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An extremal vertex operator algebra of central charge $c = 64$

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Abstract

We construct an extremal self-dual vertex operator algebra of central charge 64. It is a simple-current extension of a strongly rational fixed-point algebra U in the tensor square of the standard self-dual Barnes–Wall lattice orbifold of central charge 32. The fusion group of U is a 56-dimensional hyperbolic quadratic space over $\mathbb{F}_2$, which we identify with the minuscule $E_7(2)$-module. Its weight-two labels are the rational points of the Freudenthal variety. In the exterior-power model, the map $\mu(u \wedge v) = uv(u + v)$ on $\mathbb{F}_{256}$ defines a maximal totally singular subspace avoiding that variety and hence gives the extremal extension. We also prove $\mathrm{Aut}(U) \cong E_7(2)$ and determine the six distinct irreducible-module characters, whose five nonvacuum lowest conformal weights are 2, 3, 4, 7/2, 9/2.

1 Introduction

We construct an extremal self-dual vertex operator algebra (VOA) of central charge 64.

1.1 Definitions and main results

All VOAs and their modules are over $\mathbb{C}$. By a *strongly rational* VOA we mean a VOA V that is simple, rational, C_2-cofinite, self-contragredient, and graded as $V = \bigoplus_{n\geq 0} V_n$ with $V_0 = \mathbb{C}\mathbf{1}$. For an ordinary or twisted module M, write $h(M)$ for its lowest L_0-eigenvalue, without the shift by $c/24$. As in my thesis ([Höh95], Definition 2.1.1), a strongly rational VOA V is called *self-dual* (also called *holomorphic*) if, up to isomorphism, its adjoint module V is its only irreducible ordinary module.

Extremality is defined as in [Höh95], Definition 5.2.1: the graded character agrees with the vacuum Virasoro character through weight $\lfloor c/24 \rfloor$ or, equivalently, there are no nonvacuum Virasoro highest-weight vectors of positive weight at most $\lfloor c/24 \rfloor$. A self-dual VOA V of central charge 64 is extremal if and only if

$$V_1 = 0, \qquad V_2 = \mathbb{C}\omega,$$

where ω is the Virasoro element of V.

Theorem 1.1. *Let $W = \widetilde{V}_{\mathrm{BW}_{32}}$ be the standard self-dual (-1)-orbifold, let $A = O_2(\mathrm{Aut}(W)) \cong 2^{27}$, and let σ exchange the factors of $W \otimes W$. The fixed-point algebra*

$$U = (W \otimes W)^{\Delta A \times \langle \sigma \rangle}$$

is strongly rational of central charge 64*, with* $U_1 = 0$ *and* $U_2 = \mathbb{C}\omega$*. Its module category is pointed, with fusion quadratic space*

$$(D,\, q) \cong (\mathbb{F}_2^{56},\, q^+), \qquad q([M]) = 2h(M) \pmod 2.$$

Here q^+ *denotes a nondegenerate hyperbolic quadratic form. The natural action on* D *identifies* $\mathrm{Aut}(U)$ *with* $E_7(2)$ *in its* 56*-dimensional minuscule representation. Its five nonzero vector orbits have lowest conformal weights* 2, 3, 4, 7/2, 9/2*. The modules of weight two correspond exactly to the rational points of the Freudenthal variety, and the six module characters are given in Theorem 7.2.*

For $d \in D$, write U_d for a representative of the corresponding simple U-module, with $U_0 = U$.

Theorem 1.2. *There is a maximal totally singular subspace* $H \subset D$ *containing no label of a weight-two module. The corresponding simple-current extension*

$$V_H = \bigoplus_{d \in H} U_d$$

is strongly rational and self-dual, with

$$c(V_H) = 64, \qquad (V_H)_1 = 0, \qquad (V_H)_2 = \mathbb{C}\omega.$$

In particular, V_H *is extremal.*

Since $64 \equiv 16 \pmod{24}$, every homogeneous component $(V_H)_n$ is a conformal 3-design ([Höh08a], Theorem 3.1).

The construction follows the extension principle developed in my paper ([Höh03], Section 4, Theorems 4.2 and 4.3): a self-perpendicular isotropic subgroup of the fusion quadratic space gives a self-dual simple-current extension. We use the strongly rational formulation of [vEMS20], Theorem 4.2.

The fusion-space identification uses the lowest-weight constituents of the permutation-twisted modules for $W \otimes W$. Their mutual fusion weights recover the cubic norm of the split Albert algebra and hence the Freudenthal coordinates.

For the extension, write $E = \mathbb{F}_{256}$ as an eight-dimensional $\mathbb{F}_2$-space. In the exterior-power model $\Lambda^2 E^* \oplus \Lambda^2 E$, define

$$\mu : \Lambda^2 E \longrightarrow E, \qquad \mu(u \wedge v) = uv(u + v).$$

The subspace $\operatorname{im}\mu^* \oplus \ker\mu$ is maximal totally singular. A rank bound for its trace forms, together with the quadratic equations of the Freudenthal variety, proves that it avoids every rational rank-one vector.

The complete weight-two locus gives an upper bound $E_7(2)$ for $\mathrm{Aut}(U)$. A permutation of four copies of $V^+_{\mathrm{BW}_{16}}$ supplies the additional automorphism needed to attain that bound. The symmetry organizes the module characters into six functions.

Conventions. For a strongly rational VOA V, write $\mathrm{Rep}(V)$ for its modular tensor category of ordinary modules ([Hua08], Theorem 4.6), $\mathrm{Irr}(V)$ for the set of isomorphism classes of simple modules, and $\boxtimes_V$ for the tensor product. The notation M^* denotes the contragredient module. Self-duality is equivalent to $\mathrm{Rep}(V) \simeq \mathrm{Vec}$ and is stronger than self-contragredience $V \cong V^*$, which is already assumed in the definition of strong rationality. For the grading specified above, rationality together with C_2-cofiniteness is equivalent to regularity ([ABD04], Theorem 4.5). We write $Y(\omega, z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$. A Virasoro highest-weight vector is a nonzero homogeneous vector annihilated by every L_n, $n > 0$. For references with a preprint link, statement numbers follow that version unless indicated otherwise.

1.2 Extremal VOAs and related literature

In my thesis ([Höh95], Section 5.2), I discussed extremal self-dual VOAs of central charges 8, 16, 24, 32, 40; at that time self-duality of the indicated rank-32 and rank-40 orbifolds was still conjectural. I recorded that no extremal VOA of central charge $c \geq 48$ was then known. These examples were subsequently summarized in my conformal-design paper ([Höh08a], Example 3.2). To my knowledge, Theorem 1.2 gives the first new central-charge value for an extremal self-dual VOA since those examples.

Following Witten [Wit07], extremality was studied through modular reconstruction [Man07, MM10, MW10, Ash20], candidate characters for $c = 8m$ [AKM07], higher-genus identities [GY07, Yin08, GKV10], and modular differential equations [Gab07, Gai12, GK08]. Keller, Mathys and Zadeh ([KMZ18], Section 5.6 and Appendix B.2) treated $c = 64$; their extremal specialization gives 278511 weight-three Virasoro highest-weight vectors, in agreement with (7.10).

In my preprint ([Höh08b], Section 4, Corollary 9 and Theorem 11), I ruled out the Monster-module structures suggested by Witten at $c = 48$, under the hypotheses stated there; see also [Gai12]. Restrictions from topological modular forms [LP23] use additional hypotheses. Superconformal variants [GGK$^+$08, BDFK15, Har16] concern a different extremality condition.

Gemünden and Keller [GK19, GK21] constructed non-extremal self-dual lattice orbifolds at $c = 48$ and $c = 72$. The proposed $c = 128$ construction in [KRR26] assumes the vanishing of an anomaly 3-cocycle and has nonvacuum minimum 4; extremality would require minimum 6. Ji and Tian [JT26] construct full, nonchiral CFTs by asymmetric lattice orbifolds, with exact Virasoro-primary gaps 3 and 4 in total dimension $h + \bar{h}$ and with unbounded central charge.

1.3 Extremal even unimodular lattices

Extremal VOAs were introduced as an analogue of extremal even lattices and extremal doubly-even codes.

Known extremal even unimodular lattices include Nebe's rank-48 example [Neb14], rank-64 examples of Quebbemann [Que84] and Shimada [Shi18, Shi22], Nebe's rank-72 example [Neb12], Bachoc and Nebe's two rank-80 examples [BN98], and Sakamoto's two rank-88 examples ([Sak26], Theorem 1.1). A computational construction in rank 112 has been announced in [roi26]. These results do not directly give extremal VOAs in these central charges: a lattice VOA has a nonzero Heisenberg space in weight one, and its standard (-1)-orbifold contains the $n(n+1)/2$-dimensional quadratic Heisenberg space in weight two.

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2 The fixed-point vertex operator algebra

Let $W = \widetilde{V}_{\mathrm{BW}_{32}}$ be the standard self-dual (-1)-orbifold of the Barnes–Wall lattice VOA. Put

$$A = O_2(\mathrm{Aut}(W)), \qquad V = W \otimes W, \qquad G = \Delta A \times \langle \sigma \rangle, \qquad U = V^G,$$

where $\Delta A = \{(a, a) : a \in A\}$ and σ exchanges the two tensor factors. Thus $G \cong (\mathbb{Z}/2)^{28}$. We first record the properties of W needed to analyze U.

