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Numerical evidence for the almost Hadamard conjecture

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Abstract

We report on encouraging numerical evidence for the almost Hadamard conjecture, stating that any local maximizer of the 1-norm on U_N must have all entries of modulus $1/\sqrt{N}$, or equivalently that every complex almost Hadamard matrix $H = \sqrt{N}\,U$ must be a complex Hadamard matrix, with computations at the key values $N = 4, 5$ arising from our attempts to both prove and disprove the conjecture, and adding to the previous abstract evidence.

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Introduction

A complex Hadamard matrix is a square matrix whose entries have modulus 1 and whose rows are orthogonal:

$$|H_{ij}| = 1, \qquad HH^* = NI$$

Its rescaling $U = H/\sqrt{N}$ is a flat unitary, meaning that all its entries have modulus $1/\sqrt{N}$. These matrices attain the upper bound

$$L(U) = \|U\| = \sum_{ij} |U_{ij}| \leq N\sqrt{N}$$

The following conjecture was made in [5].

Conjecture (Almost Hadamard, AHC)**.** *Every local maximizer of the* 1*-norm on* U_N *attains the bound* $N\sqrt{N}$*.*

The starting point was the real theory [3, 7]. A real almost Hadamard matrix is a rescaled local maximizer on O_N; such matrices exist in every order and need not be Hadamard. The passage to U_N allows new directions of variation. These exclude several real families coming from designs and circulants [5], as recalled in Section 1.

In order to put this conjecture in context and illustrate its difficulty, we discuss 2 other conjectures concerning unitary groups.

Conjecture (Marcus–de Oliveira)**.** *For diagonal matrices* $A = diag(a_1, \dots, a_N)$ *and* $B = diag(b_1, \dots, b_N)$*, the inclusion*

$$\det(A + UBU^*) \in conv\left\{\prod_{i=1}^{N}(a_i + b_{\sigma(i)}) : \sigma \in S_N\right\}$$

holds for every $U \in U_N$*.*

This concerns sums of normal matrices with prescribed spectra [17, 19]: unitary diagonalization gives the displayed form. The points on the right pair the eigenvalues in every possible order. The conjecture confines the determinant to their convex hull in the complex plane. It is known through $N = 3$ [9]; $N = 4$ is the first unsettled order [11, 18].

Higher-dimensional results treat self-adjoint pairs [16], essentially self-adjoint pairs [8], scalar multiples of unitaries [14], and certain normal dilations [23]. A variational approach maximizes the projection of the determinant onto a line and studies the resulting stationarity equations on U_N [20, 11]. Its successes and obstructions [18] provide a useful comparison with the AHC.

The second conjecture concerns squared entry moduli. For a unitary U, put

$$p(U) = (|U_{ij}|^2), \qquad J_N = (1/N)_{ij}$$

The matrix $p(U)$ is bistochastic: its entries are ≥ 0 and its row and column sums are 1. Bistochastic matrices arising this way are called unistochastic.

Conjecture (Unistochastic star-shapedness)**.** *For every unistochastic matrix* M*, the segment*

$$M_t = (1 - t)M + tJ_N, \qquad 0 \leq t \leq 1$$

consists of unistochastic matrices.

This holds at $N = 3$ [12]. Rajchel, Gąsiorowski and Życzkowski proposed the order-4 conjecture in [21]; the higher-dimensional version is discussed in [22]. Partial results include rays supplied by robust Hadamard matrices [21] and the circulant subset at order 4 [22]. In every order, the larger set defined by the necessary row and column polygon inequalities, called bracelet conditions, is star-shaped [22]. This does not settle the unistochastic case.

The connection to the AHC is the strict increase

$$h(M_t) > h(M) \quad (M \neq J_N,\ t > 0), \qquad h(M) = \sum_{ij}\sqrt{M_{ij}}, \qquad L = h \circ p$$

To contradict a local maximum at a particular U, however, one needs unitaries realizing M_t that approach that same U, up to phases. Section 2.4 supplies this lifting argument at order 3. Reconstruction from moduli [2], the defect [25], and the critical loci of the modulus map [10] describe the higher-order difficulty. Random unistochastic ensembles [27], volume calculations [15], manifold optimization [1], and calibrated searches for mutually unbiased bases [13] give further context.

Our principal structural results concern U_4.

Theorem A. *If one* 2×2 *submatrix of a local maximum on* U_4 *can be made real by row and column phases, the whole unitary is flat.*

The proof uses a block variation with positive second derivative. When no 2×2 block can be made real by phases, the critical-point equations instead impose restrictions on the entry moduli.

Theorem B. *A local maximum on U_4 is flat if every row and every column contains the same multiset of entry moduli.*

In the remaining case, the reciprocal moduli satisfy a rank-2 condition and explicit polynomial identities. Combining these restrictions with a count of the possible modulus patterns gives another criterion.

Theorem C. *A nonflat local maximum on U_4 would have at least 5 distinct entry moduli.*

The last proof uses an exact enumeration of 27 patterns. All 3 proofs are in Section 3, with the remaining curvature equations and numerical evidence in Section 4.

At order 5, our explicit saddle has

$$\text{Hessian inertia} = (15, 9, 1), \qquad \text{all row and column pair curvatures} < 0$$

The inertia lists negative, zero and positive eigenvalues. A 3-column variation gives an increase, and both unstable gradient branches converge to Fourier matrices (Section 5). Thus pair tests alone can miss a saddle.

The original campaigns and the subsequent targeted search have the following sizes, each from a 7-hour run:

Order	Campaign	Ascent starts	Stationary-search starts
4	Original	83 076 896	52 237 424
5	Original	181 164 464	22 933 376
5	Targeted	8 737 840	26 213 552

At order 4 every ascent return is within $6.014 \cdot 10^{-6}$ of flat in entry moduli. Every accepted nonflat stationary return has positive curvature; the near-zero and unresolved returns lie below a necessary entry bound. At order 5 every ascent return is near flat, while the original stationary campaign left 404 581 unresolved returns. Among its saved returns, 452 nonflat stationary matrices form 40 numerical groups and have interval exclusions. Each group representative has a nearby exact critical point with certified Hessian inertia. Their norm enclosures give at least 35 distinct nonflat critical values. All 288 saved stalls are explained. One explanation is a certified positive local minimum of the squared stationarity residual: minimizing that residual need not find a critical point of L.

The targeted campaign leaves no unresolved or candidate flags under the endpoint criteria. A separate replay of the first 16 189 008 original starts, about 70.6% of that sequence, also leaves no such flags after follow-up. Its remaining 6 744 368 starts were not revisited, and the replay is not added to the table. Small-entry exclusions concern the computed endpoints, not their possible limits; matching the known invariant signatures does not prove equivalence.

These are sampled searches, not coverings of U_4 or U_5. Sections 2 and 6 separate the variational criteria from the numerical classifications and certificates. From `code011`, with NumPy, SciPy, mpmath and SymPy, `python3 reproduce.py` rechecks the identities, certificates and retained outputs in a temporary copy. Its report `data/reproduction.json` records versions, hashes and outputs for all 27 checks. `README.md` indexes the code, and every program is cited below.

1 Hadamard matrices and the conjecture

1.1 Hadamard matrices and norm maximization

The symbols $*$ and t denote conjugate transpose and transpose.

Definition 1.1. *A complex Hadamard matrix of order N is a matrix H satisfying*

$$|H_{ij}| = 1, \qquad HH^* = NI$$

It is real Hadamard when its entries belong to $\{1, -1\}$. A unitary U is flat when $|U_{ij}| = 1/\sqrt{N}$ for every entry.

The Fourier matrices provide complex examples in every order:

$$F_N = (w^{ij})_{0 \leq i,j < N}, \qquad w = e^{2\pi i/N}$$

For instance, with $w = e^{2\pi i/3}$ in the second matrix,

$$F_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad F_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & w & w^2 \\ 1 & w^2 & w \end{pmatrix}$$

Their row orthogonality follows from the geometric-series identity. For real Hadamard matrices, $N \geq 3$ requires 4 to divide N. See [24] for background.

Proposition 1.2. *The entrywise 1-norm satisfies*

$$L(U) = \|U\| = \sum_{ij} |U_{ij}| \leq N\sqrt{N} \qquad (U \in U_N)$$

Equality holds exactly when U is flat, or equivalently when $H = \sqrt{N}\,U$ is complex Hadamard.

Proof. Cauchy–Schwarz and unitarity give

$$\sum_{ij} |U_{ij}| \leq N\Big(\sum_{ij} |U_{ij}|^2\Big)^{1/2} = N\sqrt{N}$$

Equality means that all entry moduli are equal. Their squares sum to N, so their common value is $1/\sqrt{N}$. □

1.2 Almost Hadamard matrices

The real theory replaces the equality case above by a local maximum [3, 7]. The complex extension is introduced in [5].

Definition 1.3. *Write $H = \sqrt{N}\,U$. The matrix H is real almost Hadamard if U locally maximizes L on O_N, and complex almost Hadamard if U locally maximizes L on U_N.*

Compactness gives real almost Hadamard matrices in every order. "Almost" refers to local maximality, with no prescribed distance from a Hadamard matrix. The simplest nonflat examples are reflections.

Proposition 1.4. *For $N \geq 3$, the matrix*

$$K_N = \frac{1}{\sqrt{N}} \begin{pmatrix} 2-N & 2 & \cdots & 2 \\ 2 & 2-N & \cdots & 2 \\ \vdots & \vdots & \ddots & \vdots \\ 2 & 2 & \cdots & 2-N \end{pmatrix}$$

is real almost Hadamard. It is Hadamard exactly when $N = 4$.

Proof. Let $J_N = (1/N)_{ij}$ and $W_N = K_N/\sqrt{N} = 2J_N - I$. Since $J_N^2 = J_N$, the matrix W_N is orthogonal. For its sign matrix S, direct multiplication gives

$$S^t W_N = 2I + (N-4)J_N > 0$$

Near W_N the entry signs are fixed and $L(V) = Tr(S^t V)$. For every nonzero real skew-symmetric A, the curve $W_N e^{tA}$ has

$$L'(0) = 0, \qquad L''(0) = Tr((S^t W_N)A^2) < 0$$

Thus W_N is a strict local maximum on O_N. The diagonal and off-diagonal moduli agree exactly when $N - 2 = 2$. □

The same matrices fail to be complex almost Hadamard when $N \neq 4$ [5, Theorem 4.7]. This is the basic real–complex distinction: the additional unitary directions can destroy a real local maximum.

Conjecture 1.5 (Almost Hadamard conjecture)**.** *Every local maximizer of the* 1*-norm on* U_N *is flat. Equivalently, every complex almost Hadamard matrix is complex Hadamard.*

Local maximality here is unrestricted and need not be strict. Row and column phases preserve L, and Hadamard matrices can occur in families. The real theory also includes circulant and 2-entry constructions [7], other entrywise exponents [4], and polar factors of complementary Hadamard submatrices under suitable size assumptions [6].

1.3 Designs and circulants: the earlier evidence

The 2-entry constructions come from square incidence patterns of symmetric block designs. Their rows have constant weights and constant pairwise overlaps.

Definition 1.6. *An* (a, b, c) *pattern has* 2 *symbols* x, y *such that every pair of distinct rows has the following counts:*

pair	(x,x)	(x,y)	(y,x)	(y,y)
count	a	b	b	c

$$N = a + 2b + c$$

We take $a, c \geq 0$ *and* $b > 0$.

For a full-support unitary with this pattern, normalize $y > 0$ and write $x = -\varepsilon t y$, with $t > 0$ and $|\varepsilon| = 1$. Orthogonality and normalization become

$$at^2 - 2b\,Re(\varepsilon)t + c = 0, \qquad y = \big((a+b)t^2 + b + c\big)^{-1/2}$$

Criticality further restricts the phase ε.

