inputs)**.** *1. In any complex categorification every formal codegree is a d-number. For its monic minimal polynomial $t^e + a_1 t^{e-1} + \cdots + a_e$, this means $a_e^j \mid a_j^e$ for $1 \le j \le e$ [25].*

2. If $S \subseteq R$ is a fusion subring, $D/\operatorname{FPdim}(S)$ is an algebraic integer [8, Theorem 7.17.6].

3. Every character value of a complex-categorifiable fusion ring is cyclotomic [10, Theorem 8.51 and Corollary 8.53].

4. In a spherical categorification with dimension character χ, every $\chi(b_i)$ is nonzero, $\chi(b_i) = \chi(b_{i^*})$, and its global dimension $\Delta = f_\chi$ satisfies $\Delta/f \in \overline{\mathbb{Z}}$ for every formal codegree f. Its dimension field contains all formal codegrees [24, Theorem 2.13 and Corollaries 2.14–2.15].

5. A weakly integral fusion category is pseudo-unitary [10, Proposition 8.24]. If the simple Frobenius–Perron dimensions take at most two values, then the category has a pseudo-unitary Galois conjugate [28, Lemma 3.1]. Indeed the values are 1 and possibly a single $d > 1$, precisely the hypothesis of that lemma. These statements apply in every rank.

For a spherical category $\mathcal{A}$, write $F : Z(\mathcal{A}) \to \mathcal{A}$ for the forgetful functor and I for its right adjoint. With $H = \dim(\mathcal{A})$, the center formula for higher indicators reads

$$\sum_{A \in \operatorname{Irr} Z(\mathcal{A})} [I(V) : A] \dim(A) \theta_A^n = H \nu_n(V). \tag{2.5}$$

Here the twists are roots of unity, and $\nu_n(V) \in \mathbb{Q}(\zeta_n)$; see [21, Theorem 4.1] and [22, Theorem 5.1]. When the fusion ring is commutative, induction of the unit has one multiplicity-one constituent per character, of dimension H/f_χ and twist one [24, Theorems 2.5 and 2.13].

Lemma 2.9 (The central sign and its translate twists)**.** *Let $\mathcal{S}$ be a rank-four self-dual sphericalization, with central sign ε, and let $A_\chi \subset I(1)$ be the multiplicity-one constituent corresponding to a character χ. Its canonical central sign translate satisfies*

$$\theta_{\widehat{\varepsilon} \otimes A_\chi} = \chi(\varepsilon).$$

Thus ordinary constituents have translate twist $+1$ and signed constituents have translate twist -1. The constituents of $I(1)$ and their sign translates are disjoint. Moreover $\nu_n(\varepsilon) = 0$ for odd n and -1 for even n.

Proof. Central induction identifies modular characters by

$$\frac{S_{A_\chi, W}}{S_{A_\chi, 0}} = \chi(F(W))$$

[24, Example 2.9 and the paragraph preceding Theorem 2.13]. At $W = \widehat{\varepsilon}$ the ratio is $\chi(\varepsilon)$; it is also $\dim(\varepsilon)$ times the monodromy scalar. Since $\dim(\widehat{\varepsilon}) = \theta_{\widehat{\varepsilon}} = -1$ and $\theta_{A_\chi} = 1$, balancing gives the asserted twist. The constituents have dimensions $E/f_i, E/g_i > 0$, whereas their translates have the opposite dimensions, proving disjointness. For the indicator, the invariant space is zero for odd tensor powers of ε. For even powers it is one-dimensional, and the rotation is multiplied by the inverse pivotal map -1. □

2.3 The rank-two input

We use the following part of Ostrik's classification as a subcategory obstruction [26, Main Theorem, p. 178].

Proposition 2.10 (Ostrik)**.** *Let $\mathcal{D}$ be a complex fusion category with simple objects $1, X$. Then $X^2 = 1 + nX$ with $n = 0$ or 1.*

The statement concerns arbitrary complex fusion categories, so it applies before a pivotal structure has been established.

3 Indicator inequalities

Lemma 3.1 (Positive-measure inequalities)**.** *Let μ be a finite positive measure on the unit circle, with mass M and real first two moments A, B. Then*

$$0 \le M + B \le 2M, \qquad 2A^2 \le M(M + B).$$

If its first m moments all equal the real number c, then

$$\mu(\{1\}) \le \frac{M + mc}{m + 1}.$$

These statements also apply after subtracting any known positive atomic submeasure.

Proof. Writing $z = e^{it}$, one has $\int \cos^2 t \, d\mu = (M + B)/2$; Cauchy–Schwarz gives the first inequalities. The nonnegative Fejér polynomial $F_m(z) = |1 + z + \cdots + z^m|^2/(m+1)$ has value $m + 1$ at one and integral $M + mc$, proving the atom bound. □

Lemma 3.2 (Second moment)**.** *Let u, v be the ordinary and signed inverse-square codegree sums. For any signed dimension character,*

$$E(u + v - 1) \le \sum_{i=0}^{3} |\beta_i| \le 1 + \sqrt{3(E - 1)}.$$

Proof. The eight constituents of $I(1)$ have positive dimensions E/f and squared mass $H = E^2(u+v)$. Their translates by the canonical central sign object have opposite dimensions, are disjoint from them, and have twists ± 1. The center's total squared dimension is $4E^2$. The central indicator formula [21, Theorem 4.1] gives

$$\sum_A (\dim A)^2 \theta_A^2 = 4Es, \qquad s = 1 + \sum_{i=1}^{3} \nu_2(X_i^+)\beta_i.$$

Here each $\nu_2(X_i^+) = \pm 1$. Changing a lift reverses the pivotal map. This negates its spherical dimension and its second indicator, the eigenvalue of pivotal transposition on its one-dimensional space of invariant bilinear forms. Removing the sixteen known objects, the weighted sum of squared real parts of the remaining twists is

$$\tfrac{1}{2}\big((4E^2 - 2H) + (4Es - 2H)\big) = 2E^2 + 2Es - 2H \ge 0.$$

This gives the first bound, and Cauchy's inequality gives the second. □

Lemma 3.3 (Quartic trace and norm)**.** *If L has degree four, the signed rational algebra is L itself, and its four character codegrees are the conjugates of E, with multiplicity. Moreover*

$$\mathrm{Tr}_{L/\mathbb{Q}}(E) \le \sum_j f_j, \qquad \mathrm{Norm}_{L/\mathbb{Q}}(E/f) \in \mathbb{Z}_{>0}$$

for every ordinary codegree f. If v is the signed inverse-square sum, then

$$v \ge \frac{1}{2} - \frac{9}{2}\frac{\mathrm{Tr}_{L/\mathbb{Q}}(E)}{\mathrm{Norm}_{L/\mathbb{Q}}(E)}.$$

Proof. The dimension character is a surjection from a rational algebra of dimension four onto L, hence an isomorphism. The Casimir's regular trace is the sum of the codegrees. Its trace is also $\sum_i \mathrm{Tr}(T_i^2) = \sum_{i,j,k}(T_{ij}^k)^2$, at most the corresponding ordinary sum. Norm integrality is Lemma 2.2. For $q_i = 1/g_i$ the reciprocal identity gives $\sum q_i = 1$, and the four-variable Schur inequality $9e_3(q) + 2\sum q_i^2 \ge 1$ gives the result, since $e_3(q) = \mathrm{Tr}(E)/\mathrm{Norm}(E)$. For clarity, its homogeneous form is $P = e_1^3 - 4e_1e_2 + 9e_3 \ge 0$. Subtract the smallest entry t from all four variables to obtain y with one zero entry. Expansion gives $P(q) = P(y)+2te_2(y)+3t^2e_1(y)+4t^3$, and $P(y) \ge 0$ is the ordinary three-variable Schur inequality. □

3.1 Averaging the central indicators

We record once the calculation used for both quartic fields and near-group rings. Let $\mathcal{A}$ be spherical with totally real Galois dimension field K and global dimension H. In a finite Galois extension containing K, the central twists and the required roots of unity, let Av denote the average over all automorphisms σ. Write $a_A = \dim A$ for a central simple and δ_z for the unit mass at z. The measure

$$\mu_{\mathcal{A}} = \mathrm{Av}_\sigma \sum_{A \in \mathrm{Irr}\, Z(\mathcal{A})} \sigma(a_A/H)^2\, \delta_{\sigma(\theta_A)}$$

is a probability measure: $a_A = \dim F(A)$ belongs to K, and $\sum_A a_A^2 = H^2$. Enlarging the averaging field leaves it unchanged. Multiplying (2.5) by $\dim V$, summing over simple V, and using adjunction gives $\sum_A a_A^2\theta_A^n = H\sum_V(\dim V)\nu_n(V)$. Because $\nu_n(V) \in \mathbb{Q}(\zeta_n)$, the two Galois averages factor whenever $K \cap \mathbb{Q}(\zeta_n) = \mathbb{Q}$. Hence

$$\int z^n\, d\mu_{\mathcal{A}} = \sum_{V \in \mathrm{Irr}\,\mathcal{A}} \frac{\mathrm{Tr}_{K/\mathbb{Q}}(\dim V/H)}{[K:\mathbb{Q}]}\ \mathrm{Av}_\sigma\, \sigma(\nu_n(V)) \qquad (K \cap \mathbb{Q}(\zeta_n) = \mathbb{Q}). \tag{3.1}$$

Suppose now that L has degree four, so the signed rational algebra is L. We apply this identity to the sphericalization $\mathcal{A} = \mathcal{S}$, with $H = 2E$.

Proposition 3.4. *Suppose that, for every integer $1 \le n \le m$,*

$$L \cap \mathbb{Q}(\zeta_n) = \mathbb{Q}. \tag{3.2}$$

Then

$$2u + v \le \frac{3}{4} + \frac{3}{m+1}. \tag{3.3}$$

Proof. Scale the probability measure to $\mu = 4\mu_{\mathcal{S}}$. In (3.1), normalized field traces are computed by (2.3). Every term for a lift of a nonunit simple vanishes. The surviving objects $1, \varepsilon$ have dimensions $1, -1$, with $\mathrm{Tr}(1/E) = 1$. Thus $\int z^n\, d\mu = (1 - \nu_n(\varepsilon))/2$. Lemma 2.9 yields

$$\int z^n\, d\mu(z) = \begin{cases} 1, & n \text{ even}, \\ \frac{1}{2}, & n \text{ odd}. \end{cases} \tag{3.4}$$

Lemma 2.9 gives the indicator signs, the eight constituents of $I(1)$ of dimensions $E/f_i, E/g_i$, and their eight disjoint sign translates. Thus translates of the R constituents have twist $+1$, while translates of the T constituents have twist -1. These statements are preserved by every $\sigma \in G$. Summing the squared weights yields the two estimates

$$\mu(\{1\}) \geq 2u + v, \qquad \mu(\{-1\}) \geq v. \tag{3.5}$$

By Cauchy–Schwarz, $v \geq 1/4$. Consequently $\nu = \mu - \frac{1}{4}\,\mathrm{Dirac}_{-1}$ is a positive measure. Its mass is $15/4$, and its first m moments are all $3/4$. Apply Lemma 3.1 to obtain $(m+1)\mu(\{1\}) \leq (15+3m)/4$. Together with (3.5), this proves the claim. □

Proposition 3.5 (Squared moments)**.** *If $L \cap \mathbb{Q}(\zeta_{2j}) = \mathbb{Q}$ for $1 \leq j \leq a$, then*

$$u + v \leq \frac{1}{2} + \frac{3}{2(a+1)}.$$

In particular $u + v \leq 7/8$ always, and $u + v \leq 4/5$ if $\sqrt{2} \notin L$.

Proof. Push the preceding averaged measure forward by $z \mapsto z^2$. Its total mass is four, its first a moments are one, and its atom at one is at least $2(u+v)$. Apply Lemma 3.1. The first three intersections are trivial because L is totally real. The fourth can be nontrivial only if $\mathbb{Q}(\sqrt{2}) \subset L$. □

Corollary 3.6. *If the signed dimension field L is quartic, then*

$$\sqrt{2} \notin L \implies u + v \leq \frac{4}{5}, \qquad \sqrt{5} \notin L \implies 2u + v \leq \frac{9}{8}.$$

Proof. The first assertion is Proposition 3.5. For the second, the real subfield at order seven has degree three, which cannot divide four. Thus the only nontrivial possible intersection up to order seven is $\mathbb{Q}(\sqrt{5})$ at order five. Apply Proposition 3.4 with $m = 7$. □

Lemma 3.7. *Suppose the signed dimension field is $K = \mathbb{Q}(\sqrt{2}, \sqrt{5})$. With the preceding notation, one has*

$$5u + 4v \leq \frac{15}{4}.$$

Proof. Use the unsquared averaged central measure μ above. The nontrivial quadratic subfields of K have conductors $5, 8, 40$. Hence $K \cap \mathbb{Q}(\zeta_n) = \mathbb{Q}$ whenever $5 \nmid n$ and $8 \nmid n$. The same indicator calculation gives

$$\int z^n \, d\mu = \begin{cases} 1/2 & n \text{ odd}, \\ 1 & n \text{ even}, \end{cases} \qquad 5 \nmid n, \quad 8 \nmid n.$$

These identities hold at every such order: extending the averaging field leaves the measure unchanged.

Restrict μ to the fortieth roots of unity. Averaging the functions z^{40j} over j gives their indicator function, since the measure has finite root-of-unity support. Thus, for each residue n divisible by neither 5 nor 8, the restricted measure retains the displayed nth Fourier coefficient. Subtract $\nu = \frac{3}{4}\,\mathrm{Dirac}_1 + \frac{1}{4}\,\mathrm{Dirac}_{-1}$. The resulting forty weights have Fourier support only at multiples of 5 or 8. By the discrete Fourier transform they are therefore

$$r_{ij} = A_i + B_j, \qquad i \in \mathbb{Z}/8, \quad j \in \mathbb{Z}/5,$$

where a fortieth root with exponent l corresponds to $(i,j) = (l \bmod 8, l \bmod 5)$. All cells other than $(0,0)$ and $(4,0)$ are nonnegative. The forced atoms give

$$r_{00} \geq 2u + v - \frac{3}{4}, \qquad r_{40} \geq v - \frac{1}{4}.$$

The total of the forty weights is at most $4 - 1 = 3$. For an additive array there is the exact identity

$$\sum_{i,j} r_{ij} - 5r_{00} - 3r_{40} = \frac{1}{2}\sum_{j\neq 0} r_{4j} + \frac{5}{4}\sum_{\substack{i\neq 0,4\\ j\neq 0}} r_{ij} \geq 0.$$

Consequently $3 \geq 5(2u + v - 3/4) + 3(v - 1/4)$, proving the assertion. □

3.2 Comparison of quadratic codegrees

Lemma 3.8 (Quadratic comparison)**.** *In a quadratic field let D, D' be the ordinary Frobenius–Perron codegree and its conjugate. Let E, E' be a signed dimension codegree and its conjugate, with $D > E \geq E'$. Set $P = D^{-1} + D'^{-1}$, $q = E^{-1} + E'^{-1}$, $N = DD'$ and $k = EE'/N$. Then k is a positive integer,*

$$kq < P + (k-1)/D, \qquad q \leq 2P/3. \tag{3.6}$$

Proof. Norm divisibility gives $k \in \mathbb{Z}_{>0}$. Since $D > E \geq \sqrt{kN}$, compare $x + kN/x$ at E and D to obtain the strict inequality. If $k \geq 2$, then $D/D' > k$ and $1/D < P/(k+1)$, whence $q < 2P/(k+1) \leq 2P/3$. If $k = 1$, $s = D/E > 1$ is a totally positive unit, so its integral trace is at least three. Using $E \geq E'$ gives $q/P \leq 2/(s + s^{-1}) \leq 2/3$. □

4 Non-self-dual categories

4.1 Spherical descent

Theorem 4.1. *Every complex fusion category of rank four with a non-self-dual simple object admits a spherical structure.*

Write the dual pair as X, Z and the remaining nonunit simple as Y. Use the order $1, X, Y, Z$ in both sphericalization quotients. Duality fixes two basis vectors, so its trace on the complex algebra is two. It permutes primitive idempotents and sends character values to their complex conjugates. Thus exactly two characters are real. Both real and nonreal pairs are Galois stable, since reality is equivalent to $\chi(b) = \chi(b^*)$ for every basis vector. The nonreal pair therefore takes values in an imaginary quadratic field. Write one as $(1, \alpha, t, \overline{\alpha})$, with $t \in \mathbb{Z}$, and put $a = \alpha + \overline{\alpha}$ and $b = \alpha\overline{\alpha}$. Orthogonality to its conjugate gives $1 + t^2 + \alpha^2 + \bar{\alpha}^2 = 0$. Its codegree h satisfies

$$2b = 1 + a^2 + t^2, \qquad h = 2 + a^2 + 2t^2. \tag{4.1}$$

In particular a, t have opposite parity and $h \geq 3$.

For use when h is small, reciprocity gives the following common table. In the ordinary case it is Larson's parametrization [17, Proposition 2.1]. The signed case has the same six parameters in $\mathbb{Z}$.

$$\begin{aligned} X^2 &= PX + L_0Y + CZ, & XY &= QX + JY + L_0Z,\\ Y^2 &= 1 + JX + HY + JZ, & YZ &= L_0X + JY + QZ,\\ Z^2 &= CX + L_0Y + PZ, & XZ &= 1 + PX + QY + PZ. \end{aligned} \tag{4.2}$$

In either ring let β be a character with positive values on this basis. If $h = 3$, then $(a, t) = (\pm 1, 0)$, and orthogonality to β gives $1 + a\beta(X) = 0$. Thus $a = -1$, $\beta(X) = 1$ and α is a primitive cube root of unity. Substitution in the products X^2, XY, Y^2 forces $C = J = 1$ and $P = L_0 = Q = 0$. If $h = 4$, then $(a, t) = (0, \pm 1)$ and orthogonality gives $t = -1$, $\beta(Y) = 1$. Now $\alpha = \pm i$, and the same three products give $L_0 = 1$, $J = Q = H = 0$ and $P = C$. In each case just one integral parameter μ is left:

h	$(\beta(X), \beta(Y), \beta(Z))$	dimension relation	dimension codegree
3	$(1, v, 1)$	$v^2 = 3 + \mu v$	$3 + v^2$
4	$(w, 1, w)$	$w^2 = 1 + 2\mu w$	$2 + 2w^2$.

(4.3)

Lemma 4.2. *Suppose that $\mathcal{C}$ admits no spherical structure. There is a real quadratic field K and imaginary quadratic fields L, M such that*

$$R \otimes \mathbb{Q} \cong K \times L, \qquad T \otimes \mathbb{Q} \cong K \times M.$$

The formal codegrees are, respectively,

$$D, D', f, f \qquad \text{and} \qquad E, E', g, g,$$

where $D = \mathrm{FPdim}(\mathcal{C})$ is irrational, $f, g \in \mathbb{Z}_{\geq 3}$, and primes denote the nontrivial embedding of K. We can choose the lifts and a real character β of T so that all its values are positive, its codegree is E, and

$$E \geq E', \qquad E < D, \qquad E/D \text{ and } E/g \text{ are algebraic integers.} \tag{4.4}$$

Every displayed codegree is greater than one, and $f \leq D$.

Proof. If D were rational, weak integrality would give a spherical structure [10, Proposition 8.24]. Since only two characters are real, the Frobenius–Perron field equals $K = \mathbb{Q}(D)$ of degree two. Lemma 2.2 puts D in the signed dimension field, also of degree at most two. This accounts for the two real quadratic factors. The nonreal pair in each table accounts for its imaginary quadratic factor and its repeated integral codegree by (4.1). The reciprocal-codegree identity and positivity give $h > 1$ for every codegree h. Spectral domination gives $f \leq D$. The integrality, sign choices and strict inequality in (4.4) follow from Lemma 2.2. □

Lemma 4.3. *In the situation of Lemma 4.2, put*

$$n = DD', \qquad k = \frac{EE'}{DD'}.$$

Then $k \in \{1, 2\}$ and $g \in \{3, 4\}$, with $k = 2$ possible only when $g = 3$.

Proof. The norm of the totally positive algebraic integer E/D makes k a positive integer. Set

$$p = 1 - \frac{2}{f} = \frac{1}{D} + \frac{1}{D'}, \qquad q = 1 - \frac{2}{g} = \frac{1}{E} + \frac{1}{E'}.$$

The first inequality of (3.6) gives

$$kq < p + \frac{k-1}{D} \leq 1 + \frac{k-3}{D}. \tag{4.5}$$

Here we used $f \leq D$. If $k \geq 3$, then $q \geq 1/3$ and $D \geq 4$ give the contradiction

$$\frac{k}{3} < 1 + \frac{k-3}{D} \leq \frac{k+1}{4} \leq \frac{k}{3}.$$

Thus $k \leq 2$. If $k = 2$ and $g \geq 4$, the left side of (4.5) is at least one, whereas its right side is $1 - 1/D < 1$.