I recorded the expected automorphism group of W in my conformal-design paper ([Höh08a], Example 3.2), describing the group as likely to be $2^{27}{:}E_6(2)$. Shimakura ([Shi14], Theorem 2.30)

subsequently proved the group shape $2^{27}.E_6(2)$. For the detailed structural input used here we refer to Shimakura ([Shi14], Proposition 1.8, (2.1), Proposition 2.11, Theorems 2.19 and 2.30):

$$c(W) = 32, \quad W_1 = 0, \quad \dim W_2 = 139504, \quad A = O_2(\mathrm{Aut}\, W) \cong 2^{27}, \quad \mathrm{Aut}(W)/A \cong E_6(2). \tag{2.1}$$

The VOA W is strongly rational and self-dual: the lattice VOA is strongly rational, and its (-1)-twisted minimum is $32/16 = 2$. The three nontrivial $V^+_{\mathrm{BW}_{32}}$-modules have lowest weights 1, 2, 5/2, respectively. Thus the cyclic type is $4 \cdot 2 \equiv 0 \pmod 2$, all nontrivial minima are positive, and the integral twisted summand is isotropic. The cyclic orbifold theorem ([vEMS20], Theorem 5.16) therefore applies. The quotient $\mathrm{Aut}(W)/A$ acts irreducibly on A ([Shi14], Proposition 2.11) and is transitive on the 139503 nonzero A-eigenspaces in the complement of $\mathbb{C}\omega_W$, one of which is one-dimensional with a nontrivial character ([Shi14], Section 2.4, Lemma 2.14 and Theorem 2.19). Let $S \subset A^*$ be the set of nontrivial A-characters occurring in W_2. Then

$$W_2 = \mathbb{C}\omega_W \oplus \bigoplus_{\chi \in S} \mathbb{C}_\chi, \qquad |S| = 139503, \tag{2.2}$$

where $\mathbb{C}_\chi$ is a one-dimensional A-module of character χ, and S is a single $\mathrm{Aut}(W)/A$-orbit. Here and below $2^{27}.E_6(2)$ denotes group shape; no splitting of this extension is assumed.

Since G is solvable, the fixed-point theorem ([CM18], Theorem 5.24 and Corollary 5.25), together with Theorem 4.5 of [ABD04], shows that U is strongly rational. Its conformal vector is that of V, so $c(U) = 64$, and

$$U_1 = 0, \qquad U_2 = \mathbb{C}(\omega_W \otimes \mathbf{1} + \mathbf{1} \otimes \omega_W). \tag{2.3}$$

Indeed ΔA-invariance leaves just the two conformal vectors in V_2, and σ-invariance leaves their sum.

3 Twisted modules and the fusion quadratic space

3.1 The Albert module

We use the split Albert module

$$J = M_3(\mathbb{F}_2) \oplus M_3(\mathbb{F}_2) \oplus M_3(\mathbb{F}_2)$$

with invariant cubic

$$N(X, Y, Z) = \det X + \det Y + \det Z + \mathrm{tr}(XYZ). \tag{3.1}$$

This is the three-matrix model in Cohen–Cooperstein [CC88]; their minus sign equals a plus sign in characteristic two. The differential

$$a^{\#} = dN_a \in J^*$$

is always regarded as a *covector*. Pairings are the ordinary coordinate pairing $J^* \times J \to \mathbb{F}_2$. No identification of J with J^*, and no dual-adjoint convention, is needed.

In characteristic two the only nontrivial irreducible $E_6(2)$-modules of degree at most 27 are the dual modules with highest weights λ_1 and λ_6; see [Lüb01], Theorems 2.3 and 4.4, Appendix A.51. The same degree bound shows that the irreducible 27-dimensional $\mathbb{F}_2$-module A is absolutely irreducible: a nontrivial decomposition after extending scalars would have constituents of smaller nontrivial degree. After possibly composing the chosen quotient

identification with the graph automorphism, we may therefore identify A with J and A^* with J^*.

There are three nonzero orbits, described by

rank	condition	orbit size
1	$a \neq 0,\ a^{\#} = 0$	139503
2	$a^{\#} \neq 0,\ N(a) = 0$	69193488
3	$N(a) = 1$	64884736

. (3.2)

The rank-one stabilizer has shape $2^{16}.\Omega_{10}^{+}(2)$, and the rank-three stabilizer is $F_4(2)$ [CC88]. The dual module has the same orbit sizes. Consequently S in (2.2) is the dual rank-one orbit; its stabilizer is also identified in [Shi14], Corollary 2.24.

3.2 Twisted conformal weights

For the unique a-twisted W-module $W(a)$, set $\rho_a = h(W(a))$, with $\rho_0 = 0$.

Put $R = V_{\mathrm{BW}_{16}}^{+}$. Its irreducible modules R_s form a ten-dimensional plus-type quadratic fusion space (S_{10}, q_{10}), whose polar form is denoted by b_{10}. Their lowest weights are

$$h(R_s) = \begin{cases} 0, & s = 0, \\ 1, & s \neq 0,\ q_{10}(s) = 0, \\ 3/2, & q_{10}(s) = 1. \end{cases} \tag{3.3}$$

There are 527 nonzero singular labels and 496 nonsingular labels; the singular nonvacuum modules have 16-dimensional weight-one spaces. The latter dimension also follows from transitivity on nonzero singular labels and $(V_{\mathrm{BW}_{16}}^{-})_1 \cong \mathbb{C} \otimes_{\mathbb{Z}} \mathrm{BW}_{16}$; see [Shi14], Proposition 1.8. That proposition also gives rationality and C_2-cofiniteness of R. Its self-contragredience follows from the nondegenerate invariant bilinear form on the lattice VOA: averaging over the involution shows that the restriction to R is nondegenerate. Thus R is strongly rational, and all its nontrivial irreducible modules have positive lowest weight. Shimakura's decomposition ([Shi14], (2.1)) gives the full diagonal extension

$$W = \bigoplus_{s \in S_{10}} R_s \otimes R_s. \tag{3.4}$$

Permutation symmetries of simple-current extensions are treated more generally in [LY26], Section 5. Let ι be the literal flip on each summand in (3.4). It is a VOA automorphism. Indeed, after choosing the same intertwining operator in the two identical factors, every projected component of the vertex operator is a scalar multiple of $I_{s,t} \otimes I_{s,t}$, by the tensor-product theorem for intertwining operators ([ADL05], Theorem 2.10) and fusion multiplicity one. Swapping the two tensor positions commutes with this expression. Shimakura's descriptions (2.2) and (2.4) place this flip in A.

For $t \in S_{10}$, let $\gamma_t \in A$ act on the s-summand by $(-1)^{b_{10}(s,t)}$. Write

$$\mathrm{ch}_s(z) = \mathrm{tr}_{R_s} \exp(2\pi i z(L_0 - 16/24)).$$

For $g \in \mathrm{Aut}(W)$, write $Z_W(g;\, z) = \mathrm{tr}_W(g \exp(2\pi i z(L_0 - 32/24)))$. The trace function and its modular transform are

$$Z_W(\iota\gamma_t;\, z) = \sum_s (-1)^{b_{10}(s,t)}\, \mathrm{ch}_s(2z), \tag{3.5}$$

$$Z_W(\iota\gamma_t;\, -1/z) = 32\ \mathrm{ch}_t(z/2). \tag{3.6}$$

For the second identity, use $S_{s,t} = 2^{-5}(-1)^{b_{10}(s,t)}$ ([vEMS20], Theorem 3.4) and Fourier orthogonality on the group of order 2^{10}. Modular covariance and its normalization ([vEMS20], Theorem 2.3 and Proposition 5.5) identify the right-hand side of (3.6) with the character of the unique $(\iota\gamma_t)$-twisted W-module. Comparing leading exponents gives

$$\rho_{\iota\gamma_t} - \frac{32}{24} = \frac{1}{2}\left(h(R_t) - \frac{16}{24}\right), \qquad \rho_{\iota\gamma_t} = 1 + \frac{h(R_t)}{2}.$$

Thus the three possible absolute weights are 1, 3/2, 7/4.

Proposition 3.1. *The twisted lowest weights are*

$$\rho_a = \begin{cases} 1, & \operatorname{rank}(a) = 1, \\ 3/2, & \operatorname{rank}(a) = 2, \\ 7/4, & \operatorname{rank}(a) = 3. \end{cases} \qquad 4\rho_a \equiv N(a) \pmod 2.$$

For rank-one a, $\dim W(a)_1 = 32$.

Proof. The traces of $\iota\gamma_t$ on W_2, for the three types of t in (3.3), are respectively

$$8432, \qquad 240, \qquad -272.$$

The vacuum summand contributes zero flip trace in total weight two. Each nonzero singular summand contributes 16 times its sign. For $t \neq 0$, the singular Fourier sum is $16(-1)^{q_{10}(t)} - 1$; for $t = 0$, it is 527. The three distinct traces must represent the three nonzero $E_6(2)$-orbits in A.

Since $\dim W_2^A = 1$, character averaging gives

$$139504 + 139503\, t_1 + 69193488\, t_2 + 64884736\, t_3 = 2^{27}, \tag{3.7}$$

where t_i is the trace on the rank-i orbit. Among the six permutations of 8432, 240, -272, the unique solution is

$$(t_1,\, t_2,\, t_3) = (8432,\, 240,\, -272).$$

Together with (3.6), this assigns twisted weights 1, 3/2, 7/4 to ranks one, two, and three, respectively.

For $t = 0$, the leading coefficient of ch_0 is one, so (3.6) gives $\dim W(\iota)_1 = 32$. Conjugacy gives the same dimension throughout the rank-one orbit. □

3.3 An elementary abelian orbifold criterion

Let a finite group B act faithfully on a strongly rational self-dual VOA $\mathcal{V}$, with $\mathcal{V}^B$ strongly rational. The B-graded category of twisted $\mathcal{V}$-modules has one invertible simple object $\mathcal{V}(b)$ in each homogeneous component ([McR21], Theorems 4.15 and 4.17). Its associator defines a class

$$\mathfrak{a}_B(\mathcal{V}) \in H^3(B,\, \mathbb{C}^\times),$$

with trivial coefficient action. We call this the *orbifold 3-cocycle class*; the group grading is part of the definition. Restriction to the components indexed by a subgroup gives the restriction of this class.

We use the following restriction calculation for elementary abelian 2-groups.

Lemma 3.2. *Let $\Gamma = (\mathbb{Z}/2)^r$. Restriction to all subgroups of order two gives an injective map*

$$H^3(\Gamma,\, \mathbb{C}^\times) \longrightarrow \prod_{0 \neq v \in \Gamma} H^3(\langle v\rangle,\, \mathbb{C}^\times).$$

With the latter factors identified with $\mathbb{F}_2$, its image consists of the functions

$$f(v) = \sum_i c_i v_i + \sum_{i<j} c_{ij} v_i v_j + \sum_{i<j<k} c_{ijk} v_i v_j v_k. \tag{3.8}$$

Proof. The split integral homology Künneth theorem gives

$$H_1(\Gamma, \mathbb{Z}) \cong (\mathbb{Z}/2)^r, \qquad H_2(\Gamma, \mathbb{Z}) \cong (\mathbb{Z}/2)^{\binom{r}{2}}, \qquad H_3(\Gamma, \mathbb{Z}) \cong (\mathbb{Z}/2)^{r+\binom{r}{2}+\binom{r}{3}}.$$

The splitting excludes $\mathbb{Z}/4$-extensions.