Proposition 1.7. *A full-support critical unitary with an* (a, b, c) *pattern is, after multiplication by a scalar of modulus* 1*, real or flat.*

Proof. The criticality calculation of [5, Theorem 5.2] gives

$$(\varepsilon - \bar{\varepsilon})(t - 1) = 0$$

If $t = 1$ the 2 entry moduli agree. Otherwise ε is real, and orthogonality excludes $\varepsilon = -1$, leaving $\varepsilon = 1$. □

The real branches still require a second-variation test. Theorem 6.5 of [5] excludes K_N for $N \neq 4$, the Grassmannian matrices $I_N^{q,d}$, and the Paley biplane matrix P_{11}. The projective-plane examples, including the Fano plane at $q = 2$, belong to the Grassmannian family at $d = 1$ [5, Proposition 5.5]. These exclusions do not cover every construction from a design.

A circulant matrix has rows obtained by successive cyclic shifts. Its unitarity can be expressed in Fourier coordinates as

$$U = \frac{1}{N} F_N \, diag(q_0, \ldots, q_{N-1}) F_N^*, \qquad |q_k| = 1$$

For real symmetric circulants the parameters satisfy

$$q_k \in \{1, -1\}, \qquad q_k = q_{-k}$$

The method of [5, Section 7] averages second derivatives over suitable circulant self-adjoint generators. A positive average supplies an increasing direction. It excludes the explicit nonflat circulant almost Hadamard examples of [7]; further computer checks use the examples of [4]. A general averaged inequality is proposed in [5, Conjecture 7.8], for real symmetric circulants and, in an analogous form, for self-adjoint unitary circulants. It is not proved there.

This earlier evidence gives families and variation methods. The small-order study below tests the full tangent space; the order-5 model shows why restricted tests can miss an increasing direction.

2 Variations and small orders

We now pass from norm maximization to first and second derivatives. The only nonsmooth points are matrices with zero entries.

2.1 Criticality and the Hessian

Lemma 2.1. *A local maximum of L on U_N has no zero entry.*

Proof. Choose a column rotation that moves a nonzero entry into a zero position to first order. In the sum of the increments at t and $-t$, nonzero entries contribute $O(t^2)$, the chosen zero contributes a positive multiple of $|t|$, and the other zeros contribute amounts ≥ 0. This contradicts local maximality. See also [5, Theorem 1.6]. □

We can therefore work where the norm is smooth. Its derivatives are expressed in terms of the entry phases.

Definition 2.2. *A matrix has full support when none of its entries is zero. For such a unitary V, its phase matrix and associated matrix are*

$$S_{ij} = \frac{V_{ij}}{|V_{ij}|}, \qquad X = S^* V$$

The matrix V is critical for L when the first derivative vanishes along every unitary curve through V.

We use the real Frobenius inner product and norm

$$\langle A, B \rangle = Re\, Tr(A^* B), \qquad \|A\|_F^2 = \sum_{ij} |A_{ij}|^2$$

Every tangent vector at V is realized by a curve

$$V(t) = V e^{itB}, \qquad V'(0) = iVB, \qquad B = B^*$$

Thus self-adjoint matrices B parametrize all unitary directions.

Lemma 2.3. *Along Ve^{itB} the first derivative of L at zero is $-Im\,Tr(XB)$. The second derivative is*

$$Q_V(B) = \sum_{ij} \frac{Re((VB)_{ij}\bar{S}_{ij})^2}{|V_{ij}|} - Re\,Tr(XB^2)$$

Thus V is critical precisely when X is self-adjoint.

Proof. For one entry $z(t)$ put $s = z(0)/|z(0)|$. Differentiating $|z|^2 = z\bar{z}$ gives $(|z|)' = Re(z'\bar{s})$ at zero, then $(|z|)'' = Re(z''\bar{s}) + Im(z'\bar{s})^2/|z(0)|$. Substitute $z' = i(VB)_{ij}$ and $z'' = -(VB^2)_{ij}$, and sum. Testing the first derivative against all self-adjoint B gives the last assertion. □

At a critical point, Q_V is the Hessian. Local maximality requires

$$Q_V(B) \le 0 \qquad \text{for every self-adjoint } B$$

One positive value excludes a local maximum. Nonpositivity alone need not establish one, since higher-order terms can matter.

Proposition 2.4. *At full support the row and column phase orbits have dimension $2N-1$. At a critical point their tangent directions lie in the kernel of the Hessian, so the Hessian descends to a quotient of dimension $(N-1)^2$.*

Proof. The action is $V \to D_1 V D_2$, with D_1, D_2 diagonal unitaries. At full support it fixes V exactly when $D_1 = zI$ and $D_2 = z^{-1}I$. The effective action is free and its orbits have dimension $2N-1$. Phase invariance makes the derivative of L along every phase vector field identically zero. Differentiating at a critical point shows that all Hessian terms involving a phase direction vanish. Removing these directions from the N^2-dimensional tangent space gives the result. □

The horizontal Hessian is the restriction to the orthogonal complement of the phase directions. The dimensions used in our computations are

Space	General order	$N=4$	$N=5$
Full tangent space	N^2	16	25
Phase directions	$2N-1$	7	9
Horizontal space	$(N-1)^2$	9	16

A positive geodesic second derivative also excludes a local maximum without requiring an exact stationarity certificate.

2.2 Pair tests and a polar condition

For 2 columns write their entries and relative phases as

$$p_i u_i, \quad q_i v_i, \qquad p_i, q_i > 0, \quad |u_i| = |v_i| = 1, \qquad h_i = \bar{u}_i v_i$$

After removing column phases there are 2 real mixing directions. Put

$$d_i = \frac{q_i^2}{p_i} + \frac{p_i^2}{q_i}$$

These quantities determine the largest pair curvature.

Lemma 2.5. *For 2 columns as above, the quantity*

$$M_{jk} = \frac{1}{4}\left(\sum_i d_i + \left|\sum_i d_i h_i^2\right|\right) - \frac{1}{2}\sum_i (p_i + q_i)$$

is the largest value of Q_V on Frobenius-unit off-diagonal self-adjoint generators supported on these columns.

Proof. Set $B_{jk} = e^{i\theta}/\sqrt{2}$, $B_{kj} = \bar{B}_{jk}$. The first term of $Q_V(B)$ in Lemma 2.3 becomes $\frac{1}{2}\sum_i d_i\, Re(h_i e^{-i\theta})^2$. The second uses $B^2 = (E_{jj} + E_{kk})/2$. Now $2Re(z)^2 = 1 + Re(z^2)$ on the unit circle. Maximizing the resulting expression over θ gives the stated formula for M_{jk}. □

The same test applies to rows. Pair tests are inexpensive but do not exhaust the Hessian, as the order-5 example will show. Rank-1 generators give a further necessary condition.

Proposition 2.6. *At a local maximum on U_N we have $S^*U \geq I/N$. In particular S is invertible and its unitary polar factor is U.*

Proof. Write $r_{ij} = |U_{ij}|$ and $X = S^*U$. Fix a unit vector v and insert $B = vv^*$ into $Q_U(B) \leq 0$. For $a_{ij} = Re((Uv)_i \bar{v}_j \bar{S}_{ij})$ we have $\sum_j r_{ij} a_{ij} = |(Uv)_i|^2$. By Cauchy–Schwarz and $\sum_j r_{ij}^3 \leq \sum_j r_{ij}^2 = 1$ we obtain

$$v^*Xv \geq \sum_{ij} \frac{a_{ij}^2}{r_{ij}} \geq \sum_i \frac{|(Uv)_i|^4}{\sum_j r_{ij}^3} \geq \sum_i |(Uv)_i|^4 \geq \frac{1}{N}$$

Criticality gives $X = X^* = U^*S$, hence $S = UX$ is the polar decomposition. □

2.3 The modulus map and its defect

Definition 2.7. *A matrix M is bistochastic when*

$$M_{ij} \geq 0, \qquad \sum_j M_{ij} = 1, \qquad \sum_i M_{ij} = 1$$

It is unistochastic when $M = p(U)$ for a unitary U, where

$$p(U) = (|U_{ij}|^2)$$

The fiber over M consists of all unitaries with these squared moduli.

The modulus map forgets phases. A tangent variation of a bistochastic matrix has zero row and column sums, giving $(N-1)^2$ independent real coordinates. The differential dp_U records which variations can be realized to first order by a curve of unitaries.

Definition 2.8. *The defect of a full-support unitary is*

$$d(U) = (N-1)^2 - rank(dp_U)$$

It measures the missing rank of the modulus map.

This is the defect of [25]. Nonflat critical points must lie in the singular part of the modulus map:

Proposition 2.9. *At a full-support nonflat critical point of L on U_N, the modulus map $p(U) = (|U_{ij}|^2)$ has rank strictly less than $(N-1)^2$.*

Proof. The affine space of bistochastic matrices has dimension $(N-1)^2$. Write $M = p(U)$ and $h(M) = \sum_{ij} \sqrt{M_{ij}}$. Concavity gives

$$dh_M(J_N - M) \geq h(J_N) - h(M) > 0$$

At a critical point of $L = h \circ p$, the differential dh_M vanishes on the image of dp_U. That image is therefore a proper subspace of the space of matrices with zero row and column sums. □

A direction in the kernel of dp_U leaves the squared moduli fixed to first order. Along a unitary curve with this initial direction, $p'(0) = 0$ gives

$$L''(0) = \frac{1}{2} \sum_{ij} \frac{p''_{ij}(0)}{|U_{ij}|}$$

At a critical point, this second variation depends only on the initial tangent vector. Finding a positive value in this kernel, after removing phases, is one route to the AHC. Section 3.1 implements it at order 4. Averaging over the unit sphere of the horizontal kernel is another possible test, following [5]; a generally positive average is not established here.

2.4 Orders 1 to 3 and the lifting argument

Definition 2.10. *A full-support unitary is dephased when its first row and column are positive real. A path of unitaries $U(t)$ satisfying*

$$p(U(t)) = M(t)$$

is called a lift of the modulus path $M(t)$.

Row and column phases always give a dephased representative. The local lifting problem asks for representatives approaching the specified unitary, rather than another point of its fiber.

Theorem 2.11. *Every local maximum of L on U_N is flat for $N \leq 3$.*

Proof. At order 1 the assertion is automatic. At order 2, full support and dephasing reduce the unitary to

$$\begin{pmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{pmatrix}, \qquad 0 < t < \pi/2$$

Its norm $2(\cos t + \sin t)$ has its only local maximum at $t = \pi/4$.

At order 3 a local maximum U has full support. Multiply its rows and columns by phases so that the first row and column are positive. At order 3, the unistochastic set is star-shaped about J_3, and a positive unistochastic matrix determines its dephased unitary up to complex conjugation [12, 2].

Suppose $M = p(U) \neq J_3$. Choose a decreasing sequence $t_n \to 0$ and dephased unitary lifts V_n of $(1-t_n)M + t_n J_3$. Compactness of U_3 gives a convergent subsequence. Its limit lies over M and is still dephased, since every entry of M is positive. The limit is therefore U or $\bar{U}$. Conjugating the subsequence in the second case gives unitaries tending to U. Concavity and $h(J_3) > h(M)$ yield

$$L(V_n) = h((1-t_n)M + t_n J_3) \geq (1-t_n)h(M) + t_n h(J_3) > L(U)$$

This contradicts local maximality. □

The argument needs both a segment in the image of p and control of its lifts. Higher-order fibers can be more complicated: star-shapedness alone would leave the lifting question at a prescribed unitary. The direct variations below avoid this assumption.

3 Order 4: structural restrictions

We first prove Theorem A, then obtain equations for the remaining phase patterns and deduce Theorems B and C. Section 4 then treats curvature and the numerical evidence. Throughout, U has full support and $S_{ij} = U_{ij}/|U_{ij}|$.

3.1 A real submatrix forces flatness

For a column pair $p_i u_i, q_i v_i$, call the real lines through the phase quotients $h_i = \bar{u}_i v_i$ its phase lines; h and $-h$ count as 1 line. For a full-support 2×2 block put

$$j(E) = Im(\bar{E}_{11} E_{12} E_{21} \bar{E}_{22})$$

Thus $j(E) = 0$ exactly when phases make E real.

