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Vertex operator algebras from Flohr's second family realising doubles of near-group categories $\mathbb{Z}_n + n$ for even n

David E. Evans Andrew Riesen

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Abstract

We study Flohr's second family $\mathcal{W}(2,3k)$, with generating weights 2 and $3k$ and central charge $1-24k$ for $k \in \frac{1}{2}\mathbb{Z}_{>0}$. We realise these algebras and their subalgebras $\mathcal{W}(2,8k)$, generated in weights 2 and $8k$, by deforming the conformal vector of the vertex algebra of the rank-one lattice $L = \mathbb{Z}\beta$, $(\beta,\beta) = 2k$. We prove strong rationality and compute the simple modules and modular data for $\mathcal{W}(2,8k)$ at all k and $\mathcal{W}(2,3k)$ at integral k. For nonintegral k we describe the Neveu–Schwarz sector of the $\mathcal{W}(2,3k)$ superalgebra. The same conformal vector makes the lattice vertex algebra of the index-two sublattice $2L$ an extension of $\mathcal{W}(2,8k)$. Its modules in the representation category of $\mathcal{W}(2,8k)$ form a $\mathbb{Z}_2$-graded category with neutral component of near-group type $\mathbb{Z}_{4k} + 4k$. We realise the Drinfeld centres of the graded category and its neutral component by regular vertex operator algebras, using the orthogonal complement of a primitive norm-$8k$ vector in the E_8 lattice and conformal shifts to match the ribbon structures. For integral k we give a Leavitt algebra realisation of the near-group category and derive finite quantum dilogarithm identities. We also realise the centres of its quotients by central lifts of the order-$2s$ subgroups of $\mathbb{Z}_{4k}$ for $s^2 \mid k$, including twisted Haagerup–Izumi-type centres over $\mathbb{Z}_{2s}$ when $k = s^2$. We classify the Galois conjugates of the centre modular data with positive categorical dimensions, determine the Witt classes of the $\mathcal{W}(2,3k)$ representation categories for integral k, and describe the near-group module categories and their Morita duals.

1 Introduction

Flohr [20] described two series of two-generator $\mathcal{W}$-algebras of effective central charge one. At central charge $c = 1-24k$, his free-field construction gives $\mathcal{W}(2,8k)$ for $k \in \frac{1}{4}\mathbb{Z}_{>0}$ and $\mathcal{W}(2,3k)$ for $k \in \frac{1}{2}\mathbb{Z}_{>0}$. The two entries record the weights of the Virasoro field and a primary generator. Write $n = 4k$. The odd values of n form what we call the first family. The diagonal Virasoro construction of [11] gives the corresponding vertex operator algebra (VOA) V_{n+2}. For every odd $n \geq 3$, [18] establishes strong rationality and determines its irreducible modules.

Here we study the second family, with n even. In this range $\mathcal{W}(2,3k)$ contains $\mathcal{W}(2,8k)$ at the same central charge. The free-field charges in $\sqrt{2k}\,\mathbb{Z}$ from [20, §2] lead us to the rank-one lattice $L = \mathbb{Z}\beta$ with $(\beta,\beta) = 2k$. We realise both algebras in its lattice vertex algebra V_L, a superalgebra when $2k$ is odd, using the deformed conformal vector

$$\omega_\lambda = \tfrac{1}{4k}\beta(-1)^2\mathbf{1} + \beta(-2)\mathbf{1} + \lambda D^2 e^{2\beta}, \qquad \lambda \in \mathbb{C}^\times.$$

Here $\mathbf{1}$ is the vacuum, $\beta(-m)$ are the Heisenberg creation modes, D is the translation operator and $e^{2\beta}$ is the lattice state of momentum 2β.

The only nonzero momentum in ω_λ is 2β, so this vector belongs to the lattice VOA $U = V_{2L}$ of the index-two sublattice $2L$. The diagonal Virasoro algebra $W_k = V_{n+2}$ sits conformally inside

U. Writing $L(c,h)$ for the simple Virasoro module of central charge c and lowest weight h, we have

$$W_k = \bigoplus_{j\geq 0} L(c, nj(j+1)).$$

The conformal vector and a primary of weight $2n = 8k$ generate W_k, realising $\mathcal{W}(2, 8k)$. The other lattice coset $\beta + 2L$ contributes an order-two simple current J of W_k of lowest weight $3k$. Adjoining it gives $\widehat{W}_k = W_k \oplus J$, with Flohr's operator products for the generators of weights 2 and $3k$ when k is integral (Propositions 3.1–3.2). For nonintegral k the extension is a vertex operator superalgebra with odd part J (Proposition 12.5).

A singular-vector recurrence reduces C_2-cofiniteness to a differential equation, which we compare with hypergeometric equations by an explicit gauge transformation. The bound $\dim(V_{n+2}/C_2(V_{n+2})) \leq 3n + 6$ holds for every $n \geq 1$ and extends the odd-parameter result of [18] (Corollary 4.6). The Zhu-algebra calculation then gives strong rationality of W_k for every even n and of $\widehat{W}_k$ for integral k.

Lattice branching and categorical induction give the $2n + 6$ simple W_k-modules, their theta-function quotient characters and their complete modular data. For integral k, we prove that Flohr's list of $2k + 5$ simple $\widehat{W}_k$-modules is complete and compute the block of the modular S-matrix corresponding to the W_k-modules fixed by tensoring with J (Theorems 6.4, 6.6 and 12.2). When J is fermionic, we determine the characters, modular transformations and fusion rules of the $2k + 2$ Neveu–Schwarz simples (untwisted modules up to parity) and list the Ramond modules (parity-twisted modules) (Proposition 12.5).

The inclusion $W_k \subset U$ also gives a direct construction of a near-group category. As a W_k-module, $U = W_k \oplus X$, with X simple. Let $\mathcal{N}$ be the category of U-modules in the braided category $\operatorname{Rep} W_k$ of ordinary W_k-modules. Monodromy with J gives $\mathcal{N}$ a $\mathbb{Z}_2$-grading. The neutral component $\mathcal{N}_+$ has invertible simples α_g indexed by $g \in \mathbb{Z}_n$ and one further simple R, with fusion rules

$$\alpha_g R = R\alpha_g = R, \qquad R^2 = \bigoplus_{g\in\mathbb{Z}_n} \alpha_g \oplus nR.$$

Thus $\mathcal{N}_+$ is of near-group type $\mathbb{Z}_n + n$ with trivial pointed associator (Theorems 5.1 and 12.2). We compute the modular data of its Drinfeld centre $Z(\mathcal{N}_+)$, which has rank $n(n + 3)$ (Theorem 7.1 and Proposition 7.2).

Izumi and Evans–Gannon studied these fusion rules using Cuntz algebra endomorphisms [31, 32, 16]. The Leavitt algebra approach admits non-unitary realisations [17]. For integral k we recover the associator from the tetrahedral coefficients of $\mathcal{N}_+$ and obtain a Leavitt algebra realisation in the sense of Radchenko–Wheeler [44]. Its quadratic form on $\mathbb{Z}_{4k}$ and finite quantum dilogarithm are determined up to an automorphism of $\mathbb{Z}_{4k}$ (Theorem 10.1).

To realise the centres of $\mathcal{N}$ and $\mathcal{N}_+$, we adjoin a rank-seven lattice factor. For a positive integer r, let K_r be the orthogonal complement in the E_8 lattice of a primitive vector of norm $2r$. Let μ be the order-two class in K_n^*/K_n, with corresponding lattice module $V_{K_n+\mu}$. The strongly rational VOAs of CFT type

$$B_n = V_{n+2} \otimes V_{K_n}, \qquad \widetilde{U}_n = B_n \oplus (J \otimes V_{K_n+\mu})$$

have central charge $8 - 6n$. Their module categories are braided equivalent to $Z(\mathcal{N})$ and $Z(\mathcal{N}_+)$, respectively (Theorems 8.2 and 12.2). To match the twists, choose a least-norm representative κ of μ and add $\kappa(-2)\mathbf{1}$ to the conformal vector of the K_n factor. Denoting the resulting VOAs by B_n^κ and $\widetilde{U}_n^\kappa$, we obtain ribbon equivalences

$$\operatorname{Rep} B_n^\kappa \simeq Z(\mathcal{N}), \qquad \operatorname{Rep} \widetilde{U}_n^\kappa \simeq Z(\mathcal{N}_+)$$

for every even $n \geq 2$ (Proposition 8.4). The shifted VOAs are simple, regular, self-contragredient and nonnegatively graded, with weight-zero spaces of dimension greater than one.

The larger Flohr algebra $\widehat{W}_k$ reappears in the centres of central quotients of $\mathcal{N}_+$. For integral k and $s^2 \mid k$, let $\mathcal{Q}_{2s}$ be the de-equivariantisation by the central lift of the order-$2s$ subgroup of $\mathbb{Z}_{4k}$. Put $r = k/s^2$ and let $\mathcal{P}_r$ be the pointed modular category on $\mathbb{Z}_{2r}$ with quadratic form $q_r(x) = x^2/(4r)$ in $\mathbb{Q}/\mathbb{Z}$. Its reverse $\mathcal{P}_r^{\mathrm{rev}}$ has reversed braiding and inverse twists. We obtain the ribbon decomposition

$$Z(\mathcal{Q}_{2s}) \simeq \operatorname{Rep} \widehat{W}_k \boxtimes \mathcal{P}_r^{\mathrm{rev}},$$

realised by $\widehat{W}_k \otimes V_{K_r}$ (Theorems 7.1 and 7.6, Corollary 8.3). This gives the Witt class of $\operatorname{Rep} \widehat{W}_k$ as the nontrivial class of $\mathcal{P}_k$. The classes for parameters k and j agree exactly when kj is a square (Corollary 7.7).

At a square parameter $k = \nu^2$, take $s = \nu$ and write $\mathcal{F}_\nu = \mathcal{Q}_{2\nu}$. The lattice $K_1 = E_7$ supplies the inverse-semion factor $\mathcal{P}_1^{\mathrm{rev}}$, so that

$$\operatorname{Rep}\left(\widehat{W}_{\nu^2} \otimes V_{E_7}\right) \simeq Z(\mathcal{F}_\nu)$$

as ribbon categories. The invertible simples of $\mathcal{F}_\nu$ are α_g for $g \in \mathbb{Z}_{2\nu}$, and the other simples are $\alpha_g \rho$ for a chosen noninvertible simple ρ, with

$$\rho\alpha_g = \alpha_{-g}\rho, \qquad \rho^* = \alpha_\nu \rho, \qquad \rho^2 = \alpha_\nu \oplus \bigoplus_g \alpha_g \rho.$$

The pointed associator has exact order 2ν (Theorem 7.5). These are the even cyclic counterparts of the twisted Haagerup–Izumi categories of [18]. The untwisted centres are realised in [19]. The de-equivariantisation, fusion rules and pointed class order also appear in Schopieray–Wang–Yu [48, Cor. 3.12, Thm. 5.5, Prop. 5.6]. Here the calculation gives the pointed cocycle and the centre realisation within Flohr's family.

The modular data of $Z(\mathcal{N}_+)$ allow us to classify its Galois conjugates with positive categorical dimensions for integral k. Put $d = 4k(k+1)$ and $\zeta_{4d} = \exp(2\pi i/(4d))$. The entries lie in $\mathbb{Q}(\zeta_{4d})$. Two positive conjugates agree up to relabelling exactly when the ratio of their exponents ℓ in $\zeta_{4d} \mapsto \zeta_{4d}^{\ell}$ is a square modulo d. This gives $2^{\omega(k(k+1))}$ classes, where ω counts distinct prime divisors, and at least that many Morita-inequivalent pseudo-unitary conjugates of $\mathcal{N}_+$ (Theorem 7.8 and Corollary 7.9).

We describe the Morita class of $\mathcal{N}_+$ by classifying its $\tau(n)$ indecomposable module categories and $2\tau(n)$ connected separable algebras, where $\tau(n)$ counts positive divisors. We determine the dual fusion categories and their fusion rules, and the argument applies to the C* near-group categories satisfying the same structural hypotheses (Section 9). The centre also gives the Frobenius–Schur exponent $n(n+4)$ and all higher indicators. In particular, the fourth indicator of R is $1+i$, which rules out real forms for every Galois conjugate of $\mathcal{N}_+$ (Section 11 and Proposition 12.4).

Sections 3–8 construct the VOAs and identify their representation categories. Sections 9–11 classify module categories, reconstruct associators and compute indicators. Section 12 treats nonintegral k. Appendix A compares with C* examples, and Appendix B gives the recurrence and gauge coefficients.

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2 Setting and notation

This section fixes the parameters, the vertex operator algebra W_k with which we work, the status of Flohr's results, and the notions from vertex operator algebra theory used below.

2.1 Parameters

In §§2–8, §10 and §11, $k \in \mathbb{N}$ and $n = 4k$. The conformal-shift result of Proposition 8.4 is stated for every even n. In §9, n is even. In §12, $n \equiv 2 \pmod 4$. Always $m = n + 4$ and $c = 1 - 6n = 1 - 24k$, $k = n/4$. Put

$$\delta = (\sqrt{k} + \sqrt{k+1})^2 = \tfrac{1}{2}(n + 2 + \sqrt{nm}), \qquad \delta + \delta^{-1} = n + 2.$$

Since $nm = 16k(k+1)$ is not a square, δ is irrational. It is the parameter t of [18], $t > 1$, $t + t^{-1} = n + 2$. We write t for it in §4. Also $(\delta - 1)^2 = n\delta$ and $(\delta + 1)^2 = m\delta$. For $N \in \mathbb{N}$ and $a \in \mathbb{Z}$ put $\Lambda_{a,N} = \Theta_{a,N}/\eta$, $\Theta_{a,N}(\tau) = \sum_{j \in \mathbb{Z}} q^{(2Nj+a)^2/4N}$, the convention of [20, (4.4a)] and of [18].

2.2 Vertex operator algebra notions

Let V be a vertex operator algebra with conformal vector ω, $T = \omega$, and write $Y(a,z) = \sum a_j z^{-j-1}$. Modules are ordinary modules, that is, L_0 acts semisimply with finite-dimensional eigenspaces and eigenvalues bounded below in each coset of $\mathbb{C}/\mathbb{Z}$. V is of *CFT type* if $V_j = 0$ for $j < 0$ and $V_0 = \mathbb{C}\mathbf{1}$. Put $C_2(V) = \operatorname{span}\{a_{-2}b\}$. V is *C_2-cofinite* if $\dim V/C_2(V) < \infty$, and $R(V) = V/C_2(V)$ is a commutative algebra under $[a][b] = [a_{-1}b]$ [53]. Zhu's algebra $A(V) = V/O(V)$, with product $*$, is an associative algebra whose simple modules are in bijection with the simple V-modules, through their spaces of lowest weight vectors [53]. For a V-module M, $A(M) = M/O(M)$ is the $A(V)$-bimodule of Frenkel and Zhu [21]. For simple S, T it bounds the fusion rule N_{MS}^{T} by the dimension of $\operatorname{Hom}_{A(V)}(A(M) \otimes_{A(V)} S_{\mathrm{top}}, T_{\mathrm{top}})$ [21, 37]. V is *strongly rational* if it is simple, of CFT type, self-contragredient, C_2-cofinite and rational. Then $\operatorname{Rep} V$ is a modular tensor category [28], and V is strongly rational as soon as it is simple, of CFT type, self-contragredient and C_2-cofinite with semisimple $A(V)$ [39, Main Thm. 3]. A vertex operator algebra is *regular* when every weak module is a direct sum of irreducible ordinary modules. The shifted centre realisations in §8.1 are regular and self-contragredient, but have more than one weight-zero state.

2.3 The algebra W_k

For every $n \geq 1$, the diagonal Virasoro algebra

$$V_{n+2} = \bigoplus_{j \geq 0} L(c,\, nj(j+1))$$

is a simple vertex operator algebra of CFT type, strongly generated by T and a Virasoro primary Ψ of weight $2n$ [11, Thm. 3.2]. That theorem is stated for every $n \geq 1$ (for every $k \geq 3$ in the notation $k = n + 2$ there), and its proof uses only the irrationality of t. Since V_{n+2} is simple, of CFT type and has zero weight-one space, it is self-contragredient [36]. We put

$$W_k = V_{4k+2}.$$

It realises Flohr's $\mathcal{W}(2, 8k)$. The two agree in central charge, generator weights, vacuum character and characters of lowest-weight representations [11, §§1, 5]. For integral k it is the fixed-point subalgebra of $\widehat{W}_k$ under the order-two grading automorphism. The self-coupling of its weight-$8k$

generator has been compared with Flohr's formula only for $n \le 9$ [11, §§1, 5]. This closed formula is not used in the proofs. The Flohr notation refers throughout to the algebras constructed here: the identification is through central charge, generators, operator products and characters, and does not assert uniqueness of an abstract two-generator algebra from these data. Our names "first" and "second" distinguish odd and even $n = 4k$, respectively. Its simple modules are written W_k, X, J, Y, A_a, B_b. The first four are introduced in §3, and all are classified in Theorem 6.4.

2.4 Flohr's results

Flohr constructed $\mathcal{W}(2, 3k)$ and $\mathcal{W}(2, 8k)$ at $c = 1-24k$ from free-field vertex operators. From the null vectors of the generators he determined the admissible lowest weights, and gave characters, an S-matrix and fusion rules [20, §4, App. A]. These derivations are at the level of conformal field theory. They do not establish that the algebras exist as vertex operator algebras, that they are rational, or that the list of irreducible representations is complete. In this paper $\mathcal{W}(2, 3k)$ is constructed as $\widehat{W}_k = W_k \oplus J$ (Proposition 3.1), it is shown to be strongly rational (Proposition 4.7), and Flohr's list of its simple modules is shown to be complete, with his modular data (§6.3, Theorem 6.6).

Flohr's labels record only the weights of the generators, not the central charge. For instance, $\mathcal{W}(2, 24)$ occurs both as $\mathcal{W}(2, 3k)$ with $k = 8$, at $c = -191$, and as $\mathcal{W}(2, 8k)$ with $k = 3$, at $c = -71$. We therefore write W_k and $\widehat{W}_k$, both at $c = 1 - 24k$, and use Flohr's labels only when comparing with [20].

3 The lattice picture

This section realises Flohr's $\mathcal{W}(2, 3k)$, as $\widehat{W}_k = W_k \oplus J$, inside a rank-one lattice vertex algebra with a deformed conformal vector (Proposition 3.1). The decomposition $V_{\sqrt{2k}\mathbb{Z}} = \widehat{W}_k \oplus M$ with M simple is completed in Corollary 4.8, once strong rationality is known. Along the way it introduces the W_k-modules X, J, Y, all from [18, §4], whose results there are stated for every n.

The diagonal free-field charges of [20, §2] lie in $\sqrt{2k}\,\mathbb{Z}$. Let $L = \mathbb{Z}\beta$ with $(\beta, \beta) = 2k$ and give V_L the conformal vector

$$\omega = \tfrac{1}{4k}\beta(-1)^2\mathbf{1} + \beta(-2)\mathbf{1}.$$

Then the central charge is $1 - 24k$, and $e^{j\beta}$ has conformal weight $kj(j-2)$. Consequently

$$\operatorname{ch} V_L = \frac{q^{-c/24}}{(q)_\infty} \sum_{j \in \mathbb{Z}} q^{kj(j-2)} = \Lambda_{0,k},$$

and the least conformal weight of V_L is $-k$, attained at $j = 1$.

3.1 The even sublattice

Put $\beta' = 2\beta$, so $L' = \mathbb{Z}\beta' \subset L$ has $(\beta', \beta') = 8k = 2n$ and $V_L = V_{L'} \oplus V_{L'+\beta'/2}$. In terms of β',

$$\omega = \tfrac{1}{4n}\beta'(-1)^2\mathbf{1} + \tfrac{1}{2}\beta'(-2)\mathbf{1},$$

which is the conformal vector T_0 of [18, (4.1)]. Its weight-zero primary there is $u = e^{\beta'} = e^{2\beta}$. For $\lambda \neq 0$ put $\omega_\lambda = \omega + \lambda D^2 u$. Since u has charge 2, every mode of ω_λ preserves the parity of the β-charge. The automorphism σ of V_L acting by $(-1)^j$ on charge $j\beta$ fixes ω_λ, and $V_L^\sigma = V_{L'}$. Thus $U = (V_{L'}, \omega_\lambda)$ is the algebra U of [18, (4.1)] with $n = 4k$, and $M_n = V_{L'+\beta'/2}$ is its half-lattice module.

3.2 The lattice realisation

By [18, Thm. 4.1, §4.4], stated for every positive integer n:

- there is a conformal embedding $W_k \hookrightarrow U$ onto the sum of the C_1-cofinite simple Virasoro submodules;
- $U = W_k \oplus X$ and $M_n = J \oplus Y$, with X, J, Y simple W_k-modules of lowest weights 0, $3n/4 = 3k$ and $-n/4 = -k$, and $J|_{\mathrm{Vir}} = \bigoplus_{j\geq 1} L(c, n(j^2 - \frac{1}{4}))$;
- $W_k \oplus J$ is a closed simple subalgebra of the lattice abelian intertwining algebra $U \oplus M_n$, and the vacuum channel in $J \times J$ is nonzero.

3.3 Characters

By [18, §4.5],

$$\chi_{W_k} = \tfrac{1}{2}(\Lambda_{n,n} - \Lambda_{m,m}), \quad \chi_X = \tfrac{1}{2}(\Lambda_{n,n} + \Lambda_{m,m}), \quad \chi_J = \tfrac{1}{2}(\Lambda_{0,n} - \Lambda_{0,m}), \quad \chi_Y = \tfrac{1}{2}(\Lambda_{0,n} + \Lambda_{0,m}).$$

Splitting theta sums by parity gives $\Lambda_{0,k} = \Lambda_{0,n} + \Lambda_{n,n}$ and $\Lambda_{j,k} = \Lambda_{2j,n} + \Lambda_{n-2j,n}$, and the same with (k, n) replaced by $(k+1, m)$. Hence

$$\chi_{W_k} + \chi_J = \tfrac{1}{2}(\Lambda_{0,k} - \Lambda_{0,k+1}), \qquad \chi_X + \chi_Y = \tfrac{1}{2}(\Lambda_{0,k} + \Lambda_{0,k+1}),$$

which are Flohr's characters of the vacuum module of $\mathcal{W}(2, 3k)$ and of his module M [20, Table 4.1]. Their sum is $\Lambda_{0,k} = \operatorname{ch} V_L$.

3.4 The decomposition of V_L

Proposition 3.1. *Let $k \in \mathbb{N}$, $L = \mathbb{Z}\beta$ with $(\beta, \beta) = 2k$, and $\omega_\lambda = \frac{1}{4k}\beta(-1)^2\mathbf{1} + \beta(-2)\mathbf{1} + \lambda D^2 e^{2\beta}$ with $\lambda \neq 0$. Then (V_L, ω_λ) is a simple vertex operator algebra of central charge $1 - 24k$, and $W_k \oplus J$ is a simple vertex subalgebra of V_L containing ω_λ, with vacuum character $\frac{1}{2}(\Lambda_{0,k} - \Lambda_{0,k+1})$. As a W_k-module, $V_L = W_k \oplus X \oplus J \oplus Y$.*

Proof. With $\beta' = 2\beta$ and $n = 4k$, the lattice $\mathbb{Z}\beta' \cup (\mathbb{Z}\beta' + \beta'/2) = \mathbb{Z}\beta$ is even, since $(\beta, \beta) = n/2 = 2k$. So $U \oplus M_n = V_L$ is the lattice vertex algebra, which is simple, with conformal vector ω_λ. The computation of [18, §4.1] that ω_λ is a conformal vector applies verbatim, since $e^{2\beta}$ is a Virasoro primary of weight 0 for ω with regular self-OPE. The closed simple subalgebra $W_k \oplus J$ of §3.2 is a simple vertex subalgebra containing the vacuum and ω_λ, with character $\chi_{W_k} + \chi_J$. The W_k-decomposition is §3.2. □

From here on, $\widehat{W}_k = W_k \oplus J$. For $n \equiv 2 \pmod 4$, that is $k \in \frac{1}{2} + \mathbb{Z}_{\geq 0}$, the lattice $\mathbb{Z}\beta$ is odd and the same construction gives a vertex superalgebra. This is Flohr's fermionic case, treated in §12.

3.5 Generators and Flohr's $\mathcal{W}(2, 3k)$

Proposition 3.2. *Let j span the lowest-weight space J_{3k}. Then $\widehat{W}_k$ is strongly generated by T and the Virasoro primary j of weight $3k$. The singular part of the operator product $j(z)j(w)$ lies in the Virasoro vacuum module, and so is determined by c, $3k$ and the normalisation of j. In particular the coefficient of j in it vanishes.*

Proof. Measure charge in units of β. The vertex operators of V_L are those of the lattice vertex algebra, and charge is additive under them. By [18, §4.4], j has lowest-charge term $e^{-\beta}$, and Ψ, suitably normalised, has lowest-charge term $e^{-2\beta}$ [18, §4.2]. Since $Y(e^{-\beta}, z)e^{-\beta} =$

$z^{2k}(\pm e^{-2\beta} + O(z))$, the vector $j_{(-2k-1)}j$, of weight $8k$, has charge -2 component $\pm e^{-2\beta}$. It lies in $J \cdot J \subset \widehat{W}_k \cap V_{L'} = W_k$, whose weight-$8k$ space is spanned by Virasoro descendants of $\mathbf{1}$ and by Ψ, as the Virasoro summands of W_k have lowest weights 0, $8k$, $24k, \ldots$. The Virasoro descendants of $\mathbf{1}$ have charges ≥ 0, since each mode of ω_λ is a charge-preserving mode plus a multiple of a mode of $e^{2\beta}$ [18, §4.1], so preserves or raises charge. Hence $j_{(-2k-1)}j = v + \alpha\Psi$ with v a descendant of $\mathbf{1}$ and $\alpha \neq 0$.

For $m \geq 0$, $j_{(m)}j \in W_k$ has weight $6k - 1 - m < 8k$, so it is a Virasoro descendant of $\mathbf{1}$. This is the statement on the singular part, and its projection onto the vacuum Virasoro module is fixed by conformal invariance, c and the weight $3k$. Thus the λ-brackets among T and j are normally ordered polynomials in T, j and their derivatives, so the span of normally ordered monomials in T, j and their derivatives is a vertex subalgebra [33]. It contains $\Psi = \alpha^{-1}(j_{(-2k-1)}j - v)$, hence W_k, which is strongly generated by T and Ψ (§2). Its intersection with J is a nonzero W_k-submodule of the simple module J, so it contains J. □

Flohr's $\mathcal{W}(2, 3k)$ is defined by its two generating fields, the stress tensor and a primary of weight $3k$, and their operator product expansions at $c = 1 - 24k$, with vanishing self-coupling [20, §§2–3]. By Proposition 3.2, $\widehat{W}_k$ is a simple vertex operator algebra with exactly these data. This is the sense in which it realises Flohr's $\mathcal{W}(2, 3k)$. Its vacuum character is Flohr's (§3).

4 Strong rationality

This section proves that W_k and $\widehat{W}_k$ are C_2-cofinite for every k (Proposition 4.1) and then strongly rational (Proposition 4.7).

Relation with earlier work

For odd n, C_2-cofiniteness of V_{n+2} is [18, Thm. 3.1]. The level-nine singular vector of Ψ gives a differential operator that would have a nonzero polynomial solution if $R(V_{n+2})$ were infinite-dimensional, and a 2-adic recurrence, in which $n(n+4)$ is a 2-adic unit, excludes such a solution [18, §3.2]. A second proof [18, App. B.2] reduces C_2-cofiniteness to $b_{h+1} \neq 0$ for the recurrence (1) below, and proves this by a 2-adic estimate valid for $h = 2n \equiv 2 \pmod 4$. Neither 2-adic argument applies to $n = 4k$, because $n(n+4)$ is even and $h \equiv 0 \pmod 8$. Lemma 4.2 is the reduction of [18, App. B.2], with b_{h+1} replaced by any b_j, $j > h$. Lemmas 4.3 and 4.4 are new. In place of a 2-adic estimate, they transform the recurrence operator by an explicit gauge into the tensor product of two ${}_1F_2$ operators. These are the limits $x \to \infty$ of the rank-three systems used for Zhu's algebra in [18, App. B]. They then show that the relevant formal solution diverges. This argument does not use the parity of n. Proposition 4.7 is [18, Thm. B.3], whose proof we check for $n = 4k$.

4.1 C_2-cofiniteness

Recall that V is C_2-cofinite when the quotient $R(V) = V/C_2(V)$ is finite-dimensional. For W_k this quotient is a commutative algebra generated by the classes $x = [T]$ and $y = [\Psi]$, so the task is to show that x is nilpotent. The proof is by contradiction and has four steps.

1. *Reduction to a scalar sequence* (Lemma 4.2). The singular vector of Ψ at level nine gives linear relations in $R(W_k)$. They lead to a sequence b_j defined by a five-term recurrence, and W_k is C_2-cofinite as soon as $b_j \neq 0$ for some $j > 2n$.

2. *A differential equation* (Lemma 4.3). The generating series $B = \sum_j b_j w^{-j}$ is a formal solution of a differential operator L of order nine. An explicit gauge transformation identifies

L with the tensor product of two hypergeometric operators L_t and $L_{t^{-1}}$ of order three. This is the one identity in the paper that is verified by exact computer algebra.

3. *Divergence* (Lemma 4.4). Each of L_t and $L_{t^{-1}}$ has, up to scalars, one formal solution at $w = \infty$ without exponential factor, and the product of these two solutions is a divergent series.

4. *Comparison* (Lemma 4.5). Sorting formal solutions by their exponential factors shows that a Laurent polynomial solution of the tensor product system is a multiple of that product.

If W_k were not C_2-cofinite, then b_j would vanish for all $j > 2n$ and B would be a polynomial in w^{-1}. By steps (2) and (4) it would give a Laurent polynomial that is a multiple of the divergent series of step (3), which is impossible. The argument uses n only through $t > 2$. It therefore gives C_2-cofiniteness of V_{n+2} for every $n \geq 1$, with an explicit bound on $\dim R(V_{n+2})$ (Corollary 4.6).

Proposition 4.1. *For every $k \in \mathbb{N}$, the algebras W_k and $\widehat{W}_k$ are C_2-cofinite.*

The proof occupies the rest of this subsection. Put $h = 2n$ and $t = \delta$. Here $n = 4k$ gives $t > 5$, and the argument below needs only $t > 2$. In $R(W_k)$ put $x = [T]$, $y = [\Psi]$ and $R_a = [\Psi_{2h-1-2a}\Psi]$ for $0 \leq a \leq h$, $R_a = 0$ otherwise. By strong generation, $R(W_k)$ is generated by x and y. Let b_j be given by the recurrence [18, (B.3)],

$$\sum_{q=0}^{4} Q_q(h+4-j,h)\, b_{j+q-4} = 0 \quad (j \geq 1), \qquad b_0 = 1,\ b_{<0} = 0, \tag{1}$$

with $Q_q \in \mathbb{Z}[a,h]$ as in [18, App. B.1]. Explicitly, $Q_q(a,h) = P_q(2h+8-2a,h)$, where, with s an indeterminate,

$$\begin{aligned} P_0 &= (h-s+8)(h+s-8)(2h-s+8)(h^2+8h-s^2+14s-49) \\ &\qquad \cdot (2h^2-3hs+19h+s^2-14s+49)(2h^2+hs+14h-s^2+8s-16), \\ P_q &= (-2)^q \textstyle\sum_j p_{q,j}(h)\, s^j \quad (q=1,2,3), \qquad P_4 = 64h(h+8)s, \end{aligned}$$

The coefficients $p_{q,j}(h)$ are listed in Appendix B.1. In particular $Q_4(a,h) = -128h(h+8)(a-h-4)$, and Q_q has degree $9-2q$ in a with leading coefficient -512, $768(h+4)$, $-384(h+2)(h+6)$, $64(h+4)(h^2+8h+8)$, $-128h(h+8)$ for $q = 0,\dots,4$. These agree with the polynomials defined in [18, App. B.1] from the singular-vector table of [18, App. A] (`er_appB_check.py`).

Lemma 4.2. *If $b_j \neq 0$ for some $j > h$, then W_k and $\widehat{W}_k$ are C_2-cofinite.*

Proof. Commuting Ψ_a through the level-nine singular vector of Ψ gives the relations

$$\textstyle\sum_{q=0}^{4} Q_q(a,h)\, x^q R_{a-q} = 0 \qquad (a \in \mathbb{Z})$$

in $R(W_k)$ [18, (B.1)]. Their derivation does not use the parity of n. As in the proof of [18, Thm. B.1], with $T_j = x^j R_{h-j}$ they give $x^3 T_j = b_j\, x^3 R_h$ for all $j \geq 0$, and $T_j = 0$ for $j > h$. So if $b_j \neq 0$ for some $j > h$, then $x^3 R_h = 0$. Since $R_0 = \nu\mathbf{1}$ with $\nu \neq 0$, and $R_n = \alpha x^n + Cy$ with $C \neq 0$ the coefficient of Ψ in $\Psi_{h-1}\Psi$ [18, Lemma 4.4] (stated for every n), this gives $x^{h+3} = 0$ and $x^{n+3}y = 0$. With $y^2 \in \mathbb{C}x^{2n} + \mathbb{C}x^n y$, the monomials x^i, $i \leq h+2$, and $x^i y$, $i \leq n+2$, span $R(W_k)$. Finally, $\widehat{W}_k = W_k \oplus J$ is a finite sum of W_k-modules, which are C_2-cofinite since W_k is [2], and $C_2(\widehat{W}_k) \supseteq \operatorname{span}\{a_{-2}b : a \in W_k,\ b \in \widehat{W}_k\}$. □

Let $\theta = w\, d/dw$. The recurrence says that $B = \sum_{j\geq 0} b_j w^{-j}$ is a formal solution of

$$L = \sum_{q=0}^{4} w^q\, Q_q(h+q+\theta,h),$$

an operator of order 9 whose coefficient of θ^9 is the constant -512. For $T \in \{t, t^{-1}\}$ let

$$L_T = \theta(\theta + \tfrac{1}{2} - T)(\theta + 1 - 3T) - Tw(\theta + 1 - 2T),$$

the operator of ${}_1F_2(1-2T; \tfrac{3}{2} - T, 2 - 3T; Tw)$. These are the limits $x \to \infty$ of the rank-three hypergeometric systems of [18, (B.8)].

Lemma 4.3. *Put $\gamma = -(2n+2)$. Let $r_{ij} \in \mathbb{Q}(t)[w, w^{-1}]$, $0 \le i, j \le 2$, be the explicit coefficients in Appendix B.2. For solutions f of L_t and g of $L_{t^{-1}}$, in any differential extension of $\mathbb{C}(w)$, the function $\Phi = w^\gamma \sum_{i,j} r_{ij}\, \theta^i f\, \theta^j g$ satisfies $L\Phi = 0$. Define the 9×9 matrix G by*

$$w^{-\gamma}(\theta^m \Phi)_{0 \le m \le 8} = G\,(\theta^i f\, \theta^j g)_{i,j}.$$

Then G satisfies $\theta G + \gamma G + GA = CG$, with A and C the connection matrices of $L_t \otimes L_{t^{-1}}$ and of L, and

$$\det G = \frac{(t-2)^{12}(t-1)^{21}(2t-1)^{12}(t+1)^5(t+2)^2(2t+1)^2}{2^{11}\, t^{27}}\, w^9.$$

Proof. Both statements are identities in $\mathbb{Q}(t)[w, w^{-1}]$, with $h = 2n$ replaced by $2(t + t^{-1} - 2)$. After clearing powers of 2 and t from the denominators, they are polynomial identities in $\mathbb{Z}[t, w, w^{-1}]$. The supplementary exact calculation reconstructs the coefficients of Appendix B.2, forms the connection matrices and verifies the gauge identity and determinant over $\mathbb{Q}(t)$; see §A.5. In particular,

$$\theta \log \det G = \operatorname{tr} C - \operatorname{tr} A - 9\gamma = 9,$$

so the determinant is a constant multiple of w^9. □

For $t > 2$ the determinant is nonzero, so G is invertible over $\mathbb{C}[w, w^{-1}]$.

Lemma 4.4. *For $T \in \{t, t^{-1}\}$, L_T has the formal solution*

$$h_T = w^{2T-1} \sum_{i \ge 0} d_i(T)\, w^{-i}, \qquad d_i(T) = \frac{(1-2T)_i\,(\tfrac{1}{2} - T)_i\,(T)_i}{i!\, T^i},$$

and every formal solution of L_T at $w = \infty$ without exponential factor is a multiple of h_T. The product $h_t h_{t^{-1}} = w^{2n+2} \sum_N c_N w^{-N}$, $c_N = \sum_{i+j=N} d_i(t) d_j(t^{-1})$, is a divergent series.

Proof. Substituting w^{2T-1-i} into L_T gives the recurrence for $d_i(T)$. The Newton polygon of L_T at ∞ has a slope-$\frac{1}{2}$ part of length 2, giving the formal solutions with factors $e^{\pm 2\sqrt{Tw}}$, and a horizontal part of length 1. Also $(2t-1) + (2t^{-1} - 1) = 2n + 2$.

For $t > 2$ the parameters $a = 1 - 2t^{-1}$, $b = \frac{1}{2} - t^{-1}$, $c = t^{-1}$ lie in $(0, 1)$, so $d_j := d_j(t^{-1}) = (a)_j (b)_j (c)_j t^j / j! > 0$. Also $|d_i(t)| \le \delta_i := (2t-1)_i (t - \frac{1}{2})_i (t)_i / (i!\, t^i)$. For $x > 0$ there are $0 < \kappa_x \le K_x$ with $\kappa_x (i+1)^{x-1} \le (x)_i / i! \le K_x (i+1)^{x-1}$ for all $i \ge 0$. Hence, with $\sigma = a + b + c - 3 = -\frac{3}{2} - 2t^{-1} \in (-3, 0)$, there is a constant K with

$$\frac{\delta_i\, d_{N-i}}{d_N} \le K \binom{N}{i}^{-2} t^{-2i} (i+1)^{4t} \Big(\frac{N+1}{N-i+1}\Big)^{-\sigma} \qquad (0 < i \le N).$$

For $0 < i \le N/2$ the right side is at most $2^3 K\, N^{-2} t^{-2i} (i+1)^{4t}$, and for $N/2 < i \le N$ it is at most $K\, t^{-N} (N+1)^{4t+3}$. So $\sum_{0 < i \le N} \delta_i d_{N-i} / d_N \to 0$, that is $c_N = d_N (1 + o(1))$. Since $d_N \ge \kappa\, (N!)^2 t^N (N+1)^\sigma$, $|c_N|^{1/N} \to \infty$. □

We use the following facts on formal solutions at the irregular singular point $w=\infty$ [52]. A system $\theta Z = MZ$, with $M \in \mathbb{C}[w,w^{-1}]^{r\times r}$, has a fundamental matrix of formal solutions of the form

$$F(w^{1/p})\, w^{\Lambda}\, e^{Q(w^{1/p})}.$$

Here $p\in\mathbb{N}$, F is an invertible matrix of formal Laurent series in $w^{-1/p}$, and Λ is constant and commuting with Q, and Q diagonal, with entries polynomials in $w^{1/p}$ without constant term (Hukuhara–Turrittin–Levelt). For each q occurring in Q, the formal solutions with exponential factor e^{q} form a subspace, and the space of formal solutions is the direct sum of these subspaces over the distinct q.

Lemma 4.5. *1. For $T\in\{t,t^{-1}\}$, the exponential factors of L_T at ∞ are 1 and $e^{\pm 2\sqrt{Tw}}$, each with a one-dimensional space of formal solutions. That for 1 is spanned by h_T, without logarithms.*

2. The system $\theta E = AE$ of $L_t\otimes L_{t^{-1}}$ has the fundamental system of the nine products of these solutions, with exponential factors $e^{q_1+q_2}$, $q_1\in\{0,\pm 2\sqrt{tw}\}$, $q_2\in\{0,\pm 2\sqrt{w/t}\}$. These nine are distinct, and $q_1+q_2=0$ only for $q_1=q_2=0$.

3. Every solution of $\theta E = AE$ in $\mathbb{C}[w,w^{-1}]^9$ is a multiple of the vector $(\theta^i h_t\, \theta^j h_{t^{-1}})_{i,j}$.

Proof. (1) In L_T the term θ^3 has weight $(3,0)$ and $Tw\theta$ weight $(1,1)$ in (order, power of w), so the Newton polygon at ∞ has a side of slope $\frac12$ and length 2 and a horizontal side of length 1. The first gives two solutions, after the ramification $w=u^2$, with $q=\pm 2\sqrt{Tw}$, from $\theta^2\sim Tw$. The second gives solutions $w^{\rho}\sum_i d_i w^{-i}$ without exponential factor. The coefficient of $w^{\rho+1}$ forces $\rho+1-2T=0$, a single exponent, so this space is one-dimensional, without logarithms, and spanned by h_T (Lemma 4.4).

(2) If Φ_1,Φ_2 are fundamental matrices for L_t and $L_{t^{-1}}$, then $\Phi_1\otimes\Phi_2$ is one for $A=A_t\otimes 1 + 1\otimes A_{t^{-1}}$, the system of $L_t\otimes L_{t^{-1}}$ in the basis $(\theta^i f\,\theta^j g)$. Exponential factors multiply. Since $t>2$, one has $\sqrt{t}>2/\sqrt{t}$. Thus the nine coefficients 0, $\pm\sqrt{t}$, $\pm 1/\sqrt{t}$, $\pm(\sqrt{t}+1/\sqrt{t})$ and $\pm(\sqrt{t}-1/\sqrt{t})$ are distinct. Hence the nine sums q_1+q_2 are distinct, and a sum vanishes only for $q_1=q_2=0$.

(3) A Laurent polynomial solution is a formal solution with exponential factor 1, so by the direct sum decomposition it lies in the factor-1 subspace, which by (1) and (2) is spanned by $h_t\otimes h_{t^{-1}}$. □

Proof of Proposition 4.1. Suppose W_k is not C_2-cofinite. By Lemma 4.2, B is a polynomial in w^{-1}, so $Y=(\theta^m B)_m$ is a solution of $\theta Y = CY$ in $\mathbb{C}[w,w^{-1}]^9$. By Lemma 4.3, $E=w^{-\gamma}G^{-1}Y$ is a solution of $\theta E = AE$ in $\mathbb{C}[w,w^{-1}]^9$. By Lemma 4.5(3), $E=\lambda\,(\theta^i h_t\,\theta^j h_{t^{-1}})_{i,j}$ for some λ, and $E_{00}=\lambda h_t h_{t^{-1}}$ is a Laurent polynomial. By Lemma 4.4, $\lambda=0$. Then $E=0$, $Y=0$ and $B=0$, contradicting $b_0=1$. □

Corollary 4.6. *For every integer $n\ge 1$, the diagonal Virasoro vertex algebra V_{n+2} is C_2-cofinite. With $x=[T]$ and $y=[\Psi]$ in $R(V_{n+2})$,*

$$x^{2n+3}=x^{n+3}y=0, \qquad \dim_{\mathbb{C}} R(V_{n+2})\le 3n+6.$$

In particular its associated variety is a point.

For odd n, C_2-cofiniteness of V_{n+2} is [18, Thm. 3.1, App. B], and Corollary 4.6 gives a second proof and extends the bound of [18, Thm. B.1] from odd $n\ge 3$ to every positive integer n.

Proof. The preceding gauge, formal-solution, and singular-vector arguments require only $n\ge 1$, so that $t+t^{-1}=n+2$ has its larger solution $t>2$ and the displayed determinant is nonzero. The reduction of Lemma 4.2, which is valid for every n, gives $x^{h+3}=x^{n+3}y=0$ with $h=2n$. It also shows that the C_2-algebra is spanned by $1,x,\dots,x^{2n+2}$ and $y,xy,\dots,x^{n+2}y$, a total of $3n+6$ elements. A finite-dimensional connected graded algebra has a one-point reduced spectrum. □

As a numerical check, Φ of Lemma 4.3 with $f = h_t$, $g = h_{t^{-1}}$ is a multiple of B, to 60 digits for $k = 1, 2$ (`gauge_series_check.py`), and b_{h+1}, computed exactly in $\mathbb{Q}$, is nonzero for $k \le 250$.

4.2 Strong rationality

Given C_2-cofiniteness, what remains for strong rationality is rationality, and for this it suffices that Zhu's algebra $A(W_k)$ is semisimple [39, Main Thm. 3]. Zhu's algebra is a finite-dimensional commutative algebra generated by $[T]$ and $[\Psi]$. Its simple modules correspond to the simple W_k-modules, and the eigenvalue of $[T]$ is the lowest weight. So the proposition below has two parts. It shows that $A(W_k)$ has no nilpotent elements. It also locates the possible lowest weights on two explicit grids, with each nonzero value occurring at most once. The second part is used later. The grids and the multiplicity bound are the input for the classification of the simple modules in §6.

For odd n the proposition is proved in the earlier paper [18, Thm. B.3]. That proof studies each local factor of $A(V_{n+2})$ through a differential system of rank nine depending on the lowest weight, and it is long. We do not repeat it. Instead we list the six places where it uses a property of n, and check each of them for $n = 4k$.

Proposition 4.7. *For every $k \in \mathbb{N}$, Zhu's algebra $A(W_k)$ is semisimple, and W_k and $\widehat{W}_k$ are strongly rational. The lowest weight of every simple W_k-module lies in*

$$\{a^2/4n - n/4 : a \in \mathbb{Z}\} \cup \{b^2/4m - n/4 : b \in \mathbb{Z}\},$$

and each such value other than 0 is the lowest weight of at most one simple W_k-module. In particular every lowest weight is at least $-n/4 = -k$, and Y is the unique simple W_k-module of lowest weight $-k$.

Proof. In $A(W_k)$ put $x = [T]$ and $y = [\Psi]$. By strong generation, $A(W_k)$ is generated by x and y, and x is central, so $A(W_k)$ is commutative [18, §5.1]. By Proposition 4.1 it is finite-dimensional, so it is a product of local Artin algebras, each with a single residue of x. We use the local-factor argument of [18, Thm. B.3], with Proposition 4.1 in place of the odd-n 2-adic argument of [18, App. B.2]. The residues of x are rational by [41, Cor. 5.10]. The local-factor argument uses the following properties, which hold for $n = 4k$.

1. $t + t^{-1} = n + 2 \ge 6$, so $t > 4$ and the t-factors of $K(t)$ and of the exceptional minors in [18, Prop. B.2] are nonzero. The identities of [18, Prop. B.2] are over $\mathbb{Q}(t, x, w)$.