Since $\mathbb{C}^\times$ is divisible, the universal coefficient theorem identifies $H^3(\Gamma, \mathbb{C}^\times)$ with $\mathrm{Hom}(H_3(\Gamma, \mathbb{Z}), \mathbb{C}^\times)$. It is therefore elementary abelian of the displayed rank.

For x, y, z in Γ, consider the normalized multiplicative cocycles

$$(-1)^{x_i y_i z_i}, \qquad (-1)^{x_i y_j z_j} \quad (i<j), \qquad (-1)^{x_i y_j z_k} \quad (i<j<k).$$

Their exponents are trilinear over $\mathbb{F}_2$, so the cocycle equation holds directly. On the cyclic subgroup generated by v, they restrict to the standard nontrivial C_2-cocycle to the powers v_i, $v_i v_j$, $v_i v_j v_k$, respectively. In particular a normalized coboundary evaluates to 1 on (v, v, v), whereas the nontrivial normalized C_2-cocycle evaluates to -1.

The squarefree Boolean monomials of degrees one, two, and three are linearly independent as functions on $\mathbb{F}_2^r$. Thus these cocycle classes are independent; their number equals the computed dimension of H^3, so they form a basis. Both claims follow. □

For an involution g of a strongly rational self-dual VOA $\mathcal{V}$, cyclic orbifold theory gives $4h(\mathcal{V}(g)) \in \mathbb{Z}$ and assigns the type

$$t_g = 4h(\mathcal{V}(g)) \pmod 2; \tag{3.9}$$

see [vEMS20], Theorem 5.11, definition of type, and Theorem 5.13. Its identification with $\mathfrak{a}_{\langle g\rangle}(\mathcal{V})$ can be checked directly in the order-two case. Write that restriction as $s[\epsilon]$, where $\epsilon(i, j, k) = (-1)^{ijk}$. A simple object of nontrivial C_2-degree in $Z(\mathrm{Vec}_{C_2}^{\epsilon^s})$ has a projective centralizer scalar u satisfying $u^2 = (-1)^s$, by the half-braiding hexagon. Its twist square is therefore $(-1)^s$. The VOA twist square, for a constituent of $\mathcal{V}(g)$, is

$$\exp(4\pi i h) = \exp(4\pi i h(\mathcal{V}(g))) = (-1)^{4h(\mathcal{V}(g))},$$

since constituent weights differ by half-integers and $4h(\mathcal{V}(g)) \in \mathbb{Z}$ ([vEMS20], Theorem 5.11). This proves $s = t_g$. A possible inverse-cocycle convention has no effect on a class of order two. The ribbon twist here is $\exp(2\pi i h)$, not the character T-eigenvalue $\exp(2\pi i(h - c/24))$.

Proposition 3.3 (An elementary abelian orbifold criterion)**.** *Let $\mathcal{V}$ be a strongly rational self-dual VOA and let $\Gamma \subset \mathrm{Aut}(\mathcal{V})$ be a finite elementary abelian 2-group. Suppose that*

$$h(\mathcal{V}(g)) \in \tfrac{1}{2}\mathbb{Z}_{>0} \qquad (1 \neq g \in \Gamma).$$

Then $\mathcal{V}^\Gamma$ is strongly rational, every nonvacuum irreducible $\mathcal{V}^\Gamma$-module has positive lowest weight, and

$$\mathfrak{a}_\Gamma(\mathcal{V}) = 0, \qquad \mathrm{Rep}(\mathcal{V}^\Gamma) \simeq Z(\mathrm{Vec}_\Gamma)$$

as ribbon categories. In particular, its fusion group is elementary abelian of order $|\Gamma|^2$, with hyperbolic quadratic form $[M] \mapsto 2h(M) \pmod 2$.

Proof. Since Γ is solvable, fixed-point regularity ([CM18], Theorem 5.24 and Corollary 5.25), together with Theorem 4.5 of [ABD04], gives strong rationality of $\mathcal{V}^\Gamma$. Every simple $\mathcal{V}^\Gamma$-module is a constituent of some $\mathcal{V}(g)$, by [McR21], Theorems 4.15 and 4.17. The nontrivial twisted modules have positive minima by hypothesis, and only the vacuum constituent of $\mathcal{V}|_{\mathcal{V}^\Gamma}$ contains weight zero. This proves positivity.

For every nonidentity g, the cyclic type $4h(\mathcal{V}(g)) \pmod 2$ is zero. Thus Lemma 3.2 gives $\mathfrak{a}_\Gamma(\mathcal{V}) = 0$. The center-description theorem ([DNR25], Theorem 6.2(4) and Remark 7.2) now applies. Positivity identifies categorical dimensions with positive Frobenius–Perron dimensions and hence identifies the spherical structure and conformal-weight ribbon twist. The simple objects of $Z(\mathrm{Vec}_\Gamma)$ are the pairs (g, χ), with componentwise product and twist $\chi(g)$. This proves the assertions about the fusion group and quadratic form. □

3.4 Application to the doubled action

Write $d_a = (a, a)$ and $p_a = (a, a)\sigma$. The nonidentity elements of G are d_a for $a \neq 0$ and p_a for all a. Their twisted lowest weights are

$$h(V(d_a)) = 2\rho_a, \qquad h(V(p_a)) = \frac{c(W)}{16} = 2. \tag{3.10}$$

Indeed, $V(d_a) = W(a) \otimes W(a)$. Also p_a is conjugate to σ by $(a, 1)$, and a transposition twist satisfies $L_0^{\mathrm{tw}} = \frac{1}{2}L_0 + c(W)/16$, where the central charge is that of one tensor factor ([BDM02], proof of Theorem 3.9(3), Theorem 4.4). The lowest-weight space of each $V(p_a)$ is one-dimensional.

Proposition 3.4. *The orbifold 3-cocycle class $\mathfrak{a}_G(V)$ is zero, and*

$$\mathrm{Rep}(U) \simeq Z(\mathrm{Vec}_G), \qquad D = \mathrm{Irr}(U) \cong \mathbb{F}_2^{56}. \tag{3.11}$$

This is a ribbon equivalence. Every simple object is invertible, and every nonvacuum irreducible module has positive lowest weight. The binary conformal-weight form

$$q([M]) = 2h(M) \pmod 2, \qquad b(d, e) = q(d + e) + q(d) + q(e) \tag{3.12}$$

is nondegenerate and hyperbolic. In particular, $q(d) = 0$ means integral conformal weight.

Proof. Every nonidentity element of G is either d_a, $a \neq 0$, or p_a. Proposition 3.1 and (3.10) show that their twisted modules have positive half-integral minima. Proposition 3.3 therefore applies to the full group G. □

The *twisting-label homomorphism* ν gives the canonical exact sequence

$$0 \longrightarrow G^* \longrightarrow D \xrightarrow{\nu} G \longrightarrow 0, \qquad (-1)^{b(\chi, d)} = \chi(\nu(d)).$$

Here $\chi \in G^*$, $d \in D$, and $\nu([M]) = g$ when M is a constituent of $V(g)|_U$. The kernel G^* labels the eigenspaces in $V|_U$. A compatible identification $D \cong G \oplus G^*$ is noncanonical; in such coordinates the twist of (g, χ) is $\chi(g)$.

4 The Freudenthal module and the weight-two locus

The fusion category determines conformal weights modulo $\mathbb{Z}$. To identify the weight-two locus, we first determine certain absolute fusion weights.

4.1 Permutation-twisted modules and absolute fusion weights

For $a \in A$, let F_a be the unique simple U-module containing the one-dimensional subspace $V(p_a)_2$ of $V(p_a)$. Then

$$h(F_a) = 2, \quad q(F_a) = 0, \quad \nu(F_a) = p_a.$$

The automorphism $r_c = (c, 1)$ normalizes G and sends F_a to F_{a+c}.

Proposition 4.1. *For all a and b in A,*

$$h(F_a \boxtimes_U F_b) = 2\rho_{a+b}. \tag{4.1}$$

To prove the proposition, put

$$X = W^A, \qquad Y = (X \otimes X)^\sigma = V^{\mathcal{K}}, \qquad \mathcal{K} = (A \times A) \rtimes \langle\sigma\rangle.$$

Then $G \trianglelefteq \mathcal{K}$, $\mathcal{K}/G \cong A$, and each r_c fixes Y pointwise. As in Section 2, the finite 2-group orbifold theorems ([CM18], Theorem 5.24 and Corollary 5.25; [ABD04], Theorem 4.5) show that X and Y are strongly rational. Completeness of twisted-module constituents and the corresponding equivariantization equivalence follow from [McR21], Theorems 4.15 and 4.17.

Every nonvacuum X-module has positive weight: this follows from Proposition 3.1 for twisted constituents, and from $W_0 = \mathbb{C}\mathbf{1}$ for untwisted ones. The untwisted Y-modules arise from $P \otimes Q$, with P and Q in $\mathrm{Irr}(X)$; only the symmetric vacuum constituent contains weight zero. Permutation-twisted constituents have weights at least $2 + h(P)/2 > 0$. Thus every nonvacuum Y-module also has positive weight.

For the full twisted module $T_\sigma(X)$, the fusion tensor square over $X \otimes X$ is determined by [DXY25], Theorem 4.6. We need the following statement for its lowest-weight Y-constituent.

Lemma 4.2 (Lowest-weight summands of a tensor square)**.** *Let T be the simple Y-constituent of $T_\sigma(X)$ containing its one-dimensional homogeneous subspace of weight $c(X)/16$. Then $T \cong T^*$, and for every $P \in \mathrm{Irr}(X)$ the fusion product $T \boxtimes_Y T$ contains a simple summand of lowest weight $2h(P)$, arising from $(P \otimes P^*)|_Y$.*

Proof. The category $\mathrm{Rep}(Y)$ is rigid and semisimple. All modules used below restrict to finite-length ordinary Y-modules.

Assign the modules $T_\sigma(X)$, $P \otimes P^*$, $T_\sigma(X)$ to the marked points 0, 1, ∞ of the base sphere, with monodromies σ, 1, σ. The connected covering is $z = w^2$; the modules at the marked points 0, 1, -1, ∞ of the covering sphere are X, P, P^*, X. Propagation of the vacuum modules reduces the conformal block to the contragredient pairing between P and P^*. Here conformal blocks and propagation are used in the algebraic-analytic definitions of [Gui26], Sections 1.2 and 1.3.

