A weighted reciprocal Lucas sum lies outside $\mathbb{Q}(\sqrt{5})$

twinheterodox

18 November 2026

Abstract

For the even-indexed Lucas numbers L_{2n}, the reciprocal sums with numerators 1 and n^2 are known to be transcendental. We prove that the intermediate sum $\sum_{n\geq 1} n/L_{2n}$ lies outside $\mathbb{Q}(\sqrt{5})$, and that the same conclusion holds with L_{2an} in the denominator for every integer $a \geq 1$. More generally, the weight-two value $\beta_{1/b}(2) = \sum_{n\geq 1} n/(b^n + b^{-n})$ of the q-analogue of Dirichlet's beta function lies outside $\mathbb{Q}(b)$ for every real algebraic integer $b > 1$ whose other conjugates have modulus at most one.

hexagon:2611.00639v1

1 Three weights for the even Lucas numbers

The Lucas numbers satisfy $L_0 = 2$, $L_1 = 1$ and $L_{n+2} = L_{n+1} + L_n$. With the same denominators, consider the three sums

$$\sum_{n\geq 1} \frac{1}{L_{2n}}, \qquad \sum_{n\geq 1} \frac{n}{L_{2n}}, \qquad \sum_{n\geq 1} \frac{n^2}{L_{2n}}.$$

Duverney, Ke. Nishioka, Ku. Nishioka and Shiokawa proved that the first sum is transcendental [1, p. 97 of the Kyoto account]. The third is also transcendental, by the odd-weight result of Jouhet and Mosaki [2, §1, (1.1) and the discussion before Theorem 1.1]. What happens for the intermediate weight n? We prove that $1/3 + 2/7 + 3/18 + 4/47 + \cdots$ lies outside $\mathbb{Q}(\sqrt{5})$, the field generated by the roots of the Lucas recurrence.

To see the connection with beta values, write $\varphi = (1+\sqrt{5})/2$. Binet's formula $L_j = \varphi^j + (-\varphi^{-1})^j$ gives $L_{2n} = \varphi^{2n} + \varphi^{-2n}$. For $0 < q < 1$ and positive integers s, Jouhet and Mosaki define

$$\beta_q(s) = \sum_{n\geq 1} \frac{n^{s-1} q^n}{1 + q^{2n}}.$$

At $q = \varphi^{-2}$ the three sums are precisely $\beta_q(1), \beta_q(2), \beta_q(3)$. The following theorem treats the middle weight.

Theorem 1. *Let $b > 1$ be a real algebraic integer. If every conjugate of b other than b has modulus at most one, then*

$$\beta_{1/b}(2) = \sum_{n\geq 1} \frac{n}{b^n + b^{-n}} \notin \mathbb{Q}(b).$$

Corollary 2. *For every integer $a \geq 1$, the sum $\sum_{n\geq 1} n/L_{2an}$ lies outside $\mathbb{Q}(\sqrt{5})$.*

Proof. Take $b = \varphi^{2a}$. This is an algebraic integer with conjugate $\varphi^{-2a} < 1$. These two values are distinct, so b generates $\mathbb{Q}(\sqrt{5})$. Now $b^n + b^{-n} = L_{2an}$, and Theorem 1 applies. □

The base condition includes Pisot and Salem numbers. Bundschuh and Zudilin singled out the arithmetic of this q-analogue of Catalan's constant at algebraic q, writing it as the alternating series in Section 2 [3, §6, p. 121]. For an integer $b \geq 2$, Jouhet and Mosaki proved that at least one of the ten values $\beta_{1/b}(2), \beta_{1/b}(4), \ldots, \beta_{1/b}(20)$ is irrational [2, Corollary 1.3]. The integer-base theorem of uplift-universal isolates $\beta_{1/b}(2)$ [4, Theorem 1.1]. For the Lucas denominators, Wang also relates the sums with even numerator powers to derivatives of Eisenstein series [5, Lemma 5.1 and Corollary 5.2 of the preprint].

The Lucas example illustrates the obstacle at the other embeddings. Every partial sum of $\sum n/L_{2n}$ is rational. Yet if its limit belonged to $\mathbb{Q}(\sqrt{5})$, the partial sums would give no formula for its conjugate Y. Indeed, conjugation sends φ^{-2m}, which tends to zero, to φ^{2m}, which grows. We construct integer polynomials that become small at $(b, \beta_{1/b}(2))$. At every other conjugate b' we bound the same polynomials at (b', Y) for every fixed Y, even when $|b'| = 1$. The product over the embeddings then gives the contradiction to the lower bound for a nonzero integer norm.