2. t is irrational, since $n(n+4) = 16k(k+1)$ is not a square. This gives the resonance statement of [18, App. B.5] (integral exponent differences only in the class $0, n, 3n$) and the exceptional set $\mathcal{E} = \{0, 1, 3n/4, 3n/4 + 4\}$ of [18, (B.11)]. Its derivation compares coefficients in the basis $1, t$ and uses nothing else about n.

3. $\sqrt{nm}$ is irrational, which makes the zeros of the trace function $G(x)$ of [18, (B.12)] simple. Away from $\mathcal{E}$ each local factor of $A(W_k)$ is therefore $\mathbb{C}$, and lies over a single x.

4. At $x = 1$ and $x = 3n/4 + 4$ the hypergeometric parameters are $(1 - 2t, -2t, 2)$ and $(1 - 2t, t + 3, -3t - 1)$. This uses only $(t+1)^2 = mt$. The exclusion of these fibres [18, App. B.7] uses the exponents n and $3n$ of the products $(1, 1)$, $(2, 2)$, which follow from $t + t^{-1} = n + 2$, and $\deg P \le h = 2n$.

5. At $x = 0$ and $x = 3n/4$ the parameters are $(1 - 2t, 2 - 2t, 0)$ and $(1 - 2t, t - 1, 3 - 3t)$, using $(t-1)^2 = nt$. The fibres are realised by W_k, X and by J, which exist for every n [18, Thm. 4.1]. The nonvanishing of ν and of the self-coupling C is [18, Lemma 4.4], also for every n.

6. The zeros of the factors A, B in [18, (B.10)] are $x = (3t-2)/4$ and $x = (3-2t)/(4t)$, irrational. C_t vanishes at $x = 3(t-1)^2/(4t) = 3n/4$.

The higher Zhu circle relations and skew symmetry give the universal polynomial of [18, App. B.3] of degree at most $2n$. Its normalization and its relation to y use the nonzero couplings in (5). In [18, App. B.4–B.6], (1) permits the gauge transformation, (2) identifies the exceptional residues, and (3) makes the nonexceptional local factors one-dimensional. The hypergeometric analysis in [18, App. B.7–B.8] applies by (4) and (5), with the possible zeros of the determinant factors covered by (6). These steps give semisimplicity of $A(W_k)$, with two factors at $x = 0$ and at most one at each other residue. Every residue is $3n/4$ or a zero of $G(x)$, so it lies on one of the two stated grids, including $3n/4 = (2n)^2/(4n) - n/4$. For $n = 2$, the nonvanishing check (1) is made separately in §12, where $t = 2 + \sqrt{3} < 4$.

W_k is simple, self-contragredient and of CFT type (§2), so [39, Main Thm. 3] gives strong rationality. $\widehat{W}_k = W_k \oplus J$ is simple (Proposition 3.1) and of CFT type, since $h_J = 3k > 0$. Its weight-one space is zero, so it is self-contragredient [36]. It is C_2-cofinite by Proposition 4.1. The corresponding commutative algebra $\mathbf{1} \oplus J$ in $\operatorname{Rep} W_k$ is connected and simple as a module over itself, because $\widehat{W}_k$ is simple. It is therefore separable by [7, Thm. 2.26] and [35, Thm. 6.10], as in the proof of Theorem 5.1(1). Its category of local modules, which is the module category of $\widehat{W}_k$ [6], is consequently semisimple. Finally, $-n/4$ is attained by Y, and $a^2/4n - n/4 = -n/4$ or $b^2/4m - n/4 = -n/4$ only for $a = b = 0$, so Y is the only simple module of lowest weight $-n/4$. All other values are larger. □

This route does not use the main-text rationality argument of [18, §5], which does not transfer directly. The reason is that [18, Prop. 5.5] excludes a simple of lowest weight $-n/4$ because $-n/4 \notin \mathbb{Z}$ for odd n, whereas here Y has lowest weight $-k \in \mathbb{Z}$.

Corollary 4.8. *As a $\widehat{W}_k$-module, $V_{\sqrt{2k}\mathbb{Z}} = \widehat{W}_k \oplus M$, with M simple and $M \cong X \oplus Y$ as a W_k-module. M has lowest weight $-k$ and character $\frac{1}{2}(\Lambda_{0,k} + \Lambda_{0,k+1})$. Moreover $J \boxtimes X \cong Y$ in* $\operatorname{Rep} W_k$.

Proof. Lemma 12.1 applies by Proposition 4.7, so J is a simple current and $J \otimes J \cong W_k$. That lemma uses only strong rationality, the known characters and the unique minimal module, and does not use this corollary. The last statement now follows as in [18, §4.8]. The induction of J to the lattice extension U is invertible, hence simple, and maps onto $M_n = J \oplus Y$, so $U \boxtimes J \cong J \oplus Y$ and $X \boxtimes J \cong Y$. By Proposition 4.7, $\widehat{W}_k$ is rational, so the $\widehat{W}_k$-module V_L is a direct sum of simple modules, and $\widehat{W}_k$ is a summand. Let M be a complement. By Krull–Schmidt for W_k-modules, $M \cong X \oplus Y$ as a W_k-module. If M were not simple, one of its summands would be a $\widehat{W}_k$-module with underlying W_k-module X (or Y). Then J acts on it, and not by zero, since otherwise every element of $J \cdot J$, which contains a nonzero element of W_k, would act by zero, and its annihilator is a nonzero ideal of the simple algebra W_k. So there is a nonzero intertwining operator of type $\binom{X}{J\ X}$, that is, $\operatorname{Hom}(J \boxtimes X, X) \neq 0$. But $J \boxtimes X \cong Y \ncong X$, a contradiction. Similarly for Y, as $J \boxtimes Y \cong X$. □

5 The near-group category

This section shows that the W_k-modules over the lattice extension $U = W_k \oplus X$ form a $\mathbb{Z}_2$-graded fusion category whose trivial component is a near-group category of type $\mathbb{Z}_{4k} + 4k$ (Theorem 5.1).

Recall that a near-group category of type $G + \mu$ has the elements of a finite group G as its invertible simple objects, and one further simple object R, with $R \otimes R = \bigoplus_{g \in G} g \oplus \mu R$. The idea of the construction is the following. The lattice vertex operator algebra U is a finite extension $U = W_k \oplus X$ of W_k. So U is a commutative algebra A' in $\mathcal{C} = \operatorname{Rep} W_k$, and the W_k-modules with a compatible action of A' form a fusion category $\mathcal{N}$. Among its objects are the $2n$ modules of the

vertex operator algebra U, which are invertible and are known from the lattice. The theorem says that $\mathcal{N}$ has exactly two further simple objects, and that one half of $\mathcal{N}$ is a near-group category.

The proof has three parts. Part (1) combines general results on commutative algebras in modular categories with a count of ranks. Part (2) uses the simple current J to split $\mathcal{N}$ into two halves $\mathcal{N}_\pm$. Part (3) is the main step. It determines the multiplicity of R in $R\otimes R$ from two sides, by Frobenius–Perron dimensions and by a bound on a space of intertwiners (Lemma 6.2). The classification of the simple W_k-modules is not used. On the contrary, the near-group category is the main tool for that classification in §6.

By Proposition 4.7, W_k is strongly rational, so $\mathcal{C} = \operatorname{Rep} W_k$ is a modular tensor category [28]. Let $A' = U = \mathbf{1} \oplus X$, regarded as a commutative algebra in $\mathcal{C}$, and put $\mathcal{N} = \mathcal{C}_{A'}$. Write $d = \operatorname{FPdim} X$.

We record a lattice computation used here and in Theorem 9.18. Put $u = e^{\beta'}$ in U. The lowest-weight space of X is spanned by $x_0 = \mathbf{1} + 2\lambda u$ [18, (4.8)]. The lattice product gives $u_{(-1)}u = 0$, and therefore

$$(x_0)_{(-1)}x_0 = \mathbf{1} + 4\lambda u = 2x_0 - \mathbf{1}.$$

Both components $X \otimes X \to \mathbf{1}$ and $X \otimes X \to X$ of the multiplication of A' are consequently nonzero.

Theorem 5.1. *The following hold for every $k \in \mathbb{N}$.*

1. *A' is connected étale, and $\mathcal{N}$ is a fusion category of rank $2n+2$. Its invertible objects are the $2n$ local modules M_a, $a \in \mathbb{Z}/2n$, with the lattice fusion rules.*
2. *Monodromy with J is a faithful $\mathbb{Z}_2$-grading $\mathcal{N} = \mathcal{N}_+ \oplus \mathcal{N}_-$, and $\mathcal{N}_+ = \mathcal{C}'_{A'}$ is the category of A'-modules in the centraliser $\mathcal{C}'$ of J.*
3. *$\mathcal{N}_+$ has simple objects M_{2x}, $x \in \mathbb{Z}/n$, and one further simple R, with*

$$M_{2x}M_{2y} = M_{2x+2y}, \qquad M_{2x}R = RM_{2x} = R, \qquad R^* = R, \qquad R^2 = \bigoplus_{x\in\mathbb{Z}/n} M_{2x} \oplus nR.$$

Thus $\mathcal{N}_+$ is a near-group category $\mathbb{Z}_{4k} + 4k$ with cyclic group, and $\operatorname{FPdim} R = \frac{1}{2}(n + \sqrt{n(n+4)}) = \delta - 1$.

Proof. (1) A' is a commutative algebra with trivial twist in $\mathcal{C}$, since U is a vertex operator algebra extending W_k [30], and it is simple as a module over itself, since U is simple [18, Thm. 4.1]. It is connected, as $X \ncong \mathbf{1}$. By [7, Thm. 2.26], the category of modules over a simple commutative algebra in a braided fusion category is a fusion category, so $\mathcal{N}$ is fusion. Then A' is separable by [35, Thm. 6.10]. So A' is connected étale, as in [18, Lemma 6.1]. The local modules form $\operatorname{Rep} U$, the $2n$ lattice modules M_a [18, §4.1], and they are invertible. By [18, Lemma 6.2], $\operatorname{rank}\mathcal{N}$ is the sum of $\dim \operatorname{End}_{\mathcal{C}}$ of the restrictions of the M_a. By [18, Thm. 4.1] these restrictions are $\mathbf{1} \oplus X$, $J \oplus Y$, and $2n-2$ simple modules. So $\operatorname{rank}\mathcal{N} = 2 + 2 + (2n-2) = 2n+2$.

(2) $J \otimes J \cong W_k$ by [18, §4.4]. This uses C_2-cofiniteness of W_k. Monodromy with J is therefore a natural automorphism of each object, with square the identity [3, Lemma 2.13], and it is multiplicative under $\otimes$. Since $A' \in \mathcal{C}'$, it commutes with the A'-action, so on a simple A'-module it is a scalar ± 1. This gives a $\mathbb{Z}_2$-grading of $\mathcal{N}$ with trivial component $\mathcal{C}'_{A'}$. The monodromy of J with the underlying object of M_a is $(-1)^a$, from the lattice braiding [18, §4.8], so the grading is faithful and the invertibles in $\mathcal{N}_+$ are the M_{2x}.

(3) By (1) there are exactly two non-invertible simples. By (2), the even labels give n invertibles in $\mathcal{N}_+$, forming $2\mathbb{Z}/2n \cong \mathbb{Z}_n$, and the odd labels give n in $\mathcal{N}_-$. [10, Lemma 3.11] gives $\operatorname{FPdim}\mathcal{N} = \operatorname{FPdim}\mathcal{C}/(1+d)$, and [10, Cor. 3.32] gives $\operatorname{FPdim}\mathcal{C} = (1+d)^2 \cdot 2n$. Hence each component has dimension $n(1+d)$, which exceeds n. So each component contains exactly one

non-invertible simple. Call the one in $\mathcal{N}_+$ R, and put $r = \mathrm{FPdim}\, R$. Then

$$r^2 = nd.$$

Since R is the unique non-invertible simple of $\mathcal{N}_+$, it is self-dual and fixed by the invertibles. By Frobenius reciprocity each invertible occurs once in R^2, so $R^2 = \bigoplus_x M_{2x} \oplus \mu R$ for some $\mu \in \mathbb{Z}_{\geq 0}$.

The free module $F(X)$ lies in $\mathcal{N}_+$. By adjunction it contains $\mathbf{1}$ once and no other invertible, so $F(X) = \mathbf{1} \oplus bR$ with $d = 1 + br$. Also $\dim \mathrm{End}\, F(X) = 1 + N_{XX}^X$, so $b^2 = N_{XX}^X$. The vacuum and self channels of the multiplication of A' are nonzero by the lattice computation above, so $b \geq 1$. The bimodule bound of Lemma 6.2 (whose proof does not use this theorem) gives $N_{XX}^X \leq 2$, hence $b = 1$. Now $r^2 = n + \mu r$ and $r^2 = n(1 + r)$ give $\mu = n$, and r is the positive root of $r^2 = nr + n$. □

By Proposition A.4, the theorem agrees with the modular data for $k \leq 6$. The object $M_n = F(J)$ lies in $\mathcal{N}_+$ and generates the order-two subgroup of $\mathbb{Z}_{4k}$. The de-equivariantisation of $\mathcal{N}_+$ by this subgroup is the category $\mathcal{D} = (\mathrm{Rep}\, \widehat{W}_k)_A$, $A = \mathbf{1} \oplus M$ (Theorem 7.1(3)).

6 Simple modules and modular data

This section classifies the simple W_k-modules, with their characters, and identifies the modular data of $\mathrm{Rep}\, W_k$ with the theta modular data (Theorem 6.4). It then shows that Flohr's list of simple $\widehat{W}_k$-modules is complete and determines the modular data of $\mathrm{Rep}\, \widehat{W}_k$, including the block on the four fixed-point modules, which the characters do not determine (Theorem 6.6).

The difficulty is that the lattice supplies only about half of the simple modules. A simple W_k-module that occurs in a module of the lattice algebra U can be read off from the lattice. This gives W_k, X, J, Y and the modules A_a. The remaining modules B_b do not occur in any module of U. They arise only as summands of tensor products, and their number and their characters have to be determined. We therefore proceed indirectly, in three subsections. Section 6.1 records the answer that the characters suggest, a representation of $SL_2(\mathbb{Z})$ on $2n + 6$ theta functions. Section 6.2 proves that this is the answer. The missing modules are counted by inducing them to the near-group category of §5, where each becomes a multiple of a non-invertible simple object. Their characters are then determined from the modular S-matrix and the bounds on lowest weights of Proposition 4.7. Section 6.3 passes from W_k to its extension $\widehat{W}_k$ by J. This step is formal, except at the two modules fixed by J.

6.1 Modular data

This subsection describes the expected modular data, before any module beyond those of the lattice is known to exist. The $2n + 6$ functions below are closed under $\tau \mapsto -1/\tau$ and $\tau \mapsto \tau + 1$, and the matrix S by which the first transformation acts is explicit. Proposition 6.1 records, for $k \leq 6$ and by direct computation, that S has the properties of the S-matrix of a modular tensor category, and that the simple current J acts as Flohr's list of representations requires. It is a consistency check and a guide. Its statements are proved for every k in §6.2 (Theorem 6.4 and Corollary 6.5).

The theta functions $\chi_{W_k}, \chi_X, \chi_J, \chi_Y$, $\Lambda_{\alpha,n}$ $(1 \leq \alpha < n)$ and $\Lambda_{\beta,m}$ $(1 \leq \beta < m)$ span a representation of $SL_2(\mathbb{Z})$. The matrix of S in this basis is computed from the theta transformation [18, (9.1)].

Proposition 6.1 (numerical)**.** *For $n = 4k$ and $1 \leq k \leq 6$, computed from the theta functions (for all k, see Corollary 6.5):*

1. *The matrix S is symmetric and unitary, and its Verlinde coefficients (vacuum χ_{W_k}) are non-negative integers. The rank is $2n + 6$.*

2. *$J \otimes J \cong W_k$, $\theta_J = 1$, $J \otimes X = Y$, and the J-fixed simples are $\Lambda_{n/2,n}$ and $\Lambda_{m/2,m}$. The centraliser $\mathcal{C}'$ of J has rank $n+4$. Its J-orbits, with the two fixed points counted twice, number $2k+5$. This is the number of representations of $\mathcal{W}(2,3k)$ in [20, Table 4.1], the fixed points giving $a_k^{\pm}$ and $b_{k+1}^{\pm}$.*

6.2 Simple modules

For odd n the simple W_k-modules are classified in [18, Thm. 9.2]. There are $2n+6$, each with one-dimensional top space, with characters

$$\chi_{W_k},\ \chi_X,\ \chi_J,\ \chi_Y, \qquad A_a:\ \Lambda_{a,n}\ (1 \le a < n), \qquad B_b:\ \Lambda_{b,m}\ (1 \le b < m),$$

and $A(W_k) \cong \mathbb{C}^{2n+6}$ [18, §9.4]. The modular data is the theta modular data [18, §8].

For $n = 4k$ the same $2n+6$ theta functions occur (§6.1). The lattice realisation gives, for every k, the simple modules W_k, X, J, Y and the restrictions A_a of $M_{n\pm a}$, $1 \le a < n$ [18, Thm. 4.1]. Here we use the indexing of §7, in which M_s has lowest weight $s^2/4n - s/2$ modulo 1, $M_0 = U$ and $M_n = J \oplus Y$. Thus M_s restricts to $A_{|n-s|}$. These are $n+3$ of the $2n+6$. The remaining $n+3$ characters are the $\Lambda_{b,m}$. For odd n the corresponding modules are constructed in [18, §7]. Those with odd b are the summands of the complement of $\mathbf{1} \oplus X \oplus \bigoplus_x Z_x$ in $X \otimes X$, and the others are their translates by the semion. Two features of $n = 4k$ change this argument.

1. J is a boson. Indeed $\theta_J = e(3k) = 1$ and $J \otimes J \cong W_k$ [18, §4.4]. So $\langle J \rangle$ is Tannakian and does not split off. There is no analogue of the semion factorisation $\mathcal{C} \simeq \mathrm{Sem} \boxtimes \mathcal{C}_0$ of the Haagerup family [11, §1(B)], and J-translation preserves the centraliser $\mathcal{C}'$ of J.

2. J has fixed points, namely $J \otimes A_{n/2} \cong A_{n/2}$ and $J \otimes B_{m/2} \cong B_{m/2}$ (Corollary 6.5). For odd n, J acts freely.

For odd n, the classification in [18, §7] does not construct the missing modules directly. It counts them through the category of modules over $A' = \mathbf{1} \oplus X$. Each unknown simple induces to a multiple of the non-invertible simple, and the bimodule bound [18, Thm. 7.4] forces the multiple to be one. This argument carries over to $n = 4k$, using the near-group category of Theorem 5.1. Write $F\colon \mathcal{C} \to \mathcal{N} = \mathcal{C}_{A'}$ for the free-module functor, $R_+ = R$ and R_- for the non-invertible simples of $\mathcal{N}_+$ and $\mathcal{N}_-$, and $\mathcal{C}''$ for the simples with monodromy -1 with J.

Lemma 6.2. *For every $n \ge 1$, and in particular for $n = 4k$, the bimodule $A(X)$ is generated over $A(W_k)$ on the left and right by two classes, of weights 0 and 1. Consequently $N_{XS}^T \le 2$ for all simple W_k-modules S, T.*

Proof. This is [18, Thm. 7.4], stated there for odd n. Its proof uses the following, each valid for every $n \ge 1$.

1. The decomposition $X = \bigoplus_{j \ge 0} X_j$ with $0 \to M_j \to X_j \to K_j \to 0$ [18, (7.7)], where M_j is the Verma module of weight $h_j = nj(j+1)$ and K_j the irreducible Verma module of weight $g_j = h_j + (2j+1)^2$. This rests on [18, Lemmas 4.3, 4.5], proved for every n. Distinct blocks are disjoint because $n(n+4)$ is not a square.

2. The transition maps of [18, Lemma 7.3]. Their proof uses the lattice $\mathbb{Z}\beta$, $(\beta,\beta) = 2n$, the C_1-image theorem, and invariance of the form under the primary Ψ of even weight $2n$. None depends on the parity of n.

3. The weight comparisons in the induction, namely $h_{j+1} - h_j = 2n(j+1) > 2n$ for $j \ge 1$, $g_{j+1} - g_j = 2(n+4)(j+1) > 2n$, and $h_l \ne g_j$ for all l, j. The last holds because $j(j+1)$ is even, so every h_l is even and every g_j is odd, for every n. The statement "since n is odd" in [18, §7.2] is not needed.

4. $A(W_k)$ is generated by $[T]$ and $[\Psi]$, with $[T]$ central, and the Frenkel–Zhu–Li bound on intertwining operators [18, Thm. 7.4].

□

Proposition 6.3. *Let $n = 4k$. Then $\mathcal{C}$ has exactly $2n+6$ simple objects, all self-dual. Besides the $n+3$ lattice modules W_k, X, J, Y, A_a there are $m/2-1$ simples Q'_i in $\mathcal{C}'$ and $m/2$ simples Q''_j in $\mathcal{C}''$, with*

$$F(Q'_i) = R_+, \qquad F(Q''_j) = R_-, \qquad \operatorname{FPdim} Q'_i = \operatorname{FPdim} Q''_j = \delta - 1.$$

They occur, each with multiplicity one, in

$$X \otimes X = \mathbf{1} \oplus X \oplus Y \oplus \bigoplus_{x=1}^{n/2-1} A_{2x} \oplus \bigoplus_i Q'_i, \qquad X \otimes A_1 = A_1 \oplus \bigoplus_{a \text{ odd}} A_a \oplus \bigoplus_j Q''_j,$$

where A_1 occurs twice in the second decomposition. Since $m/2-1 = 2k+1$ is odd, J fixes at least one Q'_i.

Proof. $\operatorname{FPdim} X = \delta$ and $F(X) = \mathbf{1} \oplus R_+$ (proof of Theorem 5.1). The restriction of a local module M_a has dimension $\operatorname{FPdim} A' = 1+\delta$ [10, Lemma 3.11], so $F(A_a) = M_{n+a} \oplus M_{n-a} \oplus R_\pm$, with sign $(-1)^a$, by adjunction and dimensions. Also $F(J) = M_n$ and $F(Y) = M_n \oplus R_+$. Any other simple Q has no invertible summand in $F(Q)$, by adjunction, since the restrictions of the M_a are the lattice modules. Hence $F(Q) = b_Q R_\pm$ with $b_Q \geq 1$, the sign being the monodromy of Q with J (Theorem 5.1(2)). As in [18, (7.6)], $\dim \operatorname{Hom}(Q \otimes Q^*, A') = b_Q^2$, so $N_{XQ}^Q = b_Q^2 - 1$, and the bound $N_{XQ}^Q \leq 2$ of Lemma 6.2 gives $b_Q = 1$.

J is a boson and $A_1 \in \mathcal{C}''$, so monodromy with J is a faithful $\mathbb{Z}_2$-grading and $\operatorname{FPdim} \mathcal{C}' = \operatorname{FPdim} \mathcal{C}'' = n(1+\delta)^2$. In $\mathcal{C}'$ the known simples are W_k, J (dimension 1), X, Y (δ) and A_{2x}, $1 \leq x \leq n/2-1$ ($1+\delta$). Using $(\delta-1)^2 = n\delta$ and $\delta + \delta^{-1} = n+2$,

$$\sum_{Q \in \mathcal{C}'} (\delta-1)^2 = n(1+\delta)^2 - 2 - 2\delta^2 - (\tfrac{n}{2}-1)(1+\delta)^2 = (\tfrac{n}{2}+1)\, n\delta,$$

so there are $n/2+1 = m/2-1$ such Q. In $\mathcal{C}''$ the known simples are the $n/2$ modules A_a, a odd, and the same computation gives $(n(1+\delta)^2 - \frac{n}{2}(1+\delta)^2)/(n\delta) = n/2+2 = m/2$. The decompositions follow from $X \otimes A' = UF(X)$ and $A_1 \otimes A' = UF(A_1)$, with U the restriction, $U(R_+) = X \oplus Y \oplus \bigoplus_x A_{2x} \oplus \bigoplus_i Q'_i$ and $U(R_-) = \bigoplus_{a \text{ odd}} A_a \oplus \bigoplus_j Q''_j$, by adjunction. Self-duality holds as in [18, §7.4]. Indeed $A(W_k)$ is generated by $[T]$ and $[\Psi]$, and Ψ has even weight $2n$, so a simple module and its contragredient have the same top character. Finally J permutes the Q'_i with $J^2 = \mathbf{1}$, and their number is odd. □

At the level of modular data ($k \leq 6$) the Q'_i are the B_b with b even, the Q''_j those with b odd, and J fixes exactly $B_{m/2}$.

The characters of the Q's and the modular data follow. For odd n, [18, §8] recovers the block of S on the Q's from a commutative two-valued group $H/\{\pm 1\}$ with $|H| = m$ odd, which is not available here ($m = 4(k+1)$). We instead determine the characters first, from the positivity lemma [18, Lemma 9.1] and the momentum chains of [18, Lemma 5.4]. We first show that the chain can be determined from any constituent of a simple module.

Virasoro block-chain lemma. *Put $h(p) = (p^2 - (t-1)^2)/(4t)$. Let Q be a simple ordinary W_k-module with a Virasoro constituent of lowest weight $h(r(t+1))$ for $r \in \mathbb{Q} \setminus \mathbb{Z}$. As a Virasoro module, Q is a direct sum of irreducible Verma modules with lowest weights $h((r+2a)(t+1))$, $a \in \mathbb{Z}$.*

Proof. At irrational t, Feigin–Fuchs linkage pairs the reducible Verma weight $h(ut - v)$, for $u, v \in \mathbb{Z}_{>0}$, with its singular-vector weight $h(ut + v)$. All other blocks are singletons [18, §4.1 and proof of Lemma 5.4]. For $s = r + 2a$, neither sign of $s(t+1)$ has integral Kac labels, by coefficient comparison in $1, t$. Thus each block of weight $h(s(t+1))$ is a singleton whose Verma module is irreducible.

The modes of W_k preserve integral weight cosets, so simplicity puts the weights of Q in one such coset. Since Q is ordinary, these weights are bounded below and their spaces are finite-dimensional. The functorial exact block projectors on finitely generated Virasoro submodules agree under inclusion and thus extend to Q [18, Lemma 4.2]. Each block component has finite length. Indeed, a nonzero subquotient has a lowest-weight vector, whose cyclic module has a simple quotient. This weight must be one of the at most two simple lowest weights of that block. The sum of the dimensions at those weights therefore bounds the length. In a singleton block with irreducible Verma module, a basis of the lowest-weight space generates a direct sum of copies of that Verma module. The quotient has zero lowest-weight space at the only permitted weight, so it vanishes. Write $Q[r]$ for the sum of the blocks of weights $h((r+2a)(t+1))$. It is nonzero by exactness of the projectors.

Let S be a simple Virasoro summand of $Q[r]$ with momentum $s(t+1)$. Restrict the module vertex operator to the Virasoro module generated by Ψ and to S, and project to a Virasoro block B. If the projected intertwiner is nonzero, its coefficient span is a nonzero Virasoro submodule of B by the Virasoro commutator formula. It has a simple quotient by finite length, and the intertwiner to that quotient is nonzero because its coefficients span the quotient. The level-nine null-vector condition [18, (5.23)] now gives an output momentum, up to sign, of the form

$$s(t+1) + 2it - 2j, \qquad i, j \in \{-1, 0, 1\}.$$

The weights of Q are rational since $h(r(t+1)) = (mr^2 - n)/4$. For rational u, v, reducing $t^{-1} = n + 2 - t$ shows that $h(ut - v)$ is rational exactly when $u^2 = v^2$. For the output momentum this gives

$$(s + 2i)^2 - (s - 2j)^2 = 4(i + j)(s + i - j) = 0.$$

Since $s \notin \mathbb{Z}$, we have $i = -j$. The output lies in the singleton block of weight $h((s+2i)(t+1))$, so the B-component lies in $Q[r]$. Thus $Q[r]$ is stable under the modes of Ψ and T. These fields strongly generate W_k, so $Q[r] = Q$ by simplicity. □

Theorem 6.4. *Let $n = 4k$. Then:*

1. *The simple W_k-modules are W_k, X, J, Y, A_a $(1 \le a < n)$ and B_b $(1 \le b < m)$, with characters*

   $$\chi_{W_k} = \tfrac{1}{2}(\Lambda_{n,n} - \Lambda_{m,m}),\ \chi_X = \tfrac{1}{2}(\Lambda_{n,n} + \Lambda_{m,m}),\ \chi_J = \tfrac{1}{2}(\Lambda_{0,n} - \Lambda_{0,m}),\ \chi_Y = \tfrac{1}{2}(\Lambda_{0,n} + \Lambda_{0,m}),$$

   $\operatorname{ch} A_a = \Lambda_{a,n}$, $\operatorname{ch} B_b = \Lambda_{b,m}$. *All are self-dual with one-dimensional top, and $A(W_k) \cong \mathbb{C}^{2n+6}$.*

2. *$B_b \in \mathcal{C}'$ exactly when b is even, so $\{Q'_i\} = \{B_b : b \text{ even}\}$ and $\{Q''_j\} = \{B_b : b \text{ odd}\}$.*

3. *The modular data of $\mathcal{C}$ is the theta modular data of §6.1. The categorical S is the transformation matrix of these characters under $\tau \mapsto -1/\tau$, up to the normalization of [18, (8.22)], and $\theta_M = e^{2\pi i h_M}$.*

Proof. By Proposition 4.7, W_k is strongly rational, so $\mathcal{C}$ is modular and the character S-matrix S is the categorical one up to a common scalar [27]. Proposition 6.3 lists the simples as the $n+3$ lattice modules and the Q's.

(a) As in [18, Cor. 7.5], every simple is self-dual, since its top character for $[T], [\Psi]$ is preserved by contragredience (Ψ has even weight). The characters are linearly independent. Lowest weights are distinct except for W_k, X (Proposition 4.7), and $\chi_X - \chi_{W_k} = \Lambda_{m,m}$ starts at weight 1, which is not a lowest weight. So S is real, symmetric, $S^2 = 1$. The dimension lemma [18, Lemma 6.3] applies. Indeed W_k is strongly rational. Y is the unique simple module of least lowest weight $-k$ (Proposition 4.7), with one-dimensional lowest space as $A(W_k)$ is commutative. $\chi_{W_k} = \frac{1}{2}(\Lambda_{n,n} - \Lambda_{m,m})$ gives $\lim_{\epsilon\downarrow 0} e^{-\pi/12\epsilon}\chi_{W_k}(i\epsilon) = \frac{1}{2}((2n)^{-1/2} - (2m)^{-1/2}) > 0$, where $c - 24h_Y = 1$. Hence $\operatorname{FPdim} M = S_{MY}/S_{0Y}$ for every simple M, and $S_{0Y} > 0$.

(b) The lattice branching matrix B satisfies $BS = S^U B$, with S^U the Fourier matrix of $\mathbb{Z}_{2n}$, so $B^T B$ commutes with S [34, Thms. 4.1, 4.5], [18, (8.3)]. This gives $S_{XX} = S_{00}$, $S_{XA_a} = S_{0A_a}$ and $S_{XQ} = -S_{0Q}$ for every Q. Since J is a simple current with $J \otimes X = Y$ and $X \in \mathcal{C}'$, $S_{JM} = \varepsilon_M S_{0M}$ and $S_{YM} = \varepsilon_M S_{XM}$, where $\varepsilon_M = \pm 1$ is the monodromy of M with J. Hence $\operatorname{FPdim} X = S_{XY}/S_{0Y} = S_{00}/S_{0X}$. Write $d_M = S_{0M}/S_{00}$ for the categorical dimension of a simple object M, as distinct from its Frobenius–Perron dimension. Then $d_X = 1/\operatorname{FPdim} X = \delta^{-1}$, that is $d_X = \delta_- + 1 > 0$ with $\delta_\pm = \frac{1}{2}(n \pm \sqrt{nm})$, and $S_{00} > 0$.

(c) Categorical dimensions. The free-module functor is pivotal [34], so $d(Q') = d(R_+) = d_X - 1 = \delta_-$. In $\mathcal{N}$, $R_- \otimes R_- \in \mathcal{N}_+$ contains each M_{2x} once by Frobenius reciprocity, and nR_+ by dimensions. The M_{2x} have dimension 1 (cf. [18, §4.8]). So $d(R_-)^2 = n + n\delta_- = \delta_-^2$, and $d(Q'') = \pm\delta_-$, the same sign for all Q''. Since $|S_{00}|^{-2} = \dim \mathcal{C} = 2n(d_X + 1)^2$, one gets $|S_{0Q}| = |\delta_-|\, S_{00} = (2m)^{-1/2}$.

(d) Apply S to $\Lambda_{m,m} = \chi_X - \chi_{W_k}$ and $\Lambda_{0,m} = \chi_Y - \chi_J$, using

$$\Lambda_{a,m}(-1/\tau) = (2m)^{-1/2} \textstyle\sum_{b \bmod 2m} e^{-\pi i ab/m} \Lambda_{b,m}(\tau).$$

By (b), the contributions of W_k, X, J, Y are, in both cases,

$$(S_{00} - S_{0X})(\Lambda_{m,m} + \Lambda_{0,m}), \qquad S_{00} - S_{0X} = -\delta_- S_{00} = (2m)^{-1/2},$$

which equal the endpoint terms $b = 0, m$ of the theta sums, and the A_a contribute zero. What remains is

$$\sum_Q c_Q\, \chi_Q = \sqrt{2/m} \sum_{b=1}^{m-1} (-1)^b \Lambda_{b,m}, \qquad \sum_Q \varepsilon_Q c_Q\, \chi_Q = \sqrt{2/m} \sum_{b=1}^{m-1} \Lambda_{b,m}, \qquad c_Q = -2S_{0Q}.$$

Adding and subtracting,

$$\sum_{Q \in \mathcal{C}'} \pm\chi_Q = \sum_{b \text{ even}} \Lambda_{b,m}, \qquad \sum_{Q \in \mathcal{C}''} \pm\chi_Q = \sum_{b \text{ odd}} \Lambda_{b,m},$$

with one sign in each sector by (c). By [18, Lemma 9.1], the coefficient of $q^{h+n/4}$ in $\eta\chi_Q$ is a non-negative integer when $h < 0$. The leading terms of $\Lambda_{2,m}$ and $\Lambda_{1,m}$ have $h = b^2/4m - n/4 < 0$ and coefficient 1. So both signs are $+$.

(e) We first show that every one of the $m-1$ nonendpoint m-chains is represented. Put $h_b = b^2/4m - n/4$, $1 \le b < m$. For $b \le m-3$ these weights are negative; the remaining two are $h_{m-2} = 1/m$ and $h_{m-1} = \frac{1}{2} + 1/4m$. They lie outside the exceptional congruence classes $0, -n/4$ modulo $\mathbb{Z}$. Thus [18, Lemma 9.1] makes the coefficient at $q^{h_b+n/4}$ in every $\eta\chi_Q$ a nonnegative integer. The character identities in (d) say that their sum, in the sector of parity b, is exactly 1. Consequently some Q contains the Virasoro constituent of lowest weight h_b.

Here $h_b = h((b/m)(t+1))$ with $0 < b/m < 1$. The Virasoro block-chain lemma shows that Q is supported on the chain of weights $h((b/m + 2a)(t+1))$, $a \in \mathbb{Z}$. The chains for distinct b are

disjoint. Equality of weights would give $b/m + 2a = \pm(b'/m + 2a')$. The plus sign forces $b = b'$ because $|b/m - b'/m| < 1$. The minus sign is impossible because $0 < (b + b')/m < 2$.

There are $m/2 - 1$ even chains and $m/2$ odd chains, exactly the numbers of unknown simples in the two sectors given by Proposition 6.3. Each chain must therefore occur in exactly one Q, and every Q occurs this way. Multiplying (d) by η and comparing the disjoint chain supports now gives $\chi_Q = \Lambda_{b,m}$. This proves (1) and (2), including the allocation of all unknown simples to the m-grid. The Zhu statement follows as in [18, §9.5].

(f) The $2n + 6$ characters are those of §6.1. Their transformation under S and T is the theta modular data, and [27] identifies it with the categorical data, proving (3). □

Corollary 6.5. *For every k, $J \otimes A_a \cong A_{n-a}$ and $J \otimes B_b \cong B_{m-b}$. Hence the J-fixed simples are exactly $A_{n/2}$ and $B_{m/2}$, and Proposition 6.1 holds for every k.*

Proof. $F(J) = M_n$ and F is a tensor functor, so $F(J \otimes A_a) = M_n \otimes (M_{n+a} \oplus M_{n-a} \oplus R_\pm)$ contains $M_{2n+a} = M_a$. By adjunction the simple module $J \otimes A_a$ occurs in the restriction of M_a, which is A_{n-a}, so $J \otimes A_a \cong A_{n-a}$. For B_b, the fusion with the simple current J is determined by the row of S, namely $S_{J\otimes N,M} = \varepsilon_M S_{N,M}$. By Theorem 6.4, $S_{B_b,M}$ is a multiple of $\cos(\pi bb'/m)$ for $M = B_{b'}$, and $(-1)^{b'}\cos(\pi bb'/m) = \cos(\pi(m - b)b'/m)$, with $\varepsilon_{B_{b'}} = (-1)^{b'}$. On the columns W_k, X, J, Y, where $\varepsilon_M = 1$, $S_{B_b,M}$ depends on b only through $(-1)^b = (-1)^{m-b}$, as m is even. On the columns A_a it vanishes. So the rows of $J \otimes B_b$ and B_{m-b} agree, and $J \otimes B_b \cong B_{m-b}$. The remaining statements of Proposition 6.1 are properties of the theta modular data, which is the modular data of $\mathcal{C}$ for every k. □

6.3 The simple current extension $\widehat{W}_k$

We now turn to Flohr's algebra $\mathcal{W}(2, 3k)$ itself, realised as $\widehat{W}_k = W_k \oplus J$. Since J is a simple current of order two with trivial twist, the modules of $\widehat{W}_k$ are obtained from those of W_k by a standard procedure. One keeps the W_k-modules that have trivial monodromy with J, and groups them into orbits under tensoring with J. An orbit of two modules gives one simple $\widehat{W}_k$-module. A module fixed by J gives two, and these two have the same character. This recovers Flohr's list of $2k + 5$ representations with his characters, and so proves that his list is complete. What it does not give is the modular S-matrix on the four modules that come from the two fixed points, because characters cannot distinguish the two members of each pair. Theorem 6.6 determines this 4×4 block. We begin with Flohr's list.

Table 1 lists Flohr's lowest-weight representations [20, Table 4.1]. The Frobenius–Perron dimensions are read from the row of M.

Table 1: Flohr's lowest-weight modules for $\widehat{W}_k$.

Module	**Character**	**Lowest weight**	**FPdim**
$\mathbf{1}$	$\frac{1}{2}(\Lambda_{0,k} - \Lambda_{0,k+1})$	0	1
M	$\frac{1}{2}(\Lambda_{0,k} + \Lambda_{0,k+1})$	$-k$	δ
a_j, $1 \le j \le k-1$	$\Lambda_{j,k}$	$\frac{j^2}{4k} - k$	$\delta + 1$
$a_k^\pm$	$\frac{1}{2}\Lambda_{k,k}$	$\frac{k}{4} - k$	$(\delta + 1)/2$
b_j, $1 \le j \le k$	$\Lambda_{j,k+1}$	$\frac{j^2}{4(k+1)} - k$	$\delta - 1$
$b_{k+1}^\pm$	$\frac{1}{2}\Lambda_{k+1,k+1}$	$\frac{k+1}{4} - k$	$(\delta - 1)/2$

He gives an S-matrix [20, App. A], with constants fixed in [20, (B.2b)], and the diagonal T-matrix [20, (A.3)].

By Proposition 4.7 and Theorem 6.4, $\widehat{W}_k = W_k \oplus J$ is the extension of W_k by a boson of order two, and its module category is the category of local modules over $\mathbf{1} \oplus J$ in $\mathcal{C}$ [30, 6]. These are supported on $\mathcal{C}'$. Each free J-orbit gives one simple module and each fixed point gives two. By Corollary 6.5 and Proposition 6.1(2), this produces $2k+5$ simple modules. The free orbits $\{W_k, J\}$, $\{X, Y\}$ give $\mathbf{1}$ and M, the orbits of A_a and B_b in $\mathcal{C}'$ give a_j and b_j, and the fixed points $A_{n/2}$, $B_{m/2}$ give $a_k^{\pm}$ and $b_{k+1}^{\pm}$. Each of these has underlying W_k-module $A_{n/2}$ or $B_{m/2}$, with character $\Lambda_{n/2,n} = \frac{1}{2}\Lambda_{k,k}$, resp. $\Lambda_{m/2,m} = \frac{1}{2}\Lambda_{k+1,k+1}$. So Flohr's list is the complete list of simple $\widehat{W}_k$-modules, with his characters.

The characters do not separate x^+ from x^- for $x \in \{a_k, b_{k+1}\}$, so they determine only the sums $S_{M,y^+} + S_{M,y^-}$. Let S, T be the modular matrices of $\widehat{W}_k$ in the sense of [27]. That is, S is symmetric, acts on the characters, $T = \operatorname{diag} e(h_M - c/24)$, $S^2 = C$ is the permutation matrix of contragredience, and $(ST)^3 = S^2$.

Theorem 6.6. *Label the fixed-point modules suitably. Then for $x, y \in \{a_k, b_{k+1}\}$ and $\varepsilon, \varepsilon' \in \{\pm\}$,*

$$S_{x^\varepsilon, y^{\varepsilon'}} = \tfrac{1}{2}(S_{x^+,y^+} + S_{x^+,y^-}) + \tfrac{1}{2}\varepsilon\varepsilon' A_{xy}, \qquad A = \frac{i^k}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

the first term being given by the characters, and $S_{x^+,N} = S_{x^-,N}$ for every other simple N. For odd k contragredience exchanges x^+ and x^-. For even k all simple modules are self-contragredient. In particular the modular data of $\operatorname{Rep}\widehat{W}_k$ is that of [20, App. A] with the constants of [20, (B.2b)].

Proof. Let σ be the involution of $\widehat{W}_k$ that is 1 on W_k and -1 on J. Pulling back modules along σ is a ribbon autoequivalence of $\operatorname{Rep}\widehat{W}_k$, so it preserves S and T. It fixes every simple module with underlying W_k-module $y \oplus Jy$, $y \neq Jy$, and exchanges x^+ and x^-. The pullback of x^+ again has underlying W_k-module x, and an isomorphism with x^+ would be a scalar on the simple W_k-module x anticommuting with the action of J, which is nonzero. Hence $S_{x^-,y^-} = S_{x^+,y^+}$, $S_{x^-,y^+} = S_{x^+,y^-}$ and $S_{x^+,N} = S_{x^-,N}$ for N fixed by σ. This gives the stated form with $A_{xy} = S_{x^+,y^+} - S_{x^+,y^-}$, a symmetric 2×2 matrix.

On the span of $e_{x^+} - e_{x^-}$, $x \in \{a_k, b_{k+1}\}$, S acts by A, T by $T_f = \tau \operatorname{diag}(1, i)$ with $\tau = e(k/4 - 1/24)$ (the lowest weights are $k/4 - k$ and $(k+1)/4 - k$), and C, which commutes with σ, by $\operatorname{diag}(c_a, c_b)$, $c_x = \pm 1$. So $A^2 = \operatorname{diag}(c_a, c_b)$ and $(AT_f)^3 = A^2$. Write $A = \left(\begin{smallmatrix} p & q \\ q & r \end{smallmatrix}\right)$. If $q = 0$ then $(p\tau)^3 = p^2$ gives $p = \tau^{-3} = e(1/8 - 3k/4)$, whose square is $\pm i$, not c_a. So $q \neq 0$, hence $r = -p$ and $c_a = c_b =: c$. Put $B = \left(\begin{smallmatrix} p & iq \\ q & -ip \end{smallmatrix}\right)$, so $AT_f = \tau B$, $\operatorname{tr} B = (1 - i)p$ and $\det B = -ic$. Since B is not scalar and B^3 is, Cayley–Hamilton gives $(\operatorname{tr} B)^2 = \det B$, that is $p^2 = c/2$, and then $B^3 = -\operatorname{tr} B \det B$. The equation $\tau^3 B^3 = c$ becomes $p = \left((1+i)\tau^3\right)^{-1} = e(-3k/4)/\sqrt{2} = i^k/\sqrt{2}$. Therefore $c = 2p^2 = (-1)^k$ and $q = \pm i^k/\sqrt{2}$. The sign of q changes when b_{k+1}^+ and b_{k+1}^- are relabelled. The statement on contragredience is $c = (-1)^k$. □

Integrality of the fusion coefficients does not fix A. Replacing A by $-A$ also gives non-negative integer Verlinde coefficients, and it is $(ST)^3 = S^2$ that excludes it. The script `fixed_block.py` solves the two relations symbolically for $k \leq 6$ and checks for $k \leq 12$ that the resulting S is symmetric and unitary, with $S^2 = C$, $(ST)^3 = S^2$, non-negative integer fusion coefficients, and equal to Flohr's matrix.

7 Representation theory through the doubles

This section identifies the doubles of the fusion categories $\mathcal{N}$, $\mathcal{N}_+$ and $\mathcal{D} = (\operatorname{Rep}\widehat{W}_k)_A$ in terms of $\operatorname{Rep} W_k$ and $\operatorname{Rep}\widehat{W}_k$ (Theorem 7.1). It gives the modular data of the double of the near-group category in closed form (Proposition 7.2). It also shows that for every square $k = \nu^2$, $\operatorname{Rep}\widehat{W}_k$

times an inverse semion is the double of a twisted category of Haagerup–Izumi type for $\mathbb{Z}_{2\nu}$ (Theorem 7.5).

The reason for passing to doubles is the following. The categories $\operatorname{Rep} W_k$ and $\operatorname{Rep} \widehat{W}_k$ are not doubles of fusion categories, but they are close to doubles, and the precise relation explains their modular data. The mechanism is general. If B is a connected étale algebra in a modular category $\mathcal{B}$, then the double of the category of B-modules is the product of $\mathcal{B}$ with the reverse of the category of local B-modules. For $A' = \mathbf{1} \oplus X$ in $\operatorname{Rep} W_k$ the local modules are the lattice modules, which form a pointed category $\mathcal{P}$. So $Z(\mathcal{N})$ is the product of $\operatorname{Rep} W_k$ with a pointed category, and $Z(\mathcal{N}_+)$ is obtained from it by the description of the centre of the trivial component of a graded category. The same argument for $A = \mathbf{1} \oplus M$ in $\operatorname{Rep} \widehat{W}_k$ gives the double of $\mathcal{D}$. Read in the other direction, these equivalences express $\operatorname{Rep} W_k$ and $\operatorname{Rep} \widehat{W}_k$ as centralisers of pointed subcategories in doubles. Theorem 7.1 collects these statements, and Proposition 7.2 turns the description of $Z(\mathcal{N}_+)$ into closed formulas for S and T.

