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# Grothendieck rings of complex fusion categories: rank 3

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**Abstract**

Ostrik classified the pivotal fusion categories of rank three. We show that every complex fusion category of rank three admits a spherical structure and recover the five Grothendieck rings. Both steps use the mass at twist one in the Drinfeld center. For an irreducible cubic codegree polynomial, Galois averaging in the sphericalization determines the first six moments of the squared twist. For a spherical category, the first two moments give a single inequality that excludes all the remaining rings outside the list. The rank-specific descent theorem we use from the literature is Ostrik's theorem for a quadratic factor.

*GPT-6.2 Astra assisted with the research. Complete human verification is pending, and the results should be treated with caution.*

## 1 The classification and the proof

A complex fusion category $\mathcal{C}$ is a rigid, semisimple, $\mathbb{C}$-linear tensor category with finite-dimensional morphism spaces, finitely many isomorphism classes of simple objects and simple unit. Its rank is the number of simple classes. Its Grothendieck ring $K_0(\mathcal{C})$ records their tensor multiplicities and their distinguished basis. A pivotal structure is a monoidal isomorphism from the identity to the double dual. It is spherical when the two pivotal traces agree.

For nonnegative integers $k, l, m, n$ satisfying $k^2 + l^2 = kn + lm + 1$, denote by $K(k, l, m, n)$ the based ring on $1, X, Y$ with

$$X^2 = 1 + mX + kY, \qquad Y^2 = 1 + lX + nY, \qquad XY = kX + lY.$$

The elementary derivation of this presentation appears in Section 2.

**Theorem 1.1.** *Up to based-ring isomorphism, the Grothendieck rings of complex fusion categories of rank three are exactly*

$$\mathbb{Z}[C_3], \qquad K(1,0,0,0), \qquad K(1,0,1,0), \qquad K(1,0,2,0), \qquad K(1,1,1,0).$$

*Every complex fusion category of rank three admits a spherical structure. Each of the five rings has a unitary realization.*

The five rows of Table 1 specify unitary examples. A row can have several inequivalent associators.

**History.** Two early sources of examples are Bion-Nadal's construction of an $E_6$ subfactor [2] and the Tambara–Yamagami classification [12], which includes the Ising ring. Ostrik classified the rank-three categories admitting a ribbon structure [9]. Hagge and Hong classified the tensor structures on the $E_6$ ring by solving the pentagon equations [5]. Thornton proved that every near-group category, including the rank-three near-group family, admits a spherical structure [13, Theorem 1.3]. Ostrik subsequently classified all pivotal rank-three categories, up to tensor equivalence [11, Theorem 1.1]. His proof also gave spherical descent in the quadratic case and obstructions for individual cubic rings [11, Section 5]. We use his formal codegrees and central-induction results [10, 11] and the indicator formulas of Ng and Schauenburg [7, 8].

**Outline.** Sections 2 and 3 relate the three formal codegrees to the mass at twist one in the center. Section 4 applies this relation to squared twists in the sphericalization. Six moments force the cubic ring and allow its spherical structure to descend. For a reducible codegree polynomial we invoke Ostrik's quadratic descent or the canonical spherical structure in the weakly integral case. Section 5 then applies the same mass estimate to ordinary twists. A degree-two polynomial recovers both Ostrik's structural reduction and his bound for near-group multiplicities [11, Theorems 3.3 and 4.1]. Each exclusion is independent of any bound on the structure constants.

# 2 The ring algebra and sphericalization

## 2.1 The two codegree coefficients

As in [11, Proposition 3.1], a rank-three fusion ring is either $\mathbb{Z}[C_3]$, or all its basis elements are self-dual and it has the form $K(k,l,m,n)$, with basis $1, X, Y$ and multiplication