Theorem 3.1. *A local maximum on U_4 is flat if one column pair uses at most 3 phase lines. Equivalently, every nonflat local maximum has $j(E) \neq 0$ for all 36 2×2 submatrices.*

We separate the possibilities of 1, 2 and 3 phase lines. The basic variation is an imaginary rotation of 2 real columns.

Lemma 3.2. *A real unitary locally maximizing L on the complex unitary group is flat.*

Proof. Mix 2 columns by diagonal entries $\cos t$ and off-diagonal entries $i \sin t$. Their squared moduli a^2, b^2 in each row become $(1-s)a^2 + sb^2, sa^2 + (1-s)b^2$, where $s = \sin^2 t$. Strict concavity of the square root increases their sum unless $a^2 = b^2$. At a maximum every row therefore has constant moduli. □

The same rotation handles a pair whose relative phases lie on a single real line. The remaining columns determine the equality case.

Lemma 3.3. *A nonflat full-support $U \in U_4$ with a collinear phase-quotient pair has an increasing arbitrarily small 2-row or 2-column rotation.*

Proof. Make the 2 columns real. Lemma 3.2's rotation works unless they are $(a_i), (\epsilon_i a_i)$ with $a_i > 0$, $\epsilon_i = \pm 1$. Each sign has squared mass $1/2$ and must occur twice: a singleton would force zeros in the remaining columns. Reorder to obtain $(a, b, c, d)^t, (a, b, -c, -d)^t$, with $a^2 + b^2 = c^2 + d^2 = 1/2$. The remaining columns are $(e, f)W$, $W \in U_2$, where

$$e = \sqrt{2}(b, -a, 0, 0)^t, \qquad f = \sqrt{2}(0, 0, d, -c)^t$$

The first 2 rows are collinear in phase, as are the last 2. The same rotation increases L unless $a = b$ and $c = d$, making all 4 numbers $1/2$. Orthogonality of W makes the last 2 column quotients collinear. Mixing them increases L unless their moduli also agree, which forces flatness by the row norms. Transposition gives the row version. □

At a critical point, 2 phase lines reduce to the preceding 1-line case through orthogonality and the first-order equations.

Lemma 3.4. *At a full-support critical point on U_4, a row pair with exactly 2 phase lines forces a collinear column pair, and conversely by transposition.*

Proof. The equations $\sum p_i q_i h_i = \sum (p_i - q_i) h_i = 0$ split on the 2 lines, each containing 2 opposite phases. On each, products and differences agree, determining the positive moduli. Rephase the rows to

$$(a, a, b, b), \qquad (c, -c, dz, -dz), \qquad z \notin \mathbb{R}$$

If columns 1 and 2 differ in modulus on the bottom rows, their 2 remaining differences have opposite signs. Stationarity makes their bottom phase quotients equal; orthogonality then makes these real, giving collinearity. The same applies to columns 3 and 4. Otherwise each bottom row has equal moduli within both pairs and lies in the span of $(b, b, -a, -a)$ and $(d, -d, -cz, cz)$. For its coefficients λ, μ, the modulus equalities imply $Re(\bar{\lambda}\mu) = Re(z\bar{\lambda}\mu) = 0$, so $\lambda\mu = 0$. Columns 1 and 2 are again collinear. □

3 phase lines require a block variation. We first identify 8 independent modulus coordinates for the differential of the modulus map.

Lemma 3.5. *Write $U = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ in 2×2 blocks. At full support, $j(B)j(C) \neq 0$ makes dp_U surjective onto the 4 coordinates of $|A|^2$, the first column of $|B|^2$ and the first row of $|C|^2$.*

Proof. The block C is invertible, since rank 1 would give $j(C) = 0$. The left skew-adjoint generator $\left(\begin{smallmatrix} 0 & Z \\ -Z^* & 0 \end{smallmatrix}\right)$ gives $\dot{A} = ZC$, prescribing the first 4 coordinates. Keeping A fixed, a last-column mixing changes the 2 entries of $|B_{i1}|^2$ by $2Re(w\bar{B}_{i1}B_{i2})$ up to sign. These forms are independent precisely when $j(B) \neq 0$, and C stays fixed. Mixing the last rows likewise prescribes the 2 C coordinates. □

At a nonflat critical point, Proposition 2.9 bounds the rank by 8. Under this lemma it is exactly 8, and these coordinates determine every vector in its image. Put $M = p(U)$ and $h(M) = \sum \sqrt{M_{ij}}$.

Lemma 3.6. *Let U be a full-support nonflat critical point with $j(A) = 0$ and $j(B)j(C) \neq 0$. Simultaneously exchange rows $1, 2$ and $3, 4$ and likewise columns, obtaining $\tilde{M}$. There is a unitary curve with $p'(0) = 0$, $j(D) = 0$, and*

$$L''(0) = \sum_{\{x,y\}} \frac{(x-y)^2(x+y)}{xy}$$

where the 8 pairs $\{x, y\}$ are the exchanged entry moduli.

Proof. Make A real. For $E^{\#} = \left(\begin{smallmatrix} \bar{E}_{22} & -\bar{E}_{21} \\ -\bar{E}_{12} & \bar{E}_{11} \end{smallmatrix}\right)$, the operation respects products and adjoints, and $V^{\#} = \overline{\det V} V$ for $V \in U_2$. Polar decomposition and unitarity give fixed $V, W \in U_2$ with

$$B = (I - AA^*)^{1/2}V, \quad C = W(I - A^*A)^{1/2}, \quad D = -WA^*V$$

Conversely these formulas complete any strict contraction A to a unitary. Use $A(t) = \cos t\, A + i \sin t\, A^{\#}$. The products $A(t)^*A(t), A(t)A(t)^*$ are even, since the cross products for real A equal $(\det A)I$. Thus $B'(0) = C'(0) = 0$, while

$$D(t) = \cos t\, D - i \sin t\, \eta D^{\#}, \qquad \eta = \det W \det V$$

The 8 coordinates above have zero derivative; hence all of $p'(0)$ vanishes. In particular $\eta\overline{D_{11}D_{22}}$ and $\eta\overline{D_{12}D_{21}}$ are real, so $j(D) = 0$. In the diagonal blocks, squared moduli follow exactly $\cos^2 t\, M + \sin^2 t\, \tilde{M}$.

The difference $p''(0) - 2(\tilde{M} - M)$ vanishes there and has zero row and column sums. Each other block is a multiple of $\left(\begin{smallmatrix} 1 & -1 \\ -1 & 1 \end{smallmatrix}\right)$. Both variations lie in $im(dp_U)$: since $j(D) = 0$, some last-column mixing fixes $|D|^2$ to first order, while $j(B) \neq 0$ makes its change in $|B|^2$ nonzero. Transpose for C. Criticality kills these differences under dh_M; consequently $L''(0) = 2dh_M(\tilde{M} - M)$, giving the formula by pairing terms. □

The curvature formula is positive unless all exchanged moduli agree. We now rule out this equality case at a nonflat critical point.

Lemma 3.7. *Under these hypotheses, $\tilde{M} = M$ is impossible.*

Proof. The blocks A, D would have equal opposite moduli and could be made real. A negative product of their entries would cancel their column inner product, forcing respectively $j(C) = 0$ or $j(B) = 0$. Make both blocks positive by independent phases; then $\eta = \pm 1$. For squared moduli

of a block set $f(X) = \sqrt{X_{11}X_{22}} - \sqrt{X_{12}X_{21}}$. Differentiate $\det D = \eta\overline{\det A}$ along an arbitrary unitary variation. Since η' is imaginary, real parts give

$$df_{M_D}(\dot{M}_D) - \eta\, df_{M_A}(\dot{M}_A) = 0$$

This functional is nonzero on the 9-dimensional bistochastic tangent space: its value on the checkerboard variation in A is -2η. Its kernel is therefore exactly $im(dp_U)$. On blocks with equal opposite entries, the segment $M \to J_4$ scales both f values by $1-t$. Thus $J_4 - M$ belongs to this kernel, contradicting $dh_M(J_4 - M) > 0$. □

Proof of Theorem 3.1. Lemmas 3.3 and 3.4 settle 1 or 2 lines. For 3 lines place the repeated pair in A, so $j(A) = 0$ and $j(C) \neq 0$. If $j(B) = 0$, the first 2 rows use at most 2 lines. Otherwise Lemmas 3.6 and 3.7 give positive second derivative. Transposition treats row pairs. □

The block direction can mix all 4 columns; existence of an increasing single pair at every nonflat critical point remains open. For a 3-line column pair with singleton rows a, b, orthogonality and stationarity give $p_a^{-1} - q_a^{-1} = p_b^{-1} - q_b^{-1}$: project both equations onto a linear functional killing the repeated line.

If every column pair uses at most 2 lines, any local maximum is flat by Theorem 3.1. As a separate finite check, `phase_grid.py` tests all $4^9 = 262144$ dephased fourth-root phase patterns: 192 pass the necessary sign-balance test, comprising 168 real non-Hadamard patterns, 6 real and 18 nonreal Hadamard patterns. Passing this test does not prove unitary realizability.

3.2 The equations for 4 distinct phase lines

We now assume that all 36 quantities $j(E)$ from Theorem 3.1 are nonzero. The first-order equations already restrict the moduli in this case. Write $e = (1,1,1,1)^t$ and put

$$R_{ia} = \frac{1}{|U_{ia}|}, \qquad P = I - \frac{1}{4}ee^t, \qquad G = PRP$$

Thus G is obtained from the reciprocal moduli by subtracting the row and column means and adding the overall mean.

Lemma 3.8. *At a nonflat critical point under our assumption, the 4 numbers $R_{ia} - R_{ib}$ are pairwise distinct for every pair of columns a, b. The same holds for rows.*

Proof. Put $z_i = \bar{U}_{ia}U_{ib}$ and $d_i = R_{ia} - R_{ib}$. Orthogonality and criticality give, respectively,

$$\sum_i z_i = 0, \qquad \sum_i d_i z_i = 0$$

Indeed, the second equation is the (a,b) entry of $S^*U - U^*S = 0$. If $d_1 = d_2$, subtract d_1 times the first equation from the second. The vectors z_3, z_4 are linearly independent over $\mathbb{R}$, so all 4 differences agree. Since the 2 columns have equal squared norms and $r \to r^{-2}$ is strictly decreasing, their common difference is zero. In particular, for every 2 rows i, j, the row difference $R_{il} - R_{jl}$ agrees at $l = a, b$. Apply the same argument to U^t, which is also critical. All row differences are zero, and column normalization now gives $|U_{ia}| = 1/2$ everywhere. □

The distinct reciprocal differences determine 2-dimensional planes for each column pair. Compatibility between these planes gives a rank restriction on the whole matrix.

Theorem 3.9. *At a nonflat critical point of L on U_4 with 4 distinct phase lines for every pair, the centered reciprocal modulus matrix G has rank exactly 2.*

Proof. Fix a, b and put $J_{kl,ab} = Im(\bar{z}_k z_l)$. The 2 columns $Re(z), Im(z)$ span the orthogonal complement in $\mathbb{R}^4$ of e, d. Both planes have dimension 2 by our phase assumption and Lemma 3.8. The oriented area forms of complementary 2-planes are Hodge dual up to a nonzero scalar. In coordinates their 2×2 minors therefore satisfy

$$J_{kl,ab} = c_{ab}\, \epsilon_{ijkl}(d_i - d_j)$$

where i, j, k, l are distinct and ϵ_{ijkl} is the sign of this permutation. The real number c_{ab} is independent of the row indices. Exchanging a, b negates both J and d, hence $c_{ba} = c_{ab}$.

For fixed k, l, a we have $\sum_{b \neq a} J_{kl,ab} = 0$, by orthogonality of rows k, l. For the complementary rows i, j, put $v_a = R_{ia} - R_{ja}$. The preceding identity gives

$$\sum_{b \neq a} c_{ab}(v_a - v_b) = 0$$

Let H have off-diagonal entries $-c_{ab}$ and diagonal entries $\sum_{b \neq a} c_{ab}$. This nonzero real symmetric matrix kills e and every row difference of R, viewed as a column vector. It therefore kills their projections by P, which span the row space of G. This space is orthogonal to e, so $rank(G) \leq 2$.