Write $\mathcal{P} = \operatorname{Rep} U$ for the local A'-modules, that is, the lattice modules M_s, $s \in \mathbb{Z}_{2n}$, with twist $e(s^2/4n - s/2)$ and dimension $(-1)^s$ [18, §4.8] (the computation there does not use the parity of n). Let $L = \mathbb{Z}\beta$, $(\beta,\beta) = 2k$, and $V_L = \widehat{W}_k \oplus M$ as in Corollary 4.8, with $A = \mathbf{1} \oplus M$ and $\mathcal{D} = (\operatorname{Rep} \widehat{W}_k)_A$.

Theorem 7.1. *Let $n = 4k$.*

1. *$Z(\mathcal{N}) \simeq \mathcal{C} \boxtimes \mathcal{P}^{\mathrm{rev}}$.*
2. *The object $b = J \boxtimes M_n$ generates the Tannakian subcategory $\mathcal{E} \simeq \operatorname{Rep} \mathbb{Z}_2$ of $Z(\mathcal{N})$ attached to the grading $\mathcal{N} = \mathcal{N}_+ \oplus \mathcal{N}_-$, and*

$$Z(\mathcal{N}_+) \simeq (\mathcal{E}')_{\mathbb{Z}_2}, \qquad \mathcal{E}' = \mathcal{C}' \boxtimes \mathcal{P}_{\mathrm{even}} \ \oplus \ \mathcal{C}'' \boxtimes \mathcal{P}_{\mathrm{odd}},$$

   where the de-equivariantisation is by b, whose action on the simple objects is free. The rank of $Z(\mathcal{N}_+)$ is $n(n+3)$.
3. *A is connected étale in $\operatorname{Rep} \widehat{W}_k$, and $\mathcal{D} \simeq (\mathcal{N}_+)_{\mathbb{Z}_2}$ is the de-equivariantisation of the near-group category $\mathcal{N}_+$ by its order-two subgroup $\langle M_n \rangle$. It is a fusion category of rank $2k+2$ with invertibles $\mathbb{Z}_{2k}$ and two simples of dimension $\frac{1}{2}\operatorname{FPdim} R$.*
4. *$Z(\mathcal{D}) \simeq \operatorname{Rep} \widehat{W}_k \boxtimes \mathcal{C}(\mathbb{Z}_{2k}, q)^{\mathrm{rev}}$, $q(\lambda) = \lambda^2/4k$. Thus $\operatorname{Rep} \widehat{W}_k$ is the centraliser of a pointed subcategory $\mathcal{C}(\mathbb{Z}_{2k}, q)^{\mathrm{rev}}$ in the double of $\mathcal{D}$.*
5. *If k is odd, $\mathcal{C}(\mathbb{Z}_{2k}, q) \simeq \mathcal{C}(\mathbb{Z}_2, \frac{k}{4}) \boxtimes \mathcal{C}(\mathbb{Z}_k, \mu^2/k)$, the first factor a semion category, so $Z(\mathcal{D}) \simeq \operatorname{Rep} \widehat{W}_k \boxtimes \mathcal{C}(\mathbb{Z}_2, \frac{k}{4})^{\mathrm{rev}} \boxtimes \mathcal{C}(\mathbb{Z}_k, \mu^2/k)^{\mathrm{rev}}$.*

Proof. (1) A' is connected étale in the modular category $\mathcal{C}$ (Theorem 5.1), and $\mathcal{C}^{\mathrm{loc}}_{A'} = \mathcal{P}$. Apply [10, Cor. 3.30].

(2) Under (1), $x \boxtimes P$ corresponds to $F(x) \otimes_{A'} P$ with its half-braiding. For b this is $M_n \otimes M_n = \mathbf{1}$, with half-braiding on $N \in \mathcal{N}_\pm$ equal to the monodromy of J with the underlying object, that is ± 1 (Theorem 5.1(2)). So b is the grading character. It is a boson, since $\theta_J = 1$, $\theta_{M_n} = e(n/4 - n/2) = 1$, and $b \otimes b = \mathbf{1}$. By [24], $Z(\mathcal{N}_+) \simeq (\mathcal{E}')_{\mathbb{Z}_2}$. The monodromy of $x \boxtimes M_s$ with b is $\varepsilon_x (-1)^s$, which gives $\mathcal{E}'$. Since $b \otimes (x \boxtimes M_s) = Jx \boxtimes M_{s+n}$ and $s + n \neq s$, the action is free, and $\operatorname{rank} \mathcal{E}' = (n+4)n + (n+2)n$ (Theorem 6.4).

(3) V_L is a simple vertex operator algebra extending W_k (Proposition 3.1), so $\tilde{A} = W_k \oplus X \oplus J \oplus Y$ is a simple commutative algebra in $\mathcal{C}$, and A is one in $\operatorname{Rep} \widehat{W}_k$. Both are connected étale by [7, Thm. 2.26] and [35, Thm. 6.10], as in [18, Lemma 6.1]. The underlying objects of $\tilde{A}$ lie in $\mathcal{C}'$, in which J is transparent. Condensation in stages [10, Prop. 3.16] in $\mathcal{C}'$ gives

$$(\mathcal{C}')_{\tilde{A}} \simeq \big((\mathcal{C}')_{\mathbf{1} \oplus J}\big)_A \simeq (\operatorname{Rep} \widehat{W}_k)_A = \mathcal{D},$$

since every $(\mathbf{1}\oplus J)$-module in $\mathcal{C}'$ is local, and

$$(\mathcal{C}')_{\tilde{A}} \simeq ((\mathcal{C}')_{A'})_{\mathbf{1}\oplus M_n} = (\mathcal{N}_+)_{\mathbf{1}\oplus M_n},$$

since $F(J)=M_n$. The second is the de-equivariantisation by $\langle M_n\rangle$. Its invertibles are $\mathbb{Z}_n/\langle n/2\rangle \cong \mathbb{Z}_{2k}$. R is fixed by M_n and, as $H^2(\mathbb{Z}_2,\mathbb{C}^\times)=0$, splits into two simples.

(4) $\operatorname{Rep}\widehat{W}_k$ is modular (Proposition 4.7) and its local A-modules are $\operatorname{Rep} V_L$. For $\omega = \frac{1}{4k}\beta(-1)^2+\beta(-2)$ the module $V_{L+\lambda\beta/2k}$ has twist $e(\lambda^2/4k-\lambda)=e(\lambda^2/4k)$ and the lattice braiding, so $\operatorname{Rep} V_L \simeq \mathcal{C}(\mathbb{Z}_{2k},q)$. The deformation by $\lambda D^2 e^{2\beta}$ does not change this. It changes neither the vertex operators nor L_{-1} nor L_0 [18, §4.1]. The spaces of intertwining operators among the lattice modules, which are defined by the vertex operators, L_{-1} and L_0, are therefore unchanged, and so are the associativity and braiding isomorphisms obtained from their products and analytic continuation. The twist $e^{2\pi i L_0}$ is unchanged because L_0 is. Apply [10, Cor. 3.30].

(5) For odd k, $\mathbb{Z}_{2k}=\langle k\rangle\times\langle 2\rangle$. The two subgroups are orthogonal for the bicharacter $e(\lambda\lambda'/2k)$, $q(k)=k/4$ and $q(2\mu)=\mu^2/k$. □

Since the modular data of $\mathcal{C}$ is the theta modular data (Theorem 6.4), (2) gives the modular data of $Z(\mathcal{N}_+)$ in closed form. Put $u=(2n)^{-1/2}$, $v=(2m)^{-1/2}$, $e(x)=e^{2\pi i x}$, and label the simples of $Z(\mathcal{N}_+)$ by the classes $[x,s]$ of (2). Choose representatives

$$\begin{aligned} a_g &= [W_k, 2g], \quad b_g = [X, 2g] && (g\in\mathbb{Z}_n),\\ c_{a,s} &= [A_a, s] && (s\equiv a \bmod 2),\\ d_{b,s} &= [B_b, s] && (s\equiv b \bmod 2). \end{aligned}$$

with $[A_a,s]=[A_{n-a},s+n]$ and $[B_b,s]=[B_{m-b},s+n]$. The classes of J and Y are $a_{g+n/2}$ and $b_{g+n/2}$. Their numbers are n, n, $n(n-1)/2$, $n(n+3)/2$ and their Frobenius–Perron dimensions 1, δ, $\delta+1$, $\delta-1$.

Proposition 7.2. *The modular data of $Z(\mathcal{N}_+)$, $n=4k$, is*

$$\theta_{[x,s]} = e\Big(h_x - \tfrac{s^2}{4n} + \tfrac{s}{2}\Big), \qquad S_{[x,s],[y,t]} = 2u\, S^{\theta}_{xy}\,(-1)^{s+t} e\big(\tfrac{st}{2n}\big),$$

up to the normalization of S, where S^θ is the theta matrix of §6.1. Explicitly, the nonzero blocks are

$$\begin{aligned} S_{a_g,a_{g'}} &= S_{b_g,b_{g'}} = u(u+v)\,e(2gg'/n), & S_{a_g,b_{g'}} &= u(u-v)\,e(2gg'/n),\\ S_{a_g,c_{a,s}} &= S_{b_g,c_{a,s}} = 2u^2 e(gs/n), & S_{a_g,d_{b,s}} &= -S_{b_g,d_{b,s}} = -2uv\,e(gs/n),\\ S_{c_{a,s},c_{a',s'}} &= 4u^2\cos\tfrac{\pi aa'}{n}\,(-1)^{s+s'} e\Big(\tfrac{ss'}{2n}\Big), & S_{d_{b,s},d_{b',s'}} &= 4uv\cos\tfrac{\pi bb'}{m}\,(-1)^{s+s'} e\Big(\tfrac{ss'}{2n}\Big), \end{aligned}$$

and $S_{c,d}=0$. These satisfy $S=S^t$, $SS^*=1$, $S^2=C$ and $(ST)^3=S^2$, with non-negative integral Verlinde coefficients, since they are the modular data of the modular category $Z(\mathcal{N}_+)$. The script `zclosed.py` *verifies the last two independently for $k=1,2,3,5$.*

Proof. The entries of S^θ follow from $\Lambda_{a,N}(-1/\tau)=(2N)^{-1/2}\sum_{b \bmod 2N} e(-ab/2N)\Lambda_{b,N}$. For instance $S^\theta_{00}=\frac12(u+v)$, $S^\theta_{0A_a}=(-1)^a u$, $S^\theta_{0B_b}=-(-1)^b v$, $S^\theta_{A_aA_{a'}}=2u\cos(\pi aa'/n)$, $S^\theta_{B_bB_{b'}}=2v\cos(\pi bb'/m)$, $S^\theta_{AB}=0$. For $\mathcal{P}$, $S_{st}=u\,d_s d_t\,e(st/2n)$ with $d_s=(-1)^s$ and $\theta_s=e(s^2/4n-s/2)$. The factor $\mathcal{P}^{\mathrm{rev}}$ inverts θ, and the de-equivariantisation by the freely acting boson b multiplies S by 2 on orbit representatives [10]. □

Remark 7.3. The pointed part of $\mathcal{N}_+$ has trivial associator, so $\mathcal{N}_+$ is untwisted. By Proposition 7.2, the invertible objects M_{2x} of $\mathcal{N}_+$ have central lifts a_x whose twists are $q(x)=e(-x^2/n)$, $x\in\mathbb{Z}_n$. These lifts form a pointed braided category $\mathcal{C}(\mathbb{Z}_n,q)$. Since $q(x)=\beta(x,x)$ for the symmetric

bicharacter $\beta(x,y) = e(-xy/n)$ on $\mathbb{Z}_n$, the category $\mathcal{C}(\mathbb{Z}_n, q)$ is braided equivalent to $\mathrm{Vec}_{\mathbb{Z}_n}$ with trivial associator and braiding β. The forgetful functor to $\mathcal{N}_+$ is monoidal and bijective on the invertible simples, so it identifies the pointed part of $\mathcal{N}_+$ with $\mathrm{Vec}_{\mathbb{Z}_n}$ with trivial associator. The same holds after Galois conjugation. Central lifts of this form also appear for general near-group categories of type $G + |G|$ in [48, Cor. 3.12]. The VOA construction above determines the quadratic form q explicitly.

The four types correspond to the types $a_g, b_g, c_{g,h}, d_{h,\gamma}$ of the double of $G + |G|$ in [16, §4], with the same dimensions and, for odd $|G|$ there, the same numbers. The difference is in the d-labels. For odd $|G|$ the second index γ runs over a metric group of order $|G| + 4$ [16, Conj. 2], whereas here $d_{b,s}$ pairs the theta index $b \in \mathbb{Z}_{2m}/\{\pm 1\}$, $0 < b < m$, with $s \in \mathbb{Z}_{2n}$ of the same parity, modulo $(b, s) \sim (m - b, s + n)$. For odd k the nondegenerate pointed subcategory on $8\mathbb{Z}/2n\mathbb{Z} \cong \mathbb{Z}_k$ splits off, leaving a factor of rank $4(n + 3)$.

(4) is the precise sense in which the representation category of Flohr's near-group family is determined by a near-group double. By (3), $\mathcal{D}$ is the $\mathbb{Z}_2$-de-equivariantisation of the near-group category $\mathbb{Z}_{4k} + 4k$. Equivalently $\mathcal{N}_+ \simeq \mathcal{D}^{\mathbb{Z}_2}$, so $Z(\mathcal{N}_+)$ is obtained from $Z(\mathcal{D})$ by gauging the dual $\mathbb{Z}_2$-action [24]. The S-matrix of $\operatorname{Rep}\widehat{W}_k$, including its block on the four split fixed-point modules, is given by Theorem 6.6.

7.1 Twisted doubles at square parameters

This subsection asks when $\operatorname{Rep}\widehat{W}_k$ is the double of a fusion category, after a pointed factor is removed. In the first family this happens exactly at the square parameters $n = N^2$, and the fusion category is then a twisted Haagerup–Izumi category for $\mathbb{Z}_N$. The second family behaves in the same way. For every square $k = \nu^2$, $\operatorname{Rep}\widehat{W}_k$ times an inverse semion is the double of a category $\mathcal{F}_\nu$ with Haagerup–Izumi-type fusion rules for $\mathbb{Z}_{2\nu}$. It is obtained from $\mathcal{D}$ by a further de-equivariantisation by a subgroup of order ν. Section 7.2 gives the construction for every square divisor of k; the present endpoint has Haagerup–Izumi-type fusion rules. Its pointed part has a nontrivial associator, of exact order 2ν, and none of its non-invertible simple objects is self-dual. Since N is odd in the first family and 2ν is even here, the two families together cover cyclic groups of both parities. We first explain why a square is necessary. We then record a statement on duals in $\mathcal{D}$ for odd k (Lemma 7.4), which is not used in the theorem. The theorem follows. Its part on fusion rules is proved from the description of the Morita duals in §9, which does not depend on this subsection.

$\operatorname{Rep}\widehat{W}_k$ is never a Drinfeld centre. By Theorem 7.1(4) its Witt class is that of $\mathcal{C}(\mathbb{Z}_{2k}, q)$, $q(\lambda) = \lambda^2/4k$, and a Lagrangian subgroup of the cyclic group $\mathbb{Z}_{2k}$ would be generated by r with $r^2 = 2k$, but $q(r) = \frac{1}{2}$.

Multiplying by an inverse semion can remove this obstruction only when k is a square. The Witt class of $\operatorname{Rep}\widehat{W}_k \boxtimes \mathcal{C}(\mathbb{Z}_2, \frac{1}{4})^{\mathrm{rev}}$ is that of the metric group $\mathbb{Z}_{2k} \times \mathbb{Z}_2$ with form $(\lambda, \varepsilon) \mapsto \lambda^2/4k - \varepsilon^2/4$. A modular category is a Drinfeld centre exactly when its Witt class is trivial. The class of a pointed modular category is trivial exactly when its metric group has a Lagrangian subgroup, that is an isotropic subgroup L with $|L|^2$ equal to the order of the group [10]. Here the order is $4k$, so k must be a square. Theorem 7.5 gives the converse. Thus $\operatorname{Rep}\widehat{W}_k \boxtimes \mathcal{C}(\mathbb{Z}_2, \frac{1}{4})^{\mathrm{rev}}$ is a Drinfeld centre exactly when k is a square.

For odd k the Witt class of $\operatorname{Rep}\widehat{W}_k$ is $[\mathcal{C}(\mathbb{Z}_2, \frac{k}{4})][\mathcal{C}(\mathbb{Z}_k, \mu^2/k)]$ (Theorem 7.1(5)), and the second factor is trivial exactly when k is a square. This is the parallel of the criterion of [18, §6.9] for the Haagerup family.

Lemma 7.4. *Let k be odd. Then the two non-invertible simples x, y of $\mathcal{D}$ (Theorem 7.1(3)) satisfy $x^* = y$.*

Proof. Let $F\colon \operatorname{Rep}\widehat{W}_k \to \mathcal{D}$ be the free-module functor. It is a tensor functor, with adjunction

$$\operatorname{Hom}_{\mathcal{D}}(F(N), L) \cong \operatorname{Hom}(N, U(L)).$$

The invertible A-modules are the lattice modules g_λ, $\lambda \in \mathbb{Z}_{2k}$, of $V_{\sqrt{2k}\mathbb{Z}}$. The charge-parity automorphism of $V_{\sqrt{2k}\mathbb{Z}}$ fixes ω, restricts to the involution σ of $\widehat{W}_k$ (the proof of Theorem 6.6), and fixes each lattice module up to isomorphism. Since σ exchanges a_k^+ and a_k^-, and $U(g_k)$ has character $\Lambda_{k,k}$, we get $U(g_k) = a_k^+ \oplus a_k^-$. The other $U(g_\lambda)$ do not contain $a_k^\pm$. Hence $F(a_k^\pm)$ contains g_k once and no other invertible. Comparing dimensions,

$$\operatorname{FPdim} a_k^\pm = \tfrac{1}{2}(\delta + 1) = 1 + \operatorname{FPdim} x,$$

we get $F(a_k^\pm) = g_k \oplus z^\pm$, where $z^\pm \in \{x, y\}$.

By adjunction $\dim \operatorname{End} F(a_k^+) = 1 + N_{a_k^+ M}^{a_k^+}$ and $\dim \operatorname{Hom}(F(a_k^+), F(a_k^-)) = N_{a_k^- M}^{a_k^+}$. The first is 2, so $N_{a_k^+ M}^{a_k^+} = 1$. By Verlinde's formula and Theorem 6.6,

$$N_{a_k^+ M}^{a_k^+} - N_{a_k^- M}^{a_k^+} = \sum_{y \in \{a_k, b_{k+1}\}} |A_{a_k y}|^2 \frac{S_{M,y^+}}{S_{\mathbf{1},y^+}} = \tfrac{1}{2}(1-1) = 0,$$

since $S_{M,y^+}/S_{\mathbf{1},y^+}$ is $S_{X,y}/S_{W_k,y}$, which is 1 on $A_{n/2}$ and -1 on $B_{m/2}$ (proof of Theorem 6.4(b)). So $\dim \operatorname{Hom}(F(a_k^+), F(a_k^-)) = 1$ and $z^+ \neq z^-$. For odd k, $(a_k^+)^* = a_k^-$ (Theorem 6.6), so $z^- = (z^+)^*$. Hence $x^* = y$. □

The de-equivariantisation, fusion rules and pointed class order in parts (2)–(3) below also appear in Schopieray–Wang–Yu [48, Cor. 3.12, Thm. 5.5, Prop. 5.6], with $M = 2\nu$. Their notation is $K_0(\mathcal{F}_\nu) = H_1^{\alpha_\nu}(\mathbb{Z}_{2\nu})$. We give direct proofs for the categories constructed here, identify the double in terms of Flohr's algebra, and compute an explicit cocycle for the chosen half-braiding.

Theorem 7.5. *Let $\nu \geq 1$ and $k = \nu^2$, $n = 4\nu^2$. Let $H \subset \mathbb{Z}_n$ be the subgroup of order 2ν of the invertibles of $\mathcal{N}_+$, and put $\mathcal{F}_\nu = (\mathcal{N}_+)_H$.*

1. *$\mathcal{F}_\nu \simeq \mathcal{D}_{\mathbb{Z}_\nu}$, and $Z(\mathcal{F}_\nu) \simeq \operatorname{Rep}\widehat{W}_{\nu^2} \boxtimes \mathcal{C}(\mathbb{Z}_2, \frac{1}{4})^{\mathrm{rev}}$, the second factor a semion category with twist $-i$.*
2. *$\mathcal{F}_\nu$ has simple objects $\alpha_g, \alpha_g\rho$, $g \in \mathbb{Z}_{2\nu}$, with*

$$\rho\alpha_g = \alpha_{-g}\rho, \qquad \rho^* = \alpha_a\rho, \qquad \rho^2 = \alpha_a \oplus \bigoplus_{g \in \mathbb{Z}_{2\nu}} \alpha_g\rho, \qquad a = \nu,$$

   and $\dim \rho = \nu - \sqrt{\nu^2+1}$, $\operatorname{FPdim}\rho = \nu + \sqrt{\nu^2+1}$.
3. *The pointed part of $\mathcal{F}_\nu$ has associator $\Omega(a,b,c) = e\big(\bar{c}\lfloor(\bar{a}+\bar{b})/2\nu\rfloor/2\nu\big)$, with representatives $0 \leq \bar{a} < 2\nu$. Its class has exact order 2ν in $H^3(\mathbb{Z}_{2\nu}, \mathbb{C}^\times)$.*

Proof. (1) The central lifts of the invertibles M_{2x} of $\mathcal{N}_+$ have twist $e(-x^2/n)$ (Remark 7.3). On $H = 2\nu\mathbb{Z}_n$ this is $e(-j^2) = 1$, and the associated bicharacter $e(-2xy/n)$ is trivial on H. So H is Tannakian and $Z((\mathcal{N}_+)_H) \simeq (H')_H$ [10, Thm. 3.20]. H contains the element of order two, the image of J, so by Theorem 7.1(3) and condensation in stages $\mathcal{F}_\nu \simeq \mathcal{D}_{\mathbb{Z}_\nu}$. In $Z(\mathcal{D}) \simeq \operatorname{Rep}\widehat{W}_k \boxtimes \mathcal{C}(\mathbb{Z}_{2k}, q)^{\mathrm{rev}}$ the subgroup $\mathbb{Z}_\nu = 2\nu\mathbb{Z}_{2k}$ has $q(2\nu j) = j^2$, its centraliser in $\mathbb{Z}_{2k}$ is $\nu\mathbb{Z}_{2k}$, and $\nu\mathbb{Z}_{2k}/2\nu\mathbb{Z}_{2k} \cong \mathbb{Z}_2$ with $q(\nu) = \frac{1}{4}$. Hence $Z(\mathcal{F}_\nu) \simeq \operatorname{Rep}\widehat{W}_k \boxtimes \mathcal{C}(\mathbb{Z}_2, \frac{1}{4})^{\mathrm{rev}}$. Nothing in this argument depends on the parity of ν.

(2) We use the Morita duals of §9. Neither Theorem 9.7 nor Proposition 9.10 uses the present theorem, apart from the definition of $\mathcal{F}_\nu$, so there is no circularity. By Lemma 9.2(1), $\mathcal{N}_+$

satisfies (H1)–(H3). Take $h = 2\nu$. Then $h' = n/h = 2\nu$, $s = \gcd(h, h') = 2\nu$, $r = n/s^2 = 1$ and $\epsilon_h = s/2 = \nu$. Since $h^2 = n$, Proposition 9.10 identifies $\mathcal{F}_\nu = \mathcal{Q}_{2\nu}$ with the adjoint subcategory of $\mathcal{D}_{2\nu}$. By Corollary 9.8 its simple objects are the invertibles $I_{g,l}$ with $g \equiv l \pmod{2\nu}$, that is $I_{g,g}$, $g \in \mathbb{Z}_{2\nu}$, and the 2ν objects $\Xi_i = \Upsilon_{i,i+\nu}$, $i \in \mathbb{Z}_{2\nu}$. Put

$$\alpha_g = I_{g,g}, \qquad \rho = \Xi_0 = \Upsilon_{0,\nu}.$$

We read off the fusion rules from Theorem 9.7.

- By (10), $\alpha_g\alpha_{g'} = \alpha_{g+g'}$, $\alpha_g\rho = \Upsilon_{-g,\nu-g} = \Xi_{-g}$ and $\rho\alpha_g = \Upsilon_{g,\nu+g} = \Xi_g$. So the $\alpha_g\rho$ are the 2ν distinct objects Ξ_i, and $\rho\alpha_g = \alpha_{-g}\rho$.
- By the theorem $\rho^* = \Upsilon_{\nu,0}$. Since $-\nu \equiv \nu \pmod{2\nu}$ this is $\Xi_\nu = \alpha_\nu\rho$.
- In (11) take $x = u = 0$ and $y = v = \nu$. The invertible summands are the $I_{g,l}$ with $g \equiv u - y = -\nu$ and $l \equiv v - x = \nu$, that is the single object $I_{\nu,\nu} = \alpha_\nu$. The other summands are $r = 1$ copies of $\Upsilon_{a,a+\epsilon_h+y+v-x-u} = \Upsilon_{a,a+3\nu} = \Xi_a$, $a \in \mathbb{Z}_{2\nu}$. Hence $\rho^2 = \alpha_\nu \oplus \bigoplus_g \alpha_g\rho$.

For the dimensions, Theorem 9.7 gives $\operatorname{FPdim}\rho = d_+/s = d_+/2\nu$. With $n = 4\nu^2$ and $m = 4\nu^2+4$ we have $\sqrt{nm} = 4\nu\sqrt{\nu^2+1}$, so $d_\pm = 2\nu^2 \pm 2\nu\sqrt{\nu^2+1}$ and $\operatorname{FPdim}\rho = \nu + \sqrt{\nu^2+1}$. For the categorical dimension let $F\colon \mathcal{N}_+ \to \mathcal{F}_\nu$ be the free-module functor. The underlying object of each $\Upsilon_{x,y}$ is $R \otimes V$ with $\dim V = h/s = 1$, so it is R. Hence $\operatorname{Hom}(F(R), \Xi_i) = \operatorname{Hom}_{\mathcal{N}_+}(R, R)$ is one-dimensional for each i. As $\operatorname{FPdim} F(R) = d_+ = 2\nu \operatorname{FPdim}\rho$, this gives $F(R) = \bigoplus_g \alpha_g\rho$. The functor F preserves categorical dimensions, the α_g have dimension 1, and so $2\nu \dim\rho = \dim R = d_-$. Thus $\dim\rho = \nu - \sqrt{\nu^2+1}$.

(3) Put $N = 2\nu$, so $G = \mathbb{Z}_{N^2}$ and $H = N\mathbb{Z}_{N^2}$. The pointed part of $\mathcal{N}_+$ has trivial associator, and the central half-braiding is $\beta(x,y) = e(-xy/N^2)$ (Remark 7.3). For $0 \le a < N$, let $P_a = \alpha_a \mathsf{A}_H$ be the right module with basis $u_{a,r}$ in degree $a + Nr$. Its right action is translation in r, and its induced left action is $b_{Nj}u_{a,r} = e(-ja/N)u_{a,r+j}$. Write $c(a,b) = \lfloor (a+b)/N \rfloor$ and $[a+b]$ for reduction to $\{0, \dots, N-1\}$. The balanced multiplication maps are

$$f_{a,b}(u_{a,r} \otimes_{\mathsf{A}_H} u_{b,t}) = e(-rb/N)u_{[a+b],\, r+t+c(a,b)}.$$

For three factors, the right-associated composition divided by the left-associated composition has scalar $e(c\,c(a,b)/N)$. This is the associator from the left-associated product to the right-associated one, and gives the displayed positive sign in Ω, in the same convention as [18, §6.8]. For a normalised 3-cocycle ω on $\mathbb{Z}_N$, the product $I(\omega) = \prod_{b=0}^{N-1} \omega(1,b,1)$ is unchanged by coboundaries. For $\omega = d\varphi$ it is $\prod_b \varphi(b,1)\varphi(1,b+1)/(\varphi(b+1,1)\varphi(1,b)) = 1$, by reindexing $b \mapsto b+1$ in $\mathbb{Z}_N$. So I is a homomorphism $H^3(\mathbb{Z}_N, \mathbb{C}^\times) \to \mu_N$, and $I(\omega)$ primitive implies that $[\omega]$ has order N. Here $I(\Omega) = e(1/2\nu)$ is primitive. □

For $\nu = 1$ this agrees with the module category of the Lagrangian algebra $\bigoplus_{\lambda \in \nu\mathbb{Z}_{2k}} U(\lambda) \boxtimes s^{\lambda/\nu}$ in $\operatorname{Rep}\widehat{W}_k \boxtimes \mathcal{C}(\mathbb{Z}_2, \frac{1}{4})^{\mathrm{rev}}$, s the semion, computed from the modular data (`twisted_double.py`). For $\nu \le 4$ the script `hi_even_check.py` confirms the rank $2(2k+5)$ and the twists of $Z(\mathcal{F}_\nu)$, the fusion rules of part (2), and the value of $I(\Omega)$. Since $(\alpha_c\rho)^* = \rho^*\alpha_{-c} = \alpha_{\nu+c}\rho$, no simple $\alpha_c\rho$ is self-dual.

For odd ν the relation $\rho^* = \alpha_\nu\rho$ can also be obtained from Lemma 7.4, by the argument of [18, §6.7]. That argument gives $\rho^* = \alpha_a\rho$ with $2a = 0$, and the lemma excludes $a = 0$. The proof above treats both parities at once.

Theorem 7.5 is the counterpart, for the near-group family, of [18, Thm. 1.2(iv)]. There $n = N^2$ odd, $\operatorname{Rep} V_{N^2+2} \simeq \operatorname{Sem} \boxtimes Z(\mathcal{F}_N)$ with $\mathcal{F}_N$ of Haagerup–Izumi type for $\mathbb{Z}_N$ and pointed class of order N. Here $\operatorname{Rep}\widehat{W}_{\nu^2} \boxtimes \operatorname{Sem}^{\mathrm{rev}} \simeq Z(\mathcal{F}_\nu)$ with $\mathcal{F}_\nu$ of Haagerup–Izumi type, with $\rho^* = \alpha_\nu\rho$, for the even group $\mathbb{Z}_{2\nu}$ and pointed class of order 2ν. Here ν is any positive integer. Thus the two Flohr families give VOA realisations of these doubles for cyclic groups of both parities. For $\nu = 1$ ($c = -23$), $\mathcal{F}_1$ has rank 4, invertibles $\mathbb{Z}_2$ and $\operatorname{FPdim}\rho = 1 + \sqrt{2}$.

7.2 Square divisors and Witt classes

For $k \in \mathbb{Z}_{>0}$, write

$$\mathcal{B}_k = \operatorname{Rep} \widehat{W}_k, \qquad \mathcal{P}_j = \mathcal{C}(\mathbb{Z}_{2j}, q_j), \qquad q_j(x) = \frac{x^2}{4j} \pmod{\mathbb{Z}}.$$

We give $\mathcal{P}_j$ its ribbon structure with dimensions 1 and twists $e(q_j(x))$; thus $\mathcal{P}_1$ is the semion category. For $h^2 \mid n$, let $H_h = (n/h)\mathbb{Z}_n$ be the subgroup of order h in the invertibles of $\mathcal{N}_+$, with the central lifts of Remark 7.3, and let $\mathcal{Q}_h = (\mathcal{N}_+)_{H_h}$ be the category of modules over its central regular algebra. This is the notation used in Sections 9 and 11. We use the spherical structure induced from $\mathcal{N}_+$.

Theorem 7.6. *Let k, s be positive integers with $s^2 \mid k$, and put $r = k/s^2$. Then*

$$\mathcal{Q}_{2s} \simeq \mathcal{D}_{\mathbb{Z}_s}, \qquad Z(\mathcal{Q}_{2s}) \simeq \mathcal{B}_k \boxtimes \mathcal{P}_r^{\mathrm{rev}}.$$

The second equivalence preserves the ribbon structures, with the induced spherical structure on $\mathcal{Q}_{2s}$. For $s = 1$ it is Theorem 7.1(4); for $k = \nu^2$ and $s = \nu$ it gives the centre in Theorem 7.5(1).

Proof. The generator of H_{2s} is $2k/s$. The braiding of its central lifts, given in Remark 7.3, restricts to

$$\beta((2k/s)a, (2k/s)b) = e(-kab/s^2) = 1.$$

Their dimensions and twists are 1, so they form a Tannakian subcategory. Since H_{2s} contains H_2, condensation in stages and Theorem 7.1(3) identify its quotient with $\mathcal{D}_{\mathbb{Z}_s}$ [10, Prop. 3.16].

In the centre $Z(\mathcal{D}) \simeq \mathcal{B}_k \boxtimes \mathcal{P}_k^{\mathrm{rev}}$, the remaining central subgroup has order s. The category $\mathcal{B}_k$ has no nontrivial invertible objects: by Table 1, every other simple has Frobenius–Perron dimension at least $d_+/2 > 1$. Consequently this subgroup is the unique subgroup of order s in the pointed factor, namely

$$H = \langle 2k/s \rangle \subset \mathbb{Z}_{2k}.$$

The monodromy in $\mathcal{P}_k^{\mathrm{rev}}$ is $e(-xy/(2k))$. It follows that

$$H^\perp = s\mathbb{Z}_{2k}, \qquad H^\perp/H \cong \mathbb{Z}_{2r}, \qquad -q_k(sj) = -\frac{j^2}{4r} \pmod{\mathbb{Z}}.$$

The local modules in the pointed factor therefore form exactly $\mathcal{P}_r^{\mathrm{rev}}$. The centre theorem for a central algebra [10, Thm. 3.20] now gives the displayed braided equivalence.

For the ribbon assertion, all algebras condensed here are regular algebras of dimension-one bosons, with trivial twist. Their module categories carry the induced spherical structure; the module trace is the original trace divided by the dimension of the algebra. Local-module twists are inherited from the ambient ribbon category, as in [34, Thms. 1.17–1.18, Rem. 1.19]. More explicitly, the equivalence in [10, Thm. 3.20] preserves the underlying module over the central regular algebra A_{2s}. On both sides the dimension of a module M is

$$\frac{\dim_{\mathcal{N}_+} U(M)}{2s}.$$

The two induced pivotal structures therefore have the same dimensions on every simple object. Their ratio is a monoidal automorphism of the identity, whose scalar on a simple object is the ratio of these nonzero dimensions; hence the pivotal structures agree. Since the equivalence is braided, it also preserves the twists. Starting with the ribbon factorization of Theorem 7.1(4), the pointed computation above is therefore a ribbon computation throughout. □

Corollary 7.7. *For positive integers k, j, the following conditions are equivalent:*

1. *$\mathcal{B}_k \boxtimes \mathcal{P}_j^{\mathrm{rev}}$ is braided equivalent to the Drinfeld centre of a fusion category;*
2. *$[\mathcal{B}_k] = [\mathcal{B}_j]$ in the Witt group of nondegenerate braided fusion categories;*
3. *kj is a square.*

Thus the squarefree part of k determines, and is determined by, the Witt class of $\mathcal{B}_k$ within this family.

Proof. Theorem 7.1(4) gives $[\mathcal{B}_k] = [\mathcal{P}_k]$ for every k. A nondegenerate braided fusion category is Witt trivial precisely when it is a Drinfeld centre [10, Prop. 5.8], so (1) and (2) are equivalent. A pointed modular category is Witt trivial precisely when its metric group admits a Lagrangian subgroup [10, §5.3, Prop. 5.17]. For $\mathcal{P}_k \boxtimes \mathcal{P}_j^{\mathrm{rev}}$ this group has order $4kj$. A Lagrangian subgroup has squared order $4kj$, proving (2)⇒(3).

Conversely, if kj is a square, write $k = au^2$ and $j = av^2$ with the same squarefree positive integer a. In $\mathcal{P}_k$, the subgroup $\langle 2au \rangle$ has order u and is isotropic, since $q_k(2au) = a$. Its orthogonal subgroup is $u\mathbb{Z}_{2k}$, and the quotient has quadratic form

$$(\mathbb{Z}_{2a},\ t \longmapsto t^2/(4a)).$$

Thus condensation identifies its local category with $\mathcal{P}_a$. The same argument reduces $\mathcal{P}_j$ to $\mathcal{P}_a$. Condensation preserves the Witt class [10, Prop. 5.4], whence $[\mathcal{P}_k] = [\mathcal{P}_j]$ and (2) follows.

Equivalently, an explicit Lagrangian subgroup of $\mathbb{Z}_{2k} \times \mathbb{Z}_{2j}$ is

$$\{(ub, vc) : b \in \mathbb{Z}_{2au},\ c \in \mathbb{Z}_{2av},\ b \equiv c \pmod{2a}\}.$$

Its order is $2auv = 2\sqrt{kj}$ and its quadratic form is $(b^2 - c^2)/(4a)$, which is integral under the displayed congruence. This also verifies sufficiency directly, including the 2-primary part and $k = 1$ or $j = 1$. □

7.3 The positive Galois orbit of the centre

Fix $k \in \mathbb{Z}_{>0}$. We determine the positive Galois conjugates of the modular data in Proposition 7.2, up to a simultaneous relabelling preserving the unit. Throughout this subsection,

$$n = 4k, \qquad m = n + 4, \qquad d_\pm = \frac{n \pm \sqrt{nm}}{2},$$
$$L = \mathrm{lcm}(n, m) = 4k(k+1), \qquad \mathbb{K} = \mathbb{Q}\big(\sqrt{k(k+1)}\big).$$

Let $D_{\mathbb{K}}$ be the fundamental discriminant of $\mathbb{K}$ and write $\chi_{\mathbb{K}}(\ell) = \left(\frac{D_{\mathbb{K}}}{\ell}\right)$ for its quadratic character. Here $D_{\mathbb{K}} \mid L$ and $8 \mid L$. The field $\mathbb{K}$ is quadratic, since $k(k+1)$ lies strictly between k^2 and $(k+1)^2$.

We fix the normalization of Proposition 7.2 by

$$S_{\mathbf{1},\mathbf{1}} = u^2 + uv = \frac{1}{2n} + \frac{1}{2\sqrt{nm}} = \frac{1}{n + d_-^2}, \qquad T = \mathrm{diag}(\theta_X). \tag{2}$$

Thus $T_{\mathbf{1},\mathbf{1}} = 1$: T is the matrix of twists, without a VOA central-charge scalar. All entries belong to $\mathbb{Q}(\zeta_{4L})$. For $\ell \in (\mathbb{Z}/4L\mathbb{Z})^\times$, let $\sigma_\ell(\zeta_{4L}) = \zeta_{4L}^\ell$, and put $(S^{(\ell)}, T^{(\ell)}) = (\sigma_\ell(S), \sigma_\ell(T))$. We call this conjugate *positive* if $\chi_{\mathbb{K}}(\ell) = -1$, equivalently $\sigma_\ell(d_-) = d_+$. In this case

$$S^{(\ell)}_{\mathbf{1},\mathbf{1}} = \frac{1}{n + d_+^2} > 0.$$

Consequently the Galois image in this normalization is already the positively normalized matrix of the centre with its positive spherical structure. There is no additional choice of a square root of the global dimension.

Theorem 7.8. *Let $\ell, \ell' \in (\mathbb{Z}/4L\mathbb{Z})^\times$ define positive conjugates. The following conditions are equivalent.*

1. *The pairs $(S^{(\ell)}, T^{(\ell)})$ and $(S^{(\ell')}, T^{(\ell')})$ agree under a simultaneous permutation of the simple labels preserving the unit.*
2. *The pointed quadratic forms are isometric, and the multisets of twists of the dimension-d_+ simple objects centralizing all invertible objects are equal.*
3. *$\ell'\ell^{-1}$ is a square in $(\mathbb{Z}/L\mathbb{Z})^\times$.*

There are exactly

$$2^{\omega(k(k+1))}$$

positive conjugates of the full centre modular data up to such relabelling, where $\omega(t)$ denotes the number of distinct prime divisors of t.

Proof. For a positive conjugate, the dimensions of the four types of simple objects in Proposition 7.2 are respectively

$$1, \qquad 1+d_+, \qquad 2+d_+, \qquad d_+.$$

They are distinct for every $k \geq 1$, since $d_+ > 1$. The invertible objects are therefore exactly the a_g, $g \in \mathbb{Z}_n$, and their twists give the quadratic form

$$q_\ell(g) = e(-\ell g^2/n). \tag{3}$$

A simultaneous permutation as in (1) preserves dimensions and, by Verlinde's formula, the fusion rules. Its restriction to the invertibles is consequently a group automorphism of $\mathbb{Z}_n$. The forms q_ℓ and $q_{\ell'}$ are isometric exactly when $\ell'\ell^{-1}$ is a square modulo n, because every automorphism of $\mathbb{Z}_n$ is multiplication by a unit.

The dimension-d_+ objects are exactly the d-type objects. From the pointed rows of $S^{(\ell)}$,

$$\frac{S^{(\ell)}_{a_g,d_{b,s}}}{S^{(\ell)}_{a_0,d_{b,s}}} = e(\ell g s/n).$$

Thus $d_{b,s}$ centralizes every invertible precisely when $s \equiv 0 \pmod n$. Using $(b,s) \sim (m-b, s+n)$, this set has the representatives $d_{2y,0}$, $1 \leq y < m/2$. Its twist multiset is

$$\{e(\ell y^2/m) : 1 \leq y < m/2\}. \tag{4}$$

The twists in this multiset of exact order m have exponent set

$$\ell\big((\mathbb{Z}/m\mathbb{Z})^\times\big)^2.$$

Indeed $e(\ell y^2/m)$ has order m precisely when y is a unit modulo m, and the indicated interval contains one representative of each unit modulo sign. Equality of the multisets in (4) therefore implies that $\ell'\ell^{-1}$ is a square modulo m. Conversely, multiplication by a unit modulo m permutes the nonzero, non-endpoint classes of $\mathbb{Z}_m$ modulo sign, so a square change of ℓ preserves the entire multiset, including all multiplicities. Being a square modulo both n and m is equivalent to being a square modulo their least common multiple L, prime power by prime power. This proves (1)⇒(2)⇔(3). The quadratic form in (3), including its twists, is used here; its associated double-braiding bicharacter alone need not retain the same information at the prime 2.

Suppose now that (3) holds. An odd unit is a square modulo 2^a, for $a \geq 3$, exactly when it is 1 mod 8. Since $8 \mid L$, the unit $\ell'\ell^{-1}$, considered modulo $4L$, is a square modulo the 2-part of $4L$ as well as modulo every odd prime power dividing $4L$. The Chinese remainder theorem therefore gives

$$w^2 \equiv \ell'\ell^{-1} \pmod{4L}$$

with $w \in (\mathbb{Z}/4L\mathbb{Z})^\times$. Multiply every charge in Proposition 7.2 by w:

$$a_g \longmapsto a_{wg}, \qquad b_g \longmapsto b_{wg}, \qquad c_{a,s} \longmapsto c_{wa,ws}, \qquad d_{b,s} \longmapsto d_{wb,ws}.$$

In the last two expressions, the theta label is taken modulo sign in $\mathbb{Z}_{2n}$ or $\mathbb{Z}_{2m}$, respectively, and the lattice label modulo $2n$. Multiplication by w cannot send a theta label to either endpoint. It preserves parity and respects $(a,s) \sim (n-a, s+n)$ and $(b,s) \sim (m-b, s+n)$, since w is odd. Hence these formulas define a permutation π_w of the simple labels which fixes the unit.

Every quadratic exponent in T, every Fourier phase in S, and every cosine argument in Proposition 7.2 transforms by w^2 under this permutation. The parity signs and linear parity terms are unchanged. The coefficients outside these phases are rational combinations of $u^2 = 1/(2n)$ and $uv = 1/(2\sqrt{nm})$. Both positive embeddings take the latter to $-1/(2\sqrt{nm})$. Substitution in all the displayed blocks therefore gives

$$S^{(\ell')}_{X,Y} = S^{(\ell)}_{\pi_w(X),\pi_w(Y)}, \qquad T^{(\ell')}_{X,X} = T^{(\ell)}_{\pi_w(X),\pi_w(X)}.$$

This proves (3)⇒(1).

Finally, the square-class group of the units modulo L has order $4 \cdot 2^{\omega(k(k+1))-1} = 2^{\omega(k(k+1))+1}$: the 2-primary factor has order four, and every odd prime factor contributes two. The nontrivial quadratic character $\chi_{\mathbb{K}}$ factors through this group, so precisely half its classes have value -1. Every such class lifts from $(\mathbb{Z}/L\mathbb{Z})^\times$ to $(\mathbb{Z}/4L\mathbb{Z})^\times$. The equivalence already proved gives the stated count. □

Corollary 7.9. *For every $k \geq 1$, the Galois orbit of $\mathcal{N}_+$ contains at least $2^{\omega(k(k+1))}$ pairwise categorically Morita-inequivalent pseudo-unitary fusion categories.*

Proof. Each σ_ℓ extends to an automorphism of $\mathbb{C}$. For a positive embedding, Lemma 9.3 gives a spherical conjugate of $\mathcal{N}_+$ with dimensions 1 and d_+, hence a pseudo-unitary fusion category. These dimensions specify its canonical positive spherical structure [13, Prop. 8.23]. Its centre has the positive modular data used above. Categorically Morita-equivalent fusion categories have braided-equivalent centres [47], and a braided equivalence between pseudo-unitary centres preserves their canonical positive spherical structures. The inequivalent data of Theorem 7.8 therefore give the asserted number of distinct Morita classes. □

8 Vertex operator algebras for the doubles

We realise the doubles of §7 using rank-seven lattice algebras. Theorem 8.2 gives CFT-type realisations of $Z(\mathcal{N})$, $Z(\mathcal{N}_+)$ and $Z(\mathcal{D})$, and Corollary 8.3 treats all square-divisor quotients, including $\mathcal{F}_\nu$. Proposition 8.4 then corrects the spherical structures in the first two realisations by a conformal shift. The categories being realised as ordinary VOA module categories are the Drinfeld centres; the underlying fusion categories arise from algebra modules and central quotients.

