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Uniform Spectral Gap and Stability for the Spin-2 Square-Lattice AKLT Model with Free Boundaries

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Abstract

We prove a uniform spectral gap above the full ground space of the spin-2 AKLT Hamiltonian on every square-lattice rectangle with four free sides and both lengths at least two. The ground-space dimension grows exponentially with the perimeter. Starting from the classical hard-boundary coupling of Aronow and Lopatto, we compare the boundary Gram operator of one wing with the product of its marginals on the cut side and the remaining three sides. A reference system with at most eight unsaturated sites then controls a physical cut of arbitrary length. Recursion over overlapping rectangles gives the gap. We also obtain a gap on all simple tori, including those of fixed circumference. In free squares, boundary effects on fixed local observables decay exponentially with the buffer width. By the stability theorems of Nachtergaele, Sims and Young, weak quasi-local perturbations with stretched-exponentially decaying interaction terms preserve the bulk GNS gap and the finite-torus gap uniformly on all simple tori, without an aspect-ratio restriction.

1 The Hamiltonian and the boundary problem

Let $\Lambda = [W] \times [H] \subset \mathbb{Z}^2$, where $[m] = \{1, \dots, m\}$. At every site, including every corner, take the irreducible spin-2 space $V_4 = \mathrm{Sym}^4 \mathbb{C}^2$. For a nearest-neighbour edge e let p_e be the orthogonal projection onto total spin 4 at its endpoints. Write

$$H_\Lambda = \sum_{e \subset \Lambda} p_e, \qquad G_\Lambda = \mathrm{proj}\ker H_\Lambda.$$

Each internal edge has coefficient one. There are no exterior edges or boundary fields.

Classical coupling hypothesis (H). On each sufficiently large square $[-M, M]^2$, consider the classical $\mathbb{S}^2$ spin law with bond weight $1 + n_x \cdot n_y$ and a fixed exterior. For any two exteriors differing at one adjacent site there is a coupling for which the expected number of unequal interior spins is at most Δ_M, with a common $\Delta_M/(2M+1) \to 0$ as $M \to \infty$.

Theorem 1.1 (Free rectangles)**.** *Assume* (**H**)*. There exists $g > 0$ such that $H_\Lambda \geq g(\mathbf{1} - G_\Lambda)$ for all integers $W, H \geq 2$. The inequality holds on the entire physical Hilbert space, uniformly over all aspect ratios.*

Corollary 1.2. *For the spin-2 square-lattice AKLT model, the uniform lower bound of Theorem 1.1 holds on every free rectangle of side lengths at least two.*

Proof. Proposition 4.4 of Aronow and Lopatto supplies (**H**) from their window estimate (W), established via Theorems 5.9, 7.4 and 8.1 (assembled in Section 8.4) of [5]. □

We began studying open rectangles in October 2026. While this work was in progress, Aronow and Lopatto established an explicit spectral gap for doubly periodic rectangles when both circumferences are sufficiently large (of either parity), identified the ground vector on every simple torus, and proved uniqueness of the infinite-volume frustration-free state using open-box ground states [5, Theorem 1.1, Corollary 3.3 and Appendix F]. Taking only their classical hard-boundary coupling (**H**) as a modular input, the present paper establishes four results for the undecorated spin-2 square-lattice model: (i) a uniform spectral gap on every free-boundary rectangle $W, H \geq 2$, uniformly over all aspect ratios (Theorem 1.1); (ii) a uniform gap on all simple tori $L, N \geq 3$, including fixed short circumferences down to three with arbitrary length (Theorem 6.1); (iii) exponential local topological quantum order (LTQO) in operator norm on the entire open-box ground space (Theorem 7.1); and (iv) perturbative stability of both the infinite-volume bulk GNS gap and the finite-torus gap uniformly over all aspect ratios under weak quasi-local perturbations with stretched-exponentially decaying interaction terms (Corollaries 8.1 and 8.2).

On an open rectangle, the ground-space dimension grows exponentially with the perimeter as $3^4 2^{2W+2H-8}$. Its boundary vectors need not be products, the physical inner product on them depends on the rectangle, and dividing a block-ground vector by a fixed valence-bond state can leave degrees of freedom at a free edge inside the block. In our gap proof, the classical coupling controls both a bounded, two-sided comparison of the boundary Gram operator for one whole side (Proposition 3.1) and the uniform gap of a pinned reference system with at most eight unsaturated sites (Appendix A); it is this separate gapped reference system that turns the boundary comparison into a uniform contraction at a physical seam of arbitrary length (Proposition 4.1). In Section 7, the same coupling on an intact inner rectangle controls the joint classical marginal on the inner neighbour ring and yields exponential LTQO directly.

Related work. Our analysis builds on the AKLT valence-bond framework [1], boundary operators for projected entangled pair states [7], and recursive gap criteria for frustration-free Hamiltonians [6]. Kennedy, Lieb and Tasaki described finite-volume valence-bond ground states and proved exponential decay of two-point correlations on the periodic square lattice [2]. On two-dimensional degree-3 and decorated lattices, spectral gaps were established analytically on decorated hexagonal lattices by Abdul-Rahman, Lemm, Lucia, Nachtergaele and Young [15], by finite-size criteria on the hexagonal and other degree-3 lattices by Lemm, Sandvik and Wang [16] and by Pomata and Wei [17], and via boundary-state factorization on the hexagonal lattice by Kastoryano [11]; Jackson, Nachtergaele and Young proved local indistinguishability for the hexagonal and Lieb AKLT models [10]. For the undecorated square lattice, Guo, Pomata and Wei obtained a weighted finite-size sufficient condition for a spectral gap [3].

Polynomial conventions. Give V_d the inner product in which $x^{d-j}y^j$ are orthogonal of squared norm $\binom{d}{j}^{-1}$. Identifying two spinors gives a map $J : V_k \otimes V_l \to V_{k+l}$ with $JJ^* = \mathbf{1}$, by Vandermonde's identity. Thus J^*J projects onto the largest total spin and $\ker J$ comprises the multiples of $[uv] = x_u y_v - y_u x_v$ (by divisibility on proportional spinors, or by dimension). For the polynomial divisibility argument, see Kirillov and Korepin, Theorem 2 in the arXiv version of [4]. Distinct edge factors are relatively prime. On a box D, multiplication U_D by the product of all internal edge factors maps $\bigotimes_x V_{4-\deg_D(x)}$ bijectively onto its ground space. The four corners leave dimension three each, the other boundary sites dimension two each, giving the count above. An edge factor can be inserted between any two residual sites of degree at most three with squared singular values between universal positive constants. Indeed multiplication is injective on each of finitely many local spaces $V_k \otimes V_l$ and acts as the identity on all spectator variables. We use the same observation

after lowering the output degrees at endpoints of removed edges.

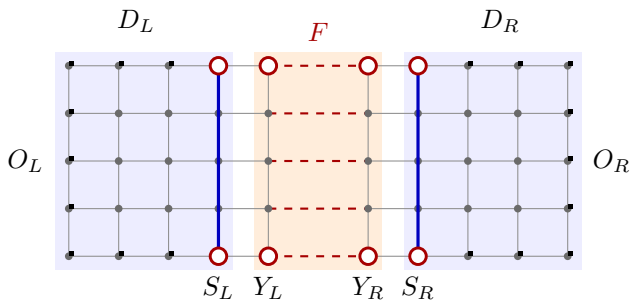

Figure 1: A vertical matching cut. The dashed edges form F. The two columns Y_L, Y_R carry their original physical spin 2 throughout. Each deep wing has one facing side S (blue), including both of its corner sites. Its three other sides, with those two corners omitted, form O. Only the reference model pins the exterior slots on O. After the matching is restored, its eight circled sites can still have free exterior slots. All slots on the original exterior are free in Theorem 1.1.

