A height criterion for irrational weight-three Lambert sums

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Abstract

For an algebraic number b with $|b| > 1$, the weight-three Lambert sum is $Z(b) = \sum_{n\geq 1} n^2/(b^n - 1)$. We prove that $Z(b)$ lies outside $\mathbb{Q}(b)$ whenever the Mahler measure $M(b)$ satisfies $\log M(b) < (352\pi^2/2889)\, r \log |b|$, where $r = 1$ for a real base and $r = 2$ otherwise. The criterion allows several conjugates outside the unit disk and applies to nonintegral bases. In particular, $Z(3+\sqrt{3}) \notin \mathbb{Q}(\sqrt{3})$, although both conjugates of the base exceed one.

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1 Two expanding conjugates

For $|b| > 1$, the weight-three Lambert sum

$$Z(b) = \sum_{n=1}^{\infty} \frac{n^2}{b^n - 1}$$

converges absolutely. With $q = 1/b$, this is a q-analogue of $\zeta(3)$. It is irrational at every integer base $b \geq 2$ [3, Theorem 1.1]. For an algebraic base b, every partial sum lies in $\mathbb{Q}(b)$. Can we show that the limit lies outside that field?

At $b = 2 + \sqrt{3}$ the answer is yes. The other conjugate, $2 - \sqrt{3}$, lies inside the unit disk, so the earlier algebraic-base theorem gives $Z(2 + \sqrt{3}) \notin \mathbb{Q}(\sqrt{3})$ [2, Theorem 1.2]. Now increase the base by one. The field stays $\mathbb{Q}(\sqrt{3})$ and the other conjugate crosses the unit circle to $3 - \sqrt{3} > 1$. We prove that $Z(3 + \sqrt{3})$ still lies outside $\mathbb{Q}(\sqrt{3})$ by quantifying how large the other conjugates can be.

For a specified algebraic base b, write its primitive integer minimal polynomial as $a_0 \prod_{j=1}^{d}(x - b_j)$, with $a_0 > 0$. Its *Mahler measure* is

$$M(b) = a_0 \prod_{j=1}^{d} \max(1, |b_j|).$$

The product counts each conjugate separately. In particular, a nonreal base and its complex conjugate contribute two factors of $|b|$.

Theorem 1.1. *Let b be algebraic with $|b| > 1$. Put $r = 1$ for real b and $r = 2$ for nonreal b. If*

$$\frac{\log M(b)}{r \log |b|} < \frac{352\pi^2}{2889}, \tag{1.1}$$

then $Z(b) \notin \mathbb{Q}(b)$.

Both roots $3 \pm \sqrt{3}$ have Mahler measure 6. To five decimal places the ratio in (1.1) is 1.15273 at $3 + \sqrt{3}$ and 7.54740 at $3 - \sqrt{3}$, against a cutoff of 1.20253. When the larger root is the base, the smaller root contributes about 15.27% as much logarithmic growth as the larger one, within the roughly 20.25% allowed by the criterion.

Corollary 1.2. *At $b = 3 + \sqrt{3}$, the real number $Z(b)$ lies outside $\mathbb{Q}(\sqrt{3})$.*

Proof. The roots of $x^2 - 6x + 6$ are b and $b' = 3 - \sqrt{3}$, so $M(b) = bb' = 6$. Since $\sqrt{3} > 12/7$, we have $b > 33/7$ and $b' < 9/7$. The inequality $9^6 < 33 \cdot 7^5$ gives $(b')^6 < b$. Thus $\log M(b) < (7/6) \log b$, and $7/6 < 352\pi^2/2889$ follows from $\pi > 31/10$. □

The same $7/6$ bound gives a family of nonintegral examples. Let b be the larger root of $2x^2 - mx + 1$ for an integer $m \geq 129$. Its discriminant $m^2 - 8$ lies strictly between $(m-1)^2$ and m^2. The lower bound on m makes the polynomial negative at $2^6 = 64$, so $2 < b^{1/6}$. The other root is $1/(2b) < 1$, and hence $M(b) = 2b < b^{7/6}$. Theorem 1.1 gives $Z(b) \notin \mathbb{Q}(b)$ in this case as well.

In general, $\log M(b) - r \log |b|$ adds together $\log a_0$ and the positive logarithms of the moduli of the remaining conjugates. Both contributions are controlled by the degree in the base of certain integer polynomials. That degree bounds their growth at the other expanding conjugates and the power of a_0 in the integer resultant of Section 3. Section 4 proves that each polynomial has the same degree in the base as its cyclotomic clearing factor and identifies its leading coefficient.