$$X^2 = 1 + mX + kY, \qquad Y^2 = 1 + lX + nY, \qquad XY = kX + lY. \tag{2.1}$$

Here $k, l, m, n$ are nonnegative integers satisfying

$$k^2 + l^2 = kn + lm + 1. \tag{2.2}$$

Here is a quick derivation. If $X^* = Y \neq X$, the product $X^2$ has no unit constituent; writing $d = \mathrm{FPdim}(X) = \mathrm{FPdim}(Y)$ gives $d^2 = (a+b)d$ for nonnegative integers $a, b$, so $d$ is an integer. The product $XY$ has one unit constituent, hence $d^2 = 1 + (c+e)d$ for integers $c, e \geq 0$; therefore $d \mid 1$ and $d = 1$, giving $\mathbb{Z}[C_3]$. Otherwise all three simples are self-dual. The involution on a based ring reverses products, so the ring is commutative. Frobenius reciprocity forces the coefficients in $XY$ to be $k, l$ as displayed; the identity $(X^2)Y = X(XY)$ gives (2.2), which also suffices for associativity.

The multiplication matrices in the ordered basis $1, X, Y$ are

$$N_X = \begin{pmatrix} 0 & 1 & 0 \\ 1 & m & k \\ 0 & k & l \end{pmatrix}, \qquad N_Y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & k & l \\ 1 & l & n \end{pmatrix}.$$

They are commuting real symmetric matrices, so their common characters $\chi_1, \chi_2, \chi_3$ are real. Their *formal codegrees* are

$$f_i = 1 + \chi_i(X)^2 + \chi_i(Y)^2.$$

Equivalently, they are the eigenvalues of multiplication by the Casimir element

$$\Omega = 1 + X^2 + Y^2 = 3 + (l+m)X + (k+n)Y.$$

Let $\tau$ extract the coefficient of the unit. For the basis $e_0 = 1, e_1 = X, e_2 = Y$ we have $\tau(e_i e_j) = \delta_{ij}$. For multiplication by $a$, the $i$th diagonal entry is $\tau(e_i a e_i)$. Thus $\mathrm{Tr}(a) = \tau(a\Omega)$ and

$$\mathrm{Tr}(a/\Omega) = \tau(a). \tag{2.3}$$

Here Tr is the regular trace of the rational ring, and $\Omega$ is invertible because its eigenvalues are positive. We shall use the same calculation in the signed ring. Taking $a = 1$ gives $\sum_i 1/f_i = 1$. The codegrees are eigenvalues of an integral matrix. Consequently their characteristic polynomial has the form

$$p(t) = \prod_{i=1}^{3}(t - f_i) = t^3 - bt^2 + ct - c. \tag{2.4}$$

We call the ring *cubic* when $p$ is irreducible over $\mathbb{Q}$. We will use the following rank-three identities:

$$b = 9 + (l+m)^2 + (k+n)^2 = 7 + 3k^2 + 3l^2 + m^2 + n^2, \tag{2.5}$$

$$c = 4b - 9 + (kl - mn)^2 + \big(2(k^2 - lm) - 1\big)^2. \tag{2.6}$$

Both identities follow by expanding $3I + (l+m)N_X + (k+n)N_Y$ and using (2.2). The determinant identity is Ostrik's identity [11, equation (12)], written as a sum of two squares. In particular,

$$c \geq 4b - 8, \tag{2.7}$$

because the last square in (2.6) is the square of an odd integer. Vieta's identities also give

$$u := \sum_{i=1}^{3} f_i^{-2} = 1 - \frac{2b}{c}. \tag{2.8}$$

## 2.2 The signed quotient

Let $\mathcal{S}$ be the sphericalization of a fusion category $\mathcal{C}$ with self-dual rank-three ring. There is a tensor functor $\pi : \mathcal{S} \to \mathcal{C}$. Each simple object $V$ of $\mathcal{C}$ has two simple lifts $V_+, V_-$ with nonzero opposite dimensions. When $V$ is self-dual, its lifts are self-dual: the dual of $V_+$ is a lift of $V$, and sphericality gives $\dim(V_+^*) = \dim(V_+)$, which rules out $V_-$. This observation is due to Thornton [13, proof of Lemma 2.3], as noted in [11, Section 5.1(a)]. Put $\varepsilon = \mathbf{1}_-$. Then

$$\varepsilon^2 \cong \mathbf{1}, \qquad \dim(\varepsilon) = -1, \qquad \varepsilon \otimes V_+ \cong V_-.$$

The construction and these properties are given in [4, Remark 3.1 and Proposition 5.14] and [11, Section 5.1]. A useful realization is the equivariantization for the canonical $C_2$-action [1, Section 5.3].