2 Two comparisons for hard classical boundaries

With unit spins $n_x \in \mathbb{S}^2$ and normalized area measure, the classical Gibbs density in a rectangle D is proportional to

$$\prod_{\{x,y\}\subset D,\ x\sim y} (1 + n_x \cdot n_y) \prod_{x\in D,\, y\notin D,\, x\sim y} (1 + n_x \cdot \eta_y).$$

All exterior spins η_y are fixed. The partition function is positive since each factor is positive almost everywhere. Throughout the following reduction we assume (**H**). Fix an odd b such that $\Delta_{(\ell-1)/2} \le b/96$ for each odd $\ell \in [b, 2b]$. Let k be the number of these lengths, so $k \ge b/2$.

Lemma 2.1. *Suppose both lengths of D are at least $2b$. There is a reversible square-block heat bath with gap at least $1/2$, for every hard exterior. If two exteriors differ only on Y, then at any $i \in D$*

$$\|\mu_i^{\eta} - \mu_i^{\eta'}\|_{\mathrm{TV}} \le \frac{b}{2} \sum_{y\in Y} \exp\{-\|i - y\|_\infty/(8b)\}.$$

Here total variation is the supremum of the absolute event difference.

Proof. In a coordinate of length n, assign the start j of an ℓ-interval the weight $1+b\mathbf{1}_{j=1}+b\mathbf{1}_{j=n-\ell+1}$. Update each contained ℓ-square by conditional resampling at the product of its two starting weights divided by kb^2. The sum of one-dimensional weights containing a given site lies between b and $4b$: an endpoint anchor suffices for the lower bound, and away from both anchors there are ℓ possible starts. For a site immediately outside the interval there are at most two starts, and endpoint bonuses occur only at the two lengths $x-1, n-x$. The sum of these outside weights across lengths is at most $2k + 2b$. Thus each site has covering rate at least one and outside-neighbour rate at most $2(4b)(2k + 2b)/(kb^2) \le 48/b$. The covering rate is at most 16 per site or 32 for blocks touching a given edge.

A single interior disagreement is erased by a block containing it. For a neighbouring block the classical input bounds expected new disagreements. The net removal rate is at least $1-(48/b)(b/96) = 1/2$. Denoting the total clock rate by λ, a single random update contracts Hamming distance by

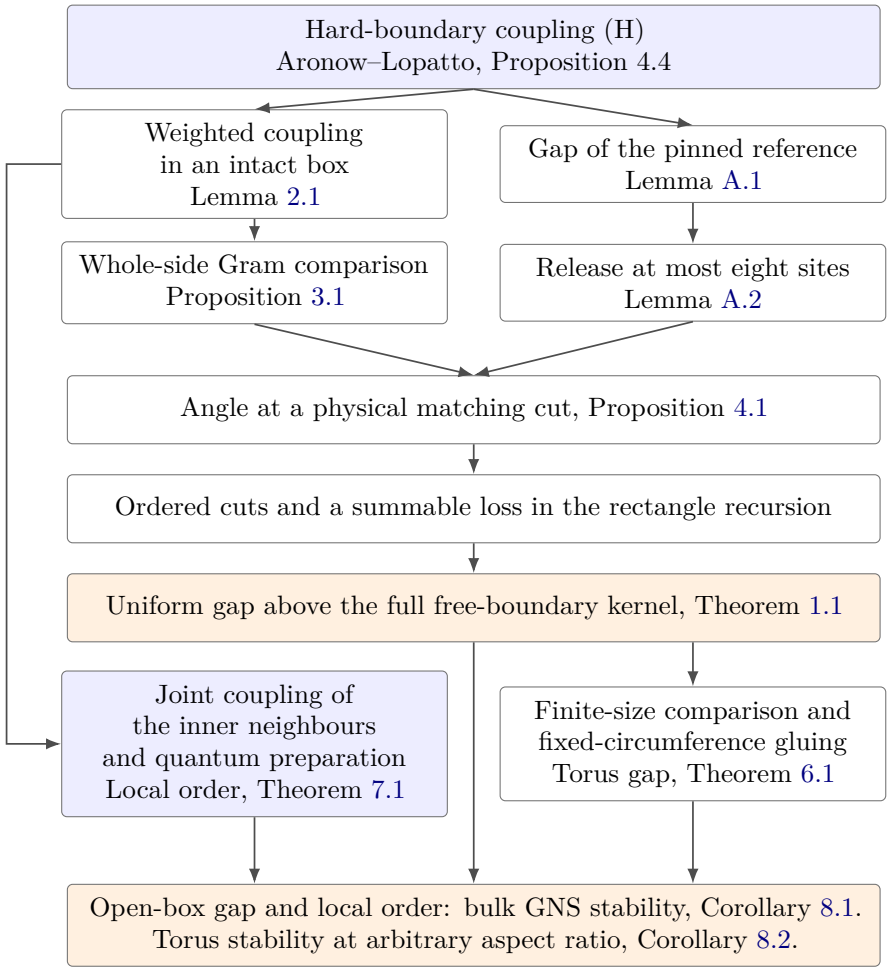

Figure 2: The comparison at a free seam needs both the boundary norm of a wing and a gapped reference system. The weighted coupling also controls the interior of an intact square. The two stability conclusions use the perturbation bounds specified in Section 8.

$1 - 1/(2\lambda)$, first for neighbouring configurations and then along a coordinate path. The Markov operator is a positive self-adjoint average of conditional expectations. A mean-zero bounded function has finite Hamming Lipschitz seminorm, so its iterates decay in supremum norm at this rate times a volume-dependent prefactor. The spectral theorem, followed by density of bounded functions in L^2, gives the stated continuous-time spectral gap.

For the marginal, put $w_x = e^{-\|x-i\|_\infty/(8b)}$ and use the sum of weights of disagreements as the distance. Within an affected block each new weight is at most $e^{1/4} w_y < (4/3) w_y$ for a disagreement at y. The corresponding interior removal rate is at least $1/3$. For a changed external spin, the normal start is forced and has weight at most $3b$, the tangential covering weight is at most $4b$. Thus the total affected rate is at most 12, injecting expected weight at most $b w_y/6$ per unit time. Couple along a coordinate path and telescope over changed exterior spins. For stationary laws, their weighted Wasserstein distance L satisfies

$$L \le (1 - 1/(3\lambda))L + (b/(6\lambda)) \sum_{y \in Y} w_y.$$

One may glue the conditional update couplings on the finite product of spheres; the weighted Hamming cost is bounded and lower semicontinuous. Since $w_i = 1$, the marginal total variation is at most L. □

3 One whole side in the physical boundary norm

In a $w \times h$ wing D, let S be a side of length h, including both corner sites, and put $O = \partial D \setminus S$. Define the trace-one Gram operator $M = U_D^* U_D / \operatorname{Tr}(U_D^* U_D)$ and its two marginal operators $M_S = \operatorname{Tr}_O M,\ M_O = \operatorname{Tr}_S M$.

