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Grothendieck rings of complex fusion categories: rank 2

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Abstract

Ostrik classified fusion categories over $\mathbb{C}$ with two simple objects. We recover the two possible Grothendieck rings from sphericalization and central induction. For a candidate rule $X^2 = 1 + nX$, the dimension divisibility of Etingof, Nikshych and Ostrik forces one lift of X to satisfy the same rule in the sphericalization. The induction matrix then expresses $-n\dim(X)$ as the sum of two roots of unity. Taking a conjugate gives $n \leq 1$. Both rings occur.

GPT-6.2 Astra assisted with the research. Complete human verification is pending, and the results should be treated with caution.

1 The classification

A complex fusion category is a rigid semisimple $\mathbb{C}$-linear tensor category with simple unit $\mathbf{1}$, finite-dimensional morphism spaces and finitely many simple objects up to isomorphism. Their number is the *rank*. The simple classes form the distinguished basis of the Grothendieck ring. All ring isomorphisms below preserve the unit and this basis. In rank one the ring is $\mathbb{Z}$. In rank two duality fixes the only nontrivial simple X, so rigidity gives $[X \otimes X : \mathbf{1}] = 1$. Thus the possible based rings are

$$\mathsf{K}_n = \mathbb{Z}[X]/(X^2 - 1 - nX), \qquad n \in \mathbb{Z}_{\geq 0}, \qquad \text{with basis } \{1, X\}.$$

Theorem 1.1 (Ostrik [5])**.** *The Grothendieck rings of complex fusion categories of rank two are precisely* K_0 *and* K_1*. Every such category admits a spherical structure.*

The two rings have different origins. The simple X in K_0 is invertible, as in the category of $\mathbb{Z}/2\mathbb{Z}$-graded vector spaces. In K_1 it is the noninvertible simple of the Fibonacci category. Its Galois conjugate, the Yang–Lee category, has the same fusion rule. In their work on rational conformal field theory, Moore and Seiberg studied the consistency equations underlying the surviving tensor structures [2]. Ostrik credits them with the classification for $n = 0, 1$ and proved the exclusion for all $n \geq 2$. His proof obtains a braiding on the category and, when $n > 0$, uses its modularity [5, Sections 2.1–2.3].

Thornton subsequently proved that a fusion category with exactly one noninvertible simple object admits a spherical structure [8, Theorem 1.3]. Ostrik observed that his lower bound on global dimensions already excludes $n \geq 2$ once sphericality is given [7, Example 4.1.2]. Here a single rational divisibility proves sphericality, and the two rows of the central induction matrix give the exclusion directly.

2 Choosing a spherical lift

A pivotal structure identifies the identity and double-dual functors by a monoidal natural isomorphism. It is *spherical* if its left and right traces coincide. The resulting dimension is additive and multiplicative, has equal values on dual objects, and is nonzero on simples. The Frobenius–Perron dimension FPdim is the unique positive dimension character on the fusion ring.

Proposition 2.1. *If $K_0(\mathcal{C}) = \mathsf{K}_n$, the sphericalization of $\mathcal{C}$ contains a spherical fusion subcategory mapped equivalently onto $\mathcal{C}$.*

Proof. In the sphericalization $\pi : \mathcal{S} \to \mathcal{C}$, each simple has two lifts, interchanged by tensoring with a central invertible object ε with $\varepsilon^2 = \mathbf{1}$ and $\dim(\varepsilon) = -1$ [1, Remark 3.1 and Proposition 5.14]. Let x be a lift of X and $d = \dim(x)$. The two lifts of X have distinct dimensions $d, -d$, so $x^* = x$. Consequently x^2 contains the unit once. Its image under π contains the unit only once, so ε does not occur. In $K_0(\mathcal{S})$ we have

$$x^2 = 1 + ax + b\varepsilon x, \qquad a + b = n, \qquad d^2 = 1 + md, \quad m = a - b.$$

Write u for the positive root of $u^2 = 1 + nu$. Both lifts of X have Frobenius–Perron dimension u, since $\mathrm{FPdim}(\varepsilon) = 1$. The global dimension of $\mathcal{S}$ is the sum of the squared spherical dimensions of its four simples. Dividing by the corresponding sum of squared Frobenius–Perron dimensions gives the algebraic integer [1, Proposition 8.22]

$$\alpha = \frac{\dim(\mathcal{S})}{\mathrm{FPdim}(\mathcal{S})} = \frac{1 + d^2}{1 + u^2}.$$

The quadratics for d and u have constant term -1, so both roots are algebraic units. Also $d - d^{-1} = m$ and $u - u^{-1} = n$. Hence

$$\left(\frac{u}{d}\alpha\right)^2 = \left(\frac{d + d^{-1}}{u + u^{-1}}\right)^2 = \frac{m^2 + 4}{n^2 + 4}. \tag{2.1}$$