If $k = 1$, the quadratic comparison (3.6) gives $q \leq 2p/3 < 2/3$, so $g < 6$. Equation (4.1) and the opposite parities of a, t rule out $g = 5$. Thus $g \in \{3, 4\}$. □

Proof of Theorem 4.1. Suppose that $\mathcal{C}$ admits no spherical structure, and use the preceding lemmas. The two possible signed tables are given by (4.3). In either row $E \geq E'$ forces $\mu \geq 0$. In the second row $\mu = 0$ would make the whole dimension character rational. Hence $w = \mu + \sqrt{\mu^2+1} > 2$ and $E' = 2 + 2/w^2 < 5/2$. Here $k = 1$, so the totally positive unit $r = D/E > 1$ has integral trace at least three. This gives $r \geq (3+\sqrt{5})/2 > E'$, contrary to $D' = E'/r > 1$.

In the first row, $v' = -3/v$ and

$$E' = 3 + 9/v^2, \qquad E + E' = \mu^2 + 12, \qquad EE' = 3(\mu^2+12). \tag{4.6}$$

Suppose $\mu > 0$. Integrality of $E/3$ and its norm give $(\mu^2+12)/3 \in \mathbb{Z}$, hence $3 \mid \mu$ and $\mu \geq 3$. Moreover

$$K = \mathbb{Q}(\sqrt{\mu^2+12}), \qquad \mu^2 + 12 = 3(3(\mu/3)^2 + 4).$$

The radicand has 3-adic valuation one, so K is neither $\mathbb{Q}(\sqrt{5})$ nor $\mathbb{Q}(\sqrt{2})$. From (4.3), $v > 7/2$ and $v^2 = 3 + \mu v > 27/2$, whence

$$E' < \frac{11}{3} < 2 + \sqrt{3}. \tag{4.7}$$

For $k = 1$, the unit $r = D/E > 1$ cannot have trace three, since that would generate $\mathbb{Q}(\sqrt{5})$. Its trace is at least four, so $r \geq 2 + \sqrt{3} > E'$, again giving $D' < 1$.

For $k = 2$, put $a_0 = E/D$ and $s = 2D/E$. The known algebraic integer a_0 has norm two, so $s = 2/a_0 = a_0'$ is an algebraic integer of norm two. It is totally positive and $s > 2$. Its integral trace $s + 2/s$ is therefore greater than three. Trace four would generate $\mathbb{Q}(\sqrt{2})$, which is excluded; thus the trace is at least five and $s > 4 > E'$. Since $D' = E'/s$, this too contradicts $D' > 1$.

Finally suppose $\mu = 0$. Then $E = E' = 6$ and $n = DD'$ is either 36 or 18. Denote the unsigned parameters corresponding to (C, H, J, L, P, Q) by $(c, e, k_0, \ell, p_0, q_0)$. Their parities are

$$c, k_0 \text{ odd}, \qquad e, \ell, p_0, q_0 \text{ even}.$$

For a nonreal character $(1, \alpha, t, \overline{\alpha})$ of R, the equations for $Y(X-Z)$ and $X^2 - Z^2$ give $t = q_0 - \ell$ and $\alpha + \overline{\alpha} = p_0 - c$. Applying (4.1) to R therefore gives

$$f = 2 + (p_0 - c)^2 + 2(q_0 - \ell)^2 \equiv 3 \pmod 8.$$

But $\mathrm{Tr}(D) = n(1 - 2/f)$ is an integer, so $f \mid 2n$. Since f is odd and $n \in \{18, 36\}$, it divides nine; the congruence and $f \geq 3$ force $f = 3$. For $n = 36$ the trace of D is 12, giving $D = D' = 6$, contrary to irrationality. For $n = 18$ the trace is 6, giving negative discriminant $36 - 72$, contrary to reality. Every possibility has now been excluded. □

Corollary 4.4. *Every non-self-dual rank-four complex fusion category has a pseudo-unitary Galois conjugate.*

Proof. It is spherical by Theorem 4.1. Duality has trace two on the based algebra and permutes primitive idempotents, so exactly two characters are real. This condition is Galois stable. If D is rational, weak integrality gives pseudo-unitarity. Otherwise the Frobenius–Perron orbit consists of both real characters: its degree is two. The spherical dimension character is real and hence belongs to this orbit. Apply Lemma 2.6. □

4.2 Two families from the repeated codegree

Larson first reduced the pseudo-unitary non-self-dual problem to two one-parameter families and determined its seven candidates [17, Sections 3–5]. We give the reduction and the exclusions needed here. Our argument starts with the nonreal character, avoiding an auxiliary parametrization of all non-self-dual rings.

Lemma 4.5. *A pseudo-unitary rank-four fusion category with a non-self-dual simple has one of the following Grothendieck rings:*

1. *the near-group ring* $\mathrm{NG}(C_3, m)$*, with basis* $1, g, g^2, \rho$ *and relations*

$$g^3 = 1, \qquad g\rho = \rho, \qquad \rho^2 = 1 + g + g^2 + m\rho \quad (m \geq 0);$$

2. *the ring* $S(m)$*, with basis* $1, g, x, y$*, where* $x^* = y$*, and*

$$g^2 = 1, \quad gx = y, \quad gy = x, \quad x^2 = y^2 = g + m(x + y), \quad xy = 1 + m(x + y) \quad (m \geq 0).$$

Proof. Let f be the repeated codegree of the nonreal characters in (4.1). Write the other two codegrees as D, D_1, with $D = \mathrm{FPdim}(\mathcal{C})$. Thus

$$D^{-1} + D_1^{-1} + 2/f = 1, \qquad D \geq f > 2.$$

Ostrik's pseudo-unitary inequality [24, Theorem 2.21] gives

$$u := D^{-2} + D_1^{-2} + 2f^{-2} \leq \frac{1}{2}(1 + D^{-1}). \tag{4.8}$$

If D is irrational, D_1 is its quadratic conjugate. Central induction makes D/f integral, so $DD_1 \geq f^2$. Therefore

$$1 - \frac{4}{f} + \frac{4}{f^2} \leq u \leq \frac{1}{2} + \frac{1}{2f},$$

which gives $f \leq 8$. Equality at $f = 8$ would require $D = f$, contrary to irrationality. If D is rational, both real codegrees are integers: they are either conjugates of one rational number or belong to two rational characters. Hence $D_1 \geq 2$, and

$$\frac{1}{2} \geq D_1^{-1} = 1 - 2/f - D^{-1} \geq 1 - 3/f$$

gives $f \leq 6$. In either case $2 < f < 8$. The expression $f = 2 + a^2 + 2t^2$, with a, t of opposite parity, leaves only

$$f = 3 : \ (a, t) = (\pm 1, 0), \qquad f = 4 : \ (a, t) = (0, \pm 1).$$

Substitution in (4.2), already carried out in (4.3), gives the two stated rings. In the first row $g = X$, $g^2 = Z$, $\rho = Y$. In the second, $g = Y$, $x = X$, $y = Z$. The remaining parameter is nonnegative in the ordinary ring. □

4.3 The near-group parameter from eleven moments

The following restriction is due to Larson [17, Theorem 4.11]. The Galois averaging argument of Proposition 3.4 gives a short proof using the same positive central measure.

Proposition 4.6. *If* $\mathrm{NG}(C_3, m)$ *admits a pseudo-unitary categorification, then* $m \in \{0, 2, 3, 6\}$.

Proof. Put $\delta = \mathrm{FPdim}(\rho)$ and $D = \mathrm{FPdim}(\mathcal{C})$. Then $\delta^2 = 3 + m\delta$ and $D = 6 + m\delta$. If δ is rational, it is an integer, and $\delta(\delta - m) = 3$ with $m \geq 0$ forces $\delta = 3$, $m = 2$. Otherwise put $K = \mathbb{Q}(\delta)$, with nontrivial embedding denoted by a prime. The codegrees are $D, D', 3, 3$. A constituent of $I(1)$ has dimension $D/3 = 2 + (m/3)\delta$; its restriction has dimension $a + b\delta$ for nonnegative integers a, b. Hence $m = 3k$. The radicand $m^2 + 12 = 3(3k^2 + 4)$ has three-adic valuation one, so K is neither $\mathbb{Q}(\sqrt{2})$ nor $\mathbb{Q}(\sqrt{5})$. These are the only real quadratic subfields of cyclotomic fields of orders at most eleven: the maximal real subfields at orders $7, 9, 11$ have odd degrees $3, 3, 5$, and the other orders are immediate. Thus $K \cap \mathbb{Q}(\zeta_n) = \mathbb{Q}$ for $1 \leq n \leq 11$.

Take $\mu = \mu_{\mathcal{C}}$ from (3.1). Since $\mathrm{Tr}_{K/\mathbb{Q}}(1/D) = 1/3$ and $\mathrm{Tr}_{K/\mathbb{Q}}(\delta/D) = 0$, its nth moment is $(1 + \overline{\nu}_n(g) + \overline{\nu}_n(g^2))/6$, where bars denote normalized Galois averages. An invertible object's indicator is zero unless its order divides n, and otherwise is an nth root of unity [28, Lemma 3.7]. Its normalized Galois average is real and at most one. Thus the moment equals $1/6$ when $3 \nmid n$ and is at most $1/2$ when $3 \mid n$.

The four constituents of $I(1)$ have twist one and give the forced atom

$$\mu(\{1\}) \geq u := \frac{1}{D^2} + \frac{1}{(D')^2} + \frac{2}{9} = \frac{1}{3} - \frac{2}{3(m^2 + 12)},$$

where $DD' = 3(m^2 + 12)$. Integrating $P(z) = |1 + z + \cdots + z^{11}|^2$, whose value at one is 144, yields

$$144u \leq \int P \, d\mu \leq 12 + 2\left(\frac{66}{6} + \frac{1}{3}(9 + 6 + 3)\right) = 46.$$

It follows that $m^2 + 12 \leq 48$, hence $m \leq 6$. Together with $3 \mid m$ and the rational case $m = 2$, this proves the claim. □

4.4 The second family and the complete list

Larson obtained $m \leq 2$ [17, Theorem 5.8]. Edie-Michell–Izumi–Penneys excluded $m = 2$ for unitary categorifications [9], and the pivotal classification of Dong–Palcoux–Plessis excludes it in the pseudo-unitary setting [7]. The two vanishing indicators of a non-self-dual simple give a direct exclusion.

Proposition 4.7. *A pseudo-unitary categorification of $S(m)$ requires $m \leq 1$.*

Proof. Suppose $m \geq 2$, set $d = m + \sqrt{m^2 + 1}$ and $D = 2 + 2d^2 = 4 + 4md$. The codegrees are D, $2+2d^{-2}$, 4, 4. Thus the three nonunit summands of $I(1)$ have dimensions $d^2, 1+md, 1+md$. Their restrictions contain the unit once, no g, and respectively $2m, m, m$ copies of x, y in total: this follows from $FI(1) = 4 \cdot 1 + 2mx + 2my$ and irrationality of d. Consequently their combined twist-one mass in $I(x) \oplus I(y)$ is

$$W = 2md^2 + 2m(1 + md) = 4m + 6m^2 d.$$

Both first and second indicators of x and y vanish. Integrating $|1 + z + z^2|^2$ against the positive twist measure of $I(x) \oplus I(y)$ therefore gives $9W \leq 3(2dD)$. But

$$3W - 2dD = 4m + (2m^2 - 8)d > 0 \qquad (m \geq 2),$$

a contradiction. □

Theorem 4.8. *The non-self-dual Grothendieck rings of rank-four complex fusion categories are precisely*

$$\mathrm{NG}(C_3, 0), \quad \mathrm{NG}(C_3, 2), \quad \mathrm{NG}(C_3, 3), \quad \mathrm{NG}(C_3, 6), \quad S(0), \quad S(1).$$

Every such category admits a spherical structure, and its Frobenius–Perron dimension is at most $24 + 12\sqrt{3} < 45$.

Proof. Corollary 4.4 gives a pseudo-unitary Galois conjugate with the same ring. Apply Lemma 4.5 and the two preceding propositions. For existence use the models and references in Table 1. Their dimension formulas are $3+\delta^2$ with $\delta = (m+\sqrt{m^2+12})/2$ in the near-group case, and $2+2d^2$ with $d = m+\sqrt{m^2+1}$ for $S(m)$. They give the stated upper bound. Sphericality was already established in Theorem 4.1, independently of these models. □

5 Categories with a nontrivial invertible object

Theorem 5.1. *Every self-dual rank-four complex fusion category with a nontrivial invertible simple object admits a spherical structure.*

Use the sphericalization notation of Section 2, choosing positive signed dimensions β_i and a maximal conjugate E. Lemma 2.2 shows that, in the absence of a spherical section,

$$E < D. \tag{5.1}$$

If D is rational, the category is weakly integral and already spherical by [10, Propositions 8.23–8.24]. We therefore restrict attention to irrational D.

Proposition 5.2. *Let $\mathcal{C}$ be a self-dual rank-four complex fusion category. Suppose that both its ordinary rational Grothendieck algebra and the signed rational algebra of its sphericalization are the product of $\mathbb{Q}$ with a totally real cubic field, and that the Frobenius–Perron and spherical dimension characters belong to the respective cubic factors. Then the ordinary rational character does not have formal codegree two.*

Proof. Suppose its codegree is two. After relabeling, the rational character is $(1,-1,0,0)$ on the basis $\mathbf{1}, G, X, Y$: orthogonality to the positive Frobenius–Perron character fixes the negative sign and gives $\mathrm{FPdim}(G) = 1$. Thus G is invertible. Its values on the three cubic characters are all one, so $G^2 = \mathbf{1}$, $GX = X$, and $GY = Y$.

Let $\mathcal{S}$ be the sphericalization, with central sign ε. Choose positive lifts G^+, X^+, Y^+ and their sign translates. The lift G^+ has dimension one, satisfies $(G^+)^2 = \mathbf{1}$, and fixes X^+, Y^+ on both sides: the alternative sign translates have the wrong nonzero dimension. In the signed quotient write G, X, Y for their classes. Its rational character is again $(1,-1,0,0)$ and has codegree two. Its multiplication has the form

$$\begin{aligned} X^2 &= 1 + G + aX + bY, \quad XY = bX + cY, \\ Y^2 &= 1 + G + cX + eY, \end{aligned}$$

with integers a, b, c, e, which may be negative. The signed Casimir is therefore

$$1 + G^2 + X^2 + Y^2 = 4 + 2G + (a+c)X + (b+e)Y.$$

Put $x = \dim X^+$, $y = \dim Y^+$, and $E = 2 + x^2 + y^2$. Since the dimension character is faithful on the cubic quotient by $G-1$, the numbers $1, x, y$ are linearly independent over $\mathbb{Q}$. Writing $p = a+c$ and $q = b+e$ gives

$$E = 6 + px + qy, \qquad E/2 = 3 + (p/2)x + (q/2)y. \tag{5.2}$$

Let $F : Z(\mathcal{S}) \to \mathcal{S}$ be forgetful and I its right adjoint. The ordinary and signed rational characters give two distinct one-dimensional characters of $K_0(\mathcal{S})$, with ε acting by $+1$ and -1, respectively. Both have formal codegree four. The central-induction theorem therefore supplies distinct constituents B_+, B_- of $I(\mathbf{1})$, each occurring once and satisfying

$$\dim B_+ = \dim B_- = \frac{\dim \mathcal{S}}{4} = \frac{E}{2}, \qquad \theta_{B_+} = \theta_{B_-} = 1.$$

Moreover $I(\mathbf{1})$ and $I(\varepsilon)$ have no common constituent: by adjunction and the induction formula their inner product is the trace of multiplication by ε, which is zero since it interchanges every pair of lifts. Hence each $F(B_\pm)$ contains the unit once and contains no ε.

Write its signed multiplicity at G as

$$r_G = [F(B_\pm) : G^+] - [F(B_\pm) : \varepsilon G^+].$$

Comparing dimensions with (5.2), using the independence of $1, x, y$, gives $1 + r_G = 3$. Thus $[F(B_\pm) : G^+] \geq 2$, so each $B_\pm$ occurs in $I(G^+)$ at least twice.

On the other hand,

$$\dim \operatorname{End}(I(G^+)) = \sum_{V \in \operatorname{Irr}(\mathcal{S})} \dim \operatorname{Hom}(VG^+, G^+V) = 8.$$

Here G^+ is invertible and commutes with every simple class; each summand equals one. The two multiplicities already contribute $2^2 + 2^2 = 8$. Consequently

$$I(G^+) = 2B_+ \oplus 2B_-.$$

Its twist is the identity, and its twist trace is $2E = \dim \mathcal{S}$. The center formula of [21, Theorem 4.1] for the first Frobenius–Schur indicator would give $\nu_1(G^+) = 1$. This contradicts $\nu_1(G^+) = 0$, since G^+ is a nontrivial simple object. □

Proof of Theorem 5.1. Assume there is no spherical section of the sphericalization. We have already excluded rational D. Since every simple is self-dual, the group of invertible simples has exponent two. If it has four elements the category is pointed and spherical. Thus it has exactly two elements $1, G$, and G either fixes or interchanges the other simples X, Y.

First suppose $GX = Y$. The simple Frobenius–Perron dimensions then have the two values $1, d$, with $d > 1$. Schopieray's theorem [28, Lemma 3.1] supplies a pseudo-unitary Galois conjugate. Its canonical spherical structure transports back along the inverse field automorphism, contradicting the assumption.

Now suppose $GX = X$ and $GY = Y$. The ordinary fixed-action table for a self-dual category is Dong's [6, Proposition 3.2]. Its signed analogue again has the character $(1, -1, 0, 0)$ on $1, G, X, Y$: the positive lift of G has dimension one, squares to the unit and fixes the positive lifts of X, Y, since their negative lifts have the opposite dimensions. Equivalently, the line spanned by $1 - G$ realizes this character in the signed algebra. Its codegree is two. The signed dimension field therefore has degree at most three, and contains the irrational D.

If it is quadratic, the general signed codegree-two obstruction, Theorem 7.2, gives a spherical structure. Its proof uses the signed table, the codegree and indicator bounds of Sections 2–3, and the quartic order criterion of Corollary 6.5. The latter uses the ordinary dimensional grading and Lemma 6.4, independently of the present theorem. Finally, if the signed field is cubic, it contains the irrational D, so both ordinary and signed rational algebras are products of $\mathbb{Q}$ and a cubic field, with the dimension characters in the cubic factors. Their rational characters have codegree two. This contradicts Proposition 5.2. Every case gives a spherical section, completing the proof. □

5.1 The Fibonacci normal form

Dong gives the ordinary table for a self-dual rank-four fusion category containing a Fibonacci subcategory [6, Proposition 3.3]. The argument below also proves that the Fibonacci relation itself forces self-duality of the remaining simples and determines the signed table.

Lemma 5.3 (Fibonacci normal form)**.** *A rank-four fusion ring containing f with $f^2 = 1 + f$ is self-dual and has a basis $1, f, X, Y$ with*

$$\begin{aligned} fX = Y, \qquad fY = X + Y, \qquad X^2 = 1 + aX + bY, \\ XY = f + bX + (a+b)Y, \qquad Y^2 = 1 + f + (a+b)X + (a+2b)Y, \end{aligned} \tag{5.3}$$

where $a, b \geq 0$ are integers. Its positive character is $(1, \phi, x, \phi x)$, where $\phi = (1+\sqrt{5})/2$ and $x^2 = 1 + (a + b\phi)x$. In a normalized signed sphericalization quotient, the same formulas hold with integers α, β satisfying $|\alpha| \leq a$, $|\beta| \leq b$, $\alpha \equiv a$ and $\beta \equiv b \pmod 2$.