Rank zero contradicts Lemma 3.8. In rank 1 we could write

$$R_{ia} = u_i + v_a + x_i y_a$$

with all parameters real, by restoring the row and column means to G. The lemma forces the x_i to be pairwise distinct, and likewise the y_a. Criticality says that iS is tangent at U, since U^*S is self-adjoint. Subtracting the phase tangents $i\, diag(u)U + iU\, diag(v)$ leaves $i\, diag(x)U\, diag(y)$ tangent. Thus $U^*\, diag(x)U$ commutes with $diag(y)$ and must be diagonal. Every column of U would be an eigenvector of $diag(x)$. Its eigenvalues are simple, whereas every column has 4 nonzero entries. □

Interpolation at the 4 distinct reciprocal differences gives a polynomial identity involving only the moduli.

Proposition 3.10. *Write $p_i = |U_{ia}|$, $q_i = |U_{ib}|$, $d_i = p_i^{-1} - q_i^{-1}$ and $D_i = \prod_{j \neq i}(d_i - d_j)$. Under the assumptions of Theorem 3.9, every column pair satisfies*

$$\sum_{i=1}^{4} p_i^2 q_i^2 D_i = 0$$

The 6 *row pairs satisfy the same identity.*

Proof. For $k = 0, 1, 2$, interpolation of t^k at the 4 distinct nodes d_i gives $\sum_i d_i^k / D_i = 0$ by comparison of cubic coefficients. The solutions of $\sum z_i = \sum d_i z_i = 0$ therefore have the form

$$z_i = (A + Bd_i)/D_i, \qquad A, B \in \mathbb{C}$$

Indeed these vectors form a 2-dimensional complex space. Now $p_i^2 q_i^2 D_i^2 = |A + Bd_i|^2$ is the value of a quadratic at d_i. The coefficient of degree 3 in its interpolating polynomial is $\sum_i p_i^2 q_i^2 D_i$, giving the formula. □

The same interpolation also reconstructs the relative phases. It leaves only finitely many conjugation choices to check.

Proposition 3.11. *In the* 4*-line case, prescribed moduli determine at most* 1 *dephased nonflat critical unitary, up to complex conjugation. There is an explicit reconstruction test.*

Proof. For each pair interpolate $p_i^2 q_i^2 D_i^2$ at d_i. By Proposition 3.10 this gives a quadratic $q(t) = a_2 t^2 + a_1 t + a_0$. 4 distinct phase lines require and, for that pair, are equivalent to

$$a_2 > 0, \qquad 4a_0 a_2 > a_1^2$$

Take $B = \sqrt{a_2}$ and $A = a_1/(2B) + i\sqrt{a_0 - a_1^2/(4a_2)}$ in $z_i = (A + Bd_i)/D_i$. This gives all pair solutions up to common phase and conjugation. Pair column 1 with each other column and make their first entries positive. The 3 conjugation choices leave at most 4 candidates up to simultaneous conjugation; test their remaining orthogonality and stationarity equations.

In fact at most 1 candidate survives. Let v span $\ker G \cap e^{\perp}$. The symmetric matrix H in the proof of Theorem 3.9 has this line as its image, so $c_{ab} = \lambda v_a v_b$ with all factors nonzero. Choose the conjugate representative with $\lambda > 0$. For each $b > 1$, the sign

$$sgn(J_{12,1b}) = sgn(v_1 v_b (R_{31} - R_{3b} - R_{41} + R_{4b}))$$

fixes the conjugation choice. This proves uniqueness. In particular, symmetric moduli force $U^t = U$: the alternative $U^t = \bar{U}$ would make U self-adjoint and give a real principal 2×2 phase product. □

3.3 Common profiles and at most 4 moduli

The rank and pair equations have concrete consequences. We first handle 2 group patterns, then show that a common multiset reduces to them.

Theorem 3.12. *Let $U \in U_4$ be a local maximum of L. Suppose that after row and column permutations its moduli have the form $|U_{ia}| = f(a - i)$ on $\mathbb{Z}_4$ or on $\mathbb{Z}_2^2$. Then U is flat.*

Proof. If U were nonflat, all $j(E)$ would be nonzero by Theorem 3.1. The reciprocal moduli have the same symmetry. In the cyclic case write their first row as (a, b, c, d). On the orthogonal complement of constants the eigenvalues of the circulant are $a - b + c - d$ and the conjugate pair $a - c \pm i(b - d)$. Rank 2 forces $a + c = b + d$. Write

$$(a, b, c, d) = (m + x, m + y, m - x, m - y)$$

Read 2 adjacent rows cyclically, with reciprocal moduli (a, b, c, d) and (b, c, d, a). Their 4 differences are $(x - y, x + y, y - x, -x - y)$. Substitution in Proposition 3.10 and multiplication by $a^2 b^2 c^2 d^2$ gives

$$64m^3 xy(x - y)(x + y) = 0$$

For example, the left side is the expansion of $c^2 d^2 D_1 + d^2 a^2 D_2 + a^2 b^2 D_3 + b^2 c^2 D_4$. Since $m > 0$, one of the 4 other factors must vanish. If $x = y$ or $x = -y$, 2 adjacent-row differences coincide. If $x = 0$ or $y = 0$, 2 opposite-row differences in $(2x, 2y, -2x, -2y)$ coincide. Both conclusions contradict Lemma 3.8.

For $\mathbb{Z}_2^2$ the 3 nonconstant eigenvalues are $a + b - c - d$, $a - b + c - d$, $a - b - c + d$. One must vanish. Relabel the 3 nonidentity elements to arrange $a + b = c + d$. The rows (a, b, c, d) and (c, d, a, b) have differences $(a - c, b - d, c - a, d - b)$. The first and last coincide, giving the same contradiction. □

A common row and column profile can be decomposed into permutation matrices. At order 4 these recover exactly the 2 group patterns just considered.

Theorem 3.13. *Suppose that each row and each column of a local maximizer on U_4 contains the same multiset of entry moduli. Then the unitary is flat.*

Proof. For a modulus occurring k times per row and column, join row i to column a when the entry has this modulus. The resulting bipartite graph is k-regular. Any set of rows has at least as many neighbors, by counting incident edges. Hall's theorem gives a perfect matching, whose removal lowers every degree by 1. Repeating this for each modulus partitions the array into 4 permutation matrices, each carrying a constant modulus. This step allows repetitions among the 4 moduli.

Permute columns to make the first matching the identity. The other 3 permutations have no fixed points. If they are all involutions, they are the 3 double transpositions, which together with the identity act regularly as the Klein 4-group. Otherwise one is a 4-cycle s. At each row i the remaining columns are $s^2(i)$ and $s^3(i)$. These edges form a single bipartite cycle of length 8. Indeed, traversing an s^3 edge and returning along an s^2 edge advances the row by s. Its only perfect matchings are the 2 alternating ones. Thus the 4 permutations are $1, s, s^2, s^3$. In either case we can label rows and columns by the group so the modulus at (i, a) depends only on $a - i$. Theorem 3.12 applies. □

In particular, a local maximizer on U_4 with at most 2 distinct entry moduli is flat. Indeed, if these moduli are $r \neq s$, the number k of entries of modulus r on any row or column satisfies $kr^2 + (4 - k)s^2 = 1$. This fixes k throughout the matrix.

With 3 distinct moduli, different row profiles can still occur. Normalization and the reciprocal-difference restrictions rule them out.

Proposition 3.14. *A local maximum on U_4 having at most 3 distinct entry moduli is flat.*

Proof. It remains to consider 3 distinct squared moduli, which we call symbols. Suppose that a nonflat local maximum exists. By Theorem 3.1 and Lemma 3.8, 2 rows can have equal symbols in at most 1 column. Also, wherever one row has a repeated symbol, another row must take different values at those positions. Both rules hold for columns.

A constant row is impossible, since another row would need 4 distinct symbols. Suppose that one row has the form (x, x, x, y). Each other row contains all 3 symbols in its first 3 positions. Normalization forces their fourth entries to have the same squared modulus z. Comparing the first row with the last column gives $3x + y = 3z + y$, hence $x = z$. Comparing with any other row now gives $3x + y = x + y + w + x$, where w is the third symbol. This forces $w = x$. We have ruled out triples in both rows and columns.

The possible multiplicities in a row are now $(2, 1, 1)$ and $(2, 2)$. 2 rows of the second kind must use the same pair x, y, since their sums agree and there are only 3 symbols. Under a row (x, x, y, y) the second such row must put distinct entries in the first pair and in the last pair. The rows then agree twice. Thus at most 1 row has type $(2, 2)$, and likewise for columns. All rows and columns of type $(2, 1, 1)$ repeat the same symbol, by normalization.

If there is a row (x, x, y, y), comparison with a row of the other type shows that its repeated symbol is the third symbol z and that $x + y = 2z$. There are 6 copies of z in the matrix. Consequently exactly 1 column has type $(2, 2)$ and contains no z. Delete this column and the row (x, x, y, y). Each row and column of the remaining 3×3 array contains 2 copies of z. Any 2 of its rows share a z in 1 column. In the deleted column 2 of these 3 rows must also agree, since only x, y occur there. This contradicts the first rule. Transposition rules out a column of type $(2, 2)$ as well. All rows and columns therefore have the same multiset, and Theorem 3.13 finishes the proof. □

4 distinct moduli require a finite classification. The count constraints reduce the problem to 27 patterns, each excluded by explicit polynomial identities.

Theorem 3.15. *Every local maximum of L on U_4 with at most 4 distinct entry moduli is flat.*

Proof. Only 4 distinct moduli and unequal profiles remain. Label symbols $0, 1, 2, 3$ in increasing reciprocal order, with squared moduli $s_0 > s_1 > s_2 > s_3$. For a sorted row or column profile w,

put $n_k(w) = \#\{w_i \geq k\}$ and $u = (s_0 - s_1, s_1 - s_2, s_2 - s_3) > 0$. Its squared norm is $4s_0 - n(w) \cdot u$, so all 8 profiles have the same scalar product with u.

For enumerating profiles, u can be taken integral with entries at most 32. In rank 2, cross 2 independent differences $n(w) - n(w_0)$: the coordinates are bounded by $2 \cdot 4^2$ and have one sign. In rank 1 a difference d has both signs; assign the sum of negative absolute entries to its positive positions, the sum of positive entries to its negative positions, and 1 to zeros. Rank zero is the common-profile case. This imposes no arithmetic restriction on the actual moduli.

There are 35 profiles. Group them by their scalar product for $u \in \{1, \ldots, 32\}^3$, retaining groups with at least 2 profiles and all 4 symbols: 157 groups remain, 93 maximal. Assemble 4 words from each maximal group, requiring columns from the same word set. Reject row pairs repeating an ordered symbol pair or agreeing twice, by Lemma 3.8; reject fewer than 4 symbols or a common profile. Canonicalize by sorting rows after every column permutation, also after transposition. Exactly 27 arrays remain.

Scale the reciprocal levels to $1 < x < y < z$. Impose equal row and column sums of inverse squares, the 12 pair equations, and

$$\det(R_{ia} - R_{1a} - R_{i1} + R_{11})_{2 \leq i,a \leq 4} = 0$$

Clear positive denominators and factor over the rationals. A factor with nonzero coefficients of one sign after $x = 1 + a, y = 1 + a + b, z = 1 + a + b + c$ is nonvanishing for $a, b, c > 0$. This check excludes 10 arrays by norms or determinant and 13 by a pair equation. In 3 more, the surviving determinant factors are $3x - y - z - 1$, $6x - 4y - z - 1$, $6x - y - 4z - 1$; each also requires $x^2(1 + y + z) - yz = 0$. Substitution for x makes this strictly positive by the same coefficient test.