Bundschuh and Väänänen used the height ratio in (1.1) for Lambert sums with periodic numerators [1, §2, (2)]. For $b = v/u$ with $u < v$ coprime positive integers, $M(b) = v$, and (1.1) becomes $\log u / \log v < 1 - 2889/(352\pi^2)$. This is the weight-three rational-parameter range recorded in [3, §7.1] and [4, entry FUT-RATIONAL], where the degree bound is used for homogenization. The stated restriction $u/v \leq 1/2$ follows here from $v > u^2$ and $v \geq 2$.

2 The moments and their determinant

We recall the polynomials whose values at conjugate bases will be multiplied. Their two arguments have different roles. The first is the base t. The second is a variable X that can be set equal to the Lambert sum at one embedding and left arbitrary at another.

First take $t > 1$ real. The Euler derivative $\theta = x\, d/dx$ sends x^j to jx^j, so applying it twice supplies the weight j^2. The denominator comes from summing on a geometric sequence. Put

$$L_t(f) = \sum_{m \geq 1} (\theta^2 f)(t^{-m}), \qquad L_t(x^j) = j^2 \sum_{m \geq 1} t^{-mj} = \frac{j^2}{t^j - 1} \quad (j \geq 1).$$

Also $L_t(1) = 0$. For f holomorphic near $[0, t^{-1}]$ the sum converges since $\theta^2 f(x) = xf'(x) + x^2 f''(x) = O_f(x)$ near zero.

To see what happens at a pole, write $G(y) = \sum_{j \geq 1} j^2 y^j = y(1+y)/(1-y)^3$ for $|y| < 1$. Summing the geometric expansions of the Lambert series gives $Z(t) = \sum_{m \geq 1} G(t^{-m})$ by absolute convergence. For the first pole $f(x) = (1 - x/t)^{-1}$, differentiating its power series gives $\theta^2 f(x) = G(x/t)$. The mth summand of $L_t(f)$ is therefore $G(t^{-(m+1)})$, skipping the first term $G(t^{-1})$. The pole at t^a skips the first a terms in the same way, giving

$$L_t((1 - x/t^a)^{-1}) = Z(t) - S_a(t), \qquad S_a(t) = \sum_{m=1}^{a} G(t^{-m}).$$

Now treat t as an indeterminate. For $K \geq 1$ put $r_a = t^a$ for $1 \leq a \leq K$ and $D(x) = \prod_{a=1}^{K} (1 - x/r_a)$. Division and partial fractions express $p(x)/D(x)$ over $\mathbb{Q}(t)$ as a polynomial plus a linear combination of $(1 - x/r_a)^{-1}$. Define $\mathcal{L}_{t,X}$ by giving the polynomial part the monomial values above and each simple fraction the value $X - S_a(t)$. For real $t > 1$, setting $X = Z(t)$ recovers L_t. These moments form the determinant

$$\Delta_{K,t}(X) = \det \left[\mathcal{L}_{t,X} \left(\frac{x^{i+j}}{D(x)} \right) \right]_{0 \leq i,j < K}.$$

Its entries are affine in X. The coefficients can have poles where two nodes coincide or $t^j = 1$. Let $C_K(t)$ be the monic product of cyclotomic polynomials specified in [2, (2.3)]. It clears the poles away from $t = 0$. Set

$$e_K = \frac{K^3 - K}{2}, \qquad d_K = \deg C_K, \qquad P_K(t, X) = C_K(t) t^{-e_K} \Delta_{K,t}(X).$$

After division by t^{e_K}, the result is still an integer polynomial. The following four properties are proved in [2, Lemmas 2.1 and 4.1, Propositions 3.1 and 5.1]. We put $\log^+ x = \log \max(1, x)$ for $x \geq 0$.