All simples of $\mathcal{S}$ are self-dual, so $K_0(\mathcal{S})$ is commutative. Its two quotients at $\varepsilon = 1$ and $\varepsilon = -1$ are respectively

$$R = K_0(\mathcal{C}), \qquad \widetilde{R} = K_0(\mathcal{S})/(\varepsilon + 1).$$

Choose lifts of $X, Y$ and denote their images in $\widetilde{R}$ by $\widetilde{X}, \widetilde{Y}$. Then $\widetilde{R} = K(\widetilde{k}, \widetilde{l}, \widetilde{m}, \widetilde{n})$, with the same multiplication formulas and associativity relation as above, but with integer, possibly negative, coefficients. For each corresponding pair of coefficients $a, \widetilde{a}$,

$$|\widetilde{a}| \leq a, \qquad \widetilde{a} \equiv a \pmod 2. \tag{2.9}$$

Indeed, $a$ is the sum and $\widetilde{a}$ the difference of the two lift multiplicities. Changing the chosen lift changes the sign of the corresponding basis element.

The coefficient-of-1 functional $\tau$ on $\widetilde{R}$ satisfies

$$\tau(\widetilde{X}_i\widetilde{X}_j) = \delta_{ij}, \qquad (\widetilde{X}_0, \widetilde{X}_1, \widetilde{X}_2) = (1, \widetilde{X}, \widetilde{Y}). \tag{2.10}$$

For example, $X_+^2$ contains $\mathbf{1}$ once and does not contain $\varepsilon$, since $X_+$ is self-dual and $\varepsilon X_+ \ncong X_+$. Thus taking the signed quotient preserves this pairing.

### 2.3 The general categorical input

For a spherical fusion category $\mathcal{D}$, write $F : Z(\mathcal{D}) \to \mathcal{D}$ for the forgetful functor from its Drinfeld center and $I$ for its right adjoint. If its Grothendieck ring is commutative, central induction gives a bijection between its ring characters $\chi$ and the simple summands $A_\chi$ of $I(\mathbf{1})$, with

$$I(\mathbf{1}) = \bigoplus_\chi A_\chi, \qquad \dim(A_\chi) = \frac{\dim(\mathcal{D})}{f_\chi}, \qquad \theta_{A_\chi} = 1. \tag{2.11}$$

These are the commutative-ring cases of Ostrik's central-induction and trivial-twist theorems [11, Theorems 2.5 and 2.13]. Two consequences will be used repeatedly. First, $\dim(\mathcal{D})/f_\chi$ is an algebraic integer, because it is an object dimension. Second, every $f_\chi$ belongs to the field of simple object dimensions: the forgetful functor preserves spherical dimensions, so both the numerator and denominator in $f_\chi = \dim(\mathcal{D})/\dim(F(A_\chi))$ belong to that field. These are [11, Corollaries 2.14–2.15].

For a fusion category without a specified pivotal structure, every formal codegree still divides its global dimension [11, Corollary 2.14].

### 2.4 The dimension field in the cubic case

**Lemma 2.1.** *Suppose that $R$ is cubic. The dimension field $L$ of $\mathcal{S}$ is a totally real cyclic cubic field, and the dimension homomorphism identifies $\widetilde{R} \otimes \mathbb{Q}$ with $L$. Set*

$$x = \dim(X_+), \qquad y = \dim(Y_+), \qquad \delta = 1 + x^2 + y^2 = \dim(\mathcal{C}),$$

*so that* $\dim(\mathcal{S}) = 2\delta$. *Then*

$$\mathrm{Tr}_{L/\mathbb{Q}}(1/\delta) = 1, \qquad \mathrm{Tr}_{L/\mathbb{Q}}(x/\delta) = \mathrm{Tr}_{L/\mathbb{Q}}(y/\delta) = 0. \tag{2.12}$$