The last array and determinant equation are

$$\begin{pmatrix} 0 & 2 & 3 & 3 \\ 2 & 1 & 2 & 3 \\ 3 & 2 & 1 & 2 \\ 3 & 3 & 2 & 0 \end{pmatrix}, \qquad xz - x + y^2 - 3yz + y + z^2 = 0$$

Rescale the reciprocal levels to $r_3 = 1$ and put $h = 1 - r_0$, $t = (1 - r_2)/h$. The determinant gives

$$(r_0, r_1, r_2, r_3) = (1 - h, 1 - h(t + t^2), 1 - ht, 1), \qquad 0 < h, t < 1, \quad t + t^2 < 1$$

Take the norm difference of rows 1 and 2 and the pair equations for rows $(1, 2), (1, 3)$. Discard sign-definite factors, homogenize with r_0, and substitute these levels. Removing constant factors and powers of $h, t - 1, t + 1$ leaves F, G, K of degrees $5, 6, 6$ in h. Their resultants have monic greatest common divisor

$$\gcd(Res_h(F, G), Res_h(F, K)) = t^{26}(t^2 + t - 1)^4$$

It cannot vanish in the required domain. This exhausts the 27 arrays. Every count, factor test and resultant is reproduced by `four_moduli.py`; `four_lines.py` checks the earlier cyclic factorization and matching classification. □

4 Order 4: curvature and numerical evidence

The restrictions above leave the case of 4 distinct phase lines and at least 5 entry moduli. We examine its remaining curvature, then give exact calibration saddles and the results of the searches.

4.1 Curvature in the remaining case

At a nonflat 4-line critical point dp_U has rank 8, by Lemma 3.5. Its kernel consists of 7 phase directions and 1 further direction, represented by $iS = iUX$: this fixes all moduli to first order and is not a phase tangent, since that would give $R_{ij} = u_i + v_j$ and $G = 0$. For an anchor (a, b) set

$$T_{ij}^{ab} = R_{ij} - R_{ib} - R_{aj} + R_{ab}, \qquad m_k^{ab} = \sum_{ij} |U_{ij}|^2 (T_{ij}^{ab})^k$$

The following formula tests this direction, even when phase lines coincide.

Proposition 4.1. *At every full-support critical point on U_4, all 16 values*

$$C_{ab} = m_3^{ab} - \frac{3}{2} m_1^{ab} m_2^{ab} + \frac{1}{2} (m_1^{ab})^3$$

agree and equal $Q_U(X) = \sum_{ij} |U_{ij}|^{-1} - Tr(X^3)$, where $X = U^* S$. A positive common value excludes a local maximum.

Proof. Put $\alpha_i = R_{ib}$, $\beta_j = R_{aj} - R_{ab}$ and $B = X - U^* \, diag(\alpha) U - diag(\beta)$. Then $(UB)_{ij} = T_{ij}^{ab} U_{ij}$ and phase invariance gives $Q_U(B) = Q_U(X)$. The self-adjoint matrix B kills the independent vectors $e_b, U^* e_a$, so it has rank at most 2. Thus

$$Tr(B) = m_1, \quad Tr(B^2) = m_2, \quad Tr(B^3) = \frac{3}{2} m_1 m_2 - \frac{1}{2} m_1^3$$

Substitute $X = B + U^* \, diag(\alpha) U + diag(\beta)$ in the second variation. The additive terms cancel against the corresponding part of $\sum_{ij} |U_{ij}| T_{ij}^2$, leaving $m_3 - Tr(B^3)$. □

In the 4-line case B has rank exactly 2: rank 1 would make the phase products in its nonzero complementary 3×3 block real. Hence $m_2 \neq m_1^2$ and $m_2 \geq m_1^2/2$. The equalities $C_{ab} = C_{11}$ give 15 necessary consistency equations, with no claim of independence or sufficiency. On the 3596 saved nonflat returns, `curvature.py` finds discrepancy below $3.3 \cdot 10^{-11}$ from the trace formula and common values in $(3.5867, 3.9118)$.

Proposition 4.2. *Bistochasticity, centered reciprocal rank 2 and nonzero mixed differences do not force a positive cubic: there are positive data M with all $C_{ab} \in (-0.008, -0.000017)$. These data have no critical unitary realization.*

Proof. An explicit certificate uses $G = 10^{-10} A D^t$, where

$$A = \begin{pmatrix} -15233 & -48634 \\ 7967 & -41084 \\ 13665 & 113540 \\ -6399 & -23822 \end{pmatrix}, \qquad D = \begin{pmatrix} 79990 & -3775 \\ -104383 & 15717 \\ 37032 & 149 \\ -12639 & -12091 \end{pmatrix}$$

Both factors are centered and have rank 2. Put $R_{ij} = G_{ij} + u_i + v_j$, $v_4 = 0$, and $M_{ij} = R_{ij}^{-2}$. Solve the 4 row and first 3 column normalization equations in the box of radius 10^{-10} around

$$(u_1, u_2, u_3, u_4) = (2.007055173083, 2.007781759308, \ 2.008127664289, 2.002764453315)$$
$$(v_1, v_2, v_3) = (-0.001933545200, -0.000421319991, -0.004970340424)$$

Let F be 1 minus these 7 sums. Take the inverse of DF at the center and round its entries to 40 significant decimals, obtaining a rational preconditioner Y. Interval arithmetic verifies $\|I - YDF\|_\infty < 4.5 \cdot 10^{-9}$ and that $x - YF(x)$ maps the box into its concentric box of radius $5 \cdot 10^{-13}$. The contraction principle gives a unique solution. On the whole box $R > 0$, all mixed differences are nonzero, and the claimed cubic bounds hold. Moreover $C_{11} \in (-0.002962, -0.002961)$ and $C_{44} \in (-0.000017406, -0.000017404)$; their inequality excludes a critical unitary by Proposition 4.1. All interval operations are in `curvature.py`. □

There is nevertheless a positive result near J_4. Write

$$H = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}, \qquad F = (e\ H), \qquad G = HAH^t$$

The 12 pair equations mean the equations of Proposition 3.10 with weights $M_{ia}M_{ib}$; they make sense even when differences coincide.

Theorem 4.3. *Let $M(t)$ be a C^2 path of positive bistochastic matrices with $M(0) = J_4$, $M'(0) \neq 0$. Suppose for all small $t > 0$ that $\mathrm{rank}(G(t)) \leq 2$, the 12 pair equations hold, and all 16 $C_{ab}(t)$ agree. Write $G(t) = tHA_0H^t + t^2HA_1H^t + o(t^2)$. After row and column permutations and possibly transposition,*

$$A_0 = \begin{pmatrix} a & b & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad a^2 + b^2 > 0, \qquad C(t) = 32(a^4 + b^4)t^4 + o(t^4) > 0$$

No unitary realization is assumed.

Proof. Normalization gives $R = 2ee^t + tG_0 + O(t^2)$ and $M = J_4 - tG_0/4 + O(t^2)$, where $G_k = HA_kH^t$. For each anchor let T, Z be the mixed differences of G_0, G_1. Put $u_k = \frac{1}{4}\sum T^k_{ij}$. The cubic coefficient is $E = u_3 - \frac{3}{2}u_1u_2 + \frac{1}{2}u_1^3$. For the matrix of these coefficients, direct expansion gives

$$(F^tEF/16)_{k0} = -96\prod_j (A_0)_{kj}, \qquad (F^tEF/16)_{0k} = -96\prod_i (A_0)_{ik} \quad (k = 1, 2, 3)$$

They vanish, so every row and column of A_0 contains a zero. The leading coefficient of a column-pair equation is $32\prod_i (A_0v)_i$, with $v = e_j \pm e_k$ up to sign. Thus any 2 columns u, v satisfy $\prod_i (u_i + v_i) = \prod_i (u_i - v_i) = 0$. They have a common zero: otherwise 2 different positions have $u_i = \pm v_i \neq 0$, forcing both third entries to vanish because each column contains a zero. Their zero sets intersect pairwise. Unless all 3 share a zero, they are the 3 distinct 2-element subsets, making A_0 a nonsingular monomial matrix. This contradicts $\det A_0 = 0$. Transposition gives a zero column as well.

For the remaining block $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, the $(1,1), (1,2), (2,1), (2,2)$ transformed cubic coefficients are $48bcd, 48acd, 48abd, 48abc$. At most 2 entries survive, and then $E = 0$. For completeness the fourth coefficient can be recovered without any asymptotic guess: expand the same cubic expression using

$$T(t) = tT + t^2Z, \qquad M(t) = J_4 - tG_0/4$$

and retain degree 4. Call the resulting anchor matrix D. If the 2 entries occupy different rows and columns, put $A_0 = diag(a, b, 0)$. Then

$$(F^tDF/16)_{33} = -48ab(a^2 + b^2) \neq 0$$

independently of A_1, contradicting equality of the anchor values. The remaining support is 1 row or column. For the displayed A_0 and arbitrary A_1, the constant coefficient is

$$(F^tDF/16)_{00} = 32(a^4 + b^4)$$

These polynomial identities are checked exactly in `flat_limit.py`. The C^2 remainders contribute $o(t^4)$, giving the conclusion. □

This result concerns regular paths in the modulus equations. It gives neither a uniform punctured neighborhood nor a global exclusion. Paths with vanishing first derivative can require fractional powers after reparametrization and are not covered.

A bounded exploratory search parametrized $G = AB^t$ with $\|G\|_F < \sqrt{3}$, balancing R by additive row and column terms. The 300 starts comprised 30 fitting cubic consistency alone, 30 fitting pair equations alone, and 120 each combining cubic consistency with a negative-value target or with the pair equations. The last 2 minimum scaled residuals were 0.04015 and 0.001853, neither below 10^{-7}. These are auxiliary residuals, not unitary criticality tests. `moduli_search.py -verify-saved` rechecks every retained endpoint; these searches exclude no additional critical unitaries.

It remains open whether normalization, reciprocal rank 2, the pair identities, cubic consistency and phase compatibility force positive common curvature at every nonflat 4-line critical point. A negative common cubic alone would not settle the full Hessian test.

4.2 2 exact calibration saddles

2 of the stationary levels in the next subsection have explicit representatives. For the first, take $a = \cos(\pi/8)/\sqrt{2}$ and $b = \sin(\pi/8)/\sqrt{2}$. An exact representative is

$$U = \begin{pmatrix} a & b & a & b \\ a & -b & -b & -a \\ b & a & -a & b \\ b & -a & -b & a \end{pmatrix}$$

The identities $2a^2 + 2b^2 = 1$ and $a^2 - 2ab - b^2 = 0$ give $UU^t = I$. Multiplication gives S^tU symmetric, proving criticality. We have $L(U) = 8(a+b) = 4\sqrt{2+\sqrt{2}}$. The imaginary column rotation in Lemma 3.2 increases L. Along a real rotation of the same columns the signs are locally constant, so the pair contributes $C\cos t + D\sin t$ with $C > 0$ and $D = 0$ by criticality. This gives a decrease.

For the highest observed nonflat level, use a bordered circulant matrix. For $0 < x < 1$ write $s = \sqrt{1-x^2}$ and take the corner entry x, with 3 equal entries $s/\sqrt{3}$ along both borders. Take the inner block circulant, with successive right shifts of the row

$$c_j(x,t) = \frac{-x + 2i\sin(t + 2\pi j/3)}{3}, \qquad j = 0, 1, 2$$

The border and constant mode carry the block $\left(\begin{smallmatrix} x & s \\ s & -x \end{smallmatrix}\right)$. The 2 nonconstant Fourier eigenvalues have modulus 1, since the sine expands into the modes e^{it} and $-e^{-it}$. This proves unitarity. At $t = 0$ the norm is

$$F(x) = 2x + 2\sqrt{3}\sqrt{1-x^2} + 2\sqrt{x^2+3}$$

We have $F''(x) = -2\sqrt{3}/(1-x^2)^{3/2} + 6/(x^2+3)^{3/2} < 0$. The derivative is positive at zero and tends to $-\infty$ at 1, giving a unique zero $x_0 = 0.6096097890\ldots$ and $F(x_0) = 7.6376151868\ldots$.