Proposition 3.1. *For each fixed $A \ge 1$ there is $C_A < \infty$ such that for every $2 \le h \le Aw$ and $w \ge 2$,*

$$C_A^{-1} M_S \otimes M_O \ \le M \le C_A M_S \otimes M_O. \tag{1}$$

Proof. The coherent-state identity $(d+1) \int |n; d\rangle\langle n; d|\, dn = \mathbf{1}$ expresses M as the expectation of the tensor product of unit coherent projectors of residual degrees $4 - \deg_D(x)$. Its probability density on all physical spins is proportional to the product of squared edge factors. Changing n_x to $-n_x$ on one sublattice makes the edge weight proportional to $1 + n_x \cdot n_y$ and applies the same variable change to the local projectors. Positivity and their unit traces are preserved.

Remove the two perimeter bonds between S and O and lower the output degree at each exposed end by one per removal. The residual source at each site is unchanged. Before trace normalization, re-inserting these factors compares the Grams in both directions with a universal constant by the local insertion estimate. Trace normalization and either partial trace preserve such a comparison, at a bounded cost. For bounded h one may instead cut all h edges between the facing column and the next column. The separated Gram is a product across $S|O$, giving (1) with constant depending only on this upper bound for h. This covers the case where either length of the inner rectangle is less than $2b$, since then $h \le A(2b+2)$. The h insertions cost at most 2^h on the unnormalized Gram. Normalizing and comparing the two true marginals as below gives a factor at most 4^{3h}.

Otherwise condition the outer ring on its two pieces s, o. The ring density of the two-edge-deleted model has the form $a(s)c(o)Z(s,o)$ up to normalization. Here Z is the partition function of the *intact* inner rectangle with hard exterior given by that ring. Each corner retains its full degree-two residual space; both corners of S occur in $a(s)$. The interior partition function is strictly positive for every fixed exterior, even at configurations where a ring weight is zero. Only its other $h - 2$

sites have an inward neighbour in the inner rectangle, one per site. Change a single S spin with an inward neighbour i from v to u, and write $f_u = 1 + n_i \cdot u$. Integrating $f_u f_v$ against the same weight with this single leg omitted gives the identity between *hard* laws

$$\frac{Z(u,o)}{Z(v,o)} = \frac{\mathbb{E}_{\mu^{v,o}} f_u}{\mathbb{E}_{\mu^{u,o}} f_v}.$$

For either law, conditional on all other spins, the density at i is the squared modulus of a product P of $t \le 4$ linear spinor factors. Multiplying a normalized degree-t monomial by x has minimum squared singular value $1/(t+1)$ over the whole space, as follows by taking ratios of the binomial norms. By unitary invariance this holds for any unit linear factor. The coherent-state normalization therefore gives $\mathbb{E} f_u \ge 2/(t+2) \ge 1/3$. Changing o to o' changes each logarithmic expectation by at most six times the marginal total variation ($0 \le f_u \le 2$).

Telescope along the S legs and apply Lemma 2.1. With $\xi = 8b$, the absolute logarithmic cross ratio of $Z(s,o), Z(s',o'), Z(s,o'), Z(s',o)$ is bounded independently of the lengths: for each adjacent side the sum is bounded by $\sum_{a\ge 0, j\ge 1} e^{-\max(a,j)/\xi}$, and for the opposite side by $h^2 e^{-(w-2)/\xi}$, bounded for $h \le Aw$. Multiply out the cross-ratio inequality, include $a(s)c(o)a(s')c(o')$, and integrate in s', o'. This compares the normalized ring law in both directions with the product of its two actual marginals, without dividing by a possibly zero perimeter weight. Integration against the same positive, trace-one coherent-projector tensors preserves the Loewner order on the full boundary space, not just on product vectors. To track the original marginals, write B^-, M^- for the unnormalized and normalized cut Grams. Each singlet insertion has squared singular values in $[1/2, 2]$: on the two degrees $k, l \le 3$ its eigenvalues in squared norm are $(t+1)(k+l+2-t)/((k+1)(l+1))$, $0 \le t \le \min(k,l)$. Thus $B^-/4 \le U_D^* U_D \le 4B^-$ and $M^-/16 \le M \le 16M^-$. Each partial trace preserves the latter order. If the cut comparison has constant K, the joint Gram and its two marginals therefore give (1) with constant $16^3 K$. □

4 From eight exposed sites to a seam of arbitrary length

In an $n \times h$ rectangle cut the matching F between columns $j, j+1$. Let P_j project onto the kernel with these edges deleted and let $T_j = \prod_{f\in F}(\mathbf{1} - p_f)$. Its factors have disjoint supports. Their simultaneous ground projector with P_j is $G = G_\Lambda$.

Proposition 4.1. *If $j-1, n-j-1 \ge \max(2, h/5)$ then $\|(T_j - G)(P_j - G)\| \le q$ for a single $q < 1$ independent of the rectangle.*

Proof. Keep the two columns next to the matching as physical columns Y_L, Y_R. Call the two remaining rectangles D_L, D_R and take their facing sides S_L, S_R as in Figure 1. In the polynomial source of the cut ground space write $\mathcal{L}$ for the residual factors of the two Y columns and the facing sides after all nonmatching bonds have been accounted for, and $\mathcal{O}$ for the residual factors on both complements O_L, O_R. Multiplication V by the vertical bonds in Y and the bonds from Y to the facing sides acts trivially on $\mathcal{O}$. Then $U = (U_{D_L} \otimes U_{D_R} \otimes \mathbf{1}_Y)V$ multiplies by every nonmatching bond exactly once and parametrizes $\operatorname{ran} P_j$ injectively. The matching acts only on the retained physical columns. Consequently $B(f) = \|Uf\|^2$ and $D(f) = \|(\mathbf{1} - T_j)Uf\|^2$ evaluate the same product Gram on Vf and on $(\mathbf{1} - T_j)Vf$, respectively. Trace-normalizing the wing Grams changes both by the same scalar.

Fill the exterior slots on each O by a fixed product polynomial $\chi = \chi_L \otimes \chi_R$. Impose this divisibility by one orthogonal complement projection per site. After restoring the matching, the only unsaturated sites in the reference system are the four ends of the two Y columns and the four

ends of the two facing sides. By Lemmas A.1 and A.2 below, the reference Hamiltonian has gap at least $g_8 > 0$. Its cut ground space is $U(\mathcal{L} \otimes \chi)$.

For a reference cut-ground vector v perpendicular to the reference full kernel, $T_j v$ stays perpendicular to it. For a nonmatching edge a, let F_a consist of the matching edges touching a, and put $T_{\mathrm{far}} = \prod_{f \notin F_a} (\mathbf{1} - p_f)$. Since $p_a v = 0$ and p_a commutes with T_{far},

$$\|p_a T_j v\|^2 \le \left\| \left(\mathbf{1} - \prod_{f \in F_a} (\mathbf{1} - p_f) \right) T_{\mathrm{far}} v \right\|^2 = \sum_{\varnothing \neq B' \subseteq F_a} \|E_{B'} v\|^2,$$

where $E_{B'}$ projects onto the pattern of violations precisely B'. Fixing any one edge in a nonempty B' shows that this pattern is charged at most six times. All onsite constraints are on O and commute with T_j. Thus $g_8 \|T_j v\|^2 \le 6 \|(\mathbf{1} - T_j) v\|^2$. For any reference cut vector this yields the energy-distance estimate with coefficient $g_8/(6 + g_8)$ after subtraction of its reference-ground component.