*The formal codegrees of $\widetilde{R}$ are the three conjugates $g_i$ of $\delta$, counted with multiplicity. Write*

$$\prod_i (t - g_i) = t^3 - \widetilde{b}t^2 + \widetilde{c}t - \widetilde{c}.$$

*Then*

$$0 < \widetilde{b} \le b, \qquad \frac{\widetilde{c}}{c} \in \mathbb{Z}_{>0}, \qquad 0 < \frac{\widetilde{b}}{\widetilde{c}} \le \frac{b}{c}. \tag{2.13}$$

*Moreover,*

$$v := \sum_i g_i^{-2} = 1 - \frac{2\widetilde{b}}{\widetilde{c}}. \tag{2.14}$$

*Proof.* The dimension homomorphism factors through $\widetilde{R} \otimes \mathbb{Q}$, so its image is a field of degree at most three. The formal codegrees of $\mathcal{S}$ are

$$2f_1, 2f_2, 2f_3, 2g_1, 2g_2, 2g_3 : \tag{2.15}$$

characters with $\varepsilon = 1$ come from $R$, and those with $\varepsilon = -1$ come from $\widetilde{R}$; the two lifts double the sum defining each codegree. Every formal codegree of a spherical category lies in its dimension field [11, Corollary 2.15]. Therefore $L$ contains all three roots of the irreducible cubic $p$. It has degree exactly three and is its splitting field. Spherical dimensions are totally real [4, Corollary 2.10], so $L$ is totally real. The map $\widetilde{R} \otimes \mathbb{Q} \to L$ is now an isomorphism.

Apply the regular-trace calculation (2.3) to the basis $1, x, y$, whose pairing is (2.10) and whose Casimir element is $\delta$. This gives (2.12). The embeddings of $L$ are precisely the three characters of the signed algebra, and its Casimir element has image $\delta$; hence its codegrees are $g_i = \sigma_i(\delta) > 0$. Their reciprocal sum is one, giving the displayed polynomial and (2.14).

Formula (2.5), valid also for the signed coefficients, gives

$$\widetilde{b} = 7 + 3\widetilde{k}^2 + 3\widetilde{l}^2 + \widetilde{m}^2 + \widetilde{n}^2 \le b.$$

Finally, formal codegrees divide the global dimension even without a pivotal structure [11, Corollary 2.14]. Thus $\delta/f_1$ is an algebraic integer in $L$. Taking its field norm yields

$$N_{L/\mathbb{Q}}(\delta/f_1) = \frac{\widetilde{c}}{c} \in \mathbb{Z}_{>0}.$$

The last inequality in (2.13) follows. □

## 3 Central induction and twist moments

Let $F : Z(\mathcal{D}) \to \mathcal{D}$ be the forgetful functor for a spherical fusion category $\mathcal{D}$, and let $I : \mathcal{D} \to Z(\mathcal{D})$ be its right adjoint. We use the induction formula [11, Proposition 2.4]

$$FI(V) \cong \bigoplus_{W \in \operatorname{Irr} \mathcal{D}} W \otimes V \otimes W^*. \tag{3.1}$$

Together with adjunction, this identifies $[I(V) : A] = [F(A) : V]$ and $\dim_{\mathbb{C}} \operatorname{End}(I(V)) = [FI(V) : V]$. Write $d = \dim(\mathcal{D})$, and for $A \in \operatorname{Irr} Z(\mathcal{D})$ put $a_A = \dim(A)$ and let $\theta_A$ be its twist. The numbers $a_A$ are real, and the twists are roots of unity. We use the identities

$$\sum_A a_A^2 = d^2, \tag{3.2}$$

$$\sum_A a_A^2 \theta_A^n = d \sum_{V \in \operatorname{Irr} \mathcal{D}} \dim(V) \nu_n(V) \qquad (n \ge 1), \tag{3.3}$$

where $\nu_n$ is the $n$th Frobenius–Schur indicator for the specified pivotal structure. To see (3.3), multiply the Ng–Schauenburg identity

$$\operatorname{tr}(\theta^n|_{I(V)}) = d\nu_n(V)$$

by $\dim(V)$ and sum over $V$. Adjunction and sphericality of the forgetful functor give $a_A = \sum_V [I(V) : A] \dim(V)$, yielding the formula. The indicator identity is [8, Theorem 4.1], and (3.2) is Müger's dimension formula for the center [6]. Also,

$$\nu_n(V) \in \mathbb{Q}(\zeta_n), \tag{3.4}$$

because the cyclic operator defining the indicator has $n$th power equal to the identity [7, Theorem 5.1]. The trace identity uses sphericality. The cyclotomicity assertion holds in any pivotal category.