To verify stationarity on all of U_4, both S and U at $t = 0$ are self-adjoint and diagonal on the 2 nonconstant modes. On the border and constant mode S has diagonal entries $1, v$ and off-diagonal entries $\sqrt{3}$, where $v = -1 - 2x/\sqrt{x^2+3}$. The 2 off-diagonal entries of S^*U differ by $s(1-v) - 2\sqrt{3}x = sF'(x)$, which vanishes at x_0. Thus S^*U is self-adjoint.

The x-direction decreases the norm. In the t-direction the varying part of the norm is $\sum_{j=0}^{2}\sqrt{x^2 + 4\sin^2(t + 2\pi j/3)}$. Putting $r = \sqrt{x^2+3}$, its second derivative at zero is

$$\frac{4}{x} - \frac{4}{r} - \frac{6}{r^3} = \frac{6(r-x)(2r+x)}{xr^3(r+x)} > 0$$

This proves the saddle assertion independently of numerical Hessian diagonalization. The 2 intermediate levels in the numerical table remain uncertified as exact critical values. The 2

representatives are matched by `models.py`; `three_lines.py` also checks the block variation and certifies column curvature $M_{23} \in (0.6092, 0.6094)$, row curvature $M_{14} \in (-0.6677, -0.6674)$ and block-curve curvature in $(1.4478, 1.4480)$ on partition $12 \mid 34$ for this calibration. Thus the singleton rows of a 3-line pair need not themselves give an increase.

4.3 Numerical evidence and its limits

A useful bound separates potential local maxima from the nonsmooth boundary. We use Frobenius-unit generators as in Lemma 2.5.

Proposition 4.4. *At a local maximum on U_N, $N \geq 2$, every entry has modulus at least*

$$b_N = \sqrt{N(N-1)^2 + 1} - (N-1)\sqrt{N}$$

In particular, $b_4 = \sqrt{37} - 6$ and $b_5 = 9 - 4\sqrt{5}$.

Proof. For an entry of modulus p, choose a partner of modulus q in its row with $q^2 \geq (1 - p^2)/(N-1)$. Choose the phase of the generator in Lemma 2.5 to agree with h_i in the selected row. Its contribution to the first term of the curvature is then $(q^2/p + p^2/q)/2$. Each column has 1-norm at most $\sqrt{N}$, giving

$$M_{jk} \geq \frac{q^2}{2p} - \sqrt{N} \geq \frac{1 - p^2}{2(N-1)p} - \sqrt{N}$$

At a local maximum $M_{jk} \leq 0$. Solving the resulting quadratic inequality gives the bound. □

We use the diagnostics

$$g(U) = \frac{1}{2}\|S^*U - U^*S\|_F, \qquad d(U) = \max_{ij} \big||U_{ij}| - 1/2\big|$$

2 Apple M3 laptops (8 GB, 3 workers each) ran 7-hour campaigns, with seeds $421701, 421702$. Ascent uses Haar starts, geodesic gradient steps and Armijo backtracking, capped at 1800 iterations. Stationary search uses damped least squares for $(S^*U - U^*S)/(2i) = 0$. 4 protocols start from Haar, phase-polar, near-real and near-flat matrices; 4 first steer Haar matrices towards $L \leq c$, $c \in \{7.40, 7.65, 7.90, 7.995\}$, with at most 24 steering and 120 root steps.

The tolerances are $d \leq 10^{-6}$ for flatness, $g \leq 10^{-12}$ for stationarity, and 10^{-8} for near-zero entries. At accepted nonflat returns both full and 9-dimensional horizontal Hessians are checked; top horizontal eigenvalue at most 10^{-7} triggers a candidate flag.

Endpoint status	Ascent	Stationary
Numerically flat	83 009 628	19 409 961
Nonflat with positive curvature	0	26 672 800
Near zero	0	6 126 915
Unresolved	67 268	27 748
Total	83 076 896	52 237 424

There were no candidate, unlisted-level, numerical-error or unitarity-error flags. All nonflat stationary returns had L within $2 \cdot 10^{-6}$ of one of the following 4 values. Repeated visits to equivalent matrices are included in the counts.

L (rounded)	Returns	Minimum top horizontal eigenvalue
7.391036	6 764 846	2.2961006
7.633239	9 497 039	2.8149691
7.633377	7 875 896	2.6601070
7.637615	2 535 019	2.8768867

Every unresolved ascent endpoint satisfies $d \le 6.01386943033244 \cdot 10^{-6}$; not all were continued, so this does not assert convergence of every trajectory. For stationary returns,

$$\max_{\text{unresolved } U} \min_{ij} |U_{ij}| = 5.6974893208468939 \cdot 10^{-7}$$

All unresolved and near-zero stationary endpoints thus fall far below $\sqrt{37} - 6 = 0.08276\ldots$. This comparison has a large numerical margin but is not an interval certificate for the entire campaign.

The initial 8192-start survey gave 3596 nonflat uphill, 4002 near-flat, 589 near-zero and 5 unresolved returns. The 4 nonflat groups have numerical positive indices $3, 3, 2, 1$. For every nonflat return, interval arithmetic proves a recorded pair curvature exceeds 0.60 throughout its coordinate box of radius 10^{-8}. This excludes local maxima in the boxes without proving existence or uniqueness of critical points there. The other 594 endpoints have a smallest entry at most $1.18 \cdot 10^{-8}$. A separate 1024-start polar-ascent control reached numerical flatness, including 11 final least-squares refinements.

The compact archives retain all 95 016 unresolved overnight endpoints, aggregate counts and extrema, and representative outputs. Independent audits recheck 99 164 retained endpoints and all 98 914 stored spectrum pairs. The sources are `search.py` and `audit.py` for the initial survey and its certificates, and `ahc_ascent.py` and `ahc_stationary.py` for the campaigns. `compact.py` exports and verifies their archives; `audit_compact.py` recomputes all retained diagnostics. The archives contain their exact generating sources and complete aggregate counts, but ordinary discarded endpoints are unavailable. These searches sample basins, not stationary points uniformly; repeated visits are not independent evidence of completeness. No covering of U_4 is claimed, and an unvisited basin remains possible.

5 Order 5: exact models and gradient geometry

5.1 A saddle invisible to pair tests

The order-5 search needs the full Hessian. Its main calibration shows why pair rotations alone cannot decide local maximality. For $0 < x < 1$, put

$$a = (1 + x)/4, \qquad b = \sqrt{1 - x^2}/2, \qquad c = (1 - x)/4$$

Let $U(x)$ have corner entry x, 4 entries b on each border, and an inner circulant block whose first row is

$$(-a,\ c - i/2,\ -a,\ c + i/2)$$

The other inner rows are successive right shifts. On the border and normalized constant inner vector, $U(x)$ acts by

$$\begin{pmatrix} x & \sqrt{1 - x^2} \\ \sqrt{1 - x^2} & -x \end{pmatrix}$$

Its other Fourier eigenvalues are $1, -1, -1$, in an appropriate order. Thus $U(x)$ is self-adjoint and unitary. Counting moduli gives

$$F(x) = L(U(x)) = 2 + 3x + 4\sqrt{1 - x^2} + 2\sqrt{(1 - x)^2 + 4}$$

Full criticality reduces here to the scalar equation $F'(x) = 0$.

Proposition 5.1. *There is a unique $x_0 \in (0, 1)$ for which $U_0 = U(x_0)$ is critical on U_5. At U_0 every 2-row and 2-column orbit has a strict local maximum modulo phases, but a 3-column variation increases L. The full Hessian has inertia $(15, 9, 1)$.*

Proof. Both $U(x)$ and its phase matrix S are self-adjoint and diagonal on the nonconstant inner Fourier modes. On the remaining 2-dimensional space write

$$S = \begin{pmatrix} 1 & 2 \\ 2 & v \end{pmatrix}, \qquad v = -2 + \frac{2(1-x)}{\sqrt{(1-x)^2+4}}$$

The difference of the off-diagonal entries of S^*U is $\sqrt{1-x^2}(1-v) - 4x = \sqrt{1-x^2}F'(x)$. Hence full criticality is equivalent to $F'(x) = 0$. We have

$$F'(x) = 3 - \frac{4x}{\sqrt{1-x^2}} - \frac{2(1-x)}{\sqrt{(1-x)^2+4}}$$

Its derivative is

$$F''(x) = -\frac{4}{(1-x^2)^{3/2}} + \frac{8}{((1-x)^2+4)^{3/2}} < 0$$

Since $F'(0) > 0$ and $F'(x) \to -\infty$ as $x \to 1$, the root is unique. It is $x_0 = 0.5374606822272723\ldots$, with $L(U_0) = 11.0911152536\ldots$.

Index the border by 0 and the inner coordinates by $1, 2, 3, 4$. Cyclic symmetry leaves 3 column-pair types; self-adjointness gives the same row tests. Lemmas 2.3 and 2.5, evaluated with outward rounding, give

Direction	Curvature enclosure
Border–inner pair	$-0.8502450 < M_{01} < -0.8502447$
Adjacent inner pair	$-0.3694043 < M_{12} < -0.3694040$
Opposite inner pair	$-0.1978585 < M_{13} < -0.1978582$
3-column generator B	$0.5674395 < Q_{U_0}(B) < 0.5674400$

Here B is self-adjoint with $B_{12} = B_{23} = i$, and all other upper and diagonal entries zero. Its Frobenius norm is 2. The pair bounds are for unit generators. The negative pair bounds and positive 3-column bound prove the local assertions.

For the full inertia take $E_{jk} + E_{kj}$ and $i(E_{jk} - E_{kj})$, in that order, on the first 8 pairs $j < k$ in lexicographic order. Polarizing Q gives a 16×16 symmetric matrix. Put $l = 0.537460682227272333271039478$. Endpoint signs enclose x_0 in $[l, l + 10^{-26}]$. Interval elimination by congruence in the indicated order gives pivot 14 in $(1.58, 1.59)$ and the other 15 pivots in $(-3.02, -0.07)$. These 16 nonzero directions, together with the 9 phase zero modes, exhaust the tangent space. The root signs, curvature bounds and elimination are reproduced by `code011/check.py`. □

5.2 Further calibrations and bordered circulants

The 4 original decimal seeds in `check.py` give the following numerical refinements; inertias use zero threshold 10^{-6}.

Seed	$L/(N\sqrt{N})$	Inertia	Top curvature	Best pair
First $N = 5$	.99109	$(14, 9, 2)$	.32297	$-.26412$
Second $N = 5$	.99202	$(15, 9, 1)$	.65312	$-.19786$
$N = 6$	.98832	$(22, 11, 3)$	.65004	$-.02548$
$N = 7$	.96837	$(27, 13, 9)$	2.49105	$+.24721$

Polar projection and horizontal Newton refinement give residuals below 10^{-14}. Positive-curvature witnesses are separately interval-certified on all 4 original coordinate boxes of radius 10^{-10}. For the second seed, row order $(5, 1, 2, 4, 3)$, column order $(2, 1, 3, 4, 5)$ and dephasing recover U_0 to $2 \cdot 10^{-15}$. For the first, orders $(3, 1, 4, 2, 5)$ and $(2, 3, 5, 4, 1)$ give a symmetric matrix invariant

under simultaneous $(2\,3)(4\,5)$, to $3 \cdot 10^{-14}$. If its positive eigenvalues are u, v and negative ones $-a_j$, a Frobenius-isotropic Gaussian direction has predicted uphill probability $2.67 \cdot 10^{-6}$. It is the angular mean of $\prod_j (1 + a_j/d)^{-1/2}$, where $d = u\cos^2\theta + v\sin^2\theta$; condition on the positive-plane radius, whose square is exponential of mean 2. This explains why random direction tests can miss a saddle.