Proposition 2.1. *Put $\gamma = 321/(64\pi^2)$ and $\varepsilon = 11/18 - \gamma > 0$. The following hold.*

1. *$P_K \in \mathbb{Z}[t, X]$, $\deg_X P_K \leq K$, and $d_K \leq \gamma K^3 + O(K^2 \log(2K))$.*
2. *For fixed $|b| > 1$, $\log |P_K(b, Z(b))| \leq -\varepsilon K^3 \log |b| + O_b(K^2 \log(2K))$, with $\log 0 = -\infty$.*
3. *For each fixed $X_0 \in \mathbb{C}$, $\max_{|t| \leq 1} \log^+ |P_K(t, X_0)| = O_{X_0}(K^2 \log^3(2K))$.*
4. *For each fixed $|b| > 1$ there are arbitrarily large K for which $P_K(b, Z(b)) \neq 0$.*

The construction and the cyclotomic exponents come from the integer-base weight-three determinant of uplift-universal-inc [3, §§1–5 and Appendix A]. The cubic scale of decay for Hankel determinants on geometric nodes already occurs in Zudilin's work on the generalized q-logarithm [5, §4, (8)]. The estimates in Proposition 2.1 use the complex-base form of the weight-three determinant, including its bound on the closed disk for every fixed second argument.

3 An integer from all the conjugates

Take $b = 3 + \sqrt{3}$ and $b' = 3 - \sqrt{3}$. If $Z(b) = u + v\sqrt{3}$ for rational u, v, its field conjugate would be $Y = u - v\sqrt{3}$. The two factors in the norm of $P_K(b, Z(b))$ would then be $P_K(b, Z(b))$ and $P_K(b', Y)$. The series converges at both roots, yet conjugation can change the behavior of a limit. For example, it sends the sequence $(2 - \sqrt{3})^n \to 0$ to $(2 + \sqrt{3})^n \to \infty$. To cover the unknown Y, we need a bound at b' for every fixed second argument. Proposition 2.1(2) gives decay for the first factor. For the second, the degree in the *base* controls growth from the unit circle out to b'.

The degree bound $\deg_t P_K \leq d_K$ from Lemma 4.1 suffices. For fixed Y, the reversed polynomial $u^{d_K} P_K(1/u, Y)$ is then holomorphic at the origin. Apply the maximum modulus principle to it at $u = 1/t$ for $|t| > 1$. Its modulus on the unit circle is bounded by Proposition 2.1(3). Together with that same estimate inside the disk this gives, for every $t \in \mathbb{C}$,

$$\log^+ |P_K(t, Y)| \leq d_K \log^+ |t| + O_Y(K^2 \log^3(2K)). \tag{3.1}$$

Thus the logarithmic cost at b' is at most $d_K \log b' + o(K^3)$. For a general base, the same degree controls the denominators.

Proof of Theorem 1.1, assuming Lemma 4.1. Suppose $\xi = Z(b)$ belongs to $\mathbb{Q}(b)$. With $d = [\mathbb{Q}(b) : \mathbb{Q}]$, we may write $\xi = g(b)/h$ for a fixed $g \in \mathbb{Z}[t]$ of degree at most $d - 1$ and a positive integer h. This representation specifies the conjugate value $g(b_j)/h$ at *every* root b_j of the minimal polynomial f. Since $\deg_X P_K \leq K$,

$$Q_K(t) = h^K P_K(t, g(t)/h) \in \mathbb{Z}[t], \qquad \deg Q_K \leq d_K + (d - 1)K.$$

Choose the unbounded sequence on which $Q_K(b) = h^K P_K(b, \xi) \neq 0$. The irreducible f then has no common root with Q_K. Their resultant is a nonzero integer, as it is the determinant of their integer Sylvester matrix. Its product expression is

$$R_K = \operatorname{Res}(f, Q_K) = a_0^{\deg Q_K} h^{dK} \prod_{j=1}^{d} P_K(b_j, g(b_j)/h). \tag{3.2}$$

Thus a_0, which may exceed one for a nonintegral base, is raised only to the degree of Q_K. The fixed denominator h contributes only $O_{b,\xi}(K)$ to $\log |R_K|$.

At b apply Proposition 2.1(2). If b is nonreal, $g(\bar{b})/h = \bar{\xi} = Z(\bar{b})$, giving a second factor with the same decay. At each of the other $d - r$ roots use (3.1) with the fixed value $Y = g(b_j)/h$. In (3.2) the leading coefficient a_0 and these remaining roots together contribute at most $d_K(\log M(b) - r \log |b|) + O_{b,\xi}(K^2 \log^3(2K))$ to the logarithm. Consequently

$$\begin{aligned} 0 \leq \log |R_K| &\leq -r\varepsilon K^3 \log |b| + d_K\big(\log M(b) - r \log |b|\big) + O_{b,\xi}(K^2 \log^3(2K)) \\ &\leq \Big(\gamma \log M(b) - \tfrac{11}{18} r \log |b|\Big) K^3 + o(K^3). \end{aligned}$$