Choose lowest-weight vectors p, p^* with $\langle p^*, p\rangle \neq 0$. The cover block is nonzero on $(\mathbf{1}, p, p^*, \mathbf{1})$. At the two unramified marked points the changes to lifted coordinates have nonzero derivatives; positive Virasoro modes annihilate lowest-weight vectors, so these changes only multiply the value by nonzero scalars. Propagation of the vacuum vectors also preserves its nonzero value.

Gui's cover theorem ([Gui26], Theorem 2.25) identifies the base and cover blocks as the *same underlying linear functional*, not merely as abstract vector spaces. Therefore the base block is nonzero on $(t, p \otimes p^*, t)$, where t spans the lowest-weight space of $T_\sigma(X)$.

By [Gui26], Theorem 4.10, this block determines a twisted intertwining operator. Restricting the vertex operators to Y gives an ordinary Y-intertwining operator by [Gui26], Remark 4.6: the relevant L_0-eigenspaces are finite-dimensional and the spectrum is bounded below. The conformal vector is unchanged. Restrict the first and third entries to T. This lowest-weight coefficient remains nonzero. Decomposing $p \otimes p^*$ among simple Y-constituents

gives a constituent M for which the three-point block with T, T is nonzero. Its lowest weight is exactly $2h(P)$: this weight occurs in M, whereas the entire ambient module $P \otimes P^*$ has no smaller weight.

Contragredient compatibility ([Gui26], Theorem 2.23) makes $T_\sigma(X)$ self-contragredient. Its nondegenerate graded pairing pairs its unique lowest-weight line with itself; hence $T^* \cong T$. The restricted three-point block therefore gives a nonzero intertwining operator of type $\binom{T}{M\ T}$. Rigidity gives

$$0 \neq \mathrm{Hom}_Y(M \boxtimes_Y T, T) \cong \mathrm{Hom}_Y(M, T \boxtimes_Y T^*).$$

By semisimplicity, M is a simple summand of $T \boxtimes_Y T$, as required. □

Proof of Proposition 4.1. Step 1: the simple-current extension $Y \subset U$. Write $W = \bigoplus_{\chi \in A^*} W_\chi$ as an X-module. Its summands are simple currents indexed by A^*, and

$$U = \bigoplus_{\chi \in A^*} (W_\chi \otimes W_\chi)^+ \tag{4.2}$$

is a simple-current extension of Y with $|A|$ summands. Indeed, $\mathcal{K}/G \cong A$ acts on U, its fixed algebra is Y, and $(c, 1)$ acts on the χ-summand by $\chi(c)$. The tensor subcategory generated by these summands is $\mathrm{Rep}(A)$ ([DNR25], Theorem 6.1). Consequently each summand in (4.2) is invertible in $\mathrm{Rep}(Y)$. Moreover U, as an algebra object in this subcategory, corresponds to $\mathrm{Fun}(A)$ ([DNR25], Theorem 6.3). In particular the multiplication of any two character summands is nonzero.

Step 2: the restriction of F_a. In the permutation-twisted module $V(\sigma) = T_\sigma(W)$, the underlying space is W, with grading $L_0^W/2 + 2$. The construction ([BDM02], (2.7), (3.13) and Theorem 3.9) lifts d_a to a and σ to $(-1)^{L_0^W}$; these commuting operators fix the lowest-weight line. By the equivariantization above, their common fixed space is $F_0 = \bigoplus_{m \geq 0} X_{2m}$. Restriction of the same construction to $X \otimes X$ identifies this space, as a Y-module, with the constituent T of $T_\sigma(X)$ containing its lowest-weight line. Since r_a fixes Y pointwise and sends F_0 to F_a,

$$\mathrm{Res}_Y^U F_a \cong T \qquad (a \in A). \tag{4.3}$$

Step 3: local induction. Every simple-current summand K of U acts nontrivially on the simple underlying object $T = \mathrm{Res}\, F_0$: otherwise the module axiom and the nonzero product $K^{-1} \boxtimes K \to \mathbf{1}$ would contradict the unit action. Thus $K \boxtimes T \cong T$. Since $\theta_K = 1$, balancing shows that T centralizes U, as do $T \boxtimes_Y T$ and all its simple summands. Free induction is consequently local and braided monoidal ([CKM24], Proposition 2.65 and Theorem 2.67). The local-module tensor product agrees with VOA fusion by [CKM24], Theorem 3.65, since Y and U are strongly rational and U is a finite object of $\mathrm{Rep}(Y)$.

By induction–restriction adjunction ([CKM24], Lemma 2.61) and (4.3), each of the $|A|$ distinct modules F_a occurs once in $\mathrm{Ind}_Y^U T$. They exhaust it because $\mathrm{Res}\,\mathrm{Ind}\, T = U \boxtimes_Y T \cong T^{\oplus |A|}$. Monoidality now gives

$$\mathrm{Ind}_Y^U T \cong \bigoplus_{a \in A} F_a, \qquad \mathrm{Ind}(T \boxtimes_Y T) \cong \left(\bigoplus_a F_a\right) \boxtimes_U \left(\bigoplus_b F_b\right).$$

Step 4: the absolute conformal weight. Fix $u \neq 0$, and choose an X-constituent P of $W(u)$ containing a lowest-weight vector. Then $h(P) = \rho_u$, and P^* also occurs in $W(u)|_X$, since $u^{-1} = u$. Lemma 4.2 gives a constituent M of $T \boxtimes_Y T$, also arising in $W(u) \otimes W(u)$, with $h(M) = 2\rho_u$. The unit summand of U gives an injective Y-map $M \to \mathrm{Res}\,\mathrm{Ind}\, M$; thus some simple U-summand N of $\mathrm{Ind}\, M$ contains a vector of weight $2\rho_u$.

We claim that $\nu(N) = d_u$. Using $Y = V^{\mathcal{K}}$, put $g = \nu(N)$. Then

$$\mathrm{Hom}_U(\mathrm{Ind}\, M, N) \cong \mathrm{Hom}_Y(M, \mathrm{Res}\, N) \neq 0.$$

Thus M occurs in both $V(g)|_Y$ and $V(d_u)|_Y$. Twisted modules labeled by different $\mathcal{K}$-conjugacy classes have disjoint simple Y-constituents ([DNR25], Theorem 5.1(1)). Since d_u is central in $\mathcal{K}$, its conjugacy class is a singleton, so $g = d_u$. The displayed induced product decomposition consequently makes N one of $F_a \boxtimes F_{a+u}$.

Every module with twisting label d_u occurs in $W(u) \otimes W(u)$, whose minimum is $2\rho_u$. Hence $h(N) = 2\rho_u$. The normalizer translations act transitively on the pairs $(a,\, a+u)$ and preserve conformal weight, proving the equality for every pair. For $u = 0$, the equality follows from $F_a^2 = U$, since D has exponent two. □

4.2 The Freudenthal coordinates

Define

$$\mathcal{E} = \mathbb{F}_2 \oplus J \oplus \mathbb{F}_2 \oplus J^*, \qquad Q(\alpha,\, a;\, \beta,\, B) = \alpha\beta + \langle B, a\rangle.$$

Its polar form is denoted by $B_{\mathcal{E}}$. For $a \in J$, put

$$f_a = (1,\, a;\, N(a),\, a^{\#}). \tag{4.4}$$

Euler's identity for a cubic in characteristic two gives $\langle a^{\#}, a\rangle = N(a)$. Cubic expansion then gives

$$Q(f_a) = 0, \qquad B_{\mathcal{E}}(f_a,\, f_b) = N(a+b). \tag{4.5}$$

On the fusion group, Propositions 3.1 and 4.1 give

$$b(F_a,\, F_b) = 4\rho_{a+b} \pmod 2 = N(a+b).$$

Proposition 4.3. *There is a unique linear isometry $\Phi : (D,\, q) \to (\mathcal{E},\, Q)$ sending F_a to f_a. For the character*

$$E_{\beta,B} : \quad d_a\sigma^s \longmapsto (-1)^{\beta s + \langle B,a\rangle}, \tag{4.6}$$

its corresponding simple module has image $(0,\, 0;\, \beta,\, B)$.

Proof. The vectors f_a span $\mathcal{E}$. Indeed, a linear functional vanishing on all of them would give an identically zero Boolean polynomial

$$c + \ell(a) + \langle a^{\#}, v\rangle + dN(a).$$

In the coordinates (3.1), N is a sum of 45 distinct squarefree cubic monomials. Each of its 27 derivatives has five quadratic monomials, and these 135 monomials are mutually distinct. The derivative polynomials are therefore independent. Constant, linear, quadratic, and cubic squarefree monomials are independent as functions on $\mathbb{F}_2^{27}$, so all coefficients vanish.

Because the form on $\mathcal{E}$ is nondegenerate, the Gram matrix of the f_a has rank 56. The equal Gram matrix of the F_a forces them to span the 56-dimensional space D. By equality of the Gram matrices, a relation among one family gives a linear combination of the other that is orthogonal to every member of that family. Nondegeneracy forces this combination to vanish. Thus the assignment extends to an isometry of polar forms. Both quadratic forms vanish on the spanning family; their difference is linear and hence zero. This proves existence and uniqueness.

The module indexed by (4.6) pairs with F_a as $\beta + \langle B, a\rangle$. The vector $(0,\, 0;\, \beta,\, B)$ has exactly these pairings with every f_a. Nondegeneracy proves the final assertion. □

4.3 The weight-two locus

For rank-one u, put

$$\beta_u(a) = \langle a^{\#}, u\rangle, \qquad H_u(a) = (a+u)^{\#} + a^{\#} + u^{\#}.$$

Since $u^{\#} = 0$, cubic expansion yields

$$f_a + f_{a+u} = (0,\, u;\, \beta_u(a),\, H_u a). \tag{4.7}$$

Lemma 4.4. *For every nonzero rank-one u, the linear map H_u has rank 10, and β_u is constant on its fibers. Consequently (4.7) gives exactly 2^{10} distinct vectors as a varies.*

Proof. The $E_6(2)$-action preserves N and is transitive on rank-one vectors. It suffices to take $u = (E_{11},\, 0,\, 0)$ in (3.1). Direct differentiation gives

$$\beta_u(X,\, Y,\, Z) = x_{22}x_{33} + x_{23}x_{32} + y_{11}z_{11} + y_{12}z_{21} + y_{13}z_{31}. \tag{4.8}$$

This consists of five disjoint hyperbolic pairs; its remaining seventeen coordinates form a radical on which it is identically zero. Its polar map is H_u, proving the claims for this u. Norm invariance gives

$$\beta_{gu}(ga) = \beta_u(a), \qquad H_{gu}(ga) = (g^{-1})^* H_u(a),$$

which transports the assertions to every rank-one vector. □

By Proposition 4.1, all 1024 corresponding distinct U-modules have weight two. The full d_u-twisted module has a lowest-weight space of dimension

$$\dim\bigl(W(u)_1 \otimes W(u)_1\bigr) = 32^2 = 1024.$$

Each of these distinct modules contributes at least one line to this space. Equality proves that they exhaust it and that each has a one-dimensional lowest-weight space.