Since $|m| \le n$, the right side is a rational algebraic integer in $(0, 1]$, and thus equals 1. We obtain $|a-b| = a+b$, so $ab = 0$. Interchanging x and εx if necessary gives $x^2 = 1+nx$, including when $n = 0$. The full additive subcategory on $\mathbf{1}, x$ is closed under products, summands and duals. The restriction of π is bijective on simple classes and induces isomorphisms between their morphism spaces, hence is a tensor equivalence. It transports the spherical structure to $\mathcal{C}$. □

3 Two roots of unity

Fix the spherical structure obtained above and put $d = \dim(X)$, so $d^2 = 1 + nd$. The Drinfeld center $\mathcal{Z}(\mathcal{C})$ consists of objects equipped with a coherent half-braiding. Let $F : \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$ be the forgetful functor and I its right adjoint. The induction formula of Etingof, Nikshych and Ostrik [1, Proposition 5.4] reads $FI(V) \cong V \oplus (X \otimes V \otimes X)$ here. Set $M_{V,B} = [F(B) : V]$, with $V = \mathbf{1}, X$ and B ranging over the central simples. Adjunction identifies $M_{V,B}$ with $[I(V) : B]$. The matrix of FI therefore gives

$$MM^{\mathsf{T}} = \begin{pmatrix} 2 & n \\ n & n^2 + 2 \end{pmatrix}. \tag{3.1}$$

This is the induction-matrix method used more generally by Morrison and Walker [3]. In rank two the matrix is forced. The column of the central unit is $(1,0)^{\mathsf{T}}$. The first diagonal entry of (3.1) leaves one more column with first entry 1, and the off-diagonal entry forces its second entry to be n. The remaining second entries have squares summing to 2. Since F sends each simple to a nonzero object, there are exactly two more columns, each $(0,1)^{\mathsf{T}}$. We recover Ostrik's four central simples and their images [5, Lemma 2.1], with inductions given by the rows of M.

B	$\mathbf{1}$	A	B_1	B_2
$F(B)$	$\mathbf{1}$	$\mathbf{1} \oplus nX$	X	X

$$I(\mathbf{1}) = \mathbf{1} \oplus A, \quad I(X) = nA \oplus B_1 \oplus B_2.$$

The twist acts trivially on $I(\mathbf{1})$ and has categorical trace zero on $I(X)$ [6, Theorem 2.5]. Write θ_i for the twist of B_i. Both are roots of unity [4, Theorem 5.5]. The dimensions in the center agree with those of the underlying objects, so $\dim(A) = 1 + nd = d^2$ and $\dim(B_i) = d$. It follows that

$$0 = \operatorname{Tr}(\theta_{I(X)}) = nd^2 + (\theta_1 + \theta_2)d, \qquad \theta_1 + \theta_2 = -nd.$$

Every algebraic conjugate of a sum of two roots of unity has absolute value at most 2. For $n > 0$, $t^2 - nt - 1$ is irreducible over $\mathbb{Q}$ since neither possible rational root $1, -1$ is a root. Its positive root $u = (n + \sqrt{n^2+4})/2$ is therefore a conjugate of d. We conclude $nu \leq 2$. Since $u > n$, this gives $n \leq 1$.

To finish Theorem 1.1, the trivial associator on $\mathbb{Z}/2\mathbb{Z}$-graded vector spaces realizes K_0. In the integer-spin subcategory of $\widehat{\mathfrak{sl}}_2$ at level 3, the two simple labels are $0, 2$, and $V_2 \otimes V_2 \cong V_0 \oplus V_2$ realizes K_1. For the tensor structures on both rings see [5, Sections 2.4–2.5].

References

[1] P. Etingof, D. Nikshych and V. Ostrik, *On fusion categories*, Ann. of Math. (2) **162** (2005), 581–642. doi:10.4007/annals.2005.162.581.

[2] G. Moore and N. Seiberg, *Classical and quantum conformal field theory*, Comm. Math. Phys. **123** (1989), 177–254. doi:10.1007/BF01238857.

[3] S. Morrison and K. Walker, *The center of the extended Haagerup subfactor has 22 simple objects*, Internat. J. Math. **28** (2017), 1750009. doi:10.1142/S0129167X17500094.

[4] S.-H. Ng and P. Schauenburg, *Frobenius–Schur indicators and exponents of spherical categories*, Adv. Math. **211** (2007), 34–71. doi:10.1016/j.aim.2006.07.017.

[5] V. Ostrik, *Fusion categories of rank 2*, with an appendix by P. Etingof, Math. Res. Lett. **10** (2003), 177–183. doi:10.4310/MRL.2003.v10.n2.a5.

[6] V. Ostrik, *Pivotal fusion categories of rank 3*, with an appendix jointly with D. Nikshych, Mosc. Math. J. **15** (2015), 373–396. doi:10.17323/1609-4514-2015-15-2-373-396.

[7] V. Ostrik, *Remarks on global dimensions of fusion categories*, with an appendix jointly with P. Etingof, Contemp. Math. **728** (2019), 169–180. doi:10.1090/conm/728/14661.

[8] J. Thornton, *On braided near-group categories* (2011). arXiv:1102.4640.

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