Lemma 8.1. *For $m \in \mathbb{N}$ write $2m-1 = x_2^2 + x_3^2 + x_4^2 + x_5^2$ and put*

$$v_m = (1, x_2, x_3, x_4, x_5, 0, 0, 0) \in D_8 \subset E_8.$$

Then v_m is primitive in E_8 with $(v_m, v_m) = 2m$, and $K_m = v_m^\perp \cap E_8$ is an even positive-definite lattice of rank 7 with discriminant form $(\mathbb{Z}_{2m},\, w \mapsto -w^2/4m)$. Hence $\operatorname{Rep} V_{K_m} \simeq \mathcal{C}(\mathbb{Z}_{2m}, w^2/4m)^{\mathrm{rev}}$ as modular categories.

Proof. The coordinate sum $1 + \sum x_i \equiv 1 + \sum x_i^2 = 2m$ is even, so $v_m \in D_8$. If $v_m = jw$ with $w \in E_8$ and $j \geq 2$, then $j = 2$ and w has the entry $\frac{1}{2}$, so all entries of w are half-integers, contradicting the entry 0 of v_m. The orthogonal complement of the primitive sublattice

$\mathbb{Z}v_m \cong \sqrt{2m}\,\mathbb{Z}$ of the unimodular lattice E_8 has discriminant form anti-isometric to that of $\mathbb{Z}v_m$ [42, Prop. 1.6.1], which is $(\mathbb{Z}_{2m}, w^2/4m)$. □

For $m = 1$, $K_1 = E_7$. Let $\mu \in K_n^*/K_n$ be the element of order two.

Theorem 8.2. *Let $n = 4k$. The following vertex operator algebras are simple, self-contragredient, of CFT type, C_2-cofinite and rational, of central charge $8 - 24k$:*

1. *$W_k \otimes V_{K_n}$, with module category braided equivalent to $Z(\mathcal{N})$;*
2. *$\widetilde{U}_n = (W_k \otimes V_{K_n}) \oplus (J \otimes V_{K_n+\mu})$, with module category braided equivalent to $Z(\mathcal{N}_+)$, the double of the near-group category $\mathbb{Z}_{4k} + 4k$;*
3. *$\widehat{W}_k \otimes V_{K_k}$, with module category braided equivalent to $Z(\mathcal{D})$.*

The equivalences in (1) and (2) are braided and compare different spherical structures; in (2) the twists differ by the character $(-1)^s$ (see below). The equivalence in (3) preserves the ribbon structure.

Proof. The local modules M_s of §7 have twist $e(s^2/4n - s/2)$ and dimension $(-1)^s$, so their self-braiding is $e(s^2/4n)$ and $\mathcal{P} \simeq \mathcal{C}(\mathbb{Z}_{2n}, s^2/4n)$ as braided categories. By Lemma 8.1, $\operatorname{Rep} V_{K_n} \simeq \mathcal{P}^{\mathrm{rev}}$ as braided categories, with M_n corresponding to $V_{K_n+\mu}$. The tensor product of strongly rational vertex operator algebras has module category the Deligne product [40], so (1) follows from Theorem 7.1(1), and (3) from Theorem 7.1(4) and Lemma 8.1 with $m = k$.

For (2), $b' = J \otimes V_{K_n+\mu}$ is a simple current of order two with trivial self-braiding, since J is a boson and μ has $-n^2/4n = -k \in \mathbb{Z}$. Its lowest weight is at least $3n/4 = 3k$, and μ has integral norm. So $\mathbf{1} \oplus b'$ is a commutative algebra, and $\widetilde{U}_n$ is a vertex operator algebra extension of CFT type [30]. Its module category is the category of local modules over $\mathbf{1} \oplus b'$ [6, Thm. 3.65], which under the equivalence above is the de-equivariantisation $(\mathcal{E}')_{\mathbb{Z}_2}$ of Theorem 7.1(2), that is $Z(\mathcal{N}_+)$. The remaining properties follow as in the proof of [19, Thm. 31.1], namely C_2-cofiniteness from that of $W_k \otimes V_{K_n}$ [2], and rationality from the semisimplicity of the category of local modules. $\widetilde{U}_n$ is simple, as b' is a simple current with nonzero product $b' \times b' \to W_k \otimes V_{K_n}$, and of CFT type. Its weight-one space is that of $W_k \otimes V_{K_n}$, namely $\mathbf{1} \otimes (V_{K_n})_1$, on which L_1 vanishes. So it is self-contragredient [36]. The same argument applies to (1) and (3). The central charge is $(1 - 24k) + 7$. □

For the equivalences in Theorem 8.2(1)–(2), the dimensions on $\mathcal{P}$ are $(-1)^s$, whereas on $\operatorname{Rep} V_{K_n}$ they are 1. Accordingly, in (2) the twists of $\operatorname{Rep} \widetilde{U}_n$ and of $Z(\mathcal{N}_+)$ (Proposition 7.2) differ by the character $(-1)^s$, and the S-matrices by $(-1)^{s+t}$, under $[x, s] \mapsto (x, s)$. In (3), the local lattice category of Theorem 7.1(4) has dimensions 1 and quadratic form q; the auxiliary lattice has the inverse ribbon data, so this equivalence is ribbon. The comparison in (2) is checked for $k \le 6$ in `voa_zng.py`, which also verifies Lemma 8.1 for $m = 4k$, $k \le 6$.

Corollary 8.3. *Let k, s be positive integers with $s^2 \mid k$, and put $r = k/s^2$. The vertex operator algebra $\widehat{W}_k \otimes V_{K_r}$ is simple, self-contragredient, of CFT type, C_2-cofinite and rational, with central charge $8 - 24k$, and*

$$\operatorname{Rep}(\widehat{W}_k \otimes V_{K_r}) \simeq Z(\mathcal{Q}_{2s})$$

as ribbon categories, with the induced spherical structure on $\mathcal{Q}_{2s}$. In particular, for every $\nu \ge 1$,

$$\operatorname{Rep}(\widehat{W}_{\nu^2} \otimes V_{E_7}) \simeq Z(\mathcal{F}_\nu).$$

Proof. The factors are strongly rational vertex operator algebras of CFT type, as in Theorem 8.2. Their tensor product has the stated properties and its module category is the ribbon Deligne product [40]. Lemma 8.1 identifies the second factor's module category with $\mathcal{P}_r^{\mathrm{rev}}$, including its dimension-one spherical structure and twists. The ribbon equivalence then follows from

Theorem 7.6. Its central charge is $(1-24k)+7 = 8-24k$. For $k=\nu^2$ and $s=\nu$, one has $r=1$, $K_1 = E_7$ and $\mathcal{Q}_{2\nu} = \mathcal{F}_\nu$. □

8.1 The spherical structure and a conformal shift

The preceding constructions already realise the centres themselves, without an additional pointed factor in the resulting module category. For $Z(\mathcal{D})$ and $Z(\mathcal{F}_\nu)$ the equivalences are ribbon, as is the general square-divisor construction. For $Z(\mathcal{N})$ and $Z(\mathcal{N}_+)$, the CFT-type realisations in Theorem 8.2 retain a parity difference in their spherical structures. The conformal-shift method used in [18, §§10–11] removes this difference as well. We give the construction for every even n; the underlying CFT-type VOA for $n \equiv 2 \pmod 4$ is supplied by Theorem 12.2(5).

Proposition 8.4. *Let $n \geq 2$ be even, and let μ be the order-two element of K_n^*/K_n. Choose a representative $\kappa \in K_n^*$ of least norm in this coset. Then*

$$2\kappa \in K_n, \qquad 0 < (\kappa,\kappa) \leq \frac{n}{2}.$$

Put $B_n = V_{n+2} \otimes V_{K_n}$ and

$$\widetilde{U}_n = B_n \oplus (J \otimes V_{K_n+\kappa}).$$

In both VOAs, replace the conformal vector ω by

$$\omega^\kappa = \omega + \mathbf{1} \otimes \kappa(-2)\mathbf{1},$$

and denote the resulting VOAs by B_n^κ and $\widetilde{U}_n^\kappa$. They are simple, regular, self-contragredient and C_2-cofinite, and are nonnegatively integer graded, with central charge

$$8 - 6n - 12(\kappa,\kappa).$$

There are ribbon equivalences

$$\operatorname{Rep} B_n^\kappa \simeq Z(\mathcal{N}), \qquad \operatorname{Rep} \widetilde{U}_n^\kappa \simeq Z(\mathcal{N}_+),$$

for the spherical structures used in Theorems 7.1 and 12.2. Neither shifted VOA is of CFT type: its weight-zero space contains the linearly independent states $\mathbf{1}$ and $\mathbf{1} \otimes e^{2\kappa}$.

Proof. We first bound the norm of the order-two coset. Use the coordinates of v_n in Lemma 8.1. Exactly three of the x_i are odd, since $\sum_{i=2}^5 x_i^2 = 2n-1 \equiv 3 \pmod 4$. Choose signs ϵ_i on these odd coordinates such that

$$S = \sum_{x_i \text{ odd}} \epsilon_i x_i, \qquad |S| \leq \max_{x_i \text{ odd}} |x_i|.$$

For three nonnegative numbers this follows by assigning opposite signs to the largest and the other two; signs of the x_i can be absorbed into the ϵ_i. Define $h_1 = -S/2$, set $h_i = \epsilon_i/2$ on the odd coordinates and $h_i = 0$ on the remaining coordinates $2 \leq i \leq 5$, and take $(h_6, h_7, h_8) = (\eta, 0, 0)$, where $\eta \in \{0,1\}$ makes $w = h + v_n/2$ have even coordinate sum. Then $w \in D_8 \subset E_8$, $h \perp v_n$, and $(w, v_n) = n$. The discriminant gluing of $\mathbb{Z}v_n$ with its orthogonal complement therefore places h in the coset μ; in particular $2h \in K_n$. Moreover

$$(h,h) = \frac{S^2+3}{4} + \eta^2 \leq \frac{n}{2} + \frac{3}{2}.$$

Since $(w,w) = (h,h) + n/2$ is even, (h,h) differs from $-n/2$ by an even integer. As n is even, the displayed upper bound forces $(h,h) \leq n/2$. The least-norm representative κ satisfies the same bound. It is nonzero because $\mu \neq 0$.

The conformal shift has

$$L^\kappa(m) = L(m) - (m+1)\kappa(m), \qquad L^\kappa(0) = L(0) - \kappa(0),$$

and changes the central charge by $-12(\kappa,\kappa)$ [12, Theorem 3.1]. On a lattice ground state of momentum p the shifted weight is

$$\frac{(p,p)}{2} - (\kappa,p) = \frac{(p-\kappa,p-\kappa)}{2} - \frac{(\kappa,\kappa)}{2}.$$

For $p \in K_n$, minimality of κ and $-\kappa + K_n = \kappa + K_n$ make this nonnegative. On $V_{K_n+\kappa}$ the least weight is $-(\kappa,\kappa)/2$. Consequently the second summand of $\widetilde{U}_n^\kappa$ has least weight

$$\frac{3n}{4} - \frac{(\kappa,\kappa)}{2} \geq \frac{n}{2} \geq 1.$$

The grading of the first summand is integral because $\kappa \in K_n^*$. On the second summand, $(\kappa,p) \in \mathbb{Z}$ because $(\kappa,\kappa) \equiv -n/2 \pmod{2\mathbb{Z}}$ and n is even; its grading is therefore integral as well. The completed-square formula proves finite-dimensional homogeneous spaces.

The state-field maps are unchanged, so simplicity and C_2-cofiniteness are unchanged. The unshifted VOAs are regular by Theorem 8.2 and Theorem 12.2(5). Every simple module restricts to a finite sum of tensor products of ordinary V_{n+2}-modules with lattice modules; for the extension these are the two summands of the local simple-current induction. The completed-square formula shows that each remains ordinary after the shift. Since the weak-module category depends only on the state-field maps, the decomposition of an arbitrary weak module into these simple modules still proves regularity.

The shifted lattice VOA is self-contragredient because $2\kappa \in K_n$ [12, Theorem 4.4]; hence so is B_n^κ. The second summand of the extension has no weight-zero vectors. The operator $L^\kappa(1)$ preserves its two summands, so its image from any weight-one vectors in the second summand is zero. It follows that

$$(\widetilde{U}_n^\kappa)_0 / L^\kappa(1)(\widetilde{U}_n^\kappa)_1 = (B_n^\kappa)_0 / L^\kappa(1)(B_n^\kappa)_1.$$

The right side has dimension one by Li's invariant-form theorem [36], because B_n^κ is simple and self-contragredient. The same theorem gives a nonzero invariant form on $\widetilde{U}_n^\kappa$; its radical is an ideal and therefore vanishes. Thus the extension is self-contragredient.

It remains to check the ribbon structure. Both conformal gradings are nonnegative and $\kappa(0)\omega = 0$, so [39, Proposition 2.17] identifies their braided tensor categories. If $\lambda_s \in K_n^*$ has discriminant label $s \in \mathbb{Z}_{2n}$ in Lemma 8.1, then

$$(\kappa,\lambda_s) \equiv -\frac{s}{2} \pmod{\mathbb{Z}}, \qquad \theta_s^\kappa = (-1)^s \theta_s.$$

Thus $\operatorname{Rep} V_{K_n}^\kappa$ has twist $e(-s^2/(4n) + s/2)$, exactly the twist of $\mathcal{P}^{\mathrm{rev}}$ in Theorem 7.1(1). The parity multiplier is 1 on the order-two extension current, since its label is n and n is even. It is also constant on each orbit $s \mapsto s+n$. Hence the same shift corrects the parity discrepancy on every simple local module of the extension. The braided equivalences of Theorem 8.2(1)–(2), or Theorem 12.2(4)–(5), therefore preserve twists after the shift. They are ribbon equivalences. Finally $2\kappa \in K_n$ gives the asserted second weight-zero state. □

9 Module categories, Morita duals and subfactors

This section classifies, for every even n, the indecomposable module categories, the connected separable algebras and the categorically Morita equivalent fusion categories of the near-group category $\mathcal{N}_+$ (Theorem 9.16). The Morita duals are denoted by $\mathcal{D}_h$, $h \mid n$. When $4 \mid n$, the

central quotient $\mathcal{D} = \mathcal{Q}_2$ of §7 is the neutral component of $\mathcal{D}_2$ (Proposition 9.10); these two categories should be distinguished. Every $\mathcal{D}_h$ has centre braided equivalent to $Z(\mathcal{N}_+)$, so the VOA realisations in §8 also realise their centres, with the ribbon structure transported along the centre equivalence. The argument uses only the fusion rules and three structural properties, (H1)–(H3) below. These also hold for the C* near-group categories $\mathbb{Z}_n + n$, n even, of [31, 16], and for those the classification describes all subfactors whose dual canonical endomorphism lies in the category (Proposition 9.24).

Strategy

The proof has five steps.

(a) A Perron–Frobenius argument shows that every indecomposable module category has exactly two orbits under the invertible objects, and reduces its fusion data to four integers subject to two equations (Theorem 9.4).

(b) The invertibles α_g and the self-intertwiners of R satisfy a Heisenberg commutation relation with a perfect bicharacter (§9.3). This gives divisibility conditions on the integers of (a) (Lemma 9.11), which reduce them to a single parameter $0 \le j \le s$ on a conic (Theorem 9.12). The endpoints $j = 0, s$ are the module categories of the subgroup algebras.

(c) For each subgroup algebra A_h we construct the dual category $\mathcal{D}_h$ of A_h-bimodules explicitly (Theorem 9.7).

(d) The internal end of a simple object of an arbitrary module category, regarded as an algebra in $\mathcal{D}_h$, lies in the neutral component of the universal grading of $\mathcal{D}_h$ (Lemma 9.13, Corollary 9.14).

(e) In that neutral component an integer row-norm identity forces $j \in \{0, s\}$ (Lemma 9.15).

No finite search and no explicit associator coefficient of $\mathcal{N}_+$ enters. Step (d) needs positivity of dimensions, which holds after a Galois conjugation.

The hypotheses and the proofs of this section require only that n be even. That $4 \mid n$ is used only in the remarks comparing the $\mathcal{D}_h$ with $\mathcal{D}$ and $\mathcal{F}_\nu$. The case $n \equiv 2 \pmod 4$ is taken up in §12.

Throughout, $e(x) = \exp(2\pi i x)$ and $d_\pm = \frac{1}{2}(n \pm \sqrt{nm})$, the two roots of $x^2 = nx + n$. So $d_+ = \delta - 1 = \mathrm{FPdim}\, R$, and d_- is the categorical dimension of R in $\mathcal{N}_+$. For a divisor h of n we write $h' = n/h$, and $\tau(n)$ for the number of divisors of n.

9.1 Hypotheses

Let n be an even positive integer. In §§2–8, except for Proposition 8.4, it is $4k$; in §12 it is congruent to 2 modulo 4. Let $\mathcal{G}$ be a spherical fusion category over $\mathbb{C}$ with simple objects α_g, $g \in \mathbb{Z}_n$, and R, and near-group fusion rules

$$\alpha_g \alpha_h = \alpha_{g+h}, \qquad \alpha_g R = R\alpha_g = R, \qquad R^2 = \bigoplus_{g \in \mathbb{Z}_n} \alpha_g \oplus nR.$$

Since R^2 contains every simple object, the universal grading group of $\mathcal{G}$ is trivial. Pivotal structures, when they exist, differ by characters of that group [14], so the spherical pivotal structure fixed here is unique. Its existence is supplied by the construction for $\mathcal{N}_+$ and by the canonical spherical structure for the C* examples; it is part of the present general setup. Consider:

(H1) the pointed part of $\mathcal{G}$ is $\mathrm{Vec}_{\mathbb{Z}_n}$ with trivial associator;

(H2) each α_g has a central lift (α_g, z_g) in $Z(\mathcal{G})$ with $z_g(\alpha_h) = \beta(g,h)$, for a nondegenerate symmetric bicharacter β on $\mathbb{Z}_n$;

(H3) the Frobenius–Schur indicator $\nu_2(R)$ is 1;

(H4) $\mathcal{G}$ is pseudo-unitary, that is $\dim \mathcal{G} = \mathrm{FPdim}\, \mathcal{G}$.

In (H2), $z_g(\alpha_h)$ is a scalar because $\alpha_g\alpha_h = \alpha_h\alpha_g = \alpha_{g+h}$ is simple. Since β is nondegenerate, $\beta(g,h) = e(-\mathsf{m}gh/n)$ with m a unit modulo n. As n is even, m is odd.

Lemma 9.1. *$\nu_2(R) = 1$ in $\mathcal{N}_+$.*

Proof. Let $F : \mathcal{C} \to \mathcal{N}$ be the free-module functor for $A' = \mathbf{1} \oplus X$. It is a pivotal tensor functor [34], and $F(X) = \mathbf{1} \oplus R$ (proof of Theorem 5.1). Recall [43] that for a self-dual object V the rotation operator E_V on $\mathrm{Hom}(\mathbf{1}, V \otimes V)$ is defined from the duality and the pivotal structure, and that for V simple it is the scalar $\nu_2(V) \in \{\pm 1\}$.

Since X is simple and self-dual, $\mathrm{Hom}(\mathbf{1}, X \otimes X)$ is spanned by a nondegenerate copairing c, and $E_X c = \nu_2(X)c$. Because F is a pivotal tensor functor, it carries the duality and pivotal structure of X to those of $F(X)$, so $E_{F(X)}F(c) = \nu_2(X)F(c)$. Now

$$\mathrm{Hom}(\mathbf{1}, F(X) \otimes F(X)) = \mathrm{Hom}(\mathbf{1}, \mathbf{1} \otimes \mathbf{1}) \oplus \mathrm{Hom}(\mathbf{1}, R \otimes R),$$

since $\mathrm{Hom}(\mathbf{1}, \mathbf{1} \otimes R) = 0$, and $E_{F(X)}$ preserves the two summands, acting by 1 on the first and by $\nu_2(R)$ on the second. The copairing $F(c)$ is nondegenerate, so both its components are nonzero. Comparing eigenvalues on the two components gives $\nu_2(X) = 1$ and $\nu_2(X) = \nu_2(R)$. □

The case $\ell = 2$ of Theorem 11.2 below gives the same value from the modular data.

Lemma 9.2. *1. $\mathcal{N}_+$, with $\alpha_g = M_{2g}$, satisfies (H1)–(H3), with $\beta(x,y) = e(-xy/n)$.*

2. Let $\mathcal{G}$ satisfy (H1)–(H3), and let $\sigma \in \mathrm{Aut}(\mathbb{C})$ with $\sigma(d(R)) = d_+$. Then the Galois conjugate $\mathcal{G}^\sigma$ satisfies (H1)–(H4). In particular this applies to $\mathcal{N}_+$ with $\sigma(d_-) = d_+$.

3. Every C near-group category of type $\mathbb{Z}_n + n$, n even, satisfies (H1)–(H4).*

Proof. (1) (H1) and (H2) are Remark 7.3, and (H3) is Lemma 9.1.

(2) The dimension $d(R)$ satisfies $d(R)^2 = nd(R) + n$, so $d(R) \in \{d_+, d_-\} \subset \mathbb{Q}(\sqrt{nm})$, and such a σ exists. The automorphism of $\mathbb{Q}(\sqrt{nm})$ exchanging $d_\pm$ extends to an automorphism of $\mathbb{C}$. The conjugate $\mathcal{G}^\sigma$ is defined in Lemma 9.3 below. It has the same fusion rules. Properties (H1)–(H3) are polynomial conditions on the structure constants, so they are preserved, with β replaced by $\sigma \circ \beta$ and $\nu_2(R) = 1$ fixed by σ. The dimension of R in $\mathcal{G}^\sigma$ is $\sigma(d(R)) = d_+$, so $\dim \mathcal{G}^\sigma = n + d_+^2 = \mathrm{FPdim}\, \mathcal{G}^\sigma$.

(3) In the description of [16, Thm. 1], α is an action of the group $\mathbb{Z}_n$ on the factor, which gives (H1). For $n' = n$ the half-braiding E_{α_g} exists, with $E_{\alpha_g}(\alpha_h) = \langle g, h\rangle$ [16, §4.3]. The pairing $\langle \cdot, \cdot \rangle$ is nondegenerate [16, Thm. 2(c)] and symmetric (the argument after [16, (2.55)]), which gives (H2). The sign s of [16, Thm. 1], defined by $S_0^* \rho(S_0) = s/d_\rho$, is $\nu_2(\rho)$, and $s = 1$ [16, Cor. 5]. This is (H3). A C* fusion category is pseudo-unitary, which is (H4). □

Galois conjugation

The classification below is proved for a pseudo-unitary category and transported to $\mathcal{G}$ by a Galois conjugation. We record what is transported.

Fix representatives X_i of the simple objects of a fusion category $\mathcal{G}$ over $\mathbb{C}$ and bases of the spaces $\mathrm{Hom}(X_k, X_i \otimes X_j)$. Then $\mathcal{G}$ is determined up to equivalence by its fusion rules and the matrices F of its associativity isomorphisms in these bases, a solution of the pentagon equations, which

are polynomial equations with integer coefficients. Two solutions define equivalent categories exactly when they are related by a permutation of the simple objects preserving the fusion rules and a change of the bases [14, Ch. 4]. For $\sigma \in \mathrm{Aut}(\mathbb{C})$, $\mathcal{G}^\sigma$ is the fusion category with the same simple objects, fusion rules and bases, and associativity matrices $\sigma(F)$.

Lemma 9.3. *Let $\mathcal{G}$ be a fusion category over $\mathbb{C}$ and $\sigma \in \mathrm{Aut}(\mathbb{C})$.*

1. *$\mathcal{G}^\sigma$ is a fusion category, well defined up to equivalence, and $(\mathcal{G}^\sigma)^{\sigma^{-1}} = \mathcal{G}$.*
2. *Applying σ to structure constants gives a bijection $Q \mapsto Q^\sigma$ from algebras in $\mathcal{G}$ to algebras in $\mathcal{G}^\sigma$ with the same underlying object, which preserves isomorphism, connectedness and separability. Its inverse is given by σ^{-1}.*
3. *It gives a bijection $\mathcal{M} \mapsto \mathcal{M}^\sigma$ from semisimple $\mathcal{G}$-module categories to semisimple $\mathcal{G}^\sigma$-module categories, with the same simple objects and the same action on the Grothendieck group, which preserves equivalence and indecomposability. The module category of Q^σ is the conjugate of that of Q, and the internal end of an object of $\mathcal{M}^\sigma$ is the conjugate of its internal end in $\mathcal{M}$.*
4. *For a connected separable algebra Q, the category of Q^σ-bimodules in $\mathcal{G}^\sigma$ is ${}_Q\mathcal{G}_Q)^\sigma$.*
5. *Applying σ to evaluation and coevaluation maps and to a pivotal structure of $\mathcal{G}$ gives the corresponding data for $\mathcal{G}^\sigma$, and sphericality is preserved. The categorical dimensions and the Frobenius–Schur indicators of $\mathcal{G}^\sigma$ are the images under σ of those of $\mathcal{G}$.*

Proof. Each structure in question is given, in the chosen bases, by finitely many matrices over $\mathbb{C}$ subject to polynomial equations with integer coefficients. Each relation between two such structures is given by matrices subject to such equations together with the nonvanishing of determinants. A field automorphism preserves both kinds of condition, and σ^{-1} gives the inverse map. In detail:

(1) $\sigma(F)$ satisfies the pentagon equations, and the triangle equations and rigidity are of the same kind. If F and F' are related by a permutation and base changes g, then $\sigma(F)$ and $\sigma(F')$ are related by the same permutation and $\sigma(g)$.

(2) An algebra structure on an object $Q = \bigoplus_i m_i X_i$ is a pair of morphisms $\mu : Q \otimes Q \to Q$ and $\eta : \mathbf{1} \to Q$, that is, matrices satisfying the associativity and unit equations, which are polynomial in the entries of μ, η and F. An isomorphism of algebras is an invertible matrix intertwining the two structures. Separability is the existence of a Q-bimodule map $Q \to Q \otimes Q$ splitting μ, again matrices subject to polynomial equations. Connectedness is the condition $m_{\mathbf{1}} = 1$ on the underlying object.

(3) A semisimple module category with simple objects m_a is given by non-negative integers $\dim \mathrm{Hom}(X_i \otimes \mathsf{m}_a, \mathsf{m}_b)$ and matrices of the module associativity constraint in chosen bases, satisfying the mixed pentagon equations. Equivalences of module categories are given by permutations of the m_a and base changes. Indecomposability is a property of the integers. The simple Q-modules and their module maps are described by matrices satisfying polynomial equations, as are the structure maps of an internal end, which is characterised by the adjunction $\mathrm{Hom}(X \otimes \mathsf{m}, \mathsf{m}) \cong \mathrm{Hom}(X, \underline{\mathrm{End}}(\mathsf{m}))$ and its compatibility with composition.

(4) A Q-bimodule structure on an object, a bimodule map, and the idempotent in $Y \otimes Y'$ whose image is $Y \otimes_Q Y'$ are given by matrices satisfying polynomial equations in the structure constants of Q and $\mathcal{G}$. So applying σ maps the simple Q-bimodules to the simple Q^σ-bimodules and the associativity matrices of ${}_Q\mathcal{G}_Q$ to those of the category of Q^σ-bimodules.

(5) Evaluation and coevaluation maps are matrices satisfying the rigidity equations. A pivotal structure is a family of nonzero scalars, one for each simple object once duality data are fixed, satisfying the equations that express monoidality. Sphericality is the equality of left and right traces. All of these are polynomial equations in the structure constants, so they are preserved

by σ. A categorical dimension is the scalar by which a composite of these maps acts on $\mathbf{1}$. A Frobenius–Schur indicator is the trace of a linear map on a Hom space whose matrix is polynomial in the same data. Hence both are transported by σ. □

All the categories, module categories and algebras considered here can be defined over $\overline{\mathbb{Q}}$. A fusion category has finitely many module categories up to equivalence, and each has structure constants in a number field after a change of bases [13]. So only the restriction of σ to $\overline{\mathbb{Q}}$ matters, and Lemma 9.3 may be read with $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ in place of $\mathrm{Aut}(\mathbb{C})$. We use $\mathrm{Aut}(\mathbb{C})$ so that algebras with arbitrary complex structure constants are covered directly.

9.2 Two orbits

The first step of the classification is combinatorial. A module category $\mathcal{M}$ over $\mathcal{G}$ gives a nimrep, that is, the Grothendieck group of $\mathcal{M}$ together with the non-negative integer matrices by which the simple objects of $\mathcal{G}$ act on it. This subsection determines the nimreps of a near-group fusion ring that can come from an indecomposable module category. The invertible objects permute the simple objects of $\mathcal{M}$. The result is that there are exactly two orbits, and that the matrix of R is described by three integers a, b, c subject to the equations (6). Only the fusion rules and the irrationality of $\sqrt{\mu^2+4n}$ are used. Many candidates remain. The later subsections use the associativity data of $\mathcal{G}$, through (H1)–(H3), to decide which of them are realised.

The following reduction holds for any near-group fusion rules. Its equations are the two-orbit case of the equations of Hannah and Ros Camacho [25, §3.2] for nimreps (non-negative integer matrix representations, a term introduced in [1]) of near-group fusion rings. We include the spectral argument that forces two orbits. A nimrep satisfies the usual rigidity condition. The action matrix of X^* is the transpose of that of X.

Theorem 9.4. *Let G be a finite group of order n, and let $\mathcal{G}$ have fusion rules $gR = Rg = R$, $R^2 = \bigoplus_{g\in G} g \oplus \mu R$, with $\sqrt{\mu^2+4n} \notin \mathbb{Q}$. Every nonzero indecomposable semisimple $\mathcal{G}$-module category has exactly two G-orbits of simple objects. If their sizes are u, v, the action of R has constant blocks*

$$N_R = \begin{pmatrix} aJ_{u,u} & bJ_{u,v} \\ bJ_{v,u} & cJ_{v,v} \end{pmatrix}, \qquad a, c \in \mathbb{Z}_{\geq 0}, \quad b \in \mathbb{Z}_{>0}, \tag{5}$$

with J the all-ones matrix, and

$$u \mid n, \qquad v \mid n, \qquad ua + vc = \mu, \qquad uv(b^2 - ac) = n. \tag{6}$$

In particular $uv \mid n$ and the rank is $u + v \leq n + 1$. Conversely, transitive G-sets of sizes u, v and integers satisfying (6) define an indecomposable nimrep by (5). Whether it lifts to a module category is a separate question.

Proof. Let $O_1, \dots, O_q$ be the G-orbits on the simple objects, of sizes u_i, and P_g the permutation matrices of the $g \in G$. The fusion rules give $N_R P_g = P_g N_R = N_R$, so N_R is constant on each block $O_i \times O_j$, say equal to a_{ij}, and rigidity ($R^* = R$) gives $a_{ij} = a_{ji}$. The orbit sizes divide n.

On the orthonormal vectors $e_i = u_i^{-1/2} \sum_{x\in O_i} x$, N_R acts by the symmetric matrix $B_{ij} = \sqrt{u_i u_j}\, a_{ij}$, and $\sum_g P_g$ acts by nI. So $B^2 = nI + \mu B$. Hence every eigenvalue of B is $\lambda_\pm = \frac{1}{2}(\mu \pm \sqrt{\mu^2+4n})$. Since the module category is indecomposable, B is irreducible. It is nonnegative, so by Perron–Frobenius its largest eigenvalue λ_+ is simple, and the remaining $q-1$ eigenvalues equal λ_-. Then

$$\sum_i u_i a_{ii} = \operatorname{tr} B = \lambda_+ + (q-1)\lambda_- = \tfrac{1}{2}\bigl(q\mu + (2-q)\sqrt{\mu^2+4n}\bigr)$$

is an integer, and since $\sqrt{\mu^2+4n}$ is irrational, $q=2$. With $a=a_{11}, b=a_{12}, c=a_{22}$, the trace and determinant of B are $ua+vc=\mu$ and $uv(ac-b^2)=\lambda_+\lambda_-=-n$, which is (6). Irreducibility gives $b>0$. As b^2-ac is a positive integer, $uv\le n$, and $u+v\le uv+1\le n+1$.

Conversely, on the two orbit-constant subspaces, in the bases of orbit sums, N_R acts by $M=\left(\begin{smallmatrix} ua & vb \\ ub & vc\end{smallmatrix}\right)$, with trace μ and determinant $-n$, so $M^2=\mu M+nI$. Since N_R has rank at most two with image in the orbit-constant vectors, this gives $N_R^2=\mu N_R+\sum_g P_g$. The relations $N_RP_g=P_gN_R=N_R$ and $N_R^{\mathsf t}=N_R$ hold by construction, and $b>0$ gives indecomposability. □

Corollary 9.5. *Let $\mathcal G$ be as in §9.1. Every connected separable algebra in $\mathcal G$ has underlying object $\bigoplus_{g\in H}\alpha_g\oplus aR$ with $H\le\mathbb Z_n$, $h=|H|$ and $0\le a\le h$, and there are $t\mid n$, $c\ge 0$ and $b>0$ with*

$$n\mid ht,\qquad c=t\Big(1-\frac{a}{h}\Big),\qquad b^2=ac+\frac{ht}{n}. \tag{7}$$

Proof. Let Q be the algebra. Its regular right module $\mathsf m$ is simple in the module category $\mathcal G_Q$, since Q is connected, and Q is the internal end of $\mathsf m$. So the multiplicity of α_g in Q is $\dim\mathrm{Hom}(\alpha_g\mathsf m,\mathsf m)$, which is 1 for g in the stabiliser H of $\mathsf m$ and 0 otherwise, and the multiplicity of R is $\dim\mathrm{Hom}(R\mathsf m,\mathsf m)$, the diagonal constant a of (5). The orbit of $\mathsf m$ has size $u=n/h$. Let the other orbit have stabiliser of order t, so $v=n/t$. Then (6) with $\mu=n$ reads $(n/h)a+(n/t)c=n$ and $(n^2/ht)(b^2-ac)=n$, which is (7). $c\ge0$ gives $a\le h$, and $uv\mid n$ gives $n\mid ht$. □

For the two orbit representatives put $x=h+ad_+$ and $y=t+cd_+$, the Frobenius–Perron dimensions of their internal ends. Then

$$xy=b^2d_+^2,\qquad \mathrm{Nm}_{\mathbb Q(d_+)/\mathbb Q}(x)=h^2+na(h-a),\qquad b^2=\frac{t(h^2+na(h-a))}{nh}. \tag{8}$$

Indeed $hc+ta=ht$ and $ac+ht/n=b^2$ by (7). Expanding xy with $d_+^2=n(1+d_+)$ gives the first identity, and the norm follows from $d_+d_-=-n$, $d_++d_-=n$. These identities hold for either root, and are necessary conditions only.

9.3 Mixed vertices

From now on let $\mathcal G$ satisfy (H1)–(H3). This subsection puts the structure maps involving both the α_g and R into a normal form, (9) below, from which the bimodules of §9.4 can be computed. We use the endomorphism notation. For morphisms f, f' and objects V,W, $f\otimes\mathrm{id}_W$ is written f and $\mathrm{id}_V\otimes f'$ is written $V(f')$, as for endomorphisms of a factor. By (H1) and $H^2(\mathbb Z_n,\mathbb C^\times)=0$ there are isomorphisms $L_g:\alpha_gR\to R$ which are coherent, $L_{g+h}=L_g\,\alpha_g(L_h)$, and we use them to identify $\alpha_gR=R$.

The bicharacter

By (H3) there is a symmetric self-duality $S_0:\mathbf 1\to R^2$. Let $U_g:R\to R\alpha_g$ be the right mate of L_g with respect to S_0 and the dualities of the α_g. The half-braiding of (H2) on R is an isomorphism $\alpha_gR\to R\alpha_g$, that is an element of $\mathrm{Hom}(R,R\alpha_g)=\mathbb CU_g$, so

$$z_g(R)=\varphi(g)U_g,\qquad \varphi(g)\in\mathbb C^\times.$$

Naturality of z_g with respect to $L_h:\alpha_hR\to R$, which is the identity in our identification, reads $z_g(R)=z_g(\alpha_hR)$. The half-braiding is monoidal, $z_g(\alpha_hR)=z_g(\alpha_h)\,\alpha_h(z_g(R))=\beta(g,h)\,\alpha_h(z_g(R))$. Hence $\varphi(g)U_g=\beta(g,h)\varphi(g)\alpha_h(U_g)$, that is $\alpha_h(U_g)=\beta(g,h)^{-1}U_g$. We put

$$\chi(a,j)=\beta(a,j)^{-1}=e(\mathsf maj/n),$$

a perfect symmetric bicharacter on $\mathbb Z_n$. For $\mathcal N_+$, $\chi(a,j)=e(aj/n)$.

Lemma 9.6. *There are bases $S_j : \alpha_j \to R^2$ and $T_j : R \to R^2$, $j \in \mathbb{Z}_n$, of the intertwiner spaces such that*

$$\begin{aligned} \alpha_a(U_j) &= \chi(a,j)U_j, \qquad U_aU_j = U_{a+j}, \\ \alpha_a(S_j) &= S_{a+j}, \qquad U_aS_j = \chi(a,j)^{-1}S_j, \qquad R(U_a)S_j = S_{j-a}, \\ \alpha_a(T_j) &= \chi(a,j)^{-1}T_j, \qquad U_aT_j = T_{j-a}, \qquad R(U_a)T_j = \frac{\varphi(j+a)}{\varphi(j)}\,T_{j+a}U_a. \end{aligned} \tag{9}$$

Moreover $\varphi(g+h) = \varphi(g)\varphi(h)\chi(g,h)$ and $\varphi(g)^2 = \chi(g,g)$, so $\varphi(g) = \varphi(1)^{g^2}$, and $\varphi(1)$ is a primitive $2n$-th root of unity.

Proof. The first identity is the computation above. The mates satisfy $U_gU_h = U_{g+h}$ because the L_g are coherent and taking mates is functorial.

The S_j. Put $S_j = \alpha_j(S_0)$, so $\alpha_a(S_j) = S_{a+j}$. The defining property of the mate gives $U_gS_0 = S_0$. Bending both legs of R and using that S_0 is symmetric gives the second mate identity $R(U_g)S_g = S_0$. Applying α_j to $U_gS_0 = S_0$ and using $\alpha_j(U_g) = \chi(j,g)U_g$ gives $U_gS_j = \chi(g,j)^{-1}S_j$. Applying α_{j-g} to $R(U_g)S_g = S_0$, with $\alpha_{j-g}R = R$, gives $R(U_g)S_j = S_{j-g}$.

The T_j. On $V = \mathrm{Hom}(R, R^2)$, of dimension n, the operators $T \mapsto \alpha_g(T)$ and $T \mapsto U_hT$ are representations of $\mathbb{Z}_n$, and $\alpha_g(U_hT) = \chi(g,h)U_h\alpha_g(T)$. Thus U_h maps the ψ-weight space of α to the $\psi\chi(\cdot,h)$-weight space. As χ is perfect, $h \mapsto \chi(\cdot,h)$ is a bijection $\mathbb{Z}_n \to \widehat{\mathbb{Z}_n}$, so all n weight spaces have the same dimension, namely 1. Choose T_0 of trivial weight and put $T_j = U_{-j}T_0$. Then $U_aT_j = T_{j-a}$ and $\alpha_a(T_j) = \chi(a,-j)T_j = \chi(a,j)^{-1}T_j$.

The function φ. A central structure on the unit is a tensor automorphism of the identity. The trivial universal grading therefore makes the kernel of the forgetful functor on invertible objects trivial. The central lift of each α_g, if it exists, is consequently unique. Hence the lifts of (H2) satisfy $(\alpha_g, z_g) \otimes (\alpha_h, z_h) \cong (\alpha_{g+h}, z_{g+h})$, and an isomorphism of central lifts of the simple object α_{g+h} is a scalar, which does not change the half-braiding. So $z_{g+h}(R) = z_g(R)\,\alpha_g(z_h(R)) = \varphi(g)\varphi(h)U_g\alpha_g(U_h) = \varphi(g)\varphi(h)\chi(g,h)U_{g+h}$. Naturality of z_g at $S_0 : \mathbf{1} \to R^2$ gives $S_0 = z_g(R^2)\alpha_g(S_0) = \varphi(g)^2R(U_g)U_gS_g$, and the two mate identities reduce this to $\varphi(g)^2\chi(g,g)^{-1} = 1$. By induction $\varphi(g) = \varphi(1)^g\chi(1,1)^{g(g-1)/2} = \varphi(1)^{g^2}$. Finally $\varphi(1)^2 = \chi(1,1) = e(\mathsf{m}/n)$ with m odd. An n-th root of unity ζ with $\zeta^2 = e(\mathsf{m}/n)$ would give $2x \equiv \mathsf{m} \pmod n$ with n even, impossible. So $\varphi(1)$ has order $2n$.

The last identity. Naturality of z_g at $T_j : R \to R^2$ gives $T_j\,\varphi(g)U_g = \varphi(g)^2\,R(U_g)\,U_g\,\alpha_g(T_j)$. Substituting $\alpha_g(T_j) = \chi(g,j)^{-1}T_j$ and $U_gT_j = T_{j-g}$ and replacing j by $j+g$ gives $R(U_g)T_j = \varphi(g)\chi(g,j)T_{j+g}U_g = \varphi(j+g)\varphi(j)^{-1}T_{j+g}U_g$. □

The same identities occur in [44, (46), (47), Lemma 4], derived there from a finite quantum dilogarithm. Here they follow from (H1)–(H3) alone.

9.4 The Morita dual for every subgroup

This subsection constructs the module categories and computes their duals. Each subgroup H of $\mathbb{Z}_n$ gives an algebra A_h in $\mathcal{G}$, the group algebra of H, and hence the module category $\mathcal{G}_{\mathsf{A}_h}$ of right A_h-modules. The fusion category that acts on the other side of this module category is the category $\mathcal{D}_h$ of A_h-bimodules. By construction it is categorically Morita equivalent to $\mathcal{G}$ and has the same double. Theorem 9.7 determines the fusion rules of $\mathcal{D}_h$ completely. It gives the simple objects, their dimensions, all products, and the order of the associator of the pointed part. The answer depends on h only through $s = \gcd(h, n/h)$. For $s = 1$ the dual is again a near-group category. For $s > 1$ it has s^2 non-invertible simple objects and a pointed part with nontrivial associator. After the theorem we derive its consequences. These are the universal

grading (Corollary 9.8), the number of distinct fusion rings (Corollary 9.9), and the identification of the neutral component with a de-equivariantisation (Proposition 9.10).

By (H1), for every divisor h of n the subgroup $H = h'\mathbb{Z}_n$ of order h gives a connected separable algebra

$$\mathsf{A}_h = \bigoplus_{a\in H} \alpha_a,$$

with multiplication given by the group law. Put

$$s = \gcd(h, h'), \qquad t = h/s, \qquad r = n/s^2, \qquad \epsilon_h = \begin{cases} 0, & r \text{ even}, \\ s/2, & r \text{ odd}, \end{cases}$$

and let $\mathcal{D}_h = {}_{\mathsf{A}_h}\mathcal{G}_{\mathsf{A}_h}$ be the category of A_h-bimodules, with tensor product $\otimes_{\mathsf{A}_h}$. Then $s^2 \mid hh' = n$. If r is odd then s is even, since $n = rs^2$ is even. If $n \equiv 2 \pmod 4$, then s is odd, $r \equiv 2 \pmod 4$, and $\epsilon_h = 0$ for every h.

Theorem 9.7. *The fusion category $\mathcal{D}_h$ has n invertible simple objects $I_{g,l}$, $(g,l) \in \mathbb{Z}_{h'} \times \mathbb{Z}_h$, and s^2 further simple objects $\Upsilon_{x,y}$, $(x,y) \in \mathbb{Z}_s^2$, of Frobenius–Perron dimension d_+/s, with*

$$I_{g,l} I_{g',l'} = I_{g+g',l+l'}, \qquad I_{g,l}\Upsilon_{x,y} = \Upsilon_{x-l,y-g}, \qquad \Upsilon_{x,y} I_{g,l} = \Upsilon_{x+g,y+l}, \tag{10}$$

$$\Upsilon_{x,y}\Upsilon_{u,v} = \bigoplus_{\substack{g\in\mathbb{Z}_{h'},\, l\in\mathbb{Z}_h \\ g\equiv u-y,\ l\equiv v-x\ (s)}} I_{g,l} \;\oplus\; r \bigoplus_{a\in\mathbb{Z}_s} \Upsilon_{a,\, a+\epsilon_h+y+v-x-u}. \tag{11}$$

In particular $\Upsilon_{x,y}^* = \Upsilon_{y,x}$. The associator of the pointed part $\mathrm{Vec}_{\mathbb{Z}_{h'}\times\mathbb{Z}_h}$ has class of exact order s. Each $\mathcal{D}_h$ is categorically Morita equivalent to $\mathcal{G}$, and $Z(\mathcal{D}_h) \simeq Z(\mathcal{G})$.

Strategy of the proof

A bimodule is either supported on the α_g or on R. The pointed ones are described by a coset and a character of H. An R-supported bimodule is $R \otimes V$, and by (9) the bimodule axioms reduce to a representation of a twisted group algebra of $H \times H$ on V, with commutator χ. Its irreducibles are counted by the radical of $\chi|_{H\times H}$. The only product that requires work is $\Upsilon_{0,0}^2$, where the sign $\varphi(k_0) = (-1)^r$ of (13) produces the shift ϵ_h. The associator of the pointed part is a carry cocycle, whose order is detected on a cyclic subgroup.

The pointed bimodules

For $g \in \mathbb{Z}_{h'}$, identified with the representatives $0, \dots, h'-1$ of $\mathbb{Z}_n/H$, and $l \in \mathbb{Z}_h$, let $I_{g,l}$ have underlying object $\bigoplus_{a\in H} \alpha_{g+a}$, left action given by translation, and right action given by translation multiplied by the character λ_l of H, $\lambda_l(h') = e(l/h)$. These are simple bimodules, since End of $I_{g,l}$ as a bimodule is $\mathbb{C}$, and every α-supported bimodule is a sum of them. A simple α-supported bimodule is supported on one coset $g + H$, and the right action of H relative to the left one is a character of H, as $H^2(H, \mathbb{C}^\times) = 0$. So there are $h'h = n$ of them. They are invertible, with the product in (10).

The R-supported bimodules

An A_h-bimodule supported on R has underlying object $R \otimes V$ for a vector space V. Its left action $\alpha_a R \otimes V = R \otimes V \to R \otimes V$ is $\mathrm{id}_R \otimes \mathsf{L}_a$, with L a representation of H on V by associativity. Its right action $R\alpha_a \otimes V \to R \otimes V$ can be written uniquely as $U_{-a} \otimes \mathsf{R}_a$, and associativity, using $U_a U_b = U_{a+b}$, makes R a representation of H. The two actions commute exactly when

$$\mathsf{L}_a \mathsf{R}_j = \chi(a,j)\, \mathsf{R}_j \mathsf{L}_a \qquad (a, j \in H), \tag{12}$$

by $\alpha_a(U_j) = \chi(a,j)U_j$. So R-supported bimodules are modules over the twisted group algebra $\mathcal{T}$ of $H \times H$ with relation (12), and bimodule maps are $\mathcal{T}$-module maps.