Proposition 4.5. *Every nonvacuum simple U-module has lowest weight at least two. The images under Φ of the labels of weight-two modules form the disjoint union*

$$\mathcal{B}_{\mathrm{unt}} = \{(0,\, 0;\, 1,\, 0)\} \cup \{(0,\, 0;\, \beta,\, B) : B \in S,\ \beta \in \mathbb{F}_2\}, \tag{4.9}$$

$$\mathcal{B}_{\mathrm{diag}} = \{(0,\, u;\, \beta_u(a),\, H_u a) : u \neq 0,\ u^{\#} = 0,\ a \in J\}, \tag{4.10}$$

$$\mathcal{B}_{\mathrm{perm}} = \{f_a : a \in J\}. \tag{4.11}$$

Put $\mathcal{B} = \mathcal{B}_{\mathrm{unt}} \sqcup \mathcal{B}_{\mathrm{diag}} \sqcup \mathcal{B}_{\mathrm{perm}}$. The three sets have respective sizes

$$279007, \qquad 142851072, \qquad 134217728,$$

and their total size is 277347807.

Proof. The ordinary module V has $V_2 = W_2 \otimes \mathbf{1} + \mathbf{1} \otimes W_2$. Its σ-eigenspaces and (2.2) give (4.9), with the vacuum constituent omitted. The preceding dimension argument gives exactly (4.10) for rank-one diagonal twisting labels. The other diagonal minima are 3 and 7/2. Every $V(p_a)$ has a one-dimensional weight-two subspace, giving (4.11). The ordinary module V has no weight-one vectors, and every nontrivially twisted module has minimum at least two. Completeness of their irreducible U-constituents excludes additional cases. □

We call $\mathcal{B}$ the *obstruction set* and also view it as a subset of D via Φ^{-1}.

4.4 The Freudenthal variety

Put $k = \overline{\mathbb{F}_2}$. Let

$$\Omega = \overline{\{[1,\, a;\, N(a),\, a^{\#}] : a \in J \otimes_{\mathbb{F}_2} k\}} \;\subset\; \mathbb{P}(\mathcal{E} \otimes_{\mathbb{F}_2} k), \tag{4.12}$$

where the bar denotes Zariski closure over k. The cubic (3.1) is the norm of the split Albert algebra: it is the first Tits construction for $M_3(\mathbb{F}_2)$ with parameter 1; see [GPR23], (12.3) and Example 12.2(b). The associated cubic embedding (4.12) is the standard open chart of the Freudenthal variety E_7/P_7 in the minuscule 56-dimensional representation. For the cubic construction, its equivalence with the weight-coordinate model, and the small-field convention, see [DSSV24], Definition 10.1, Theorem 10.4, Remark 10.5 and Section 10.4. The associated split group of type E_7 over arbitrary base rings is described in [GPR23], Sections 16–17, especially Theorem 17.4. To compare with the vector-valued Jordan adjoint, write $a^{\#_J} \in J$ for that adjoint and T_J for the nondegenerate trace pairing of the Albert algebra. The identity

$$T_J(a^{\#_J},\, -) = dN_a$$

is [GPR23], (6.4), valid in all scalar extensions. Transporting the cited construction through this trace pairing therefore gives exactly our covector convention $a^{\#} = dN_a$.

In particular, Ω is defined over $\mathbb{F}_2$. A nonzero vector whose projective class belongs to Ω will be called a *Freudenthal rank-one vector*.

Proposition 4.6. *The obstruction set $\mathcal{B}$ of Proposition 4.5 is exactly the set of rational Freudenthal rank-one vectors. Equivalently, identifying a rational projective point with its unique nonzero $\mathbb{F}_2$-representative, $\mathcal{B} = \Omega(\mathbb{F}_2)$.*

Proof. The vectors in $\mathcal{B}_{\mathrm{perm}}$ belong to the chart by definition.

Let $0 \neq u \in J$ have Albert rank one. Then $u^{\#} = 0$, and Euler's identity gives $N(u) = 0$. For an indeterminate t, cubic expansion gives the polynomial identities

$$N(a + tu) = N(a) + t\,\langle a^{\#}, u\rangle,$$
$$(a + tu)^{\#} = a^{\#} + tH_u a.$$

Consequently

$$f_{a+tu} = f_a + t(0,\, u;\, \beta_u(a),\, H_u a).$$

The point at infinity of this projective line is $[0,\, u;\, \beta_u(a),\, H_u a]$. It belongs to Ω, proving the assertion for $\mathcal{B}_{\mathrm{diag}}$.

Let $B_0 \in J^*$ be the coordinate covector extracting x_{33}. It is a weight vector of the dual minuscule E_6-module and belongs to its rank-one orbit S; compare the three-matrix model and the duality in [CC88]. For $\varepsilon \in \mathbb{F}_2$, take

$$a(t) = (\mathrm{diag}(t,\, t,\, \varepsilon),\, 0,\, 0).$$

Then

$$N(a(t)) = \varepsilon t^2, \qquad a(t)^{\#} = (\mathrm{diag}(\varepsilon t,\, \varepsilon t,\, t^2),\, 0,\, 0),$$

where the second expression lists covector coordinates in the dual coordinate basis. The leading term of $f_{a(t)}$, after multiplication by t^{-2}, gives the point $[0,\, 0;\, \varepsilon,\, B_0]$ at infinity. The $E_6(2)$-action

$$(\alpha,\, a;\, \beta,\, B) \longmapsto (\alpha,\, ga;\, \beta,\, (g^{-1})^* B)$$

preserves the chart and is transitive on S. Hence all the points $[0,\, 0;\, \varepsilon,\, B]$, $B \in S$, lie in Ω. Finally, the curve $a(t) = (tI_3,\, 0,\, 0)$, scaled by t^{-3}, gives $[0,\, 0;\, 1,\, 0]$. Thus $\mathcal{B} \subset \Omega(\mathbb{F}_2)$.

The transitive $E_7(2)$-action has point stabilizer $P_7(2)$, of order $2^{27}|E_6(2)|$. The split-group order formulas give

$$|\Omega(\mathbb{F}_2)| = \frac{|E_7(2)|}{2^{27}|E_6(2)|} = (2^5+1)(2^9+1)(2^{14}-1) = 277347807.$$

This equals $|\mathcal{B}|$ by Proposition 4.5, so the containment is an equality. □

5 An extremal self-dual extension

We now construct one maximal totally singular subspace avoiding $\mathcal{B}$. All varieties and subspaces in this section are defined over $\mathbb{F}_2$, unless otherwise stated.

5.1 The extension criterion

Corollary 5.1 (Extremality criterion)**.** *For any maximal totally singular subspace $H \subset D$, the simple-current extension*

$$V_H = \bigoplus_{d\in H} U_d$$

is strongly rational and self-dual. It is extremal if and only if $H \cap \mathcal{B} = \varnothing$.

Proof. Following [Höh03], Section 4, we use the self-dual extension associated with H. The strongly rational existence theorem ([vEMS20], Theorem 4.2), the VOA form of the Lagrangian-subgroup construction ([DMNO13], Section 5.3), applies: U is strongly rational, its nonvacuum irreducible modules have positive lowest weight, and H is isotropic and self-perpendicular. Every included module has integral lowest weight. By (2.3), the vacuum summand contributes no weight one and only the conformal vector in weight two. Proposition 4.5 lists exactly the nonvacuum integral-weight summands which contribute in these degrees. Thus V_H is extremal exactly when none is included. □

5.2 The exterior-power model

Let $E = \mathbb{F}_{256}$, viewed for the moment simply as an eight-dimensional $\mathbb{F}_2$-space. Put

$$\mathcal{M} = \Lambda^2 E^* \oplus \Lambda^2 E, \qquad Q_8(X, Y) = \langle X, Y\rangle. \tag{5.1}$$

The pairing in (5.1) is the natural perfect pairing between the two exterior powers. In a basis, X and Y are represented by alternating 8×8 matrices, and

$$Q_8(X, Y) = \sum_{0\le i<j\le 7} X_{ij}Y_{ij}.$$

In particular, this quadratic form is not $\mathrm{tr}(XY)$, which vanishes identically in characteristic two.

The standard subgroup of type A_7 in the split group of type E_7 gives its minuscule representation in the form (5.1), with the natural $\mathrm{SL}(E)$-action on $\Lambda^2 E$ and its dual on $\Lambda^2 E^*$. In the usual Euclidean realization of the weight lattice, the vectors $e_i \wedge e_j$ and $e^i \wedge e^j$ have weights $\lambda_{ij} = \varepsilon_i + \varepsilon_j - \frac{1}{4}\sum_k \varepsilon_k$ and $-\lambda_{ij}$, respectively. Let Ω_8 be the closed E_7-orbit through $[0,\, e_0 \wedge e_1]$, its highest-weight variety. We use the unique nonzero volume form over $\mathbb{F}_2$.

Lemma 5.2. *There is an $\mathbb{F}_2$-linear isomorphism*

$$\Psi : \mathcal{E} \longrightarrow \mathcal{M}$$

of the two split minuscule E_7-models which carries Ω to Ω_8. Every such isomorphism is an isometry from Q to Q_8.

Proof. The cubic and weight-coordinate constructions cited above are models over $\mathbb{F}_2$ of the same split minuscule representation. Thus an equivariant isomorphism can be chosen over $\mathbb{F}_2$; it carries the highest-weight variety to the highest-weight variety.