We use the signed determinant and the cyclotomic divisibility of the integer-base paper [4, §§2–7]. To compare conjugate specializations, we use the base as a polynomial variable and bound the normalized determinant throughout the closed unit disk. This follows the conjugate estimate for a weight-three Lambert sum [6, §§4 and 6]. Quadratic norms and conjugate determinant bounds also appear in Krisman's proof for reciprocal Fibonacci cubes [7, §5]. For an earlier use of estimates over all places for Lambert series at algebraic bases, see Bundschuh and Väänänen [8, §2]. The proof below uses partial fractions, elementary facts about algebraic integers and the maximum modulus principle.

2 One determinant for all embeddings

Fix a real $b > 1$ and set $q = 1/b$ and $\xi = \beta_q(2)$. The sum converges since its n-th term is at most nq^n. To obtain an integer polynomial in the unknown value, we need several expressions involving only that value and rational functions of the base. Expand the denominator in the positive series for $\beta_q(2)$. With $\chi(2j) = 0$ and $\chi(2j+1) = (-1)^j$, this gives

$$\frac{q^j}{1+q^{2j}} = q^j - q^{3j} + q^{5j} - \cdots = \sum_{m \geq 1} \chi(m) q^{mj} \quad (j \geq 1).$$

Write $g(y) = y/(1-y)^2$ for $y \neq 1$. When $|y| < 1$, its power series is $g(y) = \sum_{j \geq 1} j y^j$. Summing gives the identity $\xi = \sum_{k \geq 0} (-1)^k g(q^{2k+1})$ of Jouhet and Mosaki [2, §6.2]. The interchange is absolute since $\sum_{m \geq 1} q^m/(1-q^m)^2 < \infty$. At the Lucas parameter $q = \varphi^{-2}$, Binet's formula gives $g(q^m) = (\varphi^m - \varphi^{-m})^{-2} = 1/L_m^2$ for odd m. This identifies the two Lucas expansions

$$\sum_{n \geq 1} \frac{n}{L_{2n}} = \sum_{k \geq 0} \frac{(-1)^k}{L_{2k+1}^2} = 1 - \frac{1}{4^2} + \frac{1}{11^2} - \frac{1}{29^2} + \cdots . \tag{1}$$

For general q, the odd exponents give the nodes q^m, and applying $x\, d/dx$ to x^j supplies the numerator j. Define, for $f \in C^1[0, q]$,

$$L(f) = \sum_{m \geq 1} \chi(m) q^m f'(q^m).$$

The sum is bounded in modulus by $q\|f'\|_\infty/(1-q^2)$.

Removing the first odd node changes the sign of the remaining series. For example, $L((1-q^2x)^{-1}) = g(q) - \xi$. Removing the first a odd nodes gives the same value with a finite correction and sign $(-1)^a$. For a variable base t, record the finite corrections and polynomial moments by

$$S_N(t) = \sum_{m=1}^{N} \chi(m) \frac{t^m}{(t^m-1)^2}, \qquad \mu_0(t) = 0, \qquad \mu_j(t) = \frac{jt^j}{1+t^{2j}} \quad (j \geq 1). \tag{2}$$

Indeed $g(q^m) = g(b^m)$. Summing the geometric progression on odd indices gives $L(x^j) = \mu_j(b)$, and the shift just described gives

$$L((1-x/b^{2a})^{-1}) = (-1)^a(\xi - S_{2a}(b)).$$

For size $K \geq 1$ write $r_a = t^{2a}$ and $D_K(x) = \prod_{a=1}^{K}(1-x/r_a)$. On rational functions of x with denominator dividing D_K, define the linear functional $\mathcal{L}_{t,X}$ over $\mathbb{Q}(t,X)$ by

$$\mathcal{L}_{t,X}(x^j) = \mu_j(t), \qquad \mathcal{L}_{t,X}\big((1-x/r_a)^{-1}\big) = (-1)^a(X - S_{2a}(t)).$$

These values specify the functional by polynomial division and simple partial fractions. At $(t,X) = (b,\xi)$ it agrees with L. The determinant from [4, §2] is

$$\Delta_K(t,X) = \det\left[\mathcal{L}_{t,X}\left(\frac{x^{i+j}}{D_K(x)}\right)\right]_{0\leq i,j<K}. \tag{3}$$

Each entry is affine in X, so the degree in X is at most K. At $K=1$ the determinant is simply $t/(t-1)^2 - X$. At $K=3$ the bottom right entry involves $x^4/D_3(x)$, whose polynomial quotient is linear. The moments in (2) include its contribution. At this size the two computations, by polynomial division and by the pole recurrence, can be compared exactly at both Lucas embeddings. From the root of the source bundle, run `python3` checks/check_three_poles.py. Rational modifications of moments occur in Uvarov's transformation of orthogonal polynomials [9, §1] and in Krattenthaler's determinant formula [10, Theorem 1]. Here the denominator gains a new pole with each increase in K.