Let $K = \{a \in H : \chi(a,H) = 1\} = H \cap H^\perp$ be the radical of $\chi|_{H\times H}$. Since χ is perfect, $H^\perp$ is the subgroup of order $n/h = h'$, so K is the subgroup of order $\gcd(h,h') = s$, generated by $k_0 = n/s$. The elements $\mathsf{L}_a, \mathsf{R}_a$ with $a \in K$ are central in $\mathcal{T}$, and $\mathcal{T}$ is the direct sum, over the s^2 pairs of characters of K for L_K and R_K, of twisted group algebras of $(H/K)^2$ with nondegenerate commutator. Each is a full matrix algebra of size $|H/K| = t$. So there are exactly s^2 simple R-supported bimodules, each with $\dim V = t$. We label them $\Upsilon_{x,y}$, $(x,y) \in \mathbb{Z}_s^2$, by $\mathsf{L}_{k_0} = e(x/s)$ and $\mathsf{R}_{k_0} = e(y/s)$. Their Frobenius–Perron dimension in $\mathcal{D}_h$ is the dimension of the underlying object divided by $\mathrm{FPdim}\,\mathsf{A}_h = h$, that is $td_+/h = d_+/s$. For trivial central characters one may take, on $\mathbb{C}^t$, $\mathsf{L}_{h'}e_j = \zeta^j e_j$ and $\mathsf{R}_{h'}e_j = e_{j+1}$ with $\zeta = \chi(h',h') = e(\mathsf{m}h'/h)$, of order t. This is $\Upsilon_{0,0}$.

Action of the invertibles

In $I_{g,l} \otimes_{\mathsf{A}_h} \Upsilon$ the balancing identifies the summands, and the result is $R \otimes V$ with actions $\lambda_l(a)^{-1}\mathsf{L}_a$ and $\chi(g,a)^{-1}\mathsf{R}_a$. In $\Upsilon \otimes_{\mathsf{A}_h} I_{g,l}$ the actions become $\chi(g,a)\mathsf{L}_a$ and $\lambda_l(a)\mathsf{R}_a$. Evaluating at $k_0 = th'$, with $\lambda_l(k_0) = e(lt/h) = e(l/s)$ and $\chi(g,k_0) = e(\mathsf{m}g/s)$, gives (10) after replacing the label g by $\mathsf{m}g$ (for $\mathcal{N}_+$, $\mathsf{m} = 1$). In particular the invertibles act transitively on the Υ's, $\Upsilon_{x,y} = I_{-y,-x}\Upsilon_{0,0}$ and $\Upsilon_{0,0}I_{g,l} = I_{-l,-g}\Upsilon_{0,0}$.

The base square

By the last two identities it suffices to compute $\Upsilon_{0,0}^2$. Its underlying object is the image in $R^2 \otimes V \otimes V$ of the idempotent

$$P = \frac{1}{h}\sum_{a\in H} U_{-a} \otimes \mathsf{R}_a \otimes \mathsf{L}_{-a},$$

which implements the balancing $R\alpha_a \otimes V \otimes R \otimes V \to R \otimes V \otimes \alpha_a R \otimes V$. The summands form a representation of H, so P is an idempotent. Decompose $R^2 = \bigoplus_j S_j\alpha_j \oplus \bigoplus_j T_j R$ using the bases of Lemma 9.6.

Invertible channels. On the channel S_j, U_{-a} acts by $\chi(-a,j)^{-1} = \chi(a,j)$, so P restricts to $P_j = \frac{1}{h}\sum_{a\in H}\chi(a,j)\mathsf{R}_a \otimes \mathsf{L}_{-a}$ on $V \otimes V$. The subgroup K acts on $V \otimes V$ through $\mathsf{R}_a \otimes \mathsf{L}_{-a}$ trivially (both central characters are trivial for $\Upsilon_{0,0}$), so $P_j = 0$ unless $\chi(\cdot,j)$ is trivial on K, that is $s \mid j$. For $s \mid j$, H/K acts on $V \otimes V = \mathbb{C}^{t^2}$ by t copies of its regular representation, so P_j has rank t. On $\operatorname{im} P_j$ the left and right outer actions differ by $\mathsf{D}_a = \mathsf{L}_{-a} \otimes \mathsf{R}_a$. This is trivial on K, and for $a \notin K$

$$\mathrm{Tr}(\mathsf{D}_a P_j) = \frac{1}{h}\sum_{c\in H}\chi(c,j)\,\mathrm{Tr}(\mathsf{L}_{-a}\mathsf{R}_c)\,\mathrm{Tr}(\mathsf{R}_a\mathsf{L}_{-c}) = 0,$$

because a product $\mathsf{L}_x\mathsf{R}_y$ of the Weyl matrices has nonzero trace only when $x,y \in K$. So each character of H trivial on K occurs once, and the pointed part of $\Upsilon_{0,0}^2$ is $\bigoplus_{g\equiv l\equiv 0\,(s)} I_{g,l}$.

Channels through R. On the channels T_jR, (9) shows that the balancing moves T_j to T_{j-a}, so it permutes the n channels in the h' cosets $j + H$, and on each coset leaves a multiplicity space of dimension t^2. The left and right outer actions of K on it are the characters $k \mapsto \chi(k,j)^{-1}$ and $k \mapsto \varphi(k)\chi(k,j)^{-1}$. The factor $\varphi(k)$ comes from moving U_{-k} past T_j with $R(U_{-k})T_j = \varphi(j-k)\varphi(j)^{-1}T_{j-k}U_{-k}$, and the internal matrices are unchanged by K, which is central. The value at the generator is

$$\varphi(k_0) = \varphi(1)^{k_0^2} = e\Big(\frac{\tilde{\mathsf{m}}\,n^2}{2ns^2}\Big) = e\Big(\frac{\tilde{\mathsf{m}}r}{2}\Big) = (-1)^r, \tag{13}$$

where $\varphi(1) = e(\tilde{\mathsf{m}}/2n)$ with $\tilde{\mathsf{m}}$ odd by Lemma 9.6. So on each coset the two central characters differ by $(-1)^r$ at k_0, that is by the shift ϵ_h in the label. The output is a sum of $\Upsilon_{x,x+\epsilon_h}$. Each $x \in \mathbb{Z}_s$ occurs on h'/s cosets, each contributing $t^2/t = t$ copies, so with multiplicity $(h'/s)t = hh'/s^2 = r$. We have shown

$$\Upsilon_{0,0}^2 = \bigoplus_{g\equiv l\equiv 0\,(s)} I_{g,l} \;\oplus\; r \bigoplus_{a\in\mathbb{Z}_s} \Upsilon_{a,a+\epsilon_h}. \tag{14}$$

Multiplying by invertibles on both sides gives (11), and the unit occurs in $\Upsilon_{x,y}\Upsilon_{u,v}$ exactly when $(u,v) = (y,x)$, so $\Upsilon_{x,y}^* = \Upsilon_{y,x}$. As a check, $\operatorname{FPdim}\mathcal{D}_h = n + s^2(d_+/s)^2 = n + d_+^2 = \operatorname{FPdim}\mathcal{G}$.

The pointed associator

Multiplication of the I's uses the carry of the representatives, namely $g'+g'' = \{g'+g''\}+h'c(g',g'')$ with $\{\cdot\} \in \{0,\dots,h'-1\}$ and $c \in \{0,1\}$. Comparing the two balanced multiplication maps $(I_{g,l}I_{g',l'})I_{g'',l''} \to I_{g,l}(I_{g',l'}I_{g'',l''})$ gives the associator

$$\omega\big((g,l),(g',l'),(g'',l'')\big) = e\Big(\frac{l}{h}\,c(g',g'')\Big). \tag{15}$$

Upper bound. $\omega^h = 1$. For the cochain $f((g,l),(g',l')) = e(lg'/h)$, the inhomogeneous coboundary is

$$df(x,y,z) = \frac{f(y,z)f(x,y+z)}{f(x+y,z)f(x,y)} = e\Big(\frac{l'g'' + l\{g'+g''\} - (l+l')g'' - lg'}{h}\Big) = \omega(x,y,z)^{-h'},$$

for $x = (g,l)$, $y = (g',l')$, $z = (g'',l'')$. So $[\omega]^h = [\omega]^{h'} = 1$, and the order of $[\omega]$ divides s.

Lower bound. For a 3-cocycle ω on a cyclic group $\langle z\rangle$ of order N, the product $I(\omega) = \prod_{j=0}^{N-1}\omega(z,jz,z)$ is 1 for a coboundary. For $\omega = dF$ it is $\prod_j F(jz,z)F(z,(j+1)z)/(F((j+1)z,z)F(z,jz)) = 1$ by reindexing $j \mapsto j+1$. Restrict ω to $\langle z\rangle$, $z = (1,1)$, of order $N = \operatorname{lcm}(h,h') = hh'/s$. Since the first argument z has $l = 1$, (15) gives $\omega(z,jz,z) = e\big(c(\{j\},1)/h\big)$, and the carry $c(\{j\},1)$ is 1 exactly when $\{j\} = h'-1$, which happens for N/h' values of $j \in \{0,\dots,N-1\}$. So $I(\omega) = e(N/(hh')) = e(1/s)$, of order s, and $[\omega]$ has order exactly s.

Finally, A_h is connected and separable, so $\mathcal{D}_h$ is the dual of $\mathcal{G}$ with respect to the module category $\mathcal{G}_{\mathsf{A}_h}$, hence categorically Morita equivalent to $\mathcal{G}$, and $Z(\mathcal{D}_h) \simeq Z(\mathcal{G})$ [47]. This proves Theorem 9.7.

Corollary 9.8. *The universal grading group of $\mathcal{D}_h$ is $\mathbb{Z}_s$, with $\deg I_{g,l} = g - l$ and $\deg \Upsilon_{x,y} = x - y + \epsilon_h$ modulo s. The adjoint subcategory has cyclic group of invertibles of order n/s, generated by $I_{1,1}$, non-invertible simples $\Upsilon_{a,a+\epsilon_h}$, $a \in \mathbb{Z}_s$, and rank $n/s + s$.*

Proof. The degrees are additive on (10), and on (11) because $2\epsilon_h \equiv 0 \pmod s$. Every degree occurs, so this is a faithful $\mathbb{Z}_s$-grading. The adjoint subcategory is generated by the XX^*, X simple. For invertible X this is $\mathbf{1}$, and $\Upsilon_{x,y}\Upsilon_{x,y}^*$ is a translate of (14), whose simple summands are $I_{g,l}$ with $g \equiv l \equiv 0 \pmod s$ and the neutral $\Upsilon_{a,a+\epsilon_h}$. Products of the neutral Υ's give, by (11), all $I_{g,l}$ with $g \equiv l \pmod s$. This subgroup has order n/s and is generated by $(1,1)$. So the adjoint subcategory is the whole neutral component, and the grading is universal. □

Let $n = 4k$. For $h = 2$, so $s = 2$ and $r = k$, the adjoint subcategory has rank $2k+2$, invertibles $\mathbb{Z}_{2k}$ and two non-invertibles $x = \Upsilon_{0,\epsilon_2}$, $y = \Upsilon_{1,1+\epsilon_2}$ of dimension $d_+/2$, with $xx^* = \bigoplus_{j\text{ even}} I_{j,j} \oplus k\,x \oplus k\,y$, and x is self-dual exactly when k is even. For $\mathcal{G} = \mathcal{N}_+$ this subcategory is equivalent to $\mathcal{D}$, by Proposition 9.10 below. So these are the fusion rules of $\mathcal{D}$ for every k, in agreement with Theorem 7.1(3), Lemma 7.4 and, for $k \le 7$, Proposition A.1(3) under $g_\lambda \leftrightarrow I_{\lambda,\lambda}$. For $n = 4\nu^2$ and $h = 2\nu$, so $s = 2\nu$, $r = 1$ and $\epsilon_h = \nu$, the adjoint subcategory is equivalent to $\mathcal{F}_\nu$, by the same proposition, with $\rho = \Upsilon_{0,\nu}$ and $\alpha_g = I_{g,g}$. This holds for every $\nu \ge 1$, and it is the source of the fusion rules in Theorem 7.5(2).

Corollary 9.9. *The fusion rings of the $\mathcal{D}_h$ are determined by s, and distinct values of s with $s^2 \mid n$ give distinct rings, of rank $n+s^2$. Hence there are at least $\prod_{p^e \| n}(\lfloor e/2 \rfloor + 1)$ pairwise inequivalent fusion categories whose double is braided equivalent to $Z(\mathcal{G})$.*

Proof. The invertible group $\mathbb{Z}_{h'} \times \mathbb{Z}_h$ is abstractly $\mathbb{Z}_{n/s} \times \mathbb{Z}_s$. Working one prime at a time, interchange the two coordinates where needed so that the larger cyclic p-factor comes first, and make the same interchange on the two coordinates of the Υ's. This interchange commutes with the swap in (10), preserves the diagonal, and replaces ϵ_h by $-\epsilon_h = \epsilon_h$. So (10)–(11) depend, up to based isomorphism, only on n and s. Every s with $s^2 \mid n$ occurs, for $h = s$, and the ranks $n + s^2$ differ. The number of such s is the displayed product. □

For $s = 1$ the duals are again near-group categories $\mathbb{Z}_n + n$. For $\mathcal{G} = \mathcal{N}_+$ and $n > 2$ they are all equivalent to $\mathcal{N}_+$ (Theorem 9.18).

The neutral component as a de-equivariantisation

Let $h^2 \mid n$, and let $H = h'\mathbb{Z}_n$ be the subgroup of order h. Then $s = \gcd(h, h') = h$. For $a = h'i$ and $b = h'j$ in H, $\beta(a,b) = e(-\mathsf{m}\, h'^2 ij/n) = e(-\mathsf{m}\, ij\, n/h^2) = 1$. So the central lifts (α_a, z_a), $a \in H$, of (H2) braid trivially with each other. As in the proof of Lemma 9.6, $(\alpha_a, z_a) \otimes (\alpha_b, z_b) \cong (\alpha_{a+b}, z_{a+b})$, so the multiplication of A_h is a morphism in $Z(\mathcal{G})$ for the half-braiding $z_{\mathsf{A}} = \bigoplus_{a \in H} z_a$, and $(\mathsf{A}_h, z_{\mathsf{A}})$ is a commutative algebra in $Z(\mathcal{G})$, that is, $\mu \circ z_{\mathsf{A}}(\mathsf{A}_h) = \mu$, because $z_a(\alpha_b) = \beta(a,b) = 1$. The *de-equivariantisation* of $\mathcal{G}$ by H is the category $\mathcal{Q}_h = \mathcal{G}_{\mathsf{A}_h}$ of right A_h-modules, with the tensor product $\otimes_{\mathsf{A}_h}$ defined through this central structure [9, 10]. For $\mathcal{G} = \mathcal{N}_+$ this is the category $\mathcal{Q}_h$ of §11. In particular $\mathcal{Q}_2 = \mathcal{D}$ for $n = 4k$ and $\mathcal{Q}_{2\nu} = \mathcal{F}_\nu$ for $n = 4\nu^2$.

Proposition 9.10. *Let $\mathcal{G}$ satisfy* (H1)–(H3) *and let $h^2 \mid n$. Then $\mathcal{Q}_h$ is tensor equivalent to the adjoint subcategory of $\mathcal{D}_h$, that is to the neutral component of its universal grading. In particular, for $\mathcal{G} = \mathcal{N}_+$ and $n = 4k$, the neutral component of $\mathcal{D}_2$ is equivalent to $\mathcal{D}$, and for $n = 4\nu^2$ the neutral component of $\mathcal{D}_{2\nu}$ is equivalent to $\mathcal{F}_\nu$.*

Proof. Strategy. A right module over a commutative central algebra is a bimodule, the left action being the right action composed with the half-braiding. This embeds $\mathcal{Q}_h$ in $\mathcal{D}_h$ as a fusion subcategory. It contains a non-invertible simple object, hence the adjoint subcategory, and the two have the same dimension.

The embedding. Write $\mathsf{A} = \mathsf{A}_h$, $z = z_{\mathsf{A}}$ and μ for the multiplication. For a right A-module (M, r) put $l_M = r \circ z(M) : \mathsf{A} \otimes M \to M \otimes \mathsf{A} \to M$. We use that z is natural, that $z(M \otimes N) = (1_M \otimes z(N))(z(M) \otimes 1_N)$, and that μ is a morphism in $Z(\mathcal{G})$, that is $z(M)(\mu \otimes 1_M) = (1_M \otimes \mu)(z(M) \otimes 1)(1 \otimes z(M))$.

- l_M is a left action. On one hand $l_M(\mu \otimes 1) = r(1 \otimes \mu)(z(M) \otimes 1)(1 \otimes z(M))$. On the other, by naturality of z at r, $l_M(1 \otimes l_M) = r\, z(M)(1 \otimes r)(1 \otimes z(M)) = r(r \otimes 1)\, z(M \otimes \mathsf{A})(1 \otimes z(M)) = r(1 \otimes \mu)(1 \otimes z(\mathsf{A}))(z(M) \otimes 1)(1 \otimes z(M))$, and $\mu\, z(\mathsf{A}) = \mu$.
- The two actions commute. In the same way $l_M(1 \otimes r) = r(r \otimes 1) z(M \otimes \mathsf{A}) = r(1 \otimes \mu z(\mathsf{A}))(z(M) \otimes 1) = r(r \otimes 1)(z(M) \otimes 1) = r(l_M \otimes 1)$.
- A map of right modules $f : M \to N$ satisfies $f\, l_M = l_N(1 \otimes f)$, by naturality of z at f. So $\Phi : (M, r) \mapsto (M, l_M, r)$ is a functor $\mathcal{Q}_h \to \mathcal{D}_h$ which is bijective on morphism spaces.
- On $M \otimes_{\mathsf{A}} N$ the left action $l_M \otimes 1$ of the tensor product of bimodules agrees with the one defined by Φ. The latter is $(1 \otimes r_N)\, z(M \otimes N) = (1 \otimes l_N)(z(M) \otimes 1)$, and $1 \otimes l_N = r_M \otimes 1$ on $M \otimes_{\mathsf{A}} N$. The tensor product of $\mathcal{Q}_h$ is by definition that of the bimodules $\Phi(M)$, so Φ is a tensor functor, and $\Phi(\mathsf{A})$ is the unit, as $l_{\mathsf{A}} = \mu\, z(\mathsf{A}) = \mu$.

Since Φ is fully faithful, it maps simple objects to simple objects, and its essential image $\mathcal{K}$ is a full subcategory of $\mathcal{D}_h$ closed under tensor products and direct summands, that is, a fusion subcategory. Its Frobenius–Perron dimension is that of $\mathcal{Q}_h$, which is $\operatorname{FPdim}\mathcal{G}/\operatorname{FPdim}\mathsf{A} = (n+d_+^2)/h$ [10, Lemma 3.11].

The image. $(n+d_+^2)/h$ is irrational, so $\mathcal{K}$ is not pointed and contains some $\Upsilon_{x,y}$. Then it contains $\Upsilon_{x,y}\Upsilon_{x,y}^*$, hence, by the proof of Corollary 9.8, every neutral $\Upsilon_{a,a+\epsilon_h}$ and all their products, that is the whole adjoint subcategory $(\mathcal{D}_h)_{\mathrm{ad}}$. By Corollary 9.8 the universal grading is by $\mathbb{Z}_s$ and is faithful, so $\operatorname{FPdim}(\mathcal{D}_h)_{\mathrm{ad}} = \operatorname{FPdim}\mathcal{D}_h/s = (n+d_+^2)/h$, as $s=h$. A fusion subcategory containing another of the same Frobenius–Perron dimension equals it. So $\mathcal{K} = (\mathcal{D}_h)_{\mathrm{ad}}$.

The last statement follows from the identifications $\mathcal{Q}_2 = \mathcal{D}$ (Theorem 7.1(3), where the de-equivariantisation is by the central lift $a_{n/2} = [J,0]$ of M_n, the unique central lift of that object) and $\mathcal{Q}_{2\nu} = \mathcal{F}_\nu$ (Theorem 7.5). □

The orbit matrix of $\mathcal{G}_{\mathsf{A}_h}$ (Theorem 9.16 below) is $\left(\begin{smallmatrix}0 & h\\ h' & n\end{smallmatrix}\right)$, and $\operatorname{coker}\left(\begin{smallmatrix}0 & h\\ h' & n\end{smallmatrix}\right) \cong \mathbb{Z}_s \times \mathbb{Z}_{n/s}$ by its Smith normal form, the abstract invertible group of $\mathcal{D}_h$.

9.5 Arbitrary module categories

The previous subsection produced $\tau(n)$ module categories. This subsection proves that there are no others, and draws the consequences. The argument has three stages.

First, the nimrep of an arbitrary indecomposable module category is restricted. The Heisenberg relation of §9.3 forces the multiplicities in Theorem 9.4 to be divisible by explicit integers (Lemma 9.11). This reduces the possibilities to the integer points on a conic, one family for each divisor h of n (Theorem 9.12). The two end points of each family are realised by the module categories of §9.4. The interior points are not excluded by the fusion rules.

Secondly, the interior points are excluded. Here we use positivity of dimensions, which is available after a Galois conjugation. It enters through two results on algebras in graded pseudo-unitary categories (Lemma 9.13 and Corollary 9.14) and a row-norm identity in the dual category $\mathcal{D}_h$ (Lemma 9.15). The outcome is the classification of module categories, dual categories and connected separable algebras (Theorem 9.16), and of the Lagrangian algebras in the double (Corollary 9.17).

Thirdly, for the category $\mathcal{N}_+$ we decide which of the dual categories $\mathcal{D}_h$ are equivalent to each other (Theorems 9.18 and 9.20). The subsection ends with the module functors between the module categories (Theorem 9.23).

Lemma 9.11. *Let $\mathsf{m}_i, \mathsf{m}_j$ be simple objects of a semisimple $\mathcal{G}$-module category, with stabilisers $H_i, H_j \le \mathbb{Z}_n$ of orders h_i, h_j. Then*

$$\frac{h_i h_j}{\gcd(n, h_i h_j)} \;\Big|\; \dim \operatorname{Hom}(R\mathsf{m}_i, \mathsf{m}_j).$$

Proof. Choose isomorphisms $\ell_a^i : \alpha_a \mathsf{m}_i \to \mathsf{m}_i$, $a \in H_i$. They can be chosen coherent, $\ell_{a+b}^i = \ell_a^i\, \alpha_a(\ell_b^i)$, because the obstruction is a class in $H^2(H_i, \mathbb{C}^\times) = 0$. Do the same for j. On $V_{ij} = \operatorname{Hom}(R\mathsf{m}_i, \mathsf{m}_j)$ define

$$\Pi_a(f) = f \circ R(\ell_a^i) \circ (U_a)_{\mathsf{m}_i} \quad (a \in H_i), \qquad \Omega_b(f) = \ell_b^j \circ \alpha_b(f) \quad (b \in H_j),$$

where $(U_a)_{\mathsf{m}_i} : R\mathsf{m}_i \to R\alpha_a \mathsf{m}_i$. The first is a representation of H_i. Moving $R(\ell_{a'}^i)$ through U_a turns it into $R(\alpha_a(\ell_{a'}^i))$, and then $U_a U_{a'} = U_{a+a'}$ and coherence of ℓ^i give $\Pi_a \Pi_{a'} = \Pi_{a+a'}$. The second is a representation of H_j by coherence of ℓ^j. Since $\alpha_b(U_a) = \chi(b,a) U_a$,

$$\Omega_b \Pi_a = \chi(b,a)\, \Pi_a \Omega_b.$$

Decompose V_{ij} into common weight spaces of the operators Ω_b, indexed by $\widehat{H_j}$. Each Π_a maps the ψ-weight space isomorphically onto the $\psi\chi(\cdot,a)|_{H_j}$-weight space. So the image Γ of $H_i \to \widehat{H_j}$, $a \mapsto \chi(\cdot,a)|_{H_j}$, acts freely by translation, all weight spaces in a Γ-orbit have the same dimension, and $|\Gamma|$ divides $\dim V_{ij}$. With $\chi(a,b) = e(\mathsf{m}ab/n)$ and generators n/h_i, n/h_j, the image of the generator of H_i is $b \mapsto e(\mathsf{m}\,b/h_i)$ restricted to $(n/h_j)\mathbb{Z}_n$, a character of order $h_ih_j/\gcd(n,h_ih_j)$. This is $|\Gamma|$. □

Theorem 9.12. *An indecomposable semisimple $\mathcal{G}$-module category has stabilisers of orders h and h', for some $h \mid n$. With s, r as in §9.4, its constants in (5) are, for some integer j,*

$$a = \frac{h}{s}\,j, \qquad c = \frac{h'}{s}(s-j), \qquad b^2 = 1 + rj(s-j), \qquad 0 \le j \le s. \tag{16}$$

Its rank is $h + h'$. If $j = 0$ it is equivalent to $\mathcal{G}_{\mathsf{A}_h}$, and if $j = s$ to $\mathcal{G}_{\mathsf{A}_{h'}}$.

Proof. By Theorem 9.4 there are two orbits. Let their stabilisers have orders h and t, so the orbit sizes are $u = n/h$, $v = n/t$, and $uv \mid n$ gives $n \mid ht$. Put $\mathsf{q} = ht/n$. Then $uv(b^2 - ac) = n$ reads $b^2 - ac = \mathsf{q}$.

Apply Lemma 9.11. For one simple in each orbit, $\dim \mathrm{Hom}(R\mathsf{m}_i, \mathsf{m}_j) = b$ and $ht/\gcd(n,ht) = ht/n = \mathsf{q}$, so $\mathsf{q} \mid b$. For two simples in the first orbit, $\Delta_h = h^2/\gcd(n,h^2)$ divides a, and similarly $\Delta_t = t^2/\gcd(n,t^2)$ divides c. Now

$$\Delta_h\Delta_t = \mathsf{q}^2\,\frac{n^2}{\gcd(n,h^2)\gcd(n,t^2)},$$

and both gcds divide n, so $\mathsf{q}^2 \mid \Delta_h\Delta_t \mid ac$. With $\mathsf{q}^2 \mid b^2$ this gives $\mathsf{q}^2 \mid b^2 - ac = \mathsf{q}$, so $\mathsf{q} = 1$, $t = h'$, and $u = h'$, $v = h$. The rank is $h + h'$.

Now $\gcd(n,h^2) = h\gcd(h',h) = hs$, so $\Delta_h = h/s$, and likewise $\Delta_t = h'/s$. Write $a = (h/s)j$ and $c = (h'/s)j'$. The trace equation $ua + vc = n$ becomes $(n/s)(j + j') = n$, so $j' = s - j$, and $j, j' \ge 0$. The determinant equation becomes $b^2 = 1 + ac = 1 + (hh'/s^2)j(s-j) = 1 + rj(s-j)$.

If $j = 0$ then $a = 0$, so the internal end of a simple object m of the first orbit is $Q = \bigoplus_{g\in H}\alpha_g$, H its stabiliser. Its multiplication has components $\alpha_g\alpha_{g'} \to \alpha_{g+g'}$, which are nonzero because they correspond to composites of the isomorphisms $\alpha_g\mathsf{m} \cong \mathsf{m}$. So Q is a twisted group algebra of H, with a 2-cocycle that is a coboundary since H is cyclic. Hence $Q \cong \mathsf{A}_h$. As a module category is equivalent to the category of modules over the internal end of any nonzero object [14, §7.10], it is $\mathcal{G}_{\mathsf{A}_h}$. If $j = s$ then $c = 0$, and the same argument for the second orbit gives $\mathcal{G}_{\mathsf{A}_{h'}}$. □

Equivalently, (16) is the positive definite equation

$$4b^2 + r(2j - s)^2 = n + 4.$$

For $n = 4k$ and even s this is $b^2 + r(j - s/2)^2 = k + 1$. For $n = 4k$ and odd s, $4 \mid r$ and it is $b^2 + (r/4)(2j - s)^2 = k + 1$. Neither the divisibility lemma nor this reduction uses positivity. The remaining step excludes $0 < j < s$. It uses positivity, through the following lemma.

Lemma 9.13. *Let $\mathcal{E} = \bigoplus_{g\in\Gamma}\mathcal{E}_g$ be a graded pseudo-unitary fusion category and $Q = \bigoplus_g Q_g$ a connected separable algebra in $\mathcal{E}$. Then Q_e is connected and separable, and* $\mathrm{FPdim}\,Q_g = \mathrm{FPdim}\,Q_e$ *whenever $Q_g \neq 0$.*

Proof. Give $\mathcal{E}$ its canonical spherical structure, with positive dimensions. Then the regular module category of Q carries a module trace, and internal-end reconstruction makes Q a special symmetric Frobenius algebra [46, Prop. 5.8, Thm. 6.6]. We normalise its counit ϵ so that $\epsilon\eta = 1$, η the unit.

The form $\epsilon \circ \mu : Q \otimes Q \to \mathbf{1}$, μ the multiplication, is nondegenerate. As ϵ is supported on Q_e and μ is graded, it pairs Q_g only with $Q_{g^{-1}}$. In particular it is nondegenerate on Q_e. So Q_e, which is a subalgebra containing the unit, is a symmetric Frobenius algebra, and it is connected since $\mathrm{Hom}(\mathbf{1}, Q_e) \subseteq \mathrm{Hom}(\mathbf{1}, Q) = \mathbb{C}$. For its coproduct Δ_e, the map $\mu_e \Delta_e$ is a Q_e-bimodule endomorphism of Q_e, hence a scalar (it is determined by its value on the unit, and $\mathrm{Hom}(\mathbf{1}, Q_e) = \mathbb{C}$). Call it λ. Composing with ϵ and η gives $\lambda = \lambda\, \epsilon\eta = \epsilon\mu_e\Delta_e\eta$. Put $D = Q_e$, $p = \epsilon\mu_e$ and $c_0 = \Delta_e\eta$. Write $\mathrm{ev}_D : D^\vee \otimes D \to \mathbf{1}$ and $\mathrm{coev}_D : \mathbf{1} \to D \otimes D^\vee$ for the left duality maps, and let $\widetilde{\mathrm{ev}}_D : D \otimes D^\vee \to \mathbf{1}$ be the right evaluation from the pivotal structure. Nondegeneracy gives an isomorphism $f : D \to D^\vee$ with $p = \mathrm{ev}_D(f \otimes 1_D)$. The Frobenius duality gives $c_0 = (1_D \otimes f^{-1})\,\mathrm{coev}_D$. Symmetry of the pairing means that $p = \widetilde{\mathrm{ev}}_D(1_D \otimes f)$ as well. Consequently

$$\lambda = p\, c_0 = \widetilde{\mathrm{ev}}_D\, \mathrm{coev}_D = \mathrm{tr}_{\mathrm{sph}}(1_D) = \mathrm{FPdim}\, D > 0,$$

where the last equality uses the canonical spherical structure. Hence $\Delta_e / \mathrm{FPdim}\, Q_e$ splits μ_e and Q_e is separable.

Now regard Q as an algebra in $\mathcal{K} = {}_{Q_e}\mathcal{E}_{Q_e}$. It is a Q_e-bimodule, and its multiplication factors through $Q \otimes_{Q_e} Q$ by associativity. It is connected and separable there (a splitting of μ descends through the projection $Q \otimes Q \to Q \otimes_{Q_e} Q$), and its degree-$e$ part Q_e is the unit of $\mathcal{K}$. Let X be a simple summand of Q_g in $\mathcal{K}$, $g \neq e$, and $F(X) = X \otimes_{Q_e} Q$ the free module. By adjunction and the grading,

$$\mathrm{End}_Q(F(X)) \cong \mathrm{Hom}_{\mathcal{K}}(X, X \otimes_{Q_e} Q) = \mathrm{Hom}_{\mathcal{K}}(X, X \otimes_{Q_e} Q_e) = \mathbb{C},$$

since $X \otimes_{Q_e} Q_{g'}$ has degree gg', so only $g' = e$ contributes. So $F(X)$ is simple, and the nonzero map $F(X) \to Q$ adjoint to the inclusion $X \to Q$ is an isomorphism of Q-modules. Consequently $\mathrm{Hom}_{\mathcal{K}}(X, Q) \cong \mathrm{Hom}_Q(F(X), Q) \cong \mathrm{End}_Q(Q) = \mathbb{C}$, so X occurs in Q_g once. If X' is another simple summand of Q_g, then $F(X') \cong Q \cong F(X)$, while by the same degree argument $\mathrm{Hom}_Q(F(X), F(X')) \cong \mathrm{Hom}_{\mathcal{K}}(X, X') = 0$, a contradiction. So $Q_g = X$ is simple in $\mathcal{K}$. Taking $\mathrm{FPdim}_{\mathcal{K}}$ of $X \otimes_{Q_e} Q \cong Q$ gives $\mathrm{FPdim}_{\mathcal{K}}(X) = 1$. Since $\mathrm{FPdim}_{\mathcal{K}}(Y) = \mathrm{FPdim}_{\mathcal{E}}(Y) / \mathrm{FPdim}\, Q_e$ for Q_e-bimodules Y [14, §7.16], $\mathrm{FPdim}_{\mathcal{E}}\, Q_g = \mathrm{FPdim}\, Q_e$. □

Corollary 9.14. *If every non-invertible simple object of a graded pseudo-unitary fusion category $\mathcal{E}$ has the same irrational Frobenius–Perron dimension u, a connected separable algebra in $\mathcal{E}$ whose only invertible summand is the unit lies in $\mathcal{E}_e$.*

Proof. Its neutral component has dimension $1 + a_e u$ and a component of degree $g \neq e$ has dimension $a_g u$, for integers $a_e, a_g \geq 0$. If $a_g \neq 0$, Lemma 9.13 gives $1 + a_e u = a_g u$, so $u = 1/(a_g - a_e)$ would be rational. □

Lemma 9.15. *Let $\mathcal{G}$ satisfy (H1)–(H4), and let $\mathcal{E}_h = (\mathcal{D}_h)_{\mathrm{ad}}$. A connected separable algebra in $\mathcal{E}_h$ whose pointed part is the unit has underlying object $\mathbf{1}$ or $\mathbf{1} \oplus \bigoplus_{i \in \mathbb{Z}_s} \Xi_i$, where $\Xi_i = \Upsilon_{i,i+\epsilon_h}$.*

Strategy

In the neutral component the non-invertibles are permuted transitively by the cyclic group of invertibles, whose associator has exact order s. The two-orbit argument applies again. The associator forces the second orbit to have size exactly s, which reduces the module category to a single parameter A. Then the fusion rule for $\Xi_0\Xi_0^*$, read on the diagonal of the action matrix, gives an identity between sums of squares of nonnegative integers that leaves no room for $0 < A < s$.

Proof. Put $s' = n/s = sr$ and $\gamma_j = I_{j,j}$, $j \in \mathbb{Z}_{s'}$. By Theorem 9.7 and Corollary 9.8,

$$\gamma_j \Xi_i = \Xi_{i-j}, \quad \Xi_i \gamma_j = \Xi_{i+j}, \quad \Xi := \bigoplus_{i \in \mathbb{Z}_s} \Xi_i = \Xi^*,$$
$$\Xi^2 = s \bigoplus_{j \in \mathbb{Z}_{s'}} \gamma_j \oplus n\Xi, \qquad \Xi_0 \Xi_0^* = \bigoplus_{j \equiv 0\,(s)} \gamma_j \oplus r\Xi. \tag{17}$$

The associator. The pointed part of $\mathcal{E}_h$ is $\mathrm{Vec}^{\omega}_{\mathbb{Z}_{s'}}$, with ω the restriction of (15) to $\langle (1,1) \rangle$. By the proof of Theorem 9.7 its class has exact order s. Under $H^3(\mathbb{Z}_{s'}, \mathbb{C}^\times) \cong \mathbb{Z}/s'$, given by I of that proof, it has a parameter p of order s, so $\gcd(p, s') = s'/s = r$. Write $p = rw$ with $\gcd(w, s) = 1$.

Two orbits. Let Q be the algebra, and $\mathsf{m}_0 = Q$ its regular module, which is simple. The module category $\mathcal{M} = (\mathcal{E}_h)_Q$ is indecomposable. The stabiliser of m_0 under the γ_j is trivial, since $\mathrm{Hom}_Q(\gamma_j Q, Q) \cong \mathrm{Hom}(\gamma_j, Q) = 0$ for $j \neq 0$, so its orbit has size s'. By (17), the matrix of Ξ on orbit sums satisfies $M^2 = nI + nM$, and it is irreducible because the γ_j and Ξ generate $\mathcal{E}_h$. On each orbit sum $\sum_j \gamma_j$ acts by s', so $M^2 = ss'I + nM = nI + nM$, and the Perron–Frobenius and irrational-trace argument of Theorem 9.4 gives exactly two orbits. Let the second have size v, and let $A, C \geq 0$ and $B > 0$ be the block constants of N_Ξ. The determinant gives $s'v(B^2 - AC) = n = ss'$, so $v \mid s$.

The second orbit has size s. The stabiliser of a simple object m of the second orbit has order s'/v, and the isomorphisms $\gamma_j \mathsf{m} \cong \mathsf{m}$ give a 2-cochain whose coboundary is the restriction of ω. So that restriction is trivial. Restricting the class of parameter p to the subgroup of order $M = s'/v$ gives parameter $p \bmod M$ (substitute $a \mapsto (s'/M)a$ in the standard cocycle), so $M \mid p$. Write $v = s/d$, so $M = rd$. Then $rd \mid rw$ gives $d \mid w$, and $d \mid s$ with $\gcd(w, s) = 1$ gives $d = 1$. So $v = s$, and the trace and determinant give

$$C = r(s - A), \qquad B^2 = 1 + rA(s - A), \qquad 0 \leq A \leq s.$$

The row-norm identity. Label the first orbit $\mathsf{m}_j = \gamma_j \mathsf{m}_0$, $j \in \mathbb{Z}_{s'}$, and the second m'_i, $i \in \mathbb{Z}_s$. Let $f_j = \dim \mathrm{Hom}(\Xi_0 \mathsf{m}_0, \mathsf{m}_j)$ and $v_i = \dim \mathrm{Hom}(\Xi_0 \mathsf{m}_0, \mathsf{m}'_i)$. Since $\gamma_s \Xi_0 = \Xi_{-s} = \Xi_0$, f is periodic modulo s. Since $\Xi_i = \gamma_{-i} \Xi_0$, summing over i gives the block constants of N_Ξ in the row of m_0:

$$A = \sum_{i \in \mathbb{Z}_s} f_i, \qquad B = \sum_{i \in \mathbb{Z}_s} v_i.$$

Rigidity gives $N_{\Xi_0^*} = N_{\Xi_0}^{\mathsf{t}}$, so the $(\mathsf{m}_0, \mathsf{m}_0)$ entry of $N_{\Xi_0}^{\mathsf{t}} N_{\Xi_0} = N_{\Xi_0^* \Xi_0}$ is $\sum_{j \in \mathbb{Z}_{s'}} f_j^2 + \sum_i v_i^2 = r \sum_{i \in \mathbb{Z}_s} f_i^2 + \sum_i v_i^2$, by periodicity. The fusion rules also give $\Xi_0^* \Xi_0 = \Xi_0 \Xi_0^*$, so by (17) the same entry is $1 + rA$. Among the γ_j, $j \equiv 0 \pmod s$, only γ_0 fixes m_0, and N_Ξ has diagonal entry A at m_0. Hence

$$r \sum_{i \in \mathbb{Z}_s} f_i^2 + \sum_{i \in \mathbb{Z}_s} v_i^2 = 1 + rA.$$

For nonnegative integers, $\sum f_i^2 \geq \sum f_i = A$ and $\sum v_i^2 \geq \sum v_i = B$. So $1 + rA \geq rA + B$, that is $B \leq 1$, and $B = 1$. Then $1 = 1 + rA(s - A)$ gives $A \in \{0, s\}$, and equality forces every $f_i \in \{0, 1\}$. Since $\mathsf{m}_0 = Q$ is the regular module, f_i is the multiplicity of Ξ_i in Q. Indeed $\mathrm{Hom}(\Xi_0 \mathsf{m}_0, \mathsf{m}_i) = \mathrm{Hom}_Q(\Xi_0 Q, \gamma_i Q) \cong \mathrm{Hom}(\gamma_{-i} \Xi_0, Q) = \mathrm{Hom}(\Xi_i, Q)$. So $Q = \mathbf{1}$ ($A = 0$) or $Q = \mathbf{1} \oplus \bigoplus_i \Xi_i$ ($A = s$). □

Theorem 9.16. *Let $\mathcal{G}$ satisfy (H1)–(H3), with n even.*

1. *The indecomposable semisimple $\mathcal{G}$-module categories are the $\mathcal{G}_{\mathsf{A}_h}$, $h \mid n$, pairwise inequivalent. There are $\tau(n)$ of them.*
2. *Every fusion category categorically Morita equivalent to $\mathcal{G}$ is equivalent to some $\mathcal{D}_h$.*

3. *The connected separable algebras in $\mathcal{G}$, up to isomorphism, are one algebra on each of the objects*

$$\bigoplus_{g\in H}\alpha_g, \qquad \bigoplus_{g\in H}\alpha_g\oplus|H|R, \qquad H\le\mathbb{Z}_n;$$

there are $2\tau(n)$ of them. The algebra on $\bigoplus_{g\in H}\alpha_g\oplus|H|R$ is the internal end of a simple object of $\mathcal{G}_{\mathsf{A}_h}$ with $h=n/|H|$.

In particular this holds for $\mathcal{N}_+$, for its Galois conjugates, and for the C^* near-group categories $\mathbb{Z}_n+n$ with n even.

Proof. Step 1. Reduction to (H4). By Lemma 9.2(2) there is $\sigma\in\mathrm{Aut}(\mathbb{C})$ with $\mathcal{G}^\sigma$ satisfying (H1)–(H4). By Lemma 9.3, conjugation by σ is a bijection between the indecomposable module categories of $\mathcal{G}$ and of $\mathcal{G}^\sigma$, between their connected separable algebras, and between the corresponding categories of bimodules. It preserves equivalence, isomorphism, underlying objects and internal ends, and it maps A_h to the subgroup algebra of $\mathcal{G}^\sigma$ and $\mathcal{D}_h$ to the corresponding dual of $\mathcal{G}^\sigma$. So each of the statements (1)–(3) holds for $\mathcal{G}$ if it holds for $\mathcal{G}^\sigma$, and we may assume (H4). Then each $\mathcal{D}_h$ is pseudo-unitary. The global dimension [13] and the Frobenius–Perron dimension [14, §7.16] are invariant under categorical Morita equivalence, so $\dim\mathcal{D}_h=\dim\mathcal{G}=\mathrm{FPdim}\,\mathcal{G}=\mathrm{FPdim}\,\mathcal{D}_h$.

Step 2. Every module category is $\mathcal{G}_{\mathsf{A}_h}$. Let $\mathcal{M}$ be an indecomposable module category, m a simple object with stabiliser H of order h, and Q its internal end. Its pointed part $\bigoplus_{g\in H}\alpha_g$ is a subalgebra (products of invertibles are invertible), isomorphic to A_h by the argument at the end of the proof of Theorem 9.12. So Q is an A_h-bimodule, and by associativity its multiplication factors through $Q\otimes_{\mathsf{A}_h}Q$. This makes Q an algebra $\widetilde{Q}$ in $\mathcal{D}_h$ with unit $\mathsf{A}_h\to Q$. It is connected, since $\mathrm{Hom}_{\mathcal{D}_h}(\mathsf{A}_h,Q)\subseteq\mathrm{Hom}(\mathbf{1},Q)=\mathbb{C}$, and separable, since a splitting of the multiplication of Q descends through $Q\otimes Q\to Q\otimes_{\mathsf{A}_h}Q$. The α-supported part of Q is $\mathsf{A}_h=I_{0,0}$, so the unit is its only invertible summand. The non-invertible simples of $\mathcal{D}_h$ all have dimension $d_+/s\notin\mathbb{Q}$, so by Corollary 9.14 $\widetilde{Q}$ lies in $\mathcal{E}_h$, and by Lemma 9.15 it contains 0 or s of the Ξ_i. Each Υ has underlying object tR, so the multiplicity a of R in Q is 0 or $st=h$. In (16), $a=(h/s)j$, so $j\in\{0,s\}$. By Theorem 9.12, $\mathcal{M}$ is some $\mathcal{G}_{\mathsf{A}_{h''}}$.

Step 3. The categories $\mathcal{G}_{\mathsf{A}_h}$. The simple right A_h-modules supported on the α's are the free modules $\alpha_g\otimes\mathsf{A}_h$, one for each of the h' cosets of H. A module structure on R is a representation of H given by the maps $R\alpha_a\cong R$ (that is, by the U_a), so the R-supported simple modules are R_ψ, $\psi\in\widehat{H}$, each with underlying object R. By $\alpha_g(U_a)=\chi(g,a)U_a$, $\alpha_gR_\psi=R_{\psi\chi(g,\cdot)}$, and as χ is perfect the restriction $\mathbb{Z}_n\to\widehat{H}$ is onto, so $\mathbb{Z}_n$ is transitive on the h modules R_ψ. The free module $R\otimes\mathsf{A}_h=\bigoplus_\psi R_\psi$ gives $b=1$ and $a=0$ for the first orbit, and the trace equation $h'\cdot0+h\,c=n$ gives $c=h'$. So $\mathcal{G}_{\mathsf{A}_h}$ has orbit constants $(a,b,c)=(0,1,h')$, with orbits of sizes h' (diagonal 0) and h (diagonal $h'\ge1$). The orbit with zero diagonal is unique and has size h', so the $\mathcal{G}_{\mathsf{A}_h}$ are pairwise inequivalent. This proves (1), and (2) follows by taking dual categories.

Step 4. Algebras. A connected separable algebra is the internal end of a simple object of its module category, which by (1) is some $\mathcal{G}_{\mathsf{A}_h}$. A simple object of the first orbit has stabiliser H and internal end A_h. A simple object of the second orbit has stabiliser of order $n/h=h'$ and R-multiplicity $c=h'$, so its internal end has underlying object $\bigoplus_{g\in H'}\alpha_g\oplus h'R$ with $|H'|=h'$. This gives the list in (3), and each object occurs. By (H2), for each g the functor $\alpha_g\otimes-$ on $\mathcal{G}_{\mathsf{A}_h}$ is a module autoequivalence, with module constraint given by the half-braiding z_g. Module equivalences preserve internal ends as algebras. Both orbits are transitive, so the internal ends of the simple objects of one orbit are isomorphic as algebras. Their underlying objects distinguish the two orbits and the subgroup, so there are exactly $2\tau(n)$ isomorphism classes.