Proof. On the two outside simples, multiplication by f has a symmetric nonnegative integral matrix M satisfying $M^2 = I + M$. Its characteristic polynomial is $t^2 - t - 1$ by irreducibility, so the diagonal entries are $0, 1$ and the off-diagonal entry is one. The ring is commutative by Lemma 2.1. Duality commutes with M and cannot exchange the two different diagonal entries. Hence all simples are self-dual. Frobenius reciprocity and associativity give the displayed products. For a signed quotient, first change the sign of f so that $f^2 = 1 + f$. A sign change of an outside basis element normalizes its nonzero off-diagonal action coefficient to one. The same calculation then gives the signed formulas; domination and parity give the stated restrictions on α, β. □

6 Central columns and descent of the dimension field

Let $\mathcal{C}$ be a self-dual rank-four complex fusion category, $\mathcal{S}$ its sphericalization, and ε its sign object. Write $R = K_0(\mathcal{C})$ and $T = K_0(\mathcal{S})/(\varepsilon + 1)$, and put $D = \mathrm{FPdim}(\mathcal{C})$ and $2E = \dim(\mathcal{S})$. The ordinary and signed rational algebras are commutative, as explained above. Let L be the field generated by the signed spherical dimensions. It is totally real and abelian. Central induction gives $E/D \in L$, so $D \in L$.

We use the canonical normalized modular matrix $S = s/(2E)$ of $Z(\mathcal{S})$. For a one-dimensional character χ of $K_0(\mathcal{S})$, let A_χ be the corresponding simple constituent of $I(\mathbf{1})$. The central-induction theorem and modular-character formula give

$$[I(\mathbf{1}) : A_\chi] = 1, \qquad \dim A_\chi = \frac{2E}{f_\chi}, \qquad \frac{S_{A_\chi, V}}{S_{A_\chi, 0}} = \chi(F(V)). \tag{6.1}$$

In this display f_χ is the codegree in $K_0(\mathcal{S})$, twice the corresponding codegree of R or T. Here $F : Z(\mathcal{S}) \to \mathcal{S}$ is forgetful; see [24, Example 2.9 and Theorem 2.13].

Proposition 6.1 (Simultaneous Galois columns)**.** *Let L be the signed dimension field and $\sigma \in \mathrm{Gal}(L/\mathbb{Q})$. Let B_σ be the induction constituent associated with the signed dimension character followed by σ. If χ is any ordinary or signed character and f_χ its codegree in R or T, then*

$$\chi(F(B_\sigma)) = \frac{f_\chi}{\sigma(f_\chi)}.$$

Consequently the codegrees determine both its folded and signed forgetful classes, even if the rational algebras are products of fields. Their coefficient vectors w_σ, z_σ satisfy

$$w_{\sigma,0} = z_{\sigma,0} = 1, \qquad w_{\sigma,i} \geq |z_{\sigma,i}|, \qquad w_{\sigma,i} \equiv z_{\sigma,i} \pmod 2.$$

Proof. Let A_χ be the induction constituent for χ. Its dimension is E/f_χ and $S_{A_\chi,0} = 1/(2f_\chi)$. The modular-character formula gives

$$S_{B_\sigma, A_\chi} = \frac{\sigma(E/f_\chi)}{2\sigma(E)} = \frac{1}{2\sigma(f_\chi)}.$$

Symmetry of S gives the asserted character value. All characters together separate the commutative semisimple algebra, so these values determine each class. If the two lift multiplicities at X_i are m_i^+, m_i^-, the two coefficients are their sum and difference. Adjunction gives unit multiplicity one. The sign multiplicity is zero: $I(\varepsilon) = \widehat{\varepsilon} \otimes I(1)$ has negative-dimensional constituents, whereas the constituents of $I(1)$ have positive dimensions. □

Proposition 6.2 (Positive Galois columns)**.** *Suppose* $[\mathbb{Q}(D) : \mathbb{Q}] = 4$. *Identify* R *with its Frobenius–Perron image and* T *with its spherical-dimension image in the common field* $K = L = \mathbb{Q}(D)$. *For every* $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$, *the element*

$$u_\sigma = \frac{D}{\sigma(D)}$$

belongs to R *and has nonnegative integral coefficients in its distinguished basis. Write* w_σ *for its coefficient vector and* z_σ *for the signed coefficient vector of* $v_\sigma = E/\sigma(E) \in T$. *Then*

$$w_{\sigma,0} = z_{\sigma,0} = 1, \qquad w_{\sigma,i} \geq |z_{\sigma,i}|, \qquad w_{\sigma,i} \equiv z_{\sigma,i} \pmod 2.$$

Proof. Apply Proposition 6.1 to all ordinary and signed characters. In a quartic field the dimension maps are injective, so the folded and signed elements are respectively $D/\sigma(D)$ and $E/\sigma(E)$. The coefficient assertions are those of the proposition. □

Proposition 6.3 (Descent of the dimension field)**.** *If* $L \neq \mathbb{Q}(D)$, *then* $\mathcal{C}$ *admits a nontrivial faithful grading by a finite cyclic group.*

Proof. Choose a nonidentity $\sigma \in \mathrm{Gal}(L/\mathbb{Q}(D))$. Define B_σ as in the preceding proof. The same calculation applies without any injectivity assumption on the dimension maps and gives

$$\mathrm{FPdim}(F(B_\sigma)) = D/\sigma(D) = 1.$$

Adjunction gives $[F(B_\sigma) : \mathbf{1}] = 1$. Consequently $F(B_\sigma) = \mathbf{1}$ in $\mathcal{S}$. The character defining B_σ differs from the dimension character, whose constituent is the tensor unit. Thus $B_\sigma \neq \mathbf{1}$ in $Z(\mathcal{S})$.

A central structure on the tensor unit is a monoidal natural automorphism γ of $\mathrm{Id}_\mathcal{S}$, and here γ is nonidentity. Its value at ε is one. Indeed the modular-character formula evaluated at the canonical central lift $\widehat{\varepsilon}$ gives -1; on the other hand this value is $\dim(\varepsilon)$ times the monodromy scalar, namely $-\gamma_\varepsilon$. Hence $\gamma_\varepsilon = 1$.

The scalar values of γ therefore agree on the two lifts of every simple of $\mathcal{C}$. They descend to a nonidentity monoidal natural automorphism of $\mathrm{Id}_\mathcal{C}$: if an ordinary fusion coefficient is nonzero, one of its lifted coefficients is nonzero and supplies the required product relation. The finitely many scalar values form a subgroup of $\mathbb{C}^\times$, since tensor products give products and duals give inverses. This finite group is cyclic and gives the asserted faithful grading. □

Lemma 6.4. *A rank-four complex fusion category with a nontrivial faithful grading is either weakly integral or has a nontrivial invertible simple object.*

Proof. The neutral component has rank one, two, or three. In rank one, $X \otimes X^*$ is a sum of units for every simple X, and its unit multiplicity is one. Thus every simple is invertible.

In rank two, the neutral component is pointed or has Fibonacci fusion rules by Proposition 2.10. In the pointed case its global dimension is two, and equality of the dimensions of homogeneous components makes the whole category weakly integral. In the Fibonacci case write $f^2 = 1 + f$ and $\phi = (1 + \sqrt{5})/2$. A component of rank one is impossible: the integer multiplicity for multiplication by f would satisfy $m^2 = 1 + m$. Thus there is just one other component, of rank two. On its simples X, Y, multiplication by the self-dual object f has a

symmetric nonnegative integer matrix M satisfying $M^2 = I + M$. Its eigenvalues are the two roots of $t^2 - t - 1$, so its trace is one and determinant is minus one. Up to exchanging X, Y,

$$M = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \qquad \operatorname{FPdim}(Y) = \phi \operatorname{FPdim}(X).$$

Equality of component dimensions now gives $(1 + \phi^2) \operatorname{FPdim}(X)^2 = 1 + \phi^2$. Hence X is a nontrivial invertible object.

In rank three, there is a single nonneutral simple X, and the grading group has order two. Each neutral simple Y satisfies $Y \otimes X = m_Y X$, so $\operatorname{FPdim}(Y) = m_Y$ is integral. The two homogeneous components have equal Frobenius–Perron dimension; hence the total dimension is an integer. See [10, Proposition 8.20] for equality of component dimensions. □

Corollary 6.5. *If the rational Grothendieck algebra of a rank-four complex fusion category is a quartic field, its Frobenius–Perron global dimension has degree four.*

Proof. Its Frobenius–Perron character identifies the algebra with a quartic field $K = \mathbb{Q}(d_i)$. This character is injective, so the unit is the only simple of dimension one. If D were rational, its four conjugates, counted with multiplicity, would all be D. The reciprocal-codegree identity would give $D = 4$, making every simple invertible. Thus D is irrational. If $\mathbb{Q}(D) \subsetneq K$, the dimensional grading of Gannon–Schopieray [12, Section 1.2] is nontrivial since its group has order $[K : \mathbb{Q}(D)]$. Lemma 6.4 would force weak integrality or a nontrivial invertible simple, both of which we have just excluded. □

Corollary 6.6. *Every nonspherical self-dual rank-four complex fusion category satisfies $L = \mathbb{Q}(D)$.*

Proof. Otherwise the preceding proposition and lemma apply. Weakly integral fusion categories are spherical by [10, Proposition 8.24]. Theorem 5.1 gives a spherical structure in the remaining case, a contradiction. □

7 Quadratic codegree obstructions

Ostrik proved that formal codegrees are d-numbers [25], and Schopieray studied the norms and traces of formal codegrees, including finiteness at fixed norm [29]. Here the ordinary and signed codegrees occur in the same quadratic field. Comparing their two norm pairs first eliminates rational signed rows and then bounds the ordinary global dimension. The same list of short quadratic character rows applies to both tables.

Lemma 7.1 (The quadratic-unit gap)**.** *Let $0 < t < 1$ be a totally positive quadratic algebraic integer of norm one. Then $t \le (3 - \sqrt{5})/2$. Unless its field is $\mathbb{Q}(\sqrt{5})$, one has $t \le 2 - \sqrt{3}$.*

Proof. Its integral trace $t + t^{-1}$ is at least three. Trace three gives $\mathbb{Q}(\sqrt{5})$; otherwise the trace is at least four. Solve the quadratic equation and take its smaller root. □

7.1 A quadratic signed dimension field with codegree two

Theorem 7.2. *Let $\mathcal{C}$ be a self-dual rank-four complex fusion category. If the signed quotient of its sphericalization has a quadratic spherical dimension field and a rational formal codegree two, then $\mathcal{C}$ admits a spherical structure.*

We prove that a proper signed quotient is impossible. Choose a field embedding and lifts with positive dimension values, and let $E \geq E'$ be the two dimension codegrees of the signed quotient T. If the ordinary Frobenius–Perron dimension D is rational, the category is weakly integral and hence pseudo-unitary [10]. Otherwise D is quadratic in the signed dimension field K. A proper quotient has $E < D$: equality forces equality in the spectral comparison with the ordinary nonnegative table, giving a spherical section. We do not assume that the ordinary Frobenius–Perron *character* field equals its codegree field; that distinction will be addressed below.

7.1.1 The signed table and its norm restrictions

A character with codegree two is rational. Indeed its three nonunit coordinates are totally real algebraic integers whose squares sum to one at every embedding. Each nonzero coordinate has absolute norm at least one and absolute value at most one at every embedding, so is ± 1. Orthogonality to the positive row therefore puts the codegree-two row in the form $(1, -1, 0, 0)$. Orthogonality makes every other character value on X equal to one. Thus multiplication by X is a symmetric integral involution. Its negative eigenspace is already spanned by $1 - X$, so it fixes Y, Z and exchanges $1, X$. Consequently

$$\begin{aligned} &X^2 = 1, && XY = Y, && XZ = Z, \\ &Y^2 = 1 + X + aY + bZ, && YZ = bY + cZ, && Z^2 = 1 + X + cY + dZ, \end{aligned} \tag{7.1}$$

with integral coefficients and $b^2 + c^2 = 2 + ac + bd$. The other rational character is $(1, 1, r, t)$, with

$$h = 2 + r^2 + t^2 > 2, \qquad 2 + ru + tv = 0,$$

where $(1, 1, u, v)$ is the positive dimension character.

Put

$$p = D^{-1} + (D')^{-1}, \quad q = E^{-1} + (E')^{-1} = \frac{1}{2} - \frac{1}{h}, \quad k = \frac{EE'}{DD'} \in \mathbb{Z}_{>0}.$$

Central induction gives integrality of $E/D, E/2, E/h$. Each of the other two ordinary codegrees is at most D, so $p \leq 1 - 2/D$, whether or not those characters are rational. Comparison on the upper-root interval yields

$$kq < p + \frac{k-1}{D} \leq 1 + \frac{k-3}{D}. \tag{7.2}$$

Moreover

$$s_0 = \frac{kD}{E} = (E/D)' \quad \text{is integral}, \qquad \mathrm{Norm}(s_0) = k, \qquad k < s_0 < E'. \tag{7.3}$$

Here $D' = E'/s_0 > 1$ because every formal codegree exceeds one. For $k = 1$, Lemma 7.1 gives $s_0 \geq (3 + \sqrt{5})/2$; for $k = 2$, the integral trace $s_0 + 2/s_0$ is at least four (since $s_0 > 2$), so $s_0 \geq 2 + \sqrt{2}$. If $h \geq 6$, then $q \geq 1/3$ and (7.2), with $D \geq 4$, forces $k \leq 2$: for $k \geq 3$ it would give $k/3 < (k+1)/4 \leq k/3$.

For $h \geq 11$, norm divisibility gives $EE' \geq h^2$, whence

$$E' \leq \frac{4h}{h - 2 + \sqrt{(h-2)^2 - 16}} < \frac{4h}{2h - 5} \leq \frac{44}{17} < \frac{3 + \sqrt{5}}{2}.$$

This contradicts (7.3). In particular $h \leq 10$. The values $h = 5, 8, 9$ are impossible because $h - 2 = 3, 6, 7$ is not a sum of two squares. We next treat $h = 4, 6, 7, 10$ explicitly.

For $h = 4$, the entries r, t have absolute value one. Both positive are incompatible with $2 + ru + tv = 0$. Both negative give $u + v = 2$ and $E < 6$, whereas $E \geq 2/q = 8$. Thus take $(r, t) = (1, -1)$. Evaluation of (7.1) gives

$$v = u + 2, \quad (a, b, c, d) = (b - 1, b, b + 1, b + 2), \quad u^2 = (2b - 1)u + 2b + 2.$$

For $\Delta = 4b^2+4b+9$, the codegrees have trace 2Δ and norm 8Δ. Since Δ is odd, $\mathrm{Norm}(E/4) = \Delta/2$ is not an integer, a contradiction.

For $h = 6$, take $(r, t) = (0, -2)$. Then

$$v = 1, \quad (a, b, c, d) = (\mu, 1, 0, -1), \quad u^2 = 3 + \mu u, \quad E = 3 + u^2, \quad E' = 3 + 9/u^2.$$

The ordering gives $\mu \geq 0$. The trace and norm of E are $\mu^2 + 12$ and $3(\mu^2 + 12)$; integrality of $E/6$ forces $6 \mid \mu$. If $\mu > 0$, then $E' < 13/4$ and $K = \mathbb{Q}(\sqrt{\mu^2 + 12})$ is ramified at three. Hence K is neither $\mathbb{Q}(\sqrt{5})$ nor $\mathbb{Q}(\sqrt{2})$. For $k = 1$ the trace of s_0 is at least four, so $s_0 \geq 2 + \sqrt{3} > E'$. For $k = 2$ its trace is at least five, so $s_0 > 4 > E'$. Both are impossible. If $\mu = 0$, then $E = E' = 6$ and $K = \mathbb{Q}(\sqrt{3})$. Norm two is impossible in $\mathbb{Z}[\sqrt{3}]$ modulo three, so $k = 1$. The norm-one unit $1 < s_0 < 6$ has even trace four or six; trace six gives the wrong field. Thus this branch leaves only

$$D = 12 + 6\sqrt{3}. \tag{7.4}$$

For $h = 7$, integrality of $E/7$ gives $EE' = 49A$ and $\mathrm{Tr}(E/7) = 5A/2$, so A is even and $EE' \geq 98$. Consequently

$$E' \leq (35 - \sqrt{833})/2 < 13/4.$$

This excludes $k = 2$ and forces, for $k = 1$, a unit of trace three, hence $K = \mathbb{Q}(\sqrt{5})$. The absolute values of r, t are one and two. Both negative give $u + 2v = 2$ and $E < 6$, contrary to $E \geq (E + E')/2 \geq 35/2$. The two remaining sign choices yield, for an integer j,

(r, t)	(a, b, c, d)	equation for v
$(1, -2)$	$(8j - 5, 4j - 2, 2j, j - 1)$	$v^2 + (1 - 5j)v + 4j - 2 = 0$
$(-1, 2)$	$(8j + 5, 4j + 2, 2j, j + 1)$	$v^2 - (5j + 1)v - 4j - 2 = 0.$

For $\Delta = 25j^2 \pm 26j + 9$ with the corresponding sign, the codegree trace and norm are $5\Delta, 14\Delta$. Integrality of $E/7$ forces $7 \mid \Delta$, while

$$25\Delta = (25j \pm 13)^2 + 56$$

then gives $v_7(\Delta) = 1$. This field cannot be $\mathbb{Q}(\sqrt{5})$.

For $h = 10$, take $(r, t) = (2, -2)$; the other signs contradict positivity or give $u + v = 1$ and $E < 3$, whereas $E \geq 5$. Evaluation gives $v = u + 1$ and $(a, b, c, d) = (b + 1, b, b + 2, b + 1)$. With $m = b + 1$,

$$u^2 = (2m - 1)u + m + 1, \qquad \mathrm{Tr}(E) = 2(4m^2 + 5), \quad \mathrm{Norm}(E) = 5(4m^2 + 5).$$

The norm is odd, contradicting the integrality of $E/2$.

7.1.2 The codegrees two and three

Only $h = 3$ remains. Its rational row can be taken as $(1, 1, 0, -1)$, so evaluation gives

$$(a, b, c, d) = (6j, 2, 0, 1), \quad v = 2, \quad u^2 = 6 + 6ju, \quad j \geq 0,$$

$$E = 6 + u^2, \qquad E' = 6 + 36/u^2.$$

The multiple of six follows from the integral traces of $E/2$ and $E/3$. Suppose first $j > 0$. Then $E > 42$, $E' < 7$, and (7.2) gives $k \le 6$. The integral trace $t_0 = s_0 + k/s_0$ lies between $k+2$ and seven. The field $K = \mathbb{Q}(\sqrt{3(3j^2+2)})$ is ramified at three, so $t_0^2 - 4k$ has odd three-adic valuation. The entire resulting list is

$$(k, t_0, K) = (1, 4, \mathbb{Q}(\sqrt{3})),\ (1, 5, \mathbb{Q}(\sqrt{21})),\ (3, 6, \mathbb{Q}(\sqrt{6})),\ (4, 7, \mathbb{Q}(\sqrt{33})).$$

Indeed $1 \le k \le 6$ and $k+2 \le t_0 \le 7$ are finite integer intervals; imposing odd three-adic valuation gives exactly these four entries. The first two are impossible modulo three: they would require $3j^2+2$ to be a square or seven times a square. For the last, $s_0 = (7+\sqrt{33})/2 > 25/4$, while $j \ge 2$ gives $E' < 6 + 1/j^2 \le 25/4$; at $j = 1$ the field is instead $\mathbb{Q}(\sqrt{15})$. The remaining possibility is

$$\begin{gathered} k = 3, \quad K = \mathbb{Q}(\sqrt{6}), \quad 2w^2 - 3j^2 = 2, \quad u = 3j + w\sqrt{6}, \\ DD' = 48w^2, \qquad EE' = 144w^2, \qquad p = \frac{w+j}{2w}. \end{gathered} \tag{7.5}$$

The Pell equation makes j even. Since $j = 2$ would give $w^2 = 7$, one has $j \ge 4$, $w \ge 5$, and

$$(j/w)^2 = \frac{2}{3}(1 - w^{-2}) \ge \frac{16}{25}, \qquad p \ge \frac{9}{10}.$$

Apply Lemma 3.2, with U_2, V_2 the ordinary and signed inverse-square codegree sums:

$$E(U_2 + V_2 - 1) \le 1 + \sqrt{3(E-1)}. \tag{7.6}$$

Cauchy's inequality on the other two ordinary codegrees gives

$$U_2 \ge p^2 + \frac{(1-p)^2}{2} - \frac{2}{DD'} \ge \frac{163}{200} - \frac{1}{600} = \frac{61}{75}.$$

The signed reciprocal identity gives

$$V_2 = \frac{1}{4} + \frac{1}{9} + \frac{1}{36} - \frac{2}{144w^2} \ge \frac{233}{600}.$$

Hence $U_2 + V_2 - 1 \ge 121/600$. But $E \ge (E+E')/2 = EE'/12 \ge 300$, so (7.6) bounds this by less than $1/300 + 1/10 = 31/300 < 121/600$. This excludes the whole positive Pell family without restrictions on the other ordinary characters.