Both estimates below use the following observation. If $\mu$ is a finite positive measure on the unit circle $\mathbb{T}$ and $q$ is a polynomial, then

$$|q(1)|^2 \mu(\{1\}) \le \int_{\mathbb{T}} |q(z)|^2 \, d\mu(z). \tag{3.5}$$

The right side depends only on the moments through degree $\deg q$.

**The central sign.** The equivariantization model of $\mathcal{S}$ gives a central copy of $\mathrm{Rep}(C_2)$ whose sign representation lifts $\varepsilon$ [1, Definition 5.13 and Corollary 5.15]. Its underlying fusion subcategory has the usual tensor product of representations, with pivotal map on the sign equal to minus its usual pivotal map because $\dim(\varepsilon) = -1$. For even $n$, cyclic rotation on the line $\mathrm{Hom}(\mathbf{1}, \varepsilon^{\otimes n})$ is the ordinary rotation multiplied by this inverse pivotal scalar. Thus

$$\nu_{2j}(\varepsilon) = -1 \qquad (j \geq 1). \tag{3.6}$$

**Lemma 3.1** (A forced squared-twist sector)**.** *In the cubic situation of Lemma 2.1, the simple constituents of $I(\mathbf{1})$ and $I(\varepsilon)$ are twelve distinct simple objects of $Z(\mathcal{S})$. Their squared twists are all one, and the sum of their squared dimensions is*

$$2\delta^2(u+v). \tag{3.7}$$

*The same statement holds after every embedding of $L$.*

*Proof.* Since $K_0(\mathcal{S})$ is commutative, $I(\mathbf{1})$ is multiplicity-free. The formal-codegree theorem [11, Theorem 2.13] and (2.15) give its six simple constituent dimensions as

$$\frac{\delta}{f_1}, \frac{\delta}{f_2}, \frac{\delta}{f_3}, \frac{\delta}{g_1}, \frac{\delta}{g_2}, \frac{\delta}{g_3}.$$

They are positive under every real embedding and their squared sum is $\delta^2(u+v)$. Their twists are one [11, Theorem 2.5].

Write $\widehat{\varepsilon}$ for the canonical central lift of $\varepsilon$. The projection formula gives $I(\varepsilon) \cong \widehat{\varepsilon} \otimes I(\mathbf{1})$. Thus its six constituents have the opposite, negative dimensions, and are disjoint from those of $I(\mathbf{1})$. The indicator identity gives $\mathrm{tr}(\theta^2|_{I(\varepsilon)}) = -2\delta = \dim I(\varepsilon)$. All six constituent dimensions are negative, so their absolute values sum to $2\delta$. Equality in the triangle inequality forces each squared twist to equal one. Their squared dimensions give the second copy of $\delta^2(u+v)$ in (3.7). □

# 4 The cubic case

The following estimate is the step that treats an arbitrary categorification of a cubic ring. It will force $b \leq 16$, so its final arithmetic consequence can be read off without a search.

**Proposition 4.1** (Cubic codegree bound)**.** *If $K(k,l,m,n)$ is cubic and admits a complex fusion categorification, then*

$$c \leq \frac{7}{2}b. \tag{4.1}$$

*Proof.* Use the notation of Lemmas 2.1–3.1, and put

$$M = \mathbb{Q}(\zeta_{120}).$$

The fields $L$ and $M$ are linearly disjoint: $L/\mathbb{Q}$ is Galois of degree 3, whereas $[M : \mathbb{Q}] = \varphi(120) = 32$. Choose three automorphisms $\sigma_1, \sigma_2, \sigma_3$ of $\overline{\mathbb{Q}}$ restricting to the three embeddings of $L$ and fixing $M$ pointwise. Fix these choices for the rest of the proof. In particular, they fix every indicator of each of the orders

$$2, 4, 6, 8, 10, 12,$$

by (3.4). The same choices work for all six orders, since each divides 120. We apply these automorphisms only to algebraic scalar identities; no choice of a field of definition for the associators is needed.