In a single wing put $M^\chi = \langle \chi_D | M | \chi_D \rangle_O$ and $s = \langle \chi_D, M_O \chi_D \rangle > 0$. Equation (1) with $A = 5$ first gives $C_5^{-1} M_S \le M^\chi / s \le C_5 M_S$, and then compares M with $(M^\chi/s) \otimes M_O$ up to factors $C_5^{\pm 2}$. For both wings, B and D are each comparable up to $C_5^{\pm 4}$ to their reference form tensored with the same positive spectator metric, with the same normalization factor. The zero subspace is $\mathcal{K} = \ker[(\mathbf{1} - T_j) V|_{\mathcal{L}}] \otimes \mathcal{O}$, because the wing Grams are positive definite and V leaves $\mathcal{O}$ untouched. In an orthonormal basis of the spectator metric the reference estimate holds componentwise. Comparing the energy and the infimum of the squared norm over the same affine subspace gives

$$D(f) \ge \eta \inf_{k' \in \mathcal{K}} B(f - k'), \qquad \eta = C_5^{-8} g_8/(6 + g_8) > 0.$$

This quotient comparison includes arbitrary entanglement with $\mathcal{O}$ and permits the physical orthogonal complements for the two metrics to differ. The matching-pass projection T_j has norm at most $q = \sqrt{1 - \eta}$ on $\mathrm{ran}(P_j - G)$. □

5 Ordered cuts and recursion on the area

Proof of Theorem 1.1. Orient the longest side horizontally and let its length be n and the height be h. Write $R_j = P_j - G$. The next split enforces the jth matching, so $P_{j+1} \le T_j$ and $\|R_{j+1} R_j\| \le q$ for every admissible j. For $a < j < b'$ there is an exact identity $R_{b'} R_j R_a = R_{b'} R_a$. Indeed P_j is the product of the ground projectors of $[1, j]$ and $[j + 1, n]$, each taken at full height. In $P_{b'} P_j P_a$, the shorter prefix projector can be absorbed by the longer prefix and the shorter suffix by the longer suffix, commuting past disjoint intervals. Subtract G, which is below every projector involved. It follows by insertion of all intermediate cuts and idempotence that

$$\|G_{[1,b']} G_{[a+1,n]} - G\| = \|R_{b'} R_a\| \le q^{\,b' - a}.$$

Here $P_{b'} P_a = G_{[1,b']} G_{[a+1,n]}$ by the same nesting argument.

For sufficiently large n put $d = \lfloor \sqrt{n} \rfloor$ and $k = \lfloor n/(10d) \rfloor$. Place k disjoint d-column overlap strips within the central interval $[2n/5, 3n/5]$, starting just after $\lceil 2n/5 \rceil$. For an overlap $(a, b']$ use $[1, b']$ and $[a + 1, n]$ as its two child boxes. Both children have area at most $3nh/5$ and the deep wing on each side of every intermediate cut has width at least $2n/5 - 1 \ge \max(2, h/5)$. The intersection of the two child kernels is the parent kernel since all edges are covered. On its orthogonal complement the sum of the two complementary projectors is at least $1 - q^d$, by the principal-angle identity for two subspaces. If γ_{ch} is the minimum gap among all the children, summing their energy bounds over the k pairs gives

$$H_\Lambda \;\ge\; \gamma_{\mathrm{ch}} \frac{k}{k+1} (1 - q^d)(\mathbf{1} - G).$$

Indeed each original edge occurs once in each pair, with a second copy only if both endpoints are in that pair's overlap. The overlaps are disjoint. The multiplicative factor $f(n)$ satisfies $1 - f(n) \le c/\sqrt{n}$: $1/(k+1) \le 10/\sqrt{n}$ and $q^{\lfloor\sqrt{n}\rfloor}$ decays faster.

Choose a fixed cutoff so that $f(n) \ge 1/2$ beyond it and the described cuts exist. Reorient each child along its own longest axis. Along any branch, counting s steps upwards from a leaf, the area grows at least as $(5/3)^s$ and the longest side is at least $(5/3)^{s/2}$. The negative logarithms of the factors sum to at most $2c\sum_{s\ge0}(3/5)^{s/4} < \infty$, uniformly in the root. The finitely many leaf shapes have side lengths at least two and a positive least nonzero eigenvalue. Their minimum times this uniform product bound proves the theorem. Thin rectangles are covered by the same recursion using the bounded-h branch of Proposition 3.1. □

6 A consequence for periodic rectangles

Theorem 6.1. *For simple square-lattice tori of circumferences $L, N \ge 3$, the ground space is one-dimensional and the spectral gap has a common positive lower bound. This conclusion follows from the open-square gaps and finite-width quantum comparisons.*

Proof. Write g_n for the gap of the free $n \times n$ square. When $L, N \ge 2n+1$ and $n \ge 3$, translating that square gives the elementary finite-size estimate

$$\operatorname{gap}(H^{\mathrm{tor}}_{L,N}) \;\ge\; \frac{n(n-1)g_n - (n+1)}{(n-1)^2}. \tag{2}$$

We give the counting argument because the two circumferences may differ and have either parity. This is a Knabe-type finite-size comparison [8]. For weighted criteria, see [9]. Let H_x be the open-square Hamiltonian at translate x, and put $a = n(n-1)$, $c = (n-1)^2$. Every edge occurs a times, so $ag_nH \le \sum_x H_x^2$. A perpendicular pair of incident edges occurs c times and a collinear incident pair $c-1$ times. A disjoint pair occurs at most c times, by intersecting the sets of allowed starting positions in both coordinates. For parallel edges in the same line, their separation reduces the along-line count to at most $n-2$. The circumference assumption prevents two distinct short lifts of a contributing pair. Disjoint edge projectors commute, so their anticommutators are positive. For a collinear incident pair use $-\{p_e, p_f\} \le p_e + p_f$. Each edge has two collinear neighbours. Thus $\sum_x H_x^2 \le cH^2 + (a - c + 2)H = cH^2 + (n+1)H$, giving (2). Theorem 1.1 permits a fixed n for which the right side is positive.

It remains to treat finitely many short circumferences $3 \le h \le 2n$ with unrestricted length. We record the fixed-width argument. For any transverse matching F of h edges, let P be the ground projector before inserting F and $T = \prod_{f\in F}(\mathbf{1} - p_f)$. This applies to a cut across an open-ended cylinder of circumference h, and also to its end-to-end closing matching. There is a constant $c_h > 0$ such that every $v \in \operatorname{ran} P$ satisfies

$$\|(\mathbf{1} - T)v\|^2 \ge c_h \operatorname{dist}(v, \operatorname{ran} P \cap \operatorname{ran} T)^2.$$

To see this, strip from v all nonmatching edge determinants touching an endpoint of F, writing $v = Bu$. There are at most $6h$ stripped factors. All remaining divisibility constraints avoid the matching endpoints. On the resulting lower-degree polynomial spaces, fuse the two spinors at f by $\widetilde{J}_f$. The projections $\widetilde{J}_f^*\widetilde{J}_f$ have disjoint supports. If u_0 is the projection of u onto their common kernel, then $Bu_0 \in \operatorname{ran} P \cap \operatorname{ran} T$ and $\|u - u_0\|^2 \le \sum_f \|\widetilde{J}_f u\|^2$. Fusing v at f gives $B^{(f)}\widetilde{J}_f u$, where $B^{(f)}$ is B with the two endpoints identified. Each factor survives because the graph is simple and f itself was never stripped. In a three-cycle the two bonds to a common neighbour can become a

squared determinant after fusion. Its multiplication map is still injective since the polynomial is nonzero. The local multiplication maps by $B^{(f)}$ are injective. For fixed h their smallest singular values have a common positive lower bound, and the norms of the maps by B have a common finite upper bound: there are finitely many incidence and degree patterns and all other sites are spectators. Thus the physical fusion energies $\sum_f \|p_f v\|^2$ control $\operatorname{dist}(v, \operatorname{ran} P \cap \operatorname{ran} T)^2$. Finally $\sum_f p_f \le h(\mathbf{1} - T)$ gives the displayed estimate.