If $j = 0$, then $E = E' = 12$, $K = \mathbb{Q}(\sqrt{6})$. Write $s_0 = c + d\sqrt{6}$. The inequalities $0 < s_0' < 1 < s_0 < 12$ imply $1 \le c \le 6$ and $1 \le d \le 2$. Then $0 < c - d\sqrt{6} < 1$ leaves exactly $s_0 = 3 + \sqrt{6}$ of norm three or $s_0 = 5 + 2\sqrt{6}$ of norm one. Thus the remaining possibilities are

$$D = 12 + 4\sqrt{6} \quad \text{or} \quad D = 60 + 24\sqrt{6}. \tag{7.7}$$

7.1.3 Excluding the three ordinary dimensions

The ordinary Frobenius–Perron character field is $K = \mathbb{Q}(D)$: a quartic character field with quadratic global codegree is excluded by Corollary 6.5.

Lemma 7.3. *None of $12 + 6\sqrt{3}$, $12 + 4\sqrt{6}$, and $60 + 24\sqrt{6}$ can be the codegree of a rank-four Frobenius–Perron character with values in the respective quadratic field.*

Proof. Write its nonunit coordinates as $a_i + b_i\sqrt{d}$ for $d = 3, 6$. Orthogonality to the conjugate character gives $1 + |a|^2 = d|b|^2$. Comparing coefficients in the codegree yields

D	$\lvert a\rvert^2$	$\lvert b\rvert^2$	$a \cdot b$
$12 + 6\sqrt{3}$	5	2	3
$12 + 4\sqrt{6}$	5	1	2
$60 + 24\sqrt{6}$	29	5	12.

Up to a simultaneous signed permutation, in the first case $b = (1, 1, 0)$. Then $a_1 + a_2 = 3$ and $(a_1 - a_2)^2 + 2a_3^2 = 1$. In the second case $b = (1, 0, 0)$, so $a_1 = 2$ and $a_2^2 + a_3^2 = 1$. In the third case $b = (2, 1, 0)$ and eliminating a_2 gives $(5a_1 - 24)^2 + 5a_3^2 = 1$. In every case a nonunit coordinate vanishes, contrary to positivity. □

This completes the proof of Theorem 7.2.

7.2 Short signed quadratic orbits

Theorem 7.4. *Let $\mathcal{C}$ be a self-dual rank-four complex fusion category with irrational quadratic Frobenius–Perron global dimension D. Suppose its signed dimension field is $K = \mathbb{Q}(D)$. If $\mathcal{C}$ is not spherical, its signed rational algebra is $K \times K$. The two formal codegrees in the complementary character orbit are genuinely irrational, and their product is at least* 100.

The proof has three steps. We enumerate short quadratic pairs once for both tables, eliminate the rational complementary codegrees, and then exclude the five surviving signed pairs by comparing their dimensions with the ordinary ones.

7.2.1 Common quadratic comparisons

Assume throughout the proof that $\mathcal{C}$ is not spherical. Choose the signed dimension orbit $E \geq E'$ and put

$$N = DD', \qquad H = EE' = kN, \qquad p = D^{-1} + D'^{-1}, \qquad q = E^{-1} + E'^{-1}.$$

Spectral domination and saturation give $E < D$, and central induction gives $k \in \mathbb{Z}_{>0}$. Every ordinary codegree is at most D. The two outside the Frobenius–Perron orbit therefore have reciprocal sum at least $2/D$. Lemma 3.8 gives

$$p \leq 1 - \frac{2}{D}, \qquad kq < p + \frac{k-1}{D} \leq 1 + \frac{k-3}{D}. \tag{7.8}$$

If $q \geq 1/3$, then $k \leq 2$: for $k \geq 3$, use $D \geq 4$ in the last inequality. When $k = 2$ it gives $q < 1/2$. When $k = 1$, the totally positive unit $r = D/E > 1$ satisfies

$$r \geq \frac{3 + \sqrt{5}}{2}, \qquad \frac{q}{p} \leq \frac{2}{r + r^{-1}} \leq \frac{2}{3}.$$

In particular $q < 2/3$. If $q = 1/4$, then $E \geq 8$ and $D > 8$; (7.8) gives $k \leq 4$.

All simple Frobenius–Perron dimensions belong to K by Corollary 6.5; this justifies the ordinary dimension reconstructions below.

We repeatedly use the following two bounds, where u, v are the ordinary and signed reciprocal-square codegree sums:

$$u \geq p^2 + \frac{(1-p)^2}{2} - \frac{2}{N}, \qquad u + v - 1 \leq \frac{1 + \sqrt{3(E-1)}}{E}. \tag{7.9}$$

The first is Cauchy's inequality for the two complementary ordinary reciprocals, and does not assume that their characters are quadratic. The second is Lemma 3.2.

Lemma 7.5 (Small integral character rows)**.** *For a self-dual integral Frobenius table of rank four, a rational character cannot have codegree five, eight, or sixteen. Distinct rational characters cannot have codegree pair* $(3,4)$*,* $(3,12)$*, or* $(9,9)$*. A rational character orthogonal to a codegree-four character has even codegree.*

Proof. A rational character is an integral row $(1,a,b,c)$ with codegree $1+a^2+b^2+c^2$. At codegree five its nonunit row is $(\pm 2,0,0)$ up to permutation; evaluating the corresponding square would require $4 = 1 \pm 2m$. For eight and sixteen, the nonunit squared norm is 7 modulo eight and cannot be a sum of three squares. At codegrees $3, 4, 12, 9$ the absolute nonunit rows are respectively $(1,1,0),(1,1,1),(3,1,1),(2,2,0)$. The three excluded pairs cannot have nonunit inner product -1. Finally a codegree-four row is odd in every coordinate. Orthogonality modulo two makes the sum of the other row's coordinates, hence its squared norm, even. □

7.2.2 The complete short character chart

Lemma 7.6. *Consider a pair of conjugate quadratic characters of a self-dual integral Frobenius table of rank four, orthogonal to a character whose four coordinates are nonzero. Suppose their formal codegree* G *is a* d*-number and* $L = GG' < 100$*. Up to simultaneous signed permutation of the three nonunit coordinates and conjugation, their rows occur among the following possibilities. Here*

$$\chi = (1, (t_i + s_i\sqrt{d})/2)_{i=1}^{3}, \qquad \ell = G^{-1} + G'^{-1}.$$

Signs of structure constants are unrestricted; this chart applies equally to ordinary and signed tables. Entries marked $*$ *have rational codegree.*

d	t	s	L	ℓ	$\|U\|^2$
5	2 −1 −1	0 1 1	20	1/2	1
5	2 −1 1	0 1 1	25*	2/5	6
5	−3 −1 −1	1 1 1	25	3/5	6
2	2 −2 −2	0 2 2	32	1/2	1
3	0 −4 −2	0 2 2	36	2/3	2
5	−3 −1 1	1 1 1	45	1/3	2
5	−3 −1 −4	1 1 2	45	2/3	2
13	2 −3 −3	0 1 1	52	1/2	1
2	2 −2 2	0 2 2	64*	1/4	3
2	−2 −2 −6	2 2 4	64	3/4	3
2	−4 −2 0	2 2 2	72	1/3	2
5	2 −4 −4	0 2 2	80	1/2	1
3	2 −4 0	0 2 2	96	1/4	3
3	−4 −4 −6	2 2 4	96	3/4	3

The last column gives the squared norm of the primitive integer vector parallel to the nonunit part of the idempotent for this pair. In the four rows where it is one, that basis element squares to one.

Proof. Put $J = |s|^2$ and $C = t \cdot s$. Orthogonality of the conjugate rows gives

$$|t|^2 = dJ - 4, \quad G = \frac{dJ + C\sqrt{d}}{2}, \quad L = \frac{d}{4}(4J + |t \times s|^2), \quad \ell = \frac{dJ}{L}. \tag{7.10}$$

Character values are algebraic integers, so t_i, s_i are integers; they are both even if $d \not\equiv 1 \pmod 4$, and have the same coordinatewise parity otherwise. Orthogonality to a nonzero-coordinate character rules out support one for s. The reciprocal identity gives $0 < \ell < 1$, hence $dJ < L < 100$. The d-number condition is $L \mid (dJ)^2$.

Here is a bounded arithmetic check of the chart. Take s nonnegative and ordered, and take $t_i \geq 0$ wherever $s_i = 0$. If $d \not\equiv 1 \pmod 4$, write $s = 2b, t = 2a$. The bounds are

$$d \in \{2,3,6,7,10,11\}, \qquad 2 \leq |b|^2 \leq \lfloor 24/d \rfloor,$$
$$|a|^2 = d|b|^2 - 1, \qquad |b|^2 + |a \times b|^2 \leq \lfloor 24/d \rfloor.$$

If $d \equiv 1 \pmod 4$, they are

$$d \in \{5,13,17,21,29,33,37,41\}, \qquad 2 \leq J \leq \lfloor 99/d \rfloor, \qquad |t|^2 = dJ - 4.$$

For each permitted d, J, take the nonnegative ordered three-square representations of J. For each such s, take

$$|t_1| \leq \sqrt{dJ-4}, \quad |t_2| \leq \sqrt{dJ - 4 - t_1^2}, \quad t_3 = \pm\sqrt{dJ - 4 - t_1^2 - t_2^2},$$

retaining integral values with the stated parity. Apply (7.10), $dJ < L < 100$ and $L \mid (dJ)^2$. The resulting L lists, before identifying the repeated values, are

$d = 2$	$32, 64, 64, 72$
$d = 3$	$36, 96, 96$
$d = 5$	$20, 25, 25, 45, 45, 80$
$d = 13$	$52,$

and the other displayed radicands give no row. Substitution gives exactly the fourteen t, s pairs above; the remaining identifications are precisely signed permutations and conjugation. This is a finite sum-of-squares check, with every range and divisibility specified independently of any fusion-ring multiplicity.

The idempotent P projecting onto the character pair has unit coefficient ℓ and nonunit coefficients

$$P_i = \frac{d(Jt_i - Cs_i)}{2L}.$$

Clear their denominators and divide by their common gcd to obtain U. If $U = \lambda(P - \ell)$, its two eigenvalues are $\lambda(1-\ell)$ and $-\lambda\ell$; this computes the last column. In the norm-one rows $\ell = 1/2$ and $\lambda = 2$, so $U^2 = 1$. □

7.2.3 Rational complementary signed codegrees

Lemma 7.7. *The two complementary signed codegrees cannot both be rational.*

Proof. A rational signed codegree two already gives sphericality by Theorem 7.2; hence assume all rational codegrees under consideration are at least three.

First suppose the complementary characters form a quadratic orbit, with a repeated rational codegree g. Then $q = 1 - 2/g$. The preceding comparisons leave only $g = 3, 4, 5$, and $k = 1$ for $g = 4, 5$. Lemma 7.6 applies with $L = g^2$. Its only constant-codegree entry with $g \leq 5$ has $g = 5$.

For $g = 5$ the two rows, after signed permutation, are

$$(1, -1, \phi, \phi'), \qquad (1, -1, \phi', \phi), \qquad \phi = \frac{1+\sqrt{5}}{2}, \quad \phi' = 1 - \phi.$$

Evaluation of the symmetric table gives

$$(t, t, t, t-1, t, t-1, t, t-1, t-1, t), \qquad t \in \mathbb{Z}. \tag{7.11}$$

This calculation uses only the displayed character rows and the orthonormal based pairing, so it also applies to ordinary tables. The remaining characters are $(1, z+1, z, z)$, where $z^2 + (2-3t)z - t = 0$. With $\Delta = 9t^2 - 8t + 4$ their codegrees satisfy

$$E + E' = 3\Delta, \qquad EE' = 5\Delta.$$

The value $t = 0$ gives rational selected dimensions. Otherwise $\Delta \geq 5$, and the smaller root is

$$E' = \frac{10}{3 + \sqrt{9 - 20/\Delta}} \leq \frac{15 - 5\sqrt{5}}{2} < \frac{3 + \sqrt{5}}{2} \leq r.$$

This contradicts $D' = E'/r > 1$.

It remains to treat two rational characters, of integral codegrees $3 \leq g \leq h$. The small-row restrictions are Lemma 7.5. Since $q = 1 - 1/g - 1/h \geq 1/3$, we have $k \leq 2$. The conditions $q < 1/2$ for $k = 2$ and $q < 2/3$ for $k = 1$ now give

$$\begin{array}{l|l} k = 2 & (g, h) = (3, 3), \\ k = 1 & (3, 3),\ (3, h)\ (h \geq 6),\ (4, 4),\ (4, 6),\ (4, 10). \end{array}$$

For $(3, h)$ with $h \geq 6$, central integrality gives $H \geq h^2$. As $q = 2/3 - 1/h$, the smaller root satisfies

$$E' \leq \frac{2}{q + \sqrt{q^2 - 4/h^2}} \leq 9 - 3\sqrt{5} < \frac{3 + \sqrt{5}}{2} \leq r.$$

The second bound is attained at $h = 6$: both q and $q^2 - 4/h^2 = 4/9 - 4/(3h) - 3/h^2$ increase thereafter. For $(4, 6)$, integrality of $E/4, E/6$ implies that of $E/12$ by Bezout, so $E \geq 12$ and $E' \leq 2$. For $(4, 10)$ the same argument gives $E \geq 20$ and $E' \leq 5/3$. Again $D' = E'/r < 1$.

For $(4, 4)$ the rational rows are, up to signed permutation, $(1, -1, 1, -1)$ and $(1, -1, -1, 1)$. Their orthogonal complement has selected row $(1, 1, z, z)$. Character evaluation gives

$$X^2 = 1, \quad XY = Z, \quad XZ = Y, \quad Y^2 = Z^2 = 1 + t(Y + Z), \quad YZ = X + t(Y + Z).$$

Here $z^2 = 1 + 2tz$. Select the larger codegree and change both signs if necessary, so $t \geq 1, z > 1$; $t = 0$ would give rational dimensions. Then $E' = 2 + 2/z^2 < 5/2 < r$.

Finally, for $(3, 3)$ the rows are $(1, 0, 1, -1)$ and $(1, 1, -1, 0)$, up to signed permutation. Evaluation gives the tuple

$$(m - 1, m - 1, m, m, m, m, m, m, m + 1, m + 1), \qquad m \in \mathbb{Z},$$

and remaining characters $(1, z - 1, z, z + 1)$ with

$$z^2 = 3mz + 1, \qquad E = 3 + 3z^2, \qquad EE' = 9(9m^2 + 4). \tag{7.12}$$

The value $m = 0$ gives a zero simple dimension. Otherwise the larger codegree has $|z| > 3$ and $E' = 3 + 3/z^2 < 10/3$. The field $K = \mathbb{Q}(\sqrt{9m^2 + 4})$ is neither $\mathbb{Q}(\sqrt{2})$ nor $\mathbb{Q}(\sqrt{5})$, since either equality would require a square congruent to two modulo three. If $k = 1$, the positive unit r cannot have trace three, so $r \geq 2 + \sqrt{3} > E'$. If $k = 2$, put $s = 2D/E > 2$. This is a totally positive integer of norm two. Its trace is at least four, and equality would give $K = \mathbb{Q}(\sqrt{2})$. Thus its trace is at least five, so $s > 4 > E'$. In either case $D' < 1$, a contradiction. □

The selected character gives a quadratic quotient K of the semisimple rational algebra $T_{\mathbb{Q}}$, leaving a complementary algebra of dimension two. If the latter is $\mathbb{Q} \times \mathbb{Q}$, both codegrees are rational, contrary to Lemma 7.7. Otherwise it is a quadratic field K'. Its codegree G belongs to K' by its definition and to K by central induction. If $K' \neq K$, then $G \in K \cap K' = \mathbb{Q}$, giving a repeated rational codegree, again excluded by the lemma. Thus G is genuinely irrational and $K' = K$. This proves the $K \times K$ assertion in Theorem 7.4 without asserting field containment for arbitrary complementary character values.

7.2.4 A signed basis involution

Lemma 7.8. *Under the common quadratic hypotheses of Theorem 7.4, if the signed quotient has a nonunit basis element X with $X^2 = 1$, then the original category is spherical.*

Proof. Multiplication by X is a symmetric integral involution, hence a signed permutation matrix. It interchanges $1, X$. On the other two basis elements it either fixes them or interchanges them. Since every selected simple dimension is nonzero, in the fixed case the two signs equal the selected value of X. Changing its basis sign gives $XY = Y, XZ = Z$; then $(1, -1, 0, 0)$ is a codegree-two character, and Theorem 7.2 applies.

In the interchange case choose signs to obtain

$$X^2 = 1, \quad XY = Z, \quad XZ = Y, \qquad Y^2 = Z^2 = 1 + aY + bZ, \quad YZ = X + bY + aZ.$$

The selected dimensions can be taken as $(1, 1, z, z)$, with $z^2 = 1 + (a + b)z$. If z is rational, it is ± 1; central induction then makes the ordinary global dimension rational, and weak integrality gives sphericality. Otherwise select the larger signed codegree and change both signs so $z > 1$. Thus z is a norm-minus-one quadratic unit and $E = 2 + 2z^2$. Here $q = 1/2$, so (7.8) forces $k = 1$. Properness would give a totally positive unit $r = D/E > 1$.

Let $\eta > 1$ be the fundamental unit of K. The existence of z of norm minus one implies $N(\eta) = -1$. Write $\eta - \eta^{-1} = t \in \mathbb{Z}_{>0}$. If $t \geq 2$, then $z \geq \eta$, and $r \geq \eta^2 \geq 3 + 2\sqrt{2} > 3 > E' = 2 + 2/z^2$, which is impossible. If $t = 1$, then $K = \mathbb{Q}(\sqrt{5})$ and $\eta = \phi$. Write $z = \phi^{2j+1}$, $j \geq 0$. For $j \geq 1$, $E' \leq 2 + 2\phi^{-6} < \phi^2 \leq r$. For $j = 0$, only $r = \phi^2$ keeps $D' = E'/r > 1$; it gives $D = 10 + 4\sqrt{5}$ and $D^{-1} + D'^{-1} = 1$. The other ordinary reciprocal codegrees would then sum to zero. Thus no proper signed dimension exists. □

Apply the chart to the complementary signed orbit. The starred rows are excluded by Lemma 7.7, the norm-one rows by Lemma 7.8, and every row with $q = 1 - \ell \geq 2/3$ by (7.8). Only the following cases remain:

L	25	36	45	64	96
K	$\mathbb{Q}(\sqrt{5})$	$\mathbb{Q}(\sqrt{3})$	$\mathbb{Q}(\sqrt{5})$	$\mathbb{Q}(\sqrt{2})$	$\mathbb{Q}(\sqrt{3})$
q	2/5	1/3	1/3	1/4	1/4.

(7.13)

7.2.5 Ordinary dimensions needed at the small endpoints

Lemma 7.9. *None of the following can be the ordinary global dimension of the nonspherical category under consideration:*

$$5\phi^2, \quad 12 + 6\sqrt{3}, \quad 15 + 6\sqrt{5}, \quad 75 + 33\sqrt{5}, \quad 8 + 4\sqrt{2}, \quad 24 + 16\sqrt{2}, \quad 36 + 20\sqrt{3}.$$

Proof. The value $12 + 6\sqrt{3}$ is already excluded by Lemma 7.3. For the other values the simple dimensions belong to their indicated quadratic field. For $d = 2, 3$, write the three nonunit dimensions as $a_i + b_i\sqrt{d}$. Orthogonality of this character to its conjugate gives $1 + |a|^2 = d|b|^2$. If $D = A + B\sqrt{d}$, then

$$|b|^2 = \frac{A}{2d}, \qquad |a|^2 = \frac{A}{2} - 1, \qquad a \cdot b = \frac{B}{2}.$$

For $d = 5$, use $(t_i + s_i\sqrt{5})/2$ and (7.10). The resulting small integer-vector equations are recorded below. In the middle column the three entries are $(|s|^2, |t|^2, t \cdot s)$ for $d = 5$ and

$(|b|^2, |a|^2, a \cdot b)$ otherwise.

D	invariants	dimension row or consequence
$5\phi^2$	$(3, 11, 5)$	$(1, \phi, \phi, \phi^2)$
$15 + 6\sqrt{5}$	$(6, 26, 12)$	$(1, \phi, \phi^2, \phi^3)$
$75 + 33\sqrt{5}$	$(30, 146, 66)$	$(1, \phi^2, \phi^3, \phi^5)$
$8 + 4\sqrt{2}$	$(2, 3, 2)$	$(1, 1, \eta, \eta)$
$24 + 16\sqrt{2}$	$(6, 11, 8)$	$(1, \eta, \eta, \eta^2)$
$36 + 20\sqrt{3}$	$(6, 17, 10)$	$(1, \epsilon, \epsilon, 2\epsilon - 1)$.