We check the quadratic normalization after extending scalars to k. The minuscule E_7-module in characteristic two has a nonzero invariant quadratic form, unique up to a scalar; see [GG15], Remark 5.5. In the cubic model, the algebraic E_6-Levi and its central torus force such a form to be

$$c_1\alpha\beta + c_2\langle B, a\rangle.$$

Indeed, the torus weights on the four coordinates $(\alpha,\ a;\ \beta,\ B)$ can be taken to be $(-3,\ -1,\ 3,\ 1)$, and the 27-dimensional modules are irreducible duals. The invariant form vanishes on the highest-weight line by torus invariance, and hence on Ω. Evaluating it on f_a gives

$$(c_1 + c_2)N(a) = 0.$$

Thus $c_1 = c_2 \neq 0$, and this form is a scalar multiple of Q.

In the exterior-power model, invariance under the algebraic SL_8-subgroup similarly leaves only a scalar multiple of the natural pairing Q_8. Consequently Ψ preserves the two forms up to a nonzero scalar. Since Ψ, Q, Q_8 are defined over $\mathbb{F}_2$, that scalar belongs to $\mathbb{F}_2^\times = \{1\}$. □

We need only two consequences of the quadratic equations of Ω_8.

Lemma 5.3. *Over any extension field $K/\mathbb{F}_2$, every representative $(X,\ Y)$ of a point of $\Omega_8(K)$ satisfies*

$$XY = \lambda I_8 \quad \textit{for some } \lambda \in K, \qquad \operatorname{rank} X \leq 2 \iff \operatorname{rank} Y \leq 2. \tag{5.2}$$

Proof. In a volume-one basis, the root-square equations are

$$(XY)_{ij} = 0 \quad (i \neq j), \qquad \mathrm{Pf}(X_I) = \mathrm{Pf}(Y_{I^c}) \quad (|I| = 4).$$

Here I ranges over four-subsets of the eight basis indices, and X_I denotes the corresponding principal submatrix. These are the $56 + 70 = 126$ integral root quadrics of [VL16], Section 4, reduced modulo two; see also [DSSV24], Sections 10.2.1–10.2.2 and Theorem 10.6. For the root $\varepsilon_i - \varepsilon_j$, the six weight pairs give $\sum_{k\neq i,j} Y_{ik}X_{jk} = (XY)_{ji}$. For $\alpha_I = \sum_{i\in I}\varepsilon_i - \frac{1}{2}\sum_k \varepsilon_k$, $|I| = 4$, they give the three terms of $\mathrm{Pf}(Y_I)$ and the three terms of $\mathrm{Pf}(X_{I^c})$. All integral signs become plus signs modulo two, yielding the displayed equations after reindexing.

The $\mathrm{SL}(E)$-invariance of Ω_8 shows that XY is diagonal after every elementary change of basis. Under such a change, XY is conjugated. A diagonal matrix whose conjugates by all elementary transvections are diagonal has equal diagonal entries. Hence XY is scalar. Finally, an alternating matrix has rank at most two exactly when all its principal 4×4 Pfaffians vanish. The complementary-Pfaffian equations give the rank equivalence. □

5.3 The finite-field construction

Using multiplication in E, define the $\mathbb{F}_2$-linear map

$$\mu : \Lambda^2 E \longrightarrow E, \qquad \mu(u \wedge v) = uv(u + v). \tag{5.3}$$

It is well defined because $uv(u + v) = u^2v + uv^2$ is alternating and $\mathbb{F}_2$-bilinear. Use the field-trace pairing to identify E^* with E. The transpose of μ is then

$$\mu^*(a) = b_a, \qquad b_a(u,\ v) = \mathrm{Tr}_{E/\mathbb{F}_2}\big(a\,uv(u + v)\big). \tag{5.4}$$

Theorem 5.4. *The subspace*

$$\mathcal{H} = \operatorname{im}\mu^* \oplus \ker\mu \subset \Lambda^2 E^* \oplus \Lambda^2 E \tag{5.5}$$

is maximal totally singular for Q_8, and

$$\mathbb{P}(\mathcal{H})(\mathbb{F}_2) \cap \Omega_8(\mathbb{F}_2) = \varnothing.$$

Proof. For $a \neq 0$, trace invariance gives

$$b_a(u,\, v) = \operatorname{Tr}_{E/\mathbb{F}_2}((au^2 + a^{128}u^{128})v).$$

Nondegeneracy of the trace pairing implies that every radical vector u satisfies

$$0 = (au^2 + a^{128}u^{128})^2 = a^2u^4 + au.$$

This nonzero polynomial has degree four, so the radical has at most four elements and $\operatorname{rank} b_a \geq 6$. In particular μ^* is injective. Thus $\dim\operatorname{im}\mu^* = 8$ and $\dim\ker\mu = 20$. Since $\ker\mu$ annihilates $\operatorname{im}\mu^*$, the 28-dimensional space $\mathcal{H}$ is totally singular, and hence maximal.

Suppose that a nonzero rational vector $(X,\, Y) \in \mathcal{H}$ represents a point of Ω_8. Write $X = b_a$. By Lemma 5.3, $XY = \lambda I_8$. We show that $\lambda = 0$. After extending scalars to E, use the coordinates $x_i(u) = u^{2^i}$, with $i \in \mathbb{Z}/8$. They form a basis of $(E \otimes_{\mathbb{F}_2} E)^*$ by separability. Expanding the trace gives

$$X = \sum_{i\in\mathbb{Z}/8} a^{2^i}\, x_i \wedge x_{i+1}. \tag{5.6}$$

For the rational bivector $Y = \sum_r u_r \wedge v_r$, its coefficients in the dual basis satisfy

$$\begin{aligned} Y_{i,i+1} &= \sum_r (u_r^{2^i} v_r^{2^{i+1}} + v_r^{2^i} u_r^{2^{i+1}}) \\ &= \mu(Y)^{2^i} = 0. \end{aligned}$$

Here the indices are taken modulo eight, using $u^{256} = u$ for $u \in E$. Thus X is supported on the edges of an eight-cycle and Y vanishes on those edges. Every diagonal entry of XY is zero, so $\lambda = 0$.

If $X \neq 0$, the rank bound and $XY = 0$ give $\operatorname{rank} Y \leq 2$. Lemma 5.3 then gives $\operatorname{rank} X \leq 2$, a contradiction. If $X = 0$, the same lemma gives $\operatorname{rank} Y \leq 2$. Since $Y \neq 0$ is rational, it has the form $u \wedge v$ for $\mathbb{F}_2$-linearly independent u and v in E. But $Y \in \ker\mu$ gives $uv(u + v) = 0$, impossible in a field. □

Choose an isomorphism Ψ of the two split E_7-models as in Lemma 5.2, and set

$$H = (\Psi \circ \Phi)^{-1}(\mathcal{H}) \subset D. \tag{5.7}$$

Proof of Theorem 1.2. Lemma 5.2, Proposition 4.6, and Theorem 5.4 show that H is maximal totally singular and avoids $\mathcal{B}$. The extremality criterion, Corollary 5.1, therefore gives the strongly rational self-dual extremal extension $V_H = \bigoplus_{d\in H} U_d$. □

6 The automorphism group

Lemma 6.1. *The setwise stabilizer of $\mathcal{B}$ in $\mathrm{GL}_{\mathbb{F}_2}(\mathcal{E})$ is $E_7(2)$ in its 56-dimensional minuscule representation.*

Proof. By Proposition 4.6, $\mathcal{B} = \Omega(\mathbb{F}_2)$. Let $I_2(\mathcal{B})$ be the space of homogeneous quadratic forms vanishing on $\mathcal{B}$, and put $k = \overline{\mathbb{F}_2}$. We first show that $I_2(\mathcal{B}) \otimes_{\mathbb{F}_2} k = I_2(\Omega)$. In the minuscule representation every root subgroup has the form $x_\alpha(t) = 1 + tN_\alpha$, with N_α defined over $\mathbb{F}_2$ ([VL16], Lemma 2). For $v \in \mathcal{B}$, the vector $N_\alpha v$ is zero or represents the point at infinity of the root-group line $[v + tN_\alpha v] \subset \Omega$; hence $N_\alpha v \in \mathcal{B} \cup \{0\}$. Also $v + N_\alpha v = x_\alpha(1)v \in \mathcal{B}$. A quadratic form vanishing at v, $N_\alpha v$, $v + N_\alpha v$ vanishes at $v + tN_\alpha v$ for every $t \in k$. Thus $I_2(\mathcal{B}) \otimes k$ is stable under all root subgroups of $E_7(k)$. Since it vanishes at a rational highest-weight vector, it vanishes on its full projective orbit Ω, proving the equality.

Restriction of quadrics to Ω surjects onto $H^0(\Omega, \mathcal{O}_\Omega(2))$ ([BL03], Corollary 4(1)); the Weyl dimension formula for $2\lambda_7$ gives dimension 1463 for this space. Consequently

$$\dim I_2(\mathcal{B}) = \binom{57}{2} - 1463 = 133.$$

The 133 quadratic equations in [VL16], Section 4, vanish on Ω and remain linearly independent in characteristic two ([VL16], Lemma 4). They therefore span $I_2(\mathcal{B})$. Every linear transformation preserving $\mathcal{B}$ preserves the ideal they generate, whose stabilizer is the extended Chevalley group by [VL16], Theorem 1. Over $\mathbb{F}_2$ this is $E_7(2)$, since the additional torus has trivial rational points ([VL16], Section 2). The reverse inclusion follows from the minuscule action on $\Omega(\mathbb{F}_2)$. □

Theorem 6.2. *The natural action of* $\mathrm{Aut}(U)$ *on its fusion group is faithful. Under the isometry* Φ *of Proposition 4.3, its image is the split group* $E_7(2)$ *in its* 56*-dimensional minuscule representation. In particular,*

$$\mathrm{Aut}(U) \cong E_7(2).$$

Proof. Faithfulness and the upper bound. Put $V = W \otimes W$ and $\Gamma = \mathrm{Aut}(W)$. By Section 3, V is a simple-current extension of U with support $\ker \nu = G^* \subset D$; the dual grading group is the original group G. The centralizer exact sequence for simple-current extensions ([Shi07], Corollary 2.2 and (2.2)) therefore applies.