On the open-ended cylinder the resulting cut angle is bounded away from one uniformly in the length. Use the same insertion identity and disjoint-overlap average as in the proof of Theorem 1.1, always splitting the open direction. Each child has at most $3/5$ of its parent's length, and the finitely many terminal lengths give a positive seed. This proves a length-independent cylinder gap $\gamma_h > 0$. Closing the two ends adds the h matching projectors. The closed Hamiltonian dominates $\gamma_h(\mathbf{1} - P) + (\mathbf{1} - T)$, which has a gap by the angle estimate above and the two-projection decomposition. Take the minimum over the finitely many short circumferences. On a torus every site has degree four, so the product of all edge determinants exhausts every degree and spans the full kernel. □

7 Exponential local indistinguishability in open boxes

Write $B_x(r) = \{z \in \mathbb{Z}^2 : \|z - x\|_\infty \le r\}$, and let G_L denote the full ground projection of the open Hamiltonian on L. Local topological quantum order is a sufficient local-order input in stability results for frustration-free Hamiltonians [12]. An infinite-volume state is frustration-free if it annihilates every bond projection. Aronow and Lopatto proved uniqueness of this state in [5, Theorem F.1]. Their equation (F.1) bounds the total variation of the classical marginal on a fixed central set by $4(2M+1)(1-\delta)^{\lfloor M^{1/8} \rfloor}$ for sufficiently large M. Sections F.2–F.3 then use informational completeness of the coherent measurement to identify the quantum state. Here we condition on the neighbours of the support of an observable. Those spins prepare a density matrix on the support, which gives a quantitative estimate in operator norm.

Theorem 7.1 (Local topological quantum order)**.** *Let b be as in Lemma 2.1, and let ω be the unique infinite-volume frustration-free state. For every $x \in \mathbb{Z}^2$ and integers $r \ge 0$, $\ell \ge b + 1$, set $B = B_x(r)$, $R = r + \ell$ and $L = B_x(R)$. Every operator A supported on B satisfies*

$$\|G_L A G_L - \omega(A) G_L\| \le 2\|A\| \min\{1,\, 4bR\, e^{-(\ell-1)/(8b)}\}. \tag{3}$$

The estimate applies to complex observables and to the entire open-box ground space.

Proof. Let $O = \{z : \|z - x\|_\infty = R\}$, $D = L \setminus O$, and let $S = \{z \notin B : \operatorname{dist}_1(z, B) = 1\}$ be the ring of nearest neighbours of B. Then $S \subset D$, including when $r = 0$. We first bound the *joint* marginal on S of the hard-boundary classical law on D (Figure 3). The side length of D is $2R - 1 \ge 2b$. In the proof of Lemma 2.1, take $w_z = \exp[-\operatorname{dist}_\infty(z, S)/(8b)]$. Distance to S is 1-Lipschitz, so a disagreement at y can create only weights at most $e^{1/4} w_y$ in an affected block. The interior removal rate is again at least $1/3$ and the exterior injection rate at most $b w_y/6$. The same stationary coupling inequality therefore bounds the weighted Hamming Wasserstein distance by $(b/2)\sum_y w_y$, with the sum over the changed exterior spins. A mismatch anywhere on S has cost at least one. All exterior neighbours of D belong to O, which has $8R$ sites and lies at ℓ^∞-distance at least $\ell - 1$ from S. Thus, for arbitrary hard configurations η, η' on O,

$$\|\mu_D^\eta|_S - \mu_D^{\eta'}|_S\|_{\mathrm{TV}} \le \varepsilon, \qquad \varepsilon = \min\{1, 4bR\, e^{-(\ell-1)/(8b)}\}.$$

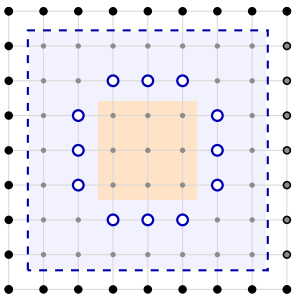

Figure 3: Conditioning on the outer layer gives a classical law on all of D, including the sites of B. Its joint marginal on S mixes the quantum states Q_s on B. The four diagonal corners just outside B do not meet it by a bond. The drawing illustrates incidence and is not to scale.

Write a ground vector on L as $U_L f$, where f is a polynomial on the outer layer O. Insert the coherent resolution of the identity on $L \setminus B$, leaving the quantum spins in B unevaluated. Write the classical spins after the sublattice sign change of [5, Appendix F.2]. For a configuration s on S, evaluating the exterior ends of all bonds crossing out of B, and retaining the singlet factors internal to B, gives a vector v_s on B. This is a product of nonzero polynomials and hence is nonzero for every s. The choices of spinor phases cancel in $Q_s = |v_s\rangle\langle v_s|/\|v_s\|^2$. The coherent identity on B gives $\|v_s\|^2 = c_B Z_B(s)$, where $Z_B(s)$ is the classical partition function with these hard neighbours and $c_B > 0$ is independent of s.

In particular, fixing η on O prepares on B the density matrix

$$\sigma_\eta = \int Q_s \, d(\mu_D^\eta|_S)(s).$$

To see the measure in this formula, multiply the squared factors on $D \setminus B$ and its bonds to O by $\|v_s\|^2$. The factor $Z_B(s)$ integrates exactly the missing sites and bonds of the intact rectangle D. Its normalization is $Z_D(\eta)$. Consequently there is a positive measure $d\tau(\eta)$ on O, independent of f and A, such that, for any two boundary polynomials,

$$\langle U_L g, A U_L f\rangle = \int \overline{g(\eta)} f(\eta) \, \operatorname{Tr}(A\sigma_\eta) \, d\tau(\eta). \tag{4}$$

Here the boundary polynomial is evaluated at the coherent spinors before the sublattice sign change, expressed as a function of η. The measure $d\tau$ is the product area measure times $Z_D(\eta)$, the bond weights within O, and a common positive normalization. Taking $A = \mathbf{1}$ gives $\int |f|^2 \, d\tau = \|U_L f\|^2$, so entangled boundary polynomials have the same representation.

Each Q_s has trace norm one, so $\|\sigma_\eta - \sigma_{\eta'}\|_1 \le 2\varepsilon$. For completeness, normalized open-box ground states on an exhaustion have a locally convergent subsequence by finite-dimensional compactness and a diagonal argument. The consistent limit annihilates every bond. If ω is any such state, its restriction to L is supported on G_L, by positivity of the local Hamiltonian. Its restriction to B is therefore a convex mixture of the σ_η by (4) and spectral decomposition. In particular $|\operatorname{Tr}(A\sigma_\eta) - \omega(A)| \le 2\varepsilon\|A\|$ for every η. Subtract the identity case of (4) multiplied by $\omega(A)$. Cauchy–Schwarz bounds the resulting matrix element by $2\varepsilon\|A\| \, \|U_L g\| \, \|U_L f\|$. Supremizing over both ground vectors proves (3). Any second frustration-free state is also supported on G_L upon restriction. Taking ℓ to infinity for fixed B gives equality on all local observables, and hence uniqueness. □

8 Stability of the gap on the infinite plane

Let $\Phi(x,n)$ be self-adjoint, supported on $B_x(n)$, for $x \in \mathbb{Z}^2$ and integers $n \geq 1$. Assume $\|\Phi(x,n)\| \leq Je^{-an^\theta}$, uniformly in x, n, with $J, a > 0$ and $0 < \theta \leq 1$. Onsite terms may be included at radius one. The family defines the perturbation of the infinite-volume dynamics with coupling s as in [13, Assumption 2.6].