Here $\eta = 1 + \sqrt{2}$ and $\epsilon = 2 + \sqrt{3}$. We justify uniqueness in this small table. For $d = 2, 3$, a vector of squared norm two is a signed permutation of $(1, 1, 0)$, and one of squared norm six is a signed permutation of $(1, 1, 2)$. The indicated dot products and companion squared norms then force respectively

$$a = (1, 1, \pm 1), \quad (1, 1, 3), \quad (2, 2, 3),$$

up to the corresponding signed permutations. For $d = 5$, squared norms three, six and thirty force $s = (1, 1, 1), (1, 1, 2), (1, 2, 5)$. Parity and the dot products force $t = (1, 1, 3), (1, 3, 4), (3, 4, 11)$. For the last case, the particularly short check is $|t - (11/5)s|^2 = 4/5$. Restoring the signs required by positive Frobenius–Perron dimensions gives the displayed rows.

An object of dimension one is invertible, so the $8 + 4\sqrt{2}$ row gives sphericality by Theorem 5.1. In either of the $5\phi^2$ and $15 + 6\sqrt{5}$ rows, an object X of dimension ϕ satisfies $X^2 = 1 + Y$ for some object Y of dimension ϕ. If $Y \neq X$, reciprocity puts X in XY; the remaining dimension is one but $[XY : 1] = 0$, an impossibility. Thus $Y = X$. Apply the normal form in Lemma 5.3. For $5\phi^2$ its remaining parameter is $x = \phi$, so $a + b\phi = 1$, giving $(a, b) = (1, 0)$; Corollary 2.7 gives sphericality. For $15 + 6\sqrt{5}$ one has $x = \phi^2$, so $a + b\phi = \phi^2 - \phi^{-2} = \sqrt{5} = -1 + 2\phi$. This forces $a = -1$, impossible already for a fusion ring.

For $75 + 33\sqrt{5}$, label the dimensions of X, Y, Z by ϕ^2, ϕ^3, ϕ^5. Evaluation of $X^2 = 1 + aX + bY + cZ$ gives $a + b + 3c = 1$ and $a + 2b + 5c = 3$, hence $a + c = -1$. This contradicts nonnegativity.

For $24 + 16\sqrt{2}$, $d_X = d_Y = \eta$ and $d_Z = \eta^2 = 1 + 2\eta$. Evaluation of X^2, Y^2 gives $c = h = 0$, $a + b = d + g = 2$. Evaluation of $XY = bX + dY + eZ$ then gives $e = 1$, $b = d = 0$. Consequently $X^2 = 1 + 2X$, excluded by Proposition 2.10. For the last row, $\epsilon^2 = 4\epsilon - 1$ and $d_Z = 2\epsilon - 1$. Evaluation of the two squares forces $X^2 = Y^2 = 1 + 2Z$, so $b = d = 0$. Evaluation of $XY = eZ$ would simultaneously require $e = 1$ and $2e = 4$, a contradiction. □

7.2.6 Excluding the five signed rows

For a totally positive quadratic d-number in $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\sqrt{3})$ or $\mathbb{Q}(\sqrt{5})$, its norm has square class in, respectively,

$$\{1, 2\}, \qquad \{1, 6\}, \qquad \{1, 5\}. \tag{7.14}$$

Indeed its conjugate quotient is a totally positive unit. The least such units are $3 + 2\sqrt{2}$, $2 + \sqrt{3}$ and ϕ^2. Separate the even and odd powers of that quotient: the original number is a rational multiple of a unit or of one plus that unit. The norms of the latter three elements have square classes two, six and five. This proves (7.14), including the restrictions it imposes on the integral ratio $k = H/N$.

The quadratic unit descriptions used below are elementary Pell reductions. In $\mathbb{Z}[\sqrt{3}]$ the prime over three is generated by $\sqrt{3}$ and the prime over two by $1 + \sqrt{3}$; in $\mathbb{Z}[\sqrt{2}]$ the prime over two is generated by $\sqrt{2}$. Divide a generator by powers of the least positive unit until

it lies in the fundamental interval. The norm equation in that interval leaves precisely the generators just stated. In $\mathbb{Q}(\sqrt{5})$ the units are $\pm\phi^n$. These assertions can also be checked directly by writing an algebraic integer and its conjugate as $z, N(z)/z$.

Complementary norm 25. The row in the chart gives, by its two orthogonality equations, selected dimensions $(1, 1, z, -1-z)$. Orthogonality to the selected conjugate gives $\mathrm{Tr}(z) + 2N(z) = -3$. Put $h = N(z) + 1 \in \mathbb{Z}$ and $A = 4h^2 + 5$. Direct substitution gives

$$E = A + 2|h|\sqrt{A}, \qquad H = 5A.$$

The selected dimension field is $\mathbb{Q}(\sqrt{5})$ precisely when $A = 5w^2$. This implies $5 \mid h$. If $h \neq 0$, then $|h| \geq 10$, since $h = 5v$ gives $w^2 = 20v^2 + 1$ and $v = \pm 1$ is impossible. Thus $E \geq 405$. Here $q = 2/5$ gives $k \leq 2$, and (7.14) excludes $k = 2$. Hence $r = D/E \geq \phi^2$. Since $D' = E'/r > 1$, we have $E' > \phi^2$, so

$$E < \frac{1}{2/5 - \phi^{-2}} < 56,$$

a contradiction. If $h = 0$, then $E = 5$ and $r = \phi^2$ is the only positive unit with $E'/r > 1$. It gives $D = 5\phi^2$, excluded by Lemma 7.9.

Complementary norm 36. The short row gives $(1, x, 1, -1)$ with $N(x) = -3$. With $\epsilon = 2 + \sqrt{3}$, select the larger square to write

$$x = \sqrt{3}\epsilon^j, \qquad E = 3 + 3\epsilon^{2j}, \qquad j \geq 0.$$

As $q = 1/3$, the norm classes force $k = 1$. Then $r = D/E \geq \epsilon$. For $j \geq 1$, $E' \leq 3 + 3\epsilon^{-2} < \epsilon$, impossible. For $j = 0$, $E = 6$, and only $r = \epsilon$ keeps $D' > 1$. It gives $D = 12 + 6\sqrt{3}$, excluded by Lemma 7.9.

Complementary norm 45. The selected dimensions are $(1, 1-z, -1-z, z)$, with $N(z) = -1$. Thus

$$|z| = \phi^{2j+1}, \qquad E = 3 + 3\phi^{4j+2}, \qquad j \geq 0.$$

Again $q = 1/3$ and the norm classes give $k = 1$. Since $E' \leq 3 + 3\phi^{-2} < \phi^4$, properness forces $r = \phi^2$. It follows that

$$p = \frac{\phi^2}{3} - \frac{\sqrt{5}}{E}.$$

The cases $j = 0, 1$ give respectively $D = 15 + 6\sqrt{5}$ and $D = 75 + 33\sqrt{5}$, already excluded. For $j \geq 2$, $E > 300$, $N \geq 1125$, and $p \geq 5/6$. Here $v = 23/45 - 2/N$, so (7.9) gives

$$u + v - 1 \geq \frac{79}{360} - \frac{4}{N} > \frac{1}{5}, \qquad u + v - 1 < \frac{1}{300} + \sqrt{\frac{3}{300}} < \frac{1}{5},$$

a contradiction.

Complementary norm 64. Orthogonality gives, after signs, the selected row $(1, 1-z, 1+z, 1)$ with $N(z) = -2$. Put $\epsilon = 3 + 2\sqrt{2}$. Then

$$z = \sqrt{2}\epsilon^j, \quad E = 4 + 4\epsilon^{2j}, \quad H = 64w^2, \qquad j \geq 0,$$

where w is the rational coefficient of ϵ^j. Here $q = 1/4$, so $k \leq 4$; the norm classes leave $k = 1, 2, 4$. For $k = 1$, $r \geq \epsilon$. If $j \geq 1$ then $E' < \epsilon$. If $j = 0$, $E = 8$ and only $r = \epsilon$ is possible; this gives $D = 24 + 16\sqrt{2}$, already excluded.

For $k = 4$, the totally positive integer $s = 4D/E > 4$ has norm four. Its ideal is (2), so $s = 2\epsilon^a$ with $a \geq 1$. But $s \geq 2\epsilon > 8 \geq E'$, contradicting $D' = E'/s > 1$. For $k = 2$, the corresponding integer $s = 2D/E > 2$ has norm two, so $s = (2+\sqrt{2})\epsilon^a$. Since $E' \leq 8$, only $a = 0$ is possible. Consequently

$$p = \frac{2+\sqrt{2}}{4} - \frac{2\sqrt{2}}{E}.$$

At $j = 0$, this gives $D = 8 + 4\sqrt{2}$, already excluded. For $j \geq 1$, $E > 100$, $H \geq 576$, $p \geq 5/6$ and $N = H/2$. Since $v = 19/32 - 2/H$, (7.9) gives

$$u + v - 1 \geq \frac{29}{96} - \frac{6}{H} \geq \frac{7}{24}, \qquad u + v - 1 < \frac{1}{100} + \sqrt{\frac{3}{100}} < \frac{37}{200},$$

again impossible.

Complementary norm 96. The selected row is $(1, 1-z, 1+z, -1)$, so $E = 4 + 2z^2$ and $N(z) = -2$. Put $\epsilon = 2 + \sqrt{3}$. Selecting the larger square gives

$$z = (1+\sqrt{3})\epsilon^j, \quad E = 4 + 4\epsilon^{2j+1}, \quad H = 96w^2, \qquad j \geq 0,$$

where w is the $\sqrt{3}$ coefficient of z. The norm classes and $k \leq 4$ leave $k = 1, 4$. For $k = 4$, the integer $s = 4D/E > 4$ has norm four and equals $2\epsilon^a$, $a \geq 1$. This exceeds $E' \leq 12 - 4\sqrt{3}$, a contradiction. Thus $k = 1$. The same bound for E' is less than ϵ^2, so $r = \epsilon$, and

$$p = \frac{\epsilon}{4} - \frac{2\sqrt{3}}{E}.$$

At $j = 0$, $D = 36 + 20\sqrt{3}$, already excluded. For $j \geq 1$, $E > 200$, $N \geq 864$, $p \geq 11/12$, and $v = 29/48 - 2/N$. Hence

$$u + v - 1 \geq \frac{43}{96} - \frac{4}{N} > \frac{2}{5}, \qquad u + v - 1 < \frac{1}{200} + \sqrt{\frac{3}{200}} < \frac{2}{5}.$$

This excludes the last row of (7.13) and proves Theorem 7.4.

7.3 The quadratic dimension bound

Suppose throughout that the category is nonspherical and D is irrational quadratic. By Corollary 6.6 the signed dimension field is $\mathbb{Q}(D)$. Keep p, q, N from (7.8), write $EE' = hN$, put $t = E/D$, and let M, L be the ordinary and signed complementary codegree products. Theorem 7.4 gives $L \geq 100$. With $F(z) = z^2 + (1-z)^2$ and u, v as in (7.9),

$$N = \frac{D}{p - 1/D}, \qquad q = \frac{t}{h}p + \frac{1 - t^2/h}{E}, \qquad E^2 \geq hN.$$

These identities allow the same second-moment comparison for both types of ordinary character orbit.

Lemma 7.10 (All norm ratios greater than one)**.** *If $h \geq 2$, then $D < 1000$, without any assumption on the ordinary complementary codegrees.*

Proof. Suppose $D \geq 1000$. Since $0 < t < 1$, its integral trace satisfies $t + h/t \geq h + 2$. Consequently

$$\frac{t}{h} \leq \frac{2}{h + 2 + \sqrt{h^2+4}} \leq 1 - \frac{1}{\sqrt{2}} < \frac{3}{10}.$$

We have $u \geq p^2+(1-p)^2/2-2/N$ by (7.9) and $v = F(q)-2/(hN)-2/L$. Also $E > \sqrt{2D} > 44$. If $E < 60$, then $p > 2D/E^2 > 5/9$, and

$$q \leq \frac{E}{2D} + \frac{1}{E} \leq \frac{7}{150}.$$

For the last inequality, the function $E/2000 + 1/E$ increases on $[\sqrt{2000}, 60]$ and has terminal value $7/150$. Thus

$$u + v - 1 \geq \frac{11}{27} + F(7/150) - 1 - \frac{23}{1000} > \frac{29}{100} > \frac{1}{44} + \sqrt{\frac{3}{44}} > \frac{1 + \sqrt{3(E-1)}}{E},$$

a contradiction.

Assume now $E \geq 60$. Set $a = \min(E/(2D), 3/10)$ and $G(p) = p^2 + (1-p)^2/2$. Since $t/h \leq a$, the formula for q gives $q \leq ap + 1/E$. The function F decreases up to $1/2$ and has $|F'| \leq 2$ on $[0,1]$, so $F(q) \geq F(ap) - 2/E$. Completing the square in $G(p) + F(ap) - 1$ and using $N > D$ gives

$$C(a) - \frac{1}{50} - \frac{3}{D} \leq \frac{3}{E} + \sqrt{\frac{3}{E}}, \qquad C(a) = \frac{1-2a}{3+4a^2}.$$

The function C decreases on $[0, 3/10]$. The following elementary comparisons contradict this inequality:

E range	left side exceeds	right side is less than
$[60, 100]$	$90/301 - 23/1000 > 11/40$	$11/40$
$[100, 200]$	$5/19 - 23/1000 > 21/100$	$21/100$
$[200, 400]$	$15/79 - 23/1000 > 7/50$	$7/50$
$[400, \infty)$	$5/42 - 23/1000 > 19/200$	$19/200$.

In each row the bound on a follows from $D \geq 1000$; the last uses $a \leq 3/10$. The right column follows by evaluating at the left endpoint. This completes the proof. □

Lemma 7.11 (Unit-norm comparison)**.** *Suppose $h = 1$ and write $U = u + 2/N$. Then*

$$U + F(tp) - 1 - \frac{1}{50} \leq \frac{3 + \sqrt{3(E-1)}}{E}.$$

If $E/E' \geq \kappa$ and $D \geq D_0$, then $E \geq \max\{D_0 t, \sqrt{D_0 \kappa / p}\}$.

Proof. Here $q = tp + (1-t^2)/E$, and convexity gives $F(q) \geq F(tp) + (4tp-2)(1-t^2)/E$. In $u + v = U + F(q) - 4/N - 2/L$, use $4E/N = 4t(p - 1/D) < 4tp$. The combined error, multiplied by E, is at most

$$4tp + (2 - 4tp)(1 - t^2) = 2 - 2t^2(1 - 2tp) \leq 2.$$

Indeed $0 < t \leq (3 - \sqrt{5})/2 < 1/2$ and $0 < p < 1$. Now apply (7.9). The last bound uses $E = tD$ and $E^2 = (E/E')N$ with $N > D/p$. □

Theorem 7.12. *Every nonspherical self-dual rank-four complex fusion category with irrational quadratic Frobenius–Perron global dimension satisfies $D < 1000$. Every fusion coefficient is at most 31.*

Proof. Suppose $D \geq 1000$. An invertible nonunit would give sphericality by Theorem 5.1. Lemma 7.10 leaves $h = 1$. We describe the possible lower bounds $H(t)$ for the left side of Lemma 7.11 and then check all of them together.

Large complementary product. If $M \geq 100$, then $U = F(p) - 2/M$. Minimizing $F(p) + F(tp) - 1$ in p by completing the square gives

$$H(t) = \frac{(1-t)^2}{2(1+t^2)} - \frac{1}{25}.$$

In the bound for E we may replace p by 1.

Short quadratic orbit. If $M < 100$ and the two ordinary complementary characters are conjugate, Lemma 7.6 applies. The $M = 20$ row forces an invertible. For the other rows, except for the repeated codegree five, substitution into $U = F(p) - 2/M$ gives the following lower bounds U_0.

p	$1/4$	$1/3$	$2/5$	$1/2$	$2/3$	$3/4$
U_0	$19/32$	$1/2$	$11/25$	$7/16$	$23/45$	$19/32$

For the repeated codegree five, $(p, U_0) = (3/5, 11/25)$. In each case use $H(t) = U_0 + F(pt) - 1 - 1/50$.

Two rational characters. Write their codegrees $g \leq j$, with $gj < 100$. A codegree-two rational row forces an invertible by orthogonality to the positive dimension row. Thus $g \geq 3$. Discard $5, 8, 16$, the pairs $(3,4), (3,12), (9,9)$, and odd j when $g = 4$, by Lemma 7.5. The pair $(4,4)$ also forces an invertible: of two distinct orthogonal sign rows, their sum has unit coordinate two and exactly one nonzero nonunit coordinate, of absolute value two. Orthogonality to the positive row forces the corresponding dimension to be one. Call the remaining set $\mathcal{P}$; it contains 52 pairs. For each, put

$$p = 1 - 1/g - 1/j, \qquad U_0 = p^2 + 1/g^2 + 1/j^2, \qquad H(t) = U_0 + F(pt) - 1 - 1/50.$$

We may take $\kappa = 1$ except for $(3,3)$, where (7.12) gives $N = 9(9m^2+4)$ with $m \neq 0$. This lies strictly between consecutive squares after division by nine. Hence $EE' = N$ is not a square, and E/E' is a nontrivial totally positive quadratic unit by Ostrik's d-number theorem [25]. We may take $\kappa = 5/2$ by Lemma 7.1.

There are two ranges for the unit t: either $t \leq 2 - \sqrt{3} < 268/1000$, or $t = (3 - \sqrt{5})/2 < 382/1000$. The latter occurs in $\mathbb{Q}(\sqrt{5})$. For rational complementary characters the discriminant of the ordinary order is Ngj and has square class five. By (7.14), gj is a square or five times a square. Among $\mathcal{P}$ this leaves $(3,3)$ and

$$\mathcal{P}_5 = \{(3,15), (3,27), (4,20), (6,6), (7,7)\}.$$

The field in the $(3,3)$ case is $\mathbb{Q}(\sqrt{9m^2+4})$, which cannot be $\mathbb{Q}(\sqrt{5})$ by reduction modulo three.

Endpoint check. All the functions H just listed decrease on the indicated intervals. The right side of Lemma 7.11 decreases for $E \geq 2$. The table gives a subdivision for each family. Its entries are the successive right endpoints in thousandths, starting at zero. For $[a,b]$ in this subdivision set

$$Z = \max\left\{ D_0 a,\ \frac{\lfloor 100\sqrt{D_0\kappa/p} \rfloor}{100} \right\}.$$

In every interval the following two rational comparisons hold:

$$ZH(b) - 3 > 0, \qquad (ZH(b) - 3)^2 > 3(Z-1).$$

Since $E \geq Z$, these comparisons contradict the lemma.

Input	D_0	Right endpoints multiplied by 1000
$M \geq 100$	1000	$60, 190, 360, 382$
Six short quadratic rows	1000	$90, 220, 340, 382$
Repeated five, $t \leq 2 - \sqrt{3}$	1000	$60, 130, 230, 268$
Repeated five, all t	3000	$150, 340, 382$
$\mathcal{P}$, $t \leq 2 - \sqrt{3}$	1000	$46, 52, 75, 140, 220, 268$
$\mathcal{P}_5$, all t	1000	$100, 240, 340, 370, 382$

For instance, for $(g, j) = (3, 6)$ on $[0, 46/1000]$, the input is $p = 1/2$, $U_0 = 7/18$, $Z = 4472/100$. Substitution gives $ZH(b) - 3 > 287/25$ and $3(Z - 1) < 263/2$. The table is reproducible by two nested loops over its stated rows and endpoints. The accompanying exact checker uses rational arithmetic and integer square roots.

Only the repeated five with $t = (3 - \sqrt{5})/2$ and $1000 \leq D < 3000$ remains. In (7.11), denote the integer parameter by m. The choice of basis signs does not affect the codegrees. We have $N = 5\Delta$, $D + D' = 3\Delta$ and $\Delta = 9m^2 - 8m + 4$. The d-number condition $N \mid (3\Delta)^2$ gives $m \equiv 1 \pmod 5$. Since $1/D' = 3/5 - 1/D > 1/2$, we have $D' < 2$ and $\Delta < 1001$. Thus $m = -9, -4, 1, 6$, with $\Delta = 805, 180, 5, 280$, respectively. The selected field $\mathbb{Q}(\sqrt{\Delta})$ equals $\mathbb{Q}(\sqrt{5})$ only for $\Delta = 180, 5$, but $D < 3\Delta < 1000$ in both cases. This proves the dimension bound. Finally every symmetric ordinary multiplication matrix has operator norm its Perron root $d_i \leq \sqrt{D} < 32$, so all entries are at most 31. □

8 Sphericality in the cubic case

The even part of $\mathrm{SU}(2)_7$ is the familiar rank-four example with a cubic dimension field [27, Section 5.3.10]. We prove that a category with a cubic dimension field already admits a spherical structure. This allows the pivotal classification to be applied after descent, without assuming a pivotal structure here.