Write

$$\zeta_1 = \omega_W \otimes \mathbf{1}, \qquad \zeta_2 = \mathbf{1} \otimes \omega_W, \qquad v = \zeta_1 - \zeta_2.$$

Since $W_1 = 0$ and $W_2^A = \mathbb{C}\omega_W$, the ΔA-fixed subspace of V_2 is $\mathbb{C}\zeta_1 \oplus \mathbb{C}\zeta_2$, and its (-1)-eigenspace under σ is $\mathbb{C}v$. For $g \in C_{\mathrm{Aut}(V)}(G)$, we thus have $g(v) = \lambda v$. The identity $v_1 v = 2(\zeta_1 + \zeta_2)$ gives $\lambda^2 = 1$. Hence g fixes or interchanges ζ_1 and ζ_2. The identities

$$W \otimes \mathbf{1} = \ker(\zeta_2)_1, \qquad \mathbf{1} \otimes W = \ker(\zeta_1)_1$$

show that g fixes or interchanges the tensor factors. Thus $g = (b_1, b_2)\sigma^\epsilon$ for b_1 and b_2 in Γ. Commutation with σ gives $b_1 = b_2 = b$, and commutation with ΔA gives $b \in C_\Gamma(A) = A$ ([Shi14], Proposition 2.18). Therefore

$$C_{\mathrm{Aut}(V)}(G) = G. \tag{6.1}$$

An automorphism of U acting trivially on D fixes the isomorphism class of every G-eigenspace in V. The centralizer exact sequence lifts it to $C_{\mathrm{Aut}(V)}(G) = G$, whose restriction to U is trivial. This proves faithfulness. Every automorphism of U preserves fusion and conformal weights, and hence q and $\mathcal{B}$. Lemma 6.1 now gives $\mathrm{Aut}(U) \le E_7(2)$.

A maximal parabolic subgroup. The subgroup

$$N_0 = (A \times 1)\Delta\Gamma \le \mathrm{Aut}(V)$$

normalizes G. Indeed, diagonal automorphisms normalize ΔA and commute with σ, while $(a,\,1)\sigma(a,\,1)^{-1} = d_a\sigma$ for $a \in A$. By the normalizer exact sequence ([Shi07], Corollary 2.2 and (2.1)), the kernel of restriction $N_0 \to \mathrm{Aut}(U)$ is $N_0 \cap G = \Delta A$. Its image P is therefore

$$P \cong N_0/\Delta A \cong A \rtimes (\Gamma/A) \cong 2^{27}{:}E_6(2).$$

The images of $A \times 1$ and $\Delta\Gamma$ have trivial intersection, and the second image is Γ/A. The class $E_{1,0}$ in (4.6), which contains v, is invariant under P. Thus P fixes $f_\infty = (0,\,0;\,1,\,0) \in \mathcal{B}$. The point stabilizer in $E_7(2)$ has order $2^{27}|E_6(2)|$, so $P = P_7(2)$. In the standard BN-pair, every subgroup containing a Borel subgroup is parabolic. Since $P_7(2)$ corresponds to omitting one simple root, P is maximal in $E_7(2)$.

A four-factor permutation. Return to $R = V^+_{\mathrm{BW}_{16}}$ and the diagonal decomposition (3.4). Put $\mathcal{T} = S_{10}^* \le A$, the group acting by characters on that decomposition, and let

$$\rho : N_\Gamma(\mathcal{T}) \longrightarrow \mathrm{Aut}(R^{\otimes 2})$$

be restriction. Define

$$C_R = \ker\bigl(\mathrm{Aut}(R) \longrightarrow \mathrm{Aut}(S_{10})\bigr), \qquad A_0 = \rho^{-1}(\Delta C_R).$$

Shimakura's descriptions give

$$C_R \cong 2^{16}, \qquad \ker\rho = \mathcal{T}, \qquad A = \langle A_0,\,\iota\rangle$$

([Shi14], Proposition 1.8(4)–(5), (2.2) and (2.4)), where ι is the literal flip already used in Section 3.

Use the same representative R_s in every tensor position. The tensor-product theorem for intertwining operators ([ADL05], Theorem 2.10) and fusion multiplicity one give each component of the vertex operator of W in the form $c_{s,t}I_{s,t}^{\otimes 2}$, where $I_{s,t}$ is an intertwining operator from R_s, R_t to R_{s+t}. It follows that

$$Z := V^{\Delta\mathcal{T}} = \bigoplus_{s\in S_{10}} R_s^{\otimes 4}$$

has corresponding components $c_{s,t}^2 I_{s,t}^{\otimes 4}$. Literal permutations of the four positions preserve these operators, the vacuum and the conformal vector. They therefore act on Z by VOA automorphisms.

For $a \in A_0$ with $\rho(a) = (c,\,c)$, choose an implementing c-intertwiner $\phi_{c,s}$ on R_s. It exists because C_R fixes each module class. Schur's lemma gives

$$a|_{R_s^{\otimes 2}} = \lambda_{a,s}\phi_{c,s}^{\otimes 2}, \qquad \Delta a|_{R_s^{\otimes 4}} = \lambda_{a,s}^2\phi_{c,s}^{\otimes 4}.$$

Thus the displayed action of ΔA_0 on Z, including its scalar factors, commutes with all four-factor permutations. The image $\overline{G}$ of G on Z is generated by this subgroup and

$$\delta = \Delta\iota = (12)(34), \qquad \sigma = (13)(24).$$

The permutation $\pi = (23)$ centralizes the image of ΔA_0 and interchanges δ and σ. It therefore induces an automorphism t of $Z^{\overline{G}} = U$.

Every element of P fixes $w = \omega_W \otimes \omega_W \in U_4$. Let η_i be the conformal vector in the i-th copy of R, whose conformal vector is ω_R. Then

$$w = (\eta_1 + \eta_2)_{-1}(\eta_3 + \eta_4),$$
$$t(w) = (\eta_1 + \eta_3)_{-1}(\eta_2 + \eta_4).$$

Their components of fourfold weight (2, 2, 0, 0) are respectively zero and $\omega_R \otimes \omega_R \otimes \mathbf{1} \otimes \mathbf{1} \neq 0$. Hence $t \notin P$. Maximality of P now gives

$$E_7(2) = \langle P, t \rangle \leq \mathrm{Aut}(U) \leq E_7(2),$$

as required. □

Changing Ψ in (5.7) conjugates H under $\mathrm{Aut}(U)$, by Lemma 6.1 and Theorem 6.2. Hence the isomorphism class of V_H is independent of this choice ([Shi07], Theorem 2.1).

The identities $\mu(cu \wedge cv) = c^3\mu(u \wedge v)$ and $\mu(u^2 \wedge v^2) = \mu(u \wedge v)^2$ show that $\Gamma L_1(E) \cong 255{:}8 \leq \mathrm{SL}(E)$ preserves $\mathcal{H}$. Its inverse image under the normalizer restriction map has kernel $H^* \cong 2^{28}$, so $\mathrm{Aut}(V_H)$ contains a subgroup of shape $2^{28}.(255{:}8)$ ([Shi07], Corollary 2.2 and (2.1)).

7 Characters

Identify the fusion space with $(\mathcal{E}, Q)$ by Proposition 4.3, and write b for its polar form. In this section $q = e^{2\pi i\tau}$ is the character variable. For $d \in D$, put

$$\chi_d(\tau) = \mathrm{tr}_{U_d}\, q^{L_0 - 8/3}, \qquad C_d(q) = q^{8/3}\chi_d(\tau).$$

Thus the exponents of C_d are absolute conformal weights. We determine these functions from the Barnes–Wall trace functions.

Let r_0, r_s, r_n be the graded dimensions $\mathrm{tr}\, q^{L_0}$ of the vacuum, a nonzero singular, and a nonsingular $R = V^+_{\mathrm{BW}_{16}}$-module, respectively. They are independent of the chosen label of each type by [Shi14], Proposition 1.8. For the two nonvacuum types, the twisted Fock-space character gives

$$\begin{aligned} p(t) &= \prod_{m\geq 1} (1 - t^{2m-1})^{-16}, \\ r_\mathrm{s}(q) &= 8\,q\big(p(\sqrt{q}) + p(-\sqrt{q})\big), \qquad r_\mathrm{n}(q) = 8\,q\big(p(\sqrt{q}) - p(-\sqrt{q})\big). \end{aligned} \tag{7.1}$$

Indeed, the twisted lattice module is $M(1)(\theta) \otimes T$, where $M(1)(\theta)$ is the symmetric algebra on the half-integer Heisenberg modes; its involution is oscillator parity ([ADL05], Section 3.1, (3.7) and Proposition 3.2). Its lowest-weight multiplicity is $\dim T = 16$, by the singular lowest-weight dimension recorded after (3.3). Thus its graded dimension is $16\,q\,p(\sqrt{q})$, and its integral and nonintegral parts give the two displayed characters.

Choose $a_j \in A$ of Albert rank j, where $a_0 = 0$, and set $w_j(q) = \mathrm{tr}_W(a_j q^{L_0})$, interpreting a_0 as the identity automorphism. With $x = j(\tau)^{1/3} = E_4/\eta^8$, the character formula ([Höh95], Satz 2.1.2) gives

$$\begin{aligned} w_0(q) &= q^{4/3}(x^4 - 992x), \\ w_1(q) &= r_0(q^2) + 527 r_\mathrm{s}(q^2) + 496 r_\mathrm{n}(q^2), \\ w_2(q) &= r_0(q^2) + 15 r_\mathrm{s}(q^2) - 16 r_\mathrm{n}(q^2), \\ w_3(q) &= r_0(q^2) - 17 r_\mathrm{s}(q^2) + 16 r_\mathrm{n}(q^2). \end{aligned} \tag{7.2}$$

The last three identities are (3.5), with the rank assignment of Proposition 3.1. Equation (3.4) determines the remaining series uniquely by

$$r_0(q)^2 = w_0(q) - 527 r_\mathrm{s}(q)^2 - 496 r_\mathrm{n}(q)^2, \qquad r_0(0) = 1. \tag{7.3}$$

Hence these formulas specify all four trace functions to every order.