3 An integer polynomial in the base

To compare the evaluations at all conjugates of b, we clear denominators before specializing t. Let $C_K(t) = \prod_{d\leq 4K} \Phi_d(t)^{E(K,d)}$, where Φ_d is the d-th cyclotomic polynomial and Appendix A lists the exponents. The arithmetic input from [4, (4.1), Proposition 4.1 and the proof of Proposition 5.1] is

$$C_K\Delta_K \in \mathbb{Z}[t,t^{-1}][X], \tag{4}$$

$$\deg C_K = (C+o(1))K^3, \quad C = \frac{1}{6} + \frac{1309}{150\pi^2}, \qquad \sum_{d\leq 4K} E(K,d) = O\Big(K^2\log(2K)\Big). \tag{5}$$

In units of $K^3 \log b$, Section 4 gives an upper exponent of $-1/3$ for $\Delta_K(b,\xi)$. Clearing adds $C \approx 1.05086$. Extracting a common power t^{c_K} saves $5/6$, giving the net exponent $-1/3 + C - 5/6 \approx -0.11580$. With c_K given by (8), we use the polynomial

$$P_K(t,X) = t^{-c_K} C_K(t)\Delta_K(t,X) \in \mathbb{Z}[t,X], \qquad c_K = \tfrac{5}{6}K^3 + O(K^2). \tag{6}$$

To verify integrality at $t=0$, observe that (4) makes each coefficient in X an integer Laurent polynomial. The base-power divisibility in [4, Proposition 3.1 and (5.2)] gives $P_K(p,X) \in \mathbb{Z}[X]$ for every positive rational prime p. If any coefficient had a lowest term at^{-h} with $h>0$, multiplying its value at any prime $p \nmid a$ by p^h would give an integer congruent to a mod p. Integrality of the value also makes that integer divisible by p, a contradiction. For $K=1$ the result reads $P_1(t,X) = t-(t-1)^2X$.

4 Decay and nonvanishing at the real base

Specialize $t = b,\ X = \xi$. The nodes on which L acts are $q, q^3, q^5, \ldots$. A first derivative vanishes at a double zero, so a polynomial can remove the early, largest nodes from the sum. For example, $(x-q)^2(x-q^3)^2$ removes the first two. At every later node its modulus is at most $q^{2+6} = q^8$. This is the repeated-root basis used in [4, §6]. The cubic gain from a geometric lattice has a fixed-moment antecedent in Zudilin's estimate [11, §4, (8)].

Take the monic polynomials

$$B_j(x) = \prod_{\ell=1}^{j} (x - q^{2\lceil \ell/2 \rceil - 1}), \qquad B_0 = 1.$$

Their root exponents sum to $A_j = \lceil j^2/2 \rceil$. Write $R_j = 2\lceil j/2 \rceil - 1$ for the largest root exponent for $j \geq 1$, and set $R_0 = 0$. Replacing monomials of degree j by B_j in both rows and columns of (3) preserves the determinant, as both basis changes are triangular with diagonal entries one. Put $W = 1/D_K$ at $t = b$. On $[0, q]$, W and W' are bounded uniformly in K. For W use $\prod_{a\geq 1}(1-q^{2a+1})^{-1}$, and for W'/W use $\sum_{a\geq 1} q^{2a}/(1-q^{2a+1})$.