Corollary 8.1 (Bulk gap under local perturbations)**.** *Under* (**H**)*, the frustration-free state ω on $\mathbb{Z}^2$ has a positive gap γ_0 in its GNS Hamiltonian. For each $0 < \gamma < \gamma_0$ there is $s_0 > 0$ such that, for real $|s| < s_0$, the spectral-flow state $\omega_s = \omega \circ \alpha_s$ [18] is a ground state of the perturbed infinite-volume dynamics and its GNS Hamiltonian has gap at least γ.*

Proof. First take any local A with $\omega(A) = 0$. For increasing open squares L containing the support and all its neighbours, let $P = G_L$. Frustration freeness gives $\omega(P) = 1$. For the local energy derivation $\delta(A) = [H_L, A]$, Theorem 1.1 yields

$$\omega(A^*\delta(A)) = \omega(A^*H_LA) \geq g\big(\omega(A^*A) - \omega(A^*PA)\big).$$

Here $\omega(A^*PA)$ is at most $\|PAP\|^2$, since ω is supported on P. It tends to zero by Theorem 7.1. This proves the GNS gap criterion [13, (2.22)] with $g > 0$.

We check the one-term-per-site convention of the stability theorem. On $\mathbb{Z}^2$ with the ℓ^∞ metric, let h_x be half the sum of the four bond projections around the unit plaquette with lower left corner x. Every bond belongs to two plaquettes, so this defines the same infinite interaction derivation. On a square $B_x(n)$, $n \geq 1$, formula (2.13) in [13] keeps the plaquettes contained in the square. Each boundary bond then has weight $1/2$ and each remaining bond has weight one. The local kernel is exactly $G_{B_x(n)}$ and the local gap is at least $g/2$ by Theorem 1.1. Ball volumes grow quadratically. Colour centres by both coordinates modulo $2n+1$. Balls of a common colour are disjoint, and there are $(2n+1)^2$ colours. Thus the exponents in Assumption 2.3 can be taken as $\nu = \zeta = 2$ and $\alpha = 0$ (with ν the volume exponent).

For the local-order condition (2.27) in that paper, put $t = m - k$ for integer outer and inner radii $m \geq k$. If $t \geq b + 1$, Theorem 7.1 bounds the norm by $8b\|A\|(k+t)e^{-(t-1)/(8b)}$. Since $k + t \leq (1+k)^2(1+t)$, one may take $G_0(t) = C_b e^{-t/(16b)}$ in (2.27), enlarging C_b to cover the trivial bound $2\|A\|$ for $0 \leq t < b + 1$. At radius zero the full one-site space is the local kernel. For real radii one takes integer parts and enlarges C_b once. All the moments of G_0 in (2.26) are finite. The interaction radius, local gaps, local order and perturbation bound now satisfy the hypotheses of [13, Theorems 2.8–2.9]. The latter theorem gives the stated infinite-volume conclusion. □

Finite tori of arbitrary aspect ratio

Corollary 8.2. *Assume* (**H**) *and fix $J, a > 0$ and $0 < \theta \leq 1$. On each simple torus $C_h \times C_n$, $3 \leq h \leq n$, let $V = \sum_{x,r} \Phi(x,r)$, where $1 \leq r \leq \lfloor n/2 \rfloor$, $\Phi(x,r)$ is self-adjoint and supported on the periodic ℓ^∞ ball of radius r at x, and $\|\Phi(x,r)\| \leq Je^{-ar^\theta}$. The terms may be chosen independently on different tori. There are $s_*, g_* > 0$, independent of h, n and of these choices, such that $H^{\mathrm{tor}} + sV$ has a unique ground state and gap at least g_* for all real $|s| \leq s_*$.*

Proof. We use the finite-volume stability theorem of Nachtergaele, Sims and Young [14, Theorem 6.8]. For tori of unrestricted shape we need two additional estimates: a common gap on the open cylinders that occur as metric balls, and local indistinguishability even when the inner observable winds around the short circle.

Uniform open-cylinder gaps. Let g be the free-rectangle bound of Theorem 1.1. Fix an integer $\ell \geq 2$ and consider $C_h \times [w]$ with $h \geq 2\ell + 1$ and $w \geq 2$. Translate an open $\ell \times w$ strip around C_h. Write $c = \ell - 1$, $\alpha = c/\ell$, and let K_j be the strip Hamiltonian with bond weights one in the periodic direction and α in the open direction. The weights are all positive, so the strip kernel is unchanged and $K_j^2 \geq \alpha g K_j$. Each open-direction bond occurs ℓ times and each periodic-direction bond occurs c times, giving $\sum_j K_j = cH$, with H the open-cylinder Hamiltonian. In $\sum_j K_j^2$ each diagonal or disjoint-pair coefficient is at most c: two open-direction bonds occur together at most ℓ times, giving weight at most $\ell\alpha^2 \leq c$, and a pair containing a periodic-direction bond occurs together at most c times. Disjoint projectors commute, so their anticommutators are positive. For incident pairs of two open-direction bonds, two periodic-direction bonds, or one of each, the coefficients are respectively $c - \alpha$, $c - 1$, and $c - \alpha$. A deficit of at most one in such an anticommutator is bounded above in the comparison by $p_e + p_f$, since $-(p_e p_f + p_f p_e) \leq p_e + p_f$. Each bond has at most six incident neighbours. Thus

$$\alpha g c H \leq \sum_j K_j^2 \leq cH^2 + 6H, \qquad \operatorname{gap}(H) \geq \alpha g - 6/c.$$

Fix ℓ so that $6/c \leq g/4$. For the finitely many shorter circumferences $3 \leq h \leq 2\ell$, use the length-independent open-cylinder gap proved in Theorem 6.1 before adding the closing matching, including its three-cycle case. Taking the minimum with $g/4$ yields a common positive gap for all these open cylinders.

Classical mixing on open cylinders. Use the actual untransformed bond weights $1 - u \cdot v$ on the sphere. Let b be the odd block scale of Section 2. For $h \geq 4b + 4$ and axial length at least $2b$, every heat-bath square of side $\ell \in [b, 2b]$ used in Lemma 2.1, with its relevant exterior neighbours, lifts to a planar square with hard exterior. The staggered spin flip is performed only on that local lift; it preserves the Hamming cost and invokes (**H**) even if h is odd. Keep the endpoint-weighted starts in the open direction and give each cyclic start weight one in the periodic direction. For each length the periodic coordinate then has covering multiplicity $\ell \in [b, 2b]$ and exactly two adjacent outside starts, within the bounds used in Lemma 2.1. With the same clock normalization the covering rate is at least one, the rate of blocks adjacent to an interior disagreement at most $48/b$, and the rate of blocks affected by a changed hard exterior spin at most 12. For a set S of observed sites use weights $w_z = \exp[-d_\infty(z, S)/(8b)]$, with open distance along the axis and cyclic distance around the cylinder. The weighted cost of each new discrepancy generated by one at z is at most $4w_z/3$. As in Lemma 2.1, this gives an interior removal rate at least $1/3$ and exterior injection at most $bw_y/6$ for a changed exterior site y. Path coupling, and then stationarity, bound the weighted cost by $(b/2)\sum_y w_y$. Any mismatch on S costs at least one, so the same bound applies to total variation of the *joint* marginal on S.