For $N = 6$, column order $(4, 2, 6, 5, 1, 3)$ and dephasing give symmetry to $3 \cdot 10^{-15}$. 9 of 20 column triples have positive curvature; the best is .32062 on original columns $(1, 4, 6)$. On that triple, the self-adjoint generator with upper entries $(1 - i, 1 + i, -1 + i)$ has $1.0604779 < Q < 1.0604782$ on the original rounding box. For $N = 7$, columns $(2, 7)$ already give an increase. No exact order-6 or order-7 critical model is asserted.

More generally, a border $x, \sqrt{1 - x^2}/\sqrt{m}$ and circulant core with Fourier eigenvalues $-x, z_1, \ldots, z_{m-1}$, $|z_k| = 1$, give a unitary of order $m + 1$. At full support, $0 < x < 1$, stationarity in x and all these phases is equivalent to full criticality: if the phase eigenvalues of the core are v_k, phase stationarity makes $\bar{v}_k z_k$ real for $k > 0$; reality of $Tr(S^*U)$ then makes v_0 real, and the remaining skew entry is $\sqrt{1 - x^2}\,\partial_x L$. Restricted phase searches alone do not give this conclusion.

5.3 The actual unstable branches

Return to the exact order-5 model $U(x)$ from Proposition 5.1. Replace its inner row by

$$(-a + \frac{i}{2}\sin t,\ c - \frac{i}{2}\cos t, -a - \frac{i}{2}\sin t,\ c + \frac{i}{2}\cos t)$$

to obtain $U(x, t)$. Its nonconstant Fourier eigenvalues are $e^{it}, -1, -e^{-it}$. Put $A = (a^2 + \sin^2 t/4)^{1/2}$, $C = (c^2 + \cos^2 t/4)^{1/2}$ and $H(x, t) = x + 4\sqrt{1 - x^2} + 8(A + C)$. This surface is the local common fixed set of $V \to PVP^{-1}$ and $V \to \bar{V}P^2$, where P cycles the 4 inner coordinates. The first equation gives a circulant block; the second makes the border block real and imposes the 3 displayed Fourier phases. Both maps preserve L, so its gradient is tangent to the surface. The induced metric and gradient equations are

$$\frac{2\,dx^2}{1 - x^2} + 2\,dt^2, \qquad \dot{x} = \frac{1 - x^2}{2} H_x, \quad \dot{t} = \frac{1}{2} H_t, \quad H_t = \sin(2t)(1/A - 1/C)$$

A fixed-corner t-rotation is a different curve: its corner stays x_0, and it cannot become flat. Starting exactly at U_0 also gives no motion. The proposition concerns the actual nonconstant gradient trajectories with backward limit U_0.

Proposition 5.2. *The 2 unstable branches through U_0 for gradient ascent converge to $U(1/\sqrt{5}, \pm t_*)$, where $t_* = \frac{1}{2}\arccos(1/\sqrt{5})$. Both limits are Fourier-equivalent.*

Proof. The restricted Hessian has opposite signs in x, t, and the full horizontal Hessian has exactly 1 positive eigenvalue. The unstable manifold theorem [26], on the phase quotient, therefore identifies these 2 surface branches with the full unstable branches. Equivariance and the trivial effective phase stabilizer fix their lifts through U_0.

Consider the positive branch in $0 < x < 1$, $0 < t < \pi/2$. On the boundary, $H \le F(x_0)$: the $t = 0$ edge is $F(x)$, the $t = \pi/2$ edge has a larger $|A^2 - C^2|$ with the same sum, and on $x = 0, 1$ use

$$H \le G(x) := x + 4\sqrt{1 - x^2} + 4\sqrt{3 + x^2}, \quad G(0) = 4 + 4\sqrt{3}, \quad G(1) = 9 < F(1/2)$$

Here also $G(0) < F(1/2) = 7/2 + 2\sqrt{3} + \sqrt{17}$. The branch has strictly larger value than $F(x_0)$ and is trapped in a compact interior superlevel set. Every forward limit point is critical. Interior criticality gives $A = C$, hence $\cos(2t) = x$, and then $G'(x) = 0$. Since $G'' < 0$, the unique solution is $x = 1/\sqrt{5}$. Reflection treats the other branch. At the positive limit, the inner row of $\sqrt{5}U$ is (w^2, w^4, w^3, w) with $w = e^{2\pi i/5}$; its cyclic shifts form the Fourier multiplication table, with row order $(2, 1, 3, 4)$ and column order $(1, 2, 4, 3)$. The negative limit is its adjoint. □

6 Order 5: numerical evidence and certification

6.1 The 7-hour campaigns

The standalone `ahc5_ascent.py` and `ahc5_stationary.py` check the full 25-dimensional Hessian and its 16-dimensional horizontal restriction at accepted nonflat stationary returns. They use flatness tolerance 10^{-6}, stationarity tolerance 10^{-10}, near-zero threshold 10^{-7} and positive-curvature threshold 10^{-7}. Up to 80 polishing steps target residual 10^{-12}. Ascent uses Haar, phase-polar, Haar-polar, near-real, near-block and near-saddle starts, with at most 1600 steps. The stationary search uses Haar, phase-polar, near-real, near-flat, near-saddle, near-block, bordered and circulant starts, and 4 steering caps $L/(5\sqrt{5}) \le .90, .95, .985, .999$, with at most 32 steering and 160 root steps. The algorithms passed the derivative, calibration and restart checks in `test_ahc5.py`. 2 Apple M3 laptops (8 GB, 3 workers each) completed 7-hour runs, with seeds 521811, 521812:

Endpoint status	Ascent	Stationary
Numerically near flat	181 164 464	2 402 576
Nonflat with positive curvature	0	18 997 063
Near zero	0	1 129 156
Unresolved	0	404 581
Total	181 164 464	22 933 376

There were no candidate, uncertain-Hessian or numerical-error flags. Every ascent endpoint satisfied $\max_{ij} \big||U_{ij}| - 1/\sqrt{5}\big| < 10^{-6}$. The smallest recorded top curvature at a nonflat stationary return was $0.206350429\ldots$; all such returns had residual below 10^{-12}. The earlier 5-minute local pilots gave 593 024 near-flat ascent returns and 105 248 stationary returns: 10 996 near-flat, 87 250 uphill, 5162 near-zero and 1840 unresolved. They are separate runs; their 3840-start test prefixes are not counted again.

One retained numerical critical point has $L = 11.081082511058\ldots$, top Hessian eigenvalue $0.206350429\ldots$, and largest row or column pair curvature $-0.226683\ldots$. Its numerical inertia is again $(15, 9, 1)$. An independent interval check proves

$$0.20635 < Q_V(B) < 0.20636$$

for a fixed rational self-adjoint generator throughout the coordinate box of radius 10^{-12} around the stored matrix. This excludes every unitary in the box as a local maximum. Theorem 6.2 certifies an exact critical point near the representative of group 39, with this inertia and negative pair curvatures. The matrix and witness are in `code011/data/ahc5-weak.json`, and the check is `code011/weak5.py`.

The near-zero returns lie below $b_5 = 9 - 4\sqrt{5} > 0.0557$ and hence cannot themselves be local maxima. The unresolved returns have residual at least $1.254 \cdot 10^{-5}$, with smallest-entry moduli reaching 0.1539657; the entry bound does not dismiss all of them. All 288 retained unresolved endpoints have numerically positive geodesic curvature, with smallest top value $3.67446\ldots$. The saved-point follow-up below explains these stalls. A subsequent partial replay addresses the original starting sequence; neither calculation asserts exhaustive coverage.

The archives `code011/overnight5/ascent-mac.zip` and `stationary-mac.zip` contain the generating sources, all-run counters and extrema, and 96 and 1076 retained endpoints. Ordinary endpoints use bounded sampling, supplemented by priority pools for weak saddles and small unresolved residuals; every candidate flag would be retained. The 2 databases total 4 460 544 bytes. `audit_ahc5.py` rechecks source hashes, counters, starts and retained endpoints, including 452 stored pairs of full and horizontal spectra and their uphill witnesses. Maximum spectral and witness discrepancies were below $6 \cdot 10^{-14}$. The audit also computes curvature at the

retained unresolved points without refining them. The unretained endpoints cannot be audited individually from these archives.

6.2 Catalogue and systematic certificates

The 452 retained nonflat stationary returns form 40 numerical groups under row and column phases, permutations and complex conjugation, using Frobenius tolerance 10^{-6}. The comparison checks all pairs of row and column permutations after dephasing; equal norms alone are not used as equivalence tests. The table gives each representative norm, its positive horizontal index p, and a lower bound q for a fixed uphill direction throughout every retained rounding box in that group. Theorem 6.2 certifies these indices and the displayed norms at exact critical points near the 40 representatives.

Group	L	p	q	Group	L	p	q
1	9.779960	6	17.279	21	10.475603	6	3.673
2	9.944272	6	3.572	22	10.475618	5	4.844
3	10.204351	7	5.864	23	10.475618	5	4.844
4	10.204611	6	4.582	24	10.475787	4	9.810
5	10.204611	6	4.582	25	10.493467	4	8.293
6	10.229297	6	6.638	26	10.599732	5	4.534
7	10.231276	5	7.045	27	10.656854	5	6.094
8	10.234788	5	3.557	28	10.658120	4	6.675
9	10.252396	6	10.519	29	10.659153	4	7.120
10	10.255519	6	3.761	30	10.662660	3	4.602
11	10.264114	6	3.507	31	10.662660	3	4.602
12	10.282845	6	4.288	32	11.000000	5	0.499
13	10.283393	5	3.930	33	11.047758	4	0.539
14	10.286257	5	6.297	34	11.048168	3	0.621
15	10.290855	5	5.992	35	11.048247	2	0.652
16	10.290855	5	2.989	36	11.061538	2	0.292
17	10.322914	4	36.252	37	11.079258	3	0.740
18	10.325913	4	6.172	38	11.080767	2	0.322
19	10.334241	4	8.302	39	11.081083	1	0.206
20	10.334241	4	8.302	40	11.091115	1	0.653

For every one of the 452 matrices, `catalogue5.py` rounds a self-adjoint uphill generator to 12 decimals and interval-evaluates $Q_U(B)$ on the real and imaginary coordinate box of radius 10^{-8}. Every lower bound exceeds .20634. These are rigorous exclusions of local maxima among unitaries in the boxes. The grouping remains numerical; these box exclusions alone do not establish critical-root existence. That additional conclusion is proved below for the 40 representatives. No completeness of the catalogue is asserted. Representatives, matching permutations and individual certificates are in `data/catalogue5.npz` and `data/catalogue5.json`.

Allowing transposition merges groups 4/5, 19/20, 22/23 and 30/31 numerically, with matching errors below $6 \cdot 10^{-11}$, leaving 36 groups. Groups 15, 16 have norms agreeing to 12 decimals, $L = 10.290855136357$, but different smallest entry moduli, $0.109049\ldots$ and $0.211227\ldots$. These comparisons are also checked by `catalogue5.py`; exact equality of these 2 norms is not asserted.

6.3 Stalled stationary searches

Write $R(U) = (S^*U - U^*S)/(2i)$ and $\Phi(U) = \|R(U)\|_F^2/2$. Restarting the residual solver on the 288 saved points, with damping reset to .01 and up to 1600 further steps, gives 3 outcomes:

Count	Outcome	Diagnostic
139	Stationary saddles	8 at $L = 10.475603$, 131 at $L = 10.475787$
134	Positive residual minimum	$L = 10.2696684693$, $\|R\|_F = 0.0164775744$
15	Near the zero-entry boundary	Smallest entry below $4.865 \cdot 10^{-7}$

All 139 in the first group had exhausted the original 160 steps and missed the polishing trigger $\|R\|_F < 10^{-5}$. Restarting needs only 7–29 steps, giving residual below 10^{-10} and top curvature above 3.67. The 15 boundary points already had an entry below $4.269 \cdot 10^{-5}$ before restarting, far below b_5. The 134 middle returns agree with one matrix up to phases, permutations and conjugation within 10^{-6}. At each, replace U by

$$U \exp\big(-0.6iR(U)/\|R(U)\|_F\big)$$

and allow 800 root steps. All reach the known level $L = 10.2641137511\ldots$, with top curvature above 3.50 and residual below 10^{-10}. Perturbations of lengths .1 and .3 returned to the residual minimum. The 3 final saddle types match already retained representatives within 10^{-6}. These are numerical observations, reproduced by `stalled5.py`, which saves the matrices, equivalences and witnesses in `data/stalled5.npz` and `data/stalled5.json`. The identity $\nabla\Phi = Hess(L)\nabla L$ explains how the residual can be stationary while L is not. The positive minimum is certified in Proposition 6.1.