On the unit circle $\mathbb{T}$, define the finite positive measure

$$\mu = \sum_{i=1}^{3} \sum_{A \in \operatorname{Irr} Z(\mathcal{S})} \frac{\sigma_i(a_A)^2}{4\sigma_i(\delta)^2} \operatorname{Dirac}_{\sigma_i(\theta_A)^2}. \tag{4.2}$$

Here $\operatorname{Dirac}_z$ is unit point mass at $z$. Every central dimension belongs to $L$, since the forgetful functor is spherical. Its conjugates are real, and $\sigma_i(\delta) > 0$. Thus the coefficients in (4.2) are nonnegative. Twists and their conjugates are roots of unity, so the support is contained in $\mathbb{T}$. Equation (3.2) gives

$$\mu(\mathbb{T}) = 3. \tag{4.3}$$

For $1 \le j \le 6$, apply $\sigma_i$ to (3.3), with $n = 2j$, and divide by $4\sigma_i(\delta)^2$. Since the indicators are fixed, summing over $i$ gives

$$\begin{aligned} \int_{\mathbb{T}} z^j \, d\mu(z) &= \frac{1}{2} \sum_{V \in \operatorname{Irr} \mathcal{S}} \nu_{2j}(V) \operatorname{Tr}_{L/\mathbb{Q}} \left( \frac{\dim(V)}{\delta} \right) \\ &= \frac{1}{2} (\nu_{2j}(\mathbf{1}) - \nu_{2j}(\varepsilon)) = 1. \end{aligned} \tag{4.4}$$

For the second equality, the four non-unit lifts have dimensions $\pm x, \pm y$ and vanish under the weighted trace by (2.12); the traces for $\mathbf{1}$ and $\varepsilon$ are 1 and $-1$. The last equality uses (3.6) and $\nu_n(\mathbf{1}) = 1$. This step does not require knowing any indicator of $X_\pm$ or $Y_\pm$.

Lemma 3.1 forces the lower bound

$$\mu(\{1\}) \ge \frac{3}{2}(u + v). \tag{4.5}$$

Indeed, in each of the three summands defining $\mu$, those twelve objects contribute $(u + v)/2$ at one. The sums $u, v$ are rational and hence independent of the embedding.

Apply (3.5) to the Fejér polynomial

$$P(z) = |1 + z + \cdots + z^6|^2 = 7 + \sum_{j=1}^{6} (7 - j)(z^j + z^{-j}) \qquad (z \in \mathbb{T}).$$

Its value at one is 49. Since the negative moments are the complex conjugates of the positive ones, (4.3) and (4.4) imply

$$49\mu(\{1\}) \le \int_{\mathbb{T}} P(z) \, d\mu(z) = 7 \cdot 3 + 2 \sum_{j=1}^{6} (7 - j) = 63.$$

Combining this with (4.5) yields

$$u + v \le \frac{6}{7}.$$

Substitute (2.8) and (2.14) to obtain

$$\frac{b}{c} + \frac{\widetilde{b}}{\widetilde{c}} \ge \frac{4}{7}. \tag{4.6}$$

But (2.13) bounds the second ratio by the first. Therefore $b/c \ge 2/7$, which is (4.1). □

*Remark* 4.2 (The role of the six indicators). The moment argument uses only the fields containing the indicators, not their values, signs or sizes. Its atom bound is sharp for the stated moment information: the measure with mass $9/7$ at 1 and mass $2/7$ at each nontrivial seventh root has mass 3 and first six moments 1. The choice of order twelve is also natural for field disjointness. At order fourteen, $\mathbb{Q}(\zeta_{14})$ already contains a cubic field, so the degree argument used above cannot be continued uniformly.