For the last sentence of the theorem use Lemma 9.2. □

Both kinds of algebra in (3) exist for every even n. The algebra on $\bigoplus_{g\in H}\alpha_g\oplus|H|R$ is

constructed as an internal end in $\mathcal{G}_{\mathsf{A}_{n/|H|}}$. For $H = 0$ this is the algebra $\mathbf{1} \oplus R$, from a simple R-supported module of $\mathcal{G}_{\mathsf{A}_n}$.

Corollary 9.17. *The Lagrangian algebras in $Z(\mathcal{G})$ are the full centres of the A_h, $h \mid n$. They are pairwise non-isomorphic, and there are $\tau(n)$ of them.*

Proof. For a fixed fusion category, indecomposable module categories up to equivalence correspond bijectively to Lagrangian algebras in its double up to isomorphism, through the full centre [10, Prop. 4.8, Rem. 4.9]. Apply this to Theorem 9.16(1). □

A fusion category whose double is braided equivalent to $Z(\mathcal{G})$ is categorically Morita equivalent to $\mathcal{G}$ [14], hence equivalent to some $\mathcal{D}_h$. Only the $\mathcal{D}_h$ with $\gcd(h, h') = 1$ have near-group fusion rules, so at most $2^{\omega(n)}$ near-group categories of cyclic type have double braided equivalent to $Z(\mathcal{G})$, where $\omega(n)$ is the number of prime divisors of n. For $\mathcal{G} = \mathcal{N}_+$ and $n > 2$ there is exactly one.

Theorem 9.18. *Let $n > 2$ be even and $\mathcal{G} = \mathcal{N}_+$ (Theorem 5.1 for $n = 4k$, Theorem 12.2 for $n \equiv 2 \pmod 4$, whose proof does not use the present theorem). Then $\mathcal{D}_h$ is tensor equivalent to $\mathcal{N}_+$ for every divisor h of n with $\gcd(h, n/h) = 1$. In particular the dual $\mathcal{D}_n$ of $\mathcal{N}_+$ with respect to the algebra $\bigoplus_{g\in\mathbb{Z}_n} \alpha_g$ is equivalent to $\mathcal{N}_+$, and $\mathcal{N}_+$ is the only near-group category, up to equivalence, whose double is braided equivalent to $Z(\mathcal{N}_+)$.*

Strategy

The isometries of the lattice factor $\mathcal{P}$ of $Z(\mathcal{N}) \simeq \mathcal{C} \boxtimes \mathcal{P}^{\mathrm{rev}}$ give $2^{\omega(n)}$ braided autoequivalences of $Z(\mathcal{N}_+)$. A braided autoequivalence carries the Lagrangian algebra L_1 of the regular module category to a Lagrangian algebra whose dual category is again $\mathcal{N}_+$. If it fixes L_1, it is induced by a tensor autoequivalence ψ of $\mathcal{N}_+$. Since the isometries act trivially on the factor $\mathcal{C}$, ψ commutes with the free-module functor, and because the algebra $A' = \mathbf{1} \oplus X$ has no nontrivial automorphism, ψ is then isomorphic to the identity functor. This is incompatible with the action of a nontrivial isometry on the invertible objects. So the $2^{\omega(n)}$ images of L_1 are distinct, and they exhaust the Lagrangian algebras of near-group type.

Proof. Notation. $F_+ : Z(\mathcal{N}_+) \to \mathcal{N}_+$ is the forgetful functor and I its right adjoint. $L_1 = I(\mathbf{1})$ is the Lagrangian algebra of the regular module category, and $\mathcal{N}_+ \simeq Z(\mathcal{N}_+)_{L_1}$, the category of L_1-modules, with F_+ corresponding to the free-module functor [10, Lemma 3.5]. In the same way, with L_h the Lagrangian algebra of $\mathcal{G}_{\mathsf{A}_h}$ (Corollary 9.17), $\mathcal{D}_h \simeq Z(\mathcal{N}_+)_{L_h}$, since L_h is the image of the corresponding algebra of $\mathcal{D}_h$ under $Z(\mathcal{D}_h) \simeq Z(\mathcal{N}_+)$ [10, Prop. 4.8]. $G : \mathcal{C}' \to Z(\mathcal{N}_+)$ is the braided functor $x \mapsto [x, 0]$, the composite of $x \mapsto x \boxtimes M_0$ with the de-equivariantisation functor of Theorem 7.1(2). By the description of the equivalences in the proof of Theorem 7.1 (Theorem 12.2(4) for $n \equiv 2 \pmod 4$), $F_+ \circ G$ is the free-module functor $F : \mathcal{C}' \to \mathcal{C}'_{A'} = \mathcal{N}_+$, as tensor functors.

The autoequivalences. Let u be an integer with $u^2 \equiv 1 \pmod{4n}$. Then $s \mapsto us$ is an automorphism of $\mathbb{Z}_{2n}$ preserving the quadratic form $e(s^2/4n)$ of the pointed braided category $\mathcal{P}$, so it is induced by a braided autoequivalence of $\mathcal{P}$ [9], and $\mathrm{id} \boxtimes u$ is a braided autoequivalence of $\mathcal{C}\boxtimes\mathcal{P}^{\mathrm{rev}}$. As u is odd, it preserves the subcategory $\mathcal{E}'$ and fixes $b = J\boxtimes M_n$. So it induces a braided autoequivalence φ_u of the de-equivariantisation $Z(\mathcal{N}_+)$, with $\varphi_u[x, s] = [x, us]$. $\varphi_u \circ G \cong G$ as tensor functors, since $\mathrm{id} \boxtimes u$ is the identity on $\mathcal{C}' \boxtimes M_0$. The classes of such u modulo $2n$ form a group Γ. Writing $n = 2^e n'$ with n' odd, u is determined modulo 2^{e+1} and modulo each prime power dividing n', with two choices ± 1 in each case. So $|\Gamma| = 2^{\omega(n)}$. Distinct elements of Γ act differently on the labels.

Elements fixing L_1. Suppose $\varphi_u(L_1) \cong L_1$ as algebras. Then φ_u induces a tensor autoequivalence ψ of $Z(\mathcal{N}_+)_{L_1} \simeq \mathcal{N}_+$, and since φ_u maps the free module $Y \otimes L_1$ to $\varphi_u(Y) \otimes L_1$, $\psi \circ F_+ \cong F_+ \circ \varphi_u$ as tensor functors. Composing with G,

$$\psi \circ F \cong F_+ \circ \varphi_u \circ G \cong F_+ \circ G = F$$

as tensor functors. The category $\mathcal{C}'_{A'}$ is a left $\mathcal{C}'$-module category, with $x \triangleright M = x \otimes M \cong F(x) \otimes_{A'} M$, and the displayed isomorphism makes ψ a $\mathcal{C}'$-module functor, namely $\psi(x \triangleright M) \cong \psi F(x) \otimes_{A'} \psi(M) \cong F(x) \otimes_{A'} \psi(M) = x \triangleright \psi(M)$, the module-functor axiom following from the monoidality of the isomorphisms. A $\mathcal{C}'$-module endofunctor of $\mathcal{C}'_{A'}$ is $- \otimes_{A'} B$ for an A'-bimodule B in $\mathcal{C}'$ [14, §7.11]. As $\psi(A') \cong A'$, $B \cong A'$ as a right module, and its left action is then given by an algebra endomorphism σ of $A' = \mathbf{1} \oplus X$, which is the identity on $\mathbf{1}$ and a scalar λ on X. The component $X \otimes X \to X$ of the multiplication of A' is nonzero by the lattice computation preceding Theorem 5.1, so $\lambda^2 = \lambda$. $\lambda \neq 0$ because ψ is an equivalence. So $\sigma = \mathrm{id}$, $B \cong A'$ as a bimodule, and ψ is isomorphic to the identity functor. In particular $\psi(\alpha_g) \cong \alpha_g$ for all $g \in \mathbb{Z}_n$. But $\alpha_g = F_+(a_g)$ with $a_g = [V, 2g]$, and $\psi(\alpha_g) \cong F_+(\varphi_u(a_g)) = F_+(a_{ug}) = \alpha_{ug}$. Hence $ug \equiv g \pmod n$ for all g, that is $u \equiv 1 \pmod n$.

So $u \equiv 1$ or $u \equiv 1 + n \pmod{2n}$. The second satisfies $u^2 \equiv 1 \pmod{4n}$ only if $n \equiv 2 \pmod 4$, and then it does not fix L_1. The simple object $[A_1, n+1]$ has $F_+[A_1, n+1] \cong (M_{n+1} \oplus M_{n-1} \oplus R) \otimes M_{n+1} \supset M_0 = \mathbf{1}$, so it is a summand of L_1 (by the proof of Proposition 6.3, $F(A_a) = M_{n+a} \oplus M_{n-a} \oplus R_\pm$). Its image $[A_1, (n+1)^2] = [A_1, 1]$ has $F_+[A_1, 1] \cong M_{n+2} \oplus M_n \oplus R$, which does not contain $\mathbf{1}$ because $n > 2$. So only the identity of Γ fixes L_1.

Conclusion. The algebras $\varphi_u(L_1)$, $u \in \Gamma$, are therefore $2^{\omega(n)}$ pairwise non-isomorphic Lagrangian algebras. Each is some L_h (Corollary 9.17), and φ_u induces a tensor equivalence $\mathcal{N}_+ \simeq Z(\mathcal{N}_+)_{L_1} \to Z(\mathcal{N}_+)_{L_h} \simeq \mathcal{D}_h$. Such an h has $\gcd(h, n/h) = 1$, since $\mathcal{D}_h$ has rank $n + \gcd(h, n/h)^2$ (Corollary 9.9). There are exactly $2^{\omega(n)}$ divisors h of n with $\gcd(h, n/h) = 1$, so every one of them occurs. The last statement follows from the paragraph before the theorem. □

Relation to earlier work

For C* near-group categories of type $G + |G|$, G any finite abelian group, the case $h = n$ is a theorem of Izumi. The subfactor $\rho(M) \rtimes G \subset M$ with principal graph $2^G 1$ is self-dual, that is, its two even parts are equivalent [32, Thm. 11.3]. In our notation this says $\mathcal{D}_n \simeq \mathcal{G}$ for these categories. Grossman, Jordan and Snyder had raised the question for $G = \mathbb{Z}_p$, and observed self-duality for $\mathbb{Z}_3$ [22, §4.2]. Izumi's proof compares Cuntz algebra models and uses the C* structure. Theorem 9.18 is the corresponding statement for the category $\mathcal{N}_+$, which is not known to be unitary after Galois conjugation, by a different argument, and for every h with $\gcd(h, n/h) = 1$, not only $h = n$. We do not know whether the case $1 < h < n$ holds for every category satisfying (H1)–(H3). For $n = 2$ the group Γ acts trivially on the invertible objects and our argument does not apply.

Equivalences among all the duals

An equivalence between $\mathcal{G}$ and one of its duals transports the subgroup algebras, and so gives equivalences among the other duals. This is made precise by the following lemma, which is a computation in the pointed part.

Lemma 9.19. *Let $\mathcal{G}$ satisfy* (H1)–(H3)*, let $h \mid n$ with $\gcd(h, h') = 1$, and let $t \mid n$. Write $t = t_1 t_2$ with $t_1 \mid h'$ and $t_2 \mid h$. The invertible objects of $\mathcal{D}_h$ form a cyclic group of order n, and the pointed part of $\mathcal{D}_h$ has trivial associator. Let $\mathsf{A}_t^{(h)}$ be the algebra of its subgroup of order t. Then the category of $\mathsf{A}_t^{(h)}$-bimodules in $\mathcal{D}_h$ is tensor equivalent to $\mathcal{D}_{t''}$, where*

$$t'' = t_1 \cdot \frac{h}{t_2}.$$

Proof. By Theorem 9.7 with $s = 1$, the invertible objects of $\mathcal{D}_h$ are the $I_{g,l}$, $(g,l) \in \mathbb{Z}_{h'} \times \mathbb{Z}_h$, a cyclic group since $\gcd(h,h') = 1$, and the associator of the pointed part has order $s = 1$. Its subgroup of order t is $T_1 \times T_2$ with $T_1 \leq \mathbb{Z}_{h'}$ of order t_1 and $T_2 \leq \mathbb{Z}_h$ of order t_2, and $\mathsf{A}_t^{(h)} = \bigoplus_{(g,l)\in T_1\times T_2} I_{g,l}$, the algebra structure being unique up to isomorphism because the group is cyclic.

An algebra in the category of A_h-bimodules is an algebra B in $\mathcal{G}$ with an algebra map $\mathsf{A}_h \to B$, and its bimodules in $\mathcal{D}_h$ are the B-bimodules in $\mathcal{G}$, with the same tensor product. So the category in question is the category of B-bimodules in $\mathcal{G}$, where B is $\mathsf{A}_t^{(h)}$ regarded as an algebra in $\mathcal{G}$. Its underlying object is a sum of invertible objects, so B is an algebra in the pointed part $\mathrm{Vec}_{\mathbb{Z}_n}$ of $\mathcal{G}$.

We determine the $\mathrm{Vec}_{\mathbb{Z}_n}$-module category of right B-modules in $\mathrm{Vec}_{\mathbb{Z}_n}$. It is the category of $\mathsf{A}_t^{(h)}$-modules in the category $\mathcal{M}$ of right A_h-modules in $\mathrm{Vec}_{\mathbb{Z}_n}$. The simple objects of $\mathcal{M}$ are the h' free modules $\alpha_x \otimes \mathsf{A}_h$, $x \in \mathbb{Z}_n/H$, and $I_{g,l}$ acts on them by $x \mapsto x + g$, since the underlying object of $I_{g,l}$ is $\bigoplus_{a\in H} \alpha_{g+a}$. So $T_1 \times T_2$ has h'/t_1 orbits on the simple objects of $\mathcal{M}$, each with stabiliser $0 \times T_2$, and the simple $\mathsf{A}_t^{(h)}$-modules supported on one orbit correspond to the characters of the cyclic group T_2. Hence the category has $(h'/t_1)\, t_2 = n/t''$ simple objects.

It is indecomposable, for the following reason. A module category is indecomposable exactly when its identity functor is a simple object of the category of module endofunctors. For the category of right B-modules, the module endofunctors are the B-bimodules in $\mathrm{Vec}_{\mathbb{Z}_n}$, and the identity functor is B itself [14]. A B-bimodule endomorphism f of B is determined by $f \circ \eta$, with η the unit, because $f = \mu(f\eta \otimes 1)$. It is in particular a map of A_h-bimodules, so $f \circ \eta$ lies in $\mathrm{Hom}_{\mathcal{D}_h}(\mathbf{1}, \mathsf{A}_t^{(h)})$. This space is one-dimensional, since $\mathsf{A}_t^{(h)}$ is the algebra of a subgroup and so is connected. Hence $\mathrm{End}_{B\text{-}B}(B) = \mathbb{C}$ and B is a simple bimodule.

The indecomposable $\mathrm{Vec}_{\mathbb{Z}_n}$-module categories are the categories of A_u-modules, $u \mid n$, of rank n/u [14]. So the categories of B-modules and of $\mathsf{A}_{t''}$-modules in $\mathrm{Vec}_{\mathbb{Z}_n}$ are equivalent as module categories, that is, B and $\mathsf{A}_{t''}$ are Morita equivalent algebras in $\mathrm{Vec}_{\mathbb{Z}_n}$, hence in $\mathcal{G}$. Morita equivalent algebras have equivalent categories of bimodules, so the category of B-bimodules in $\mathcal{G}$ is equivalent to $\mathcal{D}_{t''}$. □

Theorem 9.20. *Let $n > 2$ be even and $\mathcal{G} = \mathcal{N}_+$. For divisors $t, \tilde{t}$ of n,*

$$\mathcal{D}_t \simeq \mathcal{D}_{\tilde{t}} \quad \Longleftrightarrow \quad \gcd(t, n/t) = \gcd(\tilde{t}, n/\tilde{t}).$$

So the fusion categories categorically Morita equivalent to $\mathcal{N}_+$, equivalently those whose double is braided equivalent to $Z(\mathcal{N}_+)$, are, up to equivalence, the categories $\mathcal{D}_s$ with $s^2 \mid n$. They are pairwise inequivalent, and there are $\prod_{p^e \| n}(\lfloor e/2 \rfloor + 1)$ of them. In particular $\mathcal{D}_t \simeq \mathcal{D}_{n/t}$ for every $t \mid n$.

Proof. If $\mathcal{D}_t \simeq \mathcal{D}_{\tilde{t}}$, the ranks $n + \gcd(t, n/t)^2$ agree (Corollary 9.9). Conversely, let $\gcd(t, n/t) = \gcd(\tilde{t}, n/\tilde{t})$. For a prime p with $p^e \parallel n$ let p^{a_p} and $p^{\tilde{a}_p}$ be the exact powers of p in t and $\tilde{t}$. The hypothesis says $\min(a_p, e - a_p) = \min(\tilde{a}_p, e - \tilde{a}_p)$, so $\tilde{a}_p \in \{a_p, e - a_p\}$. Let h be the product of the p^e over the primes with $\tilde{a}_p \neq a_p$. Then $\gcd(h, n/h) = 1$, and in the notation of Lemma 9.19, $t'' = t_1\, h/t_2 = \tilde{t}$. By Theorem 9.18 there is a tensor equivalence $\psi : \mathcal{N}_+ \to \mathcal{D}_h$. It maps the invertible objects of $\mathcal{N}_+$ isomorphically onto those of $\mathcal{D}_h$, so it maps the subgroup of order t onto the subgroup of order t. An algebra structure on the sum over a cyclic subgroup is unique up to isomorphism, so $\psi(\mathsf{A}_t) \cong \mathsf{A}_t^{(h)}$ as algebras. Therefore $\mathcal{D}_t$, the category of A_t-bimodules in $\mathcal{N}_+$, is equivalent to the category of $\mathsf{A}_t^{(h)}$-bimodules in $\mathcal{D}_h$, which is $\mathcal{D}_{\tilde{t}}$ by Lemma 9.19. The remaining statements follow from Theorem 9.16(2) and Corollary 9.9. For the last, take $h = n$. □

Corollary 9.21. *For every C^* near-group category $\mathcal{G}$ of type $\mathbb{Z}_n + n$, n even, $\mathcal{D}_t \simeq \mathcal{D}_{n/t}$ for every $t \mid n$.*

Proof. $\mathcal{D}_n \simeq \mathcal{G}$ by [32, Thm. 11.3]. Apply the argument of the proof of Theorem 9.20 with $h = n$, for which Lemma 9.19 gives $t'' = n/t$. □

Remark 9.22 (Lagrangian algebras). In the labelling of Proposition 7.2, write $c_{\{p,q\}} = [A_a, s]$ with $p = \frac{1}{2}(s+n+a)$, $q = \frac{1}{2}(s+n-a)$ in $\mathbb{Z}_n$. Then $F_+(c_{\{p,q\}}) = \alpha_p \oplus \alpha_q \oplus R$ and $\theta_{c_{\{p,q\}}} = e(-pq/n)$, and a_g, b_g correspond to $p = q = g$. The underlying object of the Lagrangian algebra L_h can be found by computing the image of each simple object of $Z(\mathcal{N}_+)$ in $\mathcal{D}_h$. We do not reproduce the computation. The result is

$$\bigoplus_{(p,q)\in h'\mathbb{Z}_n \times h\mathbb{Z}_n} Y_{p,q}, \qquad Y_{g,g} = a_g \oplus b_g, \quad Y_{p,q} = c_{\{p,q\}} \ (p \neq q).$$

Thus L_h and $L_{h'}$ have the same underlying object, and are different algebras when $h \neq h'$. The isometry u acts by $(p,q) \mapsto (ep + (e-1)q, (e-1)p + eq)$, where $e = \frac{1}{2}(u+1)$ is an idempotent modulo n, and exchanges the p-parts of h and h' for the primes p at which $e \equiv 0$. This is the action on the Lagrangian algebras behind Theorems 9.18 and 9.20. For $n = 4, 6, 8, 12$ the displayed objects are fixed by S and T and have dimension $\dim \mathcal{N}_+$, and the action of Γ is as stated. For $n = 2, 4, 6, 8, 10, 12$ every permutation of the simple objects preserving S and T permutes these objects in the same way as some element of Γ (`lagrangian_orbits.py`). This remark is not used elsewhere.

Theorem 9.23. *For $h, t \mid n$, let $H, K \leq \mathbb{Z}_n$ have orders h, t, and put $a = \gcd(h,t)$, $\ell = \operatorname{lcm}(h,t)$, $e = \gcd(n, ht)$ and $u = ht/e$. The category $\mathrm{Fun}_{\mathcal{G}}(\mathcal{G}_{\mathsf{A}_h}, \mathcal{G}_{\mathsf{A}_t}) \simeq {}_{\mathsf{A}_h}\mathcal{G}_{\mathsf{A}_t}$ has $na^2/(ht)$ simple objects whose underlying object is the sum of the ℓ invertibles in a coset of $H + K$, and $e^2/(ht)$ simple objects with underlying object uR. Its rank is $(na^2 + e^2)/(ht)$. In the normalised dimension $\mathrm{FPdim}_{\mathcal{G}}(X)/\sqrt{ht}$ these have dimensions $\sqrt{ht}/a$ and $\sqrt{ht}\, d_+/e$, and the sum of the squared normalised dimensions is $n + d_+^2$. By Theorem 9.16 these are all the categories of module functors between indecomposable $\mathcal{G}$-module categories.*

Proof. A module functor $\mathcal{G}_{\mathsf{A}_h} \to \mathcal{G}_{\mathsf{A}_t}$ is $- \otimes_{\mathsf{A}_h} Y$ for an A_h–A_t-bimodule Y [14, §7.11]. In the pointed part, a simple bimodule is supported on one coset of $H + K$, which has ℓ elements, and the relative right action is a character of $H \cap K$, of order a. The pointed associator is trivial, so no projective twist occurs. There are $(n/\ell)a = na^2/(ht)$ of them. In the R-supported part the left H- and right K-actions on V satisfy the commutation relation (12) with $a \in H$, $j \in K$. The image of $H \times K \to \mathbb{C}^\times$, $(a, j) \mapsto \chi(a, j)$, has order u, and the radicals in H and K have orders h/u and t/u. So the twisted group algebra has $(h/u)(t/u) = e^2/(ht)$ irreducibles, each of dimension u. Dividing the underlying dimensions ℓ and ud_+ by $\sqrt{ht}$ gives the normalised dimensions, and $(na^2/ht)(ht/a^2) + (e^2/ht)(ht\, d_+^2/e^2) = n + d_+^2$. □

9.6 Subfactors

In a C* realisation, connected separable algebras are the Q-systems of irreducible subfactors. More precisely, every connected separable algebra in a C* fusion category is isomorphic as an algebra to a Q-system, and two Q-systems that are isomorphic as algebras are unitarily isomorphic [4, 5]. So classification up to isomorphism of algebras and up to isomorphism of Q-systems agree. Theorem 9.16 then gives the following, for the C* near-group categories $\mathbb{Z}_n + n$, n even, of [31, 16]. They exist for every even $n \leq 12$ [16, Prop. 6, Table 2]. We use the notation of [16, §2]. Thus $\mathcal{G}$ is realised by endomorphisms α_g, ρ of an infinite factor M, with $\alpha_g \rho = \rho$ and $d_\rho = d_+$.

Proposition 9.24. *Let $N \subset M$ be an irreducible subfactor of finite index whose dual canonical endomorphism θ lies in $\mathcal{G}$. Then the Q-system of $N \subset M$ is isomorphic as an algebra, and hence unitarily isomorphic as a Q-system, to that of exactly one of*

1. $M^H \subset M$, $H \leq \mathbb{Z}_n$, *of index* $|H|$, *with* $\theta = \bigoplus_{g\in H} \alpha_g$;
2. $\rho(M) \rtimes K \subset M$, $K \leq \mathbb{Z}_n$, *of index* $d_+^2/|K| = (n/|K|)(1+d_+)$, *with* $\theta = \bigoplus_{g\in H} \alpha_g \oplus |H|\rho$, $|H| = n/|K|$.

All of these are intermediate subfactors of $\rho(M) \subset M$ *[16, Cor. 2]. In case (2) the* M–M *system is* $\mathcal{G}$ *and the* N–N *system is* $\mathcal{D}_{|K|}$.

Proof. The Q-system of an irreducible subfactor is a connected separable algebra in $\mathcal{G}$ with underlying object θ, and its dimension is the index. So it is isomorphic to one of the algebras of Theorem 9.16(3), which are distinguished by their underlying objects.

The subfactors in (1) exist, since α restricts to an outer action of H, and $M^H \subset M$ has dual canonical endomorphism $\bigoplus_{g\in H} \alpha_g$. For (2), [16, Cor. 2] gives the chain $\rho(M) \subset \rho(M) \rtimes K \subset \rho(M) \rtimes \mathbb{Z}_n \subset M$ with $[M : \rho(M)] = d_\rho^2$, so $[M : \rho(M) \rtimes K] = d_\rho^2/|K|$. These subfactors are irreducible, since $(\rho(M) \rtimes K)' \cap M \subseteq \rho(M)' \cap M = \mathbb{C}$ as ρ is irreducible. Their index is irrational, so their Q-systems are of the second type in Theorem 9.16(3), and the dimension $|H|(1+d_+) = d_+^2/|K| = n(1+d_+)/|K|$ gives $|H| = n/|K|$. As $\mathbb{Z}_n$ has one subgroup of each order and there is one algebra on each object, the subfactors in (2) account for all algebras of the second type.

In case (2), $\rho \prec \theta$, and ρ generates $\mathcal{G}$, so the M–M system is $\mathcal{G}$. The N–N system is then the dual of $\mathcal{G}$ with respect to the module category of the Q-system [14], which by Theorem 9.16(3) is $\mathcal{G}_{\mathsf{A}_h}$ with $h = n/|H| = |K|$. Its dual is $\mathcal{D}_{|K|}$. □

For $K = 1$ this is the subfactor $\rho(M) \subset M$ of [16, Cor. 2], with both even systems $\mathcal{G}$. For $K = \mathbb{Z}_n$ it is the subfactor of index $1 + d_+ = \delta$ and principal graph 2^n1 of [31, §5], [16, §2.2], which is self-dual [32, Thm. 11.3], so that $\mathcal{D}_n \simeq \mathcal{G}$. By Corollary 9.21 the N–N systems for K and for the subgroup of order $n/|K|$ are equivalent. For $1 < |K| < n$ the N–N system $\mathcal{D}_{|K|}$ is not a near-group category when $\gcd(|K|, n/|K|) > 1$. For $n = 4k$ and $|K| = 2$ it is $\mathbb{Z}_2$-graded, with neutral component $\mathcal{Q}_2$ (Corollary 9.8, Proposition 9.10), the analogue of the category $\mathcal{D}$. For $\mathcal{N}_+$ itself, Proposition 9.24 applies to its positive Galois conjugate once that conjugate is shown to be a C* category. This is open (§13).

10 Scalar reconstruction and finite quantum dilogarithms

The mixed vertices of Lemma 9.6 give more than the bimodule calculations of §9. They reduce the associator of the Flohr near-group category to one scalar function. We first make this reduction in spherical tensor calculus and then identify the essential image of the algebraic Leavitt construction of Radchenko and Wheeler. Throughout this section the Flohr parameter is integral: $n = 4k$ with $k \in \mathbb{Z}_{>0}$. The scalar c below denotes the tetrahedral rotation coefficient of (27).

Theorem 10.1 (Scalar reconstruction)**.** *For every* $k \in \mathbb{Z}_{>0}$, *the near-group category* $\mathcal{N}_+$ *admits a triangular-prism presentation by scalar data* (d_-, ξ, a, c, b) *as in Proposition 10.3 below. These data determine its associator, including the multiplicity-n channel of* R^2.

There are a nondegenerate even quadratic form q_{L} *on* $\mathbb{Z}_{4k}$ *and a normalized finite quantum dilogarithm* E_{L} *for* $(\mathbb{Z}_{4k}, q_{\mathrm{L}})$ *such that*

$$\mathcal{N}_+ \simeq \mathcal{L}(q_{\mathrm{L}}, E_{\mathrm{L}}), \qquad E_{\mathrm{L}}(0) = -\frac{\sqrt{4k}}{d_-},$$

where $\mathcal{L}(q, E)$ *denotes the Radchenko–Wheeler Leavitt fusion category. The pair* $(q_{\mathrm{L}}, E_{\mathrm{L}})$ *is determined by the tensor category* $\mathcal{N}_+$ *up to an automorphism of* $\mathbb{Z}_{4k}$.

The proof uses algebraic dual bases throughout. The scalar presentation is related to the coordinates of [26]; the Leavitt construction is the purely algebraic construction of [44, §5.4]. The scalar function in a tetrahedral presentation and the intrinsic Leavitt function will be denoted separately by E_{T} and E_{L}. Essential surjectivity of the construction will suffice; we shall not need to identify these two choices of coordinates pointwise.

10.1 Algebraic normalization of the vertices

We use a slightly more general setup for the extraction argument. Let $G = \mathbb{Z}_n$, $n > 1$, and let $\mathcal{C}$ be a complex spherical fusion category with fusion rules $G + n$. Assume that its pointed subcategory is Vec_G, that its mixed interchange bicharacter is nondegenerate and symmetric, and that $\nu_2(R) = 1$. These assumptions hold for $\mathcal{N}_+$ by Lemmas 9.1, 9.2 and 9.6. Every invertible has spherical dimension 1, since $\alpha_g R = R$ and the dimension of a simple object is nonzero. Write $d = \dim R$; then

$$d^2 = n(d+1), \qquad d \neq 0. \tag{18}$$

Choose coherent isomorphisms $L_g : \alpha_g R \to R$. The obstruction to choosing them for the fixed pointed multiplication lies in $H^2(\mathbb{Z}_n, \mathbb{C}^\times) = 0$. Let $s : \mathbf{1} \to R^2$ and $e : R^2 \to \mathbf{1}$ be a symmetric self-duality, with $es = d$. Let $u_g : R \to R\alpha_g$ be the mate of L_g, characterized by

$$(1_R \otimes L_g)(u_g \otimes 1_R)s = s. \tag{19}$$

The mate operation and coherence give the right action law. Opposite mixed vertices are L_g^{-1} and u_g^{-1}, with the indicated pivotal bends. A fusion followed by its splitting is therefore the identity; its closure is 1 or d according to the simple object. This fixes the mixed theta values without choosing an inner product.

Put $V = \mathrm{Hom}(R^2, R)$. A marked trivalent vertex is an element of V read in the chosen corner; bending identifies it with $\mathrm{Hom}(\mathbf{1}, R^3)$. Write

$$D_g = (1_R \otimes L_g)(u_g \otimes 1_R) \in \mathrm{End}(R^2).$$

The three corner actions on V can be written with their domains and codomains as

$$\begin{aligned} X_g(f) &= f D_g, \\ Z_g(f) &= L_{-g}(1_{\alpha_{-g}} \otimes f)(L_{-g}^{-1} \otimes 1_R), \\ Y_g(f) &= u_g^{-1}(f \otimes 1_{\alpha_g})(1_R \otimes u_g). \end{aligned} \tag{20}$$

Associators are suppressed by coherence. The mixed cancellation and interchange relations give

$$Z_g X_h = \xi(g,h) X_h Z_g, \qquad X_g Y_h = \xi(g,h) Y_h X_g, \qquad Y_g Z_h = \xi(g,h) Z_h Y_g, \tag{21}$$

where each of X, Y, Z is a representation of G and ξ is a nondegenerate symmetric bicharacter. In the endomorphism convention of Lemma 9.6, these formulas read $X_g(f) = f U_g$, $Z_g(f) = \alpha_{-g}(f)$ and $Y_g(f) = U_{-g} f R(U_g)$, and

$$\xi = \chi^{-1}. \tag{22}$$

For example, $\alpha_{-g}(U_h) = \chi(g,h)^{-1} U_h$ proves the first commutation. Moving U_g through $R(U_h)$ proves the second; the third is its bent version. This also specifies the convention when the bicharacter is not real.

Join two identically marked vertices to obtain the bilinear theta pairing $B : V \times V \to \mathbb{C}$. It is nondegenerate: bending identifies its second argument with the trace-dual space $\mathrm{Hom}(R, R^2)$, and the trace pairing between the two spaces is nondegenerate by semisimplicity and $d \neq 0$. It is symmetric: reversing the three chosen self-dualities contributes $\nu_2(R)^3 = 1$. Equivalently,

apply the pivotal transpose twice to the two vertex maps and use cyclicity of the spherical trace. Sliding a group strand across this closure gives

$$Z_g^{\mathrm{t}_B} = Z_{-g}, \qquad X_g^{\mathrm{t}_B} = Y_{-g}. \tag{23}$$

These identities use (19) to reverse a mixed vertex; the reversal is an algebraic mate.

Lemma 10.2. *There are a basis $(v_h)_{h\in G}$ of V, an even function $a : G \to \mathbb{C}^\times$ and $c \in \mathbb{C}^\times$ such that*

$$a(0) = 1, \qquad a(-g) = a(g), \qquad a(g+h)\xi(g,h) = a(g)a(h), \tag{24}$$

$$Z_g v_h = \xi(g,h)v_h, \qquad X_g v_h = v_{h+g}, \qquad Y_g v_h = \frac{a(h)}{a(h-g)} v_{h-g}, \tag{25}$$

$$B(v_h, v_j) = a(h)\delta_{h+j,0}, \qquad v'_h = a(h)^{-1} v_{-h}, \tag{26}$$

$$P_{h,g} = c\, a(g)\xi(g,h)^{-1}, \qquad P^3 = 1, \qquad c^3 \sum_{g\in G} a(g) = \frac{1}{n}. \tag{27}$$

Here (v'_h) is the bilinearly dual basis in the bent realization, and P is cyclic change of marked corner. In the conventions of (20), Lemma 9.6 gives $a = \varphi$ for $\mathcal{N}_+$.

Proof. The commuting finite-order operators Z_g are simultaneously diagonalizable over $\mathbb{C}$. By (21), X_h permutes their weights by the character $\xi(-,h)$. Nondegeneracy and $\dim V = n$ imply that every weight occurs once. Choose v_0 in the trivial-weight line and put $v_h = X_h v_0$. The first two identities in (25) follow. The remaining commutations and the representation law for Y give scalars t_g with

$$Y_g v_h = t_g \xi(g,h)^{-1} v_{h-g}, \qquad t_{g+h} = t_g t_h \xi(g,h).$$

The first identity of (23) implies that $B(v_h, v_j) = 0$ unless $h + j = 0$. Hence $b_h^\theta = B(v_h, v_{-h})$ is nonzero for every h. The second identity and $t_g t_{-g} = \xi(g,g)$ give

$$b^\theta_{h+g} = t_g^{-1}\xi(g,h)^{-1} b^\theta_h.$$

Taking $h = 0$ gives $b^\theta_g / b^\theta_0 = t_g^{-1}$. Symmetry $b^\theta_g = b^\theta_{-g}$ then implies $t_g = t_{-g}$, and the representation law gives $t_g^2 = \xi(g,g)$. Thus $a(g) = t_g^{-1}$ is even and satisfies (24). Rescale v_0 by a square root of $(b^\theta_0)^{-1}$; its translates then give (26). This rescaling exists over $\mathbb{C}$ because $b^\theta_0 \neq 0$.

For clarity, one formula for the cyclic rotation in the symmetric self-duality convention is

$$P(f) = (e \otimes 1_R)(1_R \otimes f \otimes 1_R)(1_{R^2} \otimes s).$$

The snake identities give $P^3 = 1$. Moving the mixed vertices through the cup and cap gives $PZ_g = X_g P$ and $PX_g = Y_g P$. Consequently Pv_0 is fixed by all translations X_g, so $Pv_0 = c\sum_h v_h$. Invertibility of P gives $c \neq 0$, and (25) gives

$$Pv_g = PX_g v_0 = Y_g P v_0 = c\, a(g) \sum_h \xi(g,h)^{-1} v_h.$$

Cubing this matrix and using character orthogonality gives $P^3 = nc^3(\sum_g a(g))1_V$, proving (27).

Finally, for the basis of Lemma 9.6, let T'_h be the coordinate projections dual to T_h. The formulas there give

$$X_g(T'_h) = T'_{h+g}, \qquad Z_g(T'_h) = \chi(g,h)^{-1} T'_h, \qquad Y_g(T'_h) = \varphi(g)^{-1}\chi(g,h) T'_{h-g}.$$

Comparison with $t_g = a(g)^{-1}$ proves $a = \varphi$. □

10.2 The complete tetrahedral table

In the following diagrams an unlabelled edge is R; the position of a vertex label specifies its marked corner. A prime means the bilinear dual from (26). An invertible edge retains its orientation, and reversing it replaces g by $-g$. These conventions agree with the marked diagrams of [26, §§4–5].

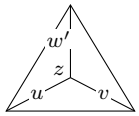
Proposition 10.3. *Under the hypotheses of §10.1, there is a function $b : G \to \mathbb{C}^\times$ for which*

$$b(0) = -\frac{1}{d}, \qquad b(g)b(-g) = \frac{a(g)}{n} - \frac{\delta_{g,0}}{d}, \tag{28}$$

$$c\,a(g) \sum_{h\in G} \xi(g,h)^{-1} b(h) = b(g), \tag{29}$$

$$\frac{a(h-g)}{n} \sum_{j\in G} \xi(h,j)^{-1} \frac{b(g-j)b(j)}{a(j)} = b(-h)b(g)b(h-g) + \frac{\delta_{h,0}\delta_{g,0}}{d}. \tag{30}$$

The all-R tetrahedral coefficients are



$$= \delta_{w,u+z} \frac{a(z)a(v)}{a(w)a(u+v)} \xi(w,u+v)^{-1} \frac{b(u+v)}{d}. \tag{31}$$

All remaining tetrahedra are the coherent mixed evaluations specified below. Together with (27) and (26), these coefficients determine the associator of $\mathcal{C}$.

Proof. Mixed evaluations. The tetrahedra with six invertible edges evaluate to 1; those whose invertible edges form a three-edge star evaluate to d. For two opposite invertible edges, with labels g on the bottom edge and h directed from the top vertex to the interior vertex, the value is $d\xi(g,h)$. For one bottom invertible edge g, with the top vertex labelled v'_j in its outside corner and the interior vertex labelled v_i in its bottom corner, the value is $\xi(g,i)\delta_{i,j}$. Indeed, the first two evaluations cancel coherent group or module multiplication with its inverse. The third leaves one mixed interchange and an R-loop. Sliding the single group edge in the fourth leaves $\xi(g,i)B(v_i,v'_j)$. These arguments also fix every nonzero normalizing factor in the mixed rows.

One scalar function. Use the vertical channel basis

$$E_{h,j} = \quad , \qquad P_x \quad (x \in G)$$



of $\mathrm{End}(R^2)$. The mixed normalization and (26) identify its algebra and trace with

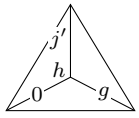
$$\mathrm{End}(R^2) = \mathbb{C}^G \oplus M_n(\mathbb{C}), \quad \mathrm{tr}(z,M) = \sum_x z_x + d\,\mathrm{Tr}(M), \quad P_x = de_x, \quad E_{h,j} = d^{-1}e_{h,j}. \tag{32}$$

In particular

$$\mathrm{tr}(P_xP_y) = d^2\delta_{x,y}, \quad \mathrm{tr}(E_{h,j}E_{u,v}) = d^{-1}\delta_{j,u}\delta_{h,v}, \quad \mathrm{tr}(P_xE_{h,j}) = 0. \tag{33}$$

The indices here are the stacking indices, so the upper vertex is the primed one.

Expand the horizontal channel $H_{0g} =$  as $\sum_{h,j} C^g_{h,j}E_{h,j} + \sum_x f^g_x P_x$. Closing it against $E_{j,h}$ gives



$$= \frac{C^g_{h,j}}{d}. \tag{34}$$

Insert a dimension-one l-loop in the left triangular face and slide it to the three boundary vertices. By (25) and $v'_j = a(j)^{-1}v_{-j}$ the multiplier is $\xi(l, h-j)$. Thus $C^g_{h,j} = 0$ for $h \neq j$. More generally, the same insertion proves that a triangular face whose marked unprimed charges sum to a nonzero element has zero evaluation.

To compare the diagonal entries, insert a h-strand between the two right external legs of H_{0g}, directed downwards. Sliding it across the g-vertex multiplies the channel by $\xi(h, g)^{-1}$. If the channel is expanded first, its h-diagonal term is the unique term carried to the 0-diagonal term; the factors at the two paired vertices cancel. Hence

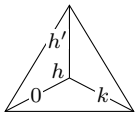
$$C^g_{h,h} = \xi(h, g)^{-1} C^g_{0,0}.$$

Define $b(g) = C^g_{0,0}$. Closing against P_x uses the one-invertible-edge evaluation and gives $d^2 f^g_x = \delta_{g,0}$. We have proved the actual channel identity

$$H_{0g} = b(g) \sum_h \xi(h, g)^{-1} E_{h,h} + d^{-2}\delta_{g,0} \sum_x P_x. \tag{35}$$

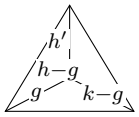
No coefficient has been introduced by comparing dimensions alone; each is obtained by the nondegenerate trace pairing (33).

Close (35) to obtain

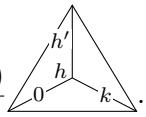


$$= \xi(h, k)^{-1} \frac{b(k)}{d}.$$

For a tetrahedron with vertex labels $(g, k-g, h', h-g)$, insert the three sides of a g-labelled invertible triangle at its lower face. Slide them to the adjacent vertices. The translations in (25) change the labels to $(0, k, h', h)$; the other two corner actions give, respectively, $a(h-g)/a(h)$ and $a(k-g)/a(k)$. Consequently



$$= \frac{a(h-g)a(k-g)}{a(h)a(k)}$$



. (36)

For general labels (u, v, w', z), the left-face charge is $u + z - w$, so the value vanishes unless $w = u + z$. In that case set $g = u$, $h = w$, $k = u + v$ in (36). This proves (31), including its support and all quadratic factors.

The rotation equation. Let $\ell(v_g) = b(g)$ be d times the tetrahedron with the other three vertices labelled v_0, using $v'_0 = v_0$ at the top. Their three marked corners point into the opposite triangular face. The order-three rotation fixing the g-vertex permutes those three marked zero vertices without changing their corner. At the fixed vertex it performs the change of corner P. Spherical isotopy therefore gives $\ell P = \ell$. Expanding Pv_g by (27) proves (29). In particular

$$c \sum_g b(g) = b(0). \tag{37}$$

The scalar equations. We give the channel calculations of the three needed prism cuts, including their dimension factors. They make explicit why the argument applies to either root of (18). For $H_{pq} =$ 

, closing against the vertical channels and using (31) gives

$$H_{pq} = \sum_{h,j} C^{p,q}_{h,j} E_{h,j} + \sum_x f^{p,q}_x P_x, \tag{38}$$

$$C^{p,q}_{h,j} = \delta_{j,h+p}\, \xi(-h, q) b(p+q),$$

$$f^{p,q}_x = d^{-2} a(p) \xi(x, q) \delta_{p+q,0}.$$

Indeed the matrix coefficient before simplifying is

$$\delta_{j,h+p}\frac{a(h)a(q)}{a(j)a(p+q)}\xi(j,p+q)^{-1}b(p+q),$$

which reduces to the displayed expression by (24). The group coefficient is the one-invertible-edge evaluation after the even permutation of the tetrahedron exchanging the lower-left with the top vertex and the lower-right with the interior vertex. Its two remaining labels are $v_p = a(p)v'_{-p}$ and v_q; it is therefore $a(p)\xi(x,q)\delta_{p+q,0}$ divided by d^2.

The horizontal invertible channel W_g has coordinates

$$W_g = \big((\xi(g,x))_x,\ a(g)X_gZ_{-g}\big) \quad \text{in } \mathbb{C}^G \oplus M_n(\mathbb{C}).$$

The mixed rows give these coordinates directly. Taking the trace in (32) gives

$$\operatorname{tr}(W_uH_{pq}) = na(u)\big(b(0)+d^{-1}\big)\delta_{p,u}\delta_{q,-u}. \tag{39}$$

The other prism cut is zero: its intermediate trivalent space has two invertibles and one R. Therefore $b(0) = -d^{-1}$.

The cut of a prism with one horizontal unit edge gives

$$\operatorname{tr}(H_{pq}H_{rs}) = d^{-1}B(v_p,v_r)B(v_q,v_s). \tag{40}$$

For $p = r = 0$, $q = g$, $s = -g$, its matrix component is $nb(g)b(-g)/d$, and its group component is $n\delta_{g,0}/d^2$. The right side is $a(g)/d$. This proves the product identity in (28). In particular every $b(g)$ is nonzero.

Here is the remaining cut, with its corner transports made explicit. Let B also denote the matrix of the theta pairing and put $A = BP$. Then

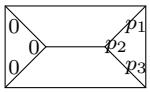
$$A_{h,j} = c\,a(h)a(j)\xi(h,j), \qquad (A^{-1})_{h,j} = \frac{\xi(h,j)^{-1}}{nc\,a(h)a(j)}, \qquad A^{\mathsf{t}} = A. \tag{41}$$

Bending a lower vertex to the opposite cut applies A; bending the paired upper vertex applies A^{-1}. Thus its effect on a matrix channel is

$$\Phi(z,M) = (z, AM^{\mathsf{t}}A^{-1}), \qquad \Phi(E_{h,j})_{i,l} = d^{-1}A_{i,j}(A^{-1})_{h,l}.$$

This is a trace-preserving anti-automorphism. In the product of three right-cut coefficients the common matrices A, A^{-1} cancel in the cyclic trace.

It suffices to use the prism



, $\qquad p_1 + p_2 + p_3 = 0.$

The two tetrahedra on its first cut, with internal dual labels v'_s, v_s, have values

$$\frac{b(-s)}{d\,a(s)}, \qquad \frac{a(p_3)a(s)\xi(p_1,p_2+s)}{d\,a(p_2+s)}b(p_2+s).$$

Putting $j = p_2 + s$ gives the first evaluation

$$\frac{a(p_3)}{d^2}\sum_j \xi(p_1,j)\frac{b(p_2-j)b(j)}{a(j)}. \tag{42}$$

For the second cut, (38) with $p = 0$ makes the R-channel matrices diagonal. Their cyclic trace is $n \prod_i b(p_i)$, and its dimension factor is d^{-2}. The group channels contribute only when all $p_i = 0$; summing their n scalar components gives n/d^3. Thus the second evaluation is

$$\frac{n}{d^2} \prod_{i=1}^{3} b(p_i) + \frac{n}{d^3} \prod_{i=1}^{3} \delta_{p_i,0}. \tag{43}$$

The two expressions evaluate the same closed spherical diagram. Set $(p_1, p_2, p_3) = (-h, g, h - g)$ and cancel n/d^2. This is exactly (30).