6.4 Daytime follow-up and targeted search

The program `ahc5_day_replay.py` replays seed 521812 with the original batches and solver. Unresolved full-support returns above $b_5 - 10^{-6}$ receive 320 restart steps, then if necessary the length-.6 perturbation and up to 800 root steps. Entry-bound exclusions and nonstationary positive curvature are counted separately. The complementary `ahc5_day_search.py`, seed 521819, uses 8 protocols: root search near all 40 representatives, near the 8 weakest, and along their soft directions; perturbed bordered and circulant starts; geodesic and polar ascent near weak saddles; and Haar root controls. Amplitudes range from 10^{-4} to 1. Both jobs check full and horizontal Hessians and retain new signatures.

Each Mac ran for 7 active hours with 2 workers and 1 BLAS thread per worker. Unresolved identifiers and diagnostics and all candidate flags are retained, with bounded pools for ordinary matrices. Data growth stops at 112 MiB, below the 128 MiB cap. Baseline replay, standalone launches, rollback, eigenbasis compatibility, locks and exact resumption are checked by `test_day5.py`. The 18-second release checks (5600 replay and 10 032 targeted starts, no unresolved or candidate returns) are prefixes, not additional starts. The completed runs give the following counts; the lower block partitions the initial stalls already included above.

Endpoint status	Replay	Targeted search
Numerically near flat	1 696 013	10 068 469
Nonflat stationary, positive curvature	13 672 742	24 303 512
Small-entry exclusion	820 253	579 411
Unresolved or candidate flags	0	0
Total starts	16 189 008	34 951 392
Initially unresolved	284 297	235 667
Restart alone gives an uphill saddle	12 306	15 090
Escape and refinement give an uphill saddle	248 455	146 623
Excluded by the entry bound	23 536	73 954

No uncertain-Hessian, unitarity or numerical-error flags occurred. Replay covers 70.59% of the original 22 933 376 starts; 6 744 368 remain unvisited, so resolution of all 404 581 original

stalls is not claimed. Its 284 297 stalls are those observed on replay, and neither replay nor follow-up iterates add independent trials. All 8 737 840 targeted ascent returns are near flat. Every accepted nonflat stationary return matches a stored signature, using sorted moduli and horizontal spectra within $2 \cdot 10^{-5}$; both runs encounter all 40, with no new signature and least top curvature 0.206350429 . . .. Signature matching is screening, not an equivalence proof or interval certificate.

The unchanged archives `code011/daytime/replay-mac.zip` and `search-mac.zip` contain 788 and 569 retained runs, with starting, raw and final matrices. Database sizes are 31.94, 26.03 MiB; sums of parent and worker peak memory are 128.5, 157.2 MiB. The program `audit_day5.py` checks hashes, database integrity, counters, retained matrices and $452 + 356$ full/horizontal spectrum pairs and uphill witnesses. Polarization independently checks each full Hessian; all spectral and witness errors are below $6 \cdot 10^{-14}$. The adjacent `replay-mac-audit.json` and `search-mac-audit.json` record these checks, which do not audit unretained matrices.

All 788 saved replay starts and 496 targeted starts regenerate within 10^{-11}. The other 67 soft-direction and 6 ascent starts depend on eigenbasis choices. Recover $B = -i \log(U_0^* U)$ from the saved start U and catalogue centre U_0. For a soft start of length ρ, Gaussian coefficients c_1, c_2, c_3 and selected eigenvalues $\lambda_1, \lambda_2, \lambda_3$, the audit checks

$$\|P_\lambda B\|_F^2 = \rho^2 \frac{\sum_{a:\lambda_a=\lambda} c_a^2}{\sum_{a=1}^3 c_a^2}$$

Here P_λ is the horizontal Hessian eigenspace projection; eigenvalues within 10^{-7} are grouped. For ascent starts it checks $B = \rho(W + .1E)/\|W + .1E\|_F$, where E is the regenerated noise and W a unit vector in the top eigenspace. Relative norm and projection discrepancies are below $2 \cdot 10^{-12}$. These are numerical compatibility checks, not identical regeneration across numerical libraries; saved starts and generating sources are unchanged.

6.5 A positive minimum of the stationarity residual

The following certificate explains the middle group in Section 6.3. It establishes an obstruction to residual minimization, without producing a local maximum of L.

Proposition 6.1. *On U_5, Φ has a strict local minimum modulo row and column phases at a full-support point V satisfying*

$$.01647 < \|R(V)\|_F < .01649$$

In particular, this point is not stationary for L.

Proof. The following integer matrix specifies a rational self-adjoint matrix A: put $A_{jj} = 10^{-12} C_{jj}$ and $A_{jk} = 10^{-12}(C_{jk} + iC_{kj})$ for $j < k$.

$$C = \begin{pmatrix} 495152000271 & 190465601450 & -262733011052 & -124814484784 & -323386252325 \\ -377050587335 & -448869470372 & 70859178042 & -55846027846 & 1182451590433 \\ 585209765500 & -541232190289 & 464444926073 & 60765711274 & -107072278033 \\ 74541284483 & -126095931572 & 461662797924 & 455696731011 & 445244851191 \\ 48142427836 & -53078770252 & 237226558258 & 964201483854 & -530101168667 \end{pmatrix}$$

From the 20 matrices $E_{jk} + E_{kj}, i(E_{jk} - E_{kj})$, ordered lexicographically by $j < k$, omit the real directions $(1,3), (2,3), (4,5)$ and the imaginary direction $(2,5)$. Call the remaining matrices $B_1, \ldots, B_{16}$. On the cube $|x_k| \le 10^{-10}$ put

$$W = (I - i(A + \sum_k x_k B_k))^{-1}, \qquad V(x) = 2W - I$$

Put $F(x) = \Phi(V(x))$ and $H(x) = D^2F(x)$. This parametrization is exactly unitary. Differentiation uses $V_j = 2iWB_jW$ and $V_{jk} = -2W(B_jWB_k + B_kWB_j)W$. The interval computation in `stalled5.py`, at 40 decimal digits, constructs rational matrices Y, P and verifies throughout the cube

$$\|Y\nabla F(0)\|_\infty < 7 \cdot 10^{-13}, \qquad \|I - YH(x)\|_\infty < .077, \qquad \|P^tH(x)P - I\|_\infty < .075$$

Here Y approximates the inverse midpoint Hessian and P the inverse transpose Cholesky factor, with entries rounded to 16 significant decimals. The map $x - Y\nabla F(x)$ contracts the cube strictly into itself, so it has a unique interior fixed point; H is positive definite. The 16 chart tangents and 9 phase tangents are independent: for their real 25×25 coordinate matrix $T(x)$, a rational preconditioner Z gives $\|I - ZT(x)\|_\infty < 2.2 \cdot 10^{-5}$. Thus the chart is transverse to the phase orbits. Finally the same intervals give $\min |V(x)_{ij}| > .1273$ and the displayed bounds on $\|R(V(x))\|_F$. Phase invariance proves the asserted strict local minimum modulo phases. □

6.6 Certified critical points

We finish by validating the critical points behind the catalogue. The certificate concerns local existence, uniqueness and inertia; the numerical grouping is not used to infer completeness or pairwise inequivalence of all 40 entries.

Theorem 6.2. *Each of the* 40 *catalogue representatives is within* 10^{-7} *in Frobenius norm of a full-support critical point of the* 1*-norm on* U_5*. This point is locally unique modulo row and column phases, and its full Hessian has inertia*

$$(16 - p, 9, p)$$

where p *is the positive index listed in the table. Its norm rounds to the displayed value. In particular, the* 1*-norm has at least* 35 *distinct nonflat critical values. For group* 39*, all row and column pair curvatures are less than* $-.22$*.*

Proof. The rational chart data are listed in `data/critical5-input.json`. For each representative an integer matrix C defines a self-adjoint matrix A by $A_{ii} = 10^{-20}C_{ii}$ and $A_{ij} = 10^{-20}(C_{ij} + iC_{ji})$ for $i < j$. The record selects 16 matrices B_a from the canonical basis of 5 diagonal matrices, followed by $E_{jk} + E_{kj}, i(E_{jk} - E_{kj})$ for $j < k$ in lexicographic order. Its indices start at 1. On the cube $|x_a| \le r = 10^{-12}$ put

$$W(x) = \left(I - i\left(A + \sum_a x_aB_a\right)\right)^{-1}, \qquad V(x) = 2W(x) - I, \qquad f(x) = L(V(x))$$

This is an exactly unitary chart. The record also gives an integer $k \in [-16, 16]$ and the unit scalar $z = (4 + ik)/(4 - ik)$. Interval bounds throughout the cube give

$$\min_{ij} |V(x)_{ij}| > .016, \qquad f(x) < 11.1 < 5\sqrt{5}, \qquad \|V(x)/z - U^{(0)}\|_F < 10^{-7}$$

Here $U^{(0)}$ is the corresponding stored representative. Thus all points in the chart have full support and are nonflat.

The derivatives used by `critical5.py` are

$$V_a = 2iWB_aW, \qquad V_{ab} = -2W(B_aWB_b + B_bWB_a)W$$

Writing $m_{ij} = |V_{ij}|$ and $t_{a,ij} = Re(\overline{V_{ij}}(V_a)_{ij})$, these give

$$f_a = \sum_{ij} \frac{t_{a,ij}}{m_{ij}}, \qquad f_{ab} = \sum_{ij} \left(\frac{Re(\overline{(V_a)_{ij}}(V_b)_{ij} + \overline{V_{ij}}(V_{ab})_{ij})}{m_{ij}} - \frac{t_{a,ij}t_{b,ij}}{m_{ij}^3} \right)$$

Put $H(x) = D^2 f(x)$. At 50 decimal digits, interval arithmetic verifies the following bounds for all 40 charts:

$$\|Y \nabla f(0)\|_\infty < 10^{-20}, \qquad \|I - YH(x)\|_\infty < .002, \qquad \|P^t H(x) P - D\|_\infty < .001$$

Here Y, P are rational matrices, proposed by midpoint inversion and diagonalization and rounded to 17 significant digits. The diagonal matrix D has $16 - p$ entries -1 and p entries 1. The first 2 bounds make $x - Y\nabla f(x)$ a contraction of the cube strictly into itself, with a unique zero of ∇f. The third fixes the inertia of H by congruence and a nonsingular homotopy to D; numerical eigenvalue signs alone are not used as proof.

To pass to full criticality, form the real 25×25 coordinate matrix $T(x)$ of the 16 self-adjoint tangent generators $-iV^* V_a$, the 5 generators E_{jj}, and the 4 generators $V^* E_{jj} V$ with $j < 5$. Coordinates are diagonal entries followed by real and imaginary parts of upper entries. A rational preconditioner gives

$$\|I - ZT(x)\|_\infty < 10^{-6}$$

Thus the chart is transverse to the phase orbits. Phase invariance turns its critical point into a critical point on U_5, locally unique modulo phases. At that point the 9 phase directions lie in the full Hessian kernel, while its restriction to the chart has the certified nondegenerate inertia. This proves the stated full inertia.

Finally, the norm enclosure for each chart lies strictly within $5 \cdot 10^{-7}$ of the displayed value. The table contains 35 different such values, whose open rounding intervals are disjoint. For group 39, interval evaluation of Lemma 2.5 on $V(x)$ and its transpose bounds all 20 pair curvatures above by $-.22$. The individual checks and enclosures are recorded in `data/critical5.json` and reproduced by `critical5.py`. □

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