### 4.1 The only cubic ring and its spherical structure

**Proposition 4.3.** *A cubic rank-three fusion ring admits a complex fusion categorification only if it is isomorphic to $K(1,1,1,0)$. Every categorification of this ring admits a spherical structure.*

*Proof.* Combine (2.7) and Proposition 4.1:

$$4b - 8 \le c \le \frac{7}{2}b, \qquad \text{hence} \qquad b \le 16.$$

If one of $k, l$ is zero, interchange $X, Y$ so that $l = 0$. Equation (2.2) becomes $k(k-n) = 1$, giving $k = 1, n = 0$. The character $X \mapsto 0$, $Y \mapsto -1$ then has formal codegree 2, contrary to irreducibility of $p$. Thus $k, l \ge 1$.

By (2.5),

$$3(k^2 + l^2) + m^2 + n^2 \le 9.$$

This forces $k = l = 1$. Associativity now gives $m + n = 1$. The two resulting choices are interchanged by exchanging $X$ and $Y$, so the ring is $K(1,1,1,0)$.

It remains to descend sphericality to the given categorification, rather than merely exhibit a different spherical categorification. For $K(1,1,1,0)$, (2.9) gives

$$\widetilde{k}, \widetilde{l}, \widetilde{m} \in \{1, -1\}, \qquad \widetilde{n} = 0.$$

Signed associativity implies $\widetilde{l}\widetilde{m} = 1$. Replacing $\widetilde{X}$ by $\widetilde{l}\widetilde{X}$ and $\widetilde{Y}$ by $\widetilde{k}\widetilde{Y}$ makes all four signed parameters equal to the original ones. For these choices of lifts, each signed multiplicity equals the corresponding unsigned multiplicity. Since these are respectively the difference and sum of the two lift multiplicities, the unwanted lift has multiplicity zero in every product. The three chosen lifts therefore form a fusion subcategory of $\mathcal{S}$.

The restriction of $\pi$ to this subcategory is a tensor equivalence with $\mathcal{C}$: it is bijective on simple objects and fully faithful. Transporting its spherical structure gives a spherical structure on $\mathcal{C}$. This is also the section criterion of [11, Section 5.1(c)]. □

### 4.2 Spherical descent in the remaining cases

**Theorem 4.4.** *Every complex fusion category of rank three admits a spherical structure.*

*Proof.* The pointed case is weakly integral. For a self-dual ring, the Frobenius–Perron dimension of the category is a formal codegree. If $p$ splits over $\mathbb{Q}$, this dimension is an integer and again the category is weakly integral. The canonical spherical structure exists in both cases [4, Propositions 8.23–8.24]. If $p$ has an irreducible quadratic factor, Ostrik's descent theorem [11, Theorem 5.1] gives a spherical structure on the original category, with no pivotality hypothesis. Proposition 4.3 handles the irreducible cubic. □

## 5 A two-moment test for the five rings

Let $K(k,l,m,n)$ be the ring of a spherical rank-three category. Put $x = \mathrm{FPdim}(X)$, $y = \mathrm{FPdim}(Y)$ and $D = 1 + x^2 + y^2$. We may use these positive dimensions in the twist identities, as in [11, Lemma 3.4]. Indeed, if $D$ is an integer, use the canonical positive spherical structure [4, Propositions 8.23–8.24]. Otherwise the spherical dimension field contains the irrational codegree $D$ by [11, Corollary 2.15]. The spherical and Frobenius–Perron dimension characters therefore both have Galois orbits of size at least two. Among three characters there is room for only one such orbit. An automorphism carrying the first character to the second carries every central dimension to the positive dimension of its underlying object and every twist to another root of unity. We use these conjugated scalar identities below.