The Frobenius–Perron dimension field means the field generated by all simple Frobenius–Perron dimensions. Rank-four fusion rings are commutative by Lemma 2.1. Suppose a complex fusion categorification exists and its dimension field K has degree three. The ordinary rational algebra is then $\mathbb{Q} \times K$, and character cyclotomicity makes K cyclic and totally real [10, Corollary 8.53]. Every character is consequently real. The based involution acts by complex conjugation on characters, so it is the identity, so every simple is self-dual. Its Frobenius–Perron codegree D is cubic as well, by the order discriminant argument in Lemma 8.3 below (applied just to the ordinary algebra).

Central induction places D in the field generated by the signed spherical dimensions. That field has degree at most four, so it has degree three and equals K. Thus the signed rational algebra is also $\mathbb{Q} \times K$, with its dimension character in the cubic factor. We write f for the codegree of the ordinary rational character.

8.1 The trace identity and a positive measure

Put $T_{\mathbb{Q}} = (K_0(\widetilde{\mathcal{C}})/(\varepsilon + 1)) \otimes \mathbb{Q}$ for the signed algebra of the sphericalization. Choose dual-compatible lifts X_i^+ of the four simples, with $X_0^+ = \mathbf{1}$, and write $X_i^- = \varepsilon \otimes X_i^+$, $x_i = \dim X_i^+$, and $E = \sum_i x_i^2$. Let ρ denote the rational character of $T_{\mathbb{Q}}$ and let $g = \sum_i \rho(X_i^+)^2$ be its codegree. The rational algebra of the sphericalization is the commutative direct sum $R_{\mathbb{Q}} \oplus T_{\mathbb{Q}}$, where $R = K_0(\mathcal{C})$. The Frobenius trace identity is

$$\mathrm{Tr}_{K/\mathbb{Q}}(x_i/E) = \delta_{i0} - \frac{\rho(X_i^+)}{g}. \tag{8.1}$$

To see this identity, let τ be the coefficient of the unit. In the orthonormal signed basis, the regular trace of multiplication by a equals $\tau(aC_T)$, where C_T is the Casimir. The rational and cubic components of C_T are g and E. Evaluating this formula on $X_i^+ C_T^{-1}$ gives the displayed identity.

Let $f_1, \dots, f_4$ be the formal codegrees of $K_0(\mathcal{C})$, and let g, E_1, E_2, E_3 be those of the signed algebra. Put

$$u = \sum_{i=1}^{4} f_i^{-2}, \qquad v = g^{-2} + \sum_{i=1}^{3} E_i^{-2}.$$

As usual, these numbers are rational. The dimension-field theorem [24, Corollary 2.15] places all the displayed codegrees in K. Throughout, codegrees are counted with their character multiplicities. Both reciprocal-codegree sums equal one and $f, g \geq 2$.

Proposition 8.1. *There is a positive, finitely supported measure ξ on the unit circle, of mass three, such that*

$$\int z^n \, d\xi(z) = \begin{cases} 1/2, & n \text{ odd}, \\ 1, & n \text{ even}, \end{cases} \qquad K \cap \mathbb{Q}(\zeta_n) = \mathbb{Q}.$$

Its atoms at 1 and -1 satisfy

$$\xi(\{1\}) \geq \frac{3}{4}(2u + v) + \frac{f-2}{f^2} + \frac{g-2}{2g^2}, \tag{8.2}$$

$$\xi(\{-1\}) \geq \frac{3}{4}v + \frac{g-2}{2g^2} \geq \frac{1}{4}. \tag{8.3}$$

Proof. Let $F : Z(\widetilde{\mathcal{C}}) \to \widetilde{\mathcal{C}}$ be forgetful and I its right adjoint. Use the canonical normalized center matrix $S = s/(2E)$. The rational character ρ corresponds to a self-dual constituent $B \neq 1$ of $I(1)$ by [24, Example 2.9 and Theorem 2.13]. Put

$$a_A = S_{0A} = \frac{\dim A}{2E} \in K, \qquad b_A = S_{BA} = \frac{\rho(F(A))}{2g} \in \mathbb{Q}.$$

Orthogonality gives

$$\sum_A a_A^2 = \sum_A b_A^2 = 1, \qquad \sum_A a_A b_A = 0. \tag{8.4}$$

Take a cyclotomic field Ω containing the modular data. For each indicator index used, enlarge it to contain ζ_n. Write $G = \mathrm{Gal}(\Omega/\mathbb{Q})$. Its signed permutations $\widehat{\sigma}$ satisfy

$$\sigma(a_A) = \epsilon_\sigma(A) a_{\widehat{\sigma}A}, \quad b_A = \epsilon_\sigma(A) b_{\widehat{\sigma}A}, \quad \theta_{\widehat{\sigma}A} = \sigma^2(\theta_A). \tag{8.5}$$

The last identity has no scalar anomaly for the canonical center normalization [5, Lemma 4.2 and Section 4.1]. Define

$$\xi = \frac{1}{|G|} \sum_{\sigma \in G} \sum_A (3\sigma^2(a_A)^2 + \sigma^2(a_A) b_A) \, \mathrm{Dirac}_{\sigma^2(\theta_A)} .$$

Here σ^2 means composition. Equation (8.4) gives mass three.

For an allowed n, the restrictions of G to K and $\mathbb{Q}(\zeta_n)$ are independent. Squaring permutes $\mathrm{Gal}(K/\mathbb{Q}) \cong C_3$; no corresponding surjectivity is required on the indicator field. Apply (2.5) and (8.1). The moment of the first term of ξ is

$$\frac{1 - \nu_n(\varepsilon)}{2} - \frac{1}{2g} \, \mathrm{Av}_{\sigma \in G} \, \sigma^2 \left(\sum_i \rho(X_i^+)[\nu_n(X_i^+) - \nu_n(X_i^-)] \right).$$

The sum in parentheses is $2g\sum_A a_A b_A \theta_A^n$ by the same center identity. The second term of ξ cancels this residual. The sign object has indicator -1 for even n and zero for odd n: its even invariant space is one-dimensional and rotation includes its inverse pivotal map -1, whereas the odd invariant space vanishes. This proves the moment formula.

Reindexing each summand by $W = \widehat{\sigma}A$ fixes the twist θ_W and gives its individual weight

$$w_W = \mathrm{Tr}_{K/\mathbb{Q}}(a_W^2) + \frac{b_W}{3}\,\mathrm{Tr}_{K/\mathbb{Q}}(a_W). \tag{8.6}$$

In particular the measure does not depend on enlarging Ω. Let $h_W = [F(W) : 1] - [F(W) : \varepsilon]$. The Frobenius trace identity says $\mathrm{Tr}(a_W) = h_W/2 - b_W$. Outside $I(1) \cup I(\varepsilon)$ one has $h_W = 0$, so $w_W = \mathrm{Tr}(a_W^2) - b_W^2/3 \geq 0$. For $W \subset I(1)$ put $t_W = \mathrm{Tr}(a_W) > 0$. All conjugate codegrees are positive, and $b_W = 1/2 - t_W$; hence

$$w_W \geq \frac{t_W^2}{3} + \frac{(1/2 - t_W)t_W}{3} = \frac{t_W}{6} > 0.$$

Its sign translate has the same weight. Thus positivity holds pointwise, without grouping atoms.

For the four ordinary constituents the b-values are $(f-3)/(2f), 1/(2f), 1/(2f), 1/(2f)$; the signed values replace f by g. This follows from $a_A = 1/(2f_\chi)$ and $b_A = 1/2 - \mathrm{Tr}(a_A)$. Summing their corrections in (8.6) gives $(f-2)/(2f^2)$ and $(g-2)/(2g^2)$, respectively. By Lemma 2.9, the ordinary sign translates have twist $+1$ and the signed translates have twist -1. Summing their weights gives (8.2)–(8.3); the final bound uses $v \geq g^{-2} + (1-g^{-1})^2/3$. □

8.2 Periodic moments

Proposition 8.2. *Suppose the ordinary and signed rational algebras are $\mathbb{Q} \times K$ as above. If F is the conductor of K, then*

$$\mathcal{L} := \frac{3}{2}(u+v) + \frac{f-2}{f^2} + \frac{g-2}{g^2} \leq 1 + \frac{2}{F} \leq \frac{9}{7}. \tag{8.7}$$

Proof. A nontrivial cyclic cubic conductor is odd and at least seven. Since $[K : \mathbb{Q}] = 3$, the field K is disjoint from $\mathbb{Q}(\zeta_n)$ whenever $F \nmid n$. Restrict ξ to the $2F$th roots and call the restriction ξ_0. For $F \nmid r$, average its moments with exponents $2Fj + r$ over $0 \leq j < M$ and let $M \to \infty$. The geometric average is zero off the $2F$th roots and one on them, so ξ_0 has the same prescribed rth moment. Its mass is at most three. Subtract $\frac{3}{4}\,\mathrm{Dirac}_1 + \frac{1}{4}\,\mathrm{Dirac}_{-1}$. The residual Fourier transform on this cyclic group is supported on $\{0, F\}$, so its weights are constant on each parity class; call them a, b. Each class has roots other than ± 1, at which these are weights of the positive measure ξ_0. Hence $a, b \geq 0$ and

$$1 + F(a+b) \leq 3, \qquad \xi(\{1\}) + \xi(\{-1\}) = 1 + a + b \leq 1 + \frac{2}{F}.$$

Now use (8.2)–(8.3). □

8.3 The rational codegrees and cubic norms

The two rational codegrees f, g are integers at least two. The ordinary value $f = 2$ is excluded by Proposition 5.2.

Lemma 8.3. *For either algebra, let r be its rational codegree and H a codegree from its cubic character orbit. There are positive integers A, s such that*

$$\mathrm{Tr}(H) = rA, \qquad \mathrm{Norm}(H) = r^3 s^2,$$

and H/r has minimal polynomial

$$z^3 - Az^2 + (r-1)s^2 z - s^2. \tag{8.8}$$

If $r \geq 3$, the inverse-square sum of the four codegrees satisfies

$$r^{-2} + \sum_{\sigma: K \hookrightarrow \mathbb{R}} \sigma(H)^{-2} \geq r^{-2} + \tfrac{1}{2}(1 - r^{-1})^2. \tag{8.9}$$

Proof. First H has degree three. If it were rational, the reciprocal-codegree identity would give $(r-1)H = 3r$, so $(r, H) = (2, 6)$ or $(4, 4)$. The order discriminant is the product of its four formal codegrees. The first case gives 432, which is not a square, contrary to the cyclic cubic field discriminant. The second gives 256; its square root would have to be divisible by the conductor F_K, whereas a nontrivial cyclic cubic conductor is odd and cannot divide sixteen.

Write the minimal polynomial of H as

$$z^3 - az^2 + (r-1)hz - rh.$$

The reciprocal identity gives this form with positive integers a, h. The d-number criterion gives $rh \mid a^3$ and $(rh)^2 \mid ((r-1)h)^3$. Thus $r^2 \mid h$ and $r \mid a$. The order discriminant $r \operatorname{Norm}(H) = r^2 h$ is a square, so $h = r^2 s^2$ and $a = rA$ for positive integers A, s. This proves (8.8). For the signed algebra the same argument applies: $2H$ is an actual formal codegree of the sphericalization, and the d-number property is unchanged by nonzero rational scaling.

Put $k = r - 1$ and $n = s^2$. The three positive reciprocal roots of (8.8) have sum k, pairwise-product sum A/n, and product $1/n$. Schur's inequality yields

$$4kA \leq k^3 n + 9.$$

If $k \geq 3$ and $4A > k^2 n$, the positive integer $4A - k^2 n$ is at least four when k or s is even, and at least three otherwise. This contradicts the displayed inequality except possibly at $k = 3$ with equality. Equality in Schur for positive numbers makes all three roots equal, contradicting their cubic degree. Hence $A \leq k^2 n/4$. For $k = 2$, Schur gives $A \leq n + 1$. At $A = n + 1$ the discriminant is $n(n-4) = s^2(s^2 - 4)$, which cannot be a nonzero square. Thus $A \leq n$ in this case too. Finally the inverse-square sum is

$$\frac{1 + k^2 - 2A/n}{r^2} \geq \frac{1}{r^2} + \frac{k^2}{2r^2},$$

as claimed. □

Theorem 8.4 (Direct cubic reduction)**.** *The only possibilities are*

$$(F, f, g) = (7, 3, 3), \quad (7, 4, 4), \quad (9, 3, 3), \qquad \operatorname{Norm}(E) = \operatorname{Norm}(D).$$

In the conductor-nine case $u = v = 1/3$, the ordinary and signed cubic codegrees agree as multisets, and the category is spherical.

Proof. For a constituent A from the ordinary cubic orbit, $a_A = 1/(2D_i)$ gives $b_A = 1/(2f)$ in this measure. Since $b_A = \rho(F(A))/(2g)$ and ρ is integral-valued, $g/f \in \mathbb{Z}_{>0}$. Put $a = \operatorname{Tr}(D)$, $b = \operatorname{Tr}(E)$, $c = \operatorname{Norm}(D)$ and $k = \operatorname{Norm}(E)/c \in \mathbb{Z}_{>0}$, using central integrality. Casimir trace domination gives $b + g \leq a + f$. For

$$H(r) = \frac{1}{2} - \frac{1}{r} + \frac{3}{2r^2}, \qquad J(r) = 1 - \frac{2}{r} + \frac{2}{r^2},$$

Lemma 8.3 and the reciprocal identities give

$$u \geq H(f), \quad u = J(f) - 2a/c, \quad v = J(g) - 2b/(kc), \quad v - u \geq J(g) - J(f).$$

If $g \geq 2f$, substitution in (8.7) yields

$$\mathcal{L} \geq \frac{3}{2} + \frac{1}{f} - \frac{1}{2f^2} - \frac{2}{g} + \frac{1}{g^2} \geq \frac{3}{2} - \frac{1}{4f^2} \geq \frac{53}{36} > \frac{9}{7},$$

a contradiction. Therefore $g = f$. Now

$$\mathcal{L} \geq \frac{3}{2} - \frac{1}{f} + \frac{1}{2f^2}.$$

This increases for $f > 1$ and equals $33/25 > 9/7$ at $f = 5$. Consequently $f = g \in \{3, 4\}$.

Lemma 8.3 makes k an integer square. If $k > 1$, then $k \geq 4$, and $b \leq a$ implies $v \geq \frac{3}{4}J(f) + \frac{1}{4}u$. Thus

$$\mathcal{L} \geq \frac{33}{16} - \frac{17}{8f} + \frac{17}{16f^2} \geq \frac{53}{36} > \frac{9}{7},$$

again impossible. Hence $k = 1$. At $f = 3$ the lower bound for $\mathcal{L}$ is $11/9$, forcing $F \leq 9$; at $f = 4$ it is $41/32$, forcing $F \leq 64/9$. The only cyclic cubic conductors in these ranges are seven and nine, as follows from primitive Dirichlet characters of order three. This gives the stated list.

At $(F, f) = (9, 3)$ equality holds throughout, so $u = v = 1/3$. Equality of norms and inverse-square sums gives $\mathrm{Tr}(E) = \mathrm{Tr}(D)$, and the reciprocal identity identifies the remaining coefficient of the two cubic polynomials. Their roots therefore agree. Choose a signed character with codegree D and apply spectral saturation in Lemma 2.2 to obtain a spherical section. □

8.4 An inert-prime obstruction for a rational codegree

Lemma 8.5. *Let R be a commutative self-dual based ring with $R_{\mathbb{Q}} \cong \mathbb{Q} \times K$, where K is a number field. Suppose the character given by projection to $\mathbb{Q}$ has formal codegree a prime p. If $L = K \otimes \mathbb{Q}_p$ is a field, the projection of $R \otimes \mathbb{Z}_p$ to L is $\mathcal{O}_L$, and $L/\mathbb{Q}_p$ is totally ramified.*

Proof. Let $A = R \otimes \mathbb{Z}_p$ and project A to an order B in $\mathcal{O}_L$, where $L = K \otimes \mathbb{Q}_p$. If ρ is the rational character, its idempotent $(1, 0) = p^{-1} \sum_i \rho(b_i) b_i$ shows that $A \cap (\mathbb{Q}_p \times 0) = p\mathbb{Z}_p \times 0$ by looking at the unit coefficient. Thus the first coordinate modulo p gives a surjection $B \to \mathbb{F}_p$ and $A = \mathbb{Z}_p \times_{\mathbb{F}_p} B$. The order B is local by lying-over from its normalization $\mathcal{O}_L$, so its maximal ideal $\mathfrak{m}$ is $B \cap \mathfrak{m}_L$ and its residue field is $\mathbb{F}_p$. If $B = \mathcal{O}_L$, this gives the claim.

Suppose that $B \neq \mathcal{O}_L$. The finite B-module $\mathcal{O}_L/B$ is nonzero and has nonzero socle. Choose $y \in \mathcal{O}_L \setminus B$ with $\mathfrak{m}y \subset B$. In fact $\mathfrak{m}y \subset \mathfrak{m}$ since y is integral. In A/pA the classes of $(p, 0)$ and $(0, py)$ are then both killed by the maximal ideal, whose inverse image is $p\mathbb{Z}_p \times \mathfrak{m}$. They are linearly independent over $\mathbb{F}_p$: a relation would give $(c, dy) \in A$ after division by p, forcing first $p \mid d$ since $y \notin B$, then $p \mid c$ by the fiber condition (here $py \in \mathfrak{m}$). But the coefficient-of-unit pairing on A/pA is associative and perfect. It identifies the annihilator of the maximal ideal with the dual of the residue field, of dimension one. This contradiction proves the claim. The general Gorenstein fiber-product criterion of Nasseh, Sather-Wagstaff, Takahashi and VandeBogert [23, Fact 2.6 and Corollary 2.7] also implies the obstruction. □

Corollary 8.6. *There is no commutative self-dual based ring whose rational algebra is $\mathbb{Q} \times \mathbb{Q}(\zeta_7 + \zeta_7^{-1})$ and whose rational formal codegree is three.*

Proof. The polynomial $x^3 + x^2 - 2x - 1$ of $\zeta_7 + \zeta_7^{-1}$ has discriminant 49 and has no root modulo three: its reduction is $x^3 + x^2 + x - 1$, taking nonzero values at all three elements of $\mathbb{F}_3$. Thus three is unramified and inert in this cubic field. Apply Lemma 8.5 with $p = 3$. □

8.5 Sphericality at rational codegree four

Lemma 8.7. *In the remaining conductor-seven case, put* $a = \mathrm{Tr}(D)$, $b = \mathrm{Tr}(E)$, *and* $c = \mathrm{Norm}(D) = \mathrm{Norm}(E)$. *Then*

$$a - b \in 64\mathbb{Z}_{\geq 0}.$$

Proof. Let B be the constituent of $I(\mathbf{1})$ corresponding to the rational signed character. Let C_R and C_T be the ordinary and signed Casimir elements. We show that the ordinary folded and signed forgetful coefficient vectors of B are respectively

$$w = C_R/4, \qquad r = C_T/4.$$

The signed cubic component of $F(B)$ is its dimension $E/4$. Its unit coefficient is one, and its sign-object coefficient is zero, because B belongs to $I(\mathbf{1})$ and the supports of $I(\mathbf{1})$ and $I(\varepsilon)$ are disjoint. The Frobenius trace formula then makes its signed rational component one, so its signed class is $C_T/4$. For an ordinary cubic induction constituent A_i, the rational signed row, already computed in Theorem 8.4, has $S_{B,A_i} = 1/8$. Symmetry and the modular character formula give

$$\chi_i(F(B)) = \frac{S_{A_i,B}}{S_{A_i,0}} = \frac{1/8}{1/(2D_i)} = D_i/4.$$

Its ordinary folded unit coefficient is likewise one, so the ordinary rational component is one and its folded class is $C_R/4$.