Lemma 7.1. *For $B \in J^*$ of rank i, including rank zero, put*

$$P_{ij} = \sum_{\substack{a \in J \\ \mathrm{rank}(a)=j}} (-1)^{\langle B,a\rangle}.$$

Then

$$P = \begin{pmatrix} 1 & 139503 & 69193488 & 64884736 \\ 1 & 8431 & 118544 & -126976 \\ 1 & 239 & -4336 & 4096 \\ 1 & -273 & 4368 & -4096 \end{pmatrix}. \tag{7.4}$$

Proof. The character of the dual rank-one orbit is $\mathrm{tr}_{W_2}(a) - 1$, so its values on the four primal orbits are (139503, 8431, 239, −273). From (7.2), the corresponding traces on W_3 are

$$(69332992,\ 126976,\ -4096,\ 4096).$$

Averaging this character and taking its inner product with the dual rank-one orbit character give multiplicity one on the trivial label and on each rank-one label. The remaining dimension is 69193488. The two remaining dual orbit sizes are 69193488 and 64884736, so nonnegative integral multiplicities force multiplicity one on rank two and zero on rank three. Subtracting the trivial and rank-one characters gives the dual rank-two orbit character; Fourier orthogonality gives the last orbit character. Finally, double counting $\sum_{\mathrm{rank}(a)=j,\,\mathrm{rank}(B)=i}(-1)^{\langle B,a\rangle}$ gives (7.4). This calculation uses only the natural duality $J^* \times J$. □

For a dual rank-i label, the graded dimension of its A-isotypic subspace in W is therefore

$$X_i(q) = 2^{-27} \sum_{j=0}^{3} P_{ij} w_j(q). \tag{7.5}$$

For an integer-power series $F(t)$, write $F^{\mathrm{ev}}(t) = (F(t) + F(-t))/2$ and $F^{\mathrm{odd}}(t) = (F(t) - F(-t))/2$.

Theorem 7.2. *The action of $\mathrm{Aut}(U) = E_7(2)$ on D has six orbits. The orbit $\mathcal{O}_1 = \mathcal{B}$ consists of the Freudenthal rank-one vectors; $\mathcal{O}_2$ and $\mathcal{O}_3$ are the other two nonzero singular orbits. The labels $+$, $-$ distinguish the two nonsingular orbits and are not signs of Q. Their sizes, absolute lowest weights, and lowest-weight multiplicities are as follows.*

Orbit	Size	$h(U_d)$	$\dim(U_d)_{h(U_d)}$
$\mathcal{O}_0 = \{0\}$	1	0	1
$\mathcal{O}_1$	277347807	2	1
$\mathcal{O}_2$	37518401085984	3	1
$\mathcal{O}_3$	35991278474747904	4	52
$\mathcal{O}_+$	18612496260661248	7/2	1
$\mathcal{O}_-$	17416300624084992	9/2	2432

Write C_i for the normalized character of one module with label in $\mathcal{O}_i$, and put $\chi_i(\tau) = q^{-8/3}C_i(q)$. Then

$$\begin{aligned} C_1(q) &= q^2 X_0^{\mathrm{ev}}(\sqrt{q}), & C_+(q) &= q^2 X_0^{\mathrm{odd}}(\sqrt{q}), \\ C_2(q) &= q^2 X_1^{\mathrm{ev}}(\sqrt{q}), & C_-(q) &= q^2 X_3^{\mathrm{odd}}(\sqrt{q}), \\ C_3(q) &= q^2 X_3^{\mathrm{ev}}(\sqrt{q}), & C_0(q) &= C_1(q) + w_0(q^2). \end{aligned} \tag{7.6}$$

Their expansions, after factoring out the leading power of q, are

$$\begin{aligned}
\chi_0(\tau) &= q^{-8/3}(1+q^2+q^3+141988\,q^4+O(q^5)),\\
\chi_1(\tau) &= q^{-8/3+2}(1+q+2484\,q^2+141988\,q^3+26371036\,q^4+O(q^5)),\\
\chi_2(\tau) &= q^{-8/3+3}(1+188\,q+76452\,q^2+25285568\,q^3+3960027018\,q^4+O(q^5)),\\
\chi_3(\tau) &= q^{-8/3+4}(52+72320\,q+25217856\,q^2\\
&\qquad +3959256576\,q^3+363041741728\,q^4+O(q^5)),\\
\chi_+(\tau) &= q^{-8/3+7/2}(1+2484\,q+1529426\,q^2\\
&\qquad +342144728\,q^3+40118653735\,q^4+O(q^5)),\\
\chi_-(\tau) &= q^{-8/3+9/2}(2432+1528320\,q+342129920\,q^2\\
&\qquad +40118506496\,q^3+2979007126400\,q^4+O(q^5)).
\end{aligned} \tag{7.7}$$

The vector $\boldsymbol{\chi}=(\chi_d)_{d\in D}$, where $\chi_d = q^{-8/3}C_d$, satisfies

$$\begin{aligned}
\chi_d(-1/\tau) &= 2^{-28}\sum_{e\in D}(-1)^{b(d,e)}\chi_e(\tau),\\
\chi_d(\tau+1) &= e^{-2\pi i(8/3)}(-1)^{Q(d)}\chi_d(\tau).
\end{aligned} \tag{7.8}$$

Proof. The five vector-stabilizer shapes occurring in the minimal module are listed in [BBR15], Proposition 2.2; their indices sum to $2^{56}-1$, so each gives one nonzero orbit. By Theorem 6.2, the character is constant on each orbit.

Character formulas. The grading and commuting lifts on $V(\sigma)$ were determined in Step 2 of the proof of Proposition 4.1. Their simultaneous eigenspaces are the irreducible U-constituents, by (3.11), and their normalized characters are $q^2X_i^{\mathrm{ev}}(\sqrt{q})$ and $q^2X_i^{\mathrm{odd}}(\sqrt{q})$. Their lowest weights, for $i=0,\,1,\,2,\,3$, are respectively

$$(2,\,3,\,4,\,4),\qquad (7/2,\,7/2,\,7/2,\,9/2),$$

as follows by expanding (7.1)–(7.5). Together with the vacuum these exhibit six different characters, so they distinguish all six orbits. In the coordinates of Proposition 4.3, these constituents have labels $(1,\,0;\,\beta,\,B)$, with β the parity and B of rank i. This proves the first five formulas in (7.6).

The symmetric and antisymmetric parts of $(W\otimes W)^{\Delta A}$ have characters whose difference is $w_0(q^2)$, by the tensor-flip trace formula. The symmetric part is U; the antisymmetric part is the module $E_{1,0}$, which belongs to $\mathcal{B}=\mathcal{O}_1$. This proves the last formula. The displayed character expansions follow by coefficient extraction from these exact identities.

Orbit sizes. Put $b_0=|\mathcal{B}|$, $e=(0,\,0;\,1,\,0)$, and $v_{\beta,B}=(1,\,0;\,\beta,\,B)$. The decomposition in Proposition 4.5 gives, when $\operatorname{rank}(B)=i$,

$$\sum_{x\in\mathcal{B}}(-1)^{b(v_{\beta,B},\,x)} = -1+32P_{i1}+(-1)^{\beta}\sum_{j=0}^{3}P_{ij}(-1)^{[j=3]}. \tag{7.9}$$

The untwisted part contributes -1. For each diagonal rank-one label, (4.8) is a hyperbolic quadratic form of rank ten, whose Gauss sum on its nondegenerate quotient is 32; this gives the middle term. The permutation part gives the last term by (4.4). On the five nonzero orbits, in the order of the table, these Fourier values are

$$\lambda_i = (8912351,\,523743,\,-545,\,15839,\,-16929).$$

The slice $b(e,\,d)=1$ consists of the irreducible constituents of the 2^{27} permutation-twisted modules. Translations r_a identify their character distributions. The intersections of this slice with the five orbits therefore have sizes $2^{27}k_i$, where

$$k_i = (1,\,139503,\,134078224,\,69332992,\,64884736).$$

Count pairs $(x,\,d)\in\mathcal{B}\times\mathcal{O}_i$ with $b(x,\,d)=1$. Transitivity on $\mathcal{B}$ gives

$$|\mathcal{O}_i|\,\frac{b_0-\lambda_i}{2} = b_0\,2^{27}k_i.$$

This yields the stated orbit sizes.

Modular transformations. The ribbon equivalence (3.11) identifies the categorical S-matrix with the Fourier matrix of $(D,\,Q)$, with normalization $|D|^{-1/2}=2^{-28}$. The modular transformation theorem ([vEMS20], Theorem 3.4) gives the first identity in (7.8); the second follows from $h(U_d)\equiv Q(d)/2 \pmod{\mathbb{Z}}$ and $c(U)=64$. □

The first nonvacuum Virasoro highest-weight vectors of U occur in weight four and span a 141986-dimensional space. This space affords the smallest nontrivial irreducible complex representation of $E_7(2)$, whose degree is listed in [LübE7].

The group of $\mathbb{F}_2$-linear permutations of D preserving all the functions χ_d is $E_7(2)$. Theorem 6.2 gives one inclusion. Conversely, preservation of the characters preserves the weight-two locus $\mathcal{O}_1=\mathcal{B}$, whose linear stabilizer is $E_7(2)$ by Lemma 6.1.

Proof of Theorem 1.1. Combine (2.3), Proposition 3.4, Proposition 4.6, and Theorems 6.2 and 7.2. □

The extremal extension. By my thesis ([Höh95], Satz 2.1.2 and (5.2)), the character of any extremal self-dual VOA of central charge 64 is a polynomial in $x=j(\tau)^{1/3}=E_4/\eta^8=q^{-1/3}(1+248\,q+4124\,q^2+\cdots)$. The extremal initial coefficients give

$$\begin{aligned} Z_{V_H}(\tau) &= \operatorname{tr}_{V_H} q^{L_0-8/3} = x^8 - 1984x^5 + 705057x^2 \\ &= q^{-8/3}(1+q^2+278512\,q^3+13996663144\,q^4+O(q^5)). \end{aligned} \tag{7.10}$$

In particular, the weight-three Virasoro highest-weight space has dimension 278511.

For any maximal totally singular H avoiding $\mathcal{O}_1$, the only nonzero orbits meeting H are $\mathcal{O}_2$ and $\mathcal{O}_3$. The weight-three coefficient and $|H|=2^{28}$ therefore give

$$|H\cap\mathcal{O}_2| = 278511, \qquad |H\cap\mathcal{O}_3| = 268156944,$$

and hence

$$Z_{V_H} = \chi_0 + 278511\,\chi_2 + 268156944\,\chi_3.$$

Here $\chi_i = q^{-8/3}C_i$, with C_i as in Theorem 7.2.

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