For the finitely many smaller h, write $B(x, y) = \prod_i (1 - x_i \cdot y_i)$ for the interaction of adjacent columns and $v(y) = \prod_i (1 - y_i \cdot y_{i+1})$ for the cyclic column. The two-step kernel $L(x, z) = \int B(x, y) v(y) B(y, z)\, dy$ satisfies, uniformly in x, z,

$$2^{-7h} \leq L(x, z) \leq 2^{3h}.$$

Indeed choose each y_i in order while avoiding the caps where a factor for x_i, z_i, or an already chosen neighbour in y is less than $1/4$. Each cap has area fraction $1/8$, and at most four caps occur at each choice. On a set of measure at least 2^{-h}, all $3h$ factors are at least $1/4$. The upper bound uses that each factor is at most two. For a given common message from the right, the two-step transition out of a column x has density proportional to $L(x, z)F(z)$, where F is a nonnegative density factor from the remaining volume. Hence every row contains the same component of mass at least 2^{-10h},

with law $F(z)\,dz/\int F$. Two copies can coalesce at this rate and then use identical future columns. Change the left end and then the right end separately. This controls the joint law of any two observed columns which are both at least a_0 steps from either end by $\min\{1, 2(1-2^{-10h})^{\lfloor a_0/2\rfloor}\}$: couple at a sampled column no later than the first observation from the changed end, and sample the remaining columns jointly thereafter. For small h the minimum of these finitely many decay rates is positive.

Local order on every torus ball. Let $d = \lfloor n/2\rfloor$, $0 \le k \le m \le d$, and A be supported in the k-ball. Let P_m be the ground projector for the sum of all bonds internal to the concentric m-ball and $\omega_{h,n}$ the unperturbed torus ground-state functional. If $2m+1 < h$, the m-ball induces an open square, and Theorem 7.1 applies, replacing the reference expectation by $\omega_{h,n}(A)$ at most doubling its error by frustration freeness. At $m = 0$ use the trivial norm bound. If $2m+1 \ge n$, the ball is the whole torus and the compression is exactly scalar by Theorem 6.1.

In the remaining case the outer ball is an open cylinder of axial length $2m+1$. Enlarge the support region to the entire inner cylinder $I = C_h \times [-k, k]$, without changing A. Let $t = m - k$ and use the trivial norm bound for $t \le 1$. Otherwise let S consist of the two columns adjacent to I and O of the two outermost columns. In the spinor representation the full kernel is the product of all internal edge determinants times a residual polynomial on O. Insert coherent resolutions on the complement of I. Given $S = s$, the normalized state prepared on I is Q_s; it depends only on s since precisely those columns carry the cross-interface determinants. Its squared normalization before division is proportional to the partition function on I with that exterior, which is positive since the bond factors are positive almost everywhere for fixed s. After integrating the remaining columns with $O = \eta$, the state is $\sigma_\eta = \int Q_s\, d(\mu_D^\eta|_S)(s)$, with D the cylinder after removing O. In fact, before normalizing the outer law, the coherent norm on I restores exactly that inner partition function. Thus, as in Section 7, for residual polynomials f, g the matrix element of A is $\int \overline{g} f\ \mathrm{Tr}(A\sigma_\eta)\, d\tau(\eta)$, where τ is positive and independent of A, f, g, and the same formula with $A = \mathbf{1}$ is the physical inner product. No global staggering is used on the cylinder.

If changing hard end columns changes the joint S law by total variation at most ϵ, then $\|\sigma_\eta - \sigma_{\eta'}\|_1 \le 2\epsilon$. The torus state restricted to the outer ball is supported in its full local kernel and hence its restriction to I averages the same family of states. The scalar difference in each integrand is at most $2\epsilon\|A\|$. Cauchy–Schwarz on the two residual polynomials gives this same bound for $\|P_m A P_m - \omega_{h,n}(A)P_m\|$, also for non-self-adjoint A. For $h \ge 4b+4$ the open cylinder D has length $2m-1 \ge 2b$. The two end columns have $2h$ spins, each at axial distance at least $t-1$ from S. The joint mixing bound therefore gives $\epsilon \le \min\{1, bhe^{-(t-1)/(8b)}\}$. Use the two-step coupling for the finitely many smaller h. Since $h \le 2m+1 \le 2(1+k)(1+t)$ in the cylinder case and the k-ball has at least $k+1$ sites, absorption of $1+t$ into the exponential and inclusion of the other ball shapes give common $C, c_0 > 0$ with

$$\|P_m A P_m - \omega_{h,n}(A)P_m\| \le |B_x(k)|\|A\| C e^{-c_0(m-k)}.$$

Thus the indistinguishability radius in [14, Definition 2.1] can be the full diameter d at every centre.

Uniform stability. For the comparison regions in [14, (6.8)–(6.12)] take the balls themselves. For $m \ge 1$ their internal-bond sums have a common positive gap by Theorems 1.1, 6.1 and the open-cylinder comparison above. Anchor each original bond with half its weight at each endpoint, so the anchoring radius is one; the comparison Hamiltonians still use all and only the original bonds supported in each region. Ball volume is at most $(2m+1)^2$. Colour the intersection graph of

radius-m balls by at most $(4m+1)^2$ colours, since intersecting balls have centres within distance $2m$. The volume and colouring exponents are two and the local-gap exponent is zero.

For the perturbation bounds (6.14)–(6.15), take $F(r) = e^{-a'r^\theta}(1+r)^{-4}$ with $a' = a/2^{1+\theta}$. If a ball of radius m contains a pair at distance r, then $r \le 2m$; the number of possible centres and the ball volume are both $O((1+m)^2)$. After division by $F(r)$ at least $e^{-am^\theta/2}$ of the original exponential decay remains to absorb the polynomial factors and sum over radii. The F-function constants are uniform by the quadratic volume bound, the triangle inequality and subadditivity of r^θ. Take $K = L = \lfloor d/3 \rfloor$ for sufficiently large d. All centres then belong to (6.6), and both cutoffs are at least $\lceil d^{1/2} \rceil$. With exponential Ω and the local gaps above, (6.38) and (6.41)–(6.43) give Assumption 6.7 of [14].

For uniformity over the entire family, suppose that no common positive threshold and gap exist. Choose tori, perturbations and $|s_j| \le 1/j$ with the two lowest eigenvalues (with multiplicity) separated by less than $1/j$. Bounded diameters are excluded by the unperturbed gap, Weyl's inequality and the bound $\|V\| \le Jhn \sum_{m\ge 1} e^{-am^\theta}$. On an unbounded subsequence the longer circle tends to infinity. Either take both circumferences growing, with centred representatives exhausting $\mathbb{Z}^2$, or take fixed shorter circumference h and exhaust $C_h \times \mathbb{Z}$. On fixed pairs and finite subsets the respective ambient metrics and the original interactions eventually agree; in both cases the subsequence satisfies the conditions of Section 6 of [14]. A common perturbation bound, not convergence of the perturbations, is what Definition 6.5(v) requires here. Absorb the sign of s_j into V_j. Theorem 6.8 gives uniform separation of the low-energy band from the rest on this subsequence for sufficiently small perturbation parameter. Its spectral projection has rank one by continuity from the unperturbed rank-one ground projection. This contradicts the chosen first gaps. Reducing the common threshold if necessary gives the stated closed interval. □

A The reference system

For every missing exterior edge choose a unit linear spinor factor at its boundary site. If the retained factors at x have product L_x, impose one onsite orthogonal projector with kernel $L_x V_{4-\deg L_x}$. With no retained factor this projector is zero. Equal factors at a corner retain their multiplicity. All original edge and onsite projections have unit coefficients. A site is saturated when every missing direction carries a factor. We prove two facts about these auxiliary Hamiltonians.