These are two coefficient vectors of the same actual object. Thus $w_i \geq |r_i|$ and $w_i \equiv r_i \pmod 2$ for every i. In the orthonormal based trace, the squared coefficient norm of a Casimir is the sum of its formal codegrees. Consequently

$$a - b = 16\big(\|w\|^2 - \|r\|^2\big) \in 64\mathbb{Z}_{\geq 0}.$$

□

Theorem 8.8. *Every category in the conductor-seven, rational-codegree-four case admits a spherical structure. Its signed and ordinary cubic codegree polynomials coincide.*

Proof. Retain a, b, c from Lemma 8.7. Here

$$u = \frac{5}{8} - \frac{2a}{c}, \qquad v = \frac{5}{8} - \frac{2b}{c}, \qquad u, v \geq \frac{11}{32}.$$

The periodic inequality is $u + v \leq 29/42$. It follows that

$$\frac{187c}{1344} \leq b \leq a \leq \frac{9c}{64}, \qquad a - b \leq \frac{c}{672}. \tag{8.10}$$

If $a > b$, the trace-gap lemma gives $c \geq 43008$.

For either triple of cubic codegrees let H be its largest element and put $x = 1/H$. Since H is at least a third of the trace, (8.10) gives $0 < x < 1/1900$. Let q_1, q_2 be the other two reciprocal codegrees. Their sum is $3/4 - x$. Also $u, v \leq 29/42 - 11/32 = 233/672$, so

$$q_1^2 + q_2^2 + x^2 \leq \frac{191}{672}, \qquad (q_1 - q_2)^2 \leq \frac{1}{168} + \frac{3}{2}x - 3x^2.$$

The last upper bound is less than $(1/12 - x)^2$, since the difference is

$$\frac{1}{1008} - \frac{5}{3}x + 4x^2 > \frac{1}{1008} - \frac{1}{1140} > 0.$$

Therefore $1/3 < q_1, q_2 < 5/12$. The two smaller codegrees of each triple lie in $(12/5, 3)$.

Align the largest signed codegree $E_{\max}$ with the ordinary Frobenius–Perron codegree D_{FP}. Central induction makes $z = E_{\max}/D_{\mathrm{FP}}$ an algebraic integer: it gives integrality against every conjugate of D, and one may then apply a field embedding sending E to $E_{\max}$. Equality of norms makes z a totally positive unit of norm one. Spectral domination gives $z \le 1$ at the chosen embedding. Its other two conjugates are ratios of numbers in $(12/5, 3)$ and are less than $5/4$. Hence

$$3 \le \mathrm{Tr}(z) < 1 + 2 \cdot \frac{5}{4} = \frac{7}{2}.$$

The trace is an integer, so it is three; equality in the arithmetic–geometric mean inequality forces every conjugate to be one. Thus $z = 1$, contradicting $a > b$. We conclude $a = b$. Equality of norms and the reciprocal identity then identifies the two cubic polynomials. Spectral saturation in Lemma 2.2 supplies the spherical structure. □

Theorem 8.9. *Every rank-four complex fusion category whose Frobenius–Perron dimension field has degree three admits a spherical structure.*

Proof. Theorem 8.4 gives sphericality in conductor nine. In conductor seven, Corollary 8.6 excludes rational codegree three, and Theorem 8.8 gives sphericality at codegree four. □

9 The quartic dimension field

Theorem 9.1. *Let $\mathcal{C}$ have rank four and self-dual simples. If $D = \mathrm{FPdim}(\mathcal{C})$ has degree four and $D \ge 950$, then $\mathcal{C}$ is spherical.*

Suppose throughout that $D \ge 950$ and that $\mathcal{C}$ has no spherical structure. By Corollary 6.6 the ordinary and signed dimension characters identify their rational algebras with the same totally real abelian quartic field $K = \mathbb{Q}(D)$. Index its embeddings by $i = 0, 1, 2, 3$, with 0 the identity. Choose the signed dimension character so that its codegree E is the largest signed codegree. Write D_i, E_i for the four conjugates, including repetitions for E. Set

$$p_i = D_i^{-1}, \qquad q_i = E_i^{-1}, \qquad d = p_0 \le d_* := 1/950, \qquad r = q_0, \qquad \alpha = E/D, \quad \alpha_i = p_i/q_i.$$

Both vectors are positive and sum to one by character orthogonality. We have $d \le r = \min_i q_i$. In addition, α_0 is the least conjugate of α. Indeed the signed and folded dimensions of the same central column are E/E_i and D/D_i (Proposition 6.2). The absolute values of the signed column coefficients are bounded by the ordinary coefficients, and the absolute signed simple dimensions by the corresponding FP dimensions. Hence $E/E_i \le D/D_i$. By central integrality and (2.4), α is a totally positive integral d-number of positive square norm h. Here the d-number property follows from that of the two codegrees: all conjugates of their ratio generate the same fractional ideal. In particular

$$\prod_i q_i = \frac{1}{h} \prod_i p_i \le \frac{d(1-d)^3}{27}. \tag{9.1}$$

We will rule out $\alpha \ne 1$. Equality $E = D$ gives the required spherical section by Lemma 2.2.

9.1 Moment bounds and the size of the trace

Write $u = \sum p_i^2$, $v = \sum q_i^2$, and $K_* = \mathbb{Q}(\sqrt{2}, \sqrt{5})$. The two moment estimates used below are

$$\begin{array}{l|c} K \ne K_* & u + v \le 4/5 \\ K = K_* & 5u + 4v \le 15/4. \end{array} \tag{9.2}$$

For the first row, Corollary 3.6 gives the assertion if $\sqrt{2} \notin K$. Otherwise $\sqrt{5} \notin K$ and $2u + v \le 9/8$, while $u \ge d^2 + (1-d)^2/3 > 13/40$. The second row is Lemma 3.7.

Lemma 9.2. *If $K \ne K_*$, then $h \in \{1, 4\}$, $r < 1/150$, and $\operatorname{Tr}_{K/\mathbb{Q}} \alpha \le 40$.*

Proof. Trace domination in Lemma 3.3 gives $e_3(q) \le e_3(p)/h$, where e_j denotes the elementary symmetric function. The three coordinates other than d have sum $1-d$, so $e_3(p) \le (1-d)^2(1+8d)/27$. Four-variable Schur gives $2v + 9e_3(q) \ge 1$. If $h \ge 9$, then

$$u + v \ge \tfrac{1}{2} + (1-d)^2(17-8d)/54 > 4/5.$$

Thus the positive square h equals one or four.

For the remaining estimates put $c = 4/5 - d_*^2 - (1-d_*)^2/3$. Then $v \le c$. If every $q_i \ge b := 1/150$, the numbers $x_i = q_i - b$ have sum $S = 1 - 4b$ and square sum at most $W = c - 2b + 4b^2$. Homogeneous Schur gives $9e_3(x) \ge S(S^2 - 2W)$. For completeness, this four-variable inequality follows from three-variable Schur by merging the two smallest variables y, z. The difference of the two Schur expressions is $yz[5(S-y-z) - 4(y+z)] \ge 0$. Expanding the shifted product gives

$$\prod_i q_i \ \ge \frac{bS(S^2-2W)}{9} + \frac{b^2(S^2-W)}{2} + b^3 S + b^4 > \frac{d_*(1-d_*)^3}{27}.$$

The last inequality is direct rational substitution, contradicting (9.1). Hence $r < b$. For each $i \ne 0$, the other two coordinates have sum at most $\sqrt{2v}$. Consequently $q_i \ge 1 - r - \sqrt{2c} > 1/40$, as $(1 - b - 1/40)^2 > 2c$. Finally $\operatorname{Tr} \alpha = d/r + \sum_{i \ne 0} p_i/q_i < 1 + 40$. The trace is integral. □

Lemma 9.3. *If $K = K_*$, then $r < 1/80$, $q_i > 1/16$ for $i \ne 0$, and $\operatorname{Tr}_{K/\mathbb{Q}} \alpha \le 16$.*

Proof. For any conjugation σ of K, the element $\sigma(\alpha) = (E/\sigma D)/(E/\sigma E)$ belongs to the signed order T. Both displayed quotients belong to T by central induction, and $E/\sigma E$ has norm one and hence its inverse belongs to T. For a nonidentity involution interchanging $0, i$ and j, k, the unit coefficient of $\alpha\sigma(\alpha)$ is a positive integer by (2.3). With $U = dp_i$, $V = p_j p_k$, this says

$$I = U(E_0 + E_i) + V(E_j + E_k) \ge 1.$$

Replacing E by D on the right gives one, since $\sum p_i = 1$. Subtracting the two expressions gives

$$I - 1 = U(\operatorname{Tr} E - \operatorname{Tr} D) + (V - U)(E_j + E_k - D_j - D_k).$$

In fact $V > U$. Otherwise $\prod p_i = UV \le U^2 < d^2$, while $r^3(1-3r) \le \prod q_i$. This last bound follows by concentrating all mass above the common lower bound r in one entry. As $x^3(1-3x)$ increases up to $1/4$, we obtain $r < 1/90$. Now $v \ge r^2 + (1-r)^2/3$ and (9.2) give $u < 49/100$. Since $u + 2p_j p_k = d^2 + p_i^2 + (p_j + p_k)^2 \ge d^2 + (1-d)^2/2$, we get $V > 1/200 - d/2 > d > U$, a contradiction. Trace domination therefore gives $E_j + E_k \ge D_j + D_k$ for each pair $j, k \ne 0$.

Choose q_1 smallest among the three coordinates other than r, put $S = r + q_1$, and write $x = p_1$. The pair bound for $2, 3$ gives $q_2 q_3 \le (p_2 + p_3)(1 - S)/4$, and hence

$$u \ge d^2 + x^2 + (1-d-x)^2/2, \qquad v \ge r^2 + q_1^2 + (1-S)^2 - (1-d-x)(1-S)/2.$$

Minimize the resulting quadratic lower bound for $5u + 4v$ over real x, and use $rq_1 \le S^2/4$. This gives

$$5u + 4v \ge \frac{21}{5} - \frac{32S}{5} + \frac{88S^2}{15} - \frac{2d}{3}(2S + 3).$$

If $S \le 3/40$, the right side is at least its value at $(S, d) = (3/40, 1/950)$, namely $15/4 + 3/3800$. Thus $S > 3/40$. By (9.1), $r^3(1-3r) < 1/(27 \cdot 950)$, so $r < 1/27$. Suppose $r \ge 1/80$. Each remaining coordinate exceeds $3/40 - r$, whence concentrating the excess in one entry gives

$$\prod q_i > \psi(r), \qquad \psi(r) = r(3/40 - r)^2(17/20 + r).$$

On $[1/80, 1/27]$ the polynomial is concave ($\psi'' = 12r^2 + 21r/5 - 39/160 < 0$), and its smaller endpoint value is $\psi(1/80) = 69/1638400 > 1/(27 \cdot 950)$. This contradicts (9.1), proving $r < 1/80$. Now $q_1 > 3/40 - 1/80 = 1/16$, and $\operatorname{Tr} \alpha < 1 + 16$. □

9.2 One square inequality for both fields

For a row (A, B) of (9.2) and positive λ, μ, complete the square at each $i \ne 0$, using $p_i = \alpha_i q_i$. Summing gives

$$Au + Bv \ge 2\lambda(1-d) + 2\mu(1-r) - \sum_{i=1}^{3} \frac{(\lambda\alpha_i + \mu)^2}{A\alpha_i^2 + B}. \tag{9.3}$$

Only the three conjugates remaining after deleting one least conjugate enter the correction. We give the full arithmetic test for this correction so that the quartic argument can be checked separately from the fusion-ring census.

Lemma 9.4 (Finite conjugate estimate)**.** *Let $\alpha \ne 1$ be a totally positive integral d-number in the totally real abelian quartic field K, of square norm and least conjugate at most one. Count conjugates with their multiplicities as embeddings of K. For each row of the table, the sum in (9.3) is less than M.*

	$\operatorname{Tr}\alpha \le$	h	A	B	λ	μ	M
$K \ne K_*$	40	$1, 4$	1	1	1/3	1/2	17/20
$K = K_*$	16	*square*	5	4	1	3	41/10

Proof. A rational α would be one. In the first row a quadratic minimal polynomial is $X^2 - tX + n$ with $3 \le t \le 20$, $n = 1, 2$, and $t^2 - 4n$ positive and nonsquare. If $n = 2$, the d-number condition requires $2 \mid t$. Each root occurs twice.

For a quartic minimal polynomial $F = X^4 - tX^3 + sX^2 - kX + h$, the abelian group fixes a partition of the four roots into two pairs. Let a, a' be their sums, b, b' their products, and put $z = b + b'$. This is a rational algebraic integer. From the elementary symmetric functions one obtains

$$A_0 = (a - a')^2 = t^2 - 4s + 4z, \quad B_0 = (b - b')^2 = z^2 - 4h, \quad (tz - 2k)^2 = A_0 B_0.$$

Thus the following integer loops cover the first row. Use $4 \le t \le 40$, $2\sqrt{h} \le z \le t^2/4 - 2\sqrt{h}$, $z < s \le 3t^2/8$. These bounds follow from AM–GM on the pairs and on the elementary symmetric sums. Test $A_0, B_0 \ge 0$, $A_0 B_0$ square and $k = (tz \pm \sqrt{A_0 B_0})/2$ integral in $[1, t^3/16]$. When $h = 4$ retain only $2 \mid t, s$ and $4 \mid k$. In general, if c_j is the jth signed elementary symmetric coefficient, the d-number property gives $h^j \mid c_j^4$: the four conjugates generate the same ideal.

The discriminants of the two quadratic factors over their pair field have sum $t^2 - 2s - 2z$ and product $J = s^2 + 2sz - 3z^2 - 4tk + 16h$. Retain positive sum and product. Remove reducible polynomials by testing monic linear and quadratic integer factors with positive roots and constant term dividing h. The pair field is $\mathbb{Q}(\sqrt{\delta})$, with $\delta = A_0$ if $A_0 > 0$ and $\delta = B_0$ otherwise. Irreducibility requires δ nonsquare. Since K splits both quadratic factors and is quadratic over the pair field, their discriminants represent the same square class there. Hence

J or $J\delta$ must be an integer square. Retain either possibility and count equal polynomials only once.

For the second row put $\phi = (1+\sqrt{5})/2$. The integral basis $1, \sqrt{2}, \phi, \sqrt{2}\phi$ follows from the coprime quadratic discriminants $8, 5$. Write

$$4\alpha = t + b\sqrt{2} + c\sqrt{5} + e\sqrt{10}.$$

Then $2 \le t \le 16$ is even, c, e are even, and $t \equiv c$, $b \equiv e \pmod 4$. Total positivity gives $2b^2, 5c^2, 10e^2 < t^2$ by adding and subtracting conjugates. Enumerate these integers, and form the polynomial of the four conjugates by putting $y = 4X - t$ in

$$4^4 F(X) = (y^2 + 5c^2 - 2b^2 - 10e^2)^2 - 5(2cy + 4be)^2.$$

Keep alternating nonzero coefficient signs (all conjugates are real), square norm h, $h^j \mid c_j^4$, and least root at most one. Remove rational constants. There are two distinct roots precisely when just one of b, c, e is nonzero.

Here are the bounds obtained from these loops. For each distinct root use its adjacent grid points of mesh 2^{-16}, duplicating the intervals when the degree is two. For ordered intervals $[a_i, b_i]$, a rational upper bound on the desired sum is $\sum_{i=1}^{3} (\lambda b_i + \mu)^2/(Aa_i^2 + B)$. A grid interval for the jth root is found by bisection on $[0, t]$. The number of roots larger than a grid point x equals the number of sign changes in the nonzero coefficients of $F(X + x)$ by Descartes' rule, which is exact when all roots are real. This specifies the entire finite check.

	deg 2	deg 4	upper bound	maximizing polynomial
$K \neq K_*$	27	291	841/1000	$X^4 - 16X^3 + 20X^2 - 8X + 1$
$K = K_*$	5	3	4068/1000	$X^2 - 3X + 1$

The counts refer to the arithmetic candidates tested, before any fusion-ring conditions. The last column gives a minimal polynomial attaining the largest rational interval bound. Both upper bounds are below the corresponding M. The accompanying script `quartic_bounds.py` implements these loops with integers and rational endpoints. □

Proof of Theorem 9.1. If $\alpha \neq 1$, Lemmas 9.2 and 9.3 allow us to apply Lemma 9.4 in (9.3). In the two cases, respectively, this yields

$$u + v > \frac{5}{3} - \frac{2}{3 \cdot 950} - \frac{1}{150} - \frac{17}{20} > \frac{4}{5}, \qquad 5u + 4v > 8 - \frac{2}{950} - \frac{6}{80} - \frac{41}{10} > \frac{15}{4}.$$

Each inequality contradicts its moment bound. Thus $E = D$, and spectral saturation gives a spherical section. □

10 The dimension bound in the nonspherical case

Proof of Theorem 1.2. Suppose $\mathcal{C}$ has no spherical structure. By Theorem 4.1 every simple is self-dual. If D were rational, it would be integral, and weak integrality gives a spherical structure by [10, Proposition 8.24]. The Frobenius–Perron character field is a factor of the rank-four rational algebra and contains $\mathbb{Q}(D)$. If D has degree three, so does the character field, and Theorem 8.9 gives sphericality.

Only degrees two and four remain. In either case Corollary 6.6 identifies the signed dimension field with $\mathbb{Q}(D)$. In degree two, Theorem 7.4 gives a product of two copies of this field for the signed algebra, with complementary codegree product at least 100. Theorem 7.12 then gives $D < 1000$. In degree four Theorem 9.1 gives $D < 950$. Finally Lemma 2.1 gives $N_{XY}^{Z} \le d_X < \sqrt{D} < 32$. As these coefficients are integers, all are at most 31. □

11 Completeness of the bounded census

The ordinary coefficients lie in $[0, M] \cap \mathbb{Z}$, with $M = 31$. We keep rings of global Frobenius–Perron dimension strictly below $\Lambda = 1000$. Dong, Palcoux and Plessis used an ordinary census in their pivotal classification [7]. Here a common linear search gives the ordinary census and the local signed lifts needed for descent.

11.1 Presentations and elimination

Use the row convention $(N_u)_{vw} = N_{uv}^{w}$ and put $B = \sum_u N_u N_u^{\mathsf{T}}$, the ordinary Casimir. Every rank-four fusion ring is commutative by Lemma 2.1. For a self-dual basis $1, x, y, z$, reciprocity gives

$$N_x = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & a & b & c \\ 0 & b & d & e \\ 0 & c & e & f \end{pmatrix}, \quad N_y = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & b & d & e \\ 1 & d & g & h \\ 0 & e & h & i \end{pmatrix}, \quad N_z = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & c & e & f \\ 0 & e & h & i \\ 1 & f & i & j \end{pmatrix}.$$

These formulas also define signed tables, when arbitrary integer coefficients are allowed. Associativity is equivalent to the following six equations:

$$ad - b^2 + bg + ch - d^2 - e^2 + 1 = 0, \tag{11.1}$$
$$ae - bc + bh + ci - de - ef = 0, \tag{11.2}$$
$$be - cd + dh - eg + ei - fh = 0, \tag{11.3}$$
$$af + bi - c^2 + cj - e^2 - f^2 + 1 = 0, \tag{11.4}$$
$$bf - ce + di - eh + ej - fi = 0, \tag{11.5}$$
$$df - e^2 + gi - h^2 + hj - i^2 + 1 = 0. \tag{11.6}$$

The six equations are obtained by comparing the three pairwise matrix commutators.

The remaining duality has a single pair. For duality $(0)(1)(23)$ the parameters are (a, b, c, d, e, f), and

$$N_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & a & b & b \\ 0 & b & c & d \\ 0 & b & d & c \end{pmatrix}, \quad N_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & b & c & d \\ 0 & d & e & f \\ 1 & c & e & e \end{pmatrix}, \quad N_3 = N_2^{\mathsf{T}}.$$

The displayed matrices commute exactly when

$$\begin{aligned} ac - b^2 + 2be - c^2 - d^2 + 1 &= 0, \\ ad - b^2 + be + bf - 2cd &= 0, \\ b(c - d) + d(e - f) &= 0, \\ -c^2 + d^2 - e^2 + f^2 - 1 &= 0. \end{aligned} \tag{11.7}$$

In both cases commutativity of the displayed matrices suffices for associativity of the proposed multiplication. Indeed, set $E = N_u N_v - \sum_w N_{uv}^{w} N_w$. Its zeroth row vanishes, and it commutes with every N_t. Its row t is the zeroth row of $N_t E = E N_t$, hence also vanishes. Thus $E = 0$, proving the full regular multiplication identities. The unit, duality, integrality, nonnegativity, and reciprocity axioms are already built into the presentations.