**Proposition 5.1.** *For $u = \sum_i f_i^{-2}$ as in (2.8),*

$$100u \le 34 + \frac{66 + 18(x+y)}{D}. \tag{5.1}$$

*Proof.* Give $\theta_A$ mass $a_A^2/D^2$ for each central simple $A$. By (3.2) this is a probability measure. The constituents of $I(\mathbf{1})$ supply mass $u$ at one by (2.11). Its first moment is $1/D$ by (3.3), since $\nu_1(V) = [V : \mathbf{1}]$. Its second moment is real and at most $(1 + x + y)/D$: the second indicator of a self-dual simple is 1 or $-1$, the trace of an involution on $\mathrm{Hom}(\mathbf{1}, V^{\otimes 2})$. Use $q(z) = 3 + 4z + 3z^2$ in (3.5). On the unit circle,

$$|q(z)|^2 = 34 + 24(z + z^{-1}) + 9(z^2 + z^{-2}).$$

Substitution gives (5.1). □

*Proof of Theorem 1.1.* Sphericality follows from Theorem 4.4. The non-self-dual case gives $\mathbb{Z}[C_3]$. For a self-dual ring, we first show that $b \ge 22$ and $D \ge 16$ cannot hold together. By (2.7) and (2.8),

$$u \ge \frac{b-4}{2b-4} \ge \frac{9}{20} \qquad (b \ge 22).$$

This gives $100u - 34 \ge 11$. But $x + y \le \sqrt{2(D-1)}$, so for $D \ge 16$ the right side of (5.1) minus 34 is smaller than $66/16 + 18/\sqrt{8} < 11$.

If $k, l > 0$ and $\max(k, l) \ge 2$, (2.5) gives $b \ge 22$. From $xy = kx + ly$ we obtain $x > l$, $y > k$ and $(x - l)(y - k) = kl$. Hence

$$x + y \ge (\sqrt{k} + \sqrt{l})^2 \ge 3 + 2\sqrt{2} > 11/2, \qquad D \ge 1 + \tfrac{1}{2}(x+y)^2 > 16.$$

This is impossible. The choice $k = l = 1$ forces $m + n = 1$ and gives $K(1, 1, 1, 0)$ after interchanging $X, Y$.

If $kl = 0$, arrange $l = 0$. Then (2.2) forces $k = 1, n = 0$, so $y = 1$, $x = (m + \sqrt{m^2 + 8})/2$, $D = 2 + x^2$ and

$$b = 10 + m^2, \qquad c = 4b - 8, \qquad u = \frac{b-4}{2b-4}.$$

As $m$ increases, $u$ increases and $(84 + 18x)/(2 + x^2)$ decreases: the latter has derivative of sign $36 - 168x - 18x^2 < 0$ for $x \ge 1$. It therefore suffices to test $m = 3$ to exclude all $m \ge 3$. Substitution in (5.1) would give

$$\frac{15}{34} \le \frac{11}{20} - \frac{9\sqrt{17}}{340}, \qquad \text{or equivalently} \qquad 9\sqrt{17} \le 37.$$

This fails since $81 \cdot 17 = 1377 > 1369 = 37^2$. This endpoint explains the coefficients in Proposition 5.1: among $1 + az + z^2$ with $a \ge 0$, the two-moment upper bound for the atom there is minimized at $a = 4/3$.

This leaves $m = 0, 1, 2$. The unitary models in Table 1 realize each surviving ring. The three near-group rings have two invertible basis elements and are distinguished by $m$. The pointed ring has three invertibles and the even $SU(2)_5$ ring has one. □

In the $SU(2)_5$ row the nontrivial basis elements $X, Y$ have highest weights $2, 4$, respectively. In the near-group rows $Y$ is the nontrivial invertible and $X$ is the remaining simple. Ostrik's tensor-equivalence classification [11, Theorem 1.1] now applies to every complex rank-three fusion category by Theorem 4.4.

| Fusion ring | A unitary realization | Reference |
|---|---|---|
| $\mathbb{Z}[C_3]$ | $\mathrm{Vec}_{C_3}$ | [11, Theorem 1.1(i)] |
| $K(1,0,0,0)$ | An Ising category | [12], [11, Thm. 1.1(iii)] |
| $K(1,0,1,0)$ | $\mathrm{Rep}(S_3)$ | [11, Theorem 1.1(iv)] |
| $K(1,0,2,0)$ | The $E_6$ even part of rank three | [2, 5], [3, Sec. 1] |
| $K(1,1,1,0)$ | The even subcategory of $SU(2)_5$ | [9, Sec. 4] |

Table 1: A unitary realization of each ring.

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