Lemma A.1. *With every site saturated, the ground state is unique and the gap is bounded below by $g_0 > 0$ for every $W, H \ge 2$ and every choice of spinor factors.*

Proof. The full product of edge determinants and onsite factors exhausts all degrees. Its normalization Ξ spans the joint kernel. The coherent integral for $|\Xi|^2$ gives exactly the classical hard-boundary law after the sublattice change of variables. Every exterior neighbour of a rectangle touches just one internal site, hence arbitrary chosen exterior factors define a legal hard configuration. Coherent evaluation divided by that of Ξ defines an isometry J_Ξ from the physical space into $L^2(\mu)$, with $J_\Xi \Xi = 1$ (the zero set has measure zero).

For an updated square R take H_R to consist of every physical edge touching R and every onsite term in R, and let P_R project onto its kernel. Dividing a vector in this kernel by Ξ leaves a function of the exterior of R: the required factors exhaust each degree in R. Writing E_R for conditional expectation given that exterior yields

$$\|(\mathbf{1} - E_R) J_\Xi \phi\|^2 \le \|(\mathbf{1} - P_R)\phi\|^2 \le g_*^{-1} \langle \phi, H_R \phi \rangle.$$

For our fixed b only finitely many local block geometries occur, including edges and corners. At each geometry the kernel is the image of multiplication by the prescribed product from a fixed

residual source. Its rank is constant for all unit pin parameters, including coincidences. The onsite projectors vary continuously, since the multiplication matrix L has full column rank and its range projector is $L(L^*L)^{-1}L^*$. Compactness thus gives a common positive g_* for the nonzero local spectra. Summing with the rates of Lemma 2.1 and using the covering bound of 32 for each term gives $H \geq (g_*/64)(\mathbf{1} - |\Xi\rangle\langle\Xi|)$ whenever both side lengths are at least $2b$.

For a fixed short side $h < 2b$ take a column as a single classical variable. Its interaction with its neighbour is $B(x, y) = \prod_{r=1}^{h}(1 + x_r \cdot y_r)$. Let $w_j(y)$ contain the within-column weights and any exterior factors. The two-step kernel

$$L_j(x, z) = \int B(x, y)w_j(y)B(y, z)\, dy$$

is continuous and everywhere positive, since each of finitely many factors has a null zero set. Compactness and the finite column geometries give $0 < m_h \leq L_j \leq M_h$. Conditional on a common right boundary, the law two columns forward has density proportional to $L_j(x, z)F(z)$ with the same nonnegative right message F for both left conditions. Both laws therefore dominate $(m_h/M_h)F(z)\, dz/\int F$. Couple every second column until agreement, then couple synchronously and fill the intermediate columns by their conditional bridges. The expected number of changed columns is at most $2M_h/m_h + 3$, uniformly in the interval length. A changed right condition is handled by reversal.

Set $D_h = 2M_h/m_h+3$ and take an odd $b_h > 12D_h$. For intervals of odd length in $[b_h, 2b_h]$, use the endpoint bonuses b_h above. Give each interval its starting weight divided by $k_h b_h$ as its continuous-time clock rate, where k_h is the number of allowed lengths. Each column is covered at rate between one and four. Its neighbouring intervals have total starting weight at most $2k_h + 2b_h \leq 6k_h$, hence total rate at most $6/b_h$. A single column discrepancy thus contracts at rate at least $1-6D_h/b_h > 1/2$. Each bond is charged at rate at most eight. The updated blocks have bounded size depending on h, so the same quantum comparison and compactness argument applies. If the length is also below $2b_h$, compactness applies to the entire rectangle. Take the minimum over the finitely many short sides. This argument obtains g_0 without using any gap assumption for the free-boundary system. □

Lemma A.2. *For each fixed K, suppose at most K boundary sites retain only some (possibly none) of their exterior factors. The gap is at least $g_K > 0$, uniformly in size, positions, spacings, and directions.*

Proof. Induct on the number j of exceptions, starting with Lemma A.1. Choose an exceptional x and let $r \in \{1, 2\}$ be its number of free exterior slots. Take S_* to be the sum of the edge terms incident to x and the current onsite terms in its closed star. Write $P = \operatorname{proj}\ker S_*$. The star constraints mean precisely divisibility by their product B_*, which includes each required determinant and each onsite polynomial. The same finite-type constant-rank argument as above gives $S_* \geq s(\mathbf{1} - P)$, uniformly over the pin parameters. Expand $p = P\psi = \sum_{a=0}^{r} p_a$ with $p_a = B_* e_a u_a$, where (e_a) are the normalized monomials in V_r at x and u_a have degree zero there. Enlarge the pin product at x by the factors of e_a to get a Hamiltonian H_a with $j - 1$ exceptions. Then p_a satisfies its enlarged onsite term and $N_a = \ker H_a \subset N = \ker H$.

For an edge e outside S_*, its endpoints do not include x. On identifying its endpoints, the product B_* becomes a polynomial $B_*^{(e)}$. Every star edge still has distinct endpoints because it contains x, and each pin factor remains nonzero. Fusion commutes with multiplication in the sense that fusing the product equals multiplying the fused factors. Multiplication by $B_*^{(e)}$ is injective, and its largest and smallest singular values have bounded ratio by compactness on a star and at most one additional edge. All other variables are spectators. In particular e may meet another exceptional site

adjacent to x. Fusion acts only on the two endpoints of e, so the distinct e_a at x remain orthogonal before multiplication by the fused star factors. Since fusion's squared norm equals the energy of the corresponding highest spin projector, there is a fixed $c > 0$ with $\|p_e p\|^2 \geq c \sum_a \|p_e p_a\|^2$. For a star-exterior onsite projector the same argument applies directly because it commutes with star multiplication and the component split. All the star terms vanish on p_a, including its new pin constraint. Consequently

$$\langle p, Hp \rangle \geq c g_{j-1} \sum_a \operatorname{dist}(p_a, N_a)^2 \geq d_j \operatorname{dist}(p, N)^2, \qquad d_j = c g_{j-1}/(r+1).$$

For the last bound, sum nearest points in N_a and use Cauchy–Schwarz.

For $\psi \perp N$, write $p = P\psi$, $u = \psi - p$ and $E = \langle \psi, H\psi \rangle$. Since $N \subset \operatorname{ran} P$, $p \perp N$ and $\|u\|^2 \leq E/s$. Projectors disjoint from the closed star commute with P and cannot have larger energy on p than on ψ. Among the remaining terms outside S_* only a bounded number D_* meet the star. Each such projector h' satisfies $\|h'p\|^2 \leq 2\|h'\psi\|^2 + 2\|u\|^2$. It follows that

$$\|\psi\|^2 \leq [s^{-1} + (2 + 2D_*/s)/d_j] E.$$

Taking minima over the finite local types closes the induction. Only $K = 8$ is used in the proof of Theorem 1.1. □

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The main results follow from the analytic estimates above together with the classical hard-boundary coupling result established in the preprint of Aronow and Lopatto [5]. As supplementary sanity checks on small volumes, exact rational calculations independently verified all 78 independent entries of each of the cut-norm and matching-energy forms in the 12-dimensional cut-source sector of the 6×2 rectangle, all 81 entries of a 2×2 wing partial trace, and all 25 entries of the central spin density operator against the inner-neighbour preparation on a 5×5 square.

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