Let $R = \max(R_i, R_j)$. At each odd $m < R$ the product B_iB_jW has a double zero at q^m. At odd $m \geq R$ each root factor satisfies $|q^m - q^h| \leq q^h$. In the Euler derivative $x\,d/dx$ a differentiated factor contributes q^m in place of q^h, gaining at least the factor q^{m-R} in the upper bound. A derivative on W contributes $O_q(q^m)$. Summing the resulting geometric bounds also covers $i = j = 0$, and gives

$$|L(B_iB_jW)| \ll_q (i+j+1)q^{A_i+A_j}.$$

Extract q^{A_i} from row i and column i. Since $2\sum_{i<K} A_i = \sum_{i<K} i^2 + \lfloor K/2 \rfloor$, Hadamard's inequality gives

$$\log|\Delta_K(b,\xi)| \leq -\tfrac{1}{3}K^3 \log b + O_b(K^2).$$

We take $\log 0 = -\infty$. For fixed $b > 1$, Möbius inversion gives $\log|\Phi_d(b)| = \phi(d)\log b + O_b(1)$, where ϕ is Euler's totient. The error is bounded uniformly in d by $\sum_{h\geq 1} |\log(1-b^{-h})|$. Equations (5) and (6) therefore give

$$\log|P_K(b,\xi)| \leq (-\delta + o(1))K^3 \log b, \qquad \delta = \tfrac{1}{3} + \tfrac{5}{6} - C = 1 - \frac{1309}{150\pi^2} > 0. \tag{7}$$

Even $\pi > 3$ gives $\delta > 41/1350$.

An upper bound allows the determinant to vanish. Since L has both signs, we get nonvanishing by interpolation. In the classical fixed-moment problem, eventual vanishing of Hankel determinants implies rationality of their generating function [12, Arts. VII and IX], [11, §4]. Here the denominator D_K changes with the size. We give the moving-pole argument of [4, Proposition 7.1] at the real base b.

Lemma 3. *There are arbitrarily large K for which $P_K(b,\xi) \neq 0$.*

Proof. The signed Cauchy transform

$$\mathcal{F}(z) = L((z-x)^{-1}) = \sum_{\substack{m\geq 1\\ m \text{ odd}}} \frac{\chi(m) q^m}{(z-q^m)^2}$$

is analytic at infinity and meromorphic on $\mathbb{C}\setminus\{0\}$. The series converges normally away from the nodes and their limit point. Every interior node $q, q^3, \ldots$ contributes a nonzero double pole, so $\mathcal{F}$ cannot be rational. If the determinants vanish eventually, the interpolation errors at successive sizes will share zeros at the exterior nodes $b^2, b^4, \ldots$, before any common factors are cancelled.

Suppose the determinants are singular at all sufficiently large sizes k. A kernel vector yields a nonzero polynomial Q_k, of degree less than k, with $L(x^j Q_k/D_k) = 0$ for $0 \leq j < k$. We use these k equations to build an approximation to $\mathcal{F}$ whose error vanishes at the exterior nodes shared with the next size. Set

$$E_k(z) = D_k(z) L\Big(\frac{Q_k(x)}{(z-x)D_k(x)}\Big), \qquad T_k(z) = \mathcal{F}(z)Q_k(z) - E_k(z).$$

The polynomial $Q_k(z)D_k(x) - D_k(z)Q_k(x)$ is divisible by $z-x$, so T_k is a polynomial of degree at most $k-1$. In the expansion of $1/(z-x)$ at infinity the first k terms give zero after applying L to their products with Q_k/D_k. The geometric series converges in $C^1[0,q]$ for $|z| > q$. Multiplication by the degree-k polynomial $D_k(z)$ thus leaves $E_k(z) = O_k(1/z)$. Also $E_k(b^{2a}) = 0$ for $1 \leq a \leq k$, because the inner transform is regular there.

Adjacent sizes share the first k exterior nodes. Consequently the polynomial

$$T_{k+1}Q_k - T_k Q_{k+1} = E_k Q_{k+1} - E_{k+1} Q_k$$

vanishes at these k nodes and has degree at most $k-1$ by its growth at infinity. It is zero. Thus all sufficiently large k give one reduced fraction $T_k/Q_k = T/Q$. Write $Q_k = V_k Q$ and $T_k = V_k T$, taking $Q = 1$ if $T = 0$. An exterior interpolation condition can disappear on reduction where the common factor V_k vanishes. The order of the error at infinity bounds how many such points there can be.

The function $E = \mathcal{F}Q - T$ is nonzero since $\mathcal{F}$ is nonrational. It is meromorphic at infinity and $V_k E = E_k = O_k(1/z)$. Hence E has a zero at infinity of some finite order $h \geq 1$, and $\deg V_k \leq h-1$. In particular $E(b^{2a}) \neq 0$ for all sufficiently large a. Each such $a \leq k$ gives a distinct zero of V_k, eventually exceeding its degree. This contradiction proves nonvanishing of $\Delta_K(b,\xi)$ along an unbounded sequence. The factors b^{-c_K} and $C_K(b)$ are nonzero. □

5 The conjugate estimate and the norm

At a conjugate base t with $|t| \le 1$, the second argument of P_K could be any fixed complex number. Bounding the factors of (6) separately can lose the cubic decay. For instance, at the Lucas conjugate $t = \varphi^{-2}$, the extracted factor $t^{-c_K} = \varphi^{2c_K}$ alone grows on the cubic scale. Near the unit circle there is also the problem that the denominators in (2) approach zero at roots of unity. Following [6, §4], we estimate on the circle of radius $1 + 1/K$. Here the denominators are separated from zero, and raising the radius to a power of order K^3 costs only $\exp(O(K^2))$. The maximum modulus principle then bounds the whole polynomial on the closed unit disk, including conjugates of modulus one.

Lemma 4. *For every fixed $Y \in \mathbb{C}$,*

$$\max_{|t|\le 1} \log^+ |P_K(t, Y)| = O_Y(K^{5/2} \log^2(2K)) = o(K^3), \qquad \log^+ v = \log \max(1, v).$$

Proof. Write $s = 1 + 1/K$ and let $|t| = s$. Then $1 < |r_a| \le e^2$. For $a \ne c$, $|1 - r_a/r_c| \ge 1 - s^{-1} = 1/(K+1)$. The partial-fraction coefficients of $1/D_K$ are $w_a = \prod_{c\ne a}(1 - r_a/r_c)^{-1}$, and $|w_a| \le (K+1)^{K-1}$. For positive m, j the reverse triangle inequality gives $|t^m - 1|,\ |1 + t^{2j}| \ge 1/K$. For $1 \le a \le K,\ 1 \le j \le 2K$, formula (2) therefore gives $S_{2a} = O(K^3)$ and $\mu_j = O(K^2)$.

To bound entries including their polynomial quotients, let $U_{a,k} = \mathcal{L}_{t,Y}(x^k/(1 - x/r_a))$. Polynomial division gives

$$U_{a,0} = (-1)^a(Y - S_{2a}), \qquad U_{a,k+1} = r_a(U_{a,k} - \mu_k).$$

Iteration for $0 \le k \le 2K - 2$ bounds $|U_{a,k}|$ by $e^{4K}(|Y| + O(K^3))$. Each entry in (3) is $\sum_a w_a U_{a,i+j}$, so determinant expansion gives $\log^+ |\Delta_K(t, Y)| = O_Y(K^2 \log(2K))$.

The factors $1 - t^{-h}$ in the Möbius product for $\Phi_d(t)$ have modulus between $1/(K+1)$ and 2. Thus

$$\log |\Phi_d(t)| \le \phi(d) \log s + \tau(d) \log(2K + 2),$$

where the divisor count satisfies $\tau(d) \le 2\sqrt{d} \le 4\sqrt{K}$ for $d \le 4K$. Summing with weights $E(K, d)$ and using (5) gives $\log^+ |C_K(t)| = O(K^{5/2} \log^2(2K))$. Finally $|t^{-c_K}| \le 1$ on this circle. The maximum modulus principle applied to the polynomial $P_K(\,\cdot\,, Y)$ proves the assertion. □

Proof of Theorem 1. Assume that $\xi = \beta_{1/b}(2)$ belongs to $F = \mathbb{Q}(b)$, and take a positive integer d such that $d\xi$ is an algebraic integer. Since b is an algebraic integer, $P_K \in \mathbb{Z}[t, X]$, and $\deg_X P_K \le K$, the number $y_K = d^K P_K(b, \xi)$ is an algebraic integer. By Lemma 3, $y_K \ne 0$ for arbitrarily large K. At those sizes the absolute norm is a positive integer.

Let ι be the chosen real embedding of F. The other embeddings $\sigma : F \hookrightarrow \mathbb{C}$ satisfy $|\sigma(b)| \le 1$. For each of them $Y = \sigma(\xi)$ is fixed as K grows, so Lemma 4 bounds its contribution to the logarithmic norm by $o(K^3)$. There are only $[F : \mathbb{Q}] - 1$ such embeddings. Using (7) at ι gives

$$\begin{aligned} 0 \le \log |N_{F/\mathbb{Q}}(y_K)| &= [F : \mathbb{Q}]\, K \log d + \log |P_K(b, \xi)| + \sum_{\sigma \ne \iota} \log |P_K(\sigma(b), \sigma(\xi))| \\ &\le -\delta(\log b) K^3 + o(K^3). \end{aligned}$$