Recovery of the associator. The invertible edges of an admissible tetrahedron form one of the following configurations: all six edges, a three-edge star, two opposite edges, one edge, or no edge. Indeed, two invertible edges at a vertex force its third edge to be invertible. Proper tetrahedral rotations reduce the first four configurations to the mixed rows above; the last is (31). The maps P and the bilinear duality prescribe every change of marked corner.

To recover a matrix of the associator, form its two fusion-tree bases and close one tree against the trace-dual of a tree from the other basis. The result is its tetrahedral coefficient, with the specified trace-normalization factors. Nondegeneracy of the pairings makes this an invertible change of coordinates. Consequently equality of the marked tetrahedral table gives equality of all associator matrices in these bases. This is also the coefficient identification in [38, Theorem 5.3]. Here all the matrices come from an existing category, so their invertibility, unit constraints and pentagon are already supplied by that category. The table recovers its tensor structure. □

10.3 The finite quantum dilogarithm equations

For a nondegenerate even quadratic form q on G write

$$\langle x, y\rangle_q = \frac{q(x+y)}{q(x)q(y)}, \qquad \widehat{E}(w) = \frac{1}{\sqrt{n}} \sum_{x \in G} \langle w, x\rangle_q^{-1} E(x),$$

using the positive real square root of n to fix the convention. The normalization needed for the Leavitt construction consists of nonzero $E(x)$ and λ satisfying

$$E(0)^2 = \sqrt{n}\, E(0) + 1, \qquad E(x)E(-x) = q(x)^{-1} \quad (x \neq 0), \tag{44}$$

$$\widehat{E}(x) = \lambda q(x) E(x), \qquad \lambda^3 = \frac{1}{\sqrt{n}} \sum_x q(x)^{-1}, \tag{45}$$

$$\langle x, y\rangle_q E(x)E(y) = -\lambda E(0) + \frac{1}{\sqrt{n}} \sum_z E(y-z)E(z)q(z)E(x-z). \tag{46}$$

The constant $-\lambda E(0)$ is nonzero. Equation (46) is the finite quantum dilogarithm identity of [44, Definition 4, equation (32)]. We retain the normalization explicitly, including when $n = 4$; this is the convention in [44, Remark 3].

Lemma 10.4. *The scalar data extracted in Proposition 10.3 give a normalized finite quantum dilogarithm by*

$$q_{\mathrm{T}} = a^{-1}, \qquad E_{\mathrm{T}} = \sqrt{n}\, b, \qquad \lambda_{\mathrm{T}} = (\sqrt{n}\, c)^{-1}, \qquad E_{\mathrm{T}}(0) = -\frac{\sqrt{n}}{d}. \tag{47}$$

Proof. The polar pairing of q_{T} is ξ by (24). Equations (18) and (28) give (44), and (27) and (29) give (45). We verify the remaining change of variables directly, writing q, E, λ for the quantities in (47).

Substitute $(g, h) = (x, -y)$ in (30) and rescale. The result is

$$\frac{1}{\sqrt{n}} \sum_z E(z)q(z)E(x-z)\xi(z,y) = q(x+y)E(-x-y)E(x)E(y) - nE(0)\delta_{x,0}\delta_{y,0}. \tag{48}$$

Put $p_x(z) = q(z)E(z)E(x-z)$. Replace y by $-w$, multiply by $\widehat{E}(w)$, and use the identity

$$q(w)E(w)E(-w) = 1 + \sqrt{n}E(0)\delta_{w,0}.$$

Together with $E(0)^2 = 1 + \sqrt{n}E(0)$ this gives, also at $x = 0$ or $w = 0$,

$$\widehat{E}(w)\widehat{p}_x(w) = E(x)\widehat{E}(w-x) + \sqrt{n}\,\lambda E(0)\delta_{w,0}.$$

Apply inverse Fourier transform in w. The product on the left becomes the normalized convolution of E with p_x, and the shift on the right becomes $\xi(x,y)E(x)E(y)$. Rearranging gives (46). □

10.4 Essential surjectivity of the Leavitt construction

We now identify the tensor category, rather than merely its scalar equations. The argument uses two injections between finite sets; it does not require a gauge identification between the tetrahedral and Leavitt presentations.

Fix $n = 4k$. Let $\mathcal{S}_n$ be the pairs (q, E) satisfying (44)–(46), modulo $\mathrm{Aut}(G)$. The scalar λ is determined by q, E, since $\lambda = \widehat{E}(0)/E(0)$. Let $\mathcal{T}_n$ be the tensor-equivalence classes of complex near-group categories with pointed part Vec_G, nondegenerate symmetric mixed interchange, and $\nu_2(R) = 1$. Every near-group category admits a spherical structure [50, Theorem 1.3]. For the fusion rule $G + n$ this structure is unique: a tensor automorphism of the identity acts on R by a scalar r satisfying $r^2 = r$, because R is a summand of R^2; hence $r = 1$, and the summands α_g of R^2 also have scalar 1. Thus the conditions defining $\mathcal{T}_n$ are conditions on its tensor-equivalence classes. Ocneanu rigidity makes $\mathcal{T}_n$ finite [13, Theorem 2.28].

Proposition 10.5. *The Radchenko–Wheeler construction induces a bijection*

$$\mathcal{S}_n \longrightarrow \mathcal{T}_n.$$

Proof. Construction and its image. For $(q, E) \in \mathcal{S}_n$ put $d = -\sqrt{n}/E(0)$. The algebraic construction of [44, §5.4, Theorem 12] gives a fusion category $\mathcal{L}(q, E)$ of type $G + n$. Its generators S_g, S'_g, T_g, T'_g satisfy algebraic inclusion–projection relations

$$S'_gS_h = T'_gT_h = \delta_{g,h}, \quad S'_gT_h = T'_gS_h = 0, \quad \sum_g S_gS'_g + \sum_g T_gT'_g = 1.$$

Its pointed part is strict, and $\alpha_h(U_g) = \langle h, g\rangle_q U_g$ makes its mixed pairing perfect. The formulas for $\rho(S_0)$ and $\rho(S'_0)$ in that construction give

$$dS'_0\rho(S_0) = d\rho(S'_0)S_0 = 1, \qquad dS'_0S_0 = d.$$

Thus S_0 and dS'_0 are a self-duality with loop d. In the unique spherical structure its dimension is $\nu_2(\rho)d$. It must satisfy the dimension equation (18); of d and $-d$, only d does. Consequently $\nu_2(\rho) = 1$ and $\dim\rho = d$. The self-dual invertible has indicator 1 in the strict pointed subcategory with dimensions 1. The resulting category therefore lies in $\mathcal{T}_n$.

All identities required by the Leavitt construction have been included in the definition of $\mathcal{S}_n$. In particular its use at $n = 4$ does not apply the $|G| > 4$ implication in [44, Theorem 5] to an unnormalized solution.

Injectivity of Leavitt reconstruction. The following invariant is the inverse of the construction [44, §5.5]. In the category choose $L_g : \alpha_g R \to R$, and choose $T_0 : R \to R^2$ in the common trivial line

$$(L_g \otimes 1_R)(1_{\alpha_g} \otimes T_0) = T_0L_g \quad (g \in G). \tag{49}$$

The perfect mixed pairing makes this line one-dimensional. Choose $u_g : R \to R\alpha_g$ so that D_g acts identically on $\mathrm{Hom}(\mathbf{1}, R^2)$, and put $T_g = D_{-g}T_0$. The endomorphism D_g is independent of

the scale of L_g: the normalization determines the reciprocal scale of u_g. Let T'_h be the coordinate projections with $T'_h T_g = \delta_{h,g} 1_R$. Then

$$\frac{E(g)}{\sqrt{n}}\, 1_R = T'_0 (T'_0 \otimes 1_R)(1_R \otimes T_g) T_0 \tag{50}$$

in a Leavitt category. Substituting the construction's formula for $\rho(T_g)$ proves this equality: the two T'_0 factors and the final T_0 select the $h = j = 0$ term, whose coefficient is $E(g)/\sqrt{n}$. The expression is unchanged by simultaneous rescaling of the T_g and reciprocal rescaling of the T'_g. It is preserved by tensor equivalences, with the induced relabelling of G. Moreover

$$q(g) = \big(E(g)E(-g)\big)^{-1} \quad (g \neq 0), \qquad q(0) = 1.$$

Thus two Leavitt categories can be tensor equivalent only if their data differ by a group automorphism. This proves the injection $\mathcal{S}_n \hookrightarrow \mathcal{T}_n$, and in particular proves that $\mathcal{S}_n$ is finite.

The reverse inequality. For each member of $\mathcal{T}_n$ choose the algebraic normalization of Lemma 10.2 and extract its tuple by Proposition 10.3. Lemma 10.4 gives an element of $\mathcal{S}_n$. If two chosen tuples have the same orbit in $\mathcal{S}_n$, relabel one of them so that their pairs $(q_{\mathrm{T}}, E_{\mathrm{T}})$ agree. Then $a = q_{\mathrm{T}}^{-1}$ and $b = E_{\mathrm{T}}/\sqrt{n}$ agree, as do

$$d = -\frac{\sqrt{n}}{E_{\mathrm{T}}(0)}, \qquad c = \frac{1}{\sqrt{n}\,\lambda_{\mathrm{T}}}, \qquad \xi(x,y) = \frac{q_{\mathrm{T}}(x+y)}{q_{\mathrm{T}}(x)q_{\mathrm{T}}(y)}.$$

Their full tetrahedral tables and corner transports therefore agree. The last part of Proposition 10.3 identifies their associators and gives a tensor equivalence. The chosen extraction is consequently an injection $\mathcal{T}_n \hookrightarrow \mathcal{S}_n$.

The finite sets have equal cardinalities, so the already constructed injection $\mathcal{S}_n \hookrightarrow \mathcal{T}_n$ is surjective. This proves the proposition. Notice that the argument uses tetrahedral tables only for categories already known to exist; it does not infer reconstruction from a subsystem of necessary scalar equations. □

Proof of Theorem 10.1. The category $\mathcal{N}_+$ satisfies the three extraction hypotheses by Lemmas 9.1, 9.2 and 9.6. Its spherical dimension is d_-. Proposition 10.3 gives its scalar presentation, and Proposition 10.5 gives a Leavitt realization. The unique spherical structure makes the dimension in that realization d_-, so $E_{\mathrm{L}}(0) = -\sqrt{n}/d_-$. Its intrinsic datum is exactly (50), followed by the product formula for q_{L}, and is unique up to $\mathrm{Aut}(G)$. □

All parameters in these scalar presentations are algebraic. For example, the set of labelled pairs in $\mathcal{S}_n$ is finite, because $\mathrm{Aut}(G)$ is finite. The possible quadratic forms take root-of-unity values, and the displayed normalized equations have algebraic coefficients. If a coordinate of E were transcendental, its orbit under $\mathrm{Aut}(\mathbb{C}/\overline{\mathbb{Q}})$ would be infinite, contrary to finiteness. The same conclusion follows from [44, Theorem 8].

Theorem 10.1 is an algebraic identification for every integral Flohr parameter. A Hermitian reflection equation and a positive definite structure are additional conditions. They are not used in its proof.

11 Frobenius–Schur exponents and indicators

This section computes the Frobenius–Schur exponent of $\mathcal{N}_+$, of its Morita duals and of its central quotients (Theorem 11.1), and all higher Frobenius–Schur indicators of R (Theorem 11.2). The fourth indicator shows that no Galois conjugate of $\mathcal{N}_+$ is defined over a real field (Corollary 11.3). Here $n = 4k$. The corresponding statements for $n \equiv 2 \pmod 4$ are in Proposition 12.4. For general stretched near-group categories, Schopieray, Wang and Yu give coprime-index indicator formulas and constraints on the exponent [48, Thm. 3.9, Cor. 3.10].

Strategy

Both quantities are read off the double. The Frobenius–Schur exponent is the order of the twist of $Z(\mathcal{N}_+)$ [43], which is explicit in Proposition 7.2. For the central quotients one first determines which simple objects of the double survive. The indicators are given by the formula of Ng and Schauenburg as a sum over the simple objects of the double, weighted by their dimensions and by the multiplicity of R in their image in $\mathcal{N}_+$. With the closed modular data, that sum is a combination of quadratic Gauss sums, which we evaluate. We use the labels $a_g, b_g, c_{a,s}, d_{b,s}$ of Proposition 7.2.

Let $L = \operatorname{lcm}(n,m) = 4k(k+1)$, so $nm = 4L$. For $h^2 \mid n$ let $H_h \le \mathbb{Z}_n$ have order h. By Remark 7.3, the central lift of $x \in \mathbb{Z}_n$ has twist $e(-x^2/n)$, and two lifts braid by $\beta(x,y)^2 = e(-2xy/n)$. For $x, y \in H_h = h'\mathbb{Z}_n$ these are $e(-h'^2 i^2/n) = e(-ni^2/h^2) = 1$ and 1, as $h^2 \mid n$. So the lifts of H_h form a Tannakian subcategory $\mathcal{T}_h \subset Z(\mathcal{N}_+)$, and we put $\mathcal{Q}_h = (\mathcal{N}_+)_{H_h}$. Then $Z(\mathcal{Q}_h)$ is the de-equivariantisation of the centraliser of $\mathcal{T}_h$ [10, Thm. 3.20]. Thus $\mathcal{Q}_1 = \mathcal{N}_+$, $\mathcal{Q}_2 = \mathcal{D}$ (Theorem 7.1(3)) and, for $k = \nu^2$, $\mathcal{Q}_{2\nu} = \mathcal{F}_\nu$ (Theorem 7.5). In this section s denotes the charge label of Proposition 7.2, not $\gcd(h,h')$. On $\mathcal{D}_h$ we use the spherical structure induced from $\mathcal{N}_+$. Twist orders are orders of $\operatorname{diag}(\theta_X)$ with $\theta_{\mathbf{1}} = 1$, without the central-charge factor of the vertex operator algebra T-matrix.

Theorem 11.1. *For every k and every $h \mid n$,*

$$\operatorname{FSexp}(\mathcal{N}_+) = \operatorname{FSexp}(\mathcal{D}_h) = 4L = 16k(k+1) = n(n+4),$$

and for $h^2 \mid n$

$$\operatorname{FSexp}(\mathcal{Q}_h) = \begin{cases} 4L, & h \text{ odd}, \\ L, & h \text{ even}, \end{cases} \qquad \operatorname{ord}\left(\theta_{\operatorname{Rep}\widehat{W}_k}\right) = L.$$

The same holds for all Galois conjugates.

Proof. $\mathcal{N}_+$ and $\mathcal{D}_h$. The Frobenius–Schur exponent of a spherical fusion category is the order of the twist of its double [43]. By Proposition 7.2, $\theta_{[x,s]} = e(h_x - s^2/4n + s/2)$. For $a_g = [W_k, 2g]$ and $b_g = [X, 2g]$ the lowest weight h_x is 0, so $\theta = e(-g^2/n)$, of order dividing n. Also

$$\theta_{c_{a,s}} = e\Big(\frac{a^2}{4n} - \frac{s^2}{4n} + \frac{s}{2}\Big), \qquad \theta_{d_{b,s}} = e\Big(\frac{b^2}{4m} - \frac{s^2}{4n} + \frac{s}{2}\Big),$$

with $a \equiv s$ and $b \equiv s \pmod 2$ (the shift $-n/4 = -k$ in h_x is an integer). All these orders divide $\operatorname{lcm}(4n, 4m) = 4L$. For $d_{1,1}$,

$$\theta_{d_{1,1}} = e\Big(\frac{1}{4m} - \frac{1}{4n} + \frac{1}{2}\Big) = e\Big(\frac{1}{2} - \frac{1}{nm}\Big) = e\Big(\frac{2L-1}{4L}\Big),$$

of order exactly $4L$, as $2L-1$ is odd. So $\operatorname{FSexp}(\mathcal{N}_+) = 4L$, and $\mathcal{D}_h$ has the same double (Theorem 9.7).

Which labels survive in $Z(\mathcal{Q}_h)$. By Proposition 7.2, $S_{a_g,[y,s]} = e(gs/n) S_{\mathbf{1},[y,s]}$ for the c- and d-types, so the monodromy $S_{a_g,Y} S_{\mathbf{1},\mathbf{1}} / (S_{\mathbf{1},a_g} S_{\mathbf{1},Y})$ of the lift $a_{h'}$ of the generator of H_h with $Y = c_{a,s}$ or $d_{b,s}$ is $e(h's/n) = e(s/h)$. So the labels in the centraliser are those with $h \mid s$. Passing to the de-equivariantisation identifies free orbits and splits fixed points, and every resulting simple object has the twist of the label it comes from. In particular $c_{2,0}$ and $d_{2,0}$ always survive, with twists $e(1/n)$ and $e(1/m)$, of orders n and m. So L divides the twist order of $Z(\mathcal{Q}_h)$.

h even. Put $r = n/h^2$ and $s = hv$. Then s is even, so the theta index is even, say $2u$, and the twists are $e(u^2/n - v^2/4r)$ and $e(u^2/m - v^2/4r)$ (the term $s/2 = hv/2$ is an integer). As $4r \mid n$, their orders divide L, and so the order is L.

h odd. Then $4 \mid r = n/h^2$, and $d_{1,h}$ survives, with twist

$$e\Big(\frac{1}{4m} - \frac{h^2}{4n} + \frac{h}{2}\Big) = e\Big(\frac{1}{4m} - \frac{1}{4r} + \frac{h}{2}\Big) = e\Big(\frac{r - m + 2hmr}{4mr}\Big).$$

Here $m - r = (h^2-1)r + 4$ with $v_2((h^2-1)r) \geq 3 + 2 = 5$, so $v_2(r-m) = 2$. $v_2(2hmr) \geq 5$, so the numerator has 2-adic valuation exactly 2. The denominator has valuation $2 + v_2(m) + v_2(r)$, so the 2-part of the order of this twist is $2^{v_2(m)+v_2(r)}$. As h is odd, $v_2(r) = v_2(n)$, so this is the 2-part of $nm = 4L$. The odd part of $4L$ divides $\operatorname{lcm}(n, m)$, the order of the labels $c_{2,0}$, $d_{2,0}$. So the order is $4L$.

$\operatorname{Rep}\widehat{W}_k$. By Theorem 7.1(4), $Z(\mathcal{Q}_2) \simeq \operatorname{Rep}\widehat{W}_k \boxtimes \mathcal{C}(\mathbb{Z}_{2k}, q)^{\mathrm{rev}}$. The images of $c_{2,0}$ and $d_{2,0}$ have charge $s = 0$, so they centralise the pointed factor and lie in $\operatorname{Rep}\widehat{W}_k$. So L divides its twist order. Conversely its twist order divides that of $Z(\mathcal{Q}_2)$, which is L by the case $h = 2$.

Galois conjugates. A Galois automorphism raises each twist to a power prime to its order, so all orders are unchanged. □

Tucker expresses near-group indicators through Gauss sums of the double [51]. Here the parity sectors must be kept. For $4 \mid N$, $\iota \in \{0, 1\}$ and $\ell \geq 1$ put

$$H_N^\iota(\ell) = \sum_{\substack{x \in \mathbb{Z}_{2N} \\ x \equiv \iota\,(2)}} e\Big(\frac{\ell x^2}{4N}\Big), \qquad g_N^\iota(\ell) = N^{-1/2} H_N^\iota(\ell),$$

with positive square roots.

Theorem 11.2. *For $\mathcal{N}_+$ with its spherical structure, $d(R) = d_-$,*

$$\nu_\ell(R) = \frac{1}{4}\sum_{\iota=0}^{1} (-1)^{\ell\iota}\Big(|g_n^\iota(\ell)|^2 - g_m^\iota(\ell)\,\overline{g_n^\iota(\ell)}\Big). \tag{51}$$

In particular $\nu_{2u}(R) = \gcd(u, n)$ for odd u, and $\nu_4(R) = 1 + i$. The indicators of a Galois conjugate are the images of (51) under the corresponding embedding. Its fourth indicator is $1 \pm i$.

Proof. The trace formula. By [43, Thm. 4.1], $\nu_\ell(R) = \dim(\mathcal{N}_+)^{-1} \sum_Y \theta_Y^\ell\, d(Y)\,[I(Y) : R]$, summed over the simple objects Y of $Z(\mathcal{N}_+)$, where $[I(Y) : R]$ is the multiplicity of R in the image of Y in $\mathcal{N}_+$. The Frobenius–Perron dimensions of the four types are $1, 1+d_+, 2+d_+, d_+$ (Proposition 7.2), and those of their images in $\mathcal{N}_+$ are the same. An image of dimension 1 is invertible, and an image of dimension $\lambda + \mu d_+$ with $\lambda, \mu \in \mathbb{Z}_{\geq 0}$ contains R exactly μ times, since d_+ is irrational. So $[I(Y) : R] = 0$ for the a-type and 1 for the b-, c- and d-types. Their spherical dimensions are $1 + d$, $2 + d$ and d, with $d = d_-$.

The three sums. Summing θ_Y^ℓ over each type gives, with $H_N^\iota = H_N^\iota(\ell)$,

$$B = \overline{H_n^0}, \qquad C = \tfrac{1}{4}\big((H_n^0 - 2)\overline{H_n^0} + (-1)^\ell H_n^1 \overline{H_n^1}\big), \qquad D = \tfrac{1}{4}\big((H_m^0 - 2)\overline{H_n^0} + (-1)^\ell H_m^1 \overline{H_n^1}\big).$$

For B, $\theta_{b_g}^\ell = e(-\ell(2g)^2/4n)$ up to an integer, summed over $g \in \mathbb{Z}_n$. This is $\overline{H_n^0}$. For C and D, the labels $c_{a,s}$ are pairs (theta index, charge) of the same parity with $1 \leq a \leq n-1$, modulo $(a, s) \sim (n - a, s + n)$. Summing over all $a \in \mathbb{Z}_{2n}$ instead counts each label four times ($\pm a$ and the identification), and adds the endpoints $a \in \{0, n\}$ with twist 1. This is the factor $\frac{1}{4}$ and the -2. The factor $e(\ell s/2)$ is $(-1)^\ell$ for odd s and 1 for even s, which gives the sign on the odd sector. The same holds for D with theta index modulo $2m$.

The formula. The numerator is $(d+1)B + (d+2)C + dD$. The endpoint terms $-2\overline{H_n^0}/4$ in C and D cancel against B, since $(d+1) - \frac{1}{2}(d+2) - \frac{1}{2}d = 0$. Dividing by $\dim \mathcal{N}_+ = n + d^2 = n(d+2)$ and using $d/(d+2) = -\sqrt{n/m}$ (from $d^2 = nd + n$ and $d < 0$) gives (51).

$\ell = 2u$, *u odd.* If $v_2(N) = 2$, then $H^0_N(2u) = \sum_{x\in\mathbb{Z}_N} e(2ux^2/N)$ vanishes (a quadratic Gauss sum modulo N with $v_2(N) = 2$ and odd multiplier), and $H^1_N(2u)$ is the full sum over $\mathbb{Z}_{2N}$, of modulus $2\sqrt{N\gcd(u,N)}$. If $v_2(N) > 2$, the shift $x \mapsto x + N/2$ multiplies the terms of $H^1_N(2u)$ by -1, so it vanishes, and $H^0_N(2u)$ has modulus $2\sqrt{N\gcd(u,N)}$. So (g^0_N, g^1_N) is $(0, z_N)$ or $(z_N, 0)$ with $|z_N|^2 = 4\gcd(u,N)$. Exactly one of n and $m = n+4$ has $v_2 = 2$, so the mixed term in (51) vanishes, and $\nu_{2u}(R) = \frac{1}{4}\cdot 4\gcd(u,n) = \gcd(u,n)$.

$\ell = 4$. Let $A_v = v^{-1/2}\sum_{x\in\mathbb{Z}_v} e(x^2/v)$, with $A_1 = 1$. Classically $A_v = 1+i, 1, 0, i$ for $v \equiv 0, 1, 2, 3 \pmod 4$. Splitting $x \in \mathbb{Z}_{8v}$ by parity gives $(g^0_{4v}(4), g^1_{4v}(4)) = 2(A_v, 1+i-A_v)$. For $v \equiv 0, 1, 2, 3 \pmod 4$ these are twice $(1+i, 0), (1, i), (0, 1+i), (i, 1)$. Each has squared norm 8, and the Hermitian pairing $g^0_{4v'}\overline{g^0_{4v}} + (-1)^4 g^1_{4v'}\overline{g^1_{4v}}$ of each with its predecessor ($v' = v+1$) is $4(1-i)$. With $n = 4k$, $m = 4(k+1)$, (51) gives $\nu_4(R) = \frac{1}{4}(8 - 4(1-i)) = 1+i$.

Galois conjugates. Indicators are traces of the pivotal rotation on $\mathrm{Hom}(\mathbf{1}, R^{\otimes\ell})$, so transporting the category and its pivotal structure by an automorphism of $\mathbb{C}$ transports them by the same automorphism. □

For $k \le 3$, (51) agrees with direct evaluation of the sum of [43, Thm. 4.1] from Proposition 7.2, for all $\ell \le 4L$ (`fs_indicator_check.py`).

Corollary 11.3. *No Galois conjugate of $\mathcal{N}_+$ is equivalent to a fusion category defined over a subfield of $\mathbb{R}$, or to its tensor-opposite category. The field of moduli of $\mathcal{N}_+$ contains $\mathbb{Q}(i)$.*

Proof. The pivotal structure is unique (§9.1), so tensor equivalences preserve indicators. A category defined over a real field is equivalent to its complex conjugate, which has $\nu_4(R) = \overline{1 \pm i} = 1 \mp i$. This is impossible. Passing to the opposite tensor product inverts the pivotal rotation, which conjugates its trace, so the same argument applies. Every automorphism of $\mathbb{C}$ fixing the equivalence class fixes $\nu_4(R) = 1 \pm i$, hence fixes i. □

12 The fermionic members of Flohr's family

We now complete the construction for $k \in \frac{1}{2} + \mathbb{Z}_{\ge 0}$, or equivalently $n = 4k \equiv 2 \pmod 4$. The order-two module J then has half-integral conformal weight, so the extension $V_{n+2} \oplus J$ is a vertex operator superalgebra. Its even subalgebra V_{n+2} remains strongly rational. The near-group category $\mathcal{N}_+$, its Drinfeld centre and the lattice VOA realising that centre are constructed from the even subalgebra and its even-lattice extension, and survive this change of parity.

There are two parts to the representation theory. Theorem 12.2 extends the simple-module classification, near-group construction and Morita theory to these parameters. Proposition 12.5 treats the superalgebra itself: it determines the Neveu–Schwarz simples, their characters and fusion rules, and lists the Ramond modules. The fourth Frobenius–Schur indicator is computed in Proposition 12.4. Together with the odd-$n \ge 3$ construction of [18], this gives near-group categories of type $\mathbb{Z}_n + n$ and VOA realisations of their centres for every $n \ge 2$.

The distinction between the two parity cases concerns the extension by J. For $4 \mid n$ it is bosonic and leads to $\widehat{W}_k$ and the central quotient $\mathcal{D}$; here J generates super vector spaces, and that order-two de-equivariantisation is unavailable (Remark 12.3). Proposition 8.4 also gives the shifted ribbon realisation of $Z(\mathcal{N}_+)$ in this parity. The scalar reconstruction of §10 and the square-parameter results are stated for integral k; no extension of those statements is asserted in this section.

12.1 The even subalgebra and its near-group category

Throughout this section $n \equiv 2 \pmod 4$, $n \geq 2$, $m = n+4$, $c = 1-6n$, $k = n/4$ and $V = V_{n+2}$. Put

$$t = \delta = \frac{n+2+\sqrt{nm}}{2}, \qquad t + t^{-1} = n+2, \qquad (t-1)^2 = nt, \qquad (t+1)^2 = mt.$$

Since $nm = (n+2)^2 - 4$ is not a square, t is irrational. We use the objects and notation of §§3–8, with V in place of W_k; in particular $U = V \oplus X$, $M_n = J \oplus Y$, $\mathcal{C} = \operatorname{Rep} V$, $A' = \mathbf{1} \oplus X$ and $\mathcal{N} = \mathcal{C}_{A'}$.

Three changes must be checked in the earlier arguments. The object J has twist -1; the lattice $\mathbb{Z}\beta$ with $(\beta,\beta) = n/2$ is odd; and the two J-fixed simples $A_{n/2}$ and $B_{m/2}$ have monodromy -1 with J. The even lattice and the algebra A' are unchanged. Moreover $J \boxtimes M_n$ is still a boson because both factors have twist -1. The first step is to prove invertibility of J without the bosonic orbifold argument.

Lemma 12.1. *Let $n \geq 1$ be such that V_{n+2} is strongly rational and Y is its unique simple module of least lowest weight $-n/4$. Then J is invertible in $\operatorname{Rep} V_{n+2}$, and $J \otimes J \cong V_{n+2}$.*

Proof. Step (a) of the proof of Theorem 6.4 uses only these two hypotheses and the character $\chi_V = \frac{1}{2}(\Lambda_{n,n} - \Lambda_{m,m})$, which is known for every n (§3). It shows that the dimension lemma [18, Lemma 6.3] applies, and gives for every simple module M,

$$\operatorname{FPdim} M = \lim_{\epsilon \downarrow 0} \frac{\chi_M(i\epsilon)}{\chi_V(i\epsilon)}.$$

By the transformation formula $\Lambda_{a,N}(-1/\tau) = (2N)^{-1/2} \sum_{b \bmod 2N} e(-ab/2N)\Lambda_{b,N}(\tau)$, whose leading term as $\tau \to i\infty$ is the term $b = 0$, one has $\Lambda_{a,N}(i\epsilon) \sim (2N)^{-1/2} e^{\pi/12\epsilon}$ for every a. Hence $\chi_J = \frac{1}{2}(\Lambda_{0,n} - \Lambda_{0,m})$ and χ_V have the same leading term $\frac{1}{2}((2n)^{-1/2} - (2m)^{-1/2})e^{\pi/12\epsilon}$, and $\operatorname{FPdim} J = 1$. A simple object of Frobenius–Perron dimension 1 in a fusion category is invertible. Finally J is self-contragredient. Indeed $A(V_{n+2})$ is generated by $[T]$ and $[\Psi]$, and Ψ has even weight $2n$, so a simple module and its contragredient have the same character of $A(V_{n+2})$ on their lowest weight spaces. Therefore $J \otimes J \cong J \otimes J^* \cong V_{n+2}$. □

The lemma applies to $n = 4k$ as well (Proposition 4.7), and there it gives a proof of $J \otimes J \cong W_k$ that does not use the orbifold theorem.

Theorem 12.2. *Let $n \equiv 2 \pmod 4$, $n \geq 2$.*

1. *V_{n+2} is strongly rational. It has exactly $2n+6$ simple modules V, X, J, Y, A_a $(1 \leq a < n)$, B_b $(1 \leq b < m)$, with the characters and lowest weights of Theorem 6.4. All are self-dual, and the modular data of $\mathcal{C}$ is the theta modular data.*

2. *$J \otimes J \cong V$, $\theta_J = -1$ and $d(J) = 1$, so J generates a copy of the category of super vector spaces. Moreover $J \otimes X \cong Y$, $J \otimes A_a \cong A_{n-a}$ and $J \otimes B_b \cong B_{m-b}$. The monodromy of J with A_a is $(-1)^a$ and with B_b is $(-1)^b$. So $\mathcal{C}'$ has the $n+4$ simple objects V, X, J, Y, A_a and B_b with a, b even, on which J acts freely, and the two J-fixed simple objects $A_{n/2}$, $B_{m/2}$ lie in $\mathcal{C}''$.*

3. *$\mathcal{N}_+ = \mathcal{C}'_{A'}$ is a near-group category of type $\mathbb{Z}_n + n$, with $\operatorname{FPdim} R = \delta - 1$ and $d(R) = \frac{1}{2}(n - \sqrt{nm})$. It satisfies (H1)–(H3) of §9.1, with $\beta(x,y) = e(-xy/n)$.*

4. *$Z(\mathcal{N}) \simeq \mathcal{C} \boxtimes \mathcal{P}^{\mathrm{rev}}$, and $Z(\mathcal{N}_+)$ is the de-equivariantisation of $\mathcal{C}' \boxtimes \mathcal{P}_{\mathrm{even}} \oplus \mathcal{C}'' \boxtimes \mathcal{P}_{\mathrm{odd}}$ by the boson $b = J \boxtimes M_n$. It has rank $n(n+3)$, and its modular data is given by the formulas of Proposition 7.2.*

5. *$\widetilde{U}_n = (V \otimes V_{K_n}) \oplus (J \otimes V_{K_n+\mu})$ is a simple, self-contragredient, C_2-cofinite, rational vertex operator algebra of CFT type and central charge $8-6n$, and its module category is braided equivalent to $Z(\mathcal{N}_+)$.*

6. *Theorems 9.7, 9.16 and 9.23 and Corollaries 9.8, 9.9 and 9.17 hold for $\mathcal{N}_+$, with $\epsilon_h = 0$ for every $h \mid n$. There are $\tau(n)$ indecomposable module categories, every categorically Morita equivalent fusion category is one of the $\mathcal{D}_h$, and there are $2\tau(n)$ connected separable algebras. The same holds for the C* near-group categories $\mathbb{Z}_n + n$, to which Proposition 9.24 applies. They exist for $n = 2, 6, 10$ [16, Prop. 6].*

Proof. (1) *Lattice data.* The statements of §§2–3 on V_{n+2}, on $U = V \oplus X$ and $M_n = J \oplus Y$, on the lowest weights 0, $3n/4$, $-n/4$ of X, J, Y, and on the four characters are quoted from [11, 18] for every $n \geq 1$. They concern the even lattice $\mathbb{Z}\beta'$, $(\beta', \beta') = 2n$, and do not involve $\mathbb{Z}\beta$.

Strong rationality. V is C_2-cofinite by Corollary 4.6. In the proof of Proposition 4.7, items (2)–(6) use only that t and $\sqrt{nm}$ are irrational, $t + t^{-1} = n + 2$, $(t-1)^2 = nt$ and $(t+1)^2 = mt$. Item (1) uses $t > 4$, which fails for $n = 2$, where $t = 2 + \sqrt{3}$. But its purpose is that the t-factors of $K(t)$ and of the two exceptional minors in [18, Prop. B.2] do not vanish. These factors are t, $t \pm 1$, $t \pm 2$, $2t \pm 1$, $2t - 3$, $3t - 2$, $3t - 4$ and $4t - 3$, and none of them vanishes at an irrational t. So Proposition 4.7 holds for V. It is strongly rational, its lowest weights lie in the two grids, each nonzero value occurs at most once, and Y is the unique simple module of least lowest weight.

J and the near-group category. By Lemma 12.1, $J \otimes J \cong V$. The proof of Corollary 4.8 then gives $J \otimes X \cong Y$, and the proof of Theorem 5.1 applies without change. Part (1) of that proof does not involve J. Part (2) uses $J \otimes J \cong V$ and the monodromy $(-1)^a$ of J with M_a, not the twist of J. Part (3) uses the bimodule bound of Lemma 6.2, proved for every $n \geq 1$. This gives the first statement of (3), with $\mathrm{FPdim}\, R = \delta - 1$.

Classification. In the proof of Proposition 6.3, monodromy with the invertible object J of order two is a $\mathbb{Z}_2$-grading of $\mathcal{C}$, faithful because $A_1 \in \mathcal{C}''$. That J is a boson is not used. The known simple objects of $\mathcal{C}'$ are V, J, X, Y and the A_a with a even, of which there are $n/2 - 1$, and those of $\mathcal{C}''$ are the $n/2$ modules A_a with a odd. These are the counts used there, so there are $m/2 - 1$ further simple objects in $\mathcal{C}'$ and $m/2$ in $\mathcal{C}''$, each inducing to $R_\pm$. (The last sentence of that proposition, that J fixes some Q'_i, is not used later, and it fails here, since $m/2 - 1$ is even.) The proof of Theorem 6.4 uses three further facts. Step (a) uses $c - 24h_Y = 1$. Step (b) uses that J is a simple current with $J \otimes X \cong Y$. Steps (d) and (e) use that the lowest weights $1/4m - n/4$ and $1/m - n/4$ of $\Lambda_{1,m}$ and $\Lambda_{2,m}$ are negative, and that there are $m/2 - 1$ even and $m/2$ odd chains. The additional coefficient comparison in step (e) uses $h_{m-2} = 1/m$ and $h_{m-1} = \frac{1}{2} + 1/4m$, which lie outside $0, -n/4$ modulo $\mathbb{Z}$ also when $n \equiv 2 \pmod 4$. All these facts hold for every even $n \geq 2$. This proves (1), and $d(R) = \frac{1}{2}(n - \sqrt{nm})$ by step (c).

(2) $J \otimes J \cong V$ is Lemma 12.1. $\theta_J = e(3n/4) = -1$. $d(J) = S_{JV}/S_{VV} = 1$ in the theta matrix. The proof of Corollary 6.5 uses only that m is even. The monodromies are read from $S_{JM} = \varepsilon_M S_{VM}$ in the theta matrix, as in Theorem 6.4(2). Since $n/2$ and $m/2$ are odd, the fixed points $A_{n/2}$ and $B_{m/2}$ lie in $\mathcal{C}''$.

(4) The proof of Theorem 7.1(1) applies as it stands. In (2) of that theorem, $b = J \boxtimes M_n$ has twist $\theta_J \theta_{M_n}^{-1}$ in $\mathcal{C} \boxtimes \mathcal{P}^{\mathrm{rev}}$, and $\theta_{M_n} = e(n/4 - n/2) = e(-n/4) = -1$. So $\theta_b = (-1)(-1) = 1$. Its dimension is $d(J)\, d(M_n) = (-1)^n = 1$, and $b \otimes b = \mathbf{1}$. So b again generates a Tannakian subcategory, it is the grading character by the argument given there, and it acts freely since $s + n \neq s$. The count is $\frac{1}{2}((n+4)n + (n+2)n) = n(n+3)$ by (2). The proof of Proposition 7.2 uses only this description and the theta matrix.

(3), *hypotheses.* By (4), the argument of Remark 7.3 applies. The central lifts $a_x = [V, 2x]$ of the invertible objects have twists $e(-x^2/n)$, which gives (H1) and (H2) with $\beta(x, y) = e(-xy/n)$. The proof of Lemma 9.1 does not involve n. It gives (H3).

(5) In the proof of Theorem 8.2, the lattice lemma holds for every index, and $\mathcal{P} \simeq \mathcal{C}(\mathbb{Z}_{2n}, s^2/4n)$

as braided categories. The module $V_{K_n+\mu}$ has twist $e(-n^2/4n) = e(-n/4) = -1$ and dimension 1, so $b' = J \otimes V_{K_n+\mu}$ has twist $(-1)(-1) = 1$ and dimension 1. By the spin–statistics relation [6, Thm. 4.1] its self-braiding is trivial. Its conformal weights lie in $(\frac{3n}{4} + \mathbb{Z}) + (\frac{1}{2} + \mathbb{Z}) = \mathbb{Z}$ and are at least $\frac{3n}{4} + \frac{1}{2} \geq 2$. So $\mathbf{1} \oplus b'$ is a commutative algebra, $\widetilde{U}_n$ is a vertex operator algebra of CFT type whose weight-one space is that of $V \otimes V_{K_n}$, and the rest of the proof of Theorem 8.2(2) applies. The central charge is $(1 - 6n) + 7$.

(6) The hypotheses and proofs of §9 require only that n be even (§9.1). Since $n = rs^2$ with $n \equiv 2 \pmod 4$, s is odd and r is even for every h, so $\epsilon_h = 0$. Lemma 9.2(3) covers the C* categories. □

For $n \equiv 2 \pmod 4$ the sign $\varphi(k_0) = (-1)^r$ of §9.4 is always 1, so every $\Upsilon_{x,y}$ has dual $\Upsilon_{y,x}$ and the neutral simple objects $\Upsilon_{a,a}$ of $\mathcal{D}_h$ are self-dual. The fusion rings of the $\mathcal{D}_h$ are again classified by the divisors s of n with $s^2 \mid n$, all of which are odd. Proposition 9.10 applies to the odd h with $h^2 \mid n$, and Theorem 9.18 applies for $n > 2$. Every $\mathcal{D}_h$ with $\gcd(h, n/h) = 1$ is equivalent to $\mathcal{N}_+$, and Theorem 9.20 gives all equivalences among the $\mathcal{D}_h$. For $n = 2$ we do not know whether $\mathcal{D}_2 \simeq \mathcal{N}_+$. For the C* categories $\mathbb{Z}_2 + 2$ this is Izumi's theorem [32, Thm. 11.3].

Remark 12.3. The central lift of the element of order two of $\mathbb{Z}_n$ is $a_{n/2}$, with twist $e(-n/4) = -1$. So the subgroup of order two of $\mathbb{Z}_n$ has no Tannakian lift to $Z(\mathcal{N}_+)$, and $\mathcal{N}_+$ has no de-equivariantisation by it. There is no analogue of $\mathcal{D}$, of Theorems 6.6 and 7.1(3)–(5), or of Theorem 8.2(3). There is no square case either, as $n \equiv 2 \pmod 4$ is not a square. The central quotients $\mathcal{Q}_h$ of §11 exist for the odd h with $h^2 \mid n$.

12.2 Frobenius–Schur exponents and indicators

For $n = 4k$ the Frobenius–Schur exponent and the indicators of R were computed in §11 from the modular data of the double. The same method applies here, because the double has the same closed modular data (Theorem 12.2(4)). Two things change in the computation. The lowest weights of the modules A_a and B_b now contribute a sign $(-1)^\ell$ to the ℓ-th powers of the twists, and certain Gauss sums vanish for odd ℓ. The outcome is that the exponent is again $n(n+4)$, that the formula for the indicators loses a sign, and that the fourth indicator is again $1 + i$, so that the category has no real form. We use the notation $H_N^\iota(\ell)$, $g_N^\iota(\ell)$ of §11.

Proposition 12.4. *Let $n \equiv 2 \pmod 4$.*

1. $\mathrm{FSexp}(\mathcal{N}_+) = \mathrm{FSexp}(\mathcal{D}_h) = n(n+4)$ *for every $h \mid n$.*
2. *For every $\ell \geq 1$,*
$$\nu_\ell(R) = \frac{1}{4} \sum_{\iota=0}^{1} \Big(|g_n^\iota(\ell)|^2 - g_m^\iota(\ell)\, \overline{g_n^\iota(\ell)} \Big).$$
3. *$\nu_4(R) = 1 + i$, and Corollary 11.3 holds. No Galois conjugate of $\mathcal{N}_+$ is defined over a subfield of $\mathbb{R}$.*

For $n = 4k$ the exponent $16k(k+1)$ of Theorem 11.1 is also $n(n+4)$, and (51) differs from (2) by the sign $(-1)^{\ell\iota}$.

Proof. (1) By Proposition 7.2, $\theta_{a_g} = \theta_{b_g} = e(-g^2/n)$, and, since $e(-n/4) = -1$,
$$\theta_{c_{a,s}} = e\Big(\frac{a^2 - s^2}{4n} + \frac{s+1}{2}\Big), \qquad \theta_{d_{b,s}} = e\Big(\frac{b^2 n - s^2 m}{4nm} + \frac{s+1}{2}\Big),$$
with $a \equiv s$ and $b \equiv s \pmod 2$. Then $4 \mid a^2 - s^2$, and $b^2 n - s^2 m = n(b^2 - s^2) - 4s^2$ is divisible by 4, so all twists have order dividing nm. For $d_{1,1}$ the twist is $e\big((n-m)/4nm\big) = e(-1/nm)$, of

order nm. The Frobenius–Schur exponent is the order of the twist of the double [43], and the $\mathcal{D}_h$ have the same double.

(2) We follow the proof of Theorem 11.2. The trace formula, the multiplicities $[I(Y):R]$ and the dimensions are unchanged, and $B = \overline{H^0_n}$. In the sums over the c- and d-types the lowest weights $a^2/4n - n/4$ and $b^2/4m - n/4$ now contribute the factor $e(-\ell n/4) = (-1)^\ell$. With the factor $e(\ell s/2)$, the even sector has sign $(-1)^\ell$ and the odd sector has sign $(-1)^\ell(-1)^\ell = 1$. The two endpoints $a \in \{0, n\}$ that are added to complete the sum have θ^ℓ equal to $(-1)^\ell e(-\ell s^2/4n)$ and $e(-\ell s^2/4n)$ respectively, with s even. So they contribute $(1+(-1)^\ell)\overline{H^0_n}$ in place of $2\overline{H^0_n}$, and the same holds for $b \in \{0, m\}$. Hence

$$C = \tfrac14\Big((-1)^\ell H^0_n\overline{H^0_n} - (1+(-1)^\ell)\overline{H^0_n} + H^1_n\overline{H^1_n}\Big),$$
$$D = \tfrac14\Big((-1)^\ell H^0_m\overline{H^0_n} - (1+(-1)^\ell)\overline{H^0_n} + H^1_m\overline{H^1_n}\Big).$$

If ℓ is even these are the expressions in the proof of Theorem 11.2 with $(-1)^\ell = 1$, and the rest of that proof gives (2). If ℓ is odd, then $H^0_N(\ell) = 0$ for $N \equiv 2 \pmod 4$. The substitution $x \mapsto x + N$ multiplies the term of an even x by $e(\ell N/4) = -1$. So $B = 0$, $C = \frac14|H^1_n|^2$ and $D = \frac14 H^1_m\overline{H^1_n}$, and dividing $(d+2)C + dD$ by $n(d+2)$ with $d/(d+2) = -\sqrt{n/m}$, as there, gives (2), in which the terms with $\iota = 0$ vanish.

(3) Write $N = 2M$ with M odd. Then

$$g^0_N(4) = N^{-1/2}\sum_{y\in\mathbb{Z}_N} e\Big(\frac{4y^2}{N}\Big) = 2N^{-1/2}\sum_{y\in\mathbb{Z}_M} e\Big(\frac{2y^2}{M}\Big) = \sqrt{2}\,\Big(\frac{2}{M}\Big)\varepsilon_M,$$

by the classical value $(\frac{2}{M})\varepsilon_M\sqrt{M}$ of the quadratic Gauss sum, where $\varepsilon_M = 1$ or i according as $M \equiv 1$ or $3 \pmod 4$. Also $g^0_N(4) + g^1_N(4) = N^{-1/2}\sum_{x\in\mathbb{Z}_{2N}} e(x^2/N) = 0$, since $x \mapsto x + M$ multiplies each term by $e(M/2) = -1$. So $g^1_N(4) = -g^0_N(4)$ and $|g^\iota_N(4)|^2 = 2$. With $n = 2M$ and $m = 2(M+2)$, (2) gives

$$\nu_4(R) = 1 - \Big(\frac{2}{M}\Big)\Big(\frac{2}{M+2}\Big)\varepsilon_{M+2}\,\overline{\varepsilon_M}.$$

For $M \equiv 1, 3, 5, 7 \pmod 8$ the pair of Jacobi symbols is $(1,-1)$, $(-1,-1)$, $(-1,1)$, $(1,1)$ and $\varepsilon_{M+2}\overline{\varepsilon_M}$ is $i, -i, i, -i$. The product is $-i$ in each case, so $\nu_4(R) = 1+i$. The proof of Corollary 11.3 then applies. □

12.3 The superalgebra and its modules

For $n \equiv 2 \pmod 4$ the module J has conformal weights in $\frac12 + \mathbb{Z}$, so $V \oplus J$ cannot be a vertex operator algebra. It is a vertex operator superalgebra with odd part J, and this is the natural meaning of Flohr's fermionic $\mathcal{W}(2, 3k)$. The representation theory of such a superalgebra has two sectors. The Neveu–Schwarz sector consists of the ordinary modules, and it corresponds to the V-modules with trivial monodromy with J. The Ramond sector consists of the modules twisted by the parity automorphism, and it corresponds to the V-modules with monodromy -1. Proposition 12.5 first realises the superalgebra inside a lattice vertex superalgebra. It then describes both sectors by induction from V, using the theory of superalgebra extensions of Creutzig, Kanade and McRae [6]. For the Neveu–Schwarz sector this gives the simple modules, their characters, the modular S-matrix and the fusion rules. For the Ramond sector it gives the list of modules only.

Let $L = \mathbb{Z}\beta$ with $(\beta,\beta) = 2k = n/2$, now an odd lattice, with the conformal vector ω_λ of Proposition 3.1. Then V_L is a vertex operator superalgebra with even part $V_{L'} = U$, $L' = 2L$, and odd part M_n, and $\widehat{W}_k = V \oplus J \subset V_L$. For $j \in \frac12\mathbb{Z}$, splitting the theta sum by parity gives $\Lambda_{j,k} = \Lambda_{2j,n} + \Lambda_{n-2j,n}$ and $\Lambda_{j,k+1} = \Lambda_{2j,m} + \Lambda_{m-2j,m}$, as in §3.

Proposition 12.5. *Let $n \equiv 2 \pmod 4$ and $k = n/4$.*

1. *$\widehat{W}_k = V \oplus J$ is a simple vertex operator superalgebra of central charge $1 - 24k$, with even part V and odd part J. It is strongly generated by T and an odd Virasoro primary of weight $3k \in \mathbb{N} + \frac{1}{2}$ whose operator product with itself has singular part in the Virasoro vacuum module. As a $\widehat{W}_k$-module, $V_L = \widehat{W}_k \oplus M$ with M simple and $M|_V = X \oplus Y$.*

2. Neveu–Schwarz sector. *The simple $\widehat{W}_k$-modules with restriction in $\mathcal{C}$ are, up to parity, the induced modules $F_J(x) = x \oplus (J \otimes x)$, x simple in $\mathcal{C}'$, and $F_J(x) \cong F_J(x')$ up to parity exactly when $x' \in \{x, J \otimes x\}$. There are $2k + 2$ of them, with characters*

$$\begin{gathered} \tfrac{1}{2}(\Lambda_{0,k} - \Lambda_{0,k+1}), \quad \tfrac{1}{2}(\Lambda_{0,k} + \Lambda_{0,k+1}), \\ \Lambda_{j,k} \ (1 \le j \le k - \tfrac{1}{2}), \quad \Lambda_{j,k+1} \ (1 \le j \le k + \tfrac{1}{2}), \qquad j \in \mathbb{Z}. \end{gathered}$$

3. *On these characters $\tau \mapsto -1/\tau$ acts by the symmetric unitary matrix $\widehat{S}_{[x],[y]} = 2S_{x,y}$, and $\tau \mapsto \tau + 2$ by $\operatorname{diag}(\theta_x^2 e(-c/12))$. The fusion rules are*

$$F_J(x) \otimes F_J(y) \cong \bigoplus_{[z]} \Big(N_{xy}^z \, F_J(z) \oplus N_{xy}^{J \otimes z} \, \Pi F_J(z) \Big),$$

$$N_{xy}^z + N_{xy}^{J \otimes z} = \sum_{[w]} \frac{\widehat{S}_{[x][w]} \widehat{S}_{[y][w]} \overline{\widehat{S}_{[z][w]}}}{\widehat{S}_{[V][w]}},$$

   where Π reverses parity, N are the fusion coefficients of $\mathcal{C}$, and the sums are over J-orbits in $\mathcal{C}'$.

4. Ramond sector. *The parity-twisted modules are the $F_J(x)$ with x simple in $\mathcal{C}''$, namely for the J-orbits of A_a and B_b with a, b odd, with characters $\Lambda_{\lambda,k}$ $(\frac{1}{2} \le \lambda \le k)$ and $\Lambda_{\lambda,k+1}$ $(\frac{1}{2} \le \lambda \le k + 1)$, $\lambda \in \mathbb{Z} + \frac{1}{2}$. For the two fixed points $x = A_{n/2}, B_{m/2}$, that is $\lambda = k$ and $\lambda = k + 1$, $F_J(x) \cong x \oplus x$ and $\operatorname{End} F_J(x)$ is spanned by the identity and an odd involution. This is a list of the twisted modules only. No tensor structure or fusion rules for the Ramond sector are determined here.*

Proof. (1) By [18, §4.4], quoted in §3.2 for every n, $V \oplus J$ is a closed simple subalgebra of the lattice abelian intertwining algebra $U \oplus M_n$, which for $n \equiv 2 \pmod 4$ is the vertex operator superalgebra V_L. J has weights in $3k + \mathbb{Z} = \frac{1}{2} + \mathbb{Z}$ and lies in the odd part M_n. The proof of Proposition 3.2 applies verbatim. Indeed $Y(e^{-\beta}, z)e^{-\beta} = z^{2k}(\pm e^{-2\beta} + O(z))$ with $2k$ an odd integer, the vector $j_{(-2k-1)}j$ has weight $8k = 2n$ and charge -2 component $\pm e^{-2\beta}$, and $j_{(m)}j$ has weight $6k - 1 - m < 8k$ for $m \ge 0$. For the last statement, $V_L = V \oplus X \oplus J \oplus Y$ as a V-module, and $J \otimes X \cong Y$, $J \otimes Y \cong X$ (Theorem 12.2(2)). The action of J on X is an intertwining operator of type $\binom{V_L}{J\ X}$, so it maps into the Y-isotypic part Y, and likewise J maps Y into X. So $M = X \oplus Y$ is a $\widehat{W}_k$-submodule and $V_L = \widehat{W}_k \oplus M$. The V-submodules of M are 0, X, Y and M, since $X \ncong Y$. If X were a $\widehat{W}_k$-submodule, J would act on it by zero. Then $J \cdot J$, which contains a nonzero element of V, would act by zero on X, and the annihilator of X would be a nonzero ideal of the simple algebra V. The same holds for Y. So M is simple.

(2) By Theorem 12.2(2), J is a simple current with $J \otimes J \cong V$, $\theta_J = -1$ and $d(J) = 1$. This is the case of [6, Thm. 4.2(3)], in which $\mathbf{1} \oplus J$ is a superalgebra S in $\mathcal{C}$ and $\widehat{W}_k$ the corresponding $\frac{1}{2}\mathbb{Z}$-graded vertex operator superalgebra. By [6, Thm. 3.65] the $\widehat{W}_k$-modules with restriction in $\mathcal{C}$ form the braided monoidal supercategory $\operatorname{Rep}^0 \mathsf{S}$ of local S-modules, and by [6, Thm. 4.3(1)] the induced module $F_J(x) = \mathsf{S} \otimes x$ is local exactly when $h_{J \otimes x} - h_J - h_x \in \mathbb{Z}$, that is when $x \in \mathcal{C}'$. The category of S-modules is semisimple, because $\mathcal{C}$ is and the multiplication of S splits as a map of bimodules, J being invertible. So every simple S-module is induced from a simple object

of $\mathcal{C}$, and simple objects induce to simple modules [6, Cor. 4.23, 4.24]. By Frobenius reciprocity [6, Lemma 2.61],

$$\mathrm{Hom}_{\mathsf{S}}(F_J(x), F_J(y)) \cong \mathrm{Hom}_{\mathcal{C}}(x, y) \oplus \mathrm{Hom}_{\mathcal{C}}(x, J \otimes y),$$

the first summand even and the second odd, which gives the statement on isomorphisms. By Theorem 12.2(2) the J-orbits in $\mathcal{C}'$ are $\{V, J\}$, $\{X, Y\}$, $\{A_a, A_{n-a}\}$ with a even, and $\{B_b, B_{m-b}\}$ with b even. There are no fixed points, so the last two families have $\frac{1}{2}(\frac{n}{2} - 1) = k - \frac{1}{2}$ and $\frac{1}{2}(\frac{m}{2} - 1) = k + \frac{1}{2}$ orbits, and the total is $2k + 2$. The characters are $\chi_x + \chi_{J\otimes x}$, which are the functions listed, by §3 and the theta identity above with $j = a/2$ or $b/2$.

(3) For $x \in \mathcal{C}'$ and y simple in $\mathcal{C}$, $S_{J\otimes x,y} = \varepsilon_y S_{x,y}$ and $S_{x,J\otimes y} = S_{x,y}$. Hence

$$\chi_{F_J(x)}(-1/\tau) = \sum_y (1 + \varepsilon_y) S_{x,y} \chi_y(\tau) = 2 \sum_{y \in \mathcal{C}'} S_{x,y} \chi_y(\tau) = \sum_{[y]} 2 S_{x,y}\, \chi_{F_J(y)}(\tau),$$

and $\widehat{S}$ is well defined on orbits and symmetric. It is unitary, since

$$\begin{aligned} \sum_{[y]} \widehat{S}_{[x][y]} \overline{\widehat{S}_{[z][y]}} &= 2 \sum_{y \in \mathcal{C}'} S_{x,y} \overline{S_{z,y}} \\ &= \sum_{y \in \mathcal{C}} (1 + \varepsilon_y) S_{x,y} \overline{S_{z,y}} \\ &= \delta_{x,z} + \delta_{J\otimes x,z}. \end{aligned}$$

By the balancing equation $\theta_{J\otimes x} = \theta_J \theta_x = -\theta_x$ for $x \in \mathcal{C}'$, so the two summands of $F_J(x)$ have conformal weights differing by a half-integer, and $\tau \mapsto \tau + 2$ acts on $\chi_{F_J(x)}$ by $\theta_x^2 e(-c/12)$. Induction is a braided tensor functor on $\mathcal{C}'$ [6, Thms. 2.67, 3.68], so

$$F_J(x) \otimes F_J(y) \cong F_J(x \otimes y) = \bigoplus_{z \in \mathcal{C}'} N_{xy}^z F_J(z),$$

and $F_J(J \otimes z) \cong \Pi F_J(z)$. This is the first formula. For the second, Verlinde's formula in $\mathcal{C}$ gives

$$N_{xy}^z + N_{xy}^{J\otimes z} = \sum_{w \in \mathcal{C}} (1 + \varepsilon_w) \frac{S_{x,w} S_{y,w} \overline{S_{z,w}}}{S_{V,w}} = 4 \sum_{[w]} \frac{S_{x,w} S_{y,w} \overline{S_{z,w}}}{S_{V,w}},$$

which is the stated expression.

(4) For $x \in \mathcal{C}''$ the induced module is not local, and is a module twisted by the parity automorphism of $\widehat{W}_k$ [6, §§1.4, 4.2.4]. The orbits and characters are found as in (2), with $\lambda = a/2$ or $b/2$. For a fixed point, $F_J(x) = x \oplus (J \otimes x) \cong x \oplus x$ and the formula for $\mathrm{Hom}_{\mathsf{S}}$ in (2) gives a two-dimensional endomorphism algebra with one-dimensional odd part. Compare [6, Lemma 4.32(2), Remark 4.33]. □

This agrees with Flohr's description of the fermionic case [20, §4, App. A]. The Neveu–Schwarz sector is invariant under S and T^2. Its S-matrix is the bosonic one, with k half-integral and the degenerate representations removed. The degenerate representations $\Lambda_{k,k}$, $\Lambda_{k+1,k+1}$ occur only in the Ramond sector [20, p. 194]. Flohr gives the S-matrix of the Neveu–Schwarz sector but states no fusion rules in the fermionic case, and for the Ramond sector he leaves open whether a Verlinde formula applies. Part (3) gives the fusion rules of the Neveu–Schwarz sector, with the parity of each summand. The Ramond sector is described in (4) only as a list of modules.

Remark 12.6 ($n = 2$ and E_6). For $n = 2$, $c = -11$ and V_4 has 10 simple modules. $\mathcal{N}_+$ has the fusion rules $\mathbb{Z}_2 + 2$ of the even part of the E_6 subfactor, with $\mathrm{FPdim}\, R = 1 + \sqrt{3}$ and $d(R) = 1 - \sqrt{3}$, and $\widetilde{U}_2$ has central charge -4. The S- and T-matrices of the double of the even part of the E_6 subfactor are given by Hong, Rowell and Wang [29, §2.1.1], with simple objects

$1, Y, X_1, \dots, X_5, U, V, W$. Apply to them the Galois automorphism $\zeta_{12} \mapsto \zeta_{12}^7$, which sends $\sqrt{3}$ to $-\sqrt{3}$ and i to $-i$. The result is, entry by entry, the matrices of Proposition 7.2 at $n = 2$, under the bijection

$$[V,0] \mapsto 1, \quad [V,2] \mapsto Y, \quad [X,0] \mapsto U, \quad [X,2] \mapsto V, \quad [A_1,1] \mapsto W,$$
$$[B_3,1] \mapsto X_1, \quad [B_1,1], [B_1,3] \mapsto X_2, X_3, \quad [B_2,2] \mapsto X_4, \quad [B_2,0] \mapsto X_5$$

(`e6_compare.py`). The automorphism $\zeta_{12} \mapsto \zeta_{12}^5$ does not match the twists. Theorem 12.2(6) gives two indecomposable module categories and four connected separable algebras, on $\mathbf{1}$, $\mathbf{1} \oplus \alpha$, $\mathbf{1} \oplus R$ and $\mathbf{1} \oplus \alpha \oplus 2R$. For the two C* categories $\mathbb{Z}_2 + 2$ [16, Prop. 6], the subfactors of Proposition 9.24 have indices 2, $2+\sqrt{3}$ and $4+2\sqrt{3}$. The second is the index of the E_6 subfactor.

For $n = 2, 6, 10$ the statements of Theorem 12.2(1)–(5) that concern modular data, Proposition 12.4(2) for all $\ell \le n(n+4)$, and the unitarity of $\widehat{S}$ and the Verlinde formula of Proposition 12.5(3) have been checked numerically (`theta_all_n.py`, `zclosed_all_n.py`, `voa_zng_all_n.py`, `checks_n2mod4.py`).

13 Concluding remarks

The scalar reconstruction gives a precise point of contact with the arithmetic construction of Radchenko and Wheeler. Put $\varepsilon = \delta = (\sqrt{k} + \sqrt{k+1})^2$, so that $\varepsilon + \varepsilon^{-1} = n + 2$. The intrinsic Leavitt function attached to the Flohr category satisfies

$$E_{\mathrm{L}}(0) = \sqrt{\varepsilon}, \qquad d_- = -\frac{\sqrt{n}}{E_{\mathrm{L}}(0)} = -\frac{\varepsilon - 1}{\varepsilon}.$$

Thus the real quadratic field and dimension normalization agree with those in the Stark-unit construction of [44]. Theorem 10.1 identifies the Flohr category with a Leavitt category for every integral parameter; identifying its intrinsic function with a particular arithmetic solution requires comparison of the full datum, not just the quadratic field or modular data.

For the arithmetic solutions, every Galois automorphism inducing the nontrivial automorphism of $\mathbb{Q}(\sqrt{n(n+4)})$ gives values of absolute value one away from zero [45, Prop. 7]. In the conventions of (53), this is the Hermitian relation after passing to the positive dimension. For the Flohr categories, the next question is therefore to compare the intrinsic scalar function with the arithmetic solutions and to identify the corresponding Hermitian structures. The algebraic reconstruction in Theorem 10.1 supplies the categorical starting point for that comparison. Appendix A records the existing candidate calculations and the precise range of their exact and numerical conclusions.

A Consistency checks and comparison with C* near-group categories

Sections A.1–A.3 record computations from modular data, for small k, of results proved above for every k. Nothing in the body of the paper depends on them. Section A.4 concerns a question left open, namely whether the positive Galois conjugate of $\mathcal{N}_+$ is one of the C* near-group categories $\mathbb{Z}_{4k} + 4k$ of Izumi and of Evans and Gannon. It gives necessary conditions, verified numerically, and exact algebraic identities for candidate solutions. No result of the paper depends on it either. In §§A.1–A.2, $\mathcal{C}$ denotes a category with the modular data of $\mathcal{W}(2, 3k)$ [20].

A.1 The category $\mathcal{D}$

This subsection checks, from Flohr's modular data alone and for $k \le 7$, the fusion rules of the category $\mathcal{D}$ that are proved for every k in Theorem 7.1(3) and Corollary 9.8.

Proposition A.1 (modular data, $k \le 7$)**.** *For $1 \le k \le 7$ the matrix $(\delta_{XY} + N_{XM}^{Y})_{X,Y}$ factorises as BB^{T} with B a non-negative integer restriction matrix [34], and the resulting fusion ring has*

1. *rank $2k+2$, invertibles g_λ, $\lambda \in \mathbb{Z}_{2k}$, and two simples x, y of dimension $\frac{1}{2}(\delta - 1)$;*
2. *local modules g_λ, with the modular data of $\mathcal{C}(\mathbb{Z}_{2k}, q)$;*
3. *$g_\lambda x = x$ or y as λ is even or odd, $x^* = g_k x$, and $x \otimes x^* = \bigoplus_{\lambda \text{ even}} g_\lambda \oplus k\,x \oplus k\,y$.*

These fusion rules agree with Theorem 7.1(3)–(4) and Lemma 7.4, as well as the case $h = 2$ of Corollary 9.8. The computation is implemented in `ca_analysis.py`.

A.2 Near-group fusion rules

This subsection shows, at the level of fusion rings, that the fusion ring of $\mathcal{D}$ found in §A.1 is that of a de-equivariantisation of a near-group category $G + 4k$ by a subgroup of order two.

Lemma A.2. *Let G be a finite abelian group of order $4k$, $z \in G$ an element of order 2, and $\bar{G} = G/\langle z \rangle$. In the near-group fusion ring $G + 4k$, with $\rho^2 = \sum_{g \in G} g + 4k\rho$, replace g by $\bar{g}$ and ρ by $x + y$. Then*

$$(x+y)^2 = 2 \sum_{\bar{g} \in \bar{G}} \bar{g} + 4k(x+y),$$

and $\operatorname{FPdim}(G + 4k) = 4k + (\delta - 1)^2 = 4k(\delta + 1) = 2 \operatorname{FPdim} \mathcal{D}$.

Proof. The displayed identity is immediate. For the dimension, use $(\delta - 1)^2 = 4k\delta$, which is equivalent to $\delta + \delta^{-1} = 4k + 2$. □

With $\bar{G} = \mathbb{Z}_{2k}$, the fusion rules of Proposition A.1(3) give, using $y = g_1 x$,

$$x \otimes x = y \otimes y = \bigoplus_{\lambda \equiv k\,(2)} g_\lambda \oplus k\,x \oplus k\,y, \qquad x \otimes y = y \otimes x = \bigoplus_{\lambda \equiv k+1\,(2)} g_\lambda \oplus k\,x \oplus k\,y,$$

and summing the four products gives exactly the identity of Lemma A.2. Thus the fusion ring of $\mathcal{D}$ is that of a de-equivariantisation along $\langle z \rangle$ of a near-group category $G + 4k$ with $G/\langle z \rangle \cong \mathbb{Z}_{2k}$. Conversely, a $\mathbb{Z}_2$-action on $\mathcal{D}$ fixing every g_λ and exchanging x and y has an equivariantisation with $4k$ invertible objects and one simple of dimension $\delta - 1$. That is, it has near-group fusion rules $G + 4k$.

The fusion rules of $\mathcal{D}$ do not determine whether $G = \mathbb{Z}_{4k}$ or $G = \mathbb{Z}_{2k} \times \mathbb{Z}_2$, nor whether such an action exists. Theorems 5.1 and 7.1(3) construct the action, with $G = \mathbb{Z}_{4k}$.

Remark A.3. For C* near-group categories of type $G + |G|$ the de-equivariantisation is computed by Izumi [32, Thm. 12.5]. Let $H \subset G$ with $H \subset H^{\perp}$. Then the invertibles are G/H, the non-invertibles are σ_χ, $\chi \in \hat{H}$, and

$$\sigma\sigma = \bigoplus_{k' \in H^{\perp}/H} \tilde{\alpha}_{k' - g_a} \oplus |H^{\perp}/H| \bigoplus_{g \in G/H^{\perp}} \tilde{\alpha}_g \sigma,$$

where $a(h) = \langle h, g_a \rangle$ for $h \in H$. Take $G = \mathbb{Z}_{4k}$, $\langle g, h \rangle = e(mgh/4k)$ and $H = \{0, 2k\}$. Then $\langle 2k, 2k \rangle = e(mk) = 1$, $H^{\perp} = 2\mathbb{Z}_{4k}$, $|H^{\perp}/H| = k$, $|G/H^{\perp}| = 2$, and $a(2k) = e(-mk/2) = (-1)^k$. So σ is self-dual exactly when k is even, and the fusion rules are those of Proposition A.1(3). Once the positive Galois conjugate of $\mathcal{N}_+$ (§5) is identified with a C* near-group category, this theorem gives the corresponding conjugate of $\mathcal{D}$ as a category. Gannon, Schopieray and Yadav give an algebraic form, without unitarity, of the related construction [32, Thm. 12.9] from Haagerup–Izumi to near-group categories [23, §6]. No such form of [32, Thm. 12.5] is used here.

A.3 Near-group fusion rules at $n = 4k$

This subsection checks, from the theta modular data and for $k \le 6$, the near-group fusion rules proved for every k in Theorem 5.1.

Proposition A.4 (modular data, $k \le 6$)**.** *For $1 \le k \le 6$ the modular data of §6.1 gives, for the modules over $A' = \mathbf{1} \oplus X$ in $\mathcal{C}'$, the near-group fusion rules $\mathbb{Z}_{4k} + 4k$ of Theorem 5.1, with local modules matching $\mathbb{Z}_n$ with $q(x) = x^2/n$ (*`neargroup_check.py`*).*

A.4 Galois conjugates and the Evans–Gannon parameters

This subsection compares the positive Galois conjugates of $\mathcal{N}_+$ with the C* near-group categories of Izumi and of Evans and Gannon, through invariants that can be computed on both sides.

The categories $\mathcal{C}$ and $\mathcal{N}_+$ of §5 are non-unitary, since $\mathrm{FPdim}\, R = \frac{1}{2}(n + \sqrt{n(n+4)})$, while at the level of modular data the categorical dimension is $\frac{1}{2}(n - \sqrt{n(n+4)}) < 0$. Positive dimensions occur after a cyclotomic automorphism σ_ℓ with $\sigma_\ell(\sqrt{k(k+1)}) = -\sqrt{k(k+1)}$. For each such ℓ we record two quantities that constrain which C* near-group category is a Galois conjugate of $\mathcal{N}_+$.

- *The bicharacter.* The central lift of M_{2x} has twist $e(-x^2/n)$, from the reversed lattice factor in $Z(\mathcal{N}) \simeq \mathcal{C} \boxtimes (\mathrm{Rep}\, U)^{\mathrm{rev}}$. By [16, (4.49)] this twist is $\langle x, x\rangle$, so after σ_ℓ the Evans–Gannon pairing is $\langle g, h\rangle = e(m_{EG} gh/n)$ with $m_{EG} \equiv -\ell \pmod n$.
- *The D-type twists.* In the centraliser of the pointed central subcategory, the objects whose underlying object is R come from $\Lambda_{2y,m}$, $1 \le y \le m/2 - 1$, with twists $e(y^2/m)$. After σ_ℓ these become $e(\ell y^2/m)$.

For example, at $k = 3$, $\ell = 17$ gives $m_{EG} = 7$ and the D-twist multiset

$$\{1, \zeta_{16}, \zeta_{16}, i, i, \zeta_{16}^9, \zeta_{16}^9\}.$$

Table 2: Positive Galois conjugates and their bicharacters.

k	n	classes of ℓ	m_{EG} mod n (each with its D-twist multiset)
1	4	2	1, 3
2	8	4	1, 3, 5, 7
3	12	4	5, 5, 7, 7
4	16	4	1, 3, 5, 7
5	20	8	1, 1, 3, 3, 11, 11, 13, 13
6	24	8	1, 5, 7, 11, 13, 17, 19, 23

Here ℓ ranges over units modulo $\mathrm{lcm}(n, m, 4r)$, where r is the squarefree part of $k(k+1)$, and two values of ℓ are in the same class when they give the same m_{EG} up to unit squares and the same D-twists. The classes come in complex-conjugate pairs $\ell \leftrightarrow -\ell$.

Comparison with [16, Table 2]. For $\mathbb{Z}_4$ the table has two rows, $m = \pm 1$, matching the two classes at $k = 1$. For $\mathbb{Z}_8$ it has eight rows, $m \in \{\pm 1, \pm 3\}$ each with $a = \pm 1$. The rows with the same m are related by the automorphism $g \mapsto 3g$ (§A.4.1), so there are four classes of categories, one for each m, matching the four classes at $k = 2$. For $\mathbb{Z}_{12}$ the four rows have, in the convention of §A.4.1 with $u = e(1/24)$, the parameters

$$(m, s, c) = (7, -1, u^7), \quad (7, +1, u^{-5}), \quad (5, +1, u^{-7}), \quad (5, -1, u^5).$$

The printed parameters of the first row, $(m,s)=(1,+1)$, do not satisfy the Gauss identity $\sum_g a(g)=\sqrt{12}\,c^{-3}$, while these do. The phases printed in that row are those of a solution with $(m,s,c)=(7,-1,u^7)$. So the four rows have $m=7,7,5,5$, matching the four classes at $k=3$. In the second class with $m=7$ ($\ell=5$), the D-twists include ζ_{16}^5, which occurs in the T_0-spectrum of §A.4.1 only on the branch $(s,c)=(+1,u^{19})$, not on the branch $(-1,u^7)$ matched to $\ell=17$. Thus, at the level of these necessary conditions, the Galois conjugates of the single VOA category account for all four $\mathbb{Z}_{12}$ rows.

A.4.1 The scalar equations and the eigenvalue test, $k\le 6$

We use Izumi's equations [31, (5.1)–(5.4), (5.11)–(5.13)], written in the convention of [16, Cor. 5]. The pairing is $\chi(g,h)=e(mgh/n)$ with m a unit mod n, and $a(g)=s^g e(-mg^2/2n)$ with $s=\pm1$. The relation $\sum_g a(g)=\sqrt{n}\,c^{-3}$ allows three choices of c. Finally $X_0=-\sqrt{n}/d$ with $d=\frac12(n+\sqrt{n(n+4)})=\delta-1=\mathrm{FPdim}\,R$, and

$$\sum_h \chi(g,h)^{-1}X_h=\sqrt{n}\,c\,a(g)X_{-g}, \tag{52}$$

$$X_gX_{-g}=a(g)^{-1}\quad(g\neq 0). \tag{53}$$

The first system is linear. We parametrise its solution space and solve the second by Newton iteration, from 400 random starting points per branch (m,s,c). Each solution is then tested against the Hermitian relation $\overline{X_g}=a(g)X_{-g}$ and the identities

$$\sum_g X_{g+y}\overline{X_g}=n\delta_{y,0}-n/d, \tag{54}$$

$$\sum_g X_{g+y}X_{g+z}\overline{X_g}=\sqrt{n}\,\chi(y,z)^{-1}X_yX_z-nc/d. \tag{55}$$

For the eigenvalue test put $b(g)=X_g/\sqrt{n}$. A D-type half-braiding with $\tau=0$ and twist ω, in the parametrisation of [16, (4.52)–(4.54)], gives $\xi\in\mathbb{C}^n$ with

$$\sum_k \xi_k=\sqrt{n}\,\omega^2c^3-n/d,\qquad \omega\xi_j=\sum_k a(j)^2b(k-j)\xi_k+\tfrac{c\sqrt{n}}{d}\,a(j)^2,$$

the second obtained by combining the last two of those equations. With $L_{jk}=a(j)^2b(k-j)$, $v_j=c\sqrt{n}\,a(j)^2/d$ and e the column of ones, put

$$T_0=\begin{pmatrix} L & 0 & v\\ (\sqrt{n}\,c^3)^{-1}e^T & 0 & \sqrt{n}/(c^3d)\\ 0^T & 1 & 0\end{pmatrix}.$$

Then $T_0(\xi,\omega,1)^T=\omega(\xi,\omega,1)^T$, so every such twist ω is an eigenvalue of T_0. The test is that the unit eigenvalues of T_0 contain, as a multiset, the D-type twists $e(\ell y^2/(n+4))$ of §A.4.

At $n=12$, $m=7$ the surviving branches are $(s,c)=(+1,u^{19})$ and $(-1,u^7)$, each with four solutions. The T_0-spectrum contains ζ_{16} only on the second and ζ_{16}^5 only on the first.

Relabelling. Put $\tilde m=m+\frac{n}{2}(1-s)$ mod $2n$. Then $a(g)=e(-\tilde m g^2/2n)$, so a branch is determined by $\tilde m$ together with c. For a unit u mod $2n$, the automorphism $g\mapsto ug$ of $\mathbb{Z}_n$ sends $\tilde m$ to $u^2\tilde m$. It sends the pairing χ_m to χ_{u^2m} and fixes c. It preserves a specified pairing exactly when $u^2\equiv 1 \pmod n$; this congruence does not hold for every unit u. Branches are compared modulo unit squares modulo $2n$, allowing the corresponding relabelling of the pairing.

If k is even, then $8\mid n$. Taking $u=1+n/2$ gives $u^2\equiv 1+n \pmod{2n}$, so the branches $(m,+1)$ and $(m,-1)$ are related by an automorphism. If k is odd, they are not.

Equivalence. By [16, Cor. 5] and [32, Thm. 9.1], two solutions define equivalent C* near-group categories exactly when they have the same c and are related by an automorphism of G. The classes are therefore the orbits of the stabiliser of $\tilde m$ in $\mathrm{Aut}(\mathbb{Z}_n)$ on the solutions of one branch in each orbit.

Proposition A.5 (numerical). *Let $1 \le k \le 6$ and $n = 4k$.*

1. *For each unit m and each sign s, solutions were found for at most one value of c; none were found for the other two values. When the linear system has a unique solution, its failure to satisfy the remaining polynomial equations gives an exact exclusion of that branch.*
2. *Every solution found satisfies the Hermitian relation and (54), (55), to within $3 \cdot 10^{-12}$. Each surviving branch produced two candidates for $n = 4, 8, 16$ and four for $n = 12, 20, 24$; the candidates found form a single orbit of the stabiliser. The number of equivalence classes represented among the candidates found is*

$$2,\ 4,\ 4,\ 4,\ 8,\ 8 \qquad (n = 4, 8, 12, 16, 20, 24).$$

3. *Fix a positive Galois conjugate σ_ℓ (§A.4). Among the surviving branches with $m \equiv m_{EG}$ up to unit squares, the D-twists of σ_ℓ lie in the T_0-spectrum for exactly one class.*
4. *The map from Galois classes of ℓ to equivalence classes in (3) is a bijection.*

For $n = 4, 12$ the counts in (2) agree with [16, Prop. 6]. For $n = 4$ this is already in [31, Ex. A.2]. There, for the pairing $\langle k, l\rangle = i^{kl}$, only $a(k) = e(-k^2/8)$ (our $s = +1$) admits solutions. There are two, and they are related by an automorphism. For $n = 8$, [16, Prop. 6] counts eight. Its Table 2 lists (m, a) with $m \in \{\pm 1, \pm 3\}$ and $a = \pm 1$, and a is the sign s here. We checked on the tabulated values j_g that the row $(m, +1)$, transported by $g \mapsto 3g$, is the row $(m, -1)$. By [16, Cor. 5] the two rows therefore define equivalent categories, and the count is four.

Table 3 gives the correspondence in (3), with $\tilde{m}$ taken modulo unit squares modulo $2n$.

Table 3: Galois classes and the corresponding near-group branches.

k	ℓ	$\tilde{m}$
1	3, 5	1, 7
2	7, 11, 13, 17	1, 5, 3, 7
3	5, 7, 17, 19	7, 5, 19, 17
4	3, 7, 13, 17	5, 1, 3, 7
5	11, 23, 31, 41, 43, 53, 61, 73	1, 13, 21, 11, 17, 7, 31, 3
6	5, 23, 31, 37, 43, 59, 65, 73	19, 1, 17, 11, 5, 13, 7, 23

At $k = 3$, $\ell = 17$ gives $\tilde{m} = 19$, the branch $(m, s) = (7, -1)$.

For odd k the D-twists alone determine the class. For even k, Galois classes whose bicharacters differ can have the same D-twists. There the bicharacter is needed as well.

The D-type half-braidings with $\tau = 0$. The eigenvalue test uses only the necessary conditions above. We now solve all of the half-braiding equations [16, (4.52)–(4.55)] (cf. [31, Thm. 6.6]) at $\tau = 0$, together with $|\xi_g| = 1$. The solution set of (4.52)–(4.53) and the relation

$$\xi_{-g} = \omega c^4 a(g)^2\, \overline{\xi_g}$$

is an affine real subspace, and we find its points with $|\xi_g| = 1$ by least squares. The printed text of (4.54)–(4.55) does not show where the complex conjugations go. We use the forms for which solutions exist, namely the conjugate on ξ_g above, and

$$\textstyle\sum_k \xi_k\, b(k-g)\, b(k-h) = c^{-2}\, b(g+h)\, \xi_g \xi_h\, \overline{a(g-h)} - c^2/d.$$

Proposition A.6 (numerical). *For each Galois class ℓ with $k \le 6$, and the class matched to it in Proposition A.5, the twists ω of the solutions found match, within numerical precision, the multiset $\{e(\ell y^2/(n+4)) : 1 \le y \le n/2 + 1\}$ of §A.4. Every candidate found satisfies (4.55) to within $4 \cdot 10^{-13}$.*

Every D-type half-braiding with $\tau = 0$ must solve the correctly specified half-braiding equations. Among the solutions found numerically, the twist multiset agrees with the VOA prediction, including multiplicities. This search does not certify that every solution has been found or establish an equivalence with the VOA category. At $k = 3$, $\ell = 17$ this is the multiset $\{1, \zeta_{16}, \zeta_{16}, i, i, \zeta_{16}^9, \zeta_{16}^9\}$. The comparison of the S-matrix block on these objects [31, Thm. 6.8], [16, (4.58)] with the VOA has not been made.

Algebraic certificates and numerical completeness. The finite-parameter evidence below separates exact algebraic polynomial identities from the numerical tests of the Hermitian condition and completeness.

Proposition A.7. *1. (Exact algebraic identities for the selected candidates, $k \le 5$.) For each representative branch listed in the certificates, the numerical candidates have reconstructed coordinates in a field $F = K(\alpha)$, $K = \mathbb{Q}(\zeta_N)$, of relative degree $r = 2$ or 4. Exact arithmetic in F verifies that these coordinates satisfy (52), (53), the specified value of X_0, and (54), (55). Their characteristic polynomials satisfy the exact conjugate-reciprocal identity described below. The supplied relative-field verifier tests the required Hermitian relation using numerical root separation. A fully exact conjugation/embedding certificate remains to be supplied in a reproducible form.*

2. (Numerical completeness, $k \le 5$.) On every branch with $n \le 20$, total-degree homotopy numerically tracks all 2^d paths of a square subsystem of d quadratic equations to well-separated finite endpoints. The resulting counts agree with Proposition A.5. Rigorous completeness would require certification of 2^d disjoint exact roots (or an equivalent exact elimination computation), not merely numerical endpoint tracking.

Method. (1) The branch is defined over K, so its set of solutions is stable under $\mathrm{Gal}(\bar{\mathbb{Q}}/K)$. We take $\alpha = X_{g_0}$ for an entry that separates the solutions. The coefficients of $\prod_j (t - \alpha^{(j)})$, and the coordinates $\beta_{g,i}$ in $X_g = \sum_i \beta_{g,i}\alpha^i$, are recognised in K by integer relations at 900 digits, subject to a height bound. All identities are then verified in exact arithmetic in $F = K[t]/(P)$, using PARI/GP, after checking that P is irreducible over K. Recognition only proposes the candidate. The verification is exact.

The Hermitian condition for $g \neq 0$ is equivalent, using (2), to $|X_g| = 1$. The relative-field script verifies exactly that the conjugate characteristic polynomial of each X_g is reciprocal. It then identifies $\overline{X_g}$ with X_g^{-1} by comparing approximate roots in floating-point arithmetic. This comparison is strong numerical evidence but not an exact root-isolation certificate. An exact repair is to construct the conjugation involution on an absolute field containing the candidates, check its action on the cyclotomic generator and all X_g algebraically, and certify the selected embedding by rigorous complex-ball root isolation.

(2) The paths are tracked by predictor–corrector continuation from $y_i^2 = 1$, and collisions are detected and the tracking repeated. This is numerical in the sense of [16, §3]. The count 2^d is Bezout's bound, and the endpoints are well separated. □

For $k = 6$ the same count holds on every branch, with one exception. On the two branches $(m, s, c) = (1, +1, e(1/24))$ and $(23, -1, e(23/24))$, one or two of the 256 paths leave every bounded region ($|y| \approx 2 \cdot 10^3$ at $t = 1$, with nonzero residual), and these are counted as roots at infinity. The exact certificate for $k = 6$ has not been obtained. Integer relations in $\mathbb{Q}(\zeta_N)$, $N = 336$, are not recognised at 1800 digits.

Consequences and limits.

1. The algebraic candidates satisfy the polynomial relations of [16, Cor. 5] and [32, (9.1)–(9.5)], with $d > 0$. The supplied numerical tests support the Hermitian relation, which is required to obtain C* near-group categories. An exact complex-conjugation certificate would upgrade each candidate to a C* near-group category and, by [31, §5], a subfactor of index $d + 1 = \delta$

with principal graph $2^{4k}1$ (index $\delta_{\mathrm{EG}}+1$ in the notation of [16], where $\delta_{\mathrm{EG}}=d_\rho=\delta-1$), and the subfactors of Proposition 9.24. The cases $k\le 3$ also occur in [16, Table 2].

2. Within the numerical search, every near-group candidate found of type $\mathbb{Z}_{4k}+4k$ with $k\le 6$ passes the bicharacter and D-twist tests for exactly one positive Galois conjugate of the category of Theorem 5.1. These are necessary conditions. For $k\le 5$, the supplied homotopy computations give numerical completeness of the branches and the algebraic candidates are exact solutions of the polynomial equations. Certified Hermitian embeddings and a rigorous root count are not yet established by the packaged scripts. The half-braiding comparison (Proposition A.6) is numerical.

3. Theorem 10.1 supplies the algebraic reconstruction of $\mathcal{N}_+$ for every positive integral k. Its positive Galois conjugates satisfy the same categorical hypotheses and also lie in the Leavitt classification of Proposition 10.5. This closes the category-to-scalar passage that was previously missing here. Identifying a conjugate with a particular candidate in Proposition A.5(3) still requires comparison of the full intrinsic scalar datum (50), up to a group automorphism. The bicharacter and D-twist tests alone do not give that comparison. A C* identification further requires the certified Hermitian structure discussed above.

A.5 Computational methods

The exact calculations in the proof of C_2-cofiniteness are documented in the *Computational Supplement*, version of 16 November 2026 at 22:46:21 EST. The supplement, source code, pinned dependencies and recorded output are in the accompanying versioned archive. With Python 3 and SymPy 1.14.0, run from the archive's `computations` directory

```
python3 gauge_symbolic_certificate.py
python3 certificates/w23k_certificates/scripts/er_appB_check.py
```

The first program builds the gauge from the printed formulas and verifies the operator identity over $\mathbb{Q}[t^{\pm1},w^{\pm1}]$. It checks the determinant of Lemma 4.3 by verifying the Abel identity and evaluating the determinant at $w=1$ over $\mathbb{Q}(t)$. The second reconstructs the recurrence polynomials from the level-nine singular-vector table [18, Table A.1, App. B.1]. Its five ratios with the printed polynomials are all 1. The programs generate all symbolic inputs from these formulas and the included transcription.

The modular-data, fusion-rule and indicator computations provide finite checks of the formulas proved in the body of the paper. The comparison with particular C* near-group candidates is separate: the algebraic relations in Proposition A.7 are exact in the stated range, whereas the half-braiding comparisons, Hermitian tests and homotopy root counts retain the numerical scope specified in that proposition and Proposition A.6. They do not establish the outstanding unitary identifications or a rigorous completeness theorem. Program names, commands and the recorded output are given with the supplementary computational material.

B The recurrence and gauge coefficients

This appendix records the coefficients used in the proof of Proposition 4.1. They are rational functions of the displayed parameters, and the identities in Lemma 4.3 are exact.

B.1 The recurrence polynomials

Use the notation of (1). Thus $Q_q(a,h) = P_q(2h+8-2a,h)$ and $P_q(s,h) = (-2)^q \sum_j p_{q,j}(h) s^j$ for $q = 1,2,3$. The nonzero coefficients are listed individually below; all other $p_{q,j}$ vanish.

$$\begin{aligned}
p_{1,0} &= -12(h-1)(h+8)(3h^5+52h^4+217h^3-760h^2-6352h-6320),\\
p_{1,1} &= 2(6h^7+150h^6+823h^5-5848h^4-69573h^3-160142h^2\\
&\qquad +203592h+449032),\\
p_{1,2} &= -12h^6+63h^5+4974h^4+39861h^3+47690h^2-348788h-573224,\\
p_{1,3} &= -21h^5-654h^4-4521h^3+7992h^2+132500h+203936,\\
p_{1,4} &= 24h^4+132h^3-3208h^2-26765h-43572,\\
p_{1,5} &= 3(2h^3+114h^2+979h+1860),\\
p_{1,6} &= -3(4h^2+53h+132),\\
p_{1,7} &= 3(h+4).
\end{aligned}$$

$$\begin{aligned}
p_{2,0} &= -24(h-1)(h+8)(h^3-7h^2-128h-32),\\
p_{2,1} &= -2(12h^5+237h^4+1725h^3+3324h^2-12400h-6256),\\
p_{2,2} &= 6h^5+111h^4+525h^3-1314h^2-13448h-9320,\\
p_{2,3} &= -3(h^4-9h^3-263h^2-1216h-1092),\\
p_{2,4} &= -6(h^3+17h^2+80h+92),\\
p_{2,5} &= 3(h+2)(h+6).
\end{aligned}$$

$$\begin{aligned}
p_{3,0} &= 8h(h-1)(h+8)^2,\\
p_{3,1} &= -2(13h^3+52h^2-424h-32),\\
p_{3,2} &= -3h^3-68h^2-376h-96,\\
p_{3,3} &= (h+4)(h^2+8h+8).
\end{aligned}$$

The factored expressions for P_0 and P_4 are given in §4.

B.2 The gauge transformation

Put

$$\ell_1 = t-2, \quad \ell_2 = t-1, \quad \ell_3 = 2t-3, \quad \ell_4 = 2t-1,$$
$$\ell_5 = 3t-4, \quad \ell_6 = 3t-2, \quad \ell_7 = 4t-3.$$

The coefficients of $\Phi = w^\gamma \sum_{i,j=0}^{2} r_{ij} \theta^i f \theta^j g$ in Lemma 4.3 are

$$\begin{aligned}
r_{00} &= \frac{3\ell_1\ell_4\ell_5\ell_7}{4t^2} + \frac{\ell_1\ell_4}{t}\,w,\\
r_{01} &= -\frac{3\ell_1\ell_3\ell_4\ell_5\ell_7}{8t^2}\,w^{-1} + \frac{\ell_4(16t-17)}{4} + \ell_4\,w,\\
r_{02} &= -\frac{3\ell_3\ell_4\ell_5\ell_7}{4t}\,w^{-1} - \frac{\ell_4(8t-9)}{2},\\
r_{10} &= \frac{3\ell_1\ell_4\ell_5\ell_6\ell_7}{8t^3}\,w^{-1} + \frac{\ell_1(17t-16)}{4t^2} - \frac{\ell_1}{t}\,w,
\end{aligned}$$

$$
\begin{aligned}
r_{11} &= -\frac{3\ell_1\ell_3\ell_4\ell_5\ell_6\ell_7}{16t^3}\,w^{-2} + \frac{15\ell_2^2(t^2-t+1)}{t^2}\,w^{-1} + \frac{8t^2-11t+8}{4t} - w,\\
r_{12} &= -\frac{3\ell_3\ell_4\ell_5\ell_6\ell_7}{8t^2}\,w^{-2} - \frac{3\ell_2(4t^2+2t-7)}{2t}\,w^{-1} + \frac{4t-7}{2},\\
r_{20} &= -\frac{3\ell_1\ell_5\ell_6\ell_7}{4t^3}\,w^{-1} - \frac{\ell_1(9t-8)}{2t^2},\\
r_{21} &= \frac{3\ell_1\ell_3\ell_5\ell_6\ell_7}{8t^3}\,w^{-2} - \frac{3\ell_2(7t^2-2t-4)}{2t^2}\,w^{-1} - \frac{7t-4}{2t},\\
r_{22} &= \frac{3\ell_3\ell_5\ell_6\ell_7}{4t^2}\,w^{-2} + \frac{15\ell_2^2}{t}\,w^{-1} + 1.
\end{aligned}
$$

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School of Computational and Mathematical Sciences, Cardiff University, Cardiff CF24 4AG, Wales, UK

Email address: EvansDE@cardiff.ac.uk

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA

Email address: a_riesen@mit.edu