Here $\log M(b) - r \log |b| \geq 0$, so the upper bound for d_K applies. Since $(11/18)/\gamma = 352\pi^2/2889$, (1.1) makes the last coefficient strictly negative. □

4 The degree in the base

Lemma 4.1. *For every $K \geq 1$, the degree of $P_K(t, X)$ in t is d_K, and its coefficient of t^{d_K} is $(-1)^{K(K-1)/2} X^K$.*

The remaining task is to see why the determinant contributes no additional power of t after division by t^{e_K}. We let t tend to $+\infty$ at fixed K. The nodes $r_a = t^a$ separate, and the polynomials that interpolate at them make the dependence on X diagonal.

Proof. Take real $t \geq 2$ and write $L = L_t = \mathcal{L}_{t,Z(t)}$ as in Section 2. Changing $Z(t)$ to Y adds $Y - Z(t)$ at each simple fraction and leaves the polynomial moments fixed. The coefficient of the simple fraction at r_a in p/D is $w_a p(r_a)$, where

$$w_a = \prod_{c \neq a} (1 - r_a/r_c)^{-1}.$$

This follows by multiplying by $1 - x/r_a$ and evaluating at r_a. Thus the change in the functional on p/D is $(Y - Z(t)) \sum_a w_a p(r_a)$, even when division gives a polynomial quotient.

Choose the degree-$(K-1)$ cardinal polynomials $\ell_a(x) = w_a D(x)/(1 - x/r_a)$, characterized by $\ell_a(r_c) = \delta_{ac}$. In the matrix with entries $\mathcal{L}_{t,Y}(\ell_a \ell_c / D)$ the preceding change contributes only to the diagonal. The full matrix is

$$(Y - Z(t))W + WHW, \qquad W = \operatorname{diag}(w_1, \ldots, w_K), \qquad H_{ac} = L\left(\frac{D(x)}{(1 - x/r_a)(1 - x/r_c)}\right).$$

This also covers $a = c$, when the fraction retains a simple pole. The evaluation matrix from monomials to node values is $V = (r_a^i)_{1 \leq a \leq K, 0 \leq i < K}$. The coefficients of the ℓ_a are the columns of V^{-1}, so returning to monomials multiplies the determinant by $(\det V)^2$. For each $a < c$ the two factors $1 - r_a/r_c$ and $1 - r_c/r_a$ have product $-(r_c - r_a)^2/(r_a r_c)$. Hence

$$(\det V)^2 \prod_a w_a = (-1)^{\binom{K}{2}} \left(\prod_a r_a\right)^{K-1} = (-1)^{\binom{K}{2}} t^{e_K}.$$

Factoring W out of the matrix and multiplying by $C_K(t)$ gives

$$P_K(t,Y) = (-1)^{\binom{K}{2}} C_K(t) \det(YI + E_t), \qquad E_t = HW - Z(t)I. \tag{4.1}$$

For $t \geq 2$ the rational function defining H_{ac} has modulus at most $4\prod_{j\geq 1}(1+2^{-j})$ on $|x| \leq 1$. Cauchy's estimates bound its first two derivatives on $|x| \leq 1/2$. Since $\theta^2 f = xf' + x^2 f''$, its second Euler derivative is $O(|x|)$ there. Summing at t^{-m} yields $H_{ac} = O(1/(t-1))$. For fixed K the weights w_a stay bounded and $Z(t) \to 0$, so $E_t \to 0$ entrywise. The determinant $\det(YI + E_t)$ is monic in Y. Each lower coefficient is a sum of principal minors of E_t of positive order and hence tends to zero. By (4.1) the coefficient of Y^K in $P_K(t,Y)$ is exactly $(-1)^{\binom{K}{2}} C_K(t)$. Every lower coefficient is $o(t^{d_K})$ and is a polynomial in t by Proposition 2.1(1), so it has degree less than d_K whenever it is nonzero. Since C_K is monic of degree d_K, the coefficient of t^{d_K} in $P_K(t,Y)$ is $(-1)^{\binom{K}{2}} Y^K$. □

Verification

A collaborating AI model reviewed the mathematics and exposition of the initial 18 November 2026 draft, taking Proposition 2.1 as a cited input. A separate AI model audited that draft for exposition.

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