Since $b > 1$ and $\delta > 0$, the right-hand side tends to minus infinity. □

A The cyclotomic factors and the power of the base

For clarity, here is the normalization behind (6). Write $v_+ = \max(v, 0)$. Define $f(0) = 0$ and $f(m) = \lfloor (m+2)^2/4 \rfloor$ for $m \geq 1$. In the first case below write $K = rd + s,\ 0 \leq s < d$. The exponents of [4, (4.1)] are

$$E(K,d) = \begin{cases} sf(r+1) + (d-s)f(r), & d \text{ odd},\ d \leq K, \\ 2(K - \lfloor (d-1)/2 \rfloor), & d \text{ odd},\ K < d \leq 2K, \\ 0, & d \text{ odd},\ d > 2K, \\ \displaystyle\sum_{j \geq 1,\ j \text{ odd}} (K - jh)_+, & d = 2h,\ h \text{ odd}, \\ \min\{K, (K-4u)_+ + (K-u-1)_+\}, & d = 4u. \end{cases}$$

The last line illustrates how the signs affect the arithmetic. At a primitive root of order $4u$, the signed proper-fraction residue has rank at most $(K-4u)_+$ because the interpolated numerator has a double zero at each cluster of coincident nodes. The polynomial quotient adds rank at most $(K-u-1)_+$. These are the two terms in the pole-order bound of [4, Proposition 4.1].

For a direct calculation of the power of t, let $\nu = \mathrm{ord}_{t=0}$. In the formal series ring set $Z_0 = \sum_{m \geq 1} \chi(m) g(t^m)$ and $H(x) = \sum_{j \geq 1} \mu_j(t) x^j$. Expanding g gives $H(x) = \sum_{m \geq 1} \chi(m) g(t^m x)$. The same shift as in Section 2 gives $S_{2a}(t) = Z_0 - (-1)^a H(r_a)$. With $U = X - Z_0$, the pole value of $\mathcal{L}_{t,X}$ is $(-1)^a U + H(r_a)$.

Put $F_K(x) = \prod_a (x - r_a)$. In $\mathcal{L}_{t,X}(x^k / F_K(x))$ the coefficient of U equals

$$-\sum_{a=1}^{K} \frac{(-1)^a r_a^{k-1}}{F_K'(r_a)}.$$

The constant part belongs to $\mathbb{Z}[[t]]$. Indeed the quotient by the monic F_K has integral coefficients and moments. The proper part contributes $-\sum_{j \geq 1} \mu_j(t) h_{k+j-K}(r_1, \ldots, r_K)$, where h_ℓ is the complete homogeneous symmetric polynomial, $h_0 = 1$, and $h_\ell = 0$ for $\ell < 0$. To see this, expand $1/F_K$ at infinity and in simple fractions. Comparing coefficients gives $\sum_a r_a^\ell / F_K'(r_a) = h_{\ell - K + 1}$ for $\ell \geq 0$. Here $\ell = k + j - 1 \geq 0$, and $\mu_j(t) \in t^j \mathbb{Z}[[t]]$ ensures convergence.

The equality $\nu(r_a - r_h) = 2\min(a, h)$ for $a \neq h$ now gives

$$\nu\left(\frac{r_a^{k-1}}{F_K'(r_a)} \right) = a^2 + (2k - 2K - 1)a \geq -w(k), \qquad w(k) = (K-k)_+ \big((K-k)_+ + 1 \big).$$

The inequality follows by minimizing the quadratic over integers $1 \leq a \leq K$. Translating U back to X preserves the bound. For a determinant term indexed by a permutation σ, convexity of w gives $2w(i + \sigma(i)) \leq w(2i) + w(2\sigma(i))$. Thus $\sum_i w(i + \sigma(i)) \leq \sum_i w(2i)$. Restoring the factor $(-1)^K t^{K(K+1)}$ in $1/D_K = (-1)^K t^{K(K+1)} / F_K$ in every row gives the coefficientwise bound $\nu(\Delta_K) \geq c_K$ with

$$c_K = K^2(K+1) - \sum_{i=0}^{K-1} w(2i) = \frac{10K^3 + 3K^2 - 10K - 3(K \bmod 2)}{12}. \tag{8}$$

The last equality follows by summing separately for even and odd K.

Verification

Two collaborating AI models reviewed the mathematics and exposition of the initial 18 November 2026 draft and reproduced its three-pole calculation, taking the cited cyclotomic estimates as inputs. A separate AI model audited that draft for exposition, and a collaborating model checked the algebraic-base extension in this revision.

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