11.2 A single bounded linear search

For any finite integer domain V, write $\mathcal{L}(k, r; V) = \{u \in V : ku = r\}$. For $k \neq 0$ this is the singleton $\{r/k\}$ if $r/k \in V$ and is empty otherwise. For $k = 0$ it is V if $r = 0$ and empty otherwise. This convention includes every free variable on a degenerate branch.

For a self-dual table use $V = \{0, \dots, M\}$. Permutations of the three nonunit basis elements are transitive on the six coordinates b, c, d, f, h, i, so we may put a minimal coordinate at b. Enumerate $a, b, c, d, e, f, h \in V$ with $b \leq c, d, f, h$. The first, second and fourth equations give, in order,

$$\begin{aligned} g &\in \mathcal{L}(b, b^2 + d^2 + e^2 - 1 - ad - ch; V), \\ i &\in \mathcal{L}(c, bc + e(d + f - a) - bh; V), \qquad i \geq b, \\ j &\in \mathcal{L}(c, c^2 + e^2 + f^2 - 1 - af - bi; V). \end{aligned}$$

Test the other three equations. Keep the lexicographically least tuple among the permutations for which the minimum is still at b. The only early dimension cutoff uses

$$B_{11} = 2 + a^2 + 2b^2 + 2c^2 + d^2 + 2e^2 + f^2 \leq D.$$

Thus any choice with $B_{11} \geq \Lambda$ can be skipped before enumerating h. The same three linear equations apply to signed lifts when V is replaced in each coordinate by its permitted parity interval.

For a dual pair enumerate $c, d, e, f \in V$ and test the last equation of (11.7). For each $b \in V$ satisfying the third, take $a \in \mathcal{L}(c, b^2 - 2be + c^2 + d^2 - 1; V)$ and test the second. Exchanging the dual pair leaves all six parameters unchanged. Distinct tuples therefore give distinct based rings.

11.3 The dimension test and the coverage conclusion

The positive dimension vector is an eigenvector of the symmetric nonnegative Casimir matrix B, with eigenvalue $D = \mathrm{FPdim}(R)$. It follows that D is its largest eigenvalue. Therefore

$$D < \Lambda \iff \Lambda I - B \text{ is positive definite} \iff \det(\Lambda I - B)_{[1,k]} > 0 \ (1 \leq k \leq 4). \quad (11.8)$$

Here $[1, k]$ denotes the leading principal k-by-k submatrix. The final equivalence is Sylvester's criterion. All determinants are integers; a zero determinant is rejected, so equality at the boundary is never admitted. The C++ enumeration uses fraction-free elimination, and the Python tensor checker recomputes the same quantities with arbitrary-precision integers.

Proposition 11.1. *The preceding procedure with $M = 31$ and $\Lambda = 1000$ emits exactly one representative of every rank-four fusion ring with $D < 1000$.*

Proof. Every rank-four fusion ring has one of the two complete presentations. All solutions of their associativity equations in the parameter box occur in the enumerated branches, and the isomorphism reduction retains exactly one representative. The acceptance condition (11.8) is equivalent to the desired strict dimension bound. Conversely, every ring below that bound has $\mu < \sqrt{D}$ by Lemma 2.1, hence $\mu \leq 31$, and therefore lies in the searched box. This proves both directions. □

The executable output has 1642 self-dual classes and 50 classes with a dual pair, for a total of 1692. As an additional implementation check, removing both dimension-pruning steps gives 16065 classes at multiplicity at most 31; an independently implemented strict dimension test retains precisely the same 1692 parameter tuples. A full small-box test examines all $3^{10} + 3^6 = 59778$ assignments at multiplicity at most two and recovers exactly the same 27 classes. The unpruned experiment shares the linear search. The small-box test uses the same tensor presentations and canonicalization.

12 Sphericality from the bounded census

Suppose that a rank-four category has no spherical structure. Theorem 1.2 puts its ordinary ring R in the census. Write P_R for the characteristic polynomial of its Casimir matrix and P_T for the corresponding polynomial of its signed quotient. Their roots, with multiplicity, are the ordinary and signed formal codegrees. The finite argument below uses only the coefficients of these polynomials and the six associativity equations. No sign choice for the spherical dimension character itself is required.

Lemma 12.1 (Two filters)**.** *For such a category, the following conditions hold.*

1. *Either P_R is irreducible quartic, or it has only linear and quadratic factors with exactly one nontrivial quadratic splitting field among them. Each irreducible factor is the minimal polynomial of a d-number.*

2. *Let $\mathcal{F}$ be the set of monic irreducible factors of $P_R P_T$. For at least one irreducible factor e of P_T,*
$$|f(0)|^{\deg e} \;\mid\; |e(0)|^{\deg f} \qquad (f \in \mathcal{F}).$$

Proof. The proof of Theorem 1.2 leaves two possibilities for $\mathbb{Q}(D)$ and identifies it with the signed dimension field L. In degree four the Casimir eigenvalue D has four distinct conjugates and these exhaust P_R. In degree two, all ordinary codegrees lie in L by Lemma 2.2, including the irrational D. The d-number assertion is Ostrik's theorem [25].

For the second assertion take e to be the minimal polynomial of the signed global codegree E. Central induction makes E/h an algebraic integer for every root h represented in $\mathcal{F}$. Take absolute norms in a number field K containing E and h. At each prime the resulting inequality is
$$\frac{v_p(|e(0)|)}{\deg e} \geq \frac{v_p(|f(0)|)}{\deg f},$$
after cancellation of $[K:\mathbb{Q}]$. This gives the divisibility. □

Proposition 12.2 (Finite reduction)**.** *The associativity, parity, domination and norm conditions below leave one possible ordinary tuple and two possible signed Casimir polynomials:*
$$\begin{aligned} N &= (2,2,2,2,3,4,2,4,4,6),\\ P_R(t) &= t^4 - 288t^3 + 2880t^2 - 9216t + 9216,\\ P_{T,1}(t) &= t^4 - 96t^3 + 1728t^2 - 9216t + 9216,\\ P_{T,2}(t) &= (t^2 - 72t + 96)(t^2 - 24t + 96). \end{aligned}$$

Exact enumeration. Start with the self-dual census. A nontrivial invertible gives sphericality by Theorem 5.1. A nonunit basis object with $X^2 = 1 + nX$ for $n \geq 2$ is excluded by Proposition 2.10. For the remaining tensors factor P_R over $\mathbb{Q}$ and apply Lemma 12.1(1). The d-number test is the coefficient divisibility in Proposition 2.8. For quadratic factors compare the squarefree parts of their discriminants.

For each surviving ten-tuple $N = (n_1, \ldots, n_{10})$, the kth signed entry ranges over $-n_k, -n_k + 2, \ldots, n_k$. In the order of (2.1), enumerate a, b, c, d, e, f, h in those ranges. One may take $a, b, c \geq 0$ since the entries $T_{111}, T_{112}, T_{113}$ transform by the independent signs of X, Y, Z. Use the three linear equations in Section 11.2 to determine g, i, j in their respective parity intervals, keeping a whole interval when the corresponding equation is $0 = 0$. Check the other three equations. If every absolute entry equals its ordinary counterpart, Lemma 2.3 gives a spherical section, so discard it. For each ordinary ring, collect the distinct remaining P_T, computed as $\det(tI - \sum_i T_i T_i^{\mathsf{T}})$, and apply Lemma 12.1(2). The following counts record the finite calculation; the last two count pairs (R, P_T).

Condition just imposed	Remaining
Self-dual ordinary rings of $D < 1000$	1642
No nontrivial invertible	1259
No forbidden rank-two subring	905
Ordinary formal codegrees are d-numbers	71
Ordinary codegree field test	41
Strict signed tuples in the chosen signs, counted over the 41 rings	1798
Distinct pairs (R, P_T)	623
Joint norm divisibility	2

The two pairs are the displayed output. All steps are integer operations or factorizations of polynomials of degree at most four over $\mathbb{Q}$. The script `verify_descent.py` implements the enumerations just described. □

Theorem 12.3. *Every complex fusion category of rank four admits a spherical structure.*

Proof. For the single ordinary candidate, P_R is irreducible, so $[\mathbb{Q}(D) : \mathbb{Q}] = 4$. Corollary 6.6 now identifies the signed dimension field L with $\mathbb{Q}(D)$. Thus Proposition 3.5 requires $u + v \leq 7/8$ for either of the two proposed signed polynomials.

For any quartic $t^4 + At^3 + Bt^2 + Ct + H$ with nonzero roots f_i, Vieta's formulas give $\sum_i f_i^{-2} = (C/H)^2 - 2B/H$. The ordinary polynomial gives $u = 3/8$. For $P_{T,1}$ and $P_{T,2}$ the respective values of v are $5/8$ and $7/12$. Thus $u + v$ is 1 or $23/24$, both strictly greater than $7/8$. □

13 The fifteen rings and sharp bounds

Corollary 13.1 (Rank-four classification)**.** *The rings in Table 1 are exactly the Grothendieck rings of complex fusion categories of rank four, up to based-ring isomorphism. Each ring has a unitary realization.*

Proof. The preceding descent gives a spherical structure on any rank-four category under consideration. In particular it is pivotal. The classification of Dong, Palcoux and Plessis [7, Theorem 1.1 and Table 1] gives exactly fifteen based rings for complex pivotal categories of rank four. The fifteen rows below are realized by the indicated categories and are pairwise nonisomorphic, as one checks from their multiplication tensors by the six permutations of the nonunit basis elements. □

Table 1 supplies one unitary realization and a construction reference for every surviving based ring. Here $\mathrm{Ad}(\mathrm{SU}(2)_k)$ denotes the adjoint subcategory, $\mathcal{H}_1$ is the rank-four Haagerup even part, and a near-group ring of type (C_3, m) satisfies $\rho^2 = \sum_{g \in C_3} g + m\rho$. The six non-self-dual models are Vec_{C_4}, the Tambara–Yamagami category on C_3, the principal even part of S', $\mathrm{Rep}(A_4)$, and the two near-group categories of parameters three and six.

Asaeda and Haagerup constructed the index-$(5 + \sqrt{13})/2$ subfactor [1]. Grossman and Snyder give the table of its rank-four even part $\mathcal{H}_1$ [13, Table 2]. The $2D2$ subfactor is due to Morrison–Penneys [20, Theorem 1.1]. The E_8 construction is due to Izumi [16]; its adjoint fusion rules are displayed in [11, Section 3.3]. The $(C_3, 6)$ example was first found by Zhengwei Liu and Noah Snyder, as credited in [15, Section 10.6].

The model check reconstructs each tensor from the group operations, the product or truncated SU(2) rules, or the cited exceptional fusion table. It identifies the corresponding ordinary census record and checks the positive character, the multiplicity and the global dimension. The cited constructions supply the unitary categories.

μ	A unitary realization	D	Reference
1	$\mathrm{Vec}_{C_2\times C_2}$	4	[19, §3.4]
1	$\mathrm{Fib}\boxtimes\mathrm{Vec}_{C_2}$	$5+\sqrt{5}$	[19, §3.4]
1	$\mathrm{Rep}(D_{10})$	10	[19, §3.4]
1	$\mathrm{Ad}(\mathrm{SU}(2)_6)$	$8+4\sqrt{2}$	[9, Theorem 3.1]
1	$\mathrm{Fib}\boxtimes\mathrm{Fib}$	$(15+5\sqrt{5})/2$	[19, §3.4]
1	$\mathrm{Ad}(\mathrm{SU}(2)_7)$	δ_7	[19, §3.4]
1	Vec_{C_4}	4	[19, §3.4]
1	Tambara–Yamagami on C_3	6	[30]
1	Principal even part of S'	$8+4\sqrt{2}$	[18, Theorem 1]
2	Principal even part of $2D2$	$20+8\sqrt{5}$	[9, Theorem 3.1]
2	Haagerup category $\mathcal{H}_1$	$(39+9\sqrt{13})/2$	[13, Table 2]
2	$\mathrm{Rep}(A_4)$	12	[17, Theorem 1.1]
3	$\mathrm{Ad}(E_8)$	δ_E	[16, 11]
3	Near-group type $(C_3,3)$	$(21+3\sqrt{21})/2$	[15, Example 9.4]
6	Near-group type $(C_3,6)$	$24+12\sqrt{3}$	[15, Theorem 10.19]

Table 1: One unitary realization of each surviving based ring, with its multiplicity μ and global Frobenius–Perron dimension D. Here D_{10} has order ten, S' has index $3+2\sqrt{2}$, and the near-group category of type $(C_3,3)$ is the Izumi–Xu category.

The numbers δ_7 and δ_E are the largest real roots of $x^3-27x^2+162x-243$ and $x^4-45x^3+375x^2-1125x+1125$, respectively. The supplementary model check verifies these polynomials, the positive dimension characters, and all tensor identifications exactly.

Corollary 13.2 (Sharp bounds)**.** *Every rank-four complex fusion category satisfies*

$$\mathrm{FPdim}(\mathcal{C})\le 24+12\sqrt{3},\qquad \max_{i,j,k}N_{ij}^{k}\le 6.$$

Both bounds are attained, and equality in either bound holds precisely for the near-group ring $\mathrm{NG}(C_3,6)$.

Proof. Apply the sharp pivotal bound of Dong, Palcoux and Plessis [7, Corollary 5.4] after Theorem 12.3. For the equality case the noninvertible simple has dimension $3+2\sqrt{3}$, so $D=3+(3+2\sqrt{3})^2=24+12\sqrt{3}$. □

Reproduction. From the `verification` directory run `python3 verify.py`. This command regenerates the ordinary census, tests every signed candidate, checks the quadratic and quartic estimates, and reconstructs the fifteen model tensors. It requires Python, SymPy and a C++17 compiler.

References

[1] M. Asaeda and U. Haagerup, *Exotic subfactors of finite depth with Jones indices* $(5+\sqrt{13})/2$ *and* $(5+\sqrt{17})/2$, Comm. Math. Phys. **202** (1999), 1–63. arXiv:math/9803044.

[2] B. Bartlett, *Fusion categories via string diagrams*, Commun. Contemp. Math. **18** (2016), no. 5, 1550080, 39 pp. doi:10.1142/S0219199715500807.

[3] P. Bruillard, *Rank 4 premodular categories*, with an appendix by P. Bruillard, C. Galindo, S.-H. Ng, J. Plavnik, E. Rowell and Z. Wang, New York J. Math. **22** (2016), 775–800. Published article.

[4] J. Dong, G. Chen and Z. Wang, *Fusion categories containing a fusion subcategory with maximal rank*, J. Algebra **604** (2022), 107–127. doi:10.1016/j.jalgebra.2022.02.020.

[5] C. Dong, X. Lin, and S.-H. Ng, *Congruence property in conformal field theory*, Algebra Number Theory **9** (2015), no. 9, 2121–2166. doi:10.2140/ant.2015.9.2121.

[6] J. Dong, *On Grothedieck rings of rank 4 self-dual fusion categories*, preprint (2025). arXiv:2505.19153v2.

[7] J. Dong, S. Palcoux and A. Plessis, *Classifying Grothendieck rings of pivotal fusion categories of rank four*, preprint, 2026. arXiv:2609.33272v2.

[8] P. Etingof, S. Gelaki, D. Nikshych and V. Ostrik, *Tensor Categories*, Mathematical Surveys and Monographs **205**, American Mathematical Society, 2015.

[9] C. Edie-Michell, M. Izumi and D. Penneys, *Classification of $\mathbb{Z}/2\mathbb{Z}$-quadratic unitary fusion categories*, with an appendix by R. Johnson, S.-H. Ng, D. Penneys, J. Roat, M. Titsworth and H. Tucker, Quantum Topol. **16** (2025), 493–568. arXiv:2108.01564v2.

[10] P. Etingof, D. Nikshych, and V. Ostrik, *On fusion categories*, Ann. of Math. (2) **162** (2005), no. 2, 581–642. doi:10.4007/annals.2005.162.581.

[11] D. E. Evans and C. Jones, *Quantum symmetries of noncommutative tori*, Comm. Math. Phys. **406** (2025), article 167. doi:10.1007/s00220-025-05337-y.

[12] T. Gannon and A. Schopieray, *Algebraic number fields generated by dimensions in fusion rings*, Commun. Number Theory Phys. **18** (2024), 705–743. doi:10.4310/CNTP.241202233757.

[13] P. Grossman and N. Snyder, *Quantum subgroups of the Haagerup fusion categories*, Comm. Math. Phys. **311** (2012), 617–643. doi:10.1007/s00220-012-1427-x.

[14] T. J. Hagge and S.-M. Hong, *Some non-braided fusion categories of rank three*, Commun. Contemp. Math. **11** (2009), 615–637. doi:10.1142/S0219199709003521.

[15] M. Izumi, *A Cuntz algebra approach to the classification of near-group categories*, in *Proceedings of the 2014 Maui and 2015 Qinhuangdao Conferences in Honour of Vaughan F. R. Jones' 60th Birthday*, Proc. Centre Math. Appl. Austral. Nat. Univ. **46** (2017), 222–343. Published chapter.

[16] M. Izumi, *On flatness of the Coxeter graph E_8*, Pacific J. Math. **166** (1994), 305–327.

[17] H. K. Larson, *Pseudo-unitary non-self-dual fusion categories of rank 4*, J. Algebra **415** (2014), 184–213. doi:10.1016/j.jalgebra.2014.05.032.

[18] Z. Liu, S. Morrison and D. Penneys, *1-supertransitive subfactors with index at most $6\frac{1}{5}$*, Comm. Math. Phys. **334** (2015), 889–922. doi:10.1007/s00220-014-2160-4.

[19] Z. Liu, S. Palcoux and Y. Ren, *Classification of Grothendieck rings of complex fusion categories of multiplicity one up to rank six*, Lett. Math. Phys. **112** (2022), article 54. arXiv:2010.10264v5.

[20] S. Morrison and D. Penneys, *2-supertransitive subfactors at index $3+\sqrt{5}$*, J. Funct. Anal. **269** (2015), no. 9, 2845–2870. doi:10.1016/j.jfa.2015.06.023.

[21] S.-H. Ng and P. Schauenburg, *Frobenius–Schur indicators and exponents of spherical categories*, Adv. Math. **211** (2007), no. 1, 34–71. doi:10.1016/j.aim.2006.07.017.

[22] S.-H. Ng and P. Schauenburg, *Higher Frobenius–Schur indicators for pivotal categories*, in *Hopf Algebras and Generalizations*, Contemp. Math. **441**, Amer. Math. Soc., Providence, RI, 2007, 63–90. doi:10.1090/conm/441/08500.

[23] S. Nasseh, S. Sather-Wagstaff, R. Takahashi, and K. VandeBogert, *Applications and homological properties of local rings with decomposable maximal ideals*, J. Pure Appl. Algebra **223** (2019), no. 3, 1272–1287. doi:10.1016/j.jpaa.2018.06.006.

[24] V. Ostrik, *Pivotal fusion categories of rank 3*, with an appendix written jointly with D. Nikshych, Mosc. Math. J. **15** (2015), no. 2, 373–396. doi:10.17323/1609-4514-2015-15-2-373-396.

[25] V. Ostrik, *On formal codegrees of fusion categories*, Math. Res. Lett. **16** (2009), no. 5, 895–901. doi:10.4310/MRL.2009.v16.n5.a11.

[26] V. Ostrik, *Fusion categories of rank 2*, with an appendix by P. Etingof, Math. Res. Lett. **10** (2003), no. 2–3, 177–183. arXiv:math/0203255.

[27] E. C. Rowell, R. Stong, and Z. Wang, *On classification of modular tensor categories*, Commun. Math. Phys. **292** (2009), no. 2, 343–389. doi:10.1007/s00220-009-0908-z.

[28] A. Schopieray, *Categorification of integral group rings extended by one dimension*, J. Lond. Math. Soc. (2) **108** (2023), 1617–1641; doi:10.1112/jlms.12789.

[29] A. Schopieray, *Norm, trace, and formal codegrees of fusion categories*, J. Algebra **568** (2021), 362–385. doi:10.1016/j.jalgebra.2020.10.014.

[30] D. Tambara and S. Yamagami, *Tensor categories with fusion rules of self-duality for finite abelian groups*, J. Algebra **209** (1998), 692–707. doi:10.1006/jabr.1998.7558.

[31] G. Vercleyen and J. K. Slingerland, *On low rank fusion rings*, J. Math. Phys. **64** (2023), 091703. doi:10.1063/5.0148848.

[32] G. Vercleyen and J. Slingerland, *On low-rank multiplicity-free complex fusion categories*, Comm. Math. Phys. **407** (2026), article 231. doi:10.1007/s00220-026